Ramsey taxation and fear of misspecification

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Abstract

A Ramsey planner knows that a representative agent distrusts a probability model for exogenous sequences of government expenditures. The representative agent expresses ambiguity aversion by using multiplier preferences of Maccheroni et al. (2006a,b) to order consumption, leisure plans. Because equilibrium prices embed the representative agent's worst case model, the planner has an incentive to manipulate them. We provide a recursive representation of the planner's problem and show how the representative agent's ambiguity aversion puts history dependence into taxes, allocations, and government debt. We use expansions in a single robustness parameter to illustrate the impacts of ambiguity aversion on the Ramsey plan.

KEY WORDS: Ramsey plan, misspecification, robustness, optimal expectations, taxes, debt, martingale.

1   Introduction

This paper withdraws complete confidence in the probability model describing histories of government expenditures from the representative household of Lucas and Stokey (1983). A Ramsey planner still trusts that probability model. The household has a set of probability measures and ranks consumption plans according to a max-min expected utility criterion as in Gilboa and Schmeidler (1989). That change in Lucas and Stokey's model alters the behavior of Ramsey allocations, taxes, and debt in promising ways for applications.

The representative household's max-min expected utility behavior builds its worst-case beliefs about probabilities into equilibrium Arrow-Debreu prices. The Ramsey planner affects taxes and prices partly by manipulating the worst-case beliefs of the household. The Ramsey planner becomes a Stackelberg leader who plays against *two* followers, the maximizing part of the representative household and the household's malevolent alter ego who chooses its worst case probability measure. Relative to the planner in Lucas and Stokey (1983), the Ramsey planner faces additional implementability constraints that summarize the activities of the household's malevolent partner.

The absence of history dependence in allocations, tax rates, and government debt is a salient feature of the Ramsey plan of Lucas and Stokey (1983). For example, with Markov government expenditures, the value of government debt at date t depends only on the date t value of the Markov state driving government expenditures. Lucas and Stokey failed to rationalize the permanent-income like predictions of Barro (1979) that put extensive history dependence into tax rates and government debt. The impression that observed time series of government debt and taxes have actually exhibited history dependence – observed series on government debt are much smoother series than the Lucas-Stokey model and more like those in Barro's model – prompted Aiyagari et al. (2002) and Battaglini and Coate (2007) to put history dependence into a Ramsey plan, in the model of Aiyagari et al. (2002), or a political-economic bargaining equilibrium, in the model of Battaglini and Coate (2007), by dropping Lucas and Stokey's assumption of complete markets. In this paper, we retain the assumption of complete markets, but find that the government manipulates the representative household's beliefs in ways that induce history dependence in allocations, tax rates, and government debt.

The assumption of rational expectations attributes a unique and fully trusted model to all agents. That assumption precludes carrying out a coherent analysis that attributes fears of misspecification to some or all agents. There remain many open questions about fruitful ways to formulate problems where some agents doubt the common model attributed to them under rational expectations. In this paper, we adopt a particular formulation of the problem that is designed to isolate the influence of the household's beliefs on equilibrium prices and, since prices and allocation reveal them, on the Ramsey planner's incentives to manipulate those beliefs.

Other contributions that share our aim of attributing misspecification fears to at least some agents include Kocherlakota and Phelan (2006), who study a mechanism design problem using a max-min expected utility criterion and Hansen and Sargent (2007, ch. 16), who formulate a linear-quadratic model in which a Stackelberg leader distrusts an approximating model while assuming that a competitive fringe of followers completely trusts it.[1] Hansen and Sargent's assumptions about the leader's and followers' concerns about misspecification in effect reverse the ones made here. In several ways, Woodford (2005) is the most interesting previous paper for us because he also uses a general equilibrium model and because of the subtle way that Woodford chose to set up the timing of events to conceal the private sector's beliefs from the government, which plays the role of Stackelberg leader in his model. In

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[1]Our work is also linked in a general sense to that of Brunnermeier et al. (2007), who study a setting in which households choose their beliefs.

Woodford's model, while both the government and the private sectors fully trust their own models, the government distrusts its knowledge of the private sector's beliefs about prices. Arranging things so that this is possible is subtle because with enough markets, equilibrium prices and allocations reveal private sector beliefs. In contrast to Woodford, we set things up with complete markets with market prices that fully reveal private sector beliefs, a feature that the Ramsey planner recognizes and exploits.

Any analysis of agents' fear of model misspecification requires a way to express those fears by representing a set of alternative models that agents think might govern the data. Along with Woodford (2005), this paper uses an approach of Hansen and Sargent (2005, 2006) and Hansen et al. (2006) that uses martingales with respect to an approximating model to define likelihood ratios that represent model misspecifications that can be difficult to detect statistically. These martingale perturbations look like multiplicative preferences shocks. In the present context, the Ramsey planner has the ability and the motives to manipulate those 'shocks'.

It would be natural to attribute specification fears to the Ramsey planner as well as to the household. We refrain from doing so in this paper in order to highlight some economic forces while keeping the analysis as simple as possible. Formulating the problem in its present form has taught us how to formulate models that also attribute specification concerns to the planner, but we postpone presenting that work to another paper.[2]

In section 2, we modify the model of Lucas and Stokey (1983) by having a representative household that entertains its fear of misspecification using the multiplier preferences of Hansen and Sargent (2005, 2006). In section 3, we characterize the Ramsey problem, showing how fear of misspecification induces history dependence in the optimal allocation, taxes, and government debt. In section 4, we identify the appropriate state variables and formulate the Ramsey problem recursively by applying the techniques of Marcet and Marimon (1998). Section 5 provides a simple example with a random fiscal shock at an arbitrary period in an otherwise deterministic environment. In section 6, we employ an expansion around the no-robustness case and exhibit the effects of concerns about misspecification on the Ramsey plan for quasi-linear utility and for the utility function of Aiyagari et al. (2002). Section 7 concludes.

2   The economy

We adopt the same physical specification of the economy made by Lucas and Stokey (1983) and Aiyagari et al. (2002). Time $t \geq 0$ is discrete and the horizon infinite. Labor is the only input into a linear technology that produces one perishable good that can be allocated to private consumption c_t or government consumption g_t. Markets are complete and competitive. The only source of uncertainty is an exogenous sequence of government expenditures g_t that potentially takes on a finite or countable number of values. Let $g^t = (g_0, ..., g_t)$ denote the history of government expenditures. Equilibrium plans for work and consumption have date t components that are measurable functions of g^t. A representative

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[2]A line of attack using martingale perturbations is set forth in Sargent (2005).

agent is endowed with one unit of leisure, works $h_t(g^t)$, and consumes $c_t(g^t)$ at history g^t for each $t \geq 0$. One unit of labor can be transformed into one unit of good. Feasible allocations satisfy

$$c_t(g^t) + g_t = h_t(g^t). \tag{1}$$

Competition makes the real wage $w_t(g^t) = 1$ for all $t \geq 0$ and any history g^t. The government finances its time t expenditures either by using a linear tax $\tau_t(g^t)$ on labor income or, in effect, by issuing a vector of state-contingent debt $b_{t+1}(g_{t+1}, g^t)$ that is sold at price $p_t(g_{t+1}, g^t)$ at history g^t and promises to pay one unit of the consumption good if government expenditures are g_{t+1} next period and zero otherwise. The one-period government budget constraint at t is

$$b_t(g^t) + g_t = \tau_t(g^t)h_t(g^t) + \sum_{g_{t+1}} p_t(g_{t+1}|g^t)b_{t+1}(g_{t+1}, g^t). \tag{2}$$

But we shall typically work with an Arrow-Debreu formulation in which all trades occur at date 0 at Arrow-Debreu history- and date-contingent prices $q_t(g^t)$. In this setting, the government faces the single intertemporal budget constraint

$$b_0 + \sum_{t=0}^{\infty}\sum_{g^t} q_t(g^t)g_t \leq \sum_{t=0}^{\infty}\sum_{g^t} q_t(g^t)\tau_t(g^t))h_t(g^t).$$

2.1 Fear of model misspecification

The representative agent and the government share an approximating model in the form of a sequence of joint densities $\pi_t(g^t)$ over sequences of histories $g^t \ \forall t \leq \infty$. The representative agent, but not the government, fears that the approximating model is misspecified in the sense that the history of government expenditures will actually be drawn from a joint density that, while different from the approximating model, is absolutely continuous with respect to the approximating model over finite time intervals. Following Hansen and Sargent (2005), we use perturbations that are martingales with respect to the approximating model to characterize model misspecifications. Thus, by the Radon-Nikodym theorem, there exists a non-negative random variable M_t with $E(M_t) = 1$ that is a measurable function of the history g^t and that has the interpretation of a change of measure. The random variable M_t, which we take as a likelihood ratio $M_t(g^t) = \frac{\tilde{\pi}_t(g^t)}{\pi_t(g^t)}$ of a distorted to the approximating density, is a martingale, i.e., $E_t M_{t+1} = M_t$ where E denotes expectation with respect to the approximating model. Here the tilde refers to a distorted model. Evidently, we can compute the mathematical expectation of a random variable $X_t(g^t)$ under a distorted measure as

$$\tilde{E}(X_t) = E(M_t X_t).$$

To attain a convenient decomposition of M_t, define

$$m_{t+1} \equiv \frac{M_{t+1}}{M_t} \text{ for } M_t > 0$$

and let $m_{t+1} \equiv 1$ when $M_t = 0$, (i.e., when the distorted model assigns zero probability to a particular history). Then

$$
\begin{aligned}
M_{t+1} &= m_{t+1} M_t \qquad &(3)\\
&= M_0 \prod_{j=1}^{t+1} m_j.
\end{aligned}
$$

The non-negative random variable m_{t+1} distorts the conditional probability of history g^{t+1} given history g^t, so that it is a conditional likelihood ratio $m_{t+1} = \frac{\tilde{\pi}_t(g_{t+1}|g^t)}{\pi_t(g_{t+1}|g^t)}$. It has to satisfy the restriction that $E_t m_{t+1} = 1$ in order to be an appropriate distortion to the conditional measure. The conditional entropy of the distortion is

$$
E(m_{t+1} \log m_{t+1} | g^t).
$$

Following Hansen and Sargent (2006), we shall in effect constrain the set of perturbations by the following constraint on a measure of discounted entropy [3]

$$
\beta E \left[\sum_{t=0}^{\infty} \beta^t M_t E(m_{t+1} \log m_{t+1} | g^t) \Big| g_0 \right] \leq \eta \qquad (4)
$$

where η measures the size of an entropy ball of models surrounding the approximating model.

2.2 Preferences

To represent fear of model misspecification, we use the multiplier preferences of Hansen and Sargent (2001) and Hansen et al. (2006) to describe how the representative consumer ranks consumption, leisure plans whose time t components are measurable functions of g^t:

$$
\min_{\{m_{t+1}, M_t\}_{t=0}^{\infty} \geq 0} \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t(g^t) U(c_t(g^t), 1 - h_t(g^t)) + \beta \theta \sum_{t=0}^{\infty} \sum_{g^t} \beta^t M_t(g^t) \varepsilon_t(m_{t+1}) \quad (5)
$$

where $U(c_t, 1 - h_t)$ is the same period utility function assumed by Lucas and Stokey (1983), the multiplier $\theta > 0$ is a penalty parameter that measures fear of model misspecification, and $\varepsilon_t(m_{t+1}) \equiv E_t m_{t+1} \ln m_{t+1}$ is relative entropy between one-step conditional distributions.[4] Along with Lucas and Stokey, we assume that $U(c, 1 - h)$ is strictly increasing, strictly concave, and thrice continuously differentiable.[5]

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[3]This constraint would be used to formulate the constraint preferences of Hansen and Sargent (2001). They discuss the relation between constraint preferences and the multiplier preferences featured in this paper and show how to construct η *ex post* as a function of the multiplier θ in (5) and other parameters.

[4]These preferences have been axiomatized and linked to ambiguity aversion by Maccheroni et al. (2006a,b).

[5]Strict concavity is not be satisfied for the quasi-linear example to be studied in subsection 6.2.

2.3 The representative household's problem

For any sequence of random variables $\{a_t\}$, let $a \equiv \{a_t(g^t)\}_{t,g^t}$. The problem of the consumer is[6]

$$\max_{c,h} \min_{M \geq 0, m \geq 0} \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t(g^t) \Big[U(c_t(g^t), 1 - h_t(g^t))$$
$$+ \theta\beta \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^{t+1}) \ln m_{t+1}(g^{t+1}) \Big]$$

subject to

$$\sum_{t=0}^{\infty} \sum_{g^t} q_t(g^t) c_t(g^t) \leq \sum_{t=0}^{\infty} \sum_{g^t} q_t(g^t)(1 - \tau_t(g^t)) h_t(g^t) + b_0 \tag{6}$$

$$c_t(g^t) \geq 0, \forall t, g^t \tag{7}$$

$$M_{t+1}(g^{t+1}) = m_{t+1}(g^{t+1}) M_t(g^t), M_0 = 1 \forall t, g^t \tag{8}$$

$$\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) m_{t+1}(g^t) = 1, \forall t, g^t \tag{9}$$

The first constraint is the intertemporal budget constraint of the household which is unique since the markets are assumed to be complete with prices at time zero $q_t(g^t)$ of Arrow-Debreu securities. The right side is the discounted present value of after tax labor income plus an initial asset position b_0 that can assume positive (denoting a government debt) or negative (denoting government assets) values.

2.4 The inner problem: choosing beliefs

The inner problem chooses (M, m) to minimize the utility of the representative household subject to the law of motion of the martingale and the restriction that the conditional distortion m integrates to unity. The optimal distortion takes the form (for the derivation see appendix A.1):

$$m_{t+1}^*(g^{t+1}) = \frac{\exp\left(-\frac{V_{t+1}(g^{t+1})}{\theta}\right)}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp\left(-\frac{V_{t+1}(g^{t+1})}{\theta}\right)}, \text{all } t \geq 0, g^t \tag{10}$$

where the asterisks denote optimal values and V_t is the utility of the household under the distorted measure, which follows the recursion

$$V_t = U(c_t, 1 - h_t) + \beta[E_t m_{t+1}^* V_{t+1} + \theta E_t m_{t+1}^* \ln m_{t+1}^*]. \tag{11}$$

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[6]We assume that uncertainty at $t = 0$ has been realized, so $\pi_0(g_0) = 1$. Thus, the distortion of the probability of the initial period is normalized to be unity, so that $M_0 \equiv 1$.

Equations (10) and (11) are the first-order conditions for the minimization problem with respect to m_{t+1} and M_t. Substituting (10) into (11) gives

$$V_t = U(c_t, 1 - h_t) + \frac{\beta}{\sigma} \ln E_t(\exp(\sigma V_{t+1})) \tag{12}$$

where $\sigma \equiv -1/\theta$. Thus, the martingale distortion evolves according to

$$M_{t+1}^* = \frac{\exp\left(\sigma V_{t+1}(g^{t+1})\right)}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)\exp\left(\sigma V_{t+1}(g^{t+1})\right)} M_t^*, \; M_0 \equiv 1 \tag{13}$$

Equation (13) asserts that the martingale distortion attaches higher probabilities to histories with low utility and lower probabilities to histories with high value. Such exponential tilting of probabilities summarizes how the representative household's distrust of the approximating model gives rise to conservative probability valuations that give rise to an indirect utility function that solves the recursion (12), which is an example of the discounted risk-sensitive preferences of Hansen and Sargent (1995).

2.5 Outer problem: choosing $\{c_t, h_t\}$ plan

An interior solution to the maximization problem of the household gives rise to the intratemporal labor supply condition

$$\frac{U_l(g^t)}{U_c(g^t)} = 1 - \tau_t(g^t) \tag{14}$$

that equates the MRS between consumption and leisure to the after tax wage rate and the intertemporal Euler equation

$$q_t(g^t) = \beta^t \pi_t(g^t) M_t^*(g^t) \frac{U_c(g^t)}{U_c(g_0)}. \tag{15}$$

Here we have normalized the price of an Arrow-Debreu security at $t = 0$ to unity $q_0(g_0) \equiv 1$. The implied price of one-period state-contingent debt (an Arrow security) is

$$p_t(g_{t+1}, g^t) = \beta \pi_{t+1}(g_{t+1}|g^t) m_{t+1}^*(g^{t+1}) \frac{U_c(g^{t+1})}{U_c(g^t)} \tag{16}$$

Remark 2.1. *The worst-case beliefs M_t^* influence the asset prices via (15). Their presence creates an avenue by which a Ramsey planner influences an allocation that is not present in Lucas and Stokey (1983).*

Definition 2.2. *A competitive equilibrium is a consumption-labor allocation (c, h), distortions to beliefs (m, M), a price system q, and a government policy (g, τ) such that (a) given $(q, \tau), (c, h)$ and (m, M) solve the household's problem, and (b) markets clear $c_t(g^t) + g_t = h_t(g^t) \forall t, g^t$.*

3 Ramsey Problem

A Ramsey planner chooses distortionary taxes on labor income and state-contingent debt at every history. In this paper, we assume that while the representative household distrusts the approximating model π, the Ramsey planner completely trusts it. The Ramsey planner chooses a competitive equilibrium allocation that maximizes the expected utility of the representative household *under the approximating model*. This assumption imposes a degree of paternalism because the planner imposes its full confidence in the model when evaluating the household's expected utility. In Karantounias et al. (2007), we study alternative sets of assumptions that allow the Ramsey planner to doubt the approximating model and also possibly instruct the planner to evaluate expected utilities using the representative household's beliefs.

We use the same primal approach employed by Lucas and Stokey (1983). The Ramsey planner chooses allocations subject to the resource constraint (1) and some implementability constraints imposed by competitive equilibrium.

Proposition 3.1. *The Ramsey planner faces the following implementability constraints*

$$\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t^*(g^t) U_c(g^t) c_t(g^t) = \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t^*(g^t) U_l(g^t) h_t(g^t)) + U_c(g_0) b_0, \quad (17)$$

the law of motion for the martingale that represents distortions to beliefs (13), and the recursion for the representative household's value function (12).

Proof. Besides the resource constraint, the CE is characterized fully by the household's two Euler equations, the intertemporal budget constraint (6) that holds with equality at an optimum, and equations (13) and (12), which describe the evolution of the endogenous beliefs of the agent. Use (14) and (15) to substitute for prices and after tax wages in the intertemporal budget constraint to obtain (17). □

Definition 3.2. *The Ramsey problem is*

$$\max_{(c,h,M^*,V)} \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) U(c_t(g^t), 1 - h_t(g^t))$$

subject to

$$\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t^*(g^t)[U_c(g^t) c_t(g^t) - U_l(g^t) h_t(g^t)] = U_c(g_0) b_0 \quad (18)$$

$$c_t(g^t) + g_t = h_t(g^t), \forall t, g^t \quad (19)$$

$$M_{t+1}^*(g^{t+1}) = \frac{\exp\left(\sigma V_{t+1}(g^{t+1})\right)}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp\left(\sigma V_{t+1}(g^{t+1})\right)} M_t^*(g^t), M_0(g_0) = 1, \forall t, g^t \quad (20)$$

$$V_t(g^t) = U(c_t(g^t), 1 - h_t(g^t)) + \frac{\beta}{\sigma} \ln \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp\left(\sigma V_{t+1}(g^{t+1})\right)$$

$$, \qquad \forall t, g^t, t \geq 1 \quad (21)$$

Remark 3.3. *The presence of the distorted beliefs in (17) changes this from the Ramsey problem studied by Lucas and Stokey (1983). The endogeneity of the representative household's worst-case beliefs about probabilities contributes two additional implementability constraints that describe their evolution. The Ramsey planner takes into account how the allocation (c, h) that he chooses affects the utility of the agent $V_t(g^t)$ and therefore the endogenous likelihood ratio $M_t^*(g^t)$ and thereby the representative households' agent's worst-case beliefs. In effect, the Ramsey problem is now a Stackelberg game with one leader and two followers, namely, the representative household and the representative household's malevolent alter ego who, by choosing a worst-case probability distortion, inspires the household to value robust decision rules.*

3.0.1 First best

By first-best, we mean the allocation that maximizes the expected utility of the household under π subject to the resource constraint (1). Note that for *any* kind of beliefs of the planner, the first-best is characterized by the condition $\frac{U_l(g^t)}{U_c(g^t)} = 1$ and the resource constraint (1). So the efficient allocation $(\hat{c}, \hat{h})$ is independent of probabilities π. The distorted beliefs of the private sector affect asset prices through (15), but not the allocation. The planner's and the household's beliefs both affect allocations in our second-best world without lump-sum taxation.

3.1 Lagrangian formulation of commitment problem

Attach the multipliers Φ, $\beta^t \pi_t(g^t) \lambda_t(g^t)$, $\beta^{t+1} \pi_{t+1}(g^{t+1}) \mu_{t+1}(g^{t+1})$, and $\beta^t \pi_t(g^t) \xi_t(g^t)$ to constraints (17), (1), (13), and (12), respectively, and form the Lagrangian

$$
\begin{aligned}
L \;=\; & \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) \Big\{ U\big(c_t(g^t), 1 - h_t(g^t)\big) - \lambda_t(g^t)\big(c_t(g^t) + g_t - h_t(g^t)\big) \\
& - \sum_{g_{t+1}} \beta \pi_{t+1}(g_{t+1}|g^t) \mu_{t+1}(g^{t+1}) \Big[M_{t+1}^*(g^{t+1}) - \frac{\exp(\sigma V_{t+1}(g^{t+1}))}{\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp(\sigma V_{t+1}(g^{t+1}))} M_t^*(g^t) \Big] \\
& - \xi_t(g^t) \Big[V_t(g^t) - U\big(c_t(g^t), 1 - h_t(g^t)\big) - \frac{\beta}{\sigma} \ln \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t) \exp(\sigma V_{t+1}(g^{t+1})) \Big] \Big\} \\
& + \Phi \Big[\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi_t(g^t) M_t^*(g^t) \Omega(c_t(g^t), h_t(g^t)) - U_c(g_0) b_0 \Big]
\end{aligned}
$$

where
$$
\Omega(c_t(g^t), h_t(g^t)) \equiv U_c(g^t) c_t(g^t) - U_l(g^t) h_t(g^t). \tag{22}
$$

First-order necessary conditions are

- $c_t(g^t), t \geq 1$:
$$
(1 + \xi_t(g^t)) U_c(g^t) + \Phi M_t^*(g^t) \Omega_c(g^t) = \lambda_t(g^t) \tag{23}
$$

- $h_t(g^t), t \geq 1$:
$$-(1 + \xi_t(g^t))U_l(g^t) + \Phi M_t^*(g^t)\Omega_h(g^t) = -\lambda_t(g^t) \tag{24}$$

- $M_t^*(g^t), t \geq 1$:
$$\mu_t(g^t) = \Phi\Omega(g^t) + \beta \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)m_{t+1}^*(g^{t+1})\mu_{t+1}(g^{t+1}) \tag{25}$$

- $V_t(g^t)$:
$$\xi_t(g^t) = \sigma m_t^*(g^t)M_{t-1}^*(g^{t-1})\left[\mu_t(g^t) - \sum_{g_t} \pi_t(g_t|g^{t-1})m_t^*(g^t)\mu_t(g^t)\right] + m_t^*(g^t)\xi_{t-1}(g^{t-1}) \tag{26}$$

- $c_0(g_0)$:
$$(1 + \xi_0)U_c(g_0) + \Phi M_0\Omega_c(g_0) = \lambda_0(g_0) + \Phi U_{cc}(g_0)b_0 \tag{27}$$

- $h_0(g_0)$:
$$-(1 + \xi_0)U_l(g_0) + \Phi M_0\Omega_h(g_0) = -\lambda_0(g_0) - \Phi U_{cl}(g_0)b_0 \tag{28}$$

We detail the derivation of the first-order condition with respect to $V_t(g^t)$ in appendix A.2. In (25) and (26), we used $m_{t+1}^*(g^{t+1}) = \exp(\sigma V_{t+1}(g^{t+1}))/\sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)\exp(\sigma V_{t+1}(g^{t+1}))$ to save notation.

Remark 3.4. *Note that in formulating the Ramsey problem, the last constraint (12) applies only from period one on since the value of the agent at $t = 0$ V_0 is not relevant to the problem due to the normalization $M_0 \equiv 1$. However, it is more convenient in constructing the Lagrangian to include it. We can set $\xi_0 = 0$ to accommodate this. Equivalently, we could maximize with respect to V_0 to get an additional first-order condition $\xi_0 = 0$.*

Remark 3.5. *Since $\xi_0 = 0, M_0 = 1$, the first-order conditions (27, 28) for (c_0, h_0) are equivalent with those for the Lucas and Stokey (1983) case where the representative consumer fears no misspecification.*

The first-order conditions (23), (24), (25), (26), (27), (28) together with equations (18), (19), (20), and (21) determine the solution to the Ramsey problem.

3.2 Characterizing the Ramsey plan

In the first-order condition (23)

$$U_c(g^t) + \xi_t(g^t)U_c(g^t) + \Phi M_t^*(g^t)\Omega_c(g^t) = \lambda_t(g^t)$$

the first term on the left represents the marginal utility the Ramsey planner gets by increasing consumption by one unit. The second term on the left takes into account how increasing consumption affects the representative household's value function, and consequently it worst-case model perturbation M_{t+1}^*. As we shall see later, the multiplier ξ_t serves as a state variable in a recursive statement of the Ramsey problem. Note that if the Ramsey planner were not to take into account that the worst-case beliefs of the representative household are endogenous, this term would be zero. The third term on the left represents the typical constraints that a competitive equilibrium allocation imposes on the Ramsey planner, but with a twist coming from the fact that the prices reflect the representative household's worst-case beliefs. The right side represents the shadow value of output. Analogous interpretations apply to the first-order condition (24) for h_t.

Derivatives of Ω with respect to c and g are

$$\Omega_c(g^t) = U_{cc}(g^t)c_t(g^t) - U_{lc}(g^t)h_t(g^t) + U_c(g^t) \qquad (29)$$
$$\Omega_h(g^t) = U_{ll}(g^t)h_t(g^t) - U_{lc}(g^t)c_t(g^t) - U_l(g^t) \qquad (30)$$

Substituting these into (23) and (24) and combining them to eliminate the shadow value of output λ_t results in

$$\begin{aligned}
\left(1 + \xi_t(g^t) + \Phi M_t^*(g^t)\right)\left(U_l(g^t) - U_c(g^t)\right) &= \Phi M_t^*(g^t)\big[U_{cc}(g^t)c_t(g^t) - U_{cl}(g^t)(c_t(g^t) + h_t(g^t)) \\
&+ U_{ll}(g^t)h_t(g^t)\big].
\end{aligned} \qquad (31)$$

This condition influences the optimal allocation by describing the optimal wedge between marginal utility of consumption and leisure ($U_l - U_c$).

Performing analogous steps for $t = 0$ and eliminating λ_0, we get

$$(1 + \Phi)(U_l(g_0) - U_c(g_0)) = \Phi\big[U_{cc}(g_0)(c_0 - b_0) - U_{cl}(g_0)(c_0 - b_0 + h_0) + U_{ll}(g_0)h_0\big]. \qquad (32)$$

Note that if the initial debt $b_0 = 0$, condition (32) would be the same as (31) since $(M_0, \xi_0) = (1, 0)$. Equation (32) shows that the consumption at time zero (and consequently labor and the tax rate) is a function of (g_0, b_0) and Φ, $c_0 = c(g_0, b_0; \Phi)$, since $M_0 = 1$ and $\xi_0 = 0$. The difference in the first period is due to the presence of the initial debt b_0 and the realization of uncertainty g_0.

Proposition 3.6. *The Ramsey allocation and taxes from period one onward are history dependent.*

Proof. Use the resource constraint (1) to substitute for h_t in (31) and solve for optimal consumption terms in terms of (g_t, M_t^*, ξ_t) and the multiplier Φ to get $c_t = c(g_t, M_t^*, \xi_t; \Phi)$. Analogously, we get $h_t = h(g_t, M_t^*, \xi_t; \Phi)$ and consequently from (14) the optimal tax rate $\tau_t = \tau(g_t, M_t^*, \xi_t; \Phi)$. Therefore, the realization of the government expenditure at time t is not sufficient to describe the optimal allocation and taxes. Instead, it depends on the history of shocks through M_t^* and ξ_t. □

Relative to the outcome in Lucas and Stokey (1983), the representative household's fear of misspecification makes the Ramsey allocation depend on two additional variables, namely, the likelihood ratio M_t^* and the multiplier ξ_t on the forward-looking constraint (12) that describes the evolution of the household's value function V_t. The multiplier ξ_t measures the shadow value to the planner of the representative household's value. It shows up in the first-order conditions (31) because increasing c_t, h_t affects V_t and therefore the household's worst-case distorted measure. The likelihood ratio M_t^* shows up in the first-order condition through its influence on equilibrium prices that become incorporated in the implementability constraint (17). These two variables are absent from Lucas and Stokey (1983), since for $\sigma = 0$ and from (10),(13) and (26), we see that $\xi_t(g^t) = 0, M_t^*(g^t) = 1, \forall t, g^t$. In this case, only the current realization of the government shock g_t determines the optimal allocation and taxes. The only intertemporal link would occur implicitly through the value of the multiplier Φ on the implementability constraint, and this by itself would impart no history dependence.

We can express the optimal tax in terms of the allocation and (M^*, ξ) as follows. Dividing (31) by $-U_c(g^t)$ and using $\tau_t = 1 - \frac{U_l}{U_c}$, we get for $t \geq 1$

$$\tau_t(g^t) = \frac{\Phi M_t^*(g^t)}{1 + \xi_t(g^t) + \Phi M_t^*(g^t)} \left[\gamma_{RA}(g^t) + \frac{U_{cl}(g^t)}{U_c(g^t)}(c_t(g^t) + h_t(g^t)) - \frac{U_{ll}(g^t)}{U_c(g^t)}h_t(g^t) \right] \quad (33)$$

where $\gamma_{RA}(g^t) \equiv -U_{cc}c/U_c$, the coefficient of relative risk aversion.

Remark 3.7. *Formula (33) shows that the planner chooses smaller tax wedges at histories that representative household thinks are less probable than does the Ramsey planner, i.e., when $M_t^*(g^t)$ is is small.*

3.2.1 Interpretation of μ_t

Fear of misspecification alters the fundamentally *static* nature of the Lucas and Stokey problem through the presence of (M_t^*, ξ_t). In order to understand better the dynamic tradeoffs involved we have to interpret the first-order conditions with respect to (M_t^*, V_t). Consider μ_t, the multiplier on the evolution equation for the likelihood ratio (13), which consequently represents the value to the planner of distorting the probability of history g^t. Replacing the definition (22) of $\Omega(g^t)$ in equation (25) gives

$$\mu_t(g^t) = \Phi[U_c(g^t)c_t(g^t) - U_l(g^t)h_t(g^t)] + \beta \sum_{g_{t+1}} \pi_{t+1}(g_{t+1}|g^t)m_{t+1}^*(g^{t+1})\mu_{t+1}(g^{t+1})$$

where $m_{t+1}^*(g^{t+1}) = \exp(\sigma V_{t+1}(g^{t+1}))/E_t \exp(\sigma V_{t+1}(g^{t+1}))$. Increasing $M_t^*(g^t)$, thereby making history g^t) more probable, has two effects: first, it affects prices at history g^t and, second, it alters the probability of next period's history. The effect on current prices for history g^t is measured by the first term on the right, which is the difference between consumption and after-tax labor income in utility terms, or, using the resource constraint (1) and $\tau_t = 1 - \frac{U_l}{U_c}$, the marginal utility of a government surplus, multiplied by the shadow value of distortionary taxation Φ. But a higher distorted probability today will lead to a higher distorted probability tomorrow, which is the second effect (see (13)). Under the distorted measure, the expected discounted marginal value of this effect is the second term on the right.

Solving forward (25) gives

$$\mu_t(g^t) = \Phi \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) \prod_{j=1}^{i} m_{t+j}^*(g^{t+j})[U_c(g^{t+i})c_{t+i}(g^{t+i}) - U_l(g^{t+i})h_{t+i}(g^{t+i})]$$

(34)

Multiplying and dividing by $U_c(g^t)$ and rewriting in terms of the after tax labor income, we get

$$\mu_t(g^t) = \Phi U_c(g^t) \sum_{i=0}^{\infty} \sum_{g^{t+i}|g^t} q_{t+i}^t(g^{t+i})[c_{t+i}(g^{t+i}) - (1 - \tau_{t+i}(g^{t+i}))h_{t+i}(g^{t+i})]$$

where

$$q_{t+i}^t(g^{t+i}) \equiv \frac{q_{t+i}(g^{t+i})}{q_t(g^t)} = \beta^i \pi_{t+i}(g^{t+i}|g^t) \prod_{j=1}^{i} m_{t+j}^*(g^{t+j}) \frac{U_c(g^{t+i})}{U_c(g^t)}$$

is the price of an Arrow-Debreu security in terms of consumption at history g^t. But from the intertemporal budget constraint at time t the above sum equals the outstanding debt of the government $b_t(g^t)$. Therefore,

$$\mu_t(g^t) = \Phi U_c(g^t)b_t(g^t).$$

(35)

Thus, if the government has outstanding debt obligations ($b_t(g^t) > 0$), then the multiplier μ_t has a positive value whenever $\Phi > 0$.

3.2.2 Dynamics of ξ_t

Consider now the first-order condition with respect to $V_t(g^t)$ which can be expressed in expectation notation as

$$\xi_t = \sigma m_t^* M_{t-1}^* [\mu_t - E_{t-1} m_t^* \mu_t] + m_t^* \xi_{t-1}, t \geq 1, \xi_0 = 0$$

(36)

where E denotes mathematical expectation under the approximating model. The dynamic tradeoffs that the planner faces are intricate. By increasing $V_t(g^t)$ he affects the household's expectations at $t-1$ for the current period t. However the planner does not choose freely any value of $V_t(g^t)$ but he is constrained by the value that he has *promised* to the household. The shadow value to the planner of the promised utility to the household is reflected by the multiplier ξ_{t-1}, and the total effect is represented by the *second* term in the right-hand side of (36). The planner's commitment to the promised value to the household, and therefore via equation (20) to the promised worst-case beliefs of the household, is the vehicle of steering the household's expectations for future histories and stems from his first-move advantage.

Furthermore, increasing $V_t(g^t)$ affects the household beliefs for current period, by decreasing the probability of this history and all future histories onward, an effect measured by the multiplier μ_t in the first term of the right-hand side. But decreasing the probability of this particular node of the event tree implies increasing the probability of the rest of the nodes with opposite effects. The total effect is measured by $\mu_t - E_{t-1} m_t^* \mu_t$, i.e. the innovation in μ_t under the distorted measure $\tilde{\pi}$, which by (35) translates to the innovation of the

13

outstanding government obligations in utility terms. This is the *first* term of the right-hand side. Optimality requires that the sum of the two effects have to equal to the shadow value of the promised utility of next period ξ_t, the left-hand side of equation (36). Note that ξ_t can take both positive and negative values because constraint (12) can bind in both directions. Besides that, ξ_t has the following property:

Lemma 3.8. *The multiplier ξ_t is a martingale under the approximating model.*

Proof. Take conditional expectation with respect to the approximating model π given history g^{t-1} in the law of motion (36) for ξ and remembering that the variables dated at t are measurable variables of the history g^t, we get

$$
\begin{aligned}
E_{t-1}\xi_t &= \sigma M_{t-1}^* E_{t-1}m_t^*[\mu_t - E_{t-1}m_t^*\mu_t] + \xi_{t-1}E_{t-1}m_t^* \\
&= \sigma M_{t-1}^*[E_{t-1}m_t^*\mu_t - E_{t-1}m_t^* \cdot E_{t-1}m_t^*\mu_t] + \xi_{t-1}E_{t-1}m_t^* \\
&= \xi_{t-1},
\end{aligned}
$$

since $E_{t-1}m_t^* = 1$. $\qquad\square$

In other words, optimality requires that the best forecast at $t-1$ under the planner's beliefs π of next period's shadow value of the worst-case beliefs of the household is the shadow value at $t-1$, ξ_{t-1}. An immediate corollary of lemma 3.8 is that the mean value of the ξ_t is zero since $E(\xi_t) = E(E_0\xi_t) = E(\xi_0) = 0$. Define the one step ahead forecast error η_t under the distorted measure of the multiplier μ_t as

$$
\eta_t \equiv \mu_t - E_{t-1}m_t^*\mu_t \tag{37}
$$

with $E_{t-1}m_t^*\eta_t = 0$. So we can rewrite (36)

$$
\xi_t = \sigma M_t^*\eta_t + m_t^*\xi_{t-1}, t \geq 1, \xi_0 = 0
$$

and by iterating backward, we can express ξ_t as

$$
\begin{aligned}
\xi_t &= \sigma M_t^*(\eta_t + ... + \eta_1) \\
&= \sigma M_t^* H_t, \tag{38}
\end{aligned}
$$

where

$$
H_t \equiv \sum_{i=1}^t \eta_i, H_0 \equiv 0
$$

is the cumulative forecast error. Therefore, if the cumulative forecast error is positive, then the shadow value of the representative household's utility is negative, since $\sigma < 0$. Furthermore, since μ_t is directly associated with the outstanding debt of the government through (35), the forecast error becomes

$$
\eta_t = \Phi[U_c(g^t)b_t(g^t) - E_{t-1}m_t^*U_c(g^t)b_t(g^t)]. \tag{39}
$$

We can rewrite η_t as

$$
\begin{aligned}
\eta_t &= \Phi \frac{U_c(g^{t-1})}{\beta} [\beta \frac{U_c(g^t)}{U_c(g^{t-1})} b_t(g^t) - E_{t-1}\beta m_t^* \frac{U_c(g^t)}{U_c(g^{t-1})} b_t(g^t)] \\
&= \Phi \frac{U_c(g^{t-1})}{\beta} [\beta \frac{U_c(g^t)}{U_c(g^{t-1})} b_t(g^t) - \sum_{g_t} p_{t-1}(g_t, g^{t-1}) b_t(g^t)]
\end{aligned}
$$

where the second line follows from the equilibrium price of state-contingent debt (16). Using the government budget constraint (2), we can substitute for the value of the portfolio of government debt to get

$$
\eta_t = \Phi \frac{U_c(g^{t-1})}{\beta} \left[\beta \frac{U_c(g^t)}{U_c(g^{t-1})} b_t(g^t) - [g_{t-1} - \tau_{t-1}(g^{t-1}) h_{t-1}(g^{t-1}) + b_{t-1}(g^{t-1})] \right] \tag{40}
$$

which shows how current debt, last period's debt, and last period's government deficit affect η_t.

4   Recursive Formulation

In this section, we restrict attention to Markov processes for the government expenditures in order to achieve a recursive formulation of the Ramsey problem along the lines of Marcet and Marimon (1998).

Assumption 4.1. *Government expenditures g_t follow a time-homogenous Markov Process $\pi_{g|g_-} \equiv Prob(g_t = g|g_{t-1} = g_-)$ with transition matrix Π.*

This problem under commitment has history dependence that adds state variables to the natural state variable g_t. Proposition (3.6) hints that the endogenous variables M_t^* and ξ_t are candidates for state variables. In this section, we show that these variables can serve as state variables in a recursive representation of the robust Ramsey plan.

To express the Ramsey problem recursively, in addition to g_t we have to include in the state the backward-looking likelihood ratio M_t^* with law of motion (13). Furthermore, the Marcet and Marimon procedure transforms the Lagrangian saddle point formulation of the commitment problem with forward-looking constraints into a saddle point functional equation by appropriately augmenting the state and defining a new period return function that takes into account restrictions to which the planner has to commit every period. A crucial element in this procedure is the multiplier ξ_t (the *co-state* variable) on the forward-looking constraint (21) that can serve as a state variable. The multiplier reflects the shadow value of the utility promises of the planner to the household that determine his worst-case model every period. Therefore, we use as the state the vector $X_t \equiv (g_t, M_t^*, \xi_t)$.

We will form the problem recursively from period one onward, due to the presence of initial debt and the realization of uncertainty at period zero. Our object of interest is the ex-ante value of the Lagrangian at period one. Let $W(X_-; \Phi)$ denote the corresponding value function in a recursive representation of the Lagrangian from period one, where the

underscore "$_-$" stands for previous period, i.e. $z_- \equiv z_{t-1}$ for any random variable z, and $\Phi > 0$ is a fixed value of the multiplier on the implementability constraint (18).

Proposition 4.2. $W(\cdot; \Phi)$ *satisfies the Bellman equation*

$$W(g_-, M_-^*, \xi_-; \Phi) = \min_{\xi_g} \max_{c_g, m_g^*, V_g} \sum_g \pi_{g|g_-} \left\{ U(c_g, 1 - c_g - g) + \Phi m_g^* M_-^* \Omega_g \right.$$

$$\left. -\xi_g(V_g - U(c_g, 1 - c_g - g)) + \xi_-(m_g^* V_g + \theta m_g^* \ln m_g^*) + \beta W(g, m_g^* M_-^*, \xi_g; \Phi) \right\}$$

where

$$\Omega_g \equiv [U_c(c_g, 1 - c_g - g) - U_l(c_g, 1 - c_g - g)]c_g - U_l(c_g, 1 - c_g - g)g$$

and

$$m_g^* = \frac{\exp\left(-\frac{V_g}{\theta}\right)}{\sum_g \pi_{g|g_-} \exp\left(-\frac{V_g}{\theta}\right)}, \forall g$$

Proof. We construct the Bellman equation in detail in appendix B. $\square$

We make several observations about the Bellman equation. The min operator reflects the minimization with respect to the multipliers ξ_g of the Lagrangian, saddle-point formulation. Note also that we adopt an *ex ante* formulation to describe the commitment problem appropriately with control variables for the planner ξ_g, c_g, m_g^*, V_g that are *state-contingent.* The reason is that by increasing, say, consumption c_g at state g, the planner increases the utility of the household at this particular state, which decreases the state's probability. However, this decrease of probability increases the probabilities of the other states. Forming the Lagrangian recursively from period one onward before the realization of uncertainty takes these effects into account.

Furthermore, note that besides the period utility U_g the return function includes the term $\Phi M_g^* \Omega_g$ that is associated with the implementability constraint (18) and describes the restrictions coming from equilibrium prices and the household's budget constraint, as well as two further terms involving the value of the state ξ_- and the current value of the multiplier ξ_g at the realization of the state g. These two terms describe in a precise sense how the planner has to commit to the shadow value ξ_- of the promised - under the distorted measure- utility $\sum_g \pi_{g|g_-}\{m_g^* V_g + \theta m_g^* \ln m_g^*\}$ to the household and how he has to consider how his actions will affect the future state ξ_g. This commitment is essential in his effort to manipulate the worst-case beliefs of the household.

The initial value of the state is $X_0 = (g_0, 1, 0)$. Thus, the value of the Lagrangian from period one onward is $W(g_0, 1, 0; \Phi)$ for the appropriate value of Φ. That $\xi_0 = 0$ reflects the fact that the planner at period one is not constrained to commit to the shadow value of his promises to the household. The value function W, as we hope the notation makes clear, is contingent on a value of the multiplier Φ. This value has to be adjusted so that the intertemporal budget constraint of the household is satisfied after solving the Bellman equation and deriving policy functions from period one onward and solving also for the period zero allocation.

4.1 Time zero.

The initial period $t = 0$ is special due to the presence of initial debt b_0 and the realization of uncertainty $\pi_0 = M_0 = 1$. The planner's problem at time zero takes the form

$$W_0(g_0, b_0; \Phi) = \max_{c_0}\{U(c_0, 1 - c_0 - g_0) + \Phi\Omega_0(c_0) - \Phi U_c(c_0, 1 - c_0 - g_0)b_0 + \beta W(g_0, 1, 0; \Phi)\},$$

which is practically the static problem

$$\max_{c_0} U(c_0, 1 - c_0 - g_0) + \Phi\Omega_0(c_0) - \Phi U_c(c_0, 1 - c_0 - g_0)b_0.$$

From the above problem, we get the policy function for $c_0(g_0, b_0; \Phi)$.

4.2 Policy functions and debt

As the Bellman equation (4.2) makes clear, we attain a time invariant representation of the policy functions.

Corollary 4.3. *The policy functions from period one onward are time-invariant functions of the state X_-*

$$
\begin{aligned}
c_g &= c_g(g_-, M_-^*, \xi_-; \Phi) \\
V_g &= V_g(g_-, M_-^*, \xi_-; \Phi) \\
m_g^* &= m_g^*(g_-, M_-^*, \xi_-; \Phi) \\
\xi_g &= \xi_g(g_-, M_-^*, \xi_-; \Phi)
\end{aligned}
$$

We will briefly consider some of the optimality conditions of the recursive problem (details are in appendix B, where we match the first-order conditions of the recursive problem with the ones coming from the Lagrangian). The first-order condition with respect to c_g delivers the same expression as (31), which by proposition (3.6) shows the dependence on the variables (M_t^*, ξ_t). This leads us to conclude that the vector of state variables (g_-, M_-^*, ξ_-) affects the optimal policy for consumption at g (and therefore labor and the tax rate), by determining the value of the state (g, M_g^*, ξ_g). Thus $c_g = c_g(g_-, M_-^*, \xi_-; \Phi) = c(g, M_g^*, \xi_g; \Phi)$.

The envelope conditions are

$$W_\xi(g_-, M_-^*, \xi_-; \Phi) = \sum_g \pi_{g|g_-}[m_g^* V_g + \theta m_g^* \ln m_g^*] \tag{41}$$

$$W_{M^*}(g_-, M_-^*, \xi_-; \Phi) = \sum_g \pi_{g|g_-}[\Phi m_g^* \Omega_g + \beta W_{M^*}(g, M^*, \xi; \Phi)]. \tag{42}$$

The first condition (41) exposes the connection between the shadow value of manipulating the worst-case model ξ and the promised utility to the household. In addition, solving (42)

forward and converting to sequence notation allows us to conclude that

$$
\begin{aligned}
W_{M^*}(g_{t-1}, M^*_{t-1}, \xi_{t-1}; \Phi) &= \Phi E_{t-1} \sum_{i=0}^{\infty} \beta^i \frac{M^*_{t+i}}{M^*_{t-1}} \Omega_{t+i} \\
&= \Phi E_{t-1} m^*_t [E_t \sum_{i=0}^{\infty} \beta^i \frac{M^*_{t+i}}{M^*_t} \Omega_{t+i}] \\
&= \Phi E_{t-1} m^*_t U_{ct} b_t \\
&= \Phi \frac{U_{c,t-1}}{\beta} B_{t-1},
\end{aligned}
\tag{43}
$$

where $B_t \equiv \sum_{g_{t+1}} p_t(g_{t+1}, g^t) b_{t+1}(g^{t+1})$, the value of the portfolio of government securities. The above expression suggests a way to calculate the optimal debt from the derivative of the value function since

$$
b_t = \frac{\Omega_t}{U_{ct}} + B_t = \frac{\Omega_t}{U_{ct}} + \frac{\beta}{\Phi U_{ct}} W_{M^*}(g_t, M^*_t, \xi_t; \Phi).
\tag{44}
$$

The right side of (44) is a function of the vector (g_t, M^*_t, ξ_t), which shows clearly how the household's fear of misspecification induces history dependence to optimal debt $b_t = b(g_t, M^*_t, \xi_t)$. This is in contrast to the no-robustness case studied by Lucas and Stokey (1983), where it would depend only on the realization of g_t. Expression (44) also makes the calculation of the value of Φ remarkably easy. Having solved for the value function W and for the consumption allocation at time zero c_0 for a given Φ, we can calculate the expression $I \equiv \frac{\Omega_0}{U_{c0}} + \frac{\beta}{\Phi U_{c0}} W_{M^*}(g_0, 1, 0; \Phi) - b_0$ and increase (decrease) Φ if I is less than (greater than) zero.

5 A simple example

Let $b_0 = 0$ and assume a government expenditure process

$$
\begin{aligned}
g_t &= 0, t \neq T \\
g_T &= g > 0 \text{ with probability } \pi \\
g_T &= 0 \text{ with probability } 1 - \pi
\end{aligned}
$$

This one-time shock structure simplifies the dynamics and allows us to highlight features coming from the representative household's fear of model misspecification. Let x^g_t denote the value of variable x at history $g^t = (0, ..., g_T = g, g_{T+1} = 0, ..., g_t = 0)$ and respectively x^0_t be the random variable x at history $g^t = (0, ..., g_T = 0, g_{T+1} = 0, ..., g_t = 0)$. Apparently, we only need to differentiate in notation for period T and afterwards.

Assumption 5.1. $U_{cl} \geq 0$

This assumption assures that the tax rate is positive in the Lucas and Stokey case that the household fears no misspecification.

5.1 Optimal policy with no fear of misspecification

We first calculate the optimal plan for $\sigma = 0$. Remember that Lucas and Stokey found that consumption is function *only* of the realization of government expenditures at the current period $c_t = c(g_t)$. Thus $c_t = h_t \equiv c^{LS} \forall t \neq T$, $c_T^0 = c^{LS}$ and $c_T^g \equiv c^{LS}(g)$ with corresponding tax rates $\tau^{LS} > 0$ and $\tau^{LS}(g) > 0$. To finance this allocation the planner accumulates *surplus* $b_t = \frac{-\tau^{LS}h^{LS}}{1-\beta}\frac{1-\beta^t}{\beta^t} < 0, t = 1, ..., T-1$. At $t = T$, when $g_T = 0$, the planner borrows an amount equal to $b_T^0 = \frac{\tau^{LS}h^{LS}}{1-\beta}$. Thereafter, this debt is rolled over. Note from the dynamic budget constraint at $t = T-1$ that

$$b_{T-1} = \tau^{LS}h^{LS} + \beta(\pi\frac{U_c(c^{LS}(g), 1 - c^{LS}(g) - g)}{U_c(c^{LS}, 1 - c^{LS})}b_T^g + (1 - \pi)b_T^0).$$

Since $b_{T-1} < 0$ and $b_T^0 > 0$, it follows that $b_T^g < 0$. Thus, the government lends to the household, insuring against the contingency $g_T = g$. At $t = T, g_T = g$ the budget constraint is

$$b_T^g + g = \tau^{LS}(g)h^{LS}(g) + \beta\frac{U_c(c^{LS}, 1 - c^{LS})}{U_c(c^{LS}(g), 1 - h^{LS}(g))}b_{T+1}^g$$

and $b_{T+1}^g = \frac{\tau^{LS}h^{LS}}{1-\beta} > 0$, which implies a deficit $\tau^{LS}(g)h^{LS}(g) - g < 0$. Therefore, the government at $t = T-1$ uses the accumulated surplus b_{T-1}, the tax revenues $\tau^{LS}h^{LS}$, and the loan from the household b_T^0 to acquire assets $b_T^g < 0$ in order to insure against the contingency $g_T = g$. If the realization of the shock at T is zero, then the government rolls over its debt forever. If the realization is positive, then the government runs a *budget deficit*, which it finances partly with its assets b_T^g and partly by issuing new debt b_{T+1}^g that it rolls over forever. Observe how state-contingent debt allows the government to implement an optimal allocation that is dependent *only* on the level of the government expenditures g, independent of the period t.

5.2 Optimal policy with fear of misspecification.

As we showed in the proposition in the second section, the optimal consumption allocation is history dependent $c_t = c(g_t, M_t^*, \xi_t)$. In order to analyze the optimal plan we need to determine the dynamics of the two state variables (M_t^*, ξ_t). The absence of uncertainty $\forall t \neq T$ makes it easy to see from (10),(13), and (26), that

$$\begin{aligned} m_t^* &= 1, \forall t \neq T \\ M_t^* &= 1, \forall t < T \text{ and } M_{T+j}^* = m_T^*, \forall j \geq 0 \end{aligned}$$

and

$$\xi_t = 0, t < T \text{and } \xi_{T+j} = \xi_T, \forall j \geq 0$$

Therefore, the optimal consumption allocation is

$$
\begin{aligned}
c_t &= h_t = c(0,1,0) \equiv \bar{c}, t = 1,..,T-1 \\
c^0_{T+j} &= c^0_T = c(0, m^{*0}_T, \xi^0_T) \equiv c^0 \forall j \geq 0 \\
c^g_T &= c(g, m^{*g}_T, \xi^g_T) \equiv c^g \\
c^g_{T+j} &= c^g_{T+1} = c(0, m^{*g}_T, \xi^g_T) \equiv \bar{c}^g \forall j \geq 1
\end{aligned}
\tag{45}
$$

with respective tax rates $(\bar{\tau}, \tau^0, \tau^g, \bar{\tau}^g)$.

Remark 5.2. *Note that because the variables (M^*_t, ξ_t) do not revert to their value before the shock, the shock has a permanent effect on the allocation and tax rate.*

The optimal distortions at the two histories are

$$
m^{*g}_T = \frac{\exp(\sigma V^g_T)}{\pi \exp(\sigma V^g_T) + (1-\pi)\exp(\sigma V^0_T)}
$$

$$
m^{*0}_T = \frac{\exp(\sigma V^0_T)}{\pi \exp(\sigma V^g_T) + (1-\pi)\exp(\sigma V^0_T)}.
$$

The continuation values have the forms

$$
V^0_T = \frac{U(c^0, 1-c^0)}{1-\beta}
\tag{46}
$$

$$
V^g_T = U(c^g, 1-c^g-g) + \frac{\beta}{1-\beta}U(\bar{c}^g, 1-\bar{c}^g).
\tag{47}
$$

The values of the multiplier ξ_T at the two histories are

$$
\xi^g_T = \sigma m^{*g}_T \eta^g_T = \sigma m^{*g}_T \left[\mu^g_T - E_{T-1}m^*_T\mu_T\right]
$$

$$
\xi^0_T = \sigma m^{*0}_T \eta^0_T = \sigma m^{*0}_T \left[\mu^0_T - E_{T-1}m^*_T\mu_T\right]
$$

It remains to find the sign of η_t, so we need to compute the innovation in the marginal utility of the government asset position. Knowing the sign of the forecast error will determine the sign of ξ_t and, therefore, through (33), the sign of the tax rate. The following lemma is very useful

Lemma 5.3. *The Ramsey outcome prescribes (a) $E_{T-1}m^*_T\mu_T < 0$ (b) $\mu^g_T < \mu^0_T$. Therefore, $\eta^g_T < 0, \eta^0_T > 0$ and consequently $\xi^g_T > 0, \xi^0_T < 0$.*

Proof. (a) The dynamic budget constraint of the government at $t = T-1$ is

$$
b_{T-1} = \bar{\tau}\bar{h} + \beta\left[\pi m^{*g}_T \frac{U_c(c^g, 1-g-c^g)}{U_c(\bar{c}, 1-\bar{c})}b^g_T + (1-\pi)m^{*0}_T \frac{U_c(c^0, 1-c^0)}{U_c(\bar{c}, 1-\bar{c})}b^0_T\right]
$$

where we have already substituted for the equilibrium price of state contingent debt from (16). But since $\mu_t = \Phi U_{ct}b_t$, we can rewrite the budget constraint as $b_{T-1} - \bar{\tau}\bar{h} = \frac{\beta}{\Phi U_c(\bar{c}, 1-\bar{c})}E_{T-1}m^*_T\mu_T$.

20

The government has accumulated a surplus until $t = T - 1$ that has the size $b_{T-1} = \frac{-\bar{\tau}\bar{h}}{1-\beta}\frac{1-\beta^{T-1}}{\beta^{T-1}} < 0$ since $\bar{\tau} > 0$ and $\Phi > 0$. This gives the desired result.

(b) Suppose the contrary, namely, $\mu_T^g \geq \mu_T^0$. Obviously, since $\min\{\mu_T^0, \mu_T^g\} = \mu_T^0 \leq E_{T-1}m_T^*\mu_T \leq \max\{\mu_T^0, \mu_T^g\} = \mu_T^g$, we have $\eta_T^0 \leq 0$ and $\eta_T^g \geq 0$ and, therefore, $\xi_T^0 \geq 0$ and $\xi_T^g \leq 0$. But from the equation for the optimal tax rate (33) we note that $\xi_T^0 \geq 0$ necessarily implies a positive tax rate $\tau^0 > 0$. But since the consumption after the realization $g_T = 0$ remains constant we can infer that the government assets are $b_t^0 = b_T^0 = \frac{\tau^0 h^0}{1-\beta} > 0, \forall t \geq T$ since $\tau^0 > 0$. But $\mu_T^0 \leq E_{T-1}m_T^*\mu_T < 0$ by part (a) of the Lemma and since $\mu_T^0 = \Phi U_c(c^0, 1-c^0)b_T^0$, it follows that $b_T^0 < 0$, which is a contradiction. $\qquad\square$

Note from the above result that we can conclude that $\mu_T^g < 0$ and therefore $b_T^g < 0$, i.e., the government lends to the household in anticipation of the contingency $g_T = g$. Furthermore, since $\xi_T^g > 0$, the tax rates at the realization $g_T = g$ and afterwards are positive $\tau^g, \bar{\tau}^g > 0$. Therefore, $b_{T+j}^g = b_{T+1}^g = \frac{\bar{\tau}^g\bar{h}^g}{1-\beta} > 0, \forall j \geq 1$ and from the dynamic budget constraint when $g_T = g$ we have

$$\tau^g h^g - g = b_T^g - \beta\frac{U_c(\bar{c}^g, 1-\bar{c}^g)}{U_c(c^g, 1-c^g-g)}b_{T+1}^g < 0$$

Thus, the government always runs a deficit at the contingency $g_T = g$. It remains to describe the policy for the contingency of $g_T = 0$ and afterwards. Since $\xi_T^0 < 0$, there is the theoretical possibility of having a subsidy $\tau^0 < 0$ instead of a tax. Thus, we have two cases:

- $\mu_T^g < E_{T-1}m_T^*\mu_T < 0 < \mu_T^0$

In this case, since $\mu_T^0 = \Phi U_c(c^0, 1-c^0)b_T^0 > 0$ we conclude that $b_T^0 = \frac{\tau^0 h^0}{1-\beta} > 0 \Rightarrow \tau^0 > 0$. Thus, even if the multiplier $\xi_T^0 < 0$, it is not negative enough to reverse the tax rate and make it a subsidy. The government borrows from the household in anticipation of the contingency $g_T = 0$ and afterward repays the interest by imposing constant taxes.

- $\mu_T^g < E_{T-1}m_T^*\mu_T < \mu_T^0 < 0$

In contrast to the previous case, we now have the possibility that $b_T^0 < 0$ since $\mu_T^0 = \Phi U_c(c^0, 1-c^0)b_T^0 < 0$. But that implies that $\tau^0 < 0$, i.e., we have a subsidy. Inspecting the formula for the optimal tax rate τ^0 (33), we see that this can happen if

$$1 + \xi_T^0 + \Phi m_T^{*0} < 0 \Rightarrow \eta_T^0 > (-1/\sigma)[1/m_T^{*0} + \Phi]$$

It is clear from the right side of the above condition that this case can't happen when $\sigma = 0$. So the debt and tax policy of the government when $g_T = 0$ are opposite from those in the first case. Here the government lends to the household against the contingency $g_T = 0$ and afterwards uses the interest on the loan in order to give a subsidy to the household, keeping the surplus constant at $b_t^0 = \tau^0 h^0/(1-\beta), t \geq T$.

5.2.1 A particular utility function.

Assume the utility function

$$U(c, 1-h) = \frac{c^{1-\gamma}}{1-\gamma} - a_h \frac{h^{1+\phi_n}}{1+\phi_n}$$

with derivatives $U_c = c^{-\gamma}, U_{cc} = -\gamma c^{-\gamma-1}, U_l = a_h h^{\phi_n}, U_{ll} = -a_h \phi_h h^{\phi_n-1}$. Note that $1/\phi_n$ is the Frisch Elasticity and that a_h controls the amount of time spent working. The labor supply condition is

$$a_h \frac{h^{\phi_n}}{c^{-\gamma}} = 1 - \tau$$

This utility function leads to uniform taxation in the Lucas and Stokey (1983) case, the tax being independent of the realization of government expenditures and equal to

$$\tau^{LS} = \frac{\Phi^{LS}(\gamma + \phi_n)}{1 + \Phi^{LS}(1 + \phi_n)}$$

This will just make stronger the contrast between the case with robustness.

When we activate the household's fear of misspecification by setting $\sigma < 0$, the formula for the tax is

$$\tau_t = \frac{\Phi M_t^*(\gamma + \phi_n)}{1 + \xi_t + \Phi M_t^*(1 + \phi_n)}$$

where Φ again measures the cost of distortionary taxation with $\sigma < 0$. Therefore, we have

$$\tau_t = \bar{\tau} = \frac{\Phi(\gamma + \phi_n)}{1 + \Phi(1 + \phi_n)}, t = 0, 1, ..., T-1$$

$$\tau_{T+j}(g) = \tau^g = \bar{\tau}^g = \frac{\Phi m_T^{*g}(\gamma + \phi_n)}{1 + \xi_T^g + \Phi m_T^{*g}(1 + \phi_n)}, j \geq 0$$

$$\tau_{T+j}^0 = \tau^0 = \frac{\Phi m_T^{*0}(\gamma + \phi_n)}{1 + \xi_T^0 + \Phi m_T^{*0}(1 + \phi_n)}, j \geq 0.$$

We use the following parameters to compute our numerical examples.

Prob$(g_T = g) = \pi$	0.6
β	1/1.01
T	4
g	0.7
γ	2
ϕ_n	1
a_h	$3^{\phi_n+\gamma} = 27$

The value of a_h induces the household to work 1/3 of its time when there are no taxes and no government expenditures. The following table includes outcomes for various θ:

	$\theta = \infty$	$\theta = 100$	$\theta = 10$	$\theta = 5$	$\theta = 1$
Φ	0.0109	0.0113	0.0124	0.0117	0.0111
V_T^0	-454.5529	-454.5500	-454.5132	-454.5015	-454.5000
V_T^g	-465.9876	-465.9974	-466.0658	-466.0838	-466.0867
m_T^{*0}	1	0.9322	0.4339	0.1542	1.5482e-005
m_T^{*g}	1	1.0452	1.3774	1.5638	1.6667
$\bar{\tau}$	τ^{LS} =0.0319	0.0332	0.0364	0.0342	0.0325
$\tau^g = \bar{\tau}^g$	τ^{LS} =0.0319	0.0346	0.0496	0.0528	0.0533
τ^0	τ^{LS} =0.0319	0.0310	0.0161	0.0054	5.1307e-007
$\bar{c}$	c^{LS} =0.3298	0.3296	0.3292	0.3295	0.3297
c^g	$c^{LS}(g)$ =0.1996	0.1994	0.1980	0.1977	0.1976
$\bar{c}^g$	c^{LS} =0.3298	0.3294	0.3277	0.3274	0.3273
c^0	c^{LS} =0.3298	0.3298	0.3315	0.3327	0.3333
$\tau_T^g h_T^g - g$	-0.6713	-0.6688	-0.6554	-0.6526	-0.6521
b_T^0	1.0621	1.0337	0.5382	0.1810	1.7273e-005
b_T^g	-0.2859	-0.2508	-0.0618	-0.0219	-0.0158

The table reveals the interesting outcome that the cost of distortionary taxation is *not monotonic* in θ. While increasing fear of misspecification by lowering θ below $+\infty$ initially increases the cost Φ, after a while Φ starts decreasing and eventually reaches a level still higher than the $\theta = +\infty$ case without robustness (last column). The reason is that with increasing doubts, the household distorts probabilities towards the event of positive government expenditures $g_T = g$ by increasing m_T^{*g} and decreasing m_T^{*0}. But the Ramsey plan prescribes lower taxes for situations with low probability and high taxes for situations with high probability. Eventually, the tax at the contingency $g_T = 0$ and onward τ^0 becomes zero (see last column of the table), which leads to practically no distortion from taxation at this history (note that consumption c^0 approaches the first best value of 1/3), which increases welfare of the planner. However, the tax τ^g is increasing, which reduces welfare. These two opposing forces lead to the non-monotonicity of Φ in θ.

6 Expansion around the no robustness case $\sigma = 0$

The representative household's fear of misspecification introduces two new state variables (M_t^*, ξ_t) in the Ramsey problem that where absent in Lucas and Stokey (1983). That there are complete markets but no capital in Lucas and Stokey's economy facilitates computing the equilibrium very easily. This allows us to expand in σ around the entire stochastic path associated with the Ramsey plan at $\sigma = 0$. Every endogenous random variable in the model is a function of the parameter σ. So the first order approximation at history g^t for every variable z_t around $\sigma = 0$ takes the form

$$z_t(g^t, \sigma) \simeq z_t(g^t, 0) + \sigma z_\sigma(g^t, 0) \tag{48}$$

where $z_t(g^t, 0)$ is the Ramsey plan of Lucas and Stokey (1983). So the expansion tells us how introducing a small fear of misspecification perturbs the Lucas and Stokey Ramsey plan. In the next subsection, we calculate the various partial derivatives z_σ and evaluate them at $\sigma = 0$. For ease of notation, we use the following convention $z_t(\sigma) \equiv z_t(g^t, \sigma)$ and $z'_t(\sigma) \equiv z_\sigma(g^t, \sigma)$. In calculating $z'_t(0)$, substantial simplification comes from the fact that $m_t^*(0) = M_t^*(0) = 1$ and $\xi_t(0) = 0$. Details of the general case derivations are relegated to appendix C.1 and the explicit calculations of the formulas for Markovian shocks to appendix C.2.

6.1 General case

The dynamics of the approximated solution are driven by the partials $(M_t^{*\prime}(0), \xi'_t(0))$ of the state variables. Observe that the *martingale* property of $M_t^*(\sigma)$ and $\xi_t(\sigma)$ is bequeathed to $(M^{*\prime}, \xi')$

$$E_{t-1}M_t^{*\prime}(\sigma) = M_{t-1}^{*\prime}(\sigma) \text{ and } E_{t-1}\xi'_t(\sigma) = \xi'_{t-1}(\sigma) \forall \sigma \leq 0$$

The law of motion for $M^{*\prime}$ for $\sigma = 0$ is

$$M_t^{*\prime}(0) = m_t^{*\prime}(0) + M_{t-1}^{*\prime}(0), M_0^{*\prime} \equiv 0 \tag{49}$$

where

$$m_t^{*\prime}(0) = V_t(0) - E_{t-1}V_t(0). \tag{50}$$

Therefore, we can approximate the conditional distortion to first order as

$$
\begin{aligned}
m_t^*(\sigma) &\simeq m_t^*(0) + \sigma m_t^{*\prime}(0) \\
&= 1 + \sigma(V_t(0) - E_{t-1}V_t(0))
\end{aligned}
$$

A negative innovation to the utility of the household prompts it to assign higher probability to this event. Knowing the Ramsey allocation of the Lucas-Stokey economy allows us immediately to calculate the approximation to the optimal distortion.

The law of motion for ξ_t is

$$\xi'_t(0) = \eta_t(0) + \xi'_{t-1}(0), \xi'_0 \equiv 0 \tag{51}$$

with

$$\eta_t(0) = \Phi(0)\big[U_{ct}(0)b_t(0) - E_{t-1}U_{ct}(0)b_t(0)\big]. \tag{52}$$

Thus, the crucial component in the evolution of the derivative of the multiplier ξ' is the innovation in the marginal utility of the bond holdings of the LS economy, weighted by the cost of distortionary taxation.

It remains for us to find $c'_t(0)$ (which is equal to $h'_t(0)$ by the resource constraint (1)) and $\Phi'(0)$. The solution for $c'_t(0) \ \forall t \geq 0$ takes the form

$$c'_t(0) = h'_t(0) = \frac{U_{lt}(0) - U_{ct}(0)}{K_t(0)}\left[(\xi'_t(0) - M_t^{*\prime}(0)) - \frac{\Phi'(0)}{\Phi(0)}\right] \tag{53}$$

where $K_t(0)$ is a function of the allocation at $\sigma = 0$, defined in the appendix. Initial conditions are $M_0^{*\prime} = \xi_0^\prime = 0$. $K_t(0)$ is an expression that is directly related to second derivative of the Lagrangian of the Lucas and Stokey problem. In particular, we are going to assume for the rest of the paper that $K_t(0) < 0$, which ensures that the second-order conditions of the Lucas and Stokey problem hold.

Lemma 6.1. *Assume that $K_t(0) < 0, \forall t \geq 0$. Then the second-order sufficient conditions of the optimal fiscal policy problem without fear of misspecification are satisfied.*

Proof. See appendix C.1. □

We can observe from the above expression that the dynamics of $c_t^\prime(0)$, and therefore (through (48)) the dynamics of c_t, are driven by the difference $(\xi^\prime - M^{*\prime})$ for small fears of model misspecification. Finding $c_t^\prime(0)$ allows us also to derive $\tau_t^\prime(0)$ and $b_t^\prime(0)$ and consequently have an expression for the approximate tax and debt policy.

$$\tau_t^\prime(0) = \alpha_t(0)\left[\xi_t^\prime(0) - M_t^{*\prime}(0) - \frac{\Phi^\prime(0)}{\Phi(0)}\right] \tag{54}$$

$$b_t^\prime(0) = \alpha_1(g^t)(\xi_t^\prime(0) - M_t^{*\prime}(0)) + \alpha_2(g^t) + \alpha_3(g^t), \tag{55}$$

where coefficients α and $\alpha_i, i = 1, 2, 3$ are defined in the appendix. Note that $\tau_t^\prime(0)$ and $b_t^\prime(0)$ depend also on $(\xi^\prime - M^{*\prime})$ due to the fact that they are functions of $c_t^\prime(0)$. The coefficient $\alpha_t(0)$ depends only on the Lucas and Stokey allocation, so it is a function only of g_t. On the other hand, coefficients $\alpha_i, i = 1, 2, 3$ depend on the whole history of shocks g^t. If we assume Markovian shocks though, then they are *solely* a function of g_t.

Finally, the expression for the derivative $\Phi^\prime(0)$ is

$$\Phi^\prime(0) = \frac{(\Phi(0))^2 \sum_{t,g^t} \beta^t \pi_t(g^t) M_t^{*\prime}(0)\Omega_t(0) + \Phi(0) \sum_{t,g^t} \beta^t \pi_t(g^t) \frac{(U_{ct}(0) - U_{lt}(0))^2}{K_t(0)}(\xi_t^\prime(0) - M_t^{*\prime}(0))}{\sum_{t,g^t} \beta^t \pi_t(g^t) \frac{\left(U_{ct}(0) - U_{lt}(0)\right)^2}{K_t(0)}}$$

$$(56)$$

The multiplier Φ encapsulates the cost of distortionary taxation. Expression (56) measures the change in the multiplier if we impute fear of misspecification to the representative household. Note that the denominator is negative due to $K_t(0) < 0$, whereas the sign of the nominator depends on the specifics of the problem. If $\Phi^\prime(0)$ is negative, then robustness *increases* the cost of distortionary taxation (remember that $\sigma < 0$).

6.2 Quasi-linear utility

We proceed with a quasi-linear example. Linearity in consumption eliminates the effects of marginal utility on labor supply and on equilibrium prices. Let the period utility of the agent take the form

$$U(c, l) = c + v(1 - l)$$

Quasi-linear utility leads to Arrow-Debreu prices $q_t = \beta^t \pi_t M_t^*$. Thus, with this preference specification, the planner cannot manipulate the marginal utility to allocate distortions over

histories. But he can still use the endogenous beliefs of the agent. For the rest assume that $v(1 - l) = v(h) = -\frac{1}{2}h^2$. Then $U_l = h, U_{ll} = -1$ and from (31) we find that the Ramsey labor allocation ans tax rate are

$$h_t = \frac{1 + \xi_t + \Phi M_t^*}{1 + \xi_t + 2\Phi M_t^*} \text{ and } \tau_t = 1 - h_t = \frac{\Phi M_t^*}{1 + \xi_t + 2\Phi M_t^*}.$$

Equations (18-21), together with (25) and (26) determine the dynamics of (ξ_t, M_t^*) and the size of Φ.

6.2.1   No fear of misspecification case ($\sigma = 0$)

The optimal plan prescribes *constant* taxes and labor for any history g^t. The lack of dependence of taxes and labor on the realization of g_t is a special result that will make more transparent the effects of the representative household's fear of misspecification. In particular,

$$
\begin{aligned}
h_t(0) &\equiv h = \frac{1 + \Phi(0)}{1 + 2\Phi(0)} \\
\tau &= 1 - h \\
c_t(0) &= h - g_t
\end{aligned}
$$

where $\Phi(0)$ represents the value of the multiplier for the Lucas and Stokey economy, where agents fully trust their model. Observe that tax revenues are constant over time and across states. Furthermore, since $\Phi(0) > 0$, $h^{LS} \in (1/2, 1]$. We can pin down labor and therefore $\Phi(0)$ from the intertemporal budget constraint. This leads to finding the root of the quadratic equation $Q(h) \equiv h^2 - h + G$, where $G \equiv (1 - \beta)[b_0 + E_0 \sum_{t=0}^{\infty} \beta^t g_t]$. We assume that G is smaller than $1/4$ in order for a solution to exist and larger than 0 in order to assume away an initial surplus that can finance the present value of government expenditures. We can discard the smaller root because it is less than $1/2$. Therefore $h = \frac{1 + \sqrt{1 - 4G}}{2}$. Having obtained the Ramsey plan for $\sigma = 0$, we can proceed to the expansion.

6.2.2   Ramsey allocation for small robustness

For the quasi-linear case the formulas of the expansion of the previous section simplify considerably, as shown in appendix C.3. In particular, we have the following partial derivatives

$$m_t^{*\prime}(0) = V_t(0) - E_{t-1}V_t(0) = -(E_t - E_{t-1})[\sum_{i=0}^{\infty} \beta^i g_{t+i}] \tag{57}$$

$$\eta_t(0) = \Phi(0)[b_t(0) - E_{t-1}b_t(0)] = -\Phi(0)(E_t - E_{t-1})[\sum_{i=0}^{\infty} \beta^i g_{t+i}] \tag{58}$$

$$c_t'(0) = h_t'(0) = \frac{\Phi(0)}{(1 + 2\Phi(0))^2}(\xi_t'(0) - M_t^{*\prime}(0)) - \frac{\Phi'(0)}{(1 + 2\Phi(0))^2} \tag{59}$$

$$\Phi'(0) = (1 - \beta)(1 + 2\Phi(0))^3 E_0 \sum_{t=0}^{\infty} \beta^t M_t^{*\prime}(0)g_t \tag{60}$$

Equations (57) and (58) show that the innovation in the present value of government expenditures determines the dynamics of the solution.[7]

To attain more concrete results, assume a Wold moving average representation for the government expenditures

$$g_t = \mu_g + \gamma(L)\varepsilon_t \tag{61}$$

where $\varepsilon_t \sim \text{iid}(0, \sigma_\varepsilon^2)$ and where we assume that $\gamma(\beta) > 0$.[8] Then the innovation in the present value of government expenditures is

$$(E_t - E_{t-1})[\sum_{i=0}^{\infty} \beta^i g_{t+i}] = \gamma(\beta)\varepsilon_t \tag{62}$$

where $\gamma(\beta)$ is the present value of the coefficients in the infinite order moving average representation. Thus, the distortion to the conditional distribution is approximately

$$
\begin{aligned}
m_t^* &= 1 - \sigma\gamma(\beta)\varepsilon_t \\
&= 1 + \frac{1}{\theta}\gamma(\beta)\varepsilon_t.
\end{aligned}
$$

This allows us to get approximate formulas for the conditional distribution of ε_t:

Result 6.2. *The conditional mean and variance of ε_t under the distorted measure are approximately equal to*

$$
\begin{aligned}
\tilde{E}_t\varepsilon_{t+1} &\equiv E_t m_{t+1}^* \varepsilon_{t+1} = \frac{1}{\theta}\gamma(\beta)\sigma_\varepsilon^2 \\
\tilde{V}ar_t(\varepsilon_{t+1}) &\equiv E_t m_{t+1}^*(\varepsilon_{t+1} - \tilde{E}_t\varepsilon_{t+1})^2 = \sigma_\varepsilon^2 + \frac{\gamma(\beta)}{\theta}E_t\varepsilon_{t+1}^3
\end{aligned}
$$

The household's worst-case probability measure has an *increased* conditional mean of government expenditure. Furthermore, if we assume that $E_t\varepsilon_{t+1}^3 = 0$, as would be the case for a normal distribution, we have

$$\tilde{V}ar_t(\varepsilon_{t+1}) = \sigma_\varepsilon^2$$

so that the conditional variance stays *constant*.

Using expression (62) for the innovation in the present value of government expenditures, we can infer the law of motion for ξ' and $M^{*'}$. In particular,

$$M_t^{*'}(0) = M_{t-1}^{*'}(0) - \gamma(\beta)\varepsilon_t$$

and

$$\xi_t'(0) = \xi_{t-1}'(0) - \Phi(0)\gamma(\beta)\varepsilon_t$$

So the law of motion for the difference $\xi' - M^{*'}$ is

$$\xi_t'(0) - M_t^{*'}(0) = \xi_{t-1}'(0) - M_{t-1}^{*'}(0) + (1 - \Phi(0))\gamma(\beta)\varepsilon_t \tag{63}$$

---

[7]The innovation to the present value of the income of a representative consumer who fears misspecification plays an important role in determining market prices of risk. For example, see Barillas et al. (2007).

[8]Note that here we have dropped the restriction that g lives on a countable space.

Remark 6.3. *The dynamics are driven by a random walk (63) with respect to the approximating measure π, with an increment that is directly associated with the innovation in the present value of government expenditures.*

Let $T_t \equiv \tau_t h_t$ denote tax revenues. Using (48), the formulas (59),(63), and the fact that $\tau_t'(0) = -h_t'(0)$, we get the following result

Result 6.4. *(Random walk components) With small household fears of model misspecification, the labor allocation, tax rate, and tax revenues follow random walks with respect to π:*

$$
h_t - h_{t-1} = -\frac{1}{\theta}\frac{\Phi(0)(1-\Phi(0))}{(1+2\Phi(0))^2}\gamma(\beta)\varepsilon_t
$$

$$
\tau_t - \tau_{t-1} = \frac{1}{\theta}\frac{\Phi(0)(1-\Phi(0))}{(1+2\Phi(0))^2}\gamma(\beta)\varepsilon_t
$$

$$
T_t - T_{t-1} = \frac{1}{\theta}\frac{\Phi(0)(1-\Phi(0))}{(1+2\Phi(0))^3}\gamma(\beta)\varepsilon_t
$$

with initial conditions $(h_0, \tau_0, T_0) = (h + \frac{1}{\theta}\frac{\Phi'(0)}{(1+2\Phi(0))^2}, \tau - \frac{1}{\theta}\frac{\Phi'(0)}{(1+2\Phi(0))^2}, \tau h - \frac{1}{\theta}\frac{\Phi'(0)}{(1+2\Phi(0))^3})$. The solution for consumption is given by $c_t = h_t - g_t$.

The household's fear of misspecification introduces non-stationarity into the Ramsey plan. Even a small fear of misspecification leads to a Ramsey plan that differs substantially from the Lucas and Stokey (1983) plan in the sense that the optimal allocation no longer simply inherits the stochastic properties of government expenditures. Note that even in the case where the government expenditures are i.i.d. ($\gamma_0 = 1, \gamma_i = 0, i \geq 1$), the labor allocation remains a random walk. Also note how the reaction of the tax rate and labor to a shock depends on the size of $\Phi(0)$. Note that $\Phi(0) \gtrless 1$ (depending on the present value of government expenditures and on initial debt b_0). If $\Phi(0) < 1$, then an innovation to government expenditures leads to a *permanent* increase in the tax rate and to a *permanent* decrease in labor. The decrease in labor is smaller than the increase in taxes, so altogether there is a permanent increase in tax revenues. All these quantities would be constant in the corresponding economy of Lucas and Stokey (1983).

It is worthwhile to analyze more carefully the effects of an innovation in the present value of government expenditures on the optimal allocation and tax rate for small fear of misspecification. Consider for example the optimal tax rate

$$
\tau_t(\sigma) = \tau + \sigma\frac{\Phi(0)}{(1+2\Phi(0))^2}\Big[M_t^{*'}(0) - \xi_t'(0) + \frac{\Phi'(0)}{\Phi(0)}\Big],
$$

or, using $M_t^{*'}(0) = (M_t^*(\sigma) - 1)/\sigma$ and $\xi_t'(0) = \xi_t(\sigma)/\sigma$

$$
\tau_t(\sigma) = \tau + \frac{\Phi(0)}{(1+2\Phi(0))^2}[(M_t^*(\sigma) - 1) - \xi_t(\sigma)] + \sigma\frac{\Phi'(0)}{(1+2\Phi(0))^2}. \tag{64}
$$

A positive innovation in the present value of government expenditures has two effects on the optimal tax rate. First, it leads to a negative innovation in utility by (57) and therefore to a positive innovation in the likelihood ratio M_t^*. This, as seen from (64) *increases* the tax rate, as noted also in remark (3.7). Second, by the logic of the intertemporal budget constraint, a positive innovation in the present value of government expenditures leads to a negative innovation in government debt b_t in order to finance all future government deficits. This means the shadow value of the worst-case model of the household ξ_t increases and consequently the optimal tax rate *decreases* by (64). The first effect comes from the *heterogeneity* of beliefs between the planner and the household, whereas the second from the planner's desire to manipulate the household's worst-case *endogenous* beliefs. The law of motion (63) for the difference between ξ' and $M^{*'}$ tells us that the two effects exactly cancel for $\Phi(0) = 1$, which would correspond to a labor allocation in the Lucas and Stokey (1983) economy of $h = 2/3$. In this borderline case, the dynamics are *muted* and the optimal labor allocation and tax rate remain constant, but at a different level than in the case without fear of misspecification. From (6.4), we see that the tax rate becomes $\tau(\sigma) = \tau - \frac{1}{\theta}\frac{\Phi'(0)}{9}$ and the labor allocation $h(\sigma) = 1 - \tau(\sigma)$.

The particular moving average representation (61) allows us to calculate explicitly the derivative of the multiplier Φ

$$\Phi'(0) = -(1 - \beta)(1 + 2\Phi(0))^3 \gamma(\beta)\sigma_\varepsilon^2 \sum_{t=0}^{\infty} \beta^t \sum_{i=0}^{t-1} \gamma_i. \tag{65}$$

If we assume that $\sum_{i=0}^{t-1} \gamma_i > 0$ ($\gamma_i > 0$ would be sufficient for this to hold) then we have the result that $\Phi'(0) < 0$, thus the household's fear of misspecification increases the cost of distortionary taxation. For example, if we assume an AR(1) process with persistence $\rho \in [0, 1)$ for g_t, then $\gamma_i = \rho^i, i = 0, 1, ..$ and $\Phi'(0) = -\beta\frac{(1+2\Phi(0))^3}{(1-\beta\rho)^2}\sigma_\varepsilon^2$.

It helps to develop a deeper understanding of the effects of model misspecification to consider the optimal debt policy implied by the above allocation. Note that when the representative household completely trusts the stochastic model, optimal debt is

$$b_t(0) = \frac{\tau h}{1 - \beta} - E_t \sum_{i=0}^{\infty} \beta^i g_{t+i}$$

Assume for example that government expenditures are i.i.d. Then

$$b_t(0) = \frac{\tau h - \mu_g}{1 - \beta} - \varepsilon_t$$

This expression makes it clear how the government uses state-contingent debt in order to smooth the distortions among histories and also how the optimal state-contingent debt inherits the i.i.d. nature of fiscal shocks. In particular, in the case of a positive (bad) shock ε_t, the government issues *less* debt, i.e. it increases its claims against the private sector, which

will be used to finance expenditures in that contingency. On the other hand, a negative shock leads to *higher* indebtedness to the private sector. So the government insures against a bad shock by buying claims against the private sector against this contingency and selling claims against the low expenditure contingency. Complete markets enable the government to do that and as long as there is full confidence in the model from both parties. Considering now the case where $\sigma < 0$, we find that for a small household concern about misspecification, the debt is approximately equal to

$$b_t(\sigma) = \frac{\tau h}{1 - \beta} - E_t \sum_{i=0}^{\infty} \beta^i g_{t+i} + \frac{1}{\theta} \frac{(1-\beta)^{-1}\Phi(0)}{(1 + 2\Phi(0))^3}(\xi_t'(0) - M_t^{*\prime}(0))$$

which in the i.i.d. case reduces to

$$b_t(\sigma) = \frac{\tau h - \mu_g}{1 - \beta} - \varepsilon_t + \frac{1}{\theta} \frac{(1-\beta)^{-1}\Phi(0)}{(1 + 2\Phi(0))^3}(\xi_t'(0) - M_t^{*\prime}(0)).$$

Despite the fact that government expenditures are i.i.d., fear of misspecification adds a unit-root component to optimal debt, an outcome that emerges from the fact that beliefs are endogenous (ξ_t) and heterogenous (M_t^*) The persistence in debt resembles the outcome in the incomplete markets-rational expectations economy of Aiyagari et al. (2002), but here all markets are open. As expected, in the case of $\Phi(0) = 1$ the two opposite effects canceling, implying - for small fear of misspecification - the *same* optimal debt as in the Lucas and Stokey economy, since $\xi_t'(0) - M_t^{*\prime}(0) = 0, \forall t \geq 0$.

6.3 Aiyagari et al. (2002) utility function

In this section, we analyze the effects of small fear of misspecification by using the utility function and shock process of Aiyagari et al. (2002):

$$U(c, l) = \frac{c^{1-\gamma} - 1}{1 - \gamma} + a_l \frac{l^{1-\psi} - 1}{1 - \psi}$$

Note that the parameter $-1/\psi$ is the elasticity of leisure with respect to after-tax wage. We follow the calibration of Aiyagari et al. and set $(\beta, \gamma, \psi, a_l) = (0.95, 0.5, 2, 1)$. Furthermore, we scale up the amount of leisure available to the household to $\bar{l} = 100$ and set the initial debt equal to zero $b_0 = 0$. Aiyagari et al. use an iid $N(30, 2.5^2)$ process for the government expenditures, which we approximate with gaussian quadrature with 11 nodes. We set the initial shock equal to the mean of government expenditures $g_0 = \bar{g} = 30$.

6.3.1 Properties of the no-robustness plan

It is useful to analyze the qualitative properties of the optimal allocation, tax and debt in the case of rational expectations, since our approximation depends on a crucial way on them. Consider for example a class of utility functions that are separable in consumption and leisure and have convex marginal utility of consumption and leisure. The Aiyagari et al. utility
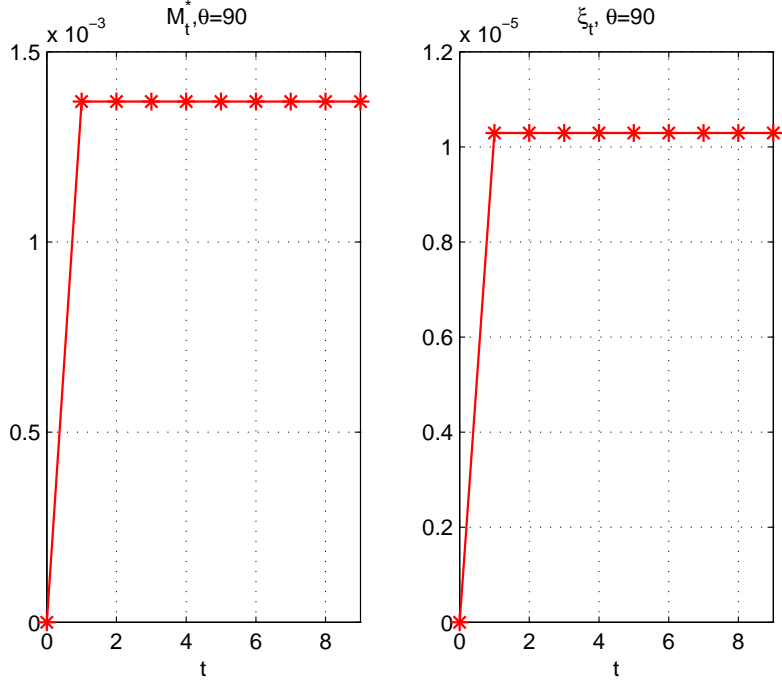
Figure 1: Impulse response functions of the two martingales to a positive fiscal shock

function which is our main object of interest is obviously member of this class. Furthermore, remember that we are working under the assumption that $K_t(0) < 0$, which ensures that the second-order conditions are satisfied. This assumption takes the form $K = T_1 - T_2 < 0$ where $T_1 \equiv (1 + 2\Phi)U_{cc} + \Phi U_{ccc}c$ and $T_2 \equiv -(1 + 2\Phi)U_{ll} + \Phi U_{lll}h > 0$. We simplified the notation of K, since it is clear that we are working the optimal solution of the no-robustness case. The following auxiliary lemma instructs us about the nature of the Lucas and Stokey plan.

Lemma 6.5. *Assume that $U_{cl} = 0, U_{ccc} > 0, U_{lll} > 0$. Then the Lucas and Stokey solution has the following properties:*

1. *Consumption c and period utility U are decreasing in g. If also g is iid, then expected discounted utility V is also decreasing in g.*

2. *If also $T_1 < 0$, then labor h is increasing in g.*

3. *If also $U_c U_{ll} T_1 + U_l U_{cc} T_2 < 0$, then the tax rate τ is increasing in g.*

4. *If also $(U_{ll}h - U_l)T_1 + (U_{cc}c + U_c)T_2 > 0$, then the government surplus in marginal utility terms Ω is decreasing in g. If also g is iid, then $y = U_c b$ and the government debt b are decreasing in g.*

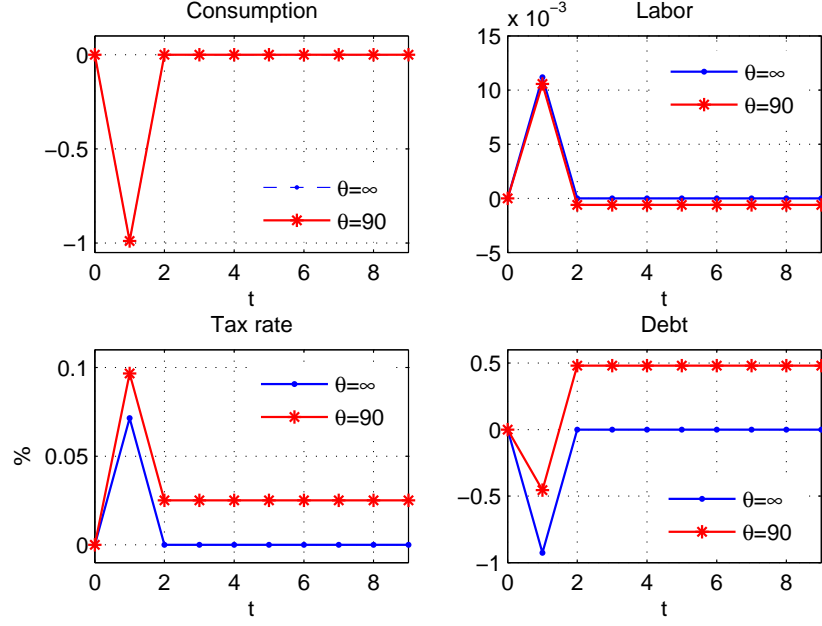*Proof.* See appendix C.4 □

31

Figure 2: Impulse response functions to a positive fiscal shock for the case of full confidence in the model and for the case of fear of misspecification with $\theta = 90$.

We can also derive the following corollary.

Corollary 6.6. *The Aiyagari et al. utility function satisfies $T_1 < 0$ and $U_c U_{ll} T_1 + U_l U_{cc} T_2 < 0$. If $\gamma < 1$, then it is ensured that $(U_{ll} h - U_l) T_1 + T_2 (U_{cc} c + U_c) > 0$.*

Proof. See appendix C.4. □

So our calculated example satisfies all the conditions of the above lemma, which allows us to understand in a deeper way the changes induced by the household's fear of misspecification.

6.3.2 Impulse response functions

At first we are going to analyze the impulse response functions to a fiscal shock at $t = 1$ for consumption, labor, the tax rate and debt under fear of misspecification. We consider the level of each variable at history $g^t = (\bar{g}, \bar{g}, ..., \bar{g})$ and at history $\hat{g}^t = (\bar{g}, g', \bar{g}, ..., \bar{g})$, and calculate the respective rate of change. We consider a positive shock and set $g' = 32.32$, which corresponds to the 7th node in our approximation scheme. Our approximate solution for small fear of misspecification is driven by the martingale derivatives $M_t^{*\prime}$ and ξ_t' which are also martingales, as noted before. Therefore, a fiscal shock at $t = 1$ has *permanent* effects as figure 1 demonstrates. A positive shock in g decreases the utility of the household and consequently increases the likelihood ratio M_t^*. Besides that, it induces a negative innovation in the marginal utility of bond holdings, which results into an increase in ξ_t.

θ	∞	120	100	90	70	50
Φ	0.0080073	0.0080095	0.0080099	0.0080102	0.008011	0.0080125
$E(\tau)$	0.3111	0.31117	0.31118	0.31118	0.3112	0.31124
$std(\tau)$	0.0018	0.0022	0.0023	0.0024	0.0026	0.0029
ρ	0	0.2639	0.3167	0.3519	0.4524	0.6334

Table 1: Approximate moments of tax rate and cost of distortionary taxation.

θ	∞	120	100	90	70	50
$E(b)$	0.0029	-0.0127	-0.0159	-0.0180	-0.0239	-0.0346
$std(b)$	2.3203	1.4276	1.2491	1.1301	0.7900	0.1779
ρ	0	-0.3849	-0.4619	-0.5132	-0.6598	-0.9238

Table 2: Approximate moments of government debt.

Furthermore, in figure 2 we can see the impulse response functions for the Lucas and Stokey economy and for the case when the household doubts it model with $\theta = 90$. At first note that in the Lucas and Stokey case all variables return to zero after the shock at $t = 1$. This highlights the *absence* of history-dependence with full confidence in the model. On the other hand, for the case of misspecification, the variables of interest do not return to zero after period $t = 1$, indicating the *history-dependence* of the optimal plan. This is not discernible for consumption, because the impulse response functions differ to a very small extent and barely discernible for labor. However, the effects of fear of misspecification are clearer for the tax rate and government debt. For each variable in the figure, the two state variables (M_t^*, ξ_t) that reflect the heterogeneity of the beliefs between the planner and the household and the endogeneity of the household's worst-case beliefs, have opposite effects. This stems from the fact that the approximate solution is driven by the difference between the two martingale derivatives, which respond in the same way to a fiscal shock. Considering for example the tax rate, an increase in the likelihood ratio is increasing τ, whereas an increase in ξ_t decreases it. It turns out that the heterogeneity effect dominates in our example, which leads to a total increase in the tax rate.

6.3.3 Approximate moments of tax rate and debt

A natural question that arises is about the effect of the household's fear of misspecification on the moments of the tax rate and government debt. In order to answer this question, we are going to treat the moments of these two variables as any other non-linear function of σ and approximate them around $\sigma = 0$. The *approximate* autocorrelations of the tax rate and debt for small fear of misspecification are of particular interest in the case of iid fiscal shocks, since they are going to shed light on the amount of history dependence that is introduced to the optimal policy. In the iid case full confidence in the model implies that the autocorrelation of the tax rate and debt is *zero*.

Proposition 6.7. *Assume that g is iid. Then the autocorrelations of the tax rate and debt for small fear of misspefication are approximately equal to*

$$\rho_\tau(\sigma) = \sigma \frac{Cov(\tau_t'(0), \tau_{t-1}(0))}{Var(\tau_t(0))}$$

$$\rho_b(\sigma) = \sigma \frac{Cov(b_t'(0), b_{t-1}(0))}{Var(b_t(0))}$$

and therefore the signs of the autocorrelations are as follows:

1. *If $U_{cl} \geq 0$, then (54) implies that*

$$\text{sign}\, \rho_\tau(\sigma) = \text{sign}\, Cov(\eta_t(0) - m_t^{*\prime}(0), \tau_t(0))$$

2. *If $E\alpha_1 < 0$, then (55) implies that*

$$\text{sign}\, \rho_b(\sigma) = \text{sign}\, Cov(\eta_t(0) - m_t^{*\prime}(0), b_t(0))$$

Proof. See appendix C.4 for the calculation of the approximate autocorrelations, along with the approximate standard deviations and means. □

Remark 6.8. *Note that the sign of the approximate autocorrelations depends on the co-variance of the tax rate or debt with the increments to the martingale derivatives $\xi_t'(0)$ and $M_t^{*\prime}(0)$. These covariances reflect the covariance of the Lucas and Stokey plan with the two separate effects that determine the optimal plan under fear of misspecification: the effect of heterogeneity of beliefs and the effect of the manipulation by the planner of the endogenous beliefs of the household.*

Note that since $m_t^{*\prime}(0) = V_t(0) - EV_t(0)$ and $\eta_t(0) = \Phi(0)(y_t(0) - Ey_t(0))$ where $y_t \equiv U_{ct}bt$, we are interested in finding the covariance of V and y with the tax rate and debt in the Lucas and Stokey plan. Lemma 6.5 instructs as about the sign of this covariances, which will consequently determine the sign of the approximate autocorrelations. Consider at first the tax rate. According to proposition 6.7, the sign of the tax autocorrelation depends on the difference of $Cov(\eta_t(0), \tau_t(0)) = \Phi(0)Cov(y_t(0), \tau_t(0))$ and $Cov(m_t^{*\prime}(0), \tau_t(0)) = Cov(V_t(0), \tau_t(0))$. When g is increasing then lemma 6.5 instructs that y and V are decreasing whereas τ is increasing. Therefore we expect both autocovariances to be negative, which let us conclude that the sign is *ambiguous*. If $|Cov(m_t^{*\prime}(0), \tau_t(0))| > |Cov(\eta_t(0), \tau_t(0))|$, i.e. if the covariance of the effect stemming from the heterogeneity of beliefs with the tax rate is stronger in absolute value than the respective covariance coming from the endogeneity of beliefs, then the sign becomes *positive*, whereas in the opposite case negative. Similarly in the case of government debt, if $E\alpha_1 < 0$ -which holds in our calculated example- then the sign depends on $Cov(\eta_t(0), b_t(0)) = \Phi(0)Cov(y_t(0), b_t(0))$ and $Cov(m_t^{*\prime}(0), b_t(0)) = Cov(V_t(0), b_t(0))$. When g is increasing, then according to the lemma y, b, V are decreasing. So both covariances of the increments with the government debt are positive. In this case, if the covariance of the heterogeneity of beliefs $Cov(m_t^{*\prime}(0), b_t(0))$ with government debt is larger than the covariance

of the effect of endogeneity with debt $Cov(\eta_t(0), b_t(0))$, the autocorrelation of government debt becomes *negative*, and positive otherwise.

We turn now to the specifics of our calculated example. Consider at first table 1 that presents the moments of the tax rate and the cost of of distortionary taxation for various degrees of misspecification. Note at first that the cost of distortionary taxation Φ is *increasing* when the representative household doubts its model but in very small amounts. Furthermore, the mean tax rate and the standard deviation are increasing as well. The most noticeable effect of the introduction of fear of misspecification is on the autocorrelation of the tax rate. The autocorrelation from zero for the no-robustness case can become even 0.6344 when $\theta = 50$. In this example, we see that the covariance coming from the heterogeneity of beliefs is larger than the respective one that originates from the desire of the planner to manipulate the household's endogenous beliefs, which leads to a positive autocorrelation. Turning now to table 2, we notice that again we have larger influence of the heterogeneity of beliefs, which results now into a *negative* autocorrelation of government debt. Furthermore, the mean debt along with the standard deviation are decreasing, i.e. the government on average accumulates more assets. The effect of lack of full confidence in the model is pretty large concerning the autocorrelation of debt. So the optimal government debt becomes a smooth and very persistent process when the household has concerns about misspecification.

6.3.4 Household's worst-case beliefs

Having computed the approximate distortion m_t^* allows us to calculate the worst-case model of the household. In figure 3 we report the conditional likelihood ratios m^* for $\theta = 50$ [9]. With full confidence in the model they would be just unity. With concerns about misspecification they become larger than unity for high levels of government expenditures an lower for low levels of g. So the household assigns higher probability mass on events that bring him low utility, i.e. events with high realizations of fiscal shocks. In figure 4 we see the actual worst-case beliefs of the household. For very high and very low g we cannot discern really the household's beliefs because the probabilities of the reference model π are practically zero. What is worth noting is that for $\theta = 50$, i.e fear of misspecification that produces high autocorrelation as shown in the previous tables, the worst-case model seems barely discernible from the reference model π.

7 Concluding remarks

We conclude by indicating some problems that can we can attack using the methods exhibited in this paper. In Karantounias et al. (2007) we study a Ramsey problem in which the planner also doubts his approximating model and so forms worst-case beliefs as part of his decision process. The planner could either have the same multiplier preferences as the household, and

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[9]Note that $m_t^*(g^t) = V_t(0) - E_{t-1}V_t(0)$ depends in the expansion only on the realization g_t due to the iid assumption about g and the history-independence of the Lucas and Stokey plan.
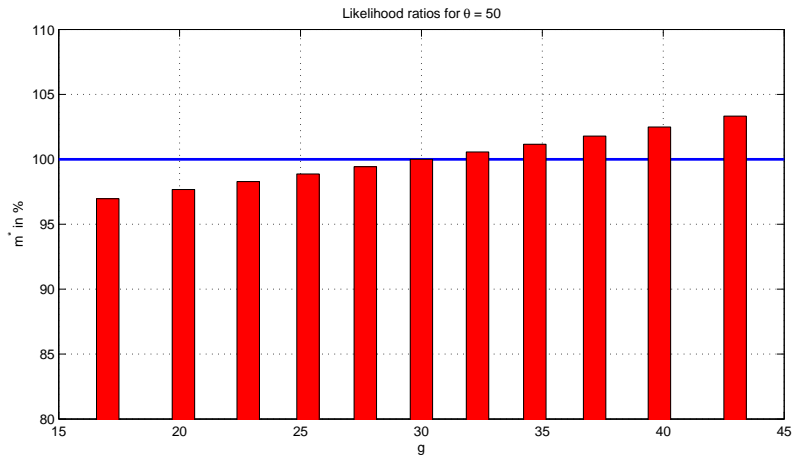
Figure 3: Conditional likelihood ratios m^* for $\theta = 50$ ($\sigma = -0.02$).
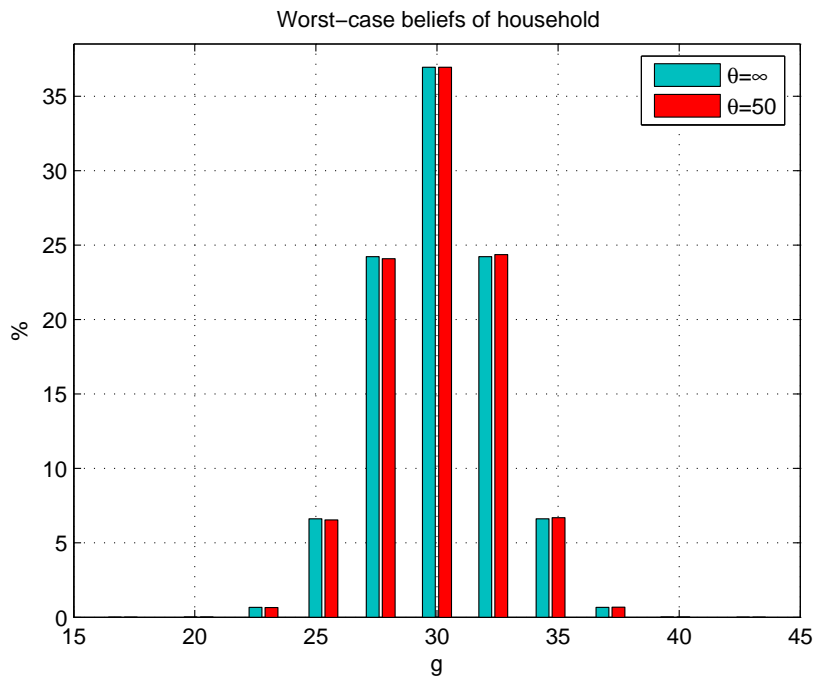

Figure 4: Household's worst-case scenario for $\theta = 50$ ($\sigma = -0.02$).

thus be purely benevolent in the sense that he does not impose his own evaluation of utility, or more generally he could have multiplier preferences that differ from the household's. In either case, the machinery developed in this paper can be used to solve the Ramsey problem. We can represent the problem recursively by using as state variables the *ratio* of beliefs between the planner and the household as well as the corresponding multiplier ξ_t on the recursion of utility of the household. The discrepancy in the fear of misspecification between the planner and the agent will have implications for the optimal tax and debt policy.

An alternative approach to the Lagrangian formulation of the Ramsey problem with fear of misspecification can be constructed by building on the approach of Kydland and Prescott (1980). The state variables according to this method would be the multipliers of the household's optimization problem, with laws of motion the corresponding Euler equations. In our setup this would mean to use the household's ex ante *utility* under the distorted measure, since this is the multiplier of the *malevolent* alter-ego of the household on the law of motion of the likelihood ratio. This approach is dual to the Marcet and Marimon (1998) approach that we followed here, where we used the *shadow* value of the utility of the household ξ_t. The Kydland and Prescott method allows complimentary insights into the structure of the problem and the interpretation of optimal plan.

We mention two other avenues for future research. First, it would be useful to attack the LQ Stackelberg monopolist/competitive fringe problem of Hansen and Sargent (2007, ch. 16) by using the martingale approach to misspecified beliefs and applying the methodology developed in this paper. This is a natural first step before proceeding to analyze the effects of fear of misspecification on optimal monetary policy in standard New-Keynesian models. Furthermore, it is interesting to explore also the effect of fear of misspecification on optimal fiscal policy over the business cycle in a setup like the complete-markets economy of Chari et al. (1994), where the choice between capital and labor income taxation is crucial. The Ramsey policies will be altered in non-trivial ways. For example, Chari et al. show that for a special class of utility functions (power utility of consumption and separability between consumption and labor) the Ramsey plan without fear of misspecification prescribes a zero tax on capital income after period zero. That will not be true with concerns about misspecification.

A   Inner problem and first-order condition

A.1   Inner problem

In this appendix we will solve the inner part of the household's problem. Assign multipliers $\beta^{t+1}\pi_{t+1}(g^{t+1})\rho_{t+1}(g^{t+1})$ and $\beta^t\pi_t(g^t)\nu_t(g^t)$ on constraints (8) and (9) respectively and remember that $M_0 \equiv 1$ and $\pi_0(g_0) = 1$. Form the Lagrangian

$$
\begin{aligned}
L \;=\; & \sum_{t=0}^{\infty}\sum_{g^t}\beta^t\pi_t(g^t)\Big\{ M_t(g^t)\big[U_t(g^t) + \theta\beta\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1})\ln m_{t+1}(g^{t+1})\big] \\
& -\sum_{g_{t+1}}\beta\pi_{t+1}(g_{t+1}|g^t)\rho_{t+1}(g^{t+1})[M_{t+1}(g^{t+1}) - m_{t+1}(g^{t+1})M_t(g^t)] \\
& -\nu_t(g^t)\big[\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1}) - 1\big]\Big\}
\end{aligned}
$$

First-order necessary conditions for an interior solution are

- $m_{t+1}(g^{t+1})$:
$$
\nu_t(g^t) = \beta\theta M_t(g^t)[1 + \ln m_{t+1}(g^{t+1})] + \beta\rho_{t+1}(g^{t+1})M_t(g^t) \tag{66}
$$

- $M_t(g^t), t \geq 1$:
$$
\begin{aligned}
\rho_t(g^t) = U_t(g^t) + \beta\Big[& \sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1})\rho_{t+1}(g^{t+1}) \\
& +\theta\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)m_{t+1}(g^{t+1})\ln m_{t+1}(g^{t+1})\Big]
\end{aligned} \tag{67}
$$

Conditions (66) and (67) together with the transversality condition
$$
\lim_{k\to\infty}\beta^k E_t M_{t+k}\rho_{t+k} = 0 \tag{68}
$$

and the constraints (8) and (9) describe the solution to the inner problem. The above conditions can be simplified as follows. Rearrange (66) to get

$$
\ln_{t+1}(g^{t+1}) = -\frac{\rho_{t+1}(g^{t+1})}{\theta} + \Big(\frac{\nu_t(g^t)}{\beta\theta M_t(g^t)} - 1\Big)
$$

or

$$
m_{t+1}(g^{t+1}) = \exp\Big(-\frac{\rho_{t+1}(g^{t+1})}{\theta}\Big)\exp\Big(\frac{\nu_t(g^t)}{\beta\theta M_t(g^t)} - 1\Big)
$$

Using the constraint that the conditional expectation of m_{t+1} has to integrate to unity (9) we can eliminate $\nu_t(g^t)$ and get

$$
m_{t+1}^*(g^{t+1}) = \frac{\exp\Big(-\frac{\rho_{t+1}^*(g^{t+1})}{\theta}\Big)}{\sum_{g_{t+1}}\pi_{t+1}(g_{t+1}|g^t)\exp\Big(-\frac{\rho_{t+1}^*(g^{t+1})}{\theta}\Big)}, \tag{69}
$$

38

where the asterisks denote optimal values. Furthermore, solving forward (67) and imposing (68) we get

$$\rho_t^*(g^t) = \sum_{i=0}^{\infty} \sum_{g^{t+i}|g^t} \beta^i \pi_{t+i}(g^{t+i}|g^t) \frac{M_{t+i}^*(g^{t+i})}{M_t^*(g^t)} \Big[U(g^{t+i}) +$$

$$\beta\theta \sum_{g_{t+i+1}|g^{t+i}} \pi_{t+i+1}(g_{t+i+1}|g^{t+i}) m_{t+i+1}^*(g^{t+i+1}) \ln m_{t+i+1}^*(g^{t+i+1}) \Big], t \geq 1$$

From the above condition we realize that $\rho_t^*(g^t) = V_t(g^t)$, i.e. the utility at history g^t. This fact together with (67) and (69) give the conditions in the text.

A.2   Derivation of first-order condition (26)

The first-order condition with respect to $V_t(g^t)$ takes the form

$$\beta^t \pi_t(g^t)\xi_t(g^t) = \beta^{t-1}\pi_{t-1}(g^{t-1})\Big[\xi_{t-1}(g^{t-1})\frac{\beta}{\sigma}\frac{\partial}{\partial V_t(g^t)} \ln \sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))$$

$$+M_{t-1}^*(g^{t-1})\frac{\partial}{\partial V_t(g^t)} \sum_{g_t} \beta\pi_t(g_t|g^{t-1})\mu_t(g^t)\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))}\Big]$$

Note that

$$\frac{\partial}{\partial V_t(g^t)} \ln \sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t)) = \frac{\sigma\pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))}{\sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))} = \sigma\pi_t(g_t|g^{t-1})m_t^*(g^t)$$

and

$$\frac{\partial}{\partial V_t(g^t)} \sum_{g_t} \beta\pi_t(g_t|g^{t-1})\mu_t(g^t)\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))} =$$

$$\beta\sigma\pi_t(g_t|g^{t-1})\Big[\mu_t(g^t)\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))}$$

$$-\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))} \sum_{g_t} \pi_t(g_t|g^{t-1})\mu_t(g^t)\frac{\exp(\sigma V_t(g^t))}{\sum_{g_t} \pi_t(g_t|g^{t-1}) \exp(\sigma V_t(g^t))}\Big]$$

$$= \beta\sigma\pi_t(g_t|g^{t-1})m_t^*(g^t)\Big[\mu_t(g^t) - \sum_{g_t} \pi_t(g_t|g^{t-1})m_t^*(g^t)\mu_t(g^t)\Big]$$

Using these two expressions and simplifying gives equation (26).

B   Recursive formulation

The saddle-point problem has the form

$$\min_{\xi} \max_{(c,m^*,M^*,V)} \tilde{L}_0$$

s.t.

$$M_{t+1}^*(g^{t+1}) = m_{t+1}^*(g^{t+1})M_t^*(g^t), M_0 \equiv 1$$

$$m_{t+1}^*(g^{t+1}) = \frac{\exp(-\frac{V_{t+1}(g^{t+1})}{\theta})}{\sum_{g_{t+1}} \pi(g_{t+1}|g_t)\exp(-\frac{V_{t+1}(g^{t+1})}{\theta})}, t \geq 0$$

where

$$
\begin{aligned}
\tilde{L}_0 &\equiv \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t)\{U_t(g^t) + \Phi M_t^*\Omega_t(g^t) \\
&\quad -\xi_t(g^t)[V_t(g^t) - U_t(g^t) - \beta \sum_{g_{t+1}} \pi(g_{t+1}|g_t)m_{t+1}^*(g^{t+1})(V_{t+1}(g^{t+1}) + \theta \ln m_{t+1}^*(g^{t+1}))]\} \\
&\quad -\Phi U_c(g_0)b_0
\end{aligned}
$$

The resource constraint (19) is already substituted in the Lagrangian. Note that we are not substituting for the optimal value of the conditional distortion m_{t+1}^* from (10). This makes the Lagrangian linear in the probability vector π and allows us to apply the Law of Iterated Expectations to get

$$\tilde{L}_0 = E_0 \sum_{t=0}^{\infty} \beta^t[(1+\xi_t)U_t + \Phi M_t^*\Omega_t - \xi_t V_t + \xi_{t-1}(m_t^* V_t + \theta m_t^* \ln m_t^*)] - \Phi U_c(g_0)b_0 \quad (70)$$

Furthermore, we decompose the Lagrangian between the period utility at time zero and the ex-ante utility of the Lagrangian at period $t = 1$, due to the initial debt b_0 and the realization of uncertainty at $t = 0$.

$$\tilde{L}_0 = U(g_0) + \Phi\Omega_0(g_0) - \Phi U_c(g_0)b_0 + \beta\tilde{L}$$

where

$$\tilde{L} \equiv E_0 \sum_{t=1}^{\infty} \beta^{t-1}[r(c_t, V_t, m_t^*, M_{t-1}^*, \xi_{t-1}, \xi_t; \Phi)] \quad (71)$$

with

$$
\begin{aligned}
&r(c_t, V_t, m_t^*, M_{t-1}^*, \xi_{t-1}, \xi_t; \Phi)(g^t) \equiv \quad (72)\\
&(1+\xi_t(g^t))U_t(g^t) + \Phi m_t^*(g^t)M_{t-1}^*(g^{t-1})\Omega_t(g^t) - \xi_t(g^t)V_t(g^t) \\
&+\xi_{t-1}(g^{t-1})[m_t^*(g^t)V_t(g^t) + \theta m_t^*(g^t)\ln m_t^*(g^t)]
\end{aligned}
$$

The augmented return function r_t doesn't depend on expectations of future variables. It depends on the controls c_t, m_t^*, V_t, ξ_t and the lagged values (M_{t-1}^*, ξ_{t-1}) which will serve as

40

a recursive state. Marcet and Marimon show how $\tilde{L}$ satisfies the functional equation in (4.2).

Furthermore, we are going to derive the first-order conditions of the functional equation, in order to verify that they match with the first-order conditions of the Lagrangian formulation. Assign the scaled multiplier $\pi_{g|g_-}\tilde{\mu}_g$ on the optimal distortion m_g^*. Form the Lagrangian

$$L = \sum_g \pi_{g|g_-} \Big\{ U(c_g, 1 - c_g - g) + \Phi m_g^* M_-^* \Omega_g$$
$$-\xi_g(V_g - U(c_g, 1 - c_g - g)) + \xi_-(m_g^* V_g + \theta m_g^* \ln m_g^*)$$
$$-\tilde{\mu}_g[m_g^* - \frac{\exp(\sigma V_g)}{\sum_g \pi_{g|g_-} \exp(\sigma V_g)} + \beta W(g, m_g^* M_-^*, \xi_g; \Phi) \Big\}$$

First-order conditions with respect to c_g, V_g, m_g^*, ξ_g

$$
\begin{aligned}
\left(1 + \xi_g + \Phi m_g^* M_-^*\right)\left(U_{l,g} - U_{c,g}\right) &= \Phi m_g^* M_-^* \big[(U_{cc} - 2U_{cl,g} + U_{ll,g})c_g \\
&\quad + (U_{ll,g} - U_{cl,g})g\big] \quad (73)\\
\xi_g &= \sigma m_g^* [\tilde{\mu}_g - \sum_g \pi_{g|g_-} m_g^* \tilde{\mu}_g] + m_g^* \xi_- \quad (74)\\
\tilde{\mu}_g &= \Phi M_-^* \Omega_g + \xi_-[V_g + \theta(1 + \ln m_g^*)] \\
&\quad + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi) M_-^* \quad (75)\\
V_g &= U_g + \beta W_\xi(g, M_g^*, \xi_g; \Phi) \quad (76)
\end{aligned}
$$

Observe that (73) is the same equation as (31), with $h_g = c_g + g$. Furthermore, using the envelope condition with respect to ξ (41), and updating one period we get the recursion of the utility of the private sector (21). It remains to show that (74) describes the appropriate law of motion. For that consider at first (75) and use the fact that $\ln m_g^* = -\frac{V_g}{\theta} - \ln \sum_g \pi_{g|g_-} \exp\left(-\frac{V_g}{\theta}\right)$ to get

$$\tilde{\mu}_g = \Phi M_-^* \Omega_g + \xi_- \theta \left[1 - \ln \sum_g \pi_{g|g_-} \exp(-\frac{V_g}{\theta})\right] + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi) M_-^*$$

Therefore

$$\tilde{\mu}_g - \sum_g \pi_{g|g_-} m_g^* \tilde{\mu}_g = M_-^* [\{\Phi\Omega_g + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi)\} - \sum_g \pi_{g|g_-} m_g^* \{\Phi\Omega_g + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi)\}]$$

since the term involving the utility under the distorted measure cancels out. Note that by (43) we have $\Phi\Omega_g + \beta W_{M^*}(g, M_g^*, \xi_g; \Phi) = \Phi(\Omega_g + U_{cg}B_g) = \Phi U_{cg}b_g$. Therefore, the increment to the martingale ξ_t is the familiar forecast error $\eta_t = \Phi(U_{ct}b_t - E_{t-1}m_t^* U_{ct}b_t)$ and (74) gives the law of motion of the martingale.

C Expansion

In evaluating the following expansions at $\sigma = 0$, bear in mind that $m_t^*(0) = M_t^*(0) = 1$ and $\xi_t(0) = 0$.

C.1 General case

Definitions and facts: For the rest bear in mind the following helpful definitions:

$$K_t(0) \equiv (1 + 2\Phi(0))\Delta_t(c_t(0)) + \Phi(0)(\Delta_t'(c_t(0))c_t(0) + \Gamma_t'(c_t(0))g_t), t \geq 1 \qquad (77)$$

$$K_0(0) \equiv (1 + 2\Phi(0))\Delta_0(c_0(0)) + \Phi(0)[\Delta_0'(c_0(0))c_0(0) + \Gamma_0'(c_0(0))g_0 - E_0'(c_0(0))b_0] (78)$$

where

$\Delta_t(c_t) \equiv U_{cct} - 2U_{clt} + U_{llt} < 0, \Gamma_t(c_t) \equiv U_{llt} - U_{lct}$ and $E_t(c_t) \equiv U_{cct} - U_{clt}$. The interpretation of notation is $\Delta_t'(c_t(0)) \equiv \frac{\partial \Delta_t}{\partial c_t}(c_t(g^t, 0))$. Also rearranging the first-order condition of the Lucas and Stokey case, which is (31) and (32) evaluated at $\sigma = 0$ (and consequently $(M_t^*(0), \xi_t(0)) = (1, 0)$), gives

$$\Delta_t(c_t(0))c_t(0) + \Gamma_t(c_t(0))g_t = \frac{1 + \Phi(0)}{\Phi(0)}(U_{lt}(0) - U_{ct}(0)) \qquad (79)$$

$$\Delta_0(c_0(0))c_0(0) + \Gamma_0(c_0(0))g_0 - E_0(c_0(0))b_0 = \frac{1 + \Phi(0)}{\Phi(0)}(U_{l0}(0) - U_{c0}(0)), \qquad (80)$$

two facts that will become handy later.

1. **Law of motion for $M^{*\prime}$.** The law of motion $\forall \sigma \leq 0$ is $M_t^*(\sigma) = m_t^*(\sigma)M_{t-1}^*(\sigma)$ with $M_0 \equiv 1$. Differentiating wrt σ we get

$$M_t^{*\prime}(\sigma) = m_t^{*\prime}(\sigma)M_{t-1}^*(\sigma) + m_t^*(\sigma)M_{t-1}^{*\prime}(\sigma)$$

and evaluating at $\sigma = 0$

$$M_t^{*\prime}(0) = m_t^{*\prime}(0) + M_{t-1}^{*\prime}(0), M_0^{*\prime} \equiv 0$$

Remember that the increment to the martingale is $m_t^*(\sigma) = \frac{\exp(\sigma V_t(\sigma))}{E_{t-1}\exp(\sigma V_t(\sigma))}$. Differentiating with respect to σ we get

$$m_t^{*\prime}(\sigma) = m_t^*(\sigma)[V_t(\sigma) + \sigma V_t'(\sigma) - E_{t-1}m_t^*(\sigma)(V_t(\sigma) + \sigma V_t'(\sigma))$$

which evaluated at $\sigma = 0$ simplifies to $m_t^{*\prime}(0) = V_t(0) - E_{t-1}V_t(0)$. Using the expression for the increment in the law of motion for $M^{*\prime}$ gives the expression at the text.

2. **Law of motion for ξ'.** Remember that $\xi_t(\sigma) = \sigma M_t^*(\sigma)\eta_t(\sigma) + m_t^*(\sigma)\xi_{t-1}(\sigma)$, where $\eta_t(\sigma) = \mu_t(\sigma) - E_{t-1}m_t^*(\sigma)\mu_t(\sigma)$. Differentiating now the law of motion of ξ_t

$$\xi_t'(\sigma) = M_t^*(\sigma)\eta_t(\sigma) + \sigma M_t^{*\prime}(\sigma)\eta_t(\sigma) + \sigma M_t^*(\sigma)\eta_t'(\sigma) + m_t^{*\prime}(\sigma)\xi_{t-1}(\sigma) + m_t^*(\sigma)\xi_{t-1}'(\sigma)$$

which at $\sigma = 0$ becomes

$$
\begin{aligned}
\xi'_t(0) &= \eta_t(0) + \xi'_{t-1}(0) \\
&= \Phi(0)\left[U_{ct}(0)b_t(0) - E_{t-1}U_{ct}(0)b_t(0)\right] + \xi'_{t-1}(0)
\end{aligned}
$$

with initial condition $\xi'_0 \equiv 0$.

3. **Expression for** $c'_t(0)$. We proceed now to the calculation of $c'_t(0)$ which is equal to $h'_t(0)$ as is clear from the resource constraint. To do we will work with (31) and (32), which determine the optimal wedge for $t \geq 1$ and $t = 0$ respectively.

   - $t \geq 1$: Substituting for $h_t = c_t + g_t$, and using the above definitions of $\Delta_t(c_t)$ and $\Gamma_t(c_t)$ (31) takes the form

   $$
   (1+\xi_t(\sigma)+\Phi(\sigma)M^*_t(\sigma))[U_{lt}(\sigma)-U_{ct}(\sigma)] = \Phi(\sigma)M^*_t(\sigma)[\Delta_t(c_t(\sigma))c_t(\sigma)+\Gamma_t(c_t(\sigma))g_t]
   $$

   Differentiating the left-hand side with respect to σ and evaluating at $\sigma = 0$ gives

   $$
   \begin{aligned}
   LHS'(0) &= [\xi'_t(0) + \Phi'(0) + \Phi(0)M^{*\prime}_t(0))][U_{lt}(0) - U_{ct}(0)] \\
   &\quad -(1 + \Phi(0))\Delta_t(c_t(0))c'_t(0)
   \end{aligned}
   $$

   Similarly, differentiate the right-hand side and set $\sigma = 0$ to get

   $$
   \begin{aligned}
   RHS'(0) &= [\Phi'(0) + \Phi(0)M^{*\prime}_t(0)][\Delta_t(c_t(0))c_t(0) + \Gamma_t(c_t(0))g_t] \\
   &\quad +\Phi(0)[\Delta'_t(c_t(0))c_t(0) + \Delta_t(c_t(0)) + \Gamma'_t(c_t(0))g_t]c'_t(0)
   \end{aligned}
   $$

   Combining the two sides and rearranging terms returns

   $$
   \begin{aligned}
   K_t(0)c'_t(0) &= \xi'_t(0)(U_{lt}(0) - U_{ct}(0)) \\
   &+(\Phi'(0) + \Phi(0)M^{*\prime}_t(0))\{U_{lt}(0) - U_{ct}(0) - [\Delta_t(c_t(0))c_t(0) + \Gamma_t(c_t(0))g_t]\}
   \end{aligned}
   $$

   The term multiplying $[\Phi'(0) + \Phi(0)M^{*\prime}_t(0)]$ can be further simplified by using fact (79), to finally get $c'_t(0) = \frac{U_{lt}(0)-U_{ct}(0)}{K_t(0)}\left[\xi'_t(0) - M^{*\prime}_t(0) - \frac{\Phi'(0)}{\Phi(0)}\right], t \geq 1$ which is expression (53) in the text.

   - $t = 0$: The difference in period zero arises from the presence of initial debt b_0. Repeating (32) here for convenience

   $$
   (1 + \Phi(\sigma))[U_{l0}(\sigma) - U_{c0}(\sigma)] = \Phi(\sigma)[\Delta_0(c_0(\sigma))c_0(\sigma) + \Gamma_0(c_0(\sigma))g_0 - E_0(c_0(\sigma))b_0]
   $$

   Differentiating over σ, evaluating at $\sigma = 0$ and using fact (80) and the definition of $K_0(0)$ allows us to derive $c'_0(0) = \frac{U_{l0}(0)-U_{c0}(0)}{K_0(0)}\left[-\frac{\Phi'(0)}{\Phi(0)}\right]$, which is the same as the expression in the text since $\xi'_0 = M^{*\prime}_0 = 0$.

4. **Expression for $\tau_t'(0)$**

The tax rate is given by $\tau_t(\sigma) = 1 - U_l(c_t(\sigma), 1 - c_t(\sigma) - g_t)/U_c(c_t(\sigma), 1 - c_t(\sigma) - g_t)$.
Differentiating with respect to σ and evaluating at $\sigma = 0$ gives

$$\tau_t'(0) = -\frac{[U_{lct}(0) - U_{llt}(0)]U_{ct}(0) - U_{lt}(0)[U_{cct}(0) - U_{clt}(0)]}{U_{ct}^2(0)} c_t'(0)$$

Using expression (53) gives

$$\tau_t'(0) = \alpha_t(0) \left[\xi_t'(0) - M_t^{*\prime}(0) - \Phi'(0)/\Phi(0)\right]$$

where

$$\alpha_t(0) \equiv -\frac{[U_{lct}(0) - U_{llt}(0)]U_{ct}(0) - U_{lt}(0)[U_{cct}(0) - U_{clt}(0)]}{U_{ct}^2(0)} \frac{U_{lt}(0) - U_{ct}(0)}{K_t(0)}$$

5. **Expression for $b_t'(0)$**

The government debt at time t as a function of σ is

$$b_t(\sigma) = \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t)(M_{t+i}^*(\sigma)/M_t^*(\sigma))\big(\Omega_{t+i}(\sigma)/U_{ct}(\sigma)\big).$$

Differentiating with respect to σ and setting $\sigma = 0$ returns

$$b_t'(0) = \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) \left(\frac{M_{t+i}^*(\sigma)}{M_t^*(\sigma)}\right)'_{\sigma=0} \frac{\Omega_{t+i}(0)}{U_{ct}(0)}$$

$$+ \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) \left(\frac{\Omega_{t+i}(\sigma)}{U_{ct}(\sigma)}\right)'_{\sigma=0}$$

Note that

$$\left(\frac{M_{t+i}^*(\sigma)}{M_t^*(\sigma)}\right)'_{\sigma=0} = M_{t+i}^{*\prime}(0) - M_t^{*\prime}(0)$$

$$\left(\frac{\Omega_{t+i}(\sigma)}{U_{ct}(\sigma)}\right)'_{\sigma=0} = (\Omega_{t+i}'(0)U_{ct}(0) - \Omega_{t+i}(0)U_{ct}'(0))/U_{ct}(0)^2,$$

with $\Omega_t'(0) = [\Delta_t(c_t(0))c_t(0) + \Gamma_t(c_t(0))g_t + U_{ct}(0) - Ul_t(0)]c_t'(0)$. Using (79) Ω' simplifies to $\Omega_t'(0) = (U_{lt}(0) - U_{ct}(0))c_t'(0)/\Phi(0)$. Furthermore, $U_{ct}'(0) = (U_{cct}(0) - U_{clt}(0))c_t'(0)$.
So the derivative of debt becomes

$$b_t'(0) = \frac{1}{U_{ct}(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t)(M_{t+i}^{*\prime}(0) - M_t^{*\prime}(0))\Omega_{t+i}(0)$$

$$+ \frac{1}{\Phi(0)U_{ct}(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t)(U_{lt+i}(0) - U_{ct+i}(0))c_{t+i}'(0)$$

$$- \frac{(U_{cct}(0) - U_{clt}(0))}{U_{ct}(0)^2} c_t'(0) \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t)\Omega_{t+i}(0)$$

Use now formula (53) for $c'_t(0)$, rewrite $\xi'_{t+i}(0) - M^{*\prime}_{t+i}(0)$ as $(\xi'_{t+i}(0) - \xi'_t(0)) - (M^{*\prime}_{t+i}(0) - M^{*\prime}_t(0)) + (\xi'_t(0) - M^{*\prime}_t(0))$, define $z_t \equiv (U_{ct} - U_{lt})^2/K_t$ and collect terms to get

$$b'_t(0) = \alpha_1(g^t)(\xi'_t(0) - M^{*\prime}_t(0)) + \alpha_2(g^t) + \alpha_3(g^t)$$

where

$$\alpha_1(g^t) \equiv \frac{1}{\Phi(0)U_{ct}(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) z_{t+i}(0)$$

$$-\frac{(U_{cct}(0) - U_{clt}(0))(U_{lt}(0) - U_{ct}(0))}{U_{ct}(0)^2 K_t(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) \Omega_{t+i}(0)$$

$$\alpha_2(g^t) \equiv \frac{\Phi'(0)}{\Phi(0)U_{ct}(0)} \Big[\frac{(U_{cct}(0) - U_{clt}(0))(U_{lt}(0) - U_{ct}(0))}{U_{ct}(0)K_t(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) \Omega_{t+i}(0)$$

$$-\frac{1}{\Phi(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) z_{t+i}(0) \Big]$$

$$\alpha_3(g^t) \equiv \frac{1}{U_{ct}(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t)(M^{*\prime}_{t+i}(0) - M^{*\prime}_t(0)) \Omega_{t+i}(0)$$

$$+\frac{1}{\Phi(0)U_{ct}(0)} \sum_{i=0}^{\infty} \beta^i \sum_{g^{t+i}|g^t} \pi_{t+i}(g^{t+i}|g^t) z_{t+i}(0)[(\xi'_{t+i}(0) - \xi'_t(0)) - (M^{*\prime}_{t+i}(0) - M^{*\prime}_t(0))]$$

The three coefficients α_i depend on the whole history of shocks through the probability measure π_t. In case of Markovian shocks, they are only *state-dependent* since z_{t+i} and Ω_{t+i} depend on the Lucas and Stokey allocation and thus are functions only of g_{t+i}, and furthermore, $\xi'_{t+i}(0) - \xi'_t(0)$ and $M^{*\prime}_{t+i}(0) - M^{*\prime}_t(0)$ depend on the sum of the increments which also inherit the Markovian property.

6. **Cost of distortionary taxation**. The above expressions express $c'_t(0)$ as a function of $\Phi'(0)$. We are going to use now the differentiated intertemporal budget constraint evaluated at $\sigma = 0$, and substitute for $c'_t(0)$ in order to derive an expression for $\Phi'(0)$ and complete the expansion. The budget constraint (IC) has the form $\sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) M^*_t(\sigma) \Omega_t(\sigma) = U_{c0}(\sigma) b_0$. Differentiating and evaluating at $\sigma = 0$ gives

$$\sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) M^{*\prime}_t(0) \Omega_t(0) + \sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) \Omega'_t(0) = E_0(c_0(0)) b_0 c'_0(0) \qquad (81)$$

By using facts (79) and (80), we find that $\Omega'_t(0) = (U_{lt}(0) - U_{ct}(0))c'_t(0)/\Phi(0), t \geq 1$ and $\Omega'_0(0) = [(U_{l0}(0) - U_{c0}(0))/\Phi(0) + E_0(c_0(0))b_0]c'_0(0)$ respectively. Using that, (81)

becomes

$$\sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) \frac{U_{ct}(0) - U_{lt}(0)}{\Phi(0)} c_t'(0) = \sum_{t=0}^{\infty} \sum_{g^t} \beta^t \pi_t(g^t) M_t^{*'}(0) \Omega_t(0)$$

Substituting expression (53) and rearranging allows us to get (56) in the text.

Proof of lemma 6.1.

Let $l(c, \Phi) \equiv U(c, 1-c-g) + \Phi\big[U_c(c, 1-c-g) - U - l(c, 1-c-g))c - U_l(c, 1-c-g)g\big]$ denote the period return in the lagrangian for the Lucas and Stokey economy for $t \geq 1$ and let $l^0(c, \Phi, b_0) \equiv U(c, 1-c-g) + \Phi\big[U_c(c, 1-c-g) - U - l(c, 1-c-g))c - U_l(c, 1-c-g)g - U_c(c, 1-c-g)b_0\big]$ denote the respective lagrangian for $t = 0$. Note that we have already substituted for labor from the resource constraint. It is easy to see that $l_{cc} = K$ and $l_{cc}^0 = K_0$ where K and K_0 the expressions in (77) and (78). We drop the 0 notation since it is clear that we work for the no-robustness case. The second-order sufficient conditions require the Hessian of the Lagrangian with respect to $c_t(g^t)$ to be negative definite on the tangent plane of the constraint space defined by $A \equiv \big\{x : \sum_{t=0}^{\infty} \sum_{g^t} \frac{\partial F(\{c\})}{\partial c_t(g^t)} x_t(g^t) = 0\big\}$, where $F(\{c\}) \equiv \sum_{t=0}^{\infty} \sum_{g^t} \pi_t(g^t)[(U_c(g^t) - U_l(g^t))c_t(g^t) - U_l(g^t)g_t] - U_{c_0}b_0$. All expressions are calculated at $\{c\}$ that satisfies the first-order conditions. The time separability of the utility function makes the Hessian diagonal, so the second order conditions take the form $\sum_{t=0}^{\infty} \beta^t \sum_{g^t} K_t(g^t) x_t^2(g^t) < 0$ for all $x \neq 0, x \in A$. It is apparent that they are satisfied if $K_t < 0, \forall t \geq 0$.

C.2 Markov case

In this section we show how to calculate the formulas for the expansion when g follows a time-invariant Markov chain with transition matrix Π (and consequently we cover also the iid case). We present formulas for the case of zero initial debt, which can be very easily modified to incorporate the more general case. In our calculations it will be crucial the main property of the Lucas and Stokey allocation: the history-independence.

At first we have to calculate the increments to martingale derivatives $M_t^{*'}$ and ξ_t'. For that we need to calculate the household's expected discounted utility V for the case of no-robustness and the bonds in marginal utility terms $y \equiv U_c b$. We are dropping the $\sigma = 0$ notation whenever we use vectors, but remember that we always referring to the Lucas and Stokey solution. The state-dependence of V and y, combined with the Markov assumption gives in vector notation

$$V = (I - \beta\Pi)^{-1} U$$
$$y = (I - \beta\Pi)^{-1} \Omega$$

We can derive the government debt by dividing each element of y with the corresponding marginal utility U_c. Let $\mathbf{m}^{*'}$ denote the matrix of the increments to the martingale $M^{*'}$ with corresponding element $\mathbf{m}^{*'}_{ij} = m^{*'}(j|i) = V_j - E(V|i)$ and let $\boldsymbol{\eta}$ denote the matrix with the

increments to ξ' with element $\boldsymbol{\eta}_{ij} = \eta(j|i) = \Phi(0)(y_j - E(y|i))$. So the increments, given the solution for V and y can be summarized as follows

$$\mathbf{m}^{*\prime} = \mathbf{1}V' - \Pi V \mathbf{1}'$$
$$\boldsymbol{\eta} = \Phi(0)(\mathbf{1}y' - \Pi y \mathbf{1}')$$

Note that the increments to the martingale derivatives inherit the Markov property. Also let $\circ$ denote element-by-element multiplication between two matrices with the same dimensions. Note that $(\Pi \circ \mathbf{m}^{*\prime})\mathbf{1} = 0$ and $(\Pi \circ \boldsymbol{\eta})\mathbf{1} = 0$, since the conditional mean of the increments is zero.

We want now to calculate $\Phi'(0)$ and the coefficients α_i of $b'_t(0)$. These expressions involve expected discounted sums of *products* of the history-dependent martingale derivatives ($M_t^{*\prime}$ or ξ'_t) with functions of the Lucas and Stokey allocation like $\Omega_t(0)$ or $z_t(0)$, that are only state-dependent. Consider for example in the expression for $\Phi'(0)$ (56) the term in the nominator $\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t) M_t^{*\prime}(0) \Omega_t(0)$. Expanding this term gives

$$\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t) M_t^{*\prime}(0) \Omega_t(0) = 0 + \beta \sum_{g_1} \pi_{(g_1|g_0)} m^{*\prime}(g_1|g_0) \Omega(g_1)$$

$$+ \beta_2 \sum_{g_1} \pi(g_1|g_0) \sum_{g_2} \pi(g_2|g_1)(m^{*\prime}(g_1|g_0) + m^{*\prime}(g_2|g_1)) \Omega(g_2)$$

$$+ \beta^3 \sum_{g_1} \pi(g_1|g_0) \sum_{g_2} \pi(g_2|g_1) \sum_{g_3} \pi(g_3|g_2)(m^{*\prime}(g_1|g_0) + m^{*\prime}(g_2|g_1) + m^{*\prime}(g_3|g_2)) \Omega(g_3) + ...$$

Let e_{g_t} denote a selector column vector with zeros everywhere and one at position i when $g_t = g_i$. Then

$$\sum_{g_1} \pi(g_1|g_0) m^{*\prime}(g_1|g_0) \Omega(g_1) = e'_{g_0}(\Pi \circ \mathbf{m}^{*\prime})\Omega$$

$$\sum_{g_1} \pi(g_1|g_0) \sum_{g_2} \pi(g_2|g_1)(m^{*\prime}(g_1|g_0) + m^{*\prime}(g_2|g_1)) \Omega(g_2) = e'_{g_0}(\Pi \circ \mathbf{m}^{*\prime})\Pi\Omega + e'_{g_0}\Pi(\Pi \circ \mathbf{m}^{*\prime})\Omega$$

$$\sum_{g_1} \pi(g_1|g_0) \sum_{g_2} \pi(g_2|g_1) \sum_{g_3} \pi(g_3|g_2)(m^{*\prime}(g_1|g_0) + m^{*\prime}(g_2|g_1) + m^{*\prime}(g_3|g_2)) \Omega(g_3)$$
$$= e'_{g_0}(\Pi \circ \mathbf{m}^{*\prime})\Pi^2\Omega + e'_{g_0}\Pi(\Pi \circ \mathbf{m}^{*\prime})\Pi\Omega + e'_{g_0}\Pi^2(\Pi \circ \mathbf{m}^{*\prime})\Omega$$

So by induction we get

$$\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t) M_t^{*\prime}(0)\Omega_t(0) = \beta e_{g_0}'(\Pi \circ \mathbf{m}^{*\prime})\Omega + \beta^2[(\Pi \circ \mathbf{m}^{*\prime})\Pi\Omega + e_{g_0}'\Pi(\Pi \circ \mathbf{m}^{*\prime})\Omega]$$

$$+\beta^3[e_{g_0}'(\Pi \circ \mathbf{m}^{*\prime})\Pi^2\Omega + e_{g_0}'\Pi(\Pi \circ \mathbf{m}^{*\prime})\Pi\Omega + e_{g_0}'\Pi^2(\Pi \circ \mathbf{m}^{*\prime})\Omega] + ...$$

$$= \beta e_{g_0}'(\Pi \circ \mathbf{m}^{*\prime})[I + \beta\Pi + \beta^2\Pi^2 + ...]\Omega + \beta^2 e_{g_0}'\Pi(\Pi \circ \mathbf{m}^{*\prime})[I + \beta\Pi + \beta^2\Pi^2 + ...]\Omega$$

$$+\beta^3 e_{g_0}'\Pi^2(\Pi \circ \mathbf{m}^{*\prime})[I + \beta\Pi + \beta^2\Pi^2 + ...]\Omega + ...$$

$$= \beta e_{g_0}'[I + \beta\Pi + \beta^2\Pi^2 + ...](\Pi \circ \mathbf{m}^{*\prime})(I - \beta\Pi)^{-1}\Omega$$

$$= \beta e_{g_0}'(I - \beta\Pi)^{-1}(\Pi \circ \mathbf{m}^{*\prime})(I - \beta\Pi)^{-1}\Omega$$

A similar calculation applies when instead of $M_t^{*\prime}$ we have ξ_t'. Therefore, the expression for $\Phi'(0)$ is

$$\Phi'(0) = \frac{\Phi(0)\beta e_{g_0}'(I - \beta\Pi)^{-1}\big[\Phi(0)(\Pi \circ \mathbf{m}^{*\prime})(I - \beta\Pi)^{-1}\Omega + [(\Pi \circ \boldsymbol{\eta}) - (\Pi \circ \mathbf{m}^{*\prime})](I - \beta\Pi)^{-1}z\big]}{e_{g_0}'(I - \beta\Pi)^{-1}z}.$$

Finally, the coefficients α_i for debt take the form

$$\alpha_1(g_t) = e_{g_t}'(I - \beta\Pi)^{-1}\Big[\frac{1}{\Phi(0)U_{ct}(0)}z - \frac{(U_{cct}(0) - U_{clt}(0))(U_{lt}(0) - U_{ct}(0))}{U_{ct}(0)^2 K_t(0)}\Omega\Big]$$

$$\alpha_2(g_t) = \frac{\Phi'(0)}{\Phi(0)U_{ct}(0)}e_{g_t}'(I - \beta\Pi)^{-1}\Big[\frac{(U_{cct}(0) - U_{clt}(0))(U_{lt}(0) - U_{ct}(0))}{U_{ct}(0)K_t(0)}\Omega - \frac{1}{\Phi(0)}z\Big]$$

$$\alpha_3(g_t) = \frac{\beta e_{g_t}'(I - \beta\Pi)^{-1}}{U_{ct}(0)}\Big[(\Pi \circ \mathbf{m}^{*\prime})(I - \beta\Pi)^{-1}\Omega + \frac{1}{\Phi(0)}[(\Pi \circ \boldsymbol{\eta}) - (\Pi \circ \mathbf{m}^{*\prime})](I - \beta\Pi)^{-1}z\Big]$$

C.3 Quasi-linear example.

The quasi-linearity in the utility function simplifies all expressions greatly. At first note that $V_t(0) = h - g_t - \frac{1}{2}h^2 + \beta E_t V_{t+1}(0)$ and $b_t(0) = \tau h - g_t + \beta E_t b_{t+1}(0)$. Calculating the innovation in $V_t(0)$ and $b_t(0)$ gives expression (57) and (58) respectively. Furthermore, by using $K_t(0) = -(1+2\Phi(0)), \forall t$ and $U_{lt} - U_{ct} = h - 1 = -\frac{\Phi(0)}{1+2\Phi(0)}$ we derive (59) and (60). The expressions in the result about random walks are derived by just applying (48) and taking first differences. In finding the tax *revenues* we use the derivative $T_t'(0) = \tau_t'(0)h + \tau h_t'(0) = (1 - 2h)h_t'(0) = -\frac{1}{1+2\Phi(0)}h_t'(0)$ since $\tau_t'(0) = -h_t'(0)$ and $\tau = 1 - h$. Furthermore, in deriving (65) from (60) we use the calculation $E_0 M_t^{*\prime}(0)g_t = -\gamma(\beta)\sigma_\varepsilon^2 \sum_{i=0}^{t-1} \gamma_i$. The worst-case conditional mean is computed from the formula $E_t m_{t+1}^* \varepsilon_{t+1} = E_t(1 + \gamma(\beta)\varepsilon_{t+1}/\theta)$. We treat the worst-case variance as any other non-linear function of σ and approximate around $\sigma = 0$. Thus $\tilde{V}ar_t(\varepsilon_{t+1}) \simeq \sigma_\varepsilon^2 + \sigma\frac{d\tilde{V}ar_t(\varepsilon_{t+1})}{d\sigma}|_{\sigma=0}$. We have $\frac{d\tilde{V}ar_t(\varepsilon_{t+1})}{d\sigma}|_{\sigma=0} = E_t m_{t+1}^{*\prime}(0)\varepsilon_{t+1}^2 = -\gamma(\beta)E_t\varepsilon_{t+1}^3$, which delivers the expression in the result. Furthermore, the coefficients for $b_t'(0)$ in C.1 simplify as follows in the quasi-linear case: $\alpha_1 = -\frac{\Phi(0)(1-\beta)^{-1}}{(1+2\Phi(0))^3}$, $\alpha_2 = \Phi'(0)(1-\beta)^{-1}(1+2\Phi(0))^{-3} = E_0 \sum_{t=0}^{\infty} \beta^t M_t^{*\prime}(0)g_t$ and $\alpha_3 = -E_t \sum_{i=0}^{\infty}(M_{t+i}^{*\prime}(0) - M_t^{*\prime}(0))g_{t+i}$.

Note that $\alpha_2 = -\alpha_3 = -\gamma(\beta)\sigma_\varepsilon^2 \sum_{i=0}^{t-1} \gamma_i$. Thus these two terms cancel out and we get the expression in the text.

C.4   AMSS utility function

In calculating the approximate solution for the Aiyagari et al. utility function, we use the formulas of appendix C.2, where in this case the transition matrix Π has identical rows due to the iid assumption.

Proof of lemma 6.5. We want to derive the properties of the solution, *given* an optimal plan. So we will apply the implicit function theorem holding Φ constant. All expressions below are evaluated at the optimal solution.

1. The combined first-order condition of the Lucas and Stokey problem, is equation (31) evaluated at $\xi_t = 0$ and $M_t^* = 1$. Differentiating with respect to g and rearranging gives us $c'(g) = \frac{T_2}{K} < 0$, since $T_2 > 0$ and $K = T_1 - T_2 < 0$. Futhermore, $U'(g) = (U_c - U_l)c'(g) - U_l < 0$, since $U_{cl} = 0$ forces the Ramsey planner to *tax* with no fear of misspecification and therefore $U_c - U_l > 0$. Furthermore, the expected discounted utility of the Lucas and Stokey solution is *state-contingent* under Markovian shocks g with transition matrix Π and takes the form $V = (I - \beta\Pi)^{-1}U$, where V and U are vectors with i component the respective expected utility V_i and period utility U_i for $g_t = g_i$. When g is iid, then this expression simplifies to $V = (I + \frac{\beta}{1-\beta}\Pi)U$ or turning back to scalar notation $V_i = U_i + \frac{\beta}{1-\beta}EU$ for each i, which allows us to conclude that V_i is decreasing in i, since U_i is decreasing.

2. Differentiating the resource constraint (19) with respect to g gives $h'(g) = c'(g) + 1 = \frac{T_1}{T_1 - T_2} > 0$, since $T_1 < 0$ by assumption.

3. The tax rate is $\tau(g) = 1 - \frac{U_l(g)}{U_c(g)}$, with derivative $\tau'(g) = [(U_c U_{ll} + U_l U_{cc})c'(g) + U_c U_{ll}]/U_c^2$. Using the solution for $c'(g) = T_2/(T_1 - T_2)$ gives $\tau'(g) = (U_c U_{ll} T_1 + U_l U_{cc} T_2)/(K U_c^2) > 0$, since the nominator is negative by assumption.

4. Calculating the derivative of Ω with respect to g and using fact (79) gives $\Omega'(g) = (U_l - U_c)c'(g)/\Phi + U_{ll}h - U_l$. Using the expression for $c'(g)$ and collecting the terms for T_1 and T_2 gives $\Omega'(g) = (\Phi(U_{ll}h - U_l)T_1 + T_2[(U_l - \Phi(U_{ll}h - U_l)) - U_c])/\Phi K$. By the first-order conditions of the Lucas and Stokey plan with respect to c and h we have $U_c + \Phi(U_{cc}c + U_c) = \lambda = U_l - \Phi(U_{ll}h - U_l)$. Therefore, $\Omega'(g) = [(U_{ll}h - U_l)T_1 + (U_{cc}c + U_c)T_2]/K < 0$, since the nominator is positive by assumption. Furthermore, the government debt vector in marginal utility terms y is given by $y = (I - \beta\Pi)^{-1}\Omega$, which in the iid case becomes (in scalar notation) $y_i = \Omega_i + \frac{\beta}{1-\beta}E\Omega$, which is decreasing in g, due to the decreasing Ω_i. Finally, $b = y/U_c$ is decreasing in g, since the nominator is decreasing in g, whereas the denominator is increasing in g due to the indirect effect of decreasing consumption and therefore increasing marginal utility (the direct effect of g is absent since it affects the denominator only through U_{cl} which is zero).

Proof of corollary 6.6. At first note that showing that $T_1 < 0$, guarantees automatically the second-order conditions of the problem since $K = T_1 - T_2 < 0$, since $T_2 > 0$. Calculating T_1 gives us $T_1 = -c^{-\gamma-1}\gamma(1+\Phi(1-\gamma))$. So the sign depends on the expression $1 + \Phi(1-\gamma)$. The tax rate for this utility function is given by the expression $\tau = \Phi(\gamma l + \psi h)/((1+\Phi)l+\Phi\psi h)$ and therefore $1-\tau = l(1+\Phi(1-\gamma))/((1+\Phi)l+\Phi\psi h) = U_l/U_c > 0$. From this we conclude that $1 + \Phi(1-\gamma) > 0$, and $T_1 < 0$. Consider now $U_c U_{ll} T_1 + U_l U_{cc} T_2$. This takes the form $c^{-2\gamma-1}l^{-\psi-1}\gamma\psi a_l[1+\Phi(1-\gamma)-a_l l^{-\psi}/c^{-\gamma}(1+2\Phi+\Phi(\psi+1)h/l)]$. By substituting for $a_l l^{-\psi}/c^{-\gamma}$ the expression for $1-\tau$ we get $U_c U_{ll} T_1 + U_l U_{cc} T_2 = -\Phi c^{-2\gamma-1}l^{-\psi-1}\gamma\psi a_l(1+\Phi(1-\gamma))\bar{l}/((1+\Phi)l+\Phi\psi h) < 0$, since $1 + \Phi(1-\gamma) > 0$. Finally, considering expression $(U_{ll}h - U_l)T_1 + (U_{cc}c + U_c)T_2$ which, if positive, ensures the negativity of $\Omega'(g)$, we note that the first term is positive since $T_1 < 0$, whereas the second term may take either sign, depending on $U_{cc}c + U_c = c^{-\gamma}(1-\gamma)$. If we assume that $\gamma < 1$, then we get the desired result.

Proof of Proposition 6.7. The autocorrelation of the tax rate is

$$\rho_\tau(\sigma) = \frac{Cov(\tau_t(\sigma), \tau_{t-1}(\sigma))}{\sqrt{Var(\tau_t(\sigma))Var(\tau_{t-1}(\sigma))}}$$

with derivative at $\sigma = 0$

$$\rho_\tau'(0) = \frac{(dCov(\tau_t(\sigma), \tau_{t-1}(\sigma))/d\sigma)_{\sigma=0}}{Var(\tau_t(0))},$$

where we used the fact that $Cov(\tau_t(0), \tau_{t-1}(0)) = 0$, since the tax rate in the Lucas and Stokey economy is state-dependent, and g is iid, and also the fact that $Var(\tau_t(0)) = Var(\tau_{t-1}(0))$. Proceeding with the calculation

$$\frac{dCov(\tau_t(\sigma), \tau_{t-1}(\sigma))}{d\sigma}\Big|_{\sigma=0} = E(\tau_t'(0) - E\tau_t'(0))(\tau_{t-1}(0) - E\tau_{t-1}(0))$$
$$+E(\tau_t(0) - E\tau_t(0))(\tau_{t-1}'(0) - E\tau_{t-1}'(0))$$
$$= E(\tau_t'(0) - E\tau_t'(0))(\tau_{t-1}(0) - E\tau_{t-1}(0)),$$

since $\tau_t(0)$ depends on g_t, whereas $\tau_{t-1}'(0)$ depends on g^{t-1}, and g is iid. Thus, $\rho_\tau(\sigma) \simeq \rho_\tau(0) + \sigma\rho_\tau'(0) = \sigma\rho_\tau'(0)$, which is the desired expression. Using now expression (54) and the facts that $\alpha_t(0)$ and the increments to the martingale derivatives $\eta_t(0)$ and $m_{t-1}^{*\prime}(0)$ depend only on the iid g_t we finally find that $dCov(\tau_t(\sigma), \tau_{t-1}(\sigma))/d\sigma|_{\sigma=0} = E\alpha Cov(\eta_{t-1}(0) - m_{t-1}^{*\prime}(0), \tau_{t-1}(0))$, where we just can change the subscript to t, since the variables are stationary. Inspecting the formula for the coefficient α in subsection C.1, we realize that $\alpha < 0$, when $U_{cl} \geq 0$. Therefore, since $\sigma < 0$, the sign of the approximate autocorrelation depends on the sign of $Cov(\eta_t(0) - m_t^{*\prime}(0), \tau_t(0))$.

Repeating similar steps for the approximate autocorrelation of debt we derive the formula stated in the proposition, which using (55) becomes

$$\rho_b(\sigma) \simeq \sigma \frac{Cov(\eta_t(0) - m_t^{*\prime}(0), b_t(0))E\alpha_1}{Var(b_t(0))}.$$

50

The sign of $E\alpha_1$ is ambiguous. Assuming $E\alpha_1 < 0$ gives the stated claim about the sign of the approximate autocorrelation of government debt.

For completeness, we include also the formulas for the rest of the approximate moments. The approximate standard deviation of the tax rate and debt are

$$std_\tau(\sigma) \simeq std_\tau(0) + \sigma \frac{Cov(\tau'_t(0), \tau_t(0))}{std_\tau(0)}$$

$$std_b(\sigma) \simeq std_b(0) + \sigma \frac{Cov(b'_t(0), b_t(0))}{std_b(0)}$$

with

$$Cov(\tau'_t(0), \tau_t(0)) = E\alpha_t(0)\big(\eta_t(0) - m^{*\prime}_t(0) - \Phi'(0)/\Phi(0)\big)(\tau_t(0) - E\tau_t(0))$$
$$Cov(b'_t(0), b_t(0)) = E[\alpha_1(g_t)(\eta_t(0) - m^{*\prime}_t(0)) + \alpha_2(g_t) + \alpha_3(g_t)](b_t(0) - Eb_t(0))$$

and the approximate means

$$E\tau_t(\sigma) \simeq E\tau_t(0) + \sigma E\alpha_t(0)\big(\eta_t(0) - m^{*\prime}_t(0) - \Phi'(0)/\Phi(0)\big)$$
$$Eb_t(\sigma) \simeq Eb_t(0) + \sigma E\big[\alpha_1(g_t)(\eta_t(0) - m^{*\prime}_t(0)) + \alpha_2(g_t) + \alpha_3(g_t)\big].$$

References

Aiyagari, S. Rao, Albert Marcet, Thomas J. Sargent, and Juha Seppala. 2002. Optimal Taxation without State-Contingent Debt. *Journal of Political Economy* 110 (6):1220–1254.

Barillas, Francisco, Lars Peter Hansen, and Thomas J. Sargent. 2007. Volatility or Doubts? Unpublished paper, University of Chicago and New York University.

Barro, Robert J. 1979. On the Determination of the Public Debt. *Journal of Political Economy* 87 (5):940–71.

Battaglini, Marco and Stephen Coate. 2007. A Dynamic Theory of Public Spending, Taxation and Debt. Discussion papers, Northwestern University, Center for Mathematical Studies in Economics and Management Science.

Brunnermeier, Markus K, Christian Gollier, and Jonathan A Parker. 2007. Optimal Beliefs, Asset Prices and the Preference for Skewed Returns. Cepr discussion papers, C.E.P.R. Discussion Papers.

Chari, V.V., Lawrence J. Christiano, and Patrick J. Kehoe. 1994. Optimal Fiscal Policy in a Business Cycle Model. *Journal of Political Economy* 102 (4):617–652.

Gilboa, I. and D. Schmeidler. 1989. Maxmin Expected Utility with Non-Unique Prior. *Journal of Mathematical Economics* 18:141–153.

Hansen, L. P. and T. J. Sargent. 2001. Robust Control and Model Uncertainty. *American Economic Review* 91:60–66.

Hansen, L. P., T. J. Sargent, G. A. Turmuhambetova, and N. Williams. 2006. Robust Control, Min-Max Expected Utility, and Model Misspecification. *Journal of Economic Theory* 128:45–90.

Hansen, Lars Peter and Thomas Sargent. 1995. Discounted Linear Exponential Quadratic Gaussian Control. *IEEE Transactions on Automatic Control* 40 (5):968–971.

Hansen, Lars Peter and Thomas J. Sargent. 2005. Robust Estimation and Control under Commitment. *Journal of Economic Theory* 124 (2):258–301.

———. 2006. Recursive Robust Estimation and Control without Commitment. Unpublished.

———. 2007. Robustness. Princeton University Press, forthcoming.

Karantounias, Anastasios G., Lars Peter Hansen, and Thomas J. Sargent. 2007. Ramsey taxation and fear of misspecification: the general case. Unpublished paper, New York University and University of Chicago.

Kocherlakota, Narayana and Christopher Phelan. 2006. Dynamic Optimal Taxation: A Robust Analysis. 2006 meeting papers, Society for Economic Dynamics.

Kydland, Finn E. and Edward C. Prescott. 1980. Dynamic optimal taxation, rational expectations and optimal control. *Journal of Economic Dynamics and Control* 2:79–91.

Lucas, Robert Jr. and Nancy L. Stokey. 1983. Optimal fiscal and monetary policy in an economy without capital. *Journal of Monetary Economics* 12 (1):55–93.

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini. 2006a. Ambiguity Aversion, Robustness, and the Variational Representation of Preferences. *Econometrica* 74 (6):1447–1498.

———. 2006b. Dynamic Variational Preferences. *Journal of Economic Theory* 128:4–44.

Marcet, Albert and Ramon Marimon. 1998. Recursive Contracts. Unpublished paper, Universitat Pompeu Fabra and European University Institute.

Sargent, Thomas J. 2005. Note on Michael Woodford's paper. Discussion prepared for conference on monetary economics at the Federal Reserve Bank of New York.

Woodford, Michael. 2005. Robustly Optimal Monetary Policy with Near Rational Expectations. Nber working papers, National Bureau of Economic Research, Inc.