Capital Misallocation and Aggregate Factor Productivity *

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Abstract

We propose a sectoral–shift theory of aggregate factor productivity for a class of economies with AK technologies, limited loan enforcement, a constant production possibilities frontier, and finitely many sectors producing the same good. Both the growth rate and factor productivity in these economies respond to random and persistent endogenous fluctuations in the sectoral distribution of physical capital which, in turn, responds to persistent and reversible shifts in relative sector productivities. Surplus capital from less productive sectors is lent to more productive sectors in the form of secured collateral loans, as in Kiyotaki–Moore (1997), and also as unsecured reputational loans suggested in Bulow–Rogoff (1989). Endogenous debt limits slow down capital reallocation, preventing the equalization of risk-adjusted equity yields across sectors. Economy-wide factor productivity and the aggregate growth rate are both negatively correlated with the dispersion of sectoral rates of return, sectoral TFP and sectoral growth rates. If sector productivities follow a symmetric two–state Markov process, many of our economies converge to a periodic cycle alternating between mild expansions and abrupt contractions.

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1 Introduction

National income accounting exercises conducted by Klenow and Rodriguez-Clare (1997), Hall and Jones (1999) and Chari, Kehoe, and McGrattan (2007) unanimously conclude that total factor productivity is of cardinal importance for both long-run growth and business cycles, including economic depressions. Klenow and Rodriguez-Clare, for example, find that at least 50% of the variation in output per worker in a sample of over 40 countries is attributable to differences in TFP. There is less agreement about the source of productivity differentials. Suggestions range widely from differences in broadly defined social infrastructure advocated by Hall and Jones, to technology adoption barriers proposed by Parente and Prescott (1999), to the labor market frictions studied in Lagos (2006).

This paper is a theoretical investigation of how credit markets frictions limit capital mobility and slow down the movement of resources from temporarily less to temporarily more productive sectors. We are pushed in this direction by much evidence connecting poor economic performance with capital misallocation. Chari et al. (2007) find that financial frictions, defined as wedges that distort the allocation of intermediate goods among firms, account for 60-80% of the US output drop in both the 1929-1933 depression and the 1979-1982 recession. Eisfeldt and Rampini (2006) point out that capital reallocation among U.S. firms–defined as sales and acquisition of property, plant and equipment–makes up nearly 25% of total investment on average. Finally, there are strong indications that macroeconomic volatility is connected with the dispersion of both sectoral productivities and sectoral rates of return on capital.[1]

Lilien (1982) was an early advocate of the importance of sectoral shocks for overall economic activity in an empirical study that connected the aggregate unemployment

[1] See Eisfeldt and Rampini (2006) on the countercyclical dispersion of capital productivity among firms and on the sectoral dispersion of TFP; Loungani, Rush, and Tave (1990) and Brainard and Cutler (1993) on the countercyclical dispersion of stock market returns across sectors; Diebold and Yilmaz (2008) on the correlation between the volatilities of stock market returns and GDP growth in a sample of 40 countries.

rate with the cross-sectional dispersion of sectoral employment. More recently, Phelan and Trejos (2000) argue in a model with labor–search frictions that sectoral reallocations are quantitatively important for business cycle dynamics. This paper puts Lilien's idea to work in a growth model with *financial frictions* and looks at the consequences for productivity and capital accumulation. It is this emphasis on sectoral shocks and productivity which separate our work from earlier literature on financial frictions as a cause of macroeconomic volatility.[2]

We describe sectoral shifts as idiosyncratic technology shocks in a class of simple economies populated by identical infinitely-lived households and consisting of finitely many sectors that produce the same consumption good. Capital is the only input in production which means that we focus on the misallocation of investment and ignore potentially larger problems stemming from imperfectly functioning labor markets.Sectoral technologies are assumed to be AK with random idiosyncratic productivities and a constant aggregate production possibility frontier, that is, a fixed value for the maximal idiosyncratic productivity. We ignore declining and expanding industries, assuming instead that all sectoral shocks are temporary and reversible.

An ideal economy of this type without any financial frictions would exploit its unchanging aggregate production possiblities to the fullest by moving all physical capital instantly to the most productive sector, and delivering to its population a constantly growing stream of aggregate output and individual consumption. In what follows, surplus capital from less productive sectors is in the form of secured collateral loans, as suggested by Kiyotaki and Moore (1997) and also in the form unsecured reputational loans, as in Bulow and Rogoff (1989). Both types of loans require endogenous debt limits which rule out default when asset markets are complete. These limits slow down capital reallocation and prevent rates of return on capital from equality across all sectors.

[2]Prominent examples are Kiyotaki and Moore (1997) on collateral constraints as a propagator and amplifier of aggregate technology shock; Matsuyama (2007) on the interplay between borrower net wealth, debt limit and investment; and Aghion, Banerjee, and Piketty (1999) on how restricted participation in credit markets contributes to macroeconomic volatility.

The rest of the paper is organized as follows. Section 2 describes a class of economies with financial frictions; section 3 gives a preview of results by explaining what happens in the worst-case scenario of financial autarky or zero capital mobility. Stationary Markov equilibria are defined in Section 4 and described in Section 5 for economies with secured loans only. Section 6 looks at the dynamics of lending when both secured and unsecured loans are traded. Section 7 presents some numerical examples connecting aggregate with sectoral variables. Extensions are discussed in Section 8 and conclusions are summarized in Section 9.

2 The environment

Consider a growth model in discrete time $t = 0, 1, 2, \ldots$ with a finite number of agent types (sectors) indexed $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ and productivity states $s \in \mathcal{S} = \{1, 2, \ldots, S\}$. Each sector comprises a continuum of agents with equal size. All agents produce the same good which is available for consumption and investment purposes. Their common preferences over consumption streams are represented by an additively separable expected utility function

$$E_0(1-\beta)\sum_{t=0}^{\infty}\beta^t \ln[c(s^t)] \ ,$$

where $s^t = (s_t, \ldots, s_0) \in \mathcal{S}^{1+t}$ is the state history in period t, and the initial state s_0 is given. The productivity state follows a Markov process with transition probability from s to s_+ equal to $\pi(s_+|s)$. In state s an agent of type i can convert capital into gross output ("resources") with linear technology $y = A^i_s k$. Resources y include current output and undepreciated capital which can be costlessly converted into the single consumption/investment good in the next period. In particular, capital investment is not sector–specific. The simplification that all agents produce the same good isolates the impact of sectoral shocks on capital reallocation while abstracting from relative price effects.

We assume that the economy's production possibility frontier is constant at $A \equiv \max_{i\in\mathcal{I}} A^i_s$ for all $s \in \mathcal{S}$. Though we do not need to impose that agent types are in some way symmetric, it simplifies the exposition to assume that every agent has access to the technological frontier sometimes and that there is always a unique most productive sector:

(A1) Every agent operates the technological frontier sometimes; that is, for each i there exists s such that $A_s^i = A$.

(A2) Not more than one agent type operates the technological frontier; that is, for each s there is exactly one i such that $A_s^i = A$.

(A3) No state is trivial; that is, every $s \in \mathcal{S}$ is in the support of the unique invariant state distribution.

Throughout this paper we focus on stationary Markov equilibria where all endogenous variables depend only on the current state vector of the economy, denoted $\sigma \equiv (\mathbf{x}, s) \in \Sigma \equiv [0,1]^I \times \mathcal{S}$, where $\mathbf{x} = (x^1, \ldots, x^I)$ is the distribution of wealth shares across agent types.

Each period, the less productive agents lend out capital to the more productive agents at gross interest rate $R(\sigma)$ in a credit market. An exogenous fraction $\lambda \in [0,1]$ of each agent's resources is pledgeable collateral which can be seized by creditors in the event of default. The value of λ is common for all producers; it depends on technological factors like the collaterizability of income and wealth, as well as on creditor rights and other aspects of economic institutions.[3] Timing within each period is as follows. First the productivity state is realized; second the credit market opens and agents decide about consumption, investment, borrowing and lending; and third, agents produce, borrowers redeem their debt, and everyone carries their wealth into the next period.

Borrowers may choose to default at the end of the period. Any agent who does so loses the collateral share of his resources to creditors and is banned from any borrowing in excess of collateral in all future periods. A defaulting agent is still allowed

[3]If resources are split into output and undepreciated capital according to $y = Ak = \tilde{A}k + (1-\delta)k$, a more general expression for collateral would be $\lambda_0 \tilde{A}k + \lambda_1(1-\delta)k$. Our simplifying assumption is that collaterizability of output and capital is the same, $\lambda_0 = \lambda_1$.

to lend, however, and also to borrow up to the discounted value of his collateralized assets. Since no uncertainty is resolved during debt contracts (that is, borrowing and debt redemption happen within the same period), there exist endogenous debt limits similar to those defined in the pure–exchange model of Alvarez and Jermann (2000). These limits are the highest values of debt that will prevent default. In the absence of collateral ($\lambda = 0$), our enforcement mechanism resembles the one discussed by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2008) who assume that defaulters are denied all credit but are still allowed to accumulate assets. With $\lambda > 0$, borrowing against collateral is always feasible and sometimes, but not always, credit limits go beyond an agent's collateral capacity and sustain a higher flow of credit. Borrowing above one's collateral is a *reputational loan* founded on a producer's desire to maintain a record of solvency and of continued access to future reputational loans.

We denote the endogenous constraint on borrower i's debt–equity ratio by $\theta^i(\sigma)$. Whenever the cost of capital $R(\sigma)$ is strictly below borrower i's marginal product A^i_s, this producer will borrow up to his debt limit, and the leveraged equity return will be $\tilde{R}^i(\sigma) = A^i_s + \theta^i(\sigma)[A^i_s - R(\sigma)]$. On the other hand, if agent i's productivity is below or equal to the capital yield $R(\sigma)$, this agent's equity return is simply $\tilde{R}^i(\sigma) = R(\sigma)$. A defaulting agent, who can only borrow against collateral, faces a maximal debt–equity ratio $\theta^i_c(\sigma) = \lambda A^i_s/[R(\sigma) - \lambda A^i_s]$, and his equity return is $\tilde{R}^i_c(\sigma) = A^i_s R(\sigma)(1-\lambda)/[R(\sigma)-\lambda A^i_s]$ when $R(\sigma) < A^i_s$, and $\tilde{R}^i_c(\sigma) = R(\sigma)$ otherwise.

It is worth noting that intra–period credit is the only traded asset in this economy. If agents were to trade insurance or contingent claims against next period's productivity state, these security markets would not open. This immediately follows from the observation that every agent's marginal utility of wealth ω is proportional to $1/\omega$, regardless of the agent's productivity state, so all agents' security demands are proportional to their wealth. That in turn implies that trade of insurance securities must be zero in equilibrium. We also do not consider a stock market distinct from the loan market. In particular, all shares in other agents' technologies are equivalent to loans and are subject to default.

The assumption of logarithmic utility implies that all agents consume a fraction $1-\beta$

of wealth, and that the expected utility of a productive borrower with end–of–period wealth ω can be expressed in the form $\ln(\omega) + V^i(\sigma)$, where $V^i(\sigma)$ is end–of–period utility of agent i with unit wealth when the current state is $\sigma = (\mathbf{x}, s)$.[4] Similarly, if agent i had defaulted in this or in some earlier period, his utility is expressed as $\ln(\omega) + V_c^i(\sigma)$ if end–of–period wealth is ω and $V_c^i(\sigma)$ denotes end–of–period utility of a unit–wealth agent of type i who has only access to collateralized loans.

3 An Example: Financial Autarky

As a first step to understand the importance of financial markets for aggregate factor productivity, we describe equilibrium in an autarkic economy without collateral in which capital is completely immobile. We suppose in particular that this economy has two sectors ($i = 1, 2$) and two states ($s = 1, 2$) with transition probabilities

$$\pi(s_+|s) = \begin{cases} \pi \in [0,1] & \text{if} \quad s_+ = s \ , \\ 1 - \pi & \text{if} \quad s_+ \neq s \ . \end{cases}$$

Sectoral productivities are

$$A_s^i = \begin{cases} A > 0 & \text{if} \quad i = s \ , \\ zA & \text{if} \quad i \neq s \ . \end{cases}$$

with $0 < z < 1$. We denote by $x \in [0, 1]$ the wealth share of the productive sector, by (Y, K) the vector of current aggregate output and capital, and by (Y_+, K_+) the future value of that vector.

Aggregate output and future capital satisfy

$$Y = AK[x + z(1 - x)], \quad K_+ = \beta Y. \tag{1}$$

The future value of the wealth share x_+ equals the ratio of the efficient producer's capital tomorrow divided by K_+. This yields the following stochastic law of motion

[4] These assertions follow from the observation that agent i's flow budget constraint takes the form $\omega^i = \tilde{R}^i(\sigma)(\omega_-^i - c^i)$ where c^i is consumption and ω^i (ω_-^i) is agent i's wealth at the end of the current period (the previous period, respectively).

for the wealth share

$$x_+ = \begin{cases} f(x) = x/[x + z(1-x)] & \text{w. prob. } \pi \ , \\ 1 - f(x) = z(1-x)/[x + z(1-x)] & \text{w. prob. } 1-\pi \ . \end{cases} \tag{2}$$

Equation (2) is graphed in Figure 1 (c) as the special case $\lambda = 0$. From equation (1), we conclude that aggregate factor productivity:

- includes a correction $x + z(1-x) < 1$ due to financial frictions;
- is lower when sectoral productivities are more dispersed, that is, for small values of z; and
- fluctuates in response to changes in the distribution of wealth between potential "borrowers" and "lenders", that is, between more productive and less productive sectors.

It is easy to check that the aggregate growth rate

$$Y_+/Y = \beta A[z + (1-z)x_+]$$

fluctuates when the wealth distribution changes, even though the aggregate production possibilities frontier is stationary. In a cross section of economies indexed on the value of z, the growth rate would be positively correlated with z and, therefore, negatively correlated with the dispersion of sectoral TFP's.

We return now to economies with active asset markets.

4 Stationary Markov equilibrium

A stationary Markov equilibrium is a list of functions

$$\left[\theta^i(\sigma), R(\sigma), \tilde{R}^i(\sigma), \tilde{R}^i_c(\sigma), v^i(\sigma), X^i(\sigma)\right]_{i\in\mathcal{I},\sigma\in\Sigma} \ . \tag{3}$$

The first four objects on that list are respectively the debt–equity limits on solvent agents, the cost of capital, and the equity returns for solvent and bankrupt agents.

The functions $v^i(\sigma) = V^i(\sigma) - V^i_c(\sigma)$ define the "penalty of default" for agent i, that is, the difference between the continuation utilities from solvency and default. Finally, the maps $x^i_+ = X^i(\sigma) : \Sigma \to [0,1]$ connect this period's state vector $(\mathbf{x}, s)$ with next period's wealth share for every agent i. Let $\mathbf{X} = (X^i)_{i \in \mathcal{I}} : \Sigma \to [0,1]^I$ be the collection of these maps.

In equilibrium, debt limits are the largest values that will deter default when any borrower with equity E is indifferent between defaulting and not defaulting:

$$\ln\left[\tilde{R}^i(\sigma)E\right] + V^i(\sigma) = \ln\left[(1-\lambda)A^i_s[1+\theta^i(\sigma)]E\right] + V^i_c(\sigma) \ .$$

Here, the right–hand side is expected utility of the defaulting agent i who leaves the default period with his unpledged wealth $(1-\lambda)A^i_s[1+\theta^i(\sigma)]E$. This equality is conveniently equivalent to

$$\theta^i(\sigma) = \frac{(e^{v^i(\sigma)} - 1 + \lambda)A^i_s}{(1-\lambda)A^i_s - e^{v^i(\sigma)}[A^i_s - R(\sigma)]} \ . \tag{4}$$

Equation (4) shows that the default–deterring debt–equity ratio is increasing in the penalty of default $v^i(\sigma)$ and in the collateral share λ. Debt–equity ratios are also decreasing in the interest rate, and equation (4) implies a lower bound on the equilibrium interest rate: the debt-equity ratio of borrower i tends to infinity when $R(\sigma)$ approaches $A^i_s[1-(1-\lambda)e^{-v^i(\sigma)}]$ from above. Intuitively, when the interest rate is low, some borrowers never opt for default. Thus their demand for loans becomes infinite which cannot be compatible with credit–market equilibrium.

Equation (4) shows that $\theta^i(\sigma)$ is larger than the collateral debt limit $\theta^i_c(\sigma) = \lambda A^i_s/[R(\sigma) - \lambda A^i_s]$ for all positive default penalties $v^i(\sigma) > 0$; it reduces to $\theta^i_c(\sigma)$ if $v^i(\sigma) = 0$. In the following, we refer to an equilibrium with $v^i(\sigma) = 0$ for all $i \in \mathcal{I}$ and $\sigma \in \Sigma$ as one of *collateral borrowing*; an equilibrium where $v^i(\sigma) > 0$ for at least some $i \in \mathcal{I}$ and $\sigma \in \Sigma$ has *reputational borrowing*: here debt–equity limits are based on collateral *and* reputation.

With aggregate capital K, agent i's equity is $x^i K$. The *supply of credit* comes from all agents with productivity $A^i_s \leq R(\sigma)$, and because agents with $A^i_s = R(\sigma)$ are indifferent between lending and borrowing at market rate $R(\sigma)$, the aggregate supply of credit (per unit of aggregate capital) is a step function, expressed as the correspondence

$$\mathrm{CS}(\sigma) = \left[\sum_{i:A_s^i < R(\sigma)} x^i \ , \ \sum_{i:A_s^i \leq R(\sigma)} x^i \right] .$$

Similarly, the demand for credit per unit of capital is the correspondence

$$\mathrm{CD}(\sigma) = \left[\sum_{i:A_s^i > R(\sigma)} \theta^i(\sigma) x^i \ , \ \sum_{i:A_s^i \geq R(\sigma)} \theta^i(\sigma) x^i \right] ,$$

and the credit market is in equilibrium if

$$\mathrm{CS}(\sigma) \cap \mathrm{CD}(\sigma) \neq \emptyset \ . \tag{5}$$

As we saw earlier, for any interest yield $R(\sigma)$, the equity return of agent i is

$$\tilde{R}^i(\sigma) = \max \left\{ A_s^i + \theta^i(\sigma) \Big[A_s^i - R(\sigma) \Big], R(\sigma) \right\} , \tag{6}$$

while the equity return of an agent who is permitted to borrow only against collateral is

$$\tilde{R}_c^i(\sigma) = \max \left\{ \frac{A_s^i R(\sigma)(1-\lambda)}{R(\sigma) - \lambda A_s^i}, R(\sigma) \right\} . \tag{7}$$

Agent i's wealth share changes from x^i to

$$x_+^i = X^i(\sigma) = \frac{\tilde{R}^i(\sigma) x^i}{\displaystyle\sum_{j \in \mathcal{I}} \tilde{R}^j(\sigma) x^j} \quad , \ \sigma = (x^1, \ldots, x^I, s) \ . \tag{8}$$

To understand this expression, suppose that total wealth is one unit today; then agent i's wealth next period is β times the numerator of (8) while total wealth is β times the denominator of (8).

Expected utilities satisfy recursive equations

$$V^i(\mathbf{x}, s) = (1-\beta) \ln(1-\beta) + \beta \sum_{s_+ \in \mathcal{S}} \pi(s_+|s) \left\{ \ln \Big[\beta \tilde{R}^i[\mathbf{X}(\mathbf{x}, s), s_+] \Big] + V^i \Big[\mathbf{X}(\mathbf{x}, s), s_+ \Big] \right\} . \tag{9}$$

Note again that V^i denotes expected utility of solvent agent i with unit wealth. In the first period, this agent consumes $c = 1 - \beta$, and so the first term on the right–hand side is the utility of first–period consumption; the other terms are discounted future payoffs. To the next period, the distribution of wealth changes from $\mathbf{x}$ to $\mathbf{x}_+ = \mathbf{X}(\mathbf{x}, s)$ and the productivity state changes from s to s_+ with probability $\pi(s_+|s)$; the agent saves a fraction β of his unit wealth, ending the period with wealth $\omega_+ = \beta\tilde{R}^i(\mathbf{x}_+, s_+)$ and utility $\ln(\omega_+) + V^i[\mathbf{x}_+, s_+]$. For an agent who has opted for default in some earlier period, the recursive equation in V_c^i is nearly identical to (9); all that changes is that the equity returns $\tilde{R}^i$ are replaced by the defaulter's lower returns $\tilde{R}_c^i$. By subtracting those equations from (9), we obtain recursive equations in the default penalties $v^i(\sigma) = V^i(\sigma) - V_c^i(\sigma)$:

$$v^i(\mathbf{x}, s) = \beta \sum_{s_+ \in \mathcal{S}} \pi(s_+|s) \left\{ \ln \frac{\tilde{R}^i[\mathbf{X}(\mathbf{x}, s), s_+]}{\tilde{R}_c^i[\mathbf{X}(\mathbf{x}, s), s_+]} + v^i\Big[\mathbf{X}(\mathbf{x}, s), s_+\Big] \right\} . \qquad (10)$$

Definition: *A stationary Markov equilibrium is a list of functions specified in (3) which satisfies equations (4)–(8) and (10) for all* $\sigma = (\mathbf{x}, s) \in \Sigma$ *and* $i \in \mathcal{I}$.

In a stationary Markov equilibrium, the state vector σ is also a sufficient statistic for the growth rate that connects aggregate current resources Y with last period's resources Y_-. In particular, current aggregate capital K equals saving βY_-, and current resources are the sum of resources across all agent types:

$$\begin{aligned} Y &= K \sum_{i \in \mathcal{I}} x^i \tilde{R}^i(\sigma) \\ &= \beta Y_- \Big\{ R(\sigma) + \sum_{i: A_s^i > R(\sigma)} [A_s^i - R(\sigma)] x^i [1 + \theta^i(\sigma)] \Big\} . \end{aligned}$$

The growth factor is

$$\frac{Y}{Y_-} = \beta \Big\{ R(\sigma) + \sum_{i: A_s^i > R(\sigma)} [A_s^i - R(\sigma)] x^i [1 + \theta^i(\sigma)] \Big\} \leq \beta A .$$

This expression has an upper bound βA achieved when no capital is misallocated.

Before we analyze stationary Markov equilibria in detail for some special cases, we state two general results. One of them says that an equilibrium with no reputational borrowing always exists and in that equilibrium all borrowing is against collateral. In particular,

Proposition 1: *There exists a unique equilibrium with collateral borrowing.*

This result generalizes earlier findings by Bulow and Rogoff (1989) and Kehoe and Levine (1993) who showed that financial autarky is an equilibrium in economies where all borrowing is reputational. Indeed, it is easy to check that $v^i(\sigma) = 0$ together with $\tilde{R}^i(\sigma) = \tilde{R}^i_c(\sigma)$ and $\theta^i(\sigma) = \theta^i_c(\sigma)$ satisfy all equilibrium equations except market clearing for any given interest rate $R(\sigma)$. Existence and uniqueness of the market–clearing interest rate is proven in the appendix.

What is the intuition for the equilibrium with collateral borrowing? If there are no reputational loans, there is no penalty of default, and therefore no borrower is permitted to borrow in excess of collateral. And conversely, when debt–equity limits just reflect collateral constraints, a good credit record is worthless because there is no default penalty. Section 4 characterizes the collateral borrowing equilibrium completely for a symmetric economy with two agent types and two states.

Our second result says that a first–best allocation can only be an equilibrium if there is enough collateral. Specifically, $\lambda \geq (I-1)/I$ is a necessary and sufficient condition to support the first best with collateral borrowing at the symmetric initial wealth distribution $x^i = 1/I$, $i \in \mathcal{I}$. Here returns are equalized, $\tilde{R}^i = A$, and the debt–equity ratio is large enough to shift all capital to the most productive sector in every state.

Can the first best also be supported by reputational borrowing when $\lambda < (I-1)/I$? Put differently, is there a first–best equilibrium where debt constraints exceeed collateral constraints? In line with earlier results by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2008), the answer to this question is no. Reputation by itself or with insufficient collateral cannot support first–best allocations. These findings are summed up in

Proposition 2: *When $\lambda \geq (I-1)/I$, the collateral–borrowing equilibrium gives rise to a first best allocation for some initial distribution of wealth. Conversely, when $\lambda < (I-1)/I$, no first best allocation can be an equilibrium with collateral borrowing or with reputational borrowing.*

The inequality $\lambda \geq (I-1)/I$ is stringent, requiring collateral to be a large proportion of available resources, that is, gross national product plus undepreciated capital. In spite of Proposition 2, we will see in Section 5 that reputational borrowing can still sometimes support production–efficient allocations, particularly in economies with very patient agents and large productivity differences.

To explore equilibrium with binding constraints in more detail, we focus for the remainder of this paper on the symmetric two–agent, two–state special case of the general environment. In particular, $A^i_s = A$ if $i = s$, and $A^i_s = zA$ if $i \neq s$, for $i \in \{1,2\}$ and $s \in \{1,2\}$, where $z < 1$ is a measure of the productivity differential. Both types are equally likely to operate the frontier technology, where π is the probability that any state $s = 1,2$ does not change from one period to the next. In this symmetric economy, stationary Markov equilibria are also symmetric. Therefore, the only relevant state variable is the share of wealth owned by the borrowing agents (short "borrower wealth"), to be denoted $x \in [0,1]$. The wealth distribution is thus $(x, 1-x)$ if $s = 1$ and $(1-x, x)$ if $s = 2$. Current rates of return and debt limits depend on borrower wealth x alone, and the productivity state s matters only for the transitional dynamics of borrower wealth.

5 Collateral borrowing

In the collateral borrowing equilibrium with zero default penalties $v^i(x,s) = 0$ and debt–equity ratio $\theta = \lambda A/[R - \lambda A]$, the market–clearing loan yield can be readily obtained as

$$R(x) = \begin{cases} zA & \text{if } x \leq 1 - \frac{\lambda}{z} \ , \\ \frac{\lambda A}{1-x} & \text{if } x \in [1 - \frac{\lambda}{z}, 1-\lambda] \ , \\ A & \text{if } x \geq 1 - \lambda \ . \end{cases}$$

When borrower wealth is below $1-\lambda/z$, credit demand is so low that the equilibrium interest rate makes unproductive lenders indifferent between production and lending. The economy is *production inefficient* because it misallocates its capital stock. When borrower wealth exceeds this threshold, all capital flows to the more productive agents and the economy becomes production efficient. For $x < 1-\lambda$, borrowers are still debt constrained and enjoy a higher equity yield than do lenders. Consumption growth rates are higher for borrowers which makes the economy *consumption inefficient*. Full efficiency in period t is attained only when borrower wealth exceeds $1-\lambda$. In what follows, we assume throughout that $\lambda < z$ so that production inefficiency remains a possibility.

The transitional dynamics of borrower wealth is described by two maps. Next period's borrower wealth is $x_+ = X_0(x)$ when the productivity state is unaltered and it is $x_+ = X_1(x) = 1 - X_0(x)$ when the productivity state changes. Using the above expressions for $R(x)$, $\theta(x) = \lambda A/[R(x) - \lambda A]$, and the borrowers' equity return $\tilde{R}(x) = A + \theta(x)[A - R(x)]$, we obtain

$$X_0(x) = \frac{\tilde{R}(x)x}{\tilde{R}(x)x + R(x)(1-x)} = \begin{cases} \dfrac{(1-\lambda)x}{(1-z)x+z-\lambda} & , \ x \leq 1 - \frac{\lambda}{z} \ , \\ 1-\lambda & , \ x \in [1-\frac{\lambda}{z}, 1-\lambda] \ , \\ x & , \ x \geq 1-\lambda \ . \end{cases}$$

Figure 1 shows the two maps X_0 and X_1 in three generic situations. It becomes evident from these graphs that the stochastic dynamics of borrower wealth must settle down on the bounded interval $[\lambda, 1-\lambda]$ or $[1-\lambda, \lambda]$ (depending on whether $\lambda \leq 1/2$ or $\lambda \geq 1/2$. Moreover, the asymptotic dynamics must be a stochastic cycle with finite support. The precise statement, which is proved in the Appendix, is

Proposition 3: *In the equilibrium with collateral borrowing and for any* $\pi \in (0,1)$ *and* $\lambda > 0$*, the dynamics of wealth* x_t *enters a finite stochastic cycle* $(x_n)_{n=1}^N$*, with probability one as* $t \to \infty$*. The cycle has the following features.*

(a) *Economies with ample collateral* $\lambda \geq 1/2$ *converge to a cycle with two states* $x_2 = 1 - x_1 \in [1-\lambda, \lambda]$*. Production is efficient, debt constraints do not bind, and aggregate output growth and individual consumption growth are constant at* βA*.*

(b) *Economies with medium collateral* $\lambda \in [z/(1+z), 1/2)$ *also converge to a cycle with two states and* $x_1 = \lambda < x_2 = 1-\lambda$*. Production is again efficient and aggregate growth is constant at* βA*. However, individual consumption* *(wealth) growth rates are volatile and borrowers are constrained in a fraction $1-\pi$ of periods. Specifically, agent i's consumption growth in state s_t is βA if $s_t = s_{t-1}$, $\beta A\lambda/(1-\lambda)$ if $i \neq s_t \neq s_{t-1}$, and $\beta A(1-\lambda)/\lambda$ if $i = s_t \neq s_{t-1}$.*

(c) *Economies with small collateral $\lambda < \frac{z}{1+z}$ converge to a cycle with generically $N = 2m$ states, with $m \geq 2$. In $2m-3$ of these states, aggregate growth is lower than βA. Cycles are typically asymmetric with booms lasting longer than recessions.*

Figure 1 illustrates the three possibilities stated in the proposition. In (a), the typical first–best equilibrium is a cycle where borrower wealth fluctuates between two states which means that every agent's wealth share is constant. Any initial wealth distribution must enter such a cycle with probability one in finitely many periods. In (b), the stochastic cycle again has only two states, but now one of them has constrained borrowers; no capital is misallocated and production is efficient in all periods. And graph (c) shows an example of a cycle with six states, with no misallocation of capital in three of them, and some misallocation in the other three. The red lines indicate the possible transitions between these states.

6 Reputational borrowing

Equilibria with reputational borrowing are not easy to describe analytically in any degree of generality. Nonetheless, it is possible to derive a few insightful results for some special cases where the asymptotic wealth dynamics settles down to a finite state space. One such case is the deterministic economy ($\pi = 0$), the other is an economy permitting simple production–efficient stochastic cycles with two states. We explore these simpler equilibria in this section.

The deterministic economy admits a steady state with binding constraints where borrower wealth is stable at some x. The wealth share of either type thus periodically alternates between x and $1-x$. This is in stark contrast to the stochastic economy where equilibria are typically cyclical and the only possible steady state is a first

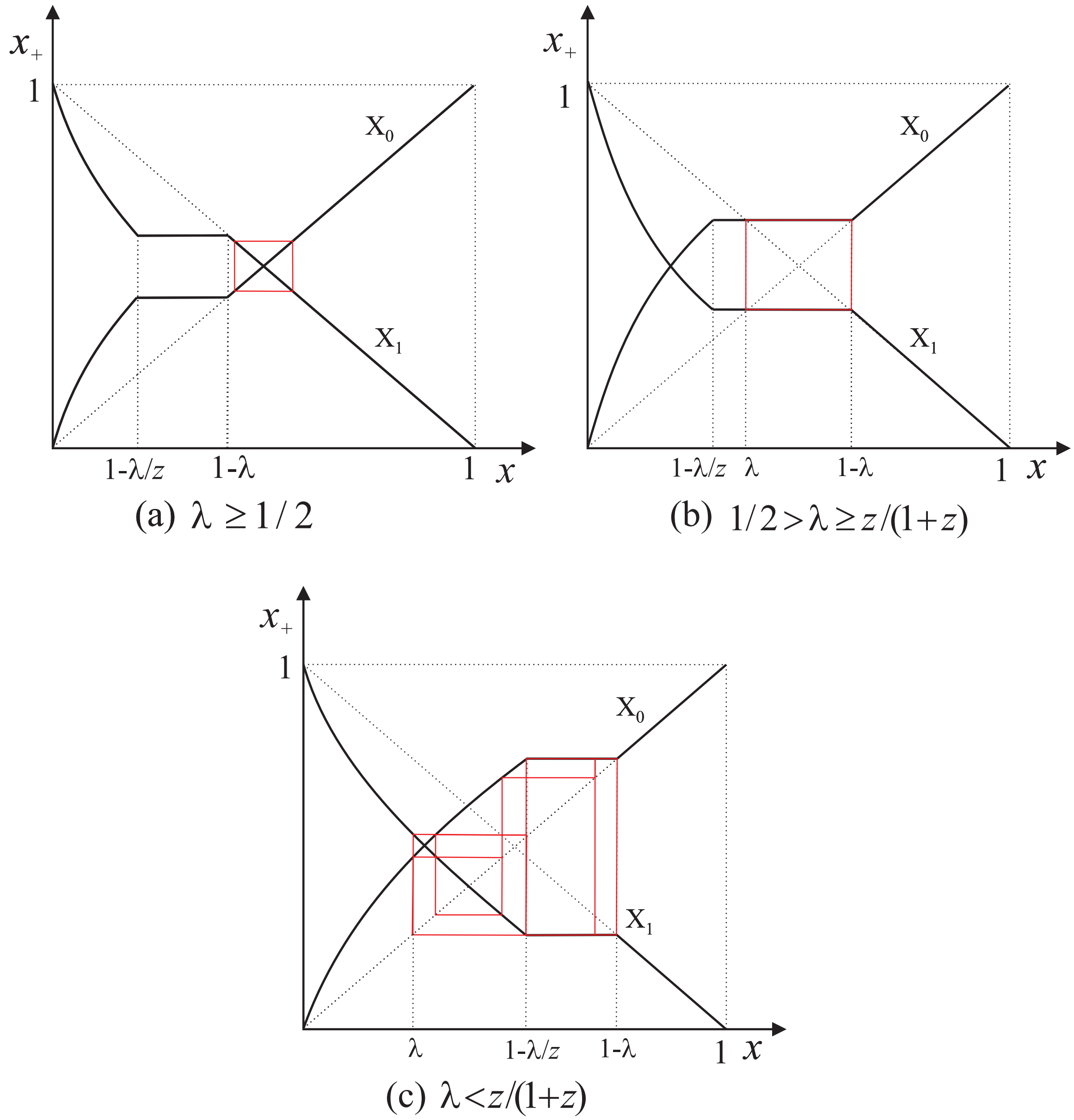

Figure 1: Asymptotic cycles with collateral borrowing.

best outcome with unconstrained borrowers achievable only if $\lambda \geq 1/2$. For $\lambda < 1/2$, borrower wealth must fluctuate permanently in a stochastic economy.

One obvious steady state in the deterministic model is the one with collateral borrowing. Although Proposition 3 requires $\pi > 0$, it is straightforward to extend the result to the deterministic case as follows. The deterministic economy has a unique steady state x with collateral borrowing which is (i) first best when $\lambda \geq 1/2$; (ii) production efficient and consumption inefficient when $z/(1+z) \leq \lambda < 1/2$; and (iii) production inefficient when $\lambda < z/(1+z)$. In Figure 1 these steady states are at the intersection of the 45^o line with the map $X_1(.)$.

For a deterministic economy with reputational borrowing, we prove the following result in the Appendix.

Proposition 4: *Let $\pi = 0$. Then there is a threshold value $\hat{\lambda} \leq \dfrac{z-\beta^2}{1-\beta^2}$ such that*

(a) *If $\beta \leq z$, there is one steady state with collateral borrowing and no steady state with reputational borrowing.*

(b) *If $\beta > z$ and $\lambda \in [\hat{\lambda}, \dfrac{\beta}{1+\beta})$, there is a production–efficient steady state with reputational borrowing which coexists with the steady state with collateral borrowing.*

(c) *If $\beta > z > \beta^2$ and $\lambda \in (\hat{\lambda}, \dfrac{z-\beta^2}{1-\beta^2})$, there is a production–inefficient steady state with reputational borrowing which coexists with the two other steady states of (b).*

To interpret these results, the inequality $\beta \leq z$ simply says that the gains from credit market participation are not high enough to support an equilibrium with reputation borrowing. Part (a) extends the well–known result of Kehoe and Levine (1993) that intertemporal financial autarky or, equivalently, pure collateral borrowing is the only equilibrium when agents are too impatient or when income fluctuations are too small. Conversely, says part (b), when $\beta > z$ reputational borrowing is feasible but now collateral may not exceed the threshold $\beta/(1+\beta)$. If the collateral value is larger than that number, the gain from borrowing above collateral is too small to prevent borrowers from defaulting. Put differently, collateral borrowing already supports efficient allocations with low leverage, so that extended credit limits add very little value. Part (c) establishes a strong form of equilibrium multiplicity. The production–inefficient steady state always co–exists with the socially more desirable production–efficient equilibrium and with the collateral borrowing equilibrium which is then also production inefficient. The explanation for equilibrium multiplicity is a *dynamic complementarity* in the endogenous borrowing limits. Borrowers' expectations of *future* credit market conditions affect their incentives to default today, and this in turn takes an impact on their *current* borrowing limits. If future constraints are tight, the payoff from solvency is modest; agents place a low value on the strategy of participating in credit markets, and their default penalty is low. In this case, current default–deterring debt limits must be low. Conversely, expectations of loose constraints in the future make participation more valuable, lessen default incentives and ease current constraints.

When agents are sufficiently patient, so that $z \leq \beta^2$, the assumptions in part (c) are not valid. Then the deterministic economy has a unique steady state with reputational borrowing, coexisting with the collateral–borrowing steady state. Figure 2 shows how steady–state loan yields vary with the collateral parameter λ when $z \leq \beta^2$.

The deterministic economy also permits an analysis of the local dynamics around the steady states. When there is only one steady state with reputational borrowing, it is locally determinate, but the collateral borrowing steady state is locally indeterminate.[5]

Proposition 5: *Let $z < \beta^2$ and $\lambda < \beta/(1+\beta)$.*

(a) *The steady state with collateral borrowing (θ^c, x^c) is locally indeterminate. Particularly, there is a continuum of equilibria (θ_t, x_t) such that $\theta_t \to \theta^c$ and x_t converges to a cycle with period two around x^c.*

(b) *The steady state with reputational borrowing is locally determinate.*

[5] In situations with two reputational borrowing steady states, the production–inefficient one turns out to be indeterminate and the steady state with collateral borrowing becomes determinate.

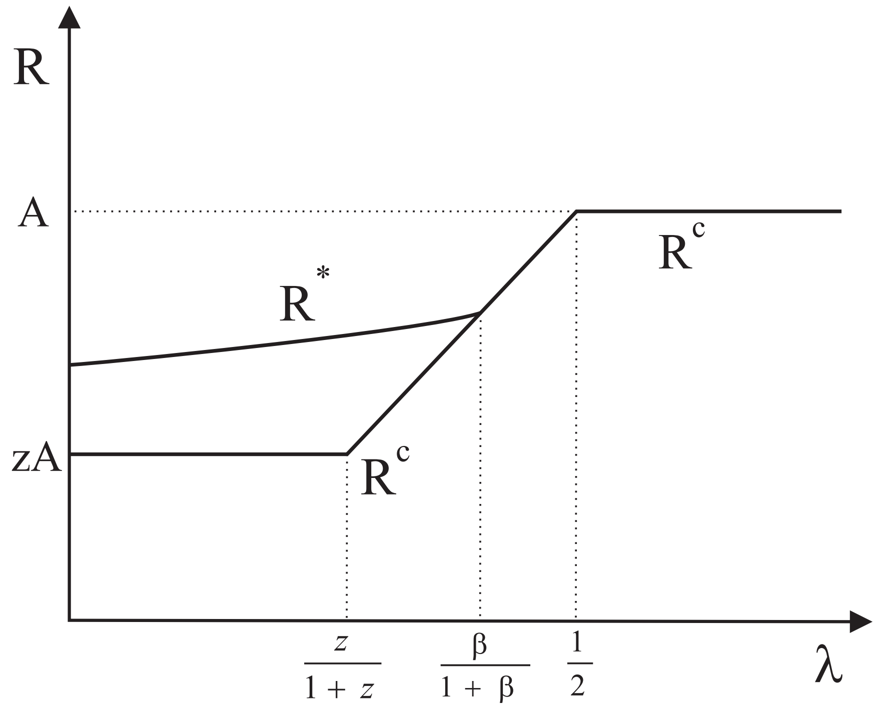

Figure 2: Steady state loan yields with collateral borrowing (R^c) and with reputational borrowing (R^*) when $z < \beta^2$.

The stochastic economy cannot have a steady state unless it is the first best, and cycles with reputational borrowing are too complex to describe analytically. Numerically we find that the qualitative features of the transition maps for borrower wealth are much like the ones shown in Figure 1, with the only difference that, unlike the collateral cycles in Figure 1(c), reputational cycles do not have finite support.

However, there are still situations where the economy has a stochastic cycle of order two which is production efficient, like the one shown in Figure 1(b). Paralleling Proposition 4, what is necessary for such cycles is that z is small relative to β: agents must be patient enough and their productivity must fluctuate sufficiently so that exclusion from reputational borrowing is a severe enough punishment. In fact, production–efficient cycles may even exist when there is no collateral at all. In the

Appendix we derive necessary and sufficient conditions for the existence of this kind of equilibrium, summarized as follows.

Proposition 6: *A production–efficient cycle with reputational borrowing:*

(a) *Does not exist for any λ if z is large, that is, if*

$$z \geq \frac{\beta(1-\pi)}{1-\beta\pi} \ .$$

(b) *Exists for intermediate λ and small z, that is, if*

$$\lambda \in \left[\underline{\lambda}, \frac{\beta(1-\pi)}{1+\beta-2\beta\pi}\right) \quad \textit{and} \quad z < \frac{\beta(1-\pi)}{1-\beta\pi} \ ,$$

for some threshold $\underline{\lambda}$ which equals zero whenever z falls below another threshold $\underline{z} < \beta(1-\pi)/(1-\beta\pi)$.

Part (a) says that production efficient cycles with reputational borrowing cannot exist if agents are too impatient or if productivity fluctuations are too small. Part (b) says the opposite: sufficiently large productivity fluctuations give rise to a production–efficient equilibrium which also requires collateral to be neither too large nor too small. For example, if collateral is large enough, production efficient (or even first best) allocations are achieved by collateral borrowing alone; a good credit reputation is then worthless. And when collateral is too small, any equilibrium with reputational borrowing must involve some states of production inefficiency. But when z is smaller than threshold $\underline{z}$, unsecured loans can support production–efficient allocations even in the complete absence of collateral.

7 Numerical examples

For a fuller description of stochastic cycles, especially ones that exhibit some misallocation of capital, we use value function iteration to isolate stationary Markov equilibria with reputational borrowing. Specifically, for arbitrary initial default penalties for agent 1 $v_0^1(.,1) > 0$ and $v_0^1(.,2) > 0$,[6] we calculate constraints and interest rates for all x using the equilibrium conditions (4)–(7) and the wealth iteration maps (8). The results are then substituted in the right–hand side of (10) to calculate new default penalties $v_1^1(.,1)$ and $v_1^1(.,2)$, and so on. Our previous results on equilibrium multiplicity imply that this map cannot be a global contraction, so one cannot expect a definitive proof that equilibrium exists. We find, however, that these iterations converge fast, and we are able to identify the theoretical equilibria in the special cases analyzed in previous sections. We conjecture that the iteration procedure generally converges to the determinate equilibrium whenever there is equilibrium multiplicity. In the deterministic economy $\pi = 0$, for example, we know that the collateral–borrowing equilibrium is determinate whenever no other equilibrium exists, and indeterminate otherwise (Proposition 5). Numerically we find indeed that value function iteration converges to the determinate equilibrium.

For a benchmark parameterization we study an economy which aims to match some features of the U.S. business cycle between 1960 and 2006 under the extreme assumption that the business cycle is driven *only* by the sectoral shocks analyzed in this paper. It is not the purpose of this exercise to conduct a full–blown quantitative analysis; such would require a much more detailed model incorporating several of the features discussed in the next section. We rather wish to illustrate how aggregate growth and volatility depend on various model parameters and how they qualitatively correlate with the sectoral dispersions of equity returns and total factor productivities. Since our stylized model does not distinguish between output and undepreciated capital, we let $g_t = Y_t/Y_{t-1} - 1$ denote the annual growth rate of current resources Y_t which describes the sum of real GDP and capital minus depreciation. We pin down the five parameters A, z, π, β and λ to match mean and standard deviation of g of about 3.3% and 0.8%, an annual autocorrelation of g of about 0.75, a mean real interest rate of 4.5% and an average share of non–financial business credit in GDP of about 55%.[7] These targets are matched at $A = 1.077$,

[6] Because of symmetry, default penalties for agent 2 are simply $v^2(x,1) = v^1(x,2)$ and $v^2(x,2) = v^1(x,1)$.

[7] The time series of g is obtained from the standard calculation of the capital stock from the

$z = 0.965$, $\pi = 0.88$, $\beta = 0.97$ and $\lambda = 0.23$. It turns out that these parameters lead to a situation of a unique equilibrium with collateral borrowing. Only when some of these parameters are changed, does reputational borrowing set in, as will be seen below. In the first–best benchmark, the economy would growth steadily at rate $\beta A - 1 \approx 4.5\%$. Hence, the economy's misallocation of capital costs about 1.2% growth per year.

Figure 3 shows how the mean and standard deviation of g change when z varies between 0.7 and 1.0. Clearly, when z is close to one, the economy is almost first best. Growth is constant at βA. But growth is also flat at βA when z is smaller than 0.76. In these situations, allocations are production efficient and supported by reputational borrowing. Indeed, we find that reputational borrowing sets in at about $z \leq 0.94$, whereas collateral borrowing is the only equilibrium at the benchmark $z = 0.965$.

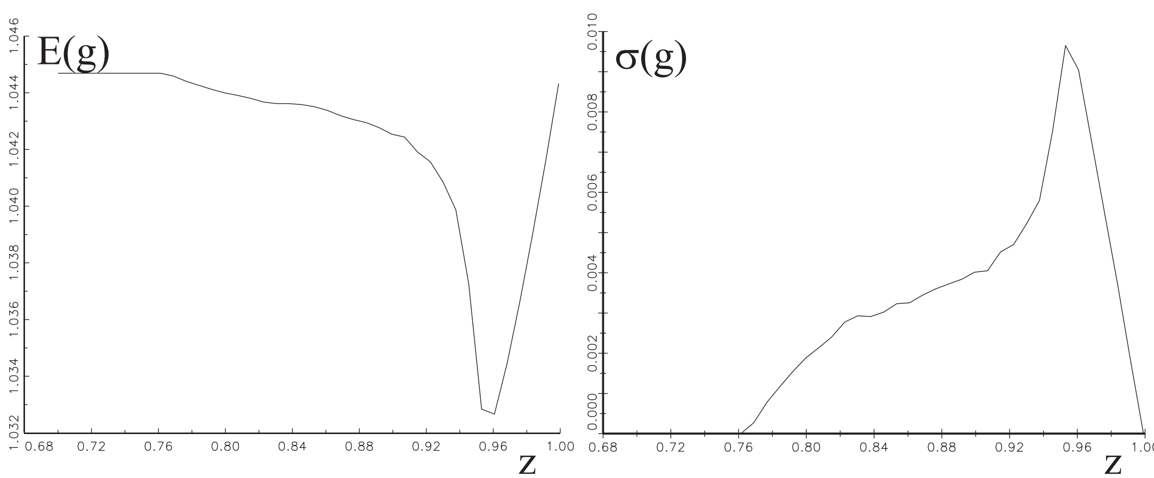

Figure 3: Mean and standard deviation of growth as z varies.

Figure 4 shows the result of the parameter variation as λ goes from zero to 1/2.

national income accounts. The interest rate is the real yield on corporate bonds, and "business credit" is all credit–market debt owed by non–financial firms, which averages around 55% dring the last decades.

From Proposition 3 follows that the economy is production efficient (so aggregate growth is constant at βA) when $\lambda \geq z/(1+z) \approx 0.491$. As the collateral share λ falls below that value, the growth rate decreases and becomes more volatile. When differences in financial development are explained by creditor protection, described as variations in the collateral share λ, our model implies that financial development correlates positively with growth and negatively with output volatility. Other differences in contract enforcement, such as those discussed in Section 7.4, are further determinants of financial development and give rise to similar implications.

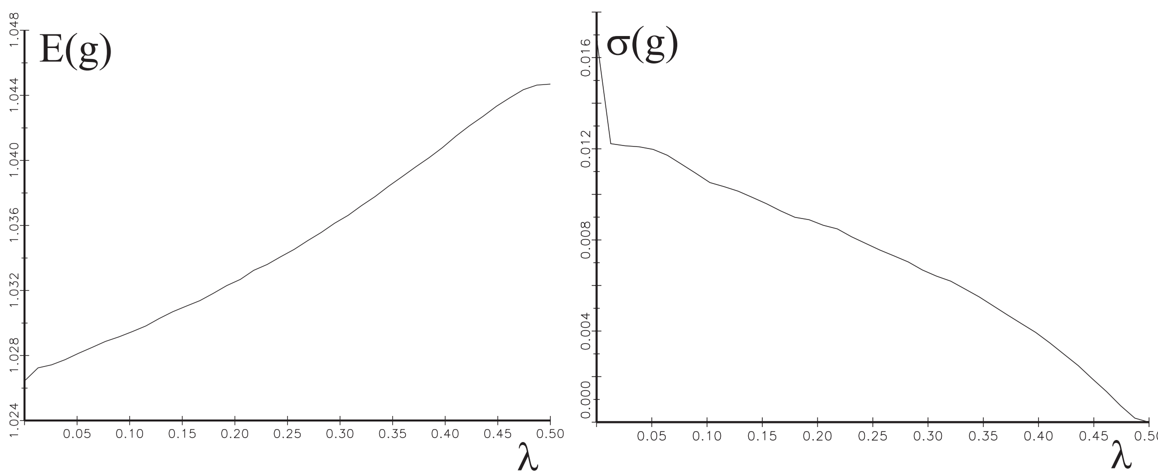

Figure 4: Mean and standard deviation of growth as λ varies.

At the benchmark calibration we calculate equity return dispersion as the spread between sectoral equity returns, measured by the weighted standard deviation between $\tilde{R}_t^1$ and $\tilde{R}_t^2$ where weights are the corresponding wealth shares. We find that this measure averages around 1.9% and has a standard deviation of 0.8%, about an order of magnitude smaller than the stock–market dispersion indices reported in Loungani, Rush, and Tave (1990) (Fig. 1). The correlation coefficient between equity–return dispersion and g_t is -0.48, which is in line with the evidence listed in the introduction. Growth is low when capital is misallocated in which case the dispersion between sectoral equity returns is large. Figure 5 shows time series for the growth rate of aggregate resources and for the dispersion of equity returns for a simulation of 100 periods at the benchmark parameter values.

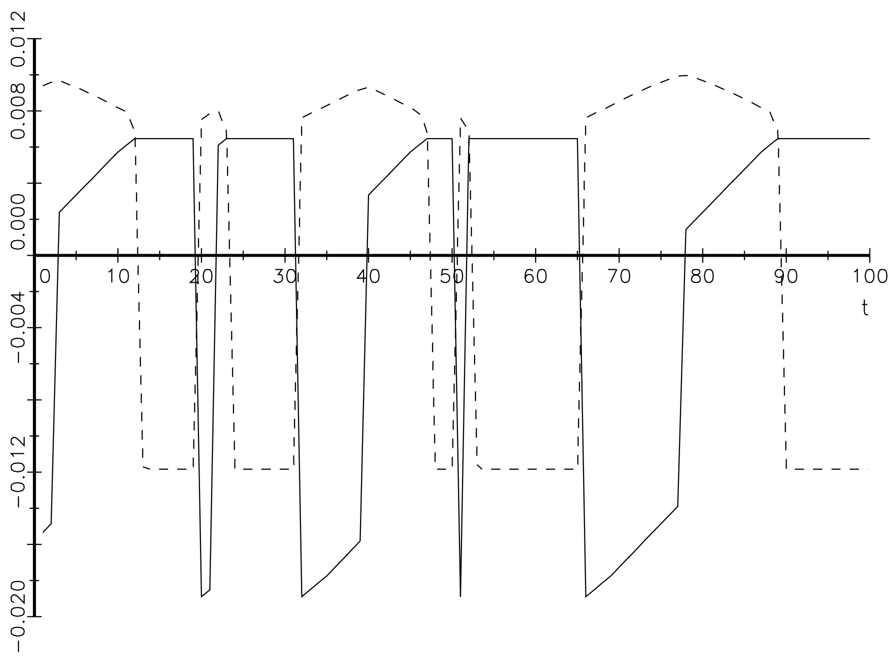

Figure 5: Simulation at benchmark parameter values: Growth rate (solid) and dispersion of equity returns (dashed), both as deviations from their mean.

In accordance with results of Eisfeldt and Rampini (2006) (Table 3), our model further produces countercyclicality of TFP dispersion across sectors. Using the standard deviation of sectoral factor productivities A and zA, weighted by their relative output shares, as a measure of TFP dispersion, we find a mean dispersion of 2.8%, a standard deviation of 1.9% and a contemporaneous correlation with growth of -0.99.

8 Extensions

8.1 Irreversible investment

Unsurprisingly, each of the two sectors is considerably more volatile than the aggregate economy. Because every stochastic cycle must enter some occasional periods of production efficiency, output in any sector falls occasionally to zero. This circumstance is not only unrealistic, it also prevents a meaningful calculation of sector growth rates. Further, it appears to be a strong abstraction to assume that all gross resources can, in the extreme, move between sectors from one period to the next. To deal with these limitations, it seems a sensible extension to augment the model by some sector specificity (investment irreversibility).[8] For simplicity, suppose that a constant fraction of resources is sector specific and cannot be employed in the other sector, and let $\Psi < 1$ be the share of resources which is usable in both sectors. Then the only change to the model is that the expression for credit supply $\mathrm{CS}(\sigma)$ is multiplied by the factor Ψ, and all other equations remain unchanged. It is important to emphasize, however, that now even the first–best economy involves fluctuations of aggregate output, as resources move sluggishly between sectors when the productive state changes.

In the numerical example, suppose that 90 percent of capital is sector specific ($\Psi = 0.1$). We find that for the benchmark parameter values, mean growth falls only slightly to 2.9% whilst its standard deviation is practically unchanged. The growth rate of either sector, however, has a standard deviation of about 9%, an order of magnitude larger than the standard deviation of aggregate growth. For larger values of Ψ, each sector's output would become even more volatile. In line with the evidence discussed in the introduction, the dispersion between sector growth rates is countercyclical; its correlation coefficient with aggregate growth is about -0.14 .

[8]There are two alternative variations achieving similar results. One is essentiality; if the two goods produced are not perfect substitutes and essential for consumers, sectoral shifts induce relative price changes which prevent that all resources are shifted to one sector. The other is complementarity between capital and another factor (labor) which is immobile between sectors.

8.2 Sectoral comovement

Another feature of the cycles characterized in the previous sections is that growth rates in the two sectors are negatively correlated. As an implication, either one sector is pro–cyclical and the other is counter–cyclical or both sectors are acyclical. The evidence however strongly supports comovement of virtually all two–digit industries with aggregate output (see e.g. Christiano and Fitzgerald (1998)). However, the absence of comovement is an artefact of the two–sector, two–state example, where business cycles are *exclusively* driven by sectoral productivity shifts between two constant productivities A and zA. When technologies are, in some way, correlated with the sectoral productivity shifts, comovement can easily be generated. To see this in an extension of the model which retains perfect symmetry between sectors and which still has a constant production possibility frontier, suppose that the technology parameter z attains one of two values z^c and z^n, depending on whether the sectoral state changes or not. That is, "change of states" has $z_t = z^c$ when $s_t \neq s_{t-1}$, and "no change of states" has $z_t = z^n$ when $s_t = s_{t-1}$. This extension thus has two sectors and four productivity states $(s_t, s_{t-1}) \in \{1, 2\}^2$. In the numerical example with the same benchmark parameter values, we find that there is comovement (i.e. positive correlations between aggregate growth and growth rates of each sector) if $z^c = 0.99 > z^n = 0.965$ (and $\Psi = 0.9$). Alternative mechanisms are possible which can generate co–movement for aggregate shocks to A or to the collateral share λ which are correlated with sectoral shifts.

8.3 Asymmetric sectors

To be discussed.

8.4 Alternative enforcement

Reputational loans in our model are supported by perpetual exclusion of defaulting agents from all borrowing in excess of collateral. Alternative punishment mechanisms are conceivable and have been explored in the literature, mostly in economic environments without production. A much more powerful enforcement of credit arrangements is obtained when defaulters can be shut out of all intertemporal trade (borrowing and lending) as is the case in the models of Kehoe and Levine (1993) and Alvarez and Jermann (2000). In their pure–exchange models with zero collateral it is well known that first–best allocations can be implemented with reputational loans provided that all agents share a *common* high discount factor and have sufficiently large income variability (see also Azariadis and Kaas (2007)). Similar results can also be obtained for our model; the only change in the model's equations is that the defaulters' equity returns $\tilde{R}^i_c(\mathbf{x}, s)$ in equations (10) must be replaced by the autarkic returns A^i_s. In the two–agent, two–state special case without collateral, it is straightforward to show that there is a first–best equilibrium at the symmetric wealth distribution iff

$$\ln\frac{1}{z} \geq \frac{(1-\beta)(1+\beta-2\beta\pi)}{\beta(1-\pi)}\ln 2 \ .$$

Thus the first best is an equilibrium if the productivity differential is sufficiently large or if agents are sufficiently patient.

On the other hand, one can also think of *weaker* punishment scenarios in our model. For example, defaulting agents may be shut out of unsecured credit for a finite number of periods before they regain unlimited access to reputational loans. Alternatively, defaulters may sometimes evade punishment and have a positive chance not to be shut out of reputational loans. Both mechanisms are rather straightforward extensions of our model. Shortened punishment periods and lower detection probabilities tighten debt limits considerably and thereby contribute to lower growth and higher aggregate TFP volatility.

9 Conclusions

This paper outlines a financial theory of aggregate factor productivity which connects the sectoral capital allocation with credit market frictions. We emphasize frictions arising from insufficient collateral for secured loans and limited enforcement of unsecured loans. Both of these lead to endogenous debt limits which slow down the reallocation of surplus capital from less productive to more productive sectors, and prevent the equalization of sectoral productivities and sectoral rates of return.

Our model is consistent with much empirical evidence suggesting that economy-wide factor productivity and economic growth are both negatively correlated with the dispersion of sectoral stock returns, the dispersion of sectoral TFP's, and the dispersion of sectoral growth rates. If we reinterpret "sectors" to be individual firms, then our results are also consistent with recent work by Hsieh and Klenow (2007) who find that industry productivity dispersion is negatively correlated with industry productivity in a panel that includes data from the U.S., China and India.

Finally, if we index countries by the fraction of collateral assets to total resources, our results are in line with Diebold and Yilmaz (2008) who find that macroeconomic volatility is positively correlated with stock market volatility in a cross section of countries.

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Appendix

Proof of Proposition 1:

It remains to show existence and uniqueness of a market–clearing interest rate $R = R(\sigma)$ with collateral borrowing for any $\sigma = (\mathbf{x}, s)$. For any $R > \lambda A \geq \lambda A_s^i$, collateral debt limits are $\theta_c^i(R) = \dfrac{\lambda A_s^i}{R - \lambda A_s^i}$. Debt limits are decreasing in the interest rate, and so is the demand for credit $\mathrm{CD}(\sigma)$; it is a downward–sloping function with finitely many upward jumps at $R = A_s^i$, it is zero at $R \geq A$ and it tends to infinity as $R \to \lambda A$. On the other hand, the supply of credit $\mathrm{CS}(\sigma)$ is a weakly increasing step function which is zero at $R = 0$ and finite at $R \geq A$. Because of these features, there exists a unique market–clearing interest rate for any σ. $\square$

Proof of Proposition 2:

A candidate first–best equilibrium has stable wealth shares $\mathbf{x}^* = (x^{i*})_{i \in \mathcal{I}}$ and an interest factor equal to the frontier productivity, $R(\mathbf{x}^*, s) = A$ for all $s \in \mathcal{S}$. With collateral borrowing ($v^i = 0$), the debt limits then follow from (4) as $\theta^i(\mathbf{x}^*, s) = \lambda/(1-\lambda)$ for all $i \in \mathcal{I}$ and $s \in \mathcal{S}$. In every productive state s, there is by assumption (A2) a unique agent $i(s)$ using the frontier technology. Because of (A3), no state is trivial, hence credit market equilibrium requires that for any $s \in \mathcal{S}$, agent $i(s)$'s maximum demand for credit does not exhaust credit supply of all other agents:

$$\frac{\lambda}{1-\lambda} x^{i(s)*} \geq \sum_{j \neq i(s)} x^{j*} = 1 - x^{i(s)*} \ , \ s \in \mathcal{S} \ ,$$

which is $\lambda \geq 1 - x^{i(s)*}$ for all s. By assumption (A1), every agent has access to the frontier technology in at least one state. Thus the first best is an equilibrium with collateral borrowing iff $\lambda \geq 1 - x^{i*}$ for all $i \in \mathcal{I}$. For this to be true *at some distribution of wealth* x^{i*}, it must hold in particular at the symmetric distribution of wealth, $x^{i*} = 1/I$. Therefore, the condition $\lambda \geq (I-1)/I$ is necessary and sufficient for the first best to be an equilibrium with collateral borrowing for some distribution of initial wealth.

Now suppose that $\lambda < (I-1)/I$ and suppose that there is a first–best equilibrium with reputational borrowing at stable wealth distribution $\mathbf{x}^* = (x^{i*})_{i \in \mathcal{I}}$ and interest yields $R(\mathbf{x}^*, s) = A$, $s \in \mathcal{S}$. But then from (6) and (7), $\tilde{R}^i(\mathbf{x}^*, s) = \tilde{R}^i_c(\mathbf{x}, s) = A$ for all $i \in \mathcal{I}$ and $s \in \mathcal{S}$, and from (10) follows that $v^i(\mathbf{x}^*, s) = 0$ for all (i, s). But this in connection with (4) implies again collateral borrowing, so debt limits are $\theta^i(\mathbf{x}^*, s) = \lambda/(1-\lambda)$, and the credit market cannot be in equilibrium, as seen above. □

Proof of Proposition 3:

Parts (a)–(b) follow simply from inspection of Figure 1 (a) and (b). To prove (c), it is useful to note the following features of the maps $X_0(x)$ and $X_1(x)$. For any $x \leq 1 - \lambda/z$, it holds that $X_1 X_1(x) = x$ and that $X_0 X_1(x) = 1 - x$.

Again it is clear from the graph that the minimum and maximum elements of the asymptotic invariant set are $\underline{x} = \lambda$ and $\overline{x} = 1 - \lambda$. Let $\ell \geq 1$ be the unique number such that $X_0^{\ell-1}(\lambda) < 1 - \lambda/z$ and $X_0^{\ell}(\lambda) \geq 1 - \lambda/z$ and suppose the last inequality is strict (a generic feature). Obviously then, $X_0^k(\lambda)$, $k = 1, \ldots, \ell$ are also elements of the asymptotic invariant set. Further elements are the ℓ wealth states $X_1 X_0^k(\lambda)$ for $k = 0, \ldots, \ell - 1$, which are all in the interval $(\lambda, 1 - \lambda/z]$ and which are generically different from the other elements. Note that $X_1 X_0^{\ell}(\lambda) = \lambda$ is not a new element of the asymptotic invariant set. To see that there are no further elements, note that any further iteration from $X_1 X_0^k(\lambda)$ can only lead either to $X_1 X_1 X_0^k(\lambda) = X_0^k(\lambda)$ or to $X_0 X_1 X_0^k(\lambda) = 1 - X_0^k(\lambda) = X_1 X_0^{k-1}(\lambda)$ which are both already elements of the asymptotic invariant set. Hence the asymptotic invariant set comprises λ, $1-\lambda$, $X_0^k(\lambda)$ for $k = 1, \ldots, \ell$ and $X_1 X_0^k(\lambda)$ for $k = 0, \ldots, \ell - 1$, which are $2\ell + 2 = 2m$ elements with $m = \ell + 1 \geq 2$. Of these, exactly the three $[1-\lambda,\ X_0^\ell(\lambda)$ and $X_1(\lambda) = 1 - \lambda/z]$ are not smaller than the threshold $1 - \lambda/z$ and have growth rates at $\beta A - 1$. All other states have growth rates below $\beta A - 1$. □

Proof of Proposition 4:

Without loss of generality, set $A = 1$ to simplify notation. Consider first a production–efficient steady state x with debt–equity constraint $\theta = (1-x)/x$ and interest rate $R \in (z, 1)$. x is a steady state if $x = X_1(x) = R(1-x)$, and hence $x = R/(1+R)$, $\theta = 1/R$, $\tilde{R} = 1 + \theta(1-R) = 1/R$, and $\tilde{R}_c = R(1-\lambda)/(R-\lambda)$. Let v and w be the default penalties for borrowing and lending agents (of both types) in steady state. From (10) follows that v and w satisfy

$$v = \beta w \; , \tag{11}$$

$$w = \beta \Big\{ \ln \Big[\frac{\tilde{R}}{\tilde{R}_c} \Big] + v \Big\} \; . \tag{12}$$

Hence,

$$v = \frac{\beta^2}{1-\beta^2} \ln \Big(\frac{\tilde{R}}{\tilde{R}^c} \Big) = \frac{\beta^2}{1-\beta^2} \ln \Big(\frac{R-\lambda}{R^2(1-\lambda)} \Big) \; .$$

On the other hand, (4) implies that

$$v = \ln \Big[(1-\lambda)(1+R) \Big] \; . \tag{13}$$

From these two equations follows that the steady–state interest rate must solve

$$(1-\lambda)^{1/\beta^2} R^2 (1+R)^{(1-\beta^2)/\beta^2} = R - \lambda \; . \tag{14}$$

Moreover, reputational borrowing requires that $v > 0$ which, from (13), implies that $R > R^c \equiv \lambda/(1-\lambda)$, where R^c is the interest rate with collateral borrowing which is always a solution of equation (14). Another solution $R^* > R^c$ exists provided that the slope of the LHS at R^c is smaller than one. This turns out to be the case if and only if $\lambda < \beta/(1+\beta)$. Now R^* indeed constitutes an equilibrium provided that $R^* > z$ and $R^* < 1$. The latter inequality follows if LHS>RHS at $R = 1$. But this turns out to be equivalent to $\lambda < 1/2$ which follows from $\lambda < \beta/(1+\beta)$. The first inequality is true either if $R^c \geq z$ (which is the same as $\lambda \geq z/(1+z)$) or if LHS<RHS at $R = z$. This last inequality is expressed as

$$\frac{(z-\lambda)^{\beta^2}}{1-\lambda} > z^{2\beta^2}(1+z)^{1-\beta^2} \ . \tag{15}$$

This inequality becomes an equality at $\lambda = z/(1+z)$, and the left–hand side is decreasing in $\lambda > \lambda_0 \equiv (z-\beta^2)/(1-\beta^2)$ and increasing in $\lambda < \lambda_0$. When $z \geq \beta$, $\lambda_0 \geq z/(1+z)$ holds, and hence (15) is violated for any $\lambda \leq z/(1+z)$; hence there is no reputational borrowing steady state in this case. When $z < \beta$, there must be a threshold $\tilde{\lambda} < \lambda_0$ where (15) holds with equality. Hence, with $\hat{\lambda} = \max(0, \tilde{\lambda})$, there exists a steady state with reputational borrowing for any $\lambda \in [\hat{\lambda}, \beta/(1+\beta))$.

Next consider a production–inefficient steady state where $R = z$, $\tilde{R} = 1 + \theta(1-z)$ and $\tilde{R}_c = z(1-\lambda)/(z-\lambda)$, and again let v be the stationary penalty of default for a borrowing agent. Now (11) and (12) can be expressed as

$$v = \frac{\beta^2}{1-\beta^2} \ln\left(\frac{[1+\theta(1-z)](z-\lambda)}{z(1-\lambda)}\right) ,$$

and (4) becomes

$$v = \ln\left[\frac{(1-\lambda)(1+\theta)}{1+\theta(1-z)}\right] \ . \tag{16}$$

Equating the two yields an equation in the debt–equity ratio,

$$1+\theta = \Big(1-\lambda/z\Big)^{\frac{\beta^2}{1-\beta^2}} \left(\frac{1+\theta(1-z)}{1-\lambda}\right)^{\frac{1}{1-\beta^2}} \ . \tag{17}$$

Here, the debt–equity ratio with collateral borrowing $\theta^c = \lambda/(z-\lambda)$ solves this equation. Another solution θ^* corresponds to a reputational–borrowing equilibrium only if $\theta^* > \theta^c$, and such a solution exists iff the slope of the RHS at θ^c is smaller than one. This is the case iff

$$\lambda < \lambda_0 = \frac{z-\beta^2}{1-\beta^2} \ .$$

The solution $\theta^* > \theta^c$ indeed gives rise to a production–inefficient equilibrium at $R = z$ if $\theta^* < (1-x^*)/x^*$ at the stationary borrower share x^* which satisfies

$$x = X_1(x) = \frac{z(1-x)}{[1+\theta^*(1-z)] + z(1-x)} ,$$

or

$$(1-z)(1+\theta^*)x^2 + 2zx - z = 0 \ .$$

Clearly this quadratic has a unique solution $x^* \in (0,1)$ for any $\theta^* > 0$. Now $\theta^* < (1-x^*)/x^*$ when $x^* < 1/(1+\theta^*)$ which is the case if the quadratic is positive at $x = 1/(1+\theta)$, which in turn is equivalent to $\theta^* < 1/z$. This condition is fulfilled whenever in (17) the LHS is smaller than the RHS at $\theta = 1/z$. But this inequality is equivalent to (15) again. Because the LHS in (15) is increasing in $\lambda \in [0, \lambda_0]$, the inequality is satisfied for all $\lambda \in (\hat{\lambda}, \lambda_0)$. Hence

$$\hat{\lambda} < \lambda < \frac{z-\beta^2}{1-\beta^2}$$

is a necessary and sufficient condition for a production inefficient steady state equilibrium with reputational borrowing. □

Proof of Proposition 5:

In the deterministic economy, the dynamic versions of the steady–state equations (11) and (12) can be simplified to

$$v_t = \beta^2 \ln\left(\frac{\tilde{R}_{t+2}}{\tilde{R}^c_{t+2}}\right) + \beta^2 v_{t+2} \ . \tag{18}$$

Suppose first that the economy is production inefficient in all periods. Then, $R_t = z$, $\tilde{R}^c_t = z(1-\lambda)/(z-\lambda)$, and from (4) follows

$$\tilde{R}_t = \frac{(1-\lambda)z}{1-\lambda-e^{v_t}(1-z)} \ .$$

Substitution into (18) yields

$$v_t = \beta^2 \ln\left(\frac{z-\lambda}{1-\lambda-e^{v_{t+2}}(1-z)}\right) + \beta^2 v_{t+2} \ .$$

Note that v_t is a forward–looking (jump) variable. Hence, the steady state $v = 0$ is locally indeterminate iff $dv_t/(dv_{t+2})|_{v=0} > 1$. But this condition turns out to be the same as

$$\lambda > \frac{z-\beta^2}{1-\beta^2} \ ,$$

which follows from $z < \beta^2$. Hence, there is an infinity of equilibria with $v_t \to 0$ (and thus $\theta_t \to \theta^c$). In the limit, the dynamics of borrower wealth becomes

$$x_{t+1} = \frac{(1-x_t)(z-\lambda)}{z-\lambda+x_t(1-z)} = X_1(x_t) \ ,$$

which satisfies $x_{t+2} = X_1^2(x_t) = x_t$. Hence, in all these equilibria, wealth converges to a cycle of periodicity two (where one of these "cycles" is the collateral–borrowing steady state).

Next consider a production–efficient economy. Here $\theta_t = (1-x_t)/x_t$, and from (4) follows

$$R_t = \frac{1-e^{-v_t}(1-\lambda)}{1-x_t} \ , \ \tilde{R}_t = \frac{1-\lambda}{e^{v_t}x_t} \ , \ \tilde{R}_t^c = \frac{[1-e^{-v_t}(1-\lambda)](1-\lambda)}{1-e^{-v_t}(1-\lambda)-\lambda(1-x_t)} \ .$$

Substitution into (18) yields

$$v_t = \beta^2 \ln\left(\frac{1-e^{-v_{t+2}}(1-\lambda)-\lambda(1-x_{t+2})}{[e^{v_{t+2}}-1+\lambda]x_{t+2}}\right) + \beta^2 v_{t+2} \ . \tag{19}$$

The dynamics of borrower wealth is

$$x_{t+1} = \frac{R_t(1-x_t)}{R_t(1-x_t)+\tilde{R}_t x_t} = 1-e^{-v_t}(1-\lambda) \ .$$

Substitution into (19) gives

$$v_t = \beta^2 \ln\left(\frac{1-e^{-v_{t+2}}(1-\lambda)-\lambda(1-\lambda)e^{-v_{t+1}}}{[e^{v_{t+2}}-1+\lambda][1-e^{-v_{t+1}}(1-\lambda)]}\right) + \beta^2 v_{t+2} \ .$$

Using $\varphi_t = e^{v_t}$, this equation is more conveniently expressed as

$$\varphi_t = \left[\frac{\varphi_{t+1}(\varphi_{t+2}-1+\lambda)-\lambda(1-\lambda)\varphi_{t+2}}{(\varphi_{t+2}-1+\lambda)(\varphi_{t+1}-1+\lambda)}\right]^{\beta^2} \ . \tag{20}$$

A steady state is a solution of

$$\varphi^{(1-\beta^2)/\beta^2} = \frac{\varphi-1+\lambda^2}{(\varphi-1+\lambda)^2} \ . \tag{21}$$

One solution is $\varphi = 1$ which (under appropriate conditions) gives rise to a steady state with collateral borrowing. A steady state with reputational borrowing must be a solution with $\varphi > 1$. Again φ_t is a forward–looking jump variable; hence a steady state is locally determinate if both eigenvalues of the backward dynamics (20) have modulus less than one, and a steady state is locally indeterminate if at least one eigenvalue has modulus larger than one. It is straightforward to calculate the determinant and trace of the Jacobian at the steady state:

$$
\begin{aligned}
D = -\frac{d\varphi_t}{d\varphi_{t+2}} &= \frac{\beta^2\lambda(1-\lambda)^2}{(\varphi-1+\lambda)(\varphi-1+\lambda^2)} , \\
T = \frac{d\varphi_t}{d\varphi_{t+1}} &= -\frac{\beta^2(\varphi-1)(1-\lambda)^2}{(\varphi-1+\lambda)(\varphi-1+\lambda^2)} .
\end{aligned}
$$

At a steady state with collateral borrowing ($\varphi = 1$), $D > 1$ if and only if $\lambda < \beta/(1+\beta)$. Hence, this steady state is indeterminate whenever there is another one with reputational borrowing. Such a steady state is determinate, provided that $D < 1$ and $D + T > -1$. At $\lambda = 0$, $D = 0$ and $T > -1$ requires that $\varphi > 1 + \beta^2$. Since $\varphi = (\varphi - 1)^{-\beta^2/(1-\beta^2)}$, this inequality is true provided that

$$
1 + \beta^2 < (\beta^2)^{-\beta^2/(1-\beta^2)} ,
$$

which is true for all $\beta^2 \in (0, 1)$. When λ increases, it can be shown numerically that both $D(\lambda)$ and $D(\lambda) + T(\lambda)$ increase (where $\varphi(\lambda) > 1$ adjusts to solve (21)), regardless of the value of β. Moreover $D(\lambda)$ converges to 1 and $T(\lambda)$ converges to zero when $\lambda \to \beta/(1 + \beta)$ (and $\varphi(\lambda) \to 1$). Therefore $D(\lambda) < 1$ and $D(\lambda) + T(\lambda) > -1$ are satisfied for any $\lambda \in [0, \beta/(1+\beta))$, and hence the steady state with reputational borrowing is locally determinate. □

Proof of Proposition 6:

A production–efficient cycle of order two has a support at $x_1 < x_2$ where x_1 applies if the productive state changes ($s_t \neq s_{t-1}$), and x_2 applies if the state stays the same. The corresponding default penalties for borrowers and lenders in these two situations are denoted v_j, w_j, $j = 1, 2$. The cycle has the following features:

- the allocation is first best at $x = x_2$, i.e. $R_2 = A$ and

$$
\theta_2 = \frac{e^{v_2} + \lambda - 1}{1 - \lambda} \geq \frac{1 - x_2}{x_2} .
$$

- the allocation is production–efficient but not first–best at $x = x_1$, so that $\theta_1 = (1 - x_1)/x_1$ and

$$R_1 = A\frac{e^{v_1} + \lambda - 1}{e^{v_1}(1 - x_1)} \in (zA, A) \ .$$

Again simplify notation by setting $A = 1$. Because the economy is first–best at x_2, $X_0(x_2) = x_2$ and $X_1(x_2) = 1 - x_2$. Hence $x_1 = 1 - x_2$. At $x = x_1$, rates of return are

$$R_1 = \frac{e^{v_1} + \lambda - 1}{e^{v_1}(1 - x_1)} < \tilde{R}_1 = \frac{1 - \lambda}{x_1 e^{v_1}} \ .$$

If the productive state is unaltered, borrower wealth must thus increase from x_1 to x_2, which implies

$$X_0(x_1) = \frac{\tilde{R}_1 x_1}{\tilde{R}_1 x_1 + R_1(1 - x_1)} = e^{-v_1}(1 - \lambda) = x_2 = 1 - x_1 \ ,$$

and therefore $x_1 = 1 - e^{-v_1}(1 - \lambda)$.

The recursive equations in default penalties are

$$\begin{aligned}
v_1 &= \beta\pi v_2 + \beta(1 - \pi)w_1 \ , \\
w_1 &= \beta\pi w_2 + \beta(1 - \pi)\Big[\ln(\tilde{R}_1/\tilde{R}_1^c) + v_1\Big] \ , \\
v_2 &= \beta\pi v_2 + \beta(1 - \pi)w_1 \ , \\
w_2 &= \beta\pi w_2 + \beta(1 - \pi)\Big[\ln(\tilde{R}_1/\tilde{R}_1^c) + v_1\Big] \ .
\end{aligned}$$

From these follows $v_1 = v_2 = v$, $w_1 = w_2 = w$, and

$$\begin{aligned}
v &= C\ln\left(\frac{\tilde{R}_1}{\tilde{R}_1^c}\right) \\
&= C\ln\left(\frac{e^v + \lambda - 1 - \lambda e^v(1 - x_1)}{(e^v + \lambda - 1)x_1 e^v}\right) \\
&= C\ln\left(\frac{e^v - 1 + \lambda^2}{(e^v + \lambda - 1)^2}\right) \ , \qquad (22)
\end{aligned}$$

with

$$C \equiv \frac{\beta^2(1 - \pi)^2}{(1 - \beta)(1 + \beta - 2\pi\beta)} \ .$$

Equation (22) has a solution $v > 0$ (necessary for reputational borrowing) provided that $\lambda^2 + 2\lambda C - C < 0$, which is equivalent to

$$\lambda < \frac{\beta(1 - \pi)}{1 + \beta - 2\beta\pi} \equiv \lambda_1 \ . \qquad (23)$$

To make sure that this is indeed a production–efficient equilibrium, it must be verified that $R_1 \in [z, 1)$ and that $\theta_2 \geq (1 - x_2)/x_2$. It is easy to show that the last condition holds with equality. $R_1 < 1$ is fulfilled provided that $e^v < 2(1 - \lambda)$. But at $e^v = 2(1 - \lambda)$ the RHS of (22) is zero; thus the equilibrium e^v must be smaller than $2(1 - \lambda)$. Hence it remains to check that $R_1 \geq z$. This is true iff $e^v \geq (1-\lambda)(1+z)$ which is valid either if $\lambda \geq z/(1+z)$ or if RHS≥LHS in equation (22) at $e^v = (1 - \lambda)(1 + z)$. Hence, $R_1 \geq z$ iff

$$\lambda \geq \frac{z}{1+z} \quad \text{or} \quad \Phi(\lambda) \equiv \frac{(z-\lambda)^C}{(1-\lambda)^{1+C}} \geq z^{2C}(1+z) \ . \tag{24}$$

The last condition holds with equality at $\lambda = z/(1 + z)$ and Φ has a maximum at $\tilde{\lambda} = z(1+C)-C$. It is straightforward to verify that $\underline{\lambda} \geq \lambda_1$ iff $z/(1+z) \geq \lambda_1$ iff $z \geq \beta(1-\pi)/(1-\beta\pi)$. Hence, if z exceeds this threshold, there is no λ satisfying both (23) and (24), and hence there cannot be a production–efficient cycle with reputational borrowing, which proves part (a). Conversely, when $z < \beta(1-\pi)/(1-\beta\pi)$, there is a unique $\hat{\lambda} < \underline{\lambda}$ such that the second condition in (24) holds with equality. In that case, any $\lambda \in [\hat{\lambda}, \lambda_1)$ gives rise to a production–efficient cycle with reputational borrowing, which proves part (b). Finally, the lower bound $\hat{\lambda}$ is non–positive provided that the second condition in (24) holds at $\lambda = 0$ which is the same as

$$1 \geq z^C(1+z) \ .$$

This condition is the same as $z \leq \underline{z}$ for another threshold $\underline{z}(\beta)$ which converges to one when $\beta \to 1$ $(C \to \infty)$. □