A CHARACTERIZATION OF VECTOR AUTOREGRESSIVE PROCESSES WITH COMMON CYCLICAL FEATURES

MASSIMO FRANCHI AND PAOLO PARUOLO

Abstract. This paper presents necessary and sufficient conditions for the existence of common cyclical features in Vector Autoregressive (VAR) process integrated of order 0, 1, 2, where the common cyclical features correspond to codependence (CD), serial correlation common features (CS), or commonality in the final equations (CE). The results are based on polynomial rank factorizations of the reversed AR polynomial around the poles of its inverse. All processes with CS structures are found to present also CE structures and vice versa. The presence of CD structures, instead, implies the presence of both CS and CE structures, but not vice versa. Characterizations of the CS, CE, CD linear combinations are given in terms of linear subspaces defined in the polynomial rank factorizations.

1. Introduction

Several macroeconomic theories predict the presence of common dynamic components in economic time series. For example, the life-cycle hypothesis and permanent income hypothesis relate current consumption to (the present value of) life-income or real wealth, hence implying common trends and cycles between these variables, see Hall (1978) and Campbell and Mankiw (1990). Similarly co-movements among national consumption aggregates are predicted by international risk-sharing, see Cavaliere, Fanelli, and Gardini (2008) and reference therein. Other economic theories with similar implications include: international equalization of interest rates, see Kugler and Neusser (1993), present value models, see Campbell and Shiller (1987), and balanced growth models, see King, Plosser, Stock, and Watson (1991).

In several of these models, commonality in dynamic behavior is implied by the first order conditions of optimizing agents. Let X_t denote a vector of observable time series, and let $X_{t-k}^{t-1} := (X_{t-k}, \ldots, X_{t-1})$; optimization usually implies that some function $y_t := g(X_t, X_{t-k}^{t-1})$ has zero expectation conditional on information available at time $t-1$, which includes X_{t-k}^{t-1}; this implies that y_t is unpredictable and hence it does not contain any cyclical component. A leading special case is when g is a linear function, which corresponds to the notion of common features introduced by Vahid and Engle (1993) and Engle and Kozicki (1993).

Date: First version: September 1, 2008. This version November 1, 2009.

JEL classification: C32; C51; C52.

Key words and phrases. Multiple time series; Common cycles; Cointegration; I(1); I(2) .

We would like to thank Franz Palm, two anonymous referees and the following people for helpful comments and discussions on previous versions of the paper: Stéphane Gregoir, Niels Haldrup, Alain Hecq, Marco Lippi, Roman Liška, Franz Palm, Hashem Pesaran, Søren Johansen, Anders Rahbek, Martin Wagner and participants of the conference 'Factor structures for panel and multivariate time series data' held in Maastricht, 18 – 20 September 2008. This paper was written while the first author was Carlo Giannini Fellow 2007-2009. Partial financial support is also acknowledged from the Italian Ministry of University grant Cofin2006-13-1140 and University of Insubria FAR 2007-2008.

Special cases of common features are common trends and common cycles. Common trends are associated with the notion of cointegration (CI) introduced in Engle and Granger (1987). The relation between CI and the existence of common trends is the subject of Granger's representation theorem, which was proved by Johansen for VAR processes integrated of order 1 and 2, I(1) and I(2). Cointegration has generated a vast literature, see Johansen (2009) for a recent summary.

Common cycles have also received considerable attention, usually within systems which also display common trends, see e.g. Kugler and Neusser (1993), Lippi and Reichlin (1994), Vahid and Issler (2002), Hecq, Palm, and Urbain (2002, 2006), Paruolo (2003, 2006), Schleicher (2007) and Cubadda, Hecq, and Palm (2009).

Several notions of common cycles have been proposed in the literature. Engle and Kozicki (1993) and Cubadda and Hecq (2001) proposed the notion of (polynomial-) serial correlation common features, here indicated as CS(d); these correspond to common factors in the AR representation, and d indicates the degree of the AR polynomial of the CS linear combination. Alternatively, Gourieroux and Peaucelle (1988) and Vahid and Engle (1997) formalized the notion of co-dependence, which requires commonality in the MA representation, i.e. collinearity in the impulse responses; we indicate it with CD(d), where d is the degree of the MA polynomial of the CD linear combination. Yet another form of common dynamics requires common factors in the set of final equations (FE), see eq. (2.7) in Zellner and Palm (1974) and Cubadda, Hecq, and Palm (2009); we refer to this notion as CE(d), where d refers to the degree of the MA polynomial in the FE of the CE linear combination.

Several aspects of the relationships among these notions for VAR processes have been investigated in the literature. Engle and Kozicki (1993) and Vahid and Engle (1993) considered implications of CI on the existence of CS(0). They noted that I(1) VARs with CI are compatible with CS(0) in the growth rate of the process, and a necessary condition for this is that the CS(0) linear combinations must belong to the orthogonal complement of the space spanned by the adjustment coefficients in the error correction term. Cubadda and Hecq (2001) and Hecq, Palm, and Urbain (2006) defined and discussed the case of CS(d) in CI I(1) VARs, where the CS linear combinations always load the contemporaneous growth rate of the process. Paruolo (2003, 2006) gave extensions to cases of I(1) and I(2) systems with CS(d) linear combinations that possibly involve both or either the growth rates and deviations from equilibria.

Some implications of CD(d) on VAR processes were discussed in Kugler and Neusser (1993), who noted that CD(d) implies the orthogonality between the CD linear combinations and some (implicitly defined) function of the AR coefficients that one encounters in the recursive calculation of the MA coefficients. Finally Cubadda, Hecq, and Palm (2009) considered the implications of CS and CI on the FE representation of VAR processes, en route to obtain the orders of the univariate ARMA(p, q) representations of single component time series. In particular they derived the implications of CS(d) on p, q for $d \geq 0$, both for stationary and CI VAR processes with I(1) variables.

This paper provides a comprehensive and unified discussion of the relationships among CS, CD, CE. We present necessary and sufficient conditions for CS, CD, CE and their possible joint occurrence for the case of I(0), I(1) and I(2) VARs. These results extend and complement the results available in the literature cited above, as discussed in details below.

We present two types of propositions, one which concerns the *existence* and the other one that concerns the *characterization* of the common features linear subspaces, both of which are based on algebraic relations between a matrix polynomial and its determinant and adjoint. The existence results consist of necessary and sufficient conditions for CS, CD, CE in terms of the order of the pole at 0 of the inverse of the reversed AR matrix polynomial. Some implications of the existence results are the following. Cubadda, Hecq, and Palm (2009) showed that CS implies CE; the existence results also give the reverse implication, i.e. that CE implies CS. In addition existence results show that CD implies CS and CE, but that the converse does not hold.

The characterization results consist of necessary and sufficient conditions that a CS, CD, CE linear combinations need to satisfy, such as belonging to certain linear subspaces, associated with the expansion of the inverse of the AR polynomial around its poles; see Franchi and Paruolo (2009). The conditions are always translated also in terms of the subspaces directly related to the VAR polynomial. These conditions consist typically in some orthogonality conditions (which can be formulated as reduced rank conditions), plus a full rank condition.

Reduced rank restrictions on VAR coefficient matrices have been proposed in the time series literature as a way to obtain parameter parsimony; special cases are the index models (IM) of Reinsel (1983) and the nested reduced rank specification (NRR) of Ahn and Reinsel (1988). The rank conditions derived here are more complicated than the ones in IM and NRR; they are motivated not by parameter parsimony but as the restrictions corresponding one-to-one to commonality in cyclical features of the system. However, in characterizing CS structure below, we show a connection between IM and NRR and CS, observing that both IM and NRR imply CS, but they are not implied by it.

The versions of the characterization conditions based on the VAR coefficient matrices will hopefully allow to devise explicit formulations of restricted VARs with CS, CD, CE features. This will then permit to conduct inference on the presence of CS, CD, CE features in VARs by likelihood-based methods. Inference aspects, however, fall outside the scope of the present paper, which focuses on the characterization of CS, CD, CE constraints. For reasons of simplicity, also the extensions of the present results to the case of VARMA processes will not be pursued here.

VAR processes with rank restrictions are a special case of observable factor models. As an example, the VAR(1) process $X_t = AB'X_{t-1} + \epsilon_t$ with A and B full column rank matrices of dimension $p \times r$, $r < p$ can be interpreted as a factor model $X_t = AF_t + \epsilon_t$ where the r factors $F_t := B'X_{t-1}$ are observable linear combinations of the past of the process. One can hence hope that the results of the paper could find application also in the analysis of the larger class of dynamic factor models.

The rest of the paper is organized as follows: Section 2 introduces notation and defines of structures of interest, Section 3 and Section 4 collect the existence and the characterization results for stationary VARs. Section 5 extends results to I(1) and I(2) systems, while Section 6 presents a numerical example. Section 7 reports conclusions. Appendices contain proofs.

In the following we employ the following notational conventions: $\mathbb{R}$ and $\mathbb{C}$ indicate real and complex numbers, $|z|$ indicates the modulus of z, $a := b$ and $b =: a$ indicate that a is defined by b. For any full column rank matrix $\gamma \in \mathbb{C}^{p\times r}$, γ^* indicates the $p \times r$ matrix of complex conjugates and γ' the conjugate transpose of γ; in case γ is real, γ' reduces to the transpose. We indicate by $\operatorname{col}(\gamma)$ the linear span of the columns of γ with coefficients in the field $\mathbb{C}$ or $\mathbb{R}$ if γ is complex or real, respectively. $\gamma_\perp$ indicates a basis of $\operatorname{col}^\perp(\gamma)$, the orthogonal complement of $\operatorname{col}(\gamma)$, $\bar{\gamma} := \gamma(\gamma'\gamma)^{-1}$ so that $P_\gamma := \bar{\gamma}\gamma' = \gamma\bar{\gamma}'$ denotes the orthogonal projector matrix onto $\operatorname{col}(\gamma)$ and $M_\gamma := I - P_\gamma$ the orthogonal projector matrix onto $\operatorname{col}^\perp(\gamma)$. For a matrix A we often employ a rank factorization of the type $A = -\alpha\beta'$ where α and β are bases of $\operatorname{col}(A)$ and $\operatorname{col}(A')$, and the negative sign is chosen for convenience in the calculations.

Any sum $\sum_{n=a}^{b} \cdot$ where $b < a$ is defined equal to 0. For any matrix polynomial $\gamma(z) := \sum_{n=0}^{d_\gamma} \gamma_n z^n$, $z \in \mathbb{C}$, $\gamma_n \in \mathbb{C}^{p \times r}$ where $0 < r \leq p$, we indicate its degree by d_γ, i.e. $d_\gamma := \deg \gamma(z)$ and $0 < d_\gamma < \infty$; when $\gamma_n \in \mathbb{R}^{p \times r}$ we say that $\gamma(z)$ has real coefficients. The representation of $\gamma(z)$ around $z = z_u$ is written as $\gamma(z) = \sum_{n=0}^{d_\gamma} \gamma_{u,n}(z - z_u)^n$, so that $\gamma_{0,n} = \gamma_n$ for $z_0 = 0$. By $\gamma_\ddagger(z)$ we indicate the reversed polynomial of $\gamma(z)$, i.e. $\gamma_\ddagger(z) := \sum_{n=0}^{d_\gamma} \gamma_{d_\gamma - n} z^n$ and its representation around $z = z_u$ is written as $\gamma_\ddagger(z) = \sum_{n=0}^{d_\gamma} \gamma_n^{(u)}(z - z_u)^n$, so that $\gamma_n^{(0)} = \gamma_{d_\Pi - n}$ for $z_0 = 0$. Finally, $1_{j,k}$ is the indicator function equal to 1 if $j = k$ and 0 otherwise.

2. Setup and definitions

In this section we introduce notation and state the autoregressive (AR), moving average (MA) and final equations (FE) representation of a VAR system. We consider the vector autoregressive process (VAR) of finite order d_Π

$$\sum_{n=0}^{d_\Pi} \Pi_n X_{t-n} = \epsilon_t \tag{1}$$

where $\Pi_n \in \mathbb{R}^{p \times p}$, $\Pi_0 = I$ and ϵ_t is a p-dimensional martingale difference sequence (with respect to the natural filtration generated by X_t) with positive definite conditional covariance matrix Ω. A leading example of this is when ϵ_t are Gaussian i.i.d. random vectors. Deterministic components D_t are omitted from (1) for ease of exposition; they could be included by replacing X_t with $X_t - D_t$ or by replacing ϵ_t with $\epsilon_t + D_t$.

Indicate the AR polynomial in (1) by $\Pi(z) := \sum_{n=0}^{d_\Pi} \Pi_n z^n$, $z \in \mathbb{C}$, and let $k(z) := \det \Pi(z)$, $K(z) := \operatorname{adj} \Pi(z)$ be respectively its characteristic and adjoint polynomials, where $\operatorname{inv} \Pi(z) = K(z)/k(z)$. Remark that, because $\Pi(z)$ has real coefficients, so do $k(z)$ and $K(z)$. It is useful to factorize the characteristic polynomial in terms of its roots; because $\Pi(0) = I$, one can write $k(z) = \prod_{u=1}^{q}(1 - w_u z)^{a_u}$, where $w_u := z_u^{-1}$ and z_u is a root of $k(z)$ with multiplicity $a_u > 0$. We also define $\rho := \min_u |z_u|$ and observe that $\rho > 0$ because $\Pi(0) = I$.

Some of the factors in $k(z)$ could be common to $K(z)$; we state the cancellation of their common factors as the following lemma, for ease of later reference. The same lemma gives also the order of the pole of $\operatorname{inv} \Pi(z)$ at $z = z_u$, labeled m_u.

Lemma 2.1 (Co-prime polynomials $G(z)$, $g(z)$ and orders m_u)**.** *One has*

$$K(z) =: G(z) \prod_{u=1}^{q} (1 - w_u z)^{b_u}, \quad 0 \leq b_u < a_u, \quad G(z_u) \neq 0,$$

where $G(z)$ has real coefficients,

$$\operatorname{inv} \Pi(z) = \frac{G(z)}{g(z)}, \quad z \in \mathbb{C} \setminus \{z_1, \dots, z_q\}, \tag{2}$$

where

$$g(z) := \prod_{u=1}^{q} (1 - w_u z)^{m_u}, \quad m_u := a_u - b_u > 0, \tag{3}$$

has real coefficients, and $\operatorname{inv} \Pi(z)$ *has a pole of order m_u at $z = z_u$.*

We remark here that $G(z)$ and $g(z)$ also have real coefficients, because if a complex root is common, so is its complex conjugate. Using this notation, we introduce the final equations (FE) form $k(L)X_t = K(L)\epsilon_t$, see e.g. Zellner and Palm (1974), Cubadda, Hecq, and Palm (2009); eliminating the common factors as in Lemma 2.1 one obtains

$$g(L)X_t = G(L)\epsilon_t, \tag{4}$$

which we refer to as the FE form of the VAR.

The power series representation of $\mathrm{inv}\,\Pi(z)$ has real coefficients, see (2), and it is written as

$$C(z) := \mathrm{inv}\,\Pi(z) = \sum_{n=0}^{d_C} C_n z^n, \quad |z| < \rho,$$

where $d_C < \infty$ if and only if $d_g = 0$, i.e. $\Pi(z)$ is unimodular. Here $C(0) = C_0 = \mathrm{inv}\,\Pi_0 = \mathrm{inv}\,I = I$. It is well known (see e.g. Brockwell and Davis, 1987, page 408) that if $\Pi(z)$ has stable roots, i.e. $\rho > 1$, then $C(z)$ is holomorphic on a disk larger than the unit disk, and the moving average (MA) form

$$X_t = C(L)\epsilon_t \tag{5}$$

corresponds to a linear process with second moments.

In order to define the common structures of interest we introduce matrix polynomials $\gamma(z) := \sum_{n=0}^{d_\gamma} \gamma_n z^n$, $\gamma_n \in \mathbb{R}^{p\times r}$, $0 < r \leq p-1$, with γ_0, γ_{d_γ} are assumed of full column rank. A matrix polynomial with this property is called of full column rank. The reason to consider full column rank matrix polynomials is that if $\gamma_0 \neq 0$ is $p \times r$ and not of full column rank, then $\gamma(z)$ can be substituted with $z^a\widetilde{\gamma}(z)$ where $a > 0$ and $\widetilde{\gamma}_0$ is of full column rank $r_1 < r$. Similarly if $\gamma_{d_\gamma} \neq 0$ is not full rank, $\gamma(z)$ can be substituted with $\widetilde{\gamma}(z)$ where $d_{\widetilde{\gamma}} < d_\gamma$ and $\widetilde{\gamma}_{d_{\widetilde{\gamma}}}$ is of full column rank $r_1 < r$. This leads to the following definitions.

Definition 2.2 (Common structures of interest)**.** *Let $\gamma(z) := \sum_{n=0}^{d_\gamma} \gamma_n z^n$, $z \in \mathbb{C}$, $\gamma_n \in \mathbb{R}^{p\times r}$, be a full column rank matrix polynomial; then*

$$X_t \in \mathrm{CS}(d_\gamma) \quad \overset{def}{\Leftrightarrow} \quad \gamma'(L)X_t = \gamma_0'\epsilon_t, \quad 0 \leq d_\gamma < d_\Pi,$$

and we say that X_t displays common serial correlation features in the AR representation (1) *of order d_γ;*

$$X_t \in \mathrm{CE}(d_\gamma) \quad \overset{def}{\Leftrightarrow} \quad g(L)\gamma_0'X_t = \gamma'(L)\epsilon_t, \quad 0 \leq d_\gamma < d_G,$$

and we say that X_t displays commonality in the FE representation (4) *of order d_γ;*

$$X_t \in \mathrm{CD}(d_\gamma) \quad \overset{def}{\Leftrightarrow} \quad \gamma_0'X_t = \gamma'(L)\epsilon_t, \quad 0 \leq d_\gamma < d_C,$$

and we say that X_t displays co-dependence in the MA representation (5) *of order d_γ.*

We observe that this definition encompasses several special cases: serial correlation common features as introduced in Engle and Kozicki (1993) correspond to the case CS(0) = CD(0). CD(d_γ) was introduced in Gourieroux and Peaucelle (1988), who considered finite-order moving averages $d_C < \infty$. CD(1) structures were studied in Vahid and Engle (1997), see also the scalar component models in Tiao and Tsay (1989). Special cases of CS are given by the following notions: polynomial serial correlation common features, defined in Cubadda and Hecq (2001) and discussed in Cubadda, Hecq, and Palm (2009); weak and strong form reduced rank structures, see Hecq, Palm, and Urbain (2006); unpredictable combinations, defined in Paruolo (2006). Finally CE structures are considered in Cubadda, Hecq, and Palm (2009).

A central role in what follows is played by reversed polynomials, which we now introduce. For any polynomial $A(z)$ we define the $\dagger$ and $\ddagger$ operators as $A_\dagger(z) := A(z^{-1}), A_\ddagger(z) := z^{d_A} A_\dagger(z)$, so that

$$A_\ddagger(z) = \sum_{n=0}^{d_A} A_{d_A - n} z^n \tag{6}$$

is the reversed polynomial, i.e. the polynomial with the same coefficients of $A(z)$ in reversed order. In the following, the representation of $A_\ddagger(z)$ around $z = z_u$ is written as $A_\ddagger(z) = \sum_{n=0}^{d_A} A_n^{(u)}(z - z_u)^n$, so that $A_n^{(0)} = A_{d_A - n}$ for $z_0 = 0$.

For example, the reversed AR polynomial $\Pi_\ddagger(z) = \sum_{n=0}^{d_\Pi} \Pi_{d_\Pi - n} z^n$ is often used to describe the stability of the VAR system (see e.g. Fuller, 1996, page 77). We note here the connection of the $\dagger$ operator (and a fortiori also of the $\ddagger$ operator) with the transformation $z \mapsto 1/z$ which maps each point into its reciprocal, so that ∞ is mapped into 0 and 0 into ∞, see Greene and Krantz (1997), Section 4.7. In particular this implies that the roots of $\det \Pi_\ddagger(z)$ are w_u, the reciprocals of the roots z_u of the characteristic polynomial $k(z) := \det \Pi(z)$, plus a root at $w_0 := 0$ which is present when Π_{d_Π} is singular, as shown in Theorem 3.1 in the next section.

3. Existence results

In this section we present existence results in Theorem 3.2, which link the degrees of the polynomials $\Pi(z)$, $g(z)$ and $G(z)$ with existence of CS, CE and CD. The necessary and sufficient conditions for existence in Theorem 3.2 involve the index

$$m_0 := d_\Pi + d_G - d_g, \tag{7}$$

which is shown in the following Theorem 3.1 to equal to order of the pole at $z = 0$ of the reversed AR polynomial $\Pi_\ddagger(z)$.

Theorem 3.1. *Let the $\ddagger$ operator be as in (6), $w_0 := 0$ and $w_u := z_u^{-1}$, $u = 1, \dots, q$; then*

$$\operatorname{inv} \Pi_\ddagger(z) = z^{-m_0} \frac{G_\ddagger(z)}{g_\ddagger(z)}, \quad z \in \mathbb{C} \setminus \{w_0, \dots, w_q\}, \tag{8}$$

where $G_\ddagger(0)$, $g_\ddagger(0) \neq 0$ and hence m_0 in (7) is the order of the pole of $\operatorname{inv} \Pi_\ddagger(z)$ at $z = 0$.

The mapping $z \mapsto 1/z$ associated with the $\ddagger$ operator reveals all the points of rank-deficiency of $\Pi(z)$, finite or at ∞; these correspond to poles of order m_u in $\operatorname{inv} \Pi_\ddagger(z)$ at $z = w_u$, for $u = 0, \dots, q$. Moreover, because $\Pi(0) = I$, the poles of $\operatorname{inv} \Pi_\ddagger(z)$ have all finite modulus. We are now in the position to formulate the existence results for CS, CE and CD structures in terms of the order m_0 in (7) and (8).

Theorem 3.2 (Existence of CS, CE, CD structures)**.** *Let m_0 be as in (7) and (8); then*

- *i) the following statements are equivalent:*
  - *i.1) Π_{d_Π} is singular;*
  - *i.2) $m_0 > 0$;*
  - *i.3) $X_t \in \mathrm{CS}(d_\gamma)$ for some $\max(0, d_\Pi - m_0) \leq d_\gamma \leq d_\Pi - 1$;*
  - *i.4) $X_t \in \mathrm{CE}(d_\gamma)$ for some $\max(0, d_G - m_0) \leq d_\gamma \leq d_G - 1$.*
- *ii) if $X_t \in \mathrm{CD}(d_\gamma)$ then $0 \leq d_\gamma \leq m_0 - d_\Pi$; moreover, i.j) holds for any $j = 1, \dots, 4$;*
- *iii) the statement i.j) for some $j = 1, \dots, 4$ does not imply $X_t \in \mathrm{CD}(d_\gamma)$.*

Theorem 3.2 states that a CS (or a CE) structure of some degree exists whenever $m_0 > 0$, i.e. when the last coefficient matrix of $\Pi(z)$ is singular. Moreover, CS and CE structures always coexist; this gives a converse to Cubadda, Hecq, and Palm (2009, Proposition 8), who show that CS implies CE. In addition, Theorem 3.2*ii*) shows that a CD structure of some degree exists only if $m_0 \geq d_\Pi$; because $d_\Pi > 0$, this implies that if a CD structure exists, then also CS (and CE) structures exist. The converse does not hold. Further one has that

$$d_\Pi - m_0 = d_g - d_G$$

provides a lower bound for d_γ in $\text{CS}(d_\gamma)$, and it reveals the highest reduction that can be achieved by a CS relation. When this difference is negative the inequality $d_\Pi - m_0 \leq d_\gamma$ is trivially satisfied because $d_\gamma \geq 0$ and it does not provide any relevant information regarding the AR part. However, If $X_t \in \text{CD}(d_\gamma)$ then $m_0 - d_\Pi = d_G - d_g$ provides an upper bound for d_γ, i.e. $0 \leq d_\gamma \leq m_0 - d_\Pi$, and it reveals the highest order of a CD relation. A similar interpretation applies to CE structures in *i*.4). Hence the difference between the degrees of the adjoint and of the determinant of $\Pi(z)$ plays and important role in distinguishing cases with only CS and CE structures from the cases where also CD are present.

4. Characterization of CS, CD and CE linear combinations

In this section we characterize CS, CD and CE linear combinations; the main results are contained in Theorem 4.2 for CS structures, Theorem 4.3 for CE structures, Theorem 4.4 for CD structures. Necessary and sufficient conditions are stated for each form of common cyclical features in terms of linear subspaces, associated with the orders $m_0, m_1, \ldots, m_q$ of the poles of $\text{inv}\, \Pi_\ddagger(z)$ defined in eqs. (3), (7), (8).

The relevant subspaces are found through the 'polynomial rank factorization' of a matrix polynomial at a given point; it consists in a sequence of m_u rank factorizations on the matrices in eq. (10) below. This definition also includes the matrices $\Pi^{(u)}_{s,j,k}$ which turn out to be useful in the analysis of CE, CD structures.

Definition 4.1 (Polynomial rank factorization and matrices $\Pi^{(u)}_{j,k}$, $\Pi^{(u)}_{s,j,k}$)**.** *Let the $\ddagger$ operator be as in* (6)*, $w_0 := 0$ and $w_u := 1/z_u$, $u = 1, \ldots, q$, $\Pi_\ddagger(z) = \sum_{n=0}^{d_\Pi} \Pi^{(u)}_n (z - w_u)^n$ and define $\alpha_{u,0}$ and $\beta_{u,0}$ of dimension $p \times r_{u,0}$, where $0 < r_{u,0} < p$, from the matrix rank factorization*

$$\Pi^{(u)}_0 = -\alpha_{u,0}\beta'_{u,0}. \tag{9}$$

For $j = 1, \ldots, m_u$, let $a_{u,j} := (\alpha_{u,0} : \cdots : \alpha_{u,j-1})$, $b_{u,j} := (\beta_{u,0} : \cdots : \beta_{u,j-1})$ and $r^{\max}_{u,j} := p - \sum_{n=0}^{j-1} r_{u,n}$ and define $\alpha_{u,j}$ and $\beta_{u,j}$ of dimension $p \times r_{u,j}$, where $0 \leq r_{u,j} < r^{\max}_{u,j}$ for $j \neq m_u$ and $0 < r_{u,m_u} = r^{\max}_{u,m_u}$, from the matrix rank factorization

$$M_{a_{u,j}} \Pi^{(u)}_{j,1} M_{b_{u,j}} = -\alpha_{u,j}\beta'_{u,j}, \tag{10}$$

where $\Pi^{(u)}_{j,k}$ is defined for $j, k \geq 1$ from the recursions

$$\Pi^{(u)}_{j,k} := \Pi^{(u)}_{j-1,k+1} + \Pi^{(u)}_{j-1,1} \sum_{n=0}^{j-2} \bar{\beta}_{u,n}\bar{\alpha}'_{u,n}\Pi^{(u)}_{n+1,k} \tag{11}$$

with initial values $\Pi^{(u)}_{0,k} := \Pi^{(u)}_{k-1}$. *Finally let* $\Pi^{(u)}_{1,j,k} := \Pi^{(u)}_{j,k}$ *and for* $s = 2, \ldots, t \leq m_u - 1$, *define* $\Pi^{(u)}_{s,j,k}$ *from the recursions*

$$\Pi^{(u)}_{s,j+1,k} := \Pi^{(u)}_{s-1,j+1,k+1} + \Pi^{(u)}_{s-1,j+1,1} \sum_{h=0}^{m_u - t} \bar{\beta}_{u,h} \bar{\alpha}'_{u,h} \Pi^{(u)}_{1,h+1,k}. \tag{12}$$

The polynomial rank factorization in Definition 4.1 gives a characterization of the set of reduced rank restrictions that are satisfied by the coefficients of a matrix polynomial whose inverse function has a pole of given order at a specific point. That is, if $\Pi_{\ddagger}(z)$ and its derivatives at $z = w_u$ satisfy those conditions, then $\operatorname{inv} \Pi_{\ddagger}(z)$ has a pole of order m_u at the same point; the converse is also true, i.e. if $\operatorname{inv} \Pi_{\ddagger}(z)$ has a pole of order m_u at $z = w_u$ then $\Pi_{\ddagger}(z)$ and its derivatives at $z = w_u$ satisfy the rank restrictions of the polynomial rank factorization at that point. Hence the polynomial rank factorization is a one to one and onto map from the structure of the matrix polynomial to the nature of the singularity of its inverse function. This result is based on the recursive algorithm developed in Franchi (2009) and further analyzed in Franchi and Paruolo (2009). The following additional remarks are in order:

Remark 1. Eq. (9), (10) define $\alpha_{u,j}$, $\beta_{u,j}$ up to a conformable change of bases of the row and column spaces; this indeterminacy does not affect the results, in the sense that the latter do not depend on the particular choice of the pair $\alpha_{u,j}$, $\beta_{u,j}$.

Remark 2. The $p \times p$ matrices $(\alpha_{u,0} : \cdots : \alpha_{u,m_u})$ and $(\beta_{u,0} : \cdots : \beta_{u,m_u})$ are non-singular with orthogonal blocks, i.e. $\alpha'_{u,j}\alpha_{u,k} = \beta'_{u,j}\beta_{u,k} = 0$ for $j \neq k$. To simplify notation, we let $a_{u,j} := (\alpha_{u,0} : \cdots : \alpha_{u,j-1})$, $a_{u,j\perp} := (\alpha_{u,j} : \cdots : \alpha_{u,m_u})$ and observe that $a_{u,j}$ is a real matrix if and only if $\operatorname{Im} w_u = 0$. Similarly we define $b_{u,j}, b_{u,j\perp}$ in terms of $\beta_{u,j}$ blocks.

Remark 3. The conditions (10) are reduced-rank conditions for $j = 1, \cdots, m_u - 1$, while the terminal condition for $j = m_u$ is a full-rank condition. The matrices $M_{a_{u,j}} = P_{a_{u,j\perp}}$, $M_{b_{u,j}} = P_{b_{u,j\perp}}$ are orthogonal projectors which successively eliminate subspaces until the terminal condition of full rank is met. In fact for $\ell, n \geq j$, $\bar{\alpha}'_{u,\ell} M_{a_{u,j}} = \bar{\alpha}'_{u,\ell} - \bar{\alpha}'_{u,\ell} P_{a_{u,j}} = \bar{\alpha}'_{u,\ell}$ and $M_{b_{u,j}} \bar{\beta}_{u,n} = \bar{\beta}_{u,n}$ so that using (10) one has $\bar{\alpha}'_{u,\ell} \Pi^{(u)}_{j,1} \bar{\beta}_{u,n} = \bar{\alpha}'_{u,\ell} M_{a_{u,j}} \Pi^{(u)}_{j,1} M_{b_{u,j}} \bar{\beta}_{u,n} = -\bar{\alpha}'_{u,\ell} \alpha_{u,j} \beta'_{u,j} \bar{\beta}_{u,n}$, or

$$\bar{a}'_{u,j\perp} \Pi^{(u)}_{j,1} \bar{b}_{u,j\perp} = \begin{pmatrix} -I_{r_{u,j}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{13}$$

Remark 4. For $w_u = 1$, and $m_u = 1$, $m_u = 2$ these conditions were derived by Johansen (1992) and are called the $I(1)$ and $I(2)$ conditions.

Remark 5. There is a duality between the polynomial rank factorizations of $\Pi_{\ddagger}(z)$ and $G_{\ddagger}(z)$. That is, let $\alpha_{u,j}$, $\beta_{u,j}$ and $\xi_{u,j}$, $\eta_{u,j}$ be respectively defined by the polynomial rank factorizations of $\Pi_{\ddagger}(z)$ and $G_{\ddagger}(z)$ at $z = w_u$; then for $j = 0, \ldots, m_u$, one has

$$\xi_{u,j} = h_u \bar{\beta}_{u,m_u - j} \qquad \text{and} \qquad \eta_{u,j} = \bar{\alpha}_{u,m_u - j}, \tag{14}$$

where $h_u := g_{\ddagger,u}(w_u) \neq 0$ is a scalar and $g_{\ddagger}(z) =: (z - w_u)^{m_u} g_{\ddagger,u}(z)$, see Franchi and Paruolo (2009) for the proof. This can be seen as a generalization of Proposition 8 in Cubadda, Hecq, and Palm (2009) about the presence of a factor structure in the adjoint under CS.

We are now in the position to give a characterization of the common structures of interest, starting from CS structures.

Theorem 4.2 (Characterization of CS structures)**.** *Let* $a_{0,j}$, $b_{0,j}$ *be defined by the polynomial rank factorization of* $\Pi_{\ddagger}(z)$ *at* $w_0 := 0$, $\Pi(z) = \sum_{n=0}^{d_\Pi} \Pi_n z^n$ *and* $\gamma(z) := \sum_{n=0}^{d_\gamma} \gamma_n z^n$ *be of full column rank and degree* $\max(0, d_\Pi - m_0) \leq d_\gamma \leq d_\Pi - 1$; *then the following statements are equivalent:*

i) $\gamma'(L)X_t = \gamma_0'\epsilon_t$;
ii) $\gamma'(z) = \gamma_0'\Pi(z)$;
iii) $\gamma_0'\Pi_{d_\gamma}$ *has full row rank and* $\gamma_0'(\Pi_{d_\gamma+1} : \cdots : \Pi_{d_\Pi}) = 0$;
iv) *let* $\ell := d_\Pi - d_\gamma$; *then*

$$\gamma_0 = a_{0,\ell\perp}\varphi_0$$

where $\varphi_0'a_{0,\ell\perp}'\Pi_{d_\gamma}$ *has full row rank and*

$$\varphi_0'a_{0,\ell\perp}'(\Pi_{d_\gamma+1}b_{0,\ell-1} : \cdots : \Pi_{d_\Pi-1}b_{0,1}) = 0.$$

Theorem 4.2 gives three equivalent characterizations of the CS structure, whose definition is reproduced in *i)*. In *ii)* one sees that once γ_0 is selected, then $\gamma(z)$ can be determined from the equality $\gamma'(z) = \gamma_0'\Pi(z)$, i.e. as $\gamma_n' = \gamma_0'\Pi_n$, $n = 1, \ldots, d_\gamma$, and $\gamma_0'\Pi_{d_\gamma}$ has full row rank. In *iii)* one sees that γ_0 is a basis of the orthogonal complement of $\mathrm{col}(\Pi_{d_\gamma+1} : \cdots : \Pi_{d_\Pi})$ and satisfies a terminal full rank condition. The former orthogonality conditions can be expressed also as $\gamma_0 \in \cap_{i=d_\gamma+1}^{d_\Pi} \mathrm{col}^\perp \Pi_i$, which is implied by the nested reduced rank specification of Ahn and Reinsel (1988) and by the index models of Reinsel (1983). We observe here that *iii)* does not imply $\mathrm{col}(\Pi_j) \supset \mathrm{col}(\Pi_{j+1})$, which corresponds to nested reduced rank specifications, or $\Pi(z) = I + \alpha(z)\beta'(z)$ in an obvious notation, which characterizes index models.

Theorem 4.2 *iv)* gives a characterization of γ_0 in terms of the coefficients of the polynomial rank factorization of $\Pi_\ddagger(z)$ at 0; first it shows that γ_0 belongs to the space spanned by the columns of $a_{0,\ell\perp} := (\alpha_{0,\ell} : \cdots : \alpha_{0,m_0})$ where $\ell := d_\Pi - d_\gamma$. Hence when $d_\gamma = d_\Pi - 1$, $\gamma_0 \in \mathrm{col}(\alpha_{0,1} : \cdots : \alpha_{0,m_0})$, while when $d_\gamma = d_\Pi - m_0 > 0$ one has $\gamma_0 = \alpha_{0,m_0}\varphi_0$. Thus the smaller d_γ in $\mathrm{CS}(d_\gamma)$, the smaller is the linear space in which γ_0 can be chosen.

This condition that γ_0 belongs to the given linear space is, however, only necessary and not sufficient in order to obtain CS; the additional condition $a_{0,\ell\perp}'(\Pi_{d_\gamma+1}b_{0,\ell-1} : \cdots : \Pi_{d_\Pi-1}b_{0,1})$ of reduced rank is needed. This determines φ_0 as a basis of $\mathrm{col}^\perp a_{0,\ell\perp}'(\Pi_{d_\gamma+1}b_{0,\ell-1} : \cdots : \Pi_{d_\Pi-1}b_{0,1})$ and completes the characterization of γ_0.

The case of CE structures is considered in the next theorem.

Theorem 4.3 (Characterization of CE structures). *Let* $a_{0,j}$, $b_{0,j}$ *be defined by the polynomial rank factorization of* $\Pi_\ddagger(z)$ *at* $w_0 := 0$, $G(z) = \sum_{n=0}^{d_G} G_n z^n$ *and* $\gamma(z) := \sum_{n=0}^{d_\gamma} \gamma_n z^n$ *be of full column rank and degree* $\max(0, d_G - m_0) \leq d_\gamma \leq d_G - 1$; *then the following statements are equivalent:*

i) $g(L)\gamma_0'X_t = \gamma'(L)\epsilon_t$;
ii) $\gamma'(z) = \gamma_0'G(z)$;
iii) $\gamma_0'G_{d_\gamma}$ *has full row rank and* $\gamma_0'(G_{d_\gamma+1} : \cdots : G_{d_G}) = 0$;
iv) *let* $\ell := m_0 - d_G + d_\gamma + 1$; *then*

$$\gamma_0 = b_{0,\ell}\varphi_0$$

where one of the following equivalent sets of conditions holds:

iv.1) $\varphi_0'b_{0,\ell}'G_{d_\gamma}$ *has full row rank and*

$$\varphi_0'b_{0,\ell}'(G_{d_\gamma+1}a_{0,\ell+1\perp} : \cdots : G_{d_G-1}a_{0,m_0\perp}) = 0;$$

iv.2) $\varphi_{0,\ell-1}'\beta_{0,\ell-1}'\beta_{0,\ell-1}\alpha_{0,\ell-1}' + H_0\bar{a}_{0,\ell\perp}'$ *has full row rank with*

$$H_0 := h_0 \sum_{j=0}^{\ell-1} \varphi_{0,j}'\bar{\alpha}_{0,j}'(\Pi_{1,j+1,1}^{(0)}\beta_{0,\ell} : \cdots : \Pi_{d_G-d_\gamma,j+1,1}^{(0)}\beta_{0,m_0})$$

and

$$\sum_{j=0}^{\ell-1} \varphi'_{0,j}\bar{\alpha}'_{0,j}(\Pi^{(0)}_{1,j+1,1}b_{0,\ell+1} : \cdots : \Pi^{(0)}_{m_0-\ell,j+1,1}b_{0,m_0}) = 0.$$

Theorem 4.3 is the dual of Theorem 4.2 for CE structures and its interpretation is exactly as above, with the role of $G(z)$ in Theorem 4.3 is the same as the one of $\Pi(z)$ in Theorem 4.2. In fact $\Pi_{\cdot}$, $a_{0,\cdot\perp}$ and $b_{0,\cdot}$ in *iv*), Theorem 4.2, are respectively replaced by $G_{\cdot}$, $b_{0,\cdot}$ and $a_{0,\cdot\perp}$ in Theorem 4.3 *iv*.1); this is a consequence of the duality result discussed in Remark 5 above, eq. (14).

Theorem 4.3 *iv*) states that γ_0 belongs to the space spanned by the columns of $b_{0,\ell} := (\beta_{0,0} : \cdots : \beta_{0,\ell-1})$ where $\ell := m_0 - d_G + d_\gamma + 1$. Similarly to the CS case, shorter CE structures correspond to smaller linear subspaces: from $\gamma_0 = (\beta_{0,0} : \cdots : \beta_{0,m_0-1})\varphi$ when $d_\gamma = d_G - 1$ to $d_\gamma = d_G - m_0 > 0$ for which $\gamma_0 = \beta_{0,0}\varphi$. As in the CS case, Theorem 4.3 *iv*.1) states also the additional condition $b'_{0,\ell}(G_{d_\gamma+1}a_{0,\ell+1\perp} : \cdots : G_{d_G-1}a_{0,m_0\perp})$.

The conditions Theorem 4.3 *iv*.1) are stated in terms of the coefficients of the reduced adjoint $G(z)$, while in Theorem 4.3 *iv*.2) the characterization is given in terms of the polynomial rank factorization of $\Pi_\ddagger(z)$ at $w_0 = 0$. The latter characterization involves blocks of the $\Pi^{(u)}_{s,j,k}$ matrices introduced in Definition 4.1. Also these conditions involve some reduced rank conditions and a full rank condition.

Finally we turn to CD structures.

Theorem 4.4 (Characterization of CD structures)**.** *Let* $w_0 := 0$*,* $w_u := 1/z_u$*,* $u = 1, \ldots, q$*, and define* $a_{u,j}$*,* $b_{u,j}$ *from the polynomial rank factorization of* $\Pi_\ddagger(z)$ *at* w_u*,* $u = 0, \ldots, q$*; furthermore let* $G(z) = \sum_{n=0}^{d_G} G_{u,n}(z - z_u)^n$*, where* $G_{0,n} := G_n$*, and* $\gamma(z) := \sum_{n=0}^{d_\gamma} \gamma_n z^n$ *be of full column rank and degree* $0 \le d_\gamma \le m_0 - d_\Pi$*. Then the following statements are equivalent:*

i) $\gamma'_0 X_t = \gamma'(L)\epsilon_t$*;*

ii) $\gamma'(z) = \gamma'_0 \operatorname{inv}\Pi(z)$*;*

iii) $\gamma'_0(G_{d_g+d_\gamma+1} : \cdots : G_{d_G}) = 0$ *and* $\gamma'_0(G_{u,d_G-m_u+1} : \cdots : G_{u,d_G}) = 0$*, for* $u = 1, \ldots, q$*;*

iv) *let* $\ell := d_\Pi + d_\gamma + 1$*; then*

$$\operatorname{col}\gamma_0 \subset (\operatorname{col} b_{0,\ell} \cap_{u=1}^{q} \operatorname{col}\beta_{u,0})$$

with representation

$$\gamma_0 = b_{0,\ell}\varphi_0 = \beta_{u,0}\varphi_u, \qquad u = 1, \ldots, q,$$

where one of the following equivalent sets of conditions holds:

iv.1) $\varphi'_0 b'_{0,\ell} G_{d_g+d_\gamma}$ *has full row rank and*

$$\varphi'_0 b'_{0,\ell}(G_{d_g+d_\gamma+1}a_{0,\ell+1\perp} : \cdots : G_{d_G-1}a_{0,m_0\perp}) = 0$$

and for $u \neq 0$*,* $\varphi'_u\beta'_{u,0}G_{u,d_G-m_u}$ *has full row rank and*

$$\varphi'_u\beta'_{u,0}(G_{u,d_G-m_u+1}a_{u,2\perp} : \cdots : G_{u,d_G-1}a_{u,m_u\perp}) = 0;$$

iv.2) $\varphi'_{0,d_\Pi+d_\gamma}\beta'_{0,d_\Pi+d_\gamma}\beta_{0,d_\Pi+d_\gamma}\alpha'_{0,d_\Pi+d_\gamma} + H_0\bar{a}'_{0,\ell\perp}$*, with* $H_0 := h_0\sum_{j=0}^{d_\Pi+d_\gamma}\varphi'_{0,j}\bar{\alpha}'_{0,j}(\Pi^{(0)}_{1,j+1,1}\beta_{0,\ell} : \cdots : \Pi^{(0)}_{m_0-\ell+1,j+1,1}\beta_{0,m_0})$*, has full row rank and*

$$\sum_{j=0}^{d_\Pi+d_\gamma} \varphi'_{0,j}\bar{\alpha}'_{0,j}(\Pi^{(0)}_{1,j+1,1}b_{0,\ell+1} : \cdots : \Pi^{(0)}_{m_0-\ell,j+1,1}b_{0,m_0}) = 0$$

and for $u \neq 0$, $\varphi_{u,0}'\beta_{u,0}'\beta_{u,0}\alpha_{u,0}' + H_u\bar{\alpha}_{u,1\perp}'$, *with* $H_u := h_u\varphi_{u,0}'\bar{\alpha}_{u,0}'(\Pi_{1,1,1}^{(u)}\beta_{u,1} : \cdots : \Pi_{m_u,1,1}^{(u)}\beta_{u,m_u})$, *has full row rank and*

$$\varphi_{u,0}'\bar{\alpha}_{u,0}'(\Pi_{1,1,1}^{(u)}b_{u,2} : \cdots : \Pi_{m_u-1,1,1}^{(u)}b_{u,m_u}) = 0.$$

Theorem 4.4 has the same structure as Theorems 4.2 and 4.3. Three equivalent formulations of the CD structures as defined in Theorem 4.4 *i*). The equality in Theorem 4.4 *ii*) has, however, an important difference from its counterpart of *ii*) in Theorems 4.2 and 4.3. In fact, see Lemma 2.1, $\operatorname{inv}\Pi(z)$ has poles at the characteristic roots while $\gamma'(z) = \gamma_0' \operatorname{inv}\Pi(z)$ is a matrix polynomial and hence it has no poles; this can be the case if and only if γ_0 cancels the principal part of $\operatorname{inv}\Pi(z)$ so that $\gamma'(z) = \gamma_0'R(z)$, see (37) in the Appendix. Because $R(z)$ is a matrix polynomial of degree $d_G - d_g$ and $d_G - d_g = m_0 - d_\Pi$, see (7), this gives an alternative explanation of why CD structures have degree $0 \le d_\gamma \le m_0 - d_\Pi$.

Next we turn to Theorem 4.4 *iii*) and recall that the principal part of $\operatorname{inv}\Pi(z)$ in (37) in the Appendix is $\sum_{u=1}^q \frac{B_u(z)}{(1-w_uz)^{m_u}}$, where $B_u(z) = \sum_{n=0}^{m_u-1} B_{u,n}(1 - w_uz)^n$; because $B_{u,n}$ is a linear combination with scalar weights of $G_{u,n}$ and $\gamma_0'B_u(z) = 0$ if and only if $\gamma_0'B_{u,n} = 0$ then γ_0 cancels the principal part if and only if $\gamma_0'(G_{u,d_G-m_u+1} : \cdots : G_{u,d_G}) = 0$, $u = 1, \ldots, q$. The role of $\gamma_0'(G_{d_g+d_\gamma+1} : \cdots : G_{d_G}) = 0$ is then to determine the length d_γ of a CD structure.

The conditions in Theorem 4.4 *iii*) are of the same type of the conditions in Theorem 4.3 *iii*), with the difference that they not only involve the coefficients of the Taylor expansion of $G(z)$ around $z = 0$ but also the ones of the expansions around each characteristic roots.

Theorem 4.4 *iv*.1) is derived applying Theorem 4.3 *iv*.1) at each characteristic root by letting $d_{\gamma_u} := d_G - m_u$, $u = 1, \ldots, q$, and $d_{\gamma_0} := d_\gamma - d_g$; then $v_u := \beta_{u,0}\varphi_u$ and the additional reduced rank condition $\beta_{u,0}'(G_{u,d_G-m_u+1}a_{u,2\perp} : \cdots : G_{u,d_G-1}a_{u,m_u\perp}) = \varphi_{u\perp}\eta_u'$ holds for $u = 1, \ldots, q$ and in the same way $v_0 := b_{0,\ell}\varphi_0$ with $\varphi_0'b_{0,\ell}'(G_{d_G-\ell+1}a_{\ell+1\perp} : \cdots : G_{d_G-1}a_{m_0\perp}) = 0$. Because γ_0 must satisfy the conditions in Theorem 4.4 *iii*) then it must be a basis of $\cap_{u=0}^q \operatorname{col} v_u$. Conversely, if γ_0 is basis of $\cap_{u=0}^q \operatorname{col} v_u$, then point Theorem 4.4 *iii*) holds.

Finally we observe that $v_u := \beta_{u,0}\varphi_u$ does not vary with d_γ, i.e. the length of a CD structure restricts γ_0 only through $v_0 := b_{0,\ell}\varphi_0$ where $\ell := d_\Pi + d_\gamma + 1$. As above, shorter CD structures restrict the portion of space in which v_0 can be found; it is interesting to note that when $d_\gamma = m_0 - d_\Pi$ one has $\ell = m_0 + 1$, i.e. $v_0 = (\beta_0 : \cdots : \beta_{m_0})\varphi_0$ which means that the only restrictions are the ones at the characteristic roots. In the other cases one has $v_0 = (\beta_0 : \cdots : \beta_{\ell-1})\varphi_0$ up to the limiting case $d_\gamma = 0$ in which $v_0 = (\beta_0 : \cdots : \beta_{d_\Pi})\varphi_0$.

As in the preceding theorems, in Theorem 4.4 *iv*.2) we translate the conditions on left null spaces of the G coefficients into their counterpart in terms of the coefficients introduced in Definition 4.1 and, as in Theorem 4.4 *iv*.1), the conditions are given by a reduced rank and a full rank condition.

5. I(1) and I(2) systems

In this section we show how the results given for stationary VARs can be directly extended to VAR systems with I(1) and I(2) variables. The main idea is that Johansen's representation theorems for I(1) and I(2) VAR systems (see Johansen, 1996, Chapter 4) imply that one can transform the original system variables into a stationary VAR process of the same dimension. The results of the previous sections then apply to the transformed system.

Consider first an I(1) VAR(d_A) process $A(L)Y_t = u_t$ with error correction representation

$$\Gamma(L)\Delta Y_t = \alpha_0\beta_0'Y_{t-1} + u_t \tag{15}$$

where $\Delta := 1-L$, $\Gamma(L) = I - \sum_{i=1}^{d_\Gamma} \Gamma_i L^i$, $d_A = d_\Gamma + 1$. The VAR polynomial $A(z) = I(1-z)\Gamma(z) - \alpha_0\beta_0' z$ has polynomial rank factorization around the point $z_1 = 1$ of the type $A_0^{(1)} = -\alpha_0\beta_0'$, $M_{a_1} A_{1,1}^{(1)} M_{b_1} = -\alpha_1\beta_1'$ with α_0, β_0 and α_1, β_1 of full column ranks equal to r and $p-r$ respectively. The stationary transformation of system Y_t in (15) for I(1) systems can be found in Johansen (1996) pages 50-53, and it has been used in Paruolo (2003) in the context of CS(d). We re-state it for ease of reference in the following lemma.

Lemma 5.1 (I(1) VAR). *Let $A(L)Y_t = u_t$ be a VAR(d_A) of dimension p, with $A_0^{(1)} = -\alpha_0\beta_0'$, $M_{a_1} A_{1,1}^{(1)} M_{b_1} = -\alpha_1\beta_1'$, with α_0, β_0 and α_1, β_1 of full column ranks equal to r and $p-r$ respectively, and define*

$$X_t := (Y_t'\beta_0 : \Delta Y_t'\beta_1)',$$

still of dimension $p \times 1$; then X_t follows a VAR process

$$\Pi(L)X_t = \varepsilon_t$$

with $\varepsilon_t := (\beta_0 : \beta_1)'u_t$, $d_\Pi = d_A$ and $\Pi_{d_\Pi} = (\Pi_{d_\Pi,0} : 0_{p\times(p-r)})$, where we have partitioned the Π_j matrices in blocks of columns conformable with the two components of X_t. Moreover the characteristic roots of $\Pi(z)$ are the same as the ones of $A(z)$, except for the $p-r$ characteristic roots at $z = 1$ of $A(z)$ that are absent from of $\Pi(z)$.

A similar result applies to the I(2) VAR(d_A) process $A(L)Y_t = u_t$ in (15) above, as proved in Theorem 3 in Paruolo (2006), to which we refer for a proof. We reproduce here a part of its statement in the present notation.

Lemma 5.2 (I(2) VAR). *Let $A(L)Y_t = u_t$ be a VAR(d_A) of dimension p, with $A_0^{(1)} = -\alpha_0\beta_0'$, $M_{a_1} A_{1,1}^{(1)} M_{b_1} = -\alpha_1\beta_1'$, $M_{a_2} A_{2,1}^{(1)} M_{b_2} = -\alpha_2\beta_2'$, with α_0, β_0, and α_1, β_1 and α_2, β_2 of full column ranks equal to r_0 and r_1 and $p - r_0 - r_1$ respectively, and define*

$$X_t := (Y_t'\beta_0 + \Delta Y_t'\beta_2\delta' : \Delta Y_t'\beta_1 : \Delta^2 Y_t'\beta_2)',$$

$\delta := -\bar{\alpha}_0' A_{1,1}^{(1)} \bar{\beta}_2$ where X_t is still of dimension $p \times 1$; then X_t follows a VAR process

$$\Pi(L)X_t = \varepsilon_t$$

with $\varepsilon_t := (\beta_0 + \beta_2\delta' : \beta_1 : \beta_2)'u_t$, $d_\Pi = d_A$ and $\Pi_{d_\Pi} = (\Pi_{d_\Pi,0} : 0_{p\times(p-r_1)})$, $\Pi_{d_\Pi,0}\delta = -\Pi_{d_\Pi-1,2}$, where we have partitioned the Π_j matrices in blocks of columns conformable with the three component of X_t. Moreover the characteristic roots of $\Pi(z)$ are the same as the ones of $A(z)$, except for the $2(p - r_0 - r_1) + r_1$ characteristic roots at $z = 1$ of $A(z)$ that are absent from of $\Pi(z)$.

Lemmas 5.1 and 5.2 show that the results in Sections 3 and 4 apply to the systems X_t derived from the original variables Y_t. The transformations from Y_t to X_t depend only on CI coefficients, and they do not alter the stationary roots of the system, which are the ones associated with cycles.

We observe that this allows to have CS, CD, CE structures both in the ΔY_t *and* in $\beta_0' Y_t$ in I(1) systems. This enlarges a common tenet that all CS must involve ΔY_t; here they may involve only $\beta_0' Y_t$. Similarly CS, CD, CE structures are here defined for I(2) systems in $\Delta^2 Y_t$, $\beta_1' \Delta Y_t$ and $\beta_0' \Delta Y_t + \delta\beta_2' \Delta Y_t$. This type of enlargement was suggested by Paruolo (2003, 2006) for CS systems in I(1) and I(2) systems, and it is applied here also to CD and CE structures.

We note that Π_{d_Π} in Lemmas 5.1 and 5.2 are singular, and hence Theorem 3.2 *i*.1) implies that all I(1) and I(2) VAR processes at least present CS and CE; obviously, all remaining characterizations can be further applied, possibly leading also to CD structures.

6. A numerical example

For $\ell = \omega, \psi$, let $1_{\ell,\psi} = 0$ if $\ell = \omega$ and $1_{\ell,\psi} = 1$ if $\ell = \psi$ and consider $\Pi^\ell(L)X_t^\ell = \epsilon_t$, where

$$\Pi^\ell(z) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3 & 4 \\ -6 & -5 + 1_{\ell,\psi} \end{pmatrix} z - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 : 1)\, z^2,$$

with MA representations $X_t^\ell = \sum_{n=0}^\infty C_n^\ell \epsilon_{t-n}$, where

$$C_0^\omega = C_0^\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_1^\omega = C_1^\psi = \frac{1}{3}\begin{pmatrix} -3 & -4 \\ 6 & 5 - 1_{\ell,\psi} \end{pmatrix},$$

and for $n = 2, 3, \ldots$, C_n^ψ is non-singular and

$$C_n^\omega = \frac{c_n}{18}\begin{pmatrix} -7 \\ 11 \end{pmatrix}(3 : 1) \text{ where } c_n := \left(\frac{2}{3}\right)^{n-2}.$$

We want to characterize the CS, CE and CD structures of the two processes. First we compute the polynomial rank factorizations of $\Pi_\ddagger^\ell(z)$ at 0, see Definition 4.1; these give $\alpha_0^\ell = (1 : 1)'$, $\beta_0^\ell = \frac{1}{2}(1 : 1)'$, $\alpha_1^\omega = \alpha_2^\omega = \beta_1^\omega = \beta_2^\omega = (0 : 0)'$ and $\alpha_3^\omega = \alpha_1^\psi = (1 : -1)'$, $\beta_3^\omega = \frac{1}{3}(-1 : 1)'$, $\beta_1^\psi = \frac{1}{12}(-1 : 1)'$ so that $m_0^\omega = 3$, $m_0^\psi = 1$. Then, see Theorem 4.2, $X_t^\ell \in \mathrm{CS}(1)$ with $\gamma_0 = \alpha_3^\omega = \alpha_1^\psi = (1 : -1)'$ and, see Theorem 4.3, $X_t^\ell \in \mathrm{CE}(1)$ with $\gamma_0 = \beta_0^\ell = \frac{1}{2}(1 : 1)'$. This shows that the processes share the same common serial correlation features in the AR representation and the same commonality in the FE representation, see Definition 2.2. Finally we consider co-dependence in the MA representation; while $m_0^\psi < d_\Pi$ implies that X_t^ψ does not display it, $m_0^\omega > d_\Pi$ does not rule it out for X_t^ω. Because $\det \Pi^\omega(z) = -\frac{1}{3}(2z-3)$, one has $z_1 = \frac{3}{2}$ and $m_1^\omega = 1$, and the polynomial rank factorization of $\Pi_\ddagger^\omega(z)$ at $z = \frac{2}{3}$ gives $\alpha_{1,0}^\omega = (-\frac{1}{3} : 1)'$, $\beta_{1,0}^\omega = \frac{1}{6}(11 : 7)'$ and $\alpha_{1,1}^\omega = (3 : 1)'$, $\beta_{1,1}^\omega = \frac{4}{1275}(7 : -11)'$. Hence $X_t^\omega \in \mathrm{CD}(1)$ with $\gamma_0 = \beta_{1,0}^\omega = \frac{1}{6}(11 : 7)'$, see Theorem 4.4. We remark that these results have been directly derived from the polynomial rank factorization of the AR polynomial at $w_0 := 0$, $w_1 := \frac{2}{3}$, so that only the knowledge of $\Pi(z)$ is needed. Note that the two processes are indistinguishable from the perspective of the column spaces of the VAR coefficients, i.e. the intersection of their left null spaces does not fully determine the types of commonality in the process. Next we illustrate the duality result in (14); first we compute

$$k^\omega(z) := \det \Pi^\omega(z) = -\frac{1}{3}(2z - 3), \qquad k^\psi(z) := \det \Pi^\psi(z) = -\frac{1}{6}(z^3 - 2z^2 + 2z - 6)$$

and

$$K^\ell(z) := \operatorname{adj} \Pi^\ell(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -5 + 1_{\ell,\psi} & -4 \\ 6 & 3 \end{pmatrix} z - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1 : -1)\, z^2,$$

so that using $d_{k^\omega} = 1$, $d_{k^\psi} = 3$, $d_{K^\ell} = 2$, and (7) one immediately gets $m_0^\omega = 3$, $m_0^\psi = 1$. The polynomial rank factorization of $K_\ddagger^\omega(z)$ at 0 gives $\xi_0^\omega = (1 : -1)'$, $\eta_0^\omega = \frac{1}{2}(1 : -1)'$, $\xi_1^\omega = \xi_2^\omega = \eta_1^\omega = \eta_2^\omega = (0 : 0)'$ and $\xi_3^\omega = (1 : 1)'$, $\eta_3^\omega = -\frac{1}{3}(1 : 1)'$. Hence $\xi_0^\omega = -\frac{2}{3}\bar\beta_3^\omega$, $\eta_0^\omega = \bar\alpha_3^\omega$, $\xi_3^\omega = \bar\beta_0^\omega$ and $\eta_3^\omega = -\frac{2}{3}\bar\alpha_0^\omega$ and similar results holds for $\ell = \psi$ and for the other roots, see the comment on the presence of a factor structure in the adjoint in Cubadda, Hecq, and Palm (2009), Section 2.4.

7. Conclusion

The present paper characterizes the restrictions on the VAR coefficients that correspond 1-to-1 to the presence of common dynamic features of the CS, CD and CD type. These characterizations are associated with the polynomial rank factorizationof the reversed AR polynomial around its characteristic roots. The given characterizations equally apply to stationary VAR and VARs with

I(1) and I(2) variables. These conditions extend and complement the ones that are already available in the literature.

The approach adopted is an algebraic one, based on properties of a matrix polynomial and its adjoint and determinant. The duality results employed here may have a separate interest for other contexts where the inversion of matrix polynomials have application.

Appendix A. Proofs

Proof of Lemma 2.1. Because $\det \Pi(z_u) = 0$ one has $0 \le \operatorname{rank} \Pi(z_u) \le p-1$; when $0 \le \operatorname{rank} \Pi(z_u) < p-1$ one has $\operatorname{adj} \Pi(z_u) = 0$ and thus each entry of $\operatorname{adj} \Pi(z)$ contains the factor $(1-z/z_u)^{b_u}$ for some $0 < b_u < a_u$; if $\operatorname{rank} \Pi(z_u) = p-1$ then $\operatorname{adj} \Pi(z_u) \neq 0$ and thus $b_u = 0$. If $\operatorname{Im} z_u \neq 0$ then the same applies to z_u^*. Let $g(z) := \prod_{u=1}^{q} (1-z/z_u)^{m_u} =: (1-z/z_u)^{m_u} g_u(z)$; because $\operatorname{inv} \Pi(z) := \frac{\operatorname{adj} \Pi(z)}{\det \Pi(z)}$ and $G(z_u), g_u(z_u) \neq 0$ one has the last statement. This completes the proof. □

Proof of Theorem 3.1. Consider a characteristic root z_u; because $\Pi_\ddagger(1/z) = z^{-d_\Pi}\Pi(z)$ and $\Pi(z_u)$ is singular, then $z = w_u := 1/z_u$ is a point of rank-deficiency for $\Pi_\ddagger(z)$, i.e. w_u is a root of $\det \Pi_\ddagger(z)$. Because the same holds for any characteristic root, one then finds the factor $\prod_{u=1}^{q} (z-w_u)^{a_u}$, $a_u > 0$, in $\det \Pi_\ddagger(z)$. Moreover, we observe that $\lim_{|z|\to\infty} z^{-d_\Pi}\Pi(z) = \Pi_{d_\Pi} = \Pi_\ddagger(0)$; hence if Π_{d_Π} is singular then 0 is the point of rank-deficiency for $\Pi_\ddagger(z)$ which corresponds to the point of rank-deficiency at ∞ for $\Pi(z)$. This shows that

$$\det \Pi_\ddagger(z) = z^{a_0} k_\ddagger(z), \qquad k_\ddagger(0) \neq 0,$$

where $a_0 > 0$ and

$$\operatorname{adj} \Pi_\ddagger(z) = z^{b_0} K_\ddagger(z), \qquad K_\ddagger(0) \neq 0,$$

where $0 \le b_0 < a_0$. This implies $\operatorname{inv} \Pi_\ddagger(z) = z^{-c_0} \frac{G_\ddagger(z)}{g_\ddagger(z)}$, $z \in \mathbb{C} \setminus \{w_0, \dots, w_q\}$, where $G_\ddagger(0), g_\ddagger(0) \neq 0$ and $c_0 := a_0 - b_0 > 0$. Next we wish to show that $c_0 = m_0$. By the definition of the † and ‡ operators one has $\operatorname{inv} \Pi_\dagger(z) = z^{d_\Pi} \operatorname{inv} \Pi_\ddagger(z)$; hence, substituting for $\operatorname{inv} \Pi_\ddagger(z)$ one finds

$$\operatorname{inv} \Pi_\dagger(z) = z^{d_\Pi - c_0} \frac{G_\ddagger(z)}{g_\ddagger(z)}, \quad z \in \mathbb{C} \setminus \{w_0, \dots, w_q\}, \tag{16}$$

and because $\operatorname{inv} \Pi(z) = \frac{G(z)}{g(z)}$, one also has

$$\operatorname{inv} \Pi_\dagger(z) = \frac{G_\dagger(z)}{g_\dagger(z)} = z^{d_g - d_G} \frac{G_\ddagger(z)}{g_\ddagger(z)}, \quad z \in \mathbb{C} \setminus \{w_0, \dots, w_q\}. \tag{17}$$

Equating (16) and (17) one finds $d_\Pi - c_0 = d_g - d_G$, i.e. $c_0 = d_\Pi + d_G - d_g =: m_0$. This completes the proof. □

Proof of Theorem 3.2. The identity $\Pi(z)G(z) = G(z)\Pi(z) = g(z)I$ implies $\Pi_\ddagger(z)G_\ddagger(z) = G_\ddagger(z)\Pi_\ddagger(z) = z^{m_0} g_\ddagger(z) I$ so that

$$\det \Pi_{d_\Pi} = 0 \Leftrightarrow \det G_{d_G} = 0 \Leftrightarrow m_0 > 0 \tag{18}$$

where $\Pi_{d_\Pi}, G_{d_G} \neq 0$.

i.1) $\Leftrightarrow$ *i.2)*. See Theorem 3.1.

i.2) $\Rightarrow$ *i.3)*. If $m_0 > 0$ then, see (18), there exists $\gamma_0 \neq 0$ such that $\gamma_0' \Pi(z) = \gamma'(z)$ where $0 \le d_\gamma \le d_\Pi - 1$. Post-multiplying by $G(z)$ one finds $g(z)\gamma_0' = \gamma'(z)G(z)$; comparing degrees of the l.h.s. and the r.h.s. one has $d_g \le d_\gamma + d_G$, i.e. $d_\gamma \ge d_g - d_G = d_\Pi - m_0$.

i.3) $\Rightarrow$ *i.4)*. If $\det \Pi_{d_\Pi} = 0$ then, see (18), $\det G_{d_G} = 0$ and there exists $\gamma_0 \neq 0$ such that $\gamma_0' G(z) = \gamma'(z)$ where $0 \le d_\gamma \le d_G - 1$. Post-multiplying by $\Pi(z)$ one finds $g(z)\gamma_0' = \gamma'(z)\Pi(z)$; comparing degrees of the l.h.s. and the r.h.s. one has $d_g \le d_\gamma + d_\Pi$, i.e. $d_\gamma \ge d_g - d_\Pi = d_G - m_0$.

i.4) $\Rightarrow$ *i.1)*. If $\det G_{d_G} = 0$ then $\det \Pi_{d_\Pi} = 0$, see (18).

ii) By definition, $X_t \in \mathrm{CD}(d_\gamma)$ if and only if $\gamma_0' C(z) = \gamma'(z)$, i.e. $\gamma_0' G(z) = g(z)\gamma'(z)$. Comparing degrees of the l.h.s. and the r.h.s. one has $d_G \ge d_g + d_\gamma$ which implies $d_G - d_g = m_0 - d_\Pi \ge d_\gamma \ge 0$.

iii) $m_0 > 0$ does not imply $m_0 \ge d_\Pi$. This completes the proof. □

In Lemma A.1 below we present a result that will be used in the proof of Theorem 4.2.

Lemma A.1 (Π-Cancellations)**.** *Let the* $\ddagger$ *operator be as in* (6), $w_0 := 0$ *and* $w_u := 1/z_u$, $u = 1, \dots, q$, *define* $\Pi_\ddagger(z) = \sum_{n=0}^{d_\Pi} \Pi_n^{(u)}(z - w_u)^n$ *and let* $a_{u,j}$, $b_{u,j}$ *and* $\Pi_{j,k}^{(u)}$ *be as in Definition* 4.1*; then for* $0 \le n \le m_u - 1$, *the following statements are equivalent:*

- *i)* $v_u' \Pi_\ddagger(z) = (z - w_u)^{n+1} \zeta_u'(z)$ *and* $\zeta_u(w_u)$ *has full column rank;*
- *ii)* $v_u'(\Pi_{0,1}^{(u)} : \cdots : \Pi_{n,1}^{(u)}) = 0$ *and* $v_u' \Pi_{n+1,1}^{(u)}$ *has full row rank;*
- *iii)* $v_u = a_{u,n+1\perp} \varphi_u$ *where*
  - *iii.*1) $\varphi_u' a_{u,n+1\perp}' (\Pi_{1,1}^{(u)} b_{u,1} : \cdots : \Pi_{n,1}^{(u)} b_{u,n}) = 0$ *and* $\varphi_u' a_{u,n+1\perp}' \Pi_{n+1,1}^{(u)}$ *has full row rank;*
  - *iii.*2) $\varphi_u' a_{u,n+1\perp}' (\Pi_1^{(n)} b_{u,1} : \cdots : \Pi_n^{(u)} b_{u,n}) = 0$ *and* $\varphi_u' a_{u,n+1\perp}' \Pi_{n+1}^{(u)}$ *has full row rank.*

Proof. First we observe that *i)* holds if and only if $v_u'(\Pi_0^{(u)} : \cdots : \Pi_n^{(u)}) = 0$ and $v_u' \Pi_{n+1}^{(u)}$ has full row rank. Next we write $\Pi_{j,k}^{(u)}$ in (11) as $\Pi_{j,k}^{(u)} := \Pi_{j-1,k+1}^{(u)} + \Pi_{j-1,1}^{(u)} b_{u,j-1}(\times)$ where $(\times)$ depends on j, k and we recall that $\Pi_{0,k}^{(u)} := \Pi_{k-1}^{(u)}$, $\Pi_{1,k}^{(u)} = \Pi_k^{(u)}$.

i) $\Leftrightarrow$ *ii)*. We proceed by induction, assuming that $v_u' \Pi_j^{(u)} = 0$ implies $v_u' \Pi_{j,1}^{(u)} = 0$ for $j = 0, \dots, \ell - 1$ and proving it for $j = \ell$, where $1 \le \ell \le n$. Hence $v_u' \Pi_{1,k}^{(u)} = 0$ for $k = 0, \dots, \ell - 1$ holds by the induction assumption and one has $v_u' \Pi_{2,k}^{(u)} = v_u' \Pi_{1,k+1}^{(u)} = 0$ for $k = 1, \dots, \ell - 2$. By successive applications of the same argument one finds $v_u' \Pi_{j,k}^{(u)} = 0$ for $k = 1, \dots, \ell - j$; next consider (11) for $j = \ell$, $k = 1$ and note that

(19)
$$v_u' \Pi_{\ell,1}^{(u)} = v_u' \Pi_{\ell-1,2}^{(u)} + \underbrace{v_u' \Pi_{\ell-1,1}^{(u)}}_{=0} b_{u,\ell-1}(\times) = v_u' \Pi_{\ell-2,3}^{(u)} + \underbrace{v_u' \Pi_{\ell-2,1}^{(u)}}_{=0} b_{u,\ell-2}(\times) = \cdots = v_u' \Pi_{1,\ell}^{(u)} = v_u' \Pi_\ell^{(u)}$$

by the induction assumption; hence $v_u' \Pi_\ell^{(u)} = 0$ implies $v_u' \Pi_{\ell,1}^{(u)} = 0$ and this completes the proof by induction. Because $v_u' \Pi_{n+1,1}^{(u)} = v_u' \Pi_{n+1}^{(u)}$ one has $v_u' \Pi_{n+1}^{(u)}$ of full row rank if and only if $v_u' \Pi_{n+1,1}^{(u)}$ has full row rank. This completes the proof of sufficiency. Next assume that $v_u' \Pi_{j,1}^{(u)} = 0$ implies $v_u' \Pi_j^{(u)} = 0$ for $j = 0, \dots, \ell - 1$ and use (19) to complete the induction. This completes the proof.

ii) $\Leftrightarrow$ *iii.*1). If $v_u' \Pi_{j,1}^{(u)} = 0$ for $0 \le j \le n$, then $v_u' \Pi_{j,1}^{(u)} P_{b_{u,j}} = 0$ and $v_u' \Pi_{j,1}^{(u)} M_{b_{u,j}} = 0$; from $v_u' \Pi_{j,1}^{(u)} M_{b_{u,j}} = 0$ one has $v_u = a_{u,n+1\perp} \varphi_u$ for some $\varphi_u \ne 0$ because $v_u = a_{u,n+1} \xi$ for some $\xi \ne 0$ contradicts $v_u'(\Pi_{0,1}^{(u)} : \cdots : \Pi_{n,1}^{(u)}) = 0$, see (13). Combining $v_u' \Pi_{j,1}^{(u)} P_{b_{u,j}} = 0$ and $v_u = a_{u,n+1\perp} \varphi_u$ one finds $\varphi_u' a_{u,n+1\perp}' \Pi_{j,1}^{(u)} b_{u,j} = 0$ for $0 \le j \le n$. This completes the proof of sufficiency. Next assume $v_u = a_{u,n+1\perp} \varphi_u$ where $\varphi_u' a_{u,n+1\perp}' (\Pi_{0,1}^{(u)} : \cdots : \Pi_{n,1}^{(u)} b_{u,n}) = 0$ and $\varphi_u' a_{u,n+1\perp}' \Pi_{n+1,1}^{(u)}$ has full row rank; then $v_u' \Pi_{j,1}^{(u)} = \varphi_u' a_{u,n+1\perp}' \Pi_{j,1}^{(u)} P_{b_{u,j}} + \varphi_u' a_{u,n+1\perp}' \Pi_{j,1}^{(u)} M_{b_{u,j}} = \varphi_u' a_{u,n+1\perp}' \Pi_{j,1}^{(u)} P_{b_{u,j}}$ for $0 \le j \le n$ because $a_{u,n+1\perp}' M_{a_{u,j+1}} = a_{u,n+1\perp}'$ and (13). Because $\varphi_u' a_{u,n+1\perp}' (\Pi_{0,1}^{(u)} : \cdots : \Pi_{n,1}^{(u)} b_{u,n}) = 0$ one then has $\varphi_u' a_{u,n+1\perp}' \Pi_{j,1}^{(u)} P_{b_{u,j}} = 0$ for $0 \le j \le n$. Hence $v_u' \Pi_{j,1}^{(u)} = 0$ for $0 \le j \le n$ and $v_u' \Pi_{n+1,1}^{(u)}$ of full row rank. This completes the proof.

*iii.*1) $\Leftrightarrow$ *iii.*2). We proceed by induction, assuming that $v_u' \Pi_{j,1}^{(u)} b_{u,j} = 0$ implies $v_u' \Pi_j^{(u)} b_{u,j} = 0$ for $j = 0, \dots, \ell - 1$ and proving it for $j = \ell$, where $1 \le \ell \le n$. Consider (11) for $j = \ell$, $k = 1$ and note that

(20)
$$v_u' \Pi_{\ell,1}^{(u)} = v_u' \Pi_{\ell-1,2}^{(u)} + \underbrace{v_u' \Pi_{\ell-1,1}^{(u)} b_{u,\ell-1}}_{=0}(\times) = v_u' \Pi_{\ell-2,3}^{(u)} + \underbrace{v_u' \Pi_{\ell-2,1}^{(u)} b_{u,\ell-2}}_{=0}(\times) = \cdots = v_u' \Pi_{1,\ell}^{(u)} = v_u' \Pi_\ell^{(u)}$$

by the induction assumption; hence $v_u' \Pi_{\ell,1}^{(u)} b_{u,\ell} = 0$ implies $v_u' \Pi_\ell^{(u)} b_{u,\ell} = 0$ and this completes the proof of sufficiency. Next assume that $v_u' \Pi_j^{(u)} b_{u,j} = 0$ implies $v_u' \Pi_{j,1}^{(u)} b_{u,j} = 0$ for $j = 0, \dots, \ell - 1$ and use (20) to complete the induction. This completes the proof. $\square$

Proof of Theorem 4.2. One has $X_t \in \mathrm{CS}(d_\gamma)$ if and only if $\gamma_0'\Pi(z) = \gamma'(z)$, $z \in \mathbb{C}$, where $\max(0, d_\Pi - m_0) \leq d_\gamma \leq d_\Pi - 1$ and $\gamma(z)$ has full column rank. We map z into z^{-1} and write the last equation as

$$\gamma_0'\Pi_\ddagger(z) = z^{d_\Pi - d_\gamma}\zeta_0'(z), \quad z \in \mathbb{C},$$

where $\zeta_0(z) := \gamma_\ddagger(z)$. This fits the assumptions of Lemma A.1 with $u := 0$, $v_0 := \gamma_0$ and $n := d_\Pi - d_\gamma - 1$. Note that $v_0 \in \mathbb{R}^{p\times r}$, where $r = \sum_{j=d_\Pi-d_\gamma}^{m_0} r_{0,j}$. This completes the proof. □

For the proofs of Theorem 4.3, 4.4 we apply the results of Corollary A.2 A.6; the former is a consequence of Lemma A.1 and the duality result in (14), the latter of Lemma A.3, A.4 and A.5 below.

Corollary A.2 (G-Cancellations). *Let the $\ddagger$ operator be as in (6), $w_0 := 0$ and $w_u := 1/z_u$, $u = 1, \ldots, q$, define $G_\ddagger(z) = \sum_{n=0}^{d_G} G_n^{(u)}(z - w_u)^n$ and let $a_{u,j}$, $b_{u,j}$ be as in Definition 4.1; then for $0 \leq n \leq m_u - 1$, the following statements are equivalent:*

- *i)* $v_u'G_\ddagger(z) = (z - w_u)^{n+1}\zeta_u'(z)$ *and* $\zeta_u(w_u)$ *has full column rank;*
- *ii)* $v_u = b_{u,m_u-n}\varphi_u$ *where* $\varphi_u'b_{u,m_u-n}'(G_1^{(u)}a_{u,m_u\perp} : \cdots : G_n^{(u)}a_{u,m_u-n+1\perp}) = 0$ *and* $\varphi_u'b_{u,m_u-n}'G_{n+1}^{(u)}$ *has full row rank.*

Proof. Replace Π with G in the proof of Lemma A.1 and use the duality result in (14). This completes the proof. □

Lemma A.3. *Let $\alpha_{u,j}$, $\beta_{u,j}$, $\Pi_{j,k}^{(u)}$ and $\xi_{u,j}$, $\eta_{u,j}$, $G_{j,k}^{(u)}$ be respectively defined by the polynomial rank factorizations of $\Pi_\ddagger(z)$ and $G_\ddagger(z) = \sum_{n=0}^{d_G} G_n^{(u)}(z - w_u)^n$ at $z = w_u$, $g_\ddagger(z) =: (z - w_u)^{m_u}g_{\ddagger,u}(z)$ and $h_u := g_{\ddagger,u}(w_u) \neq 0$; then for $0 \leq j \leq n \leq m_u$, one has*

$$\beta_{u,j}'G_{n-j}^{(u)} = \bar\alpha_{u,j}'\sum_{k=1}^{n-j}\Pi_{j+1,k}^{(u)}G_{n-j-k}^{(u)} - 1_{n,m_u}h_u\bar\alpha_{u,j}', \tag{21}$$

where $1_{n,m_u}$ is the indicator function. Moreover, for $0 \leq j \leq s-1 \leq n-1 \leq m_u - 1$, one has

$$\beta_{u,j}'G_{t,s-j}^{(u)}\alpha_{u,m_u-t} = \bar\alpha_{u,j}'\sum_{k=1}^{s-j-1}\Pi_{j+1,k}^{(u)}G_{t,s-j-k}^{(u)}\alpha_{u,m_u-t} - 1_{t+s-1,m_u}h_u\bar\alpha_{u,j}'\alpha_{u,m_u-t}, \tag{22}$$

Proof. See Franchi and Paruolo (2009). □

Lemma A.4. *For $t = 1, \ldots, n \leq m_u - 1$, one has*

$$v_u'G_t^{(u)}\alpha_{u,m_u-t+1} = 0 \Leftrightarrow \sum_{j=0}^{m_u-n-1}\varphi_{u,j}'\bar\alpha_{u,j}'\Pi_{j+1,1}^{(u)}\beta_{u,m_u-t+1} = 0. \tag{23}$$

Proof. First we show that (23) holds for $t = 1$; let $n \leq m_u - 1$, $n - j = 1$ in (21) to get

$$\beta_{u,j}'G_1^{(u)} = \bar\alpha_{u,j}'\Pi_{j+1,1}^{(u)}G_0^{(u)}; \tag{24}$$

substitute for $G_0^{(u)} = -h_u\bar\beta_{u,m_u}\bar\alpha_{u,m_u}'$, see (14), pre and post-multiply by $\varphi_{u,j}'$ and α_{u,m_u} respectively to find $\varphi_{u,j}'\beta_{u,j}'G_1^{(u)}\alpha_{u,m_u} = -h_u\varphi_{u,j}'\bar\alpha_{u,j}'\Pi_{j+1,1}^{(u)}\bar\beta_{u,m_u}$; hence (23) holds for $t = 1$, because $v_u = b_{u,m_u-n}\varphi_u = \sum_{j=0}^{m_u-n-1}\beta_{u,j}\varphi_{u,j}$. Next we proceed by induction, assuming that (23) holds for $t = 1, \ldots, \ell - 1$ and proving it for $t = \ell$, where $1 \leq \ell \leq n$. For $s - j = 1$, (22) gives

$$\beta_{u,j}'G_{t,1}^{(u)}\alpha_{u,m_u-t} = -1_{t+j,m_u}h_u\bar\alpha_{u,j}'\alpha_{u,m_u-t} \tag{25}$$

and for $n \leq m_u - 1$, $s - j = 2$ it gives

$$\beta_{u,j}'G_{t,2}^{(u)}\alpha_{u,m_u-t} = \bar\alpha_{u,j}'\Pi_{j+1,1}^{(u)}G_{t,1}^{(u)}\alpha_{u,m_u-t}. \tag{26}$$

Because $G^{(u)}_{t,1}\alpha_{u,m_u-t} = \sum_{j=0}^{m_u} \bar\beta_{u,j}\beta'_{u,j}G^{(u)}_{t,1}\alpha_{u,m_u-t} =$

$$-h_u\bar\beta_{u,m_u-t} + \sum_{j=m_u-t+1}^{m_u} \bar\beta_{u,j}\beta'_{u,j}G^{(u)}_{t,1}\alpha_{u,m_u-t} \tag{27}$$

follows from (25), (26) is then rewritten as

$$\beta'_{u,j}G^{(u)}_{t,2}\alpha_{u,m_u-t} = -h_u\bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}\bar\beta_{u,m_u-t} + \bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}b_{u,m_u-t+1\perp}(\times)$$

where $(\times)$ depends on j,t. This implies

$$v'_uG^{(u)}_{t,2}\alpha_{u,m_u-t} = -h_u\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}\bar\beta_{u,m_u-t} + \sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}b_{u,m_u-t+1\perp}(\times) \tag{28}$$

From the counterpart of (20) for $G^{(u)}_{\cdot,\cdot}$ and the duality result in (14) one has $v'_uG^{(u)}_{t,2} = v'_uG^{(u)}_{t+1}$; moreover, the induction assumption implies $\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}\beta_{u,m_u-t+1} = 0$ for $t = 1,\dots,\ell-1$ so that $\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}b_{u,m_u-\ell+2\perp} = 0$. Hence for $t = \ell-1$, (28) gives

$$v'_uG^{(u)}_{\ell}\alpha_{u,m_u-\ell+1} = -h_u\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{j+1,1}\bar\beta_{u,m_u-\ell+1},$$

i.e. (23) for $t = \ell$. This completes the proof. □

Lemma A.5. *Let $\Pi^{(u)}_{s,j,k}$ be as in Definition 4.1 and assume*

$$\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{s,j+1,1}b_{u,m_u-t+1\perp} = 0;$$

then for $t = 1,\dots,n \le m_u-1$, one has

$$v'_uG^{(u)}_{t,s+1}\alpha_{u,m_u-t} = -h_u\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{s,j+1,1}\bar\beta_{u,m_u-t}.$$

Proof. Assume that for $\ell = 1,\dots,\kappa$, one has

$$v'_uG^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} = \sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\sum_{k=1}^{\kappa-\ell+1}\Pi^{(u)}_{\ell,j+1,k}G^{(u)}_{t,\kappa+2-\ell-k}\alpha_{u,m_u-t}; \tag{29}$$

then for $\ell = \kappa$, one has

$$v'_uG^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} = \sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{\kappa,j+1,1}G^{(u)}_{t,1}\alpha_{u,m_u-t}.$$

Because $G^{(u)}_{t,1}\alpha_{u,m_u-t} = -h_u\bar\beta_{u,m_u-t} + b_{u,m_u-t+1\perp}\bar b'_{u,m_u-t+1\perp}G^{(u)}_{t,1}\alpha_{u,m_u-t}$, see (27), one then has $v'_uG^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} =$

$$-h_u\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{\kappa,j+1,1}\bar\beta_{u,m_u-t} + \sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{\kappa,j+1,1}b_{u,m_u-t+1\perp}\bar b'_{u,m_u-t+1\perp}G^{(u)}_{t,1}\alpha_{u,m_u-t}.$$

Hence $\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{\kappa,j+1,1}b_{u,m_u-t+1\perp} = 0$ implies

$$v'_uG^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} = -h_u\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar\alpha'_{u,j}\Pi^{(u)}_{\kappa,j+1,1}\bar\beta_{u,m_u-t}.$$

Next we show that (29) holds. Let $t+s-1 \leq m_u-1$, $s-j=\kappa+1$ in (22), pre-multiply by $\varphi'_{u,j}$ and sum over j to get

$$v'_u G^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} = \sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\sum_{k=1}^{\kappa}\Pi^{(u)}_{j+1,k}G^{(u)}_{t,\kappa+1-k}\alpha_{u,m_u-t};$$

this shows (29) for $\ell=1$ because $\Pi^{(u)}_{1,j,k} := \Pi^{(u,j,k}$. Next we proceed by induction assuming (29) for $\ell = 1,\ldots,\tau$ and showing it for $\ell=\tau+1\leq\kappa$. For $\ell=\tau$, (29) is rewritten as

$$v'_u G^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} = \sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\Pi^{(u)}_{\tau,j+1,1}G^{(u)}_{t,\kappa+1-\tau}\alpha_{u,m_u-t}+$$

$$\sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\sum_{k=1}^{\kappa-\tau}\Pi^{(u)}_{\tau,j+1,k+1}G^{(u)}_{t,\kappa+1-\tau-k}\alpha_{u,m_u-t} =: A+B \text{ (say)};$$

next insert $I=\sum_{h=0}^{m_u}\bar{\beta}_{u,h}\beta'_{u,h}$ between $\Pi^{(u)}_{\tau,j+1,1}$ and $G^{(u)}_{t,\kappa+1-\tau}$ to find

$$A = \sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\Pi^{(u)}_{\tau,j+1,1}\sum_{h=0}^{m_u}\bar{\beta}_{u,h}\beta'_{u,h}G^{(u)}_{t,\kappa+1-\tau}\alpha_{u,m_u-t} =$$

$$\sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\Pi^{(u)}_{\tau,j+1,1}\sum_{h=0}^{m_u-t}\bar{\beta}_{u,h}\beta'_{u,h}G^{(u)}_{t,\kappa+1-\tau}\alpha_{u,m_u-t},$$

where the second equality follows from $\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar{\alpha}'_{u,j}\Pi^{(u)}_{\tau,j+1,1}b_{u,m_u-t+1\perp}=0$. Next let $t+s-1\leq m_u-1$, $s-j=\kappa+1-\tau$ in (22) to get

$$\beta'_{u,j}G^{(u)}_{t,\kappa+1-\tau}\alpha_{u,m_u-t} = \bar{\alpha}'_{u,j}\sum_{k=1}^{\kappa-\tau}\Pi^{(u)}_{j+1,k}G^{(u)}_{t,\kappa+1-\tau-k}\alpha_{u,m_u-t},$$

so that

$$A = \sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\sum_{k=1}^{\kappa-\tau}\left(\Pi^{(u)}_{\tau,j+1,1}\sum_{h=0}^{m_u-t}\bar{\beta}_{u,h}\bar{\alpha}'_{u,h}\Pi^{(u)}_{h+1,k}\right)G^{(u)}_{t,\kappa+1-\tau-k}\alpha_{u,m_u-t}.$$

Hence $A+B=$

$$v'_u G^{(u)}_{t,\kappa+1}\alpha_{u,m_u-t} = \sum_{j=0}^{m_u-n-1} \varphi'_{u,j}\bar{\alpha}'_{u,j}\sum_{k=1}^{\kappa-\tau}\Pi^{(u)}_{\tau+1,j+1,k}G^{(u)}_{t,\kappa+1-\tau-k}\alpha_{u,m_u-t},$$

where

$$\Pi^{(u)}_{\tau+1,j+1,k} := \Pi^{(u)}_{\tau,j+1,k+1} + \Pi^{(u)}_{\tau,j+1,1}\sum_{h=0}^{m_u-t}\bar{\beta}_{u,h}\bar{\alpha}'_{u,h}\Pi^{(u)}_{1,h+1,k},$$

see (12); hence (29) holds for $\ell=\tau+1$. This completes the proof. □

Corollary A.6. *Let $\Pi^{(u)}_{s,j,k}$ be as in Definition 4.1 and $v_u := b_{u,m_u-n}\varphi_u$; then for $1\leq s\leq n\leq m_u-1$, one has*

$$v'_u(G^{(u)}_1 a_{u,m_u\perp} : \cdots : G^{(u)}_n a_{u,m_u-n+1\perp}) = 0 \Leftrightarrow \sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar{\alpha}'_{u,j}\Pi^{(u)}_{s,j+1,1}b_{u,m_u-n+s}=0. \tag{30}$$

Moreover,

$$v'_u G^{(u)}_{n+1} = -\varphi'_{u,m_u-n-1}\beta'_{u,m_u-n-1}\beta_{u,m_u-n-1}\alpha'_{u,m_u-n-1} - H_u\bar{a}'_{u,m_u-n\perp}, \tag{31}$$

with $H_u := h_u\sum_{j=0}^{m_u-n-1}\varphi'_{u,j}\bar{\alpha}'_{u,j}(\Pi^{(u)}_{1,j+1,1}\beta_{u,m_u-n} : \cdots : \Pi^{(u)}_{n+1,j+1,1}\beta_{u,m_u})$.

Proof. The counterpart of (20) for $G^{(u)}_{\cdot,\cdot}$ and the duality result in (14) imply $v_u' G^{(u)}_{t,s+1} = v_u' G^{(u)}_{t+s}$; then, see Lemma A.5, for $1 \le s \le t \le n \le m_u - 1$, one has

$$v_u' G^{(u)}_t \alpha_{u,m_u-t+s} = -h_u \sum_{j=0}^{m_u-n-1} \varphi_{u,j}' \bar{\alpha}_{u,j}' \Pi^{(u)}_{s,j+1,1} \beta_{u,m_u-t+s}. \tag{32}$$

Write $v_u'(G^{(u)}_1 a_{u,m_u\perp} : \cdots : G^{(u)}_n a_{u,m_u-n+1\perp}) = 0$, where $v_u := b_{u,m_u-n}\varphi_u$, as $v_u' G^{(u)}_t \alpha_{u,m_u-t+s} = 0$, for $1 \le s \le t \le n \le m_u - 1$. This proves (30). Next write $v_u' G^{(u)}_{n+1} = v_u' G^{(u)}_{n+1} P_{a_{u,m_u-n}} + v_u' G^{(u)}_{n+1} M_{a_{u,m_u-n}} =: A + B$ (say). The duality result in (14) together with the counterpart of (13) for G imply

$$A = -\varphi_{u,m_u-n-1}' \beta_{u,m_u-n-1}' \beta_{u,m_u-n-1} \alpha_{u,m_u-n-1}'$$

and using (32) one finds

$$B = -h_u \sum_{j=0}^{m_u-n-1} \varphi_{u,j}' \bar{\alpha}_{u,j}' (\Pi^{(u)}_{1,j+1,1} \beta_{u,m_u-n} : \cdots : \Pi^{(u)}_{n+1,j+1,1} \beta_{u,m_u}) \bar{a}_{u,m_u-n\perp}'.$$

This completes the proof. □

Proof of Theorem 4.3. One has $X_t \in \mathrm{CE}(d_\gamma)$ if and only if $\gamma_0' G(z) = \gamma'(z)$, $z \in \mathbb{C}$, where $\max(0, d_G - m_0) \le d_\gamma \le d_\Pi - 1$ and $\gamma(z)$ has full column rank. We map z into z^{-1} and write the last equation as

$$\gamma_0' G_\ddagger(z) = z^{d_G - d_\gamma} \zeta_0'(z), \quad z \in \mathbb{C},$$

where $\zeta_0(z) := \gamma_\ddagger(z)$. This fits the assumptions of Corollary A.2 with $u := 0$, $v_0 := \gamma_0$ and $n := d_G - d_\gamma - 1$. Note that $v_0 \in \mathbb{R}^{p\times r}$, where $r = \sum_{j=0}^{m_0-d_G+d_\gamma} r_{0,j}$. This proves $i) - iv.1)$; in order to prove $iv.2)$ apply Corollary A.6 for $n := d_G - d_\gamma - 1$. This completes the proof. □

In Lemma A.8, A.9, below we present results that will be used in the proof of Theorem of 4.4. Before that, Lemma A.7 below states that the left null space of a pair of complex conjugate matrices contains real vectors.

Lemma A.7 (Real left null space). *Let $A \in \mathbb{C}^{p\times p}$; then $v'A = v'A^* = 0$ for some $v \in \mathbb{C}^p$ if and only if $(\operatorname{Re} v)'(\operatorname{Re} A : \operatorname{Im} A) = (\operatorname{Im} v)'(\operatorname{Re} A : \operatorname{Im} A) = 0$. One can choose to represent the intersection of the left null spaces of $\operatorname{col} A$ and $\operatorname{col} A^*$, i.e. $\mathcal{V} := \{v \in \mathbb{C}^p : v'A = v'A^* = 0\}$ with the real left null space of $\operatorname{col}(\operatorname{Re} A : \operatorname{Im} A)$, namely $\mathcal{U} := \{u \in \mathbb{R}^p : u'(\operatorname{Re} A : \operatorname{Im} A) = 0\}$, in the sense that $\mathcal{U} = \mathcal{V} \cap \mathbb{R}^p$ and $\mathcal{V} = \mathcal{U} + \mathrm{i}\mathcal{U}$ (or $\mathcal{V} = \mathcal{U} \oplus \mathrm{i}\mathcal{U}$).*

Proof. Because $v'A = v'A^* = 0$ one has $0 = v'\frac{1}{2}(A + A^*) = v' \operatorname{Re} A$ and $0 = v'\frac{\mathrm{i}}{2}(-A + A^*) = v' \operatorname{Im} A$; this implies $\operatorname{Re} v'(\operatorname{Re} A : \operatorname{Im} A) = \operatorname{Im} v'(\operatorname{Re} A : \operatorname{Im} A) = 0$. Reading these implications in reverse order one obtains the reverse implications. The relation $\mathcal{V} = \mathcal{U} + \mathrm{i}\mathcal{U}$ readily follows. This completes the proof. □

The next result is an extension of the expansion formula of Johansen (1931).

Lemma A.8. *Let $f(z)$, $z \in \mathbb{C}$, be holomorphic for $z \in U \subseteq \mathbb{C}$, and consider n distinct points $z_1, \ldots, z_n$ in U; then one has*

$$f(z) = b(z) + v(z),$$

where $b(z)$ is a polynomial such that for $s = 0, \dots, \ell_u - 1 \geq 0$ and $u = 1, \dots, n$, one has $b^{(s)}(z_u) = f^{(s)}(z_u)$ and $v(z)$ is a remainder. In particular,

$$v(z) := a(z)r(z), \qquad a(z) := \prod_{u=1}^{n}(z - z_u)^{\ell_u},$$

$$b(z) := \sum_{u=1}^{n} a_u(z)b_u(z), \qquad a_u(z) := \frac{a(z)}{(z - z_u)^{\ell_u}},$$

$$b_u(z) := \sum_{s=0}^{\ell_u - 1} b_{u,s}(z - z_u)^s, \qquad b_{u,s} := \frac{g_u^{(s)}(z_u)}{s!}, \qquad g_u(z) := \frac{f(z)}{a_u(z)}.$$

The degrees of $a(z)$ and $b(z)$ are respectively $d_a = \sum_{u=1}^{n} \ell_u$ and $d_b = d_a - 1$. Finally $1/a(z)$ and $r(z)$ are co-prime.

Proof. See Franchi and Paruolo (2009). □

We apply Lemma A.8 letting $f(z) := G_{\ddagger}(z), a(z) := z^h g_{\ddagger}(z)$; this gives

$$G_{\ddagger}(z) = D(z) + z^h g_{\ddagger}(z)S(z), \qquad D(z) := \sum_{u=0}^{q} a_u(z)D_u(z). \tag{33}$$

Lemma A.9 (C-Cancellations). *Let the $\ddagger$ operator be as in (6), $w_0 := 0$ and $w_u := 1/z_u$, $u = 1, \dots, q$, define $G_{\ddagger}(z) = \sum_{n=0}^{d_G} G_n^{(u)}(z - w_u)^n$, and let $a_{u,j}$, $b_{u,j}$ and $\Pi_{s,j,k}^{(u)}$ be as in Definition 4.1; then for $0 \leq h \leq m_0 - d_{\Pi}$, the following statements are equivalent:*

- *i)* $v' G_{\ddagger}(z) = z^h g_{\ddagger}(z)\zeta'(z)$, *where* $\zeta'(z) := v' S(z)$;
- *ii)* $v'(G_0^{(u)} : \cdots : G_{\ell_u - 1}^{(u)}) = 0$, *where* $u = 0, \dots, q$, $\ell_0 := h$ *and* $\ell_u := m_u$, $u \neq 0$;
- *iii)* *let* $\ell := m_0 - h + 1$; *then*

$$\operatorname{col} v = \cap_{u=0}^{q} \operatorname{col} v_u \in \mathbb{R}^{p \times r},$$

where $v_0 = b_{0,\ell}\varphi_0$, $v_u = \beta_{u,0}\varphi_u$, $u \neq 0$,

iii.1) *and where* $\varphi_0' b_{0,\ell}'(G_1^{(0)} a_{0,m_0\perp} : \cdots : G_{h-1}^{(0)} a_{0,\ell+1\perp}) = 0$ *and* $\varphi_0' b_{0,\ell}' G_h^{(0)}$ *of full row rank and for* $u \neq 0$, $\varphi_u' \beta_{u,0}'(G_1^{(u)} a_{u,m_u\perp} : \cdots : G_{m_u - 1}^{(u)} a_{u,2\perp}) = 0$ *and* $\varphi_u' \beta_{u,0}' G_{m_u}^{(u)}$ *of full row rank.*

iii.2) *and where* $\sum_{j=0}^{m_0 - h} \varphi_{0,j}' \bar{\alpha}_{0,j}'(\Pi_{1,j+1,1}^{(0)} b_{0,m_0-h+2} : \cdots : \Pi_{h-1,j+1,1}^{(0)} b_{0,m_0}) = 0$ *and*

$$-\varphi_{0,m_0-h}' \beta_{0,m_0-h}' \beta_{0,m_0-h} \alpha_{0,m_0-h}' - H_0 \bar{\alpha}_{0,m_0-h+1\perp}',$$

with $H_0 := h_0 \sum_{j=0}^{m_0-h} \varphi_{0,j}' \bar{\alpha}_{0,j}'(\Pi_{1,j+1,1}^{(0)} \beta_{0,m_0-h+1} : \cdots : \Pi_{h,j+1,1}^{(0)} \beta_{0,m_0})$, *has full row rank and for* $u \neq 0$, $\varphi_{u,0}' \bar{\alpha}_{u,0}'(\Pi_{1,1,1}^{(u)} b_{u,2} : \cdots : \Pi_{m_u-1,1,1}^{(u)} b_{u,m_u}) = 0$ *and*

$$-\varphi_{u,0}' \beta_{u,0}' \beta_{u,0} \alpha_{u,0}' - H_u \bar{\alpha}_{u,1\perp}',$$

with $H_u := h_u \varphi_{u,0}' \bar{\alpha}_{u,0}'(\Pi_{1,1,1}^{(u)} \beta_{u,1} : \cdots : \Pi_{m_u,1,1}^{(u)} \beta_{u,m_u})$, *has full row rank.*

Proof. i) $\Leftrightarrow$ *ii)* Pre-multiplying both sides of (33) by v' one has $v' G_{\ddagger}(z) = z^h g_{\ddagger}(z)\zeta'(z)$, where $\zeta'(z) := v' S(z)$, if and only if $v' D(z) = 0$; next we show that

$$v' D(z) = 0 \Leftrightarrow v' D_u(z) = 0, \quad u = 0, \dots, q. \tag{34}$$

The proof of ($\Leftarrow$) is immediate; to see that ($\Rightarrow$) also holds, assume there exists j such that $v' D_j(z) \neq 0$ and $v' D(z) = 0$. Then

$$v' a_j(z) D_j(z) = -v' \sum_{u=0, u \neq j}^{q} a_u(z) D_u(z),$$

where $v'a_j(z_j)D_j(z_j) \neq 0$ and $a_u(z_j)D_u(z_j) = 0$ for $u \neq j$. Hence we reach a contradiction and conclude that there cannot be such j. This completes the proof of (34). Next we observe that because $D_u(z) = \sum_{n=0}^{\ell_u-1} D_{u,n}(z-w_u)^n$ one has

$$v'D_u(z) = 0 \Leftrightarrow v'D_{u,n} = 0, \quad n = 0, \dots, \ell_u - 1,$$

and further we note that $D_{u,n} = \sum_{j=0}^{n} b_{u,j} G_{n-j}^{(u)}$, where $b_{u,j}$ is a scalar. Hence one has

$$v'D_{u,n} = 0, \quad n = 0, \dots, \ell_u - 1 \Leftrightarrow v'(G_0^{(u)} : \dots : G_{\ell_u-1}^{(u)}) = 0.$$

This completes the proof.

ii) $\Leftrightarrow$ *iii.*1) The result is found by applying Corollary A.2 at w_u, $u = 0, \dots, q$, letting $n := \ell_u - 1$, where $\ell_0 := h$ and $\ell_u := m_u$, $u \neq 0$; this gives $v_u'(G_0^{(u)} : \dots : G_{\ell_u-1}^{(u)}) = 0$ and $v_u' G_{\ell_u}^{(u)}$ of full row rank if and only if

$$v_u = b_{u,m_u-\ell_u+1}\varphi_u \text{ where } \varphi_u' b_{u,m_u-\ell_u+1}'(G_1^{(u)} a_{u,m_u\perp} : \dots : G_{\ell_u-1}^{(u)} a_{u,m_u-\ell_u+2\perp}) = 0 \tag{35}$$

and $\varphi_u' b_{u,m_u-n}' G_{\ell_u}^{(u)}$ of full row rank. Because v must satisfy (35) for $u = 0, \dots, q$ then it must be a basis of $\cap_{u=0}^{q} \operatorname{col} v_u$. Conversely, if v is basis of $\cap_{u=0}^{q} \operatorname{col} v_u$, then it satisfies (35) for $u = 0, \dots, q$. Because v belongs to the left null space of pairs of complex conjugate matrices, Lemma A.7 applies and one can choose $v \in \mathbb{R}^p$. This completes the proof.

*iii.*1) $\Leftrightarrow$ *iii.*2) The result is found by applying Corollary A.6 at w_u, $u = 0, \dots, q$, letting $n := \ell_u - 1$, where $\ell_0 := h$ and $\ell_u := m_u$, $u \neq 0$. This completes the proof. ☐

Proof of Theorem 4.4. One has $X_t \in \mathrm{CD}(d_\gamma)$ if and only if $\gamma_0' \frac{G(z)}{g(z)} = \gamma'(z)$, $z \in \mathbb{C}$, where $0 \leq d_\gamma \leq m_0 - d_\Pi$ has full column rank. We map z into z^{-1} and write the last equation as

$$\gamma_0' \frac{G_\ddagger(z)}{z^{m_0-d_\Pi-d_\gamma} g_\ddagger(z)} = \zeta'(z), \quad z \in \mathbb{C},$$

where $\zeta(z) := \gamma_\ddagger'(z)$. This fits the assumptions of Lemma A.9 with $v := \gamma_0$ and $h := m_0 - d_\Pi - d_\gamma$. This completes the proof. ☐

In order to find the Laurent series representation of $\operatorname{inv}\Pi(z)$, see (37) below, we apply Lemma A.8 to $f(z) := G(z)$ and $a(z) := \prod_{u=1}^{q}(z - z_u)^{m_u} = c_\rho g(z)$, where $c_\rho := \prod_{u=1}^{q}(-z_u)^{-m_u}$; this gives

$$G(z) = \widetilde{B}(z) + a(z)\widetilde{R}(z), \qquad \widetilde{B}(z) := \sum_{u=1}^{q} a_u(z)\widetilde{B}_u(z), \tag{36}$$

where the matrix polynomials $\widetilde{B}(z) := b(z)$, $\widetilde{R}(z) := r(z)$ have degrees $d_{\widetilde{B}} = \sum_{u=1}^{q} m_u - 1$, $d_{\widetilde{R}} = d_G - d_g$ respectively. Dividing both sides of (36) by $g(z)$, one has

$$\operatorname{inv}\Pi(z) = \sum_{u=1}^{q} \frac{B_u(z)}{(1 - w_u z)^{m_u}} + R(z), \quad z \in \mathbb{C} \setminus \{z_1, \dots, z_q\}, \tag{37}$$

where $B_u(z) := (-z_u)^{m_u} c_\rho \widetilde{B}_u(z)$ and $R(z) := c_\rho \widetilde{R}(z)$.

Proof of Lemma 5.1. Let $B = (\beta_0 : \beta_1)'$, $D(z) := \operatorname{diag}(I_r, (1-z)I_{p-r})$ and $X_t := D(z)BY_t$ and pre-multiply (15) by B to obtain $\widetilde{A}(L)BY_t = Bu_t$ where $\widetilde{A}(z) := BA(z)B^{-1}$ with $B^{-1} = (\bar{\beta} : \bar{\beta}_1)$

$$\widetilde{A}(z) = \begin{pmatrix} \beta_0'\Gamma(z)\bar{\beta}(1-z) - \beta_0'\alpha z & \beta'\Gamma(z)\bar{\beta}_1 \\ \beta_1'\Gamma(z)\bar{\beta}(1-z) - \beta_1'\alpha z & \beta_1'\Gamma(z)\bar{\beta}_1 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & (1-z)I_{p-r} \end{pmatrix} =: \Pi(z)D(z)$$

This shows that $\Pi(L)X_t = \varepsilon_t$. One sees that the last $p - r$ columns of $\Pi(z)$ have degree $d_\Gamma = d_\Pi - 1$, which implies $\Pi_{d_\Pi} = (\Pi_{d_\Pi,0} : 0_{p\times(p-r)})$. By properties of the determinant one sees that

$\det \widetilde{A}(z) = \det \Pi(z) \det D(z)$, so that the roots of $\det \widetilde{A}(z) = \det B \det A(z) \det B^{-1} = \det A(z)$ include the roots of $\det \Pi(z)$ plus the roots of $\det D(z)$, which are $p - r$ roots at $z = 1$. $\square$

References

Ahn, S., and C. Reinsel (1988) Nested Reduced Rank Autoregressive Models for Multiple Time Series. *Journal of the American Statistical Association*, 83, 849–56.

Brockwell, P., and R. Davis (1987) *Time Series: Theory and Methods.* Springer-Verlag.

Campbell, J. Y., and N. G. Mankiw (1990) Permanent Income, Current Income, and Consumption. *Journal of Business & Economic Statistics*, 8(3), 265–279.

Campbell, J. Y., and R. J. Shiller (1987) Cointegration and Tests of Present Value Models. *The Journal of Political Economy*, 95, 1062–1088.

Cavaliere, G., L. Fanelli, and A. Gardini (2008) International dynamic risk sharing. *Journal of Applied Econometrics*, 23, 1–16.

Cubadda, G., and A. Hecq (2001) On non-contemporaneous short-run comovements. *Economics Letters*, 73, 389–397.

Cubadda, G., A. Hecq, and F. Palm (2009) Studying Co-movements in Large Multivariate Models Without Multivariate Modelling. *Journal of Econometrics*, 148, 25–35.

Engle, R., and C. Granger (1987) Co-integration and Error Correction: Representation, Estimation, and Testing. *Econometrica*, 55, 251–276.

Engle, R., and S. Kozicki (1993) Testing for common features. *Journal of Business and Economic Statistics*, 11, 369–395.

Franchi, M. (2009) A representation theory for polynomial cofractionality in vector autoregressive models. *Econometric Theory*, forthcoming.

Franchi, M., and P. Paruolo (2009) Subspace duality in the inversion of square matrix polynomials. Mimeo.

Fuller, W. A. (1996) *Introduction To Statistical Time Series.* John Wiley & Sons.

Gourieroux, C., and I. Peaucelle (1988) Detecting a long run relationship (with application to the P.P.P. hypothesis). *CEPREMAP*, Working Paper 8902.

Greene, R., and S. Krantz (1997) *Function theory of one complex variable.* John Wiley & Sons.

Hall, R. E. (1978) Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence. *The Journal of Political Economy*, 86(6), 971–987.

Hecq, A., F. Palm, and J. Urbain (2002) Separation, Weak Exogeneity and P-T Decompositions in Cointegrated VAR Systems with Common Features. *Econometric reviews*, 21,3, 273–307.

——— (2006) Common cyclical features analysis in VAR models with cointegration. *Journal of Econometrics*, 132, 117–141.

Johansen, P. (1931) Über osculierende Interpolation. *Skandinavisk Aktuarietidsskrift*, 14, 231–237.

Johansen, S. (1992) A representation of vector autoregressive processes integrated of order 2. *Econometric Theory*, 8, 188–202.

——— (1996) *Likelihood-based Inference in Cointegrated Vector Auto-Regressive Models.* Oxford University Press.

——— (2009) Cointegration. Overview and Development. in *T.G. Andersen, R. Davis, J-P. Kreiss, and T. Mikosch (eds.) Handbook of Financial Time Series*, pp. 671–693. Springer, New York.

King, R. G., C. I. Plosser, J. H. Stock, and M. W. Watson (1991) Stochastic Trends and Economic Fluctuations. *The American Economic Review*, 81(4), 819–840.

Kugler, P., and K. Neusser (1993) International Real Interest Rate Equalization: A Multivariate Time-Series Approach. *Journal of Applied Econometrics*, 8(2), 163–174.

Lippi, M., and L. Reichlin (1994) Common and uncommon trends and cycles. *European Economic Review*, 38, 624–635.

Paruolo, P. (2003) Common dynamics in I(1) systems. Discussion paper, WP Facoltà di Economia 2003/33, Università dell'Insubria, Varese, Italy. Invited paper at the conference 'Common features in Maastricht', Maastricht 14-16 December 2003.

——— (2006) Common trends and cycles in I(2) VAR systems. *Journal of Econometrics*, 132, 143–168.

Reinsel, C. (1983) Some results on multivariate autoregressive index models. *Biometrika*, 70, 145–56.

Schleicher, C. (2007) Codependence in Cointegrated Autoregressive Models. *Journal of Applied Econometrics*, 22, 137–159.

Tiao, G., and R. Tsay (1989) Model specification in multivariate time series (with discussion). *Journal of the Royal Statistical Society*, Series B 51, 157–213.

Vahid, F., and R. Engle (1993) Common trends and common cycles. *Journal of Applied Econometrics*, 8, 341–360.

——— (1997) Codependent cycles. *Journal of Econometrics*, 80, 199–221.

Vahid, F., and J. Issler (2002) The importance of common cyclical features in VAR analysis: A Monte Carlo study. *Journal of Econometrics*, 109, 341–363.

Zellner, A., and F. Palm (1974) Time series analysis and simultaneous equations econometric models. *Journal of Econometrics*, 2, 17–54.

(Massimo Franchi) Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università di Roma "La Sapienza", Italy

E-mail address: massimo.franchi@uniroma1.it

(Paolo Paruolo) European Commission, Joint Research Centre, Institute for the Protection and Security of the Citizen & Università dell'Insubria, Varese, Italy

E-mail address: paolo.paruolo@jrc.ec.europa.eu