Identification of the long-run β structure

A graduate course in the Cointegrated VAR model: Special topics in Rome

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Identification when data are nonstationary

Two different identification problems: identification of the long-run structure (i.e., of the cointegration relations) and identification of the short-run structure (i.e., of the equations of the system). The former is about imposing long-run economic structure on the unrestricted cointegration relations, the latter is about imposing short-run dynamic adjustment structure on the equations for the differenced process.

The (short-run) reduced-form representation:

$$\Delta x_t = \Gamma_1 \Delta x_{t-1} + \alpha\beta' x_{t-1} + \Phi D_t + \varepsilon_t, \quad \varepsilon_t \sim IN(0, \Omega) \tag{1}$$

and then pre-multiply (1) with a nonsingular $p \times p$ matrix A_0 to obtain the so called (short-run) structural-form representation (2):

$$A_0 \Delta x_t = A_1 \Delta x_{t-1} + a\beta' x_{t-1} + \tilde{\Phi} D_t + v_t, \quad v_t \sim IN(0, \Sigma). \tag{2}$$

where $\lambda_{RF} = \{\Gamma_1, \alpha, \beta, \Phi, \Omega\}$ and $\lambda_{SF} = \{A_0, A_1, a, \beta, \tilde{\Phi}, \Sigma\}$ are unrestricted.

To distinguish between parameters of the long-run and the short-run structure, we partition $\lambda_{RF} = \{\lambda_{RF}^S, \lambda_{RF}^L\}$, where $\lambda_{RF}^S = \{\Gamma_1, \alpha, \Phi, \Omega\}$ and $\lambda_{RF}^L = \{\beta\}$ and $\lambda_{SF} = \{\lambda_{SF}^S, \lambda_{SF}^L\}$, where $\lambda_{SF}^S = \{A_0, A_1, a, \tilde{\Phi}, \Sigma\}$ and $\lambda_{SF}^L = \{\beta\}$. The relation between λ_{RF}^S and λ_{SF}^S is given by:

$$\Gamma_1 = A_0^{-1} A_1,\ \alpha = A_0^{-1} a,\ \varepsilon_t = A_0^{-1} v_t,\ \ \Phi = A_0^{-1} \tilde{\Phi},\ \Omega\ = A_0^{-1} \Sigma\, A_0'^{-1}.$$

The short-run parameters of the reduced form, λ_{RF}^S, are uniquely defined, whereas those of the structural form, λ_{SF}^S, are not, without imposing $p(p-1)$ just-identifying restrictions. The long-run parameters β are uniquely defined based on the normalization of the eigenvalue problem. This need not coincide with an economic identification, and in general we need to impose $r(r-1)$ just-identifying restrictions on β. Because the long-run parameters remain unaltered under linear transformations of the VAR model, β is the same both in both forms and identification of the long-run structure can be done based on either the reduced form or the structural form.

Three aspects of identification

- generic (formal) identification, which is related to a statistical model
- empirical (statistical) identification, which is related to the actual estimated parameter values, and
- economic identification, which is related to the economic interpretability of the estimated coefficients of a formally and empirically identified model.

Identifying restrictions on all cointegration relations

As before, R_i denotes a $p1 \times m_i$ restriction matrix and $H_i = R_{i\perp}$ a $p1 \times s_i$ design matrix $(m_i + s_i = p1)$ so that H_i is defined by $R_i' \, H_i = 0$. Thus, there are m_i restrictions and consequently s_i parameters to be estimated in the i th relation. The cointegrating relations are assumed to satisfy the restrictions $R_i'\beta_i = 0$, or equivalently $\beta_i = H_i\varphi_i$ for some s_i-vector φ_i, that is

$$\beta = (H_1\varphi_1, ..., H_r\varphi_r), \tag{3}$$

The linear restrictions do not specify a normalization of the vectors β_i. The rank condition requires that the first cointegration relation, for example, is identified if

$$\text{rank}(R_1'\beta_1, ..., R_1'\beta_r) = \text{rank}(R_1'H_1\varphi_1, ..., R_1'H_r\varphi_r) = r - 1. \tag{4}$$

This implies that no linear combination of $\beta_2, ..., \beta_r$ can produce a vector that "looks like" the coefficients of the first relation

Formulation of identifying hypotheses and identification rank conditions

$$\begin{bmatrix} \beta^c_{11} & -\beta^c_{11} & 0 & \beta^c_{12} & -\beta^c_{12} \\ 0 & \beta^c_{21} & \beta^c_{22} & 0 & \beta^c_{23} \\ 0 & 0 & 0 & \beta^c_{31} & \beta^c_{32} \end{bmatrix} \begin{bmatrix} m^r_t \\ y^r_t \\ \Delta p_t \\ R_{m,t} \\ R_{b,t} \end{bmatrix} \tag{5}$$

The number of restrictions m_i and the number of free parameters s_i in each beta! The rank conditions are given by:

Relation	$R_{i.j}$	Relation	$R_{i.jk}$
1.2	3	1.23	3
1.3	1		
2.1	2	2.13	2
2.2	1		
3.1	1	3.12	3
3.2	2		

Normalization

The parameters $(\beta_{11}^c, \beta_{12}^c)$, $(\beta_{21}^c, \beta_{22}^c, \beta_{23}^c)$ and $(\beta_{31}^c, \beta_{32}^c)$ are defined up to a factor of proportionality, and one can always normalize on one element in each vector without changing the likelihood:

$$\begin{bmatrix} 1 & -1 & 0 & \beta_{12}^c/\beta_{11}^c & -\beta_{12}^c/\beta_{11}^c \\ 0 & 1 & \beta_{22}^c/\beta_{21}^c & 0 & \beta_{23}^c/\beta_{21}^c \\ 0 & 0 & 0 & 1 & \beta_{32}^c/\beta_{31}^c \end{bmatrix} \begin{bmatrix} m_t^r \\ y_t^r \\ \Delta p_t \\ R_{m,t} \\ R_{b,t} \end{bmatrix} \tag{6}$$

When normalizing β_i^c by diving through with a non-zero element β_{ij}^c, the corresponding α_i^c vector is multiplied by the same element. Thus, normalization does not change $\Pi = \alpha_i^c \beta_i^{c\prime} = \alpha\beta'$ and we can choose whether to normalize or not. However, when identifying restrictions have been imposed on the long-run structure, it is only possible to get standard errors of $\hat{\beta}_{ij}$ when each cointegration vector has been properly normalized.

Calculation of degrees of freedom

Given that the restrictions are identifying the degrees of freedom can be calculated from the following formula:

$$\nu = \sum (m_i - (r - 1)).$$

Consider the above example where s_i is the number of free coefficients in β_i^c, and $m_i = p - s_i$ the total number of restrictions on vector β_i^c. The degrees of freedom are calculated as:

s_i m_i	$s_1 = 2$ $m_1 = 3$	$s_2 = 3$ $m_2 = 2$	$s_3 = 2$ $m_3 = 3$
$r - 1$	2	2	2
$m_i - (r - 1)$	1	0	1

so the degrees of freedom are $\nu = 2$. Some restrictions may not be identifying (for example the same restriction on all cointegration relations), but are nevertheless testable restrictions.

Just-identifying restrictions

One can always transform the long-run matrix $\Pi = \alpha\beta'$ by a nonsingular $r \times r$ matrix Q in the following way: $\Pi = \alpha QQ^{-1}\beta' = \tilde{\alpha}\tilde{\beta}'$, where $\tilde{\alpha} = \alpha Q$ and $\tilde{\beta} = \beta Q'^{-1}$. We will now demonstrate how to choose the matrix Q so that it imposes $r-1$ just-identifying restrictions on each β_i. An example of a just identified long-run reduced form structure can be found as follows:

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \\ \dots & \dots & \dots \\ \beta_{41} & \beta_{42} & \beta_{43} \\ \beta_{51} & \beta_{52} & \beta_{53} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \dots \\ \beta_2 \end{bmatrix}; \quad \beta_1^{-1}\begin{bmatrix} \beta_1 \\ \dots \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dots & \dots & \dots \\ \tilde{\beta}_{41} & \tilde{\beta}_{42} & \tilde{\beta}_{43} \\ \tilde{\beta}_{51} & \tilde{\beta}_{52} & \tilde{\beta}_{53} \end{bmatrix}$$

We choose the design matrix $Q = [\beta_1]$ where β_1 is a $(r \times r)$ nonsingular matrix defined by $\beta' = [\beta_1, \beta_2]$. In this case $\alpha\beta' = \alpha(\beta_1\beta_1^{-1\prime}\beta') = \alpha[I, \tilde{\beta}]$ where I is the $(r \times r)$ unit matrix and $\tilde{\beta} = \beta_1^{-1\prime}\beta_2$ is a $r \times (p-r)$ matrix of full rank.

The above example for $x_t = [x'_{1t}, x'_{2t}]'$, where $x'_{1t} = [x_{1t}, x_{2t}, x_{3t}]$ and $x'_{2t} = [x_{4t}, x_{5t}]$, would describe an economic application where the three variables in x_{1t} are 'endogenous' and the two in x_{2t} are 'exogenous'. Furthermore, if we decompose $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $\alpha_2 = 0$, then $\beta' x_t$ does not appear in the equation for $\Delta x_{1,t}$ and $x_{2,t}$ is weakly exogenous for β. In this case, efficient inference on the long-run relations can be conducted in the conditional model of $\Delta x_{1,t}$, given $\Delta x_{2,t}$. When 'endogenous' and 'exogenous' are given an economic interpretation this corresponds to the triangular representation suggested by Phillips (1990). Note that the latter requires that $\alpha_2 = 0$, which is a testable hypothesis.

	$\mathcal{H}_{S.1}$			$\mathcal{H}_{S.2}$		
	$\hat{\beta}_1$	$\hat{\beta}_1$	$\hat{\beta}_3$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
m^r	**1.0**	0.0	0.0	**1.0**	0.0	0.0
y^r	**−0.94** [−6.55]	**0.04** [3.24]	**0.01** [2.06]	**−1.0**	**0.03** [3.81]	**0.04** [4.80]
Δp	0.0	**1.0**	0.0	0.0	**1.0**	**1.0**
R_m	0.0	0.0	**1.0**	−4.70 [−1.44]	**−0.54** [−4.53]	**0.32** [2.99]
R_b	3.04 [1.51]	0.20 [1.16]	**−0.63** [−7.03]	**5.99** [2.40]	**0.54** [4.53]	0.0
D_s831	**−0.27** [−8.08]	**0.01** [5.11]	**−0.01** [−5.12]	**−0.24** [−7.46]	**0.02** [6.58]	**0.01** [5.14]
	α_1	α_2	α_3	α_1	α_2	α_3
Δm_t^r	**−0.22**	*	**2.98**	**−0.22**	**−2.47**	*
Δy_t^r	0.05	*	**−1.84**	0.05	**1.75**	**−2.04**
$\Delta^2 p_t$	*	**−0.82**	*	*	*	**−1.12**
$\Delta R_{m,t}$	*	*	−0.09	*	**0.12**	−0.09
$\Delta R_{b,t}$	*	*	0.13	*	**−0.15**	**0.17**

Over-identifying restrictions

Consider the structure:

$$\mathcal{H}_{S.3} : \beta = (H_1\varphi_1, H_2\varphi_2, H_3\varphi_3),$$

where

$$H_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Which are the $\beta-$relations?

	$\mathcal{H}_{S.3}$			$\mathcal{H}_{S.4}$		
	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
m^r	**1.0**	0.0	0.0	**1.0**	0.0	0.0
y^r	**−1.0**	**0.03** [3.67]	0.0	**−1.0**	**0.03** [4.07]	0.0
Δp	0.0	**1.0**	**−0.20** [−3.95]	0	**1.0**	0.0
R_m	0.0	0.0	**1.0**	**−13.27** [−5.70]	0.0	**1.0**
R_b	0.0	0.0	**−0.80** [−15.65]	**13.27** [5.70]	0.0	**−0.81** [−10.58]
D_s831	**−0.34** [−13.60]	**0.01** [5.46]	**−0.01** [−10.67]	**−0.15** [−5.19]	**0.01** [5.30]	**−0.01** [−4.77]
	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$
Δm_t^r	**−0.21** [−4.74]	*	**3.38** [3.21]	**−0.23** [−4.89]	*	*
Δy_t^r	**0.06** [2.27]	−0.44 [−1.59]	**−1.40** [−2.21]	0.05 [1.84]	*	*
$\Delta^2 p_t$	*	**−0.84** [−5.33]	*	*	**−0.79** [−5.39]	*
$\Delta R_{m,t}$	*	*	−0.07 [−1.54]	*	0.03 [1.77]	**−0.08** [−2.29]
ΔR	*	0.05	**0.13**	*	*	**0.15**

Table: The rank conditions for identifiction

$r_{i.j}$	$\mathcal{H}_{S.3}$	$\mathcal{H}_{S.4}$	$r_{i.jg}$	$\mathcal{H}_{S.3}$	$\mathcal{H}_{S.4}$
1.2	2	2	1.23	4	3
1.3	2	1			
2.1	1	2	2.13	3	3
2.3	2	2			
3.1	1	1	3.12	3	3
3.2	2	2			

The degrees of freedom in the test of overidentifying restrictions are given by $\nu = \Sigma_i(m_i - r + 1)$, where m_i is the number of restrictions on β_i. The degrees of freedom for $\mathcal{H}_{S.3}$ are calculated as:

$$\nu = \sum_{i=1}^{r} m_i - (r-1) = (4-2) + (3-2) + (3-2) = 2 + 1 + 1 = 4.$$

The corresponding LR test statistic became $\chi^2(4) = 4.05$ with a p-value of 0.40, so the structure can be accepted.
The degrees of freedom of the hypothesis $\mathcal{H}_{S.4}$are calculated as:

$$\nu = \sum_{i=1}^{r} m_i - (r-1) = (3-2) + (3-2) + (3-2) = 3.$$

The test statistic became $\chi^2(3) = 2.84$ with a p-value of 0.42. Thus, both $\mathcal{H}_{S.3}$ and $\mathcal{H}_{S.4}$ are acceptable long-run structures with almost the same p-value. Which one should be chosen?

Lack of identification

	$\mathcal{H}_{S.5}$			$\mathcal{H}_{S.6}$		
	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
m^r	**1.0**	0.0	0.0	**1.0**	0.0	0.0
y^r	**−0.82** [−8.56]	0.0	0.0	**−1.0**	0.0	0.04 [NA]
Δp	0.0	**1.0**	0.0	0.0	0.0	**1.0**
R_m	**−24.40** [−7.74]	1.26 [NA]	**1.0**	0.0	**1.0**	1.59 [NA]
R_b	**24.40** [7.74]	−1.35 [NA]	**−0.81** [−11.83]	0.0	**−0.89** [−13.65]	−1.09 [NA]
$D_s 831$	−0.04 [−0.93]	0.00 [NA]	**−0.01** [−5.11]	**−0.34** [−13.54]	**−0.01** [−6.09]	0.00 [NA]
	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$
Δm_t^r	**−0.24** [−4.95]	*	**−2.47** [−2.11]	**−0.22** [−4.91]	**4.02** [3.47]	*
Δy_t^r	*	*	*	0.05 [1.78]	*	*
$\Delta^2 p_t$	*	**−0.70** [−5.02]	0.77 [1.86]	*	**0.84** [2.09]	**−0.81** [−5.50]
$\Delta R_{m,t}$	*	0.03 [1.83]	**−0.16** [−3.17]	*	*	*
$\Delta R_{b,t}$	*	*	0.12 [1.75]	*	*	*

Table: The rank conditions for identifiction

$r_{i.j}$	$\mathcal{H}_{S.5}$	$\mathcal{H}_{S.6}$	$r_{i.jg}$	$\mathcal{H}_{S.5}$	$\mathcal{H}_{S.6}$
1.2	2	2	1.23	2	4
1.3	1	4			
2.1	2	1	2.13	2	3
2.3	0	2			
3.1	2	1	3.12	3	1
3.2	1	0			