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The Optimum Quantity of Money with Borrowing Constraints

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Abstract

We provide an analytical characterization of the optimal anticipated monetary policy in an economy where agents have a precautionary savings motive due to random production opportunities and the presence of borrowing constraints. Non storable production makes intrinsically useless outside money valuable to insure consumption. We show that the choice of the optimal money growth rate trades off insurance vs. incentives to produce: an expansionary policy provides liquidity to borrowing constrained agents, but distorts production incentives. The joint presence of uncertainty and borrowing constraints implies that the Friedman rule leads to autarkic allocations. If the utility function satisfies Inada conditions then the optimal money growth rate is strictly positive and finite.

JEL classification: E50.

Keywords: Incomplete markets, Friedman rule, monetary policy.

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1 Introduction

We evaluate optimal monetary policy in an economy where agents face random oscillations in their production opportunities which fluctuate between productive and unproductive periods. The economy is populated by agents of two types defining their productive state. Types are assumed to be perfectly negatively correlated, so that at each point in time only one type is productive. As in the seminal work of Scheinkman and Weiss (1986) agents face a borrowing constraint: since the state is not observable agents cannot issue private debt. Because of this market incompleteness, money (an intrinsically useless object) may serve a fundamental insurance role and be valued.

We extend Scheinkman and Weiss's analysis, which assumes a constant money supply, by letting the government choose a perfectly anticipated monetary growth rate, implemented through lump-sum money transfers. As in Levine (1991) we assume that the government does not know which agent is productive, so that the transfers are equal across agents.[1] In this economy money is the only savings instrument: an unproductive agent consumes exchanging money for goods with the productive agent. As the state can be reversed, the value of money is positive for productive agents too, who are hence willing to trade their production for money. A known feature of the Scheinkman-Weiss economy is that rich productive agents will be relatively less interested in trading goods for money. It follows that trade volumes, aggregate production, and the price of money depend on the distribution of wealth (i.e. shares of money holdings) which evolves through time following the history of shocks.

We provide an analytical characterization of the price of money and aggregate production, as functions of the money growth rate parameter and the wealth distribution. Moreover, we characterize the dynamics of the wealth distribution as a function of money growth and the history of shocks. These objects give a complete description of the dynamics of this economy.

Since the money growth rate affects the distribution of wealth, monetary policy has

[1]See Kehoe, Levine, and Woodford (1990) for a thorough discussion of this assumption and in particular Levine (1991) for a careful derivation of the equal-treatment restriction from first principles.

real effects and the choice of the optimal anticipated policy involves a tradeoff between two margins: the first is that a monetary expansion provides insurance to agents who incur in a long spell of unproductive periods, and end-up having low money holdings and little consumption. The second margin is the classic cost of inflation: an expansionary policy lowers the return on money, lowering productive agents' incentives to produce in exchange for money. The choice of the optimal money growth rate trades off insurance vs. production incentives.

Overview of the analysis

Section 2 defines the economic environment: the agent's utility functions, production possibilities, and the monetary transfer scheme. With fluctuating productive opportunities, the agents face an insurance problem. It is shown that without uncertainty the Townsend (1980), Bewley (1980) result on the optimality of the "Friedman rule" holds: agents are fully insured and the efficient complete markets allocation can be sustained. We then introduce uncertainty in production opportunities, and explore what the government can do using a non-monetary instrument (direct taxation) under various environments. Assuming the government does not know which agent is productive leaves no role for direct taxation. In the same environment, the government can instead effectively use monetary policy. In a way, monetary policy can do something that direct taxation cannot do.

Section 3 defines a monetary equilibrium, and the agent's optimality conditions, in the stochastic environment. In a similar fashion to Bewley (1983), it is shown in Section 4 that under a contractionary monetary policy, the unique ergodic set of money holdings is such that markets shut down and therefore there is no monetary equilibrium. The reasoning behind the result is simple: due to the individual uncertainty, the agents need to hold large amounts of money to satisfy their tax needs. If an agent is relatively poor, she will fail to comply her tax obligations with positive probability, as her relative wealth will decrease as long as she is unproductive. We show that the there is no stationary monetary equilibrium where tax obligation are always met. Under a contractionary policy, the wealth distribution is degenerate at a single value and consumption allocations are those of autarky.

To prove that an expansionary policy is optimal, Section 5 characterizes analytically the equilibrium quantities and prices as a function of the growth rate of money and the distribution of money holdings. Section 6 derives the invariant distribution of money holdings. The optimality of an expansionary policy is discussed in Section 7, introducing an ex-ante welfare measure. It is assumed throughout that Inada conditions hold and that money is the only asset. Noting that the consumption of unproductive agents is zero under autarky implies that the ex-ante expected utility of contractionary policy, as well as that of an hyper-expansionary policy, approaches minus infinity as the unconditional probability of being unproductive is strictly positive.[2] Some analysis is required to see what happens in the case with a constant money supply: while it is easily seen that the consumption of the unproductive agent drifts towards zero as she remains unproductive, one needs to characterize the "probability mass" of histories where consumption approaches zero, to see whether they matter for ex-ante welfare. It is shown that this mass is non-negligible: this implies that the expected ex-ante utility with constant money approaches minus infinity. We finally show that for finite, strictly positive, inflation rates the expected utility is bounded below and therefore conclude that a finite and strictly positive expansionary monetary policy is optimal.

Section 8 departs from the assumption of perfectly negatively correlated shocks, and considers an economy in which a continuum of agents features uncorrelated shocks, along the lines of Bewley (1977) and Lucas (1992). We maintain the assumptions of incomplete markets and uncertain production at the individual level and show that, as in the previous economy, the optimal monetary policy is expansionary. Section 9 summarizes the main findings, discusses key differences with respect to related papers and future work.

[2]It is easily seen that the equilibrium allocation converges to autarky as inflation diverges to infinity.

Related Literature

A few previous contributions discuss environments where a flat monetary expansion is efficient in an economy with incomplete markets and where money serves an essential role. The seminal paper in this line is Levine (1991), who considers an endowment economy where the agents' utility function change randomly according to whether they are "buyers" or "sellers", a state that follows an exogenous Markov process. Levine's develops an analytical argument which shows that an expansionary monetary policy can attain the first best. Three assumptions are crucial for this result: uncertainty, constant individual endowment, and bounded utility functions. Uncertainty is key because, as in Scheinkman and Weiss and our model, it creates a demand for insurance on the part of "unlucky" agents. Constant endowment implies that monetary policy, through either inflation or deflation, cannot affect productions decisions, which simplifies the analysis. In Levine's model sellers sell their entire endowment, which amounts to a restriction on the agents marginal utilities of the set of feasible trades. Because of this assumption monetary policy can provide insurance at no cost since altering the relative price has no effect on welfare in the corner solution.

Kehoe, Levine, and Woodford (1990) extend Levine's setting to allow for internal solutions where sellers do not necessarily sell all their endowment. For reasons of tractability, they restrict attention to equilibria in which what happens in each period is *independent of history* (two state markov equilibria). Under these assumptions the distribution of money is degenerate: "sellers" always end the period owning *the whole* money stock. In this setup optimal monetary policy has a cost (it distorts the sellers' choices) and a benefit, namely that it provides some insurance. Optimal monetary policy can be either expansionary or contractionary. They use numerical solutions of the model to study how the optimal inflation varies with parameters' values.[3]

[3] Green and Zhou (2005) also study a Bewley-type setup using mechanism design theory. They restrict the feasible set of policies to model monetary equilibria and compare the allocations obtained under alternative "institutional mechanisms". One of their examples features agents who differ in their marginal utilities and endowments which vary randomly, as in Levine's. It is shown that, because utility functions are bounded, the optimal allocation is to assign all the consumption to agents with high marginal utility, which can be

Compared to these papers, our contribution is to analyze the question of the optimal policy in the context of a production economy, and to focus, as Scheinkman and Weiss, on equilibria in which the decisions in each period depend on the whole history of shocks, as summarized by the distribution of money holdings. We analytically characterize the non-degenerate distribution of money holdings, and provide a characterization of the the ex-ante efficient anticipated monetary policy. The concluding section of our paper discusses some key assumptions that explain the differences in results between these papers.

Our work is also related to Imrohoroglu (1992) who evaluates the welfare costs of inflation in an economy with borrowing constraints and where money is essential for trading. In the model a continuum of agents are faced with idiosyncratic shocks with no aggregate uncertainty that define if agents are employed, getting goods y, or unemployed, getting θy with $\theta < 1$. The consumption goods are homogeneous and therefore agents want to trade to smooth consumption. In order to do so, they are endowed with some fraction of money and trade in a centralized market. For a calibration of the model, Imrohoroglu estimates that an expansionary monetary policy is costly. The calibrated value for θ is crucial for this result (θ is calibrated to be 1/4 from previous work by her on the topic). For small enough values for θ the marginal utility of consumption when unemployed would become arbitrarily large and therefore inflations, that serve as a transfer to poor agents, would become optimal (in Section 8 we discuss a similar model and show that a strictly positive inflation rate is optimal). Our model departs from Imrohoroglu's in several ways. First, she has an endowment economy while we deal with a production economy which adds an extra margin where inflation plays a role. Moreover, in her model when an agent is unemployed receives some endowment that overturns Inada conditions, while in our model an unproductive agent receives nothing so Inada conditions become of first order importance. It follows that in our setup a monetary expansion, that among other implies a transfer from rich to poor agents, provides insurance to unproductive agents. That is, an expansion can be understood as the

done through an expansionary monetary policy.

provision of some endowment to the unlucky agents. Second, her assumption that shocks are idiosyncratic together with a centralized market for goods implies that the price of money is independent of the distribution of money holdings. In our model, because we are dealing with agents which shocks are correlated, the price of money is not orthogonal to the distribution of money holdings. Therefore, even though both economies face no aggregate uncertainty, her economy exhibits a constant steady state while ours has a stationary distribution with cycles.

A related question on an optimal scheme for monetary transfers in a search theoretic framework is evaluated in Berentsen, Camera, and Waller (2005), which show that a one time money injection may increase ex-ante welfare. The analysis is done in the search model of money of Lagos and Wright (2005), where a non-degenerate distribution of money holdings within a period provides a role for monetary policy. An important difference with our setup is that here the distribution of wealth at the end of the period is degenerate at a single value and therefore the monetary policy has no effect on the dynamics of the economy. This difference is important if we want to allow the monetary authority to choose the money growth. As noted by Berensten, Camera, and Waller, in the long run money is neutral in their model and therefore the optimal money growth is to deflate following Friedman's prescription.[4]

In a broad sense the mechanism that we study is related to a recent strand of literature dealing with the interplay between the agents' liquidity and the business cycle, these include Kiyotaki and Moore (2008), Guerrieri and Lorenzoni (2009), Chamley (2010), Brunnermeier and Sannikov (2011). Our results are also related to Aiyagari (1995) who studies a neoclassical

[4] Another related papers is Bhattacharya, Haslag, and Martin (2005), which studies several environments where the monetary policy can have redistributive effects and show that the Friedman rule might not be *ex-post* optimal. In particular, related to our work, they show that in an example of Townsend (1980) deterministic Turnpike model ex-post optimality is not attained by the Friedman rule. The different welfare criterion (ex-post versus ex-ante) is key here: In Section 2.3 we consider such a deterministic economy and show that the Friedman rule is ex-ante optimal. Edmond (2002) and Akyol (2004) also discuss the possibility of an efficient expansionary policy. Both papers study an endowment economy and use numerical simulations to show that positive inflation may be optimal. A main difference between these papers and ours (and Levine's) is that money in these papers serves no essential role, it is valued only because it enters the utility function of agents. This difference is important because, as explained by Kehoe, Levine, and Woodford (1990), it does not allow the redistributional consequences of monetary policy to be analyzed.

growth model with borrowing constraints, as in Bewley (1977). Aiyagari shows that a positive capital tax rate is optimal because agents are over-saving due to precautionary motives and therefore the capital stock is too high. Because the government faces no incomplete markets, an optimal policy involves taxing capital and using the proceedings to produce a public good that all consumers enjoy, independently of their idiosyncratic shock. Our finding that the optimal policy involves an expansionary policy is reminiscent of Aiyagari's result that the return on the savings should be taxed. However, the model and the mechanism underlying these results are different. In our setup there is no public good through which the government can "insure" the agents. This is important because under our maintained assumption the government *does not know* who is productive, and hence it has no ability to tax productive agents to redistribute to unproductive agents.[5] Another difference with Aiyagari is that the economy we consider has cycles: aggregate production, the real interest rate, and the value of money fluctuate through time. Because of these features, the method that we use to characterize the optimal policy is different.

2 The model

We consider a model of two types of infinitely long-lived individuals, indexed by $i = 1, 2$. It assumed that at each point in time only one type of agent can produce. The productive agent transforms labor into consumption one for one, the unproductive agents cannot produce. The productivity of labor is state dependent: the duration of productivity spells is random, and is exponentially distributed with mean duration $1/\lambda > 0$. Money, an intrinsically useless piece of paper, is distributed at each time t between the two types, so that $m_t^1 + m_t^2 = m_t$. We assume that monetary policy controls μ, the growth rate of the money supply:

$$m_t = m_0 e^{\mu t} \qquad \text{with } m_0 \text{ given.} \tag{1}$$

[5] We discuss in Section 2.2 how the government might actually support first best through taxation if it had the ability to commit to trigger policies.

Let $\rho > 0$ denote the time discount rate, ω denote a history of shocks and money supply levels, and $s(t, \omega) = \{1, 2\}$ be an indicator function denoting which agent is productive for a given history ω and current time t. Agent of type i chooses consumption c^i, labor supply l^i, and depletion of money balances $\dot{m}^i$, in order to maximize her (time-separable) expected discounted utility,

$$\max_{\{c^i(t,\omega), l^i(t,\omega), \dot{m}^i(t,\omega)\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \int_0^{\infty} e^{-\rho t} \left[u\left(c^i(t,\omega)\right) - l^i(t,\omega) \right] \, dt \right\} \tag{2}$$

subject to the constraints

$$\dot{m}^i(t,\omega) \leq \left[l^i(t,\omega) + \tau(t,\omega) - c^i(t,\omega) \right] / \tilde{q}(t,\omega) \qquad \text{if } s(t,\omega) = i \tag{3}$$

$$\dot{m}^i(t,\omega) \leq \left[\tau(t,\omega) - c^i(t,\omega) \right] / \tilde{q}(t,\omega) \text{ and } \; l^i(t,\omega) = 0 \qquad \text{if } s(t,\omega) = j \tag{4}$$

$$m^i(t,\omega) \geq 0 \quad l^i(t,\omega) \geq 0 \quad c^i(t,\omega) \geq 0 \tag{5}$$

where $\tilde{q}(t, \omega)$ denotes the price of money, i.e. the inverse of the consumption price level, and $\tau(t, \omega)$ denotes a government lump-sum transfer to the agent, and expectations are taken with respect to the processes s and m conditional on time $t = 0$.

A monetary policy with $\mu > 0$ is called expansionary, a policy with $\mu < 0$ is called contractionary. For any history ω the monetary policy μ determines the transfers to the agents τ_t through the government budget constraint,

$$\tilde{q}_t \; \mu \; m_t = 2\tau_t \tag{6}$$

which states that transfers must be financed by printing money. Note that here the government cannot differentiate transfers across agent-types. This follows from the assumption that the identity of the productive type is not known to the government. Section 2.2 shows the importance of this assumption within the context of our framework.

2.1 Competitive equilibrium allocation with complete markets

It is immediate to see that assuming an ex-ante equal probability of each state ($s = 1, 2$) and no borrowing constraints, the efficient allocation prescribes the same constant level of consumption, $\bar{c}$, for both types of agents, where $\bar{c}$ solves $u'(\bar{c}) = 1$. The inverse price level increases at the rate of the time preference. It also follows that aggregate output is constant at the level $2\bar{c}$.

2.2 Fiscal policy

A central assumption in our analysis is that the government does not know $s(t, \omega)$, i.e. the identity of the productive type. It is useful to explore the consequences of relaxing this assumption to better understand the nature of the monetary policy problem. Without loss of generality, given the symmetry of the states, let us assume type 1 is productive and type 2 is not productive, and label their respective consumption levels by $c^i, \quad i = 1, 2$. Also, for simplicity, let us set the money supply equal to zero in what follows.

We begin by assuming that the government observes the identity of the productive type and is able to tax productive agents and transfer resources to unproductive agents. We consider two taxing technologies. The first one is lump sum taxes: in this case the productive agent pays a flat tax $\bar{\phi} = \bar{c}$, and the government uses the proceedings to finance the consumption of the unproductive agent. It is immediate that under these assumptions the complete markets allocation can be replicated. Alternatively, consider a setup where the only available taxes are distortionary, say proportional to production: then the transfer to the unproductive agent is $\tau = \phi l$. For a generic tax rate $\phi \in (0, 1)$ the consumption of the two agents solves $u'(c^1) = \frac{1}{1-\phi}, u'(c^2) = \frac{1}{\phi}$. An ex-ante optimal policy, maximizing the expected utility of the two types with equal weights, gives $\phi = 1/2$. Under this setting the government fiscal policy provides insurance, consumption is constant through time, though the level of consumption is smaller than under complete markets.

Let us next consider a government who does not know the type's identities. In this case,

Table 1: Optimal fiscal policy under alternative feasibility assumptions

type known		type not known	
lump-sum-tax	distortionary tax	Gvt. commitment	No commitment
$u'(\bar{c}^1) = u'(\bar{c}^2) = 1$	$u'(\tilde{c}^1) = u'(\tilde{c}^2) = 2$	$u'(\bar{c}^1) = u'(\bar{c}^2) = 1$	$u'(\bar{c}^1) = 1 \quad , \quad u'(0^2) = \infty$

the efficient stationary allocation with $c = \bar{c}$ at all times for all types can be sustained if the government has the ability to commit to a trigger policy. Suppose the government credibly announces: "whoever is productive must pay an amount $\bar{c}$ to the government, who will then distribute it to unproductive agents. If at any point in time the resources are not enough to pay for the transfers, the transfer scheme will be shut down and the economy will be left in autarky forever". If the threat is credible then it is in the interest of every individual agent to comply, because deviating from it implies that the agent consumption is zero when unproductive which, due to Inada conditions, delivers an expected utility of $-\infty$.

In what follows, we do not let the government have these special powers: we assume the government does not know the identity of productive types, and that it cannot commit to trigger policies. In such a situation fiscal policy, i.e. direct taxation, is powerless. The resource allocation is autarkic, and individuals experience inefficient fluctuations in utility. The various outcomes sustainable under alternative fiscal policy assumptions are summarized in Table 1. We next study the powers of monetary policy, under the same assumptions of type-ignorance and no-commitment.

2.3 A non-stochastic model with productivity cycles

To clarify the mechanisms under analysis we solve the model with no uncertainty. Assume the evolution of labor productivity is known by both agents and perfectly forecastable. In particular, suppose each agent is productive for T periods, and then becomes unproductive for the next T periods. Without loss of generality, for the characterization of the stationary equilibria, let us assume that the economy starts in period $t = 1$ with agent 1 being productive and agent 2 owning all the money, so that $m^1_{t=1} = 0$.

Note the lagrange multiplier is hod -1 in m_t: $\tilde{\gamma}_t = \frac{1}{m_t}\gamma_t$. Assume is $u(c) = c^{1-\theta}/(1-\theta)$, i.e. CRRA with risk aversion θ. The FOC for c_t gives: $\gamma_t = \ q_t \ u'(c)$. The Euler equation for $\dot{m}_t$ gives

$$\frac{\dot{\tilde{\gamma}}_t}{\tilde{\gamma}_t} = \rho \qquad \text{or} \qquad \rho \ = \frac{\dot{\gamma}}{\gamma} \ - \frac{\dot{m}}{m} \quad = \quad \frac{\dot{q}}{q} - \theta\frac{\dot{c}}{c} - \mu$$

Notice that this is solved by the "Friedman rule"

$$c^i_t = \bar{c} \quad , \quad \frac{\dot{c}}{c} = \dot{q} = 0, \quad \mu = -\rho$$

where the level of q is pinned down by the budget constraint

$$m_0 + \int_0^T \dot{m}_t \ dt = \int_0^T \frac{\bar{c}}{\tilde{q}_t} \ dt \quad \text{ which gives } \quad q = \frac{4 \ \bar{c}}{\rho} \ \frac{1 - e^{-\rho T}}{1 + e^{-\rho T}}$$

This result shows that without uncertainty this economy replicates Townsend (1980), Bewley (1980) result on the optimality of the "Friedman rule".

3 Monetary policy in the stochastic model

We look for an equilibrium where $\tilde{q}(t,\omega) = \tilde{q}(m(t,\omega), m^i(t,\omega), s(t,\omega))$. With a slight abuse of notation this implies $c^i(t,\omega) = c^i(m(t,\omega), m^i(t,\omega), s(t,\omega))$, $l^i(t,\omega) = l^i(m(t,\omega), m^i(t,\omega), s(t,\omega))$, and $\dot{m}^i(t,\omega) = \dot{m}^i(m(t,\omega), m^i(t,\omega), s(t,\omega))$. That is, we are looking for an equilibrium that depends solely on three states: the level of the money supply, the distribution of money holdings, and the current state. Next we argue that the state vector can be further simplified.

Let x_t denote the fraction of outstanding money balances held by agents of type 1 at time t. The state of the problem is defined by two variables: x and s, the current state of nature. Why is not the aggregate amount of money part of the state? There are two things to consider. First, is an agent able to figure out her fraction of money at time $t + dt$ only by knowing her fraction at time t? (this is not obvious because the monetary policy is independent of the fraction of money held by each agent). And second, in the case of a contractionary monetary policy, is it enough for an agent to know her fraction of outstanding money balances to figure out if she is able to cover her tax liabilities? The next paragraphs deal with these issues.

First of all note that the monetary policy is such that an agent needs to know only how many bills she will receive from the government (in the case of an expansionary policy) or how many bills she needs to forego (in the contractionary case). The amount of bills that each agent receives is $m_t \frac{\mu}{2}$, where m_t are the outstanding money balances at time t. For simplicity, given the symmetry of the problem, we focus on agent 1. If she has m_t^1 bills, then money balances after the transfer are $m_t^1 + m_t \frac{\mu}{2} = m_t \left(x_t + \frac{\mu}{2} \right)$ which shows that the evolution of the money share is independent of the outstanding money balances.

The second question is, for the case where $\mu < 0$, if it is enough to know the fraction of money held in order to figure out whether the tax obligations can be met. From agent 1's perspective this requires $m_t^1 \geq -m_t \frac{\mu}{2}$ or $x_t \geq -\frac{\mu}{2}$.

We conjecture that for a given distribution of money x and a given state s, the price of money (the inverse of the price of consumption) is homogeneous of degree -1 in m, namely

$$\tilde{q}(m, x, s) = \frac{1}{m}\, q(x, s)$$

Note that this equation states that the Quantity Theory of Money applies in this setup. We look for equilibria where $q(x, s)$ is symmetric, i.e. $q\,(x, 1) = q\,(1 - x, 2)$, for each $x \in [0, 1]$ Next we define a monetary equilibrium.

Definition 1 *A monetary equilibrium is a price function* $\tilde{q}(m, x, s) = \frac{1}{m}\, q(x, s)$*, with* $q : [0, 1] \times \{1, 2\} \rightarrow \mathbb{R}^+$ *with* $q\,(x, 1) = q\,(1 - x, 2)$*, and a stochastic process* $x(t, \omega)$ *with values in* $[0, 1]$*, such that a consumer* i *maximizes expected discounted utility (equation (2)) subject to the constraints (equations (3), (4), and (5)) with* $q(t, \omega) = q(x, s)$ *and the government budget constraint (equation (6)) is balanced.*

Without loss of generality, consider the law of motion for the share of money held by type 1 when unproductive. Using this agent budget constraint in equation (4) gives

$$\dot{x}_t(x,2) = \frac{\dot{m}_t^1}{m_t} - \frac{m_t^1}{(m_t)^2}\dot{m}_t = \frac{\tau - c^1(x,2)}{q(x,2)} - \mu\ x_t = \mu\left(\frac{1}{2} - x\right) - \frac{c^1(x,2)}{q(x,2)} \tag{7}$$

so that $\dot{x}_t$ is independent of the outstanding aggregate amount of money m_t.

Let $\tilde{\gamma}(m,x,s)$ denote the costate variable, i.e. the lagrange multiplier for $\dot{m}^i$ in the problem defined in (2). Hence $\tilde{\gamma}(m,x,s)$ measures the marginal value of the fraction of money when money supply is m, agent 1 holds a share x of it and the current state is s. The first order conditions with respect to $l(t,\omega)$ and $c(t,\omega)$ give

$$\tilde{\gamma}(m,x,1) \;=\; \tilde{q}(m,x,1) \quad \text{and} \quad \tilde{\gamma}(m,x,2) = \tilde{q}(m,x,2)\ u'\left(c^1(x,2)\right)$$

It follows that under the assumed homogeneity of $\tilde{q}(m,x,1)$ w.r.t. m the lagrange multiplier $\tilde{\gamma}$ is also homogenous, i.e. $\tilde{\gamma}(m,x,1) = \gamma(x,1)/m$.[6] We can then rewrite the first order conditions as

$$\gamma(x,1) \;=\; q(x,1) \quad \text{and} \quad \gamma(x,2) \;=\; q(x,2)\ u'\left(c^1(x,2)\right) \tag{8}$$

Also, keep in mind that if $s(t,\omega) = 1$, $c^1(t,\omega) = \bar{c}$, where $\bar{c}$ solves $u'(\bar{c}) = 1$. It is shown in Appendix A that the lagrange multipliers $\gamma(x,s)$ solve the following system of differential equations

$$\gamma_x(x,1)\ \dot{x}(x,1) \;=\; (\rho+\lambda+\mu)\gamma(x,1) - \lambda\gamma(x,2) \tag{9}$$

$$\gamma_x(x,2)\ \dot{x}(x,2) \;=\; (\rho+\lambda+\mu)\gamma(x,2) - \lambda\gamma(x,1) \tag{10}$$

It is immediate that when the money supply is constant ($\mu = 0$) these equations coincide

[6] This can be seen by reformulating the problem with the constraints specified in real terms, using the lagrange multiplier γ for this new constraint in the place of $\tilde{\gamma}$. It is immediate from this formulation that the f.o.c. for l_t and c_t are independent of the *level* of the money supply m_t.

with the ones analyzed by Scheinkman and Weiss (1986).

4 No monetary equilibrium with contractionary policy

We prove a result related to Bewley (1983). Bewley shows that there is no monetary equilibrium if the interest rate is lower than the discount rate in a neoclassical growth model with incomplete markets where agents face idiosyncratic shocks and where there is a lump sum tax obligation that has to be covered by the agents..

In our model, agents fully commit to pay taxes. Consider the case where agent 1 has fraction of money balances x_t and the current state of the economy is $s(t, \omega) = 2$. If x_t is low enough, given that $\lambda > 0$, there exists a positive probability that the agent will fail to comply with the monetary authorities. Consider the case where $x_t = 1$. As the agent can always choose not to consume she will be able to comply with her tax obligations with probability 1. This implies that there exists a threshold $\bar{x}$ such that the agent is able to cover her lifetime tax needs with probability one. Note that the threshold must be independent of the current state s_t as with positive probability the states are reversed. In the next lemma we characterize this threshold.

Lemma 1 *If $\mu < 0$, for any state of the world $s(t, \omega)$, there is a unique threshold: $\bar{x} = 1/2$. Moreover, there is a unique ergodic set where $x_t = \frac{1}{2}$, $\forall$ t.*

See Appendix B for the proof. Intuitively, given the uncertain duration of the productivity spell, the only value of money holdings that ensures compliance with tax obligations is $x = 1/2$. At this point, for any history of shocks, the identical lump-sum (negative) transfers reduce the money holdings of both agents proportionally, leaving the wealth distribution unaffected. This leads us to

Proposition 1 *In the ergodic set, for $\mu < 0$ there is no stationary monetary equilibrium. Furthermore, there is no trade and only productive agents enjoy positive consumption.*

The proof of Proposition 1 follows from noting that Lemma 1 implies no trade in the ergodic set. Productive agents have an unsatisfied demand for money and unproductive ones have an unsatisfied demand for consumption goods. Therefore, the supply and demand of assets is not clearing for any price level as no trade is the only possible outcome independently of the outstanding price.

5 Expansionary monetary policy

We now turn to the case where $\mu > 0$. As done by Scheinkman and Weiss (1986), we specialize to the case with logarithmic utility function, $u(c) = \ln(c)$, that will let us characterize analytically many results. Hayek (1996) presents equilibrium existence and uniqueness results for the more general case of a CRRA utility preferences with risk aversion greater than the log case. We will show that the optimal monetary policy, $\hat{\mu}$ is expansionary and finite.

We start by discussing the boundary conditions for this problem. The first boundary occurs when agent 1 is not productive and her money holdings go to zero. In this case the agent spends the whole money transfer to finance her consumption. The budget constraint gives $c_t^1 = \tau_t = q(0,2)\mu/2$. Using equation (8) for the case of log utility gives

$$\gamma(0,2) = \frac{2}{\mu} \qquad \text{with} \qquad \lim_{\mu \downarrow 0} \gamma(0,2) = \infty \tag{11}$$

where the limit obtains because of Inada conditions. This is an important result in our analysis. If the money supply is growing it provides an upper bound to the marginal utility of money holdings because the price of money is always finite in an equilibrium and therefore the agent enjoys positive consumption even when she is very poor. When there is no money growth (i.e. $\mu \downarrow 0$), the agent is not able to consume in poverty and therefore Inada conditions imply that the marginal utility of money approaches infinity.

The second boundary condition occurs when agent 1 is productive and holds all of the money. We argue next that in this case we have

$$\lambda\,\gamma(1,2) = (\rho+\lambda+\mu)\,\gamma(1,1) \tag{12}$$

which follows from evaluating equation (9) at $x = 1$ with $\dot{x}(x,1) = 0$. The reason that x remains constant is the following. The unproductive agent, who has a zero share of money, will spend all the money she receives from the government transfer in consumption goods. Appendix C gives a formal proof of this statement. Intuitively, as the unproductive agent is impatient and there is a positive probability of becoming productive, saving a part of the transfer when $x = 0$ would be optimal if she expected to remain unproductive and that the value of money will go up in the future. In a rational expectations equilibrium, however, conditional on remaining unproductive this agent expects the value of money to go down. The amount of consumption goods bought by a unit if money, measured by $q(1-x,1)$, is increasing in x, the money holdings of the unproductive agent. Hence as $x \downarrow 0$, i.e. as the unproductive agent runs out of money, the real value of money falls.

We want to find expressions for the evolution of x over time. Using the budget constraint of the unproductive agent, equation (4), the government budget constraint, equation (6), and the first order condition equation (8) give

$$\dot{x}(x,2) = \mu\left(\frac{1}{2} - x\right) - \frac{1}{\gamma(x,2)} \qquad \text{and} \qquad \dot{x}(x,1) = \mu\left(\frac{1}{2} - x\right) + \frac{1}{\gamma(1-x,2)} \tag{13}$$

where the second expression follows by noting that $\dot{x}(x,1) + \dot{x}(1-x,2) = 0$. This equation shows that the optimal change of real money holdings, on top of the government transfer, depends on the value of money for the unproductive agent relative to the value of money for the productive agent: $\gamma(x,2)/q(x,2) = \gamma(x,2)/q(1-x,1)$. Notice from the FOC that the smaller is the consumption level of the unproductive agent, the higher is the value of $\gamma(x,2)/q(x,2)$, hence the smaller will be the (absolute) real value of money transfers. In other words, as an agent becomes poorer, she is willing to transfer less money.

Using the expressions for $\dot{x}(x,s)$ in equations (9) and (10) gives:

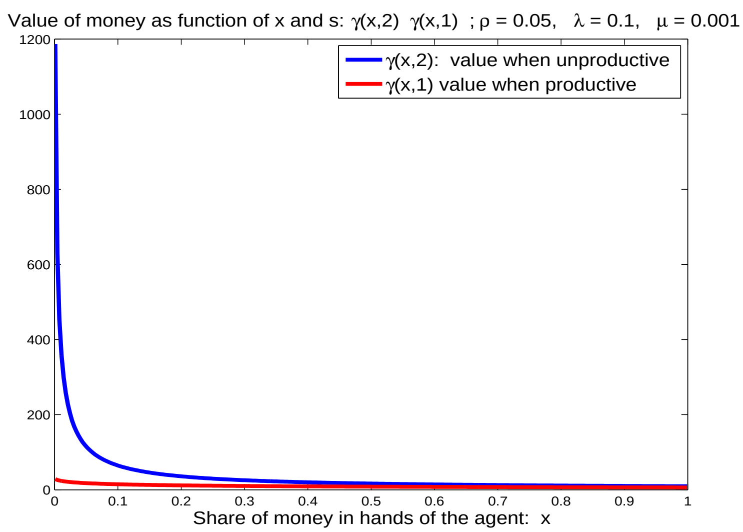
$$\gamma_x(x,1)\left[\mu\left(\frac{1}{2}-x\right)+\frac{1}{\gamma(1-x,2)}\right]=(\rho+\lambda+\mu)\gamma(x,1)-\lambda\gamma(x,2) \tag{14}$$

$$\gamma_x(x,2)\left[\mu\left(\frac{1}{2}-x\right)-\frac{1}{\gamma(x,2)}\right]=(\rho+\lambda+\mu)\gamma(x,2)-\lambda\gamma(x,1) \tag{15}$$

with the boundary conditions given in equations (11) and (12). This is a system of delay differential equations.[7] The next lemma characterizes the functions $\gamma(x,1)$ and $\gamma(x,2)$ that solve this system:

Lemma 2 *Let $\gamma_x(x,1)<0$ and consider $x\in(0,1)$, then $0<\gamma(x,1)<\gamma(x,2)$, $\gamma_x(x,2)<0$, $\dot{x}(x,2)<0$, and $\lim_{x\downarrow 0}\dot{x}(x,2)=0$.*

See Appendix C for the proof. The lemma establishes a sufficient condition for a monetary equilibrium. In particular, the fact that both $\gamma(x,s)$ functions are decreasing in x ensures, as in Scheinkman and Weiss (1986), that prices are positive and finite, which is required for a monetary equilibrium.[8] It also proves that for an unproductive agent it is indeed optimal to spend all the money transfer $\lim_{x\downarrow 0}\dot{x}(x,2)=0$.



[7] Notice in particular that the delay is non-constant, which prevents an analytical solution in closed form. Notice however that the functions $\gamma(x,1),\gamma(x,2)$ are analytical, and that the above system allows to completely characterize these functions given initial values for $\gamma(\frac{1}{2},1),\gamma(\frac{1}{2},2)$.

[8] The analysis is analogous to the "phase diagram" discussed in their paper, in Figure 1.

We now discuss the optimal consumption when unproductive. Notice that

$$c^1(x,2) = \frac{\gamma(1-x,1)}{\gamma(x,2)} \tag{16}$$

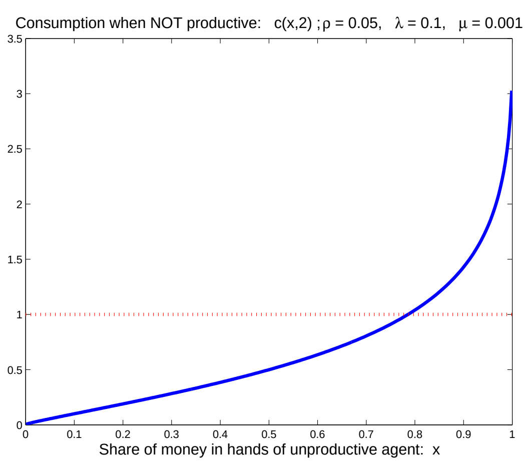
which gives

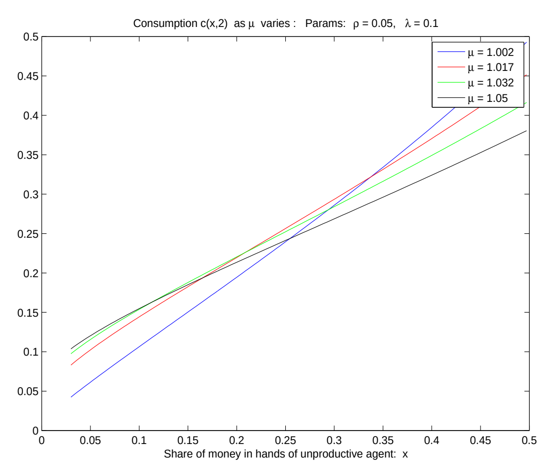
$$\frac{\partial c^1(x,2)}{\partial x} = -\frac{\gamma_x(1-x,1)}{\gamma(x,2)} - \frac{\gamma_x(x,2)\,\gamma(1-x,1)}{(\gamma(x,2))^2} > 0 \tag{17}$$

The properties of the $\gamma(x,s)$ functions discussed above imply that the consumption of the unproductive agent is increasing in x. Hence for any given money growth rate $\mu > 0$ consumption is smallest when the agent money holdings are zero, in particular at $x = 0$ we have

$$c^1(0,2) = \frac{\mu}{2}\gamma(1,1) \quad , \quad \text{with} \quad \lim_{\mu \downarrow 0} \frac{\mu}{2}\gamma(1,1) = 0 \tag{18}$$

where the limit obtains from the agent's budget constraint. This is important for the welfare analysis that will follow because it shows that monetary transfers provide the unproductive agent with a lower bound to the consumption level. Without transfers, an agent with no money cannot consume.[9]





[9]At the other extreme, as the fraction of money in the hands of the unproductive agent increases, her consumption satisfies $c^1(1,2) = \frac{\gamma(0,1)}{\gamma(1,1)}\,\frac{\lambda}{\rho+\lambda+\mu}$. As $\frac{\gamma(0,1)}{\gamma(1,1)} > 1$, this equation implies that there are values of $\frac{\lambda}{\rho+\lambda+\mu}$ such that $c^1(1,2) > 1$. Numerical analysis shows that this is more likely as $\mu \downarrow 0$. When this happens the consumption of a non-productive agent who has all the money is *bigger* than the consumption of the productive agent (that is constant at 1). The economics of this result is that the flip side of this situation is that the productive agent owns almost zero money, and is eager to accumulate money to insure against the possibility of a switch of the state. So the price of money is high (i.e. the price of consumption is low), and the productive agent is willing to produce a lot to refill his low money holdings.

We now study the local behavior for the consumption of the unproductive agent when money holdings and money growth are near zero, which will be useful in the welfare analysis. To this end we define the ball $B_\varepsilon \equiv \{z \text{ such that } z \in (0, \varepsilon)\}$. The next lemma characterizes the local behavior for consumption when the agent is unproductive, $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$, where ε and ε' are arbitrarily small.

Lemma 3 *$\lim_{x \downarrow 0} \frac{c_x^1(x,2)x}{c^1(x,2)} = Q_1$, where $Q_1 = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu}\left(1 - \frac{\gamma(0,1)}{\gamma(0,2)}\right) > 1$ and finite $\forall \ \mu > 0$. Moreover, for $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$, the following approximation holds: $\ln c^1(x,2) \approx Q_0 + Q_1 \ln x$.*

See Appendix D for the proof. Lemma 3 states that when money holdings x and money growth μ are "small", the elasticity of consumption when unproductive with respect to money holdings is constant and bigger than one: a 1% reduction of money holdings causes a more than proportional reduction in consumption. An important implication is that utility is locally logarithmic in money holdings x.

6 Invariant distribution of money holdings

Let $F(x,s)$ denote the CDF for the share of money holdings in state s with density $f(x,s) = \frac{\partial F(x,s)}{\partial x}$. The density function of the invariant distribution is derived from the usual Kolmogorov Forward Equation (KFE) after imposing for stationarity. Appendix E derives the KFE for our model with poisson jumps in the state, which gives

$$0 = f_x(x,s_i)\dot{x}(x,s_i) + f(x,s_i)\frac{\partial \dot{x}(x,s_i)}{\partial x} + \lambda\left[f(x,s_i) - f(x,s_{-i})\right] \tag{19}$$

where $s_i = 1, 2$ denotes the current state and s_{-i} the other state. It is immediate that the densities satisfy $f(x,2) = f(1-x,1)$ since, given the assumed symmetry of the shocks, for each agent with money $x = x'$ and $s = 2$ there is another agent with money $1 - x'$ and $s = 1$. This property allows us to concentrate the analysis on only one density: $f(x,2)$. Using the expressions for $\dot{x}(x,s)$ derived in equation (13) gives

$$\frac{f_x(x,2)}{f(x,2)} = \frac{\gamma(x,2)\,(\lambda-\mu) - \frac{(\rho+\lambda+\mu)\gamma(x,2)-\lambda\gamma(x,1)}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)}}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)} - \frac{\lambda}{\mu\left(\frac{1}{2}-x\right)+\frac{1}{\gamma(1-x,2)}} \equiv \Omega(x) \tag{20}$$

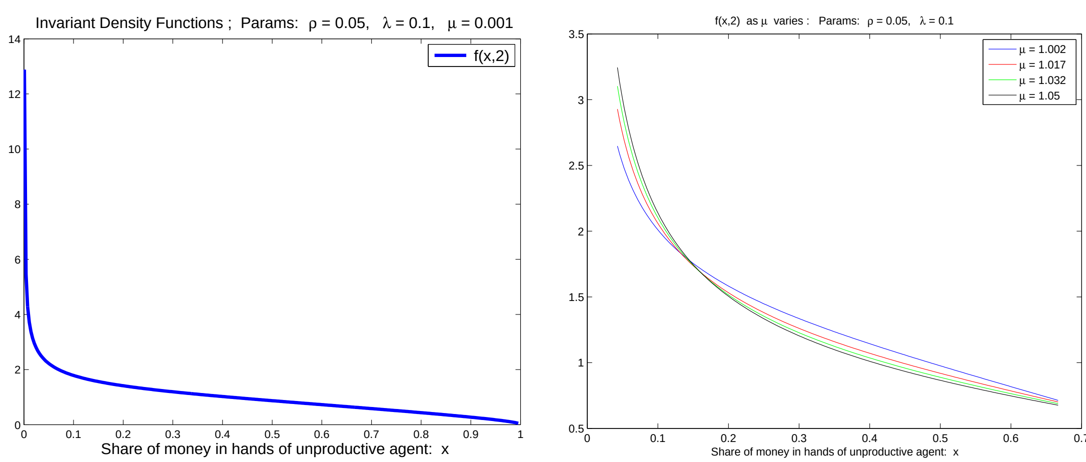
It follows that

$$f(x,2) = Ce^{\int_{1/2}^{x}\Omega(z)\ dz} \qquad \text{where} \qquad C = \left[\int_0^1 e^{\int_{1/2}^{x}\Omega(z)\ dz}\ dx\right]^{-1} \tag{21}$$

where the constant C ensures that $\int_0^1 f(x,2)\ dx = 1$. The next lemma establishes useful properties of the density function for money holdings:

Lemma 4 *Let* $Q_2 \equiv \frac{\lambda}{\frac{\mu}{2}+\frac{1}{\gamma(1,2)}}$*, with* $Q_2 > 0$ *and finite* $\forall\ \mu > 0$*. For* $x \in B_\varepsilon$*:* $f(x,2) = Ge^{\left(\frac{Q_1}{2x}-Q_2 x\right)}$*,* $G > 0$*, with (i)* $\lim_{x\downarrow 0} f(x,2) = +\infty$*, (ii)* $\lim_{x\downarrow 0} f_x(x,2) = -\infty$*, (iii) decreasing in* x*, and (iv) convex in* x*. Moreover* $\lim_{x\uparrow 1} f(x,2) = 0$*.*

The details of the proof can be found in Appendix F.



Remark 1 *The density* $f(x,2)$ *is continuous in* μ*.*

Remark 1 follows as from equation (20) we can see that $\frac{f_x(x,2)}{f(x,2)}$ is a continuous function of growth rate of money μ because it is a composition of continuous functions on μ. This implies that, for any μ, the density of unproductive agents has an asymptote at infinity as x

converges to zero from above. In the next section we will show that this implies that there is a non zero mass of agents arbitrarily close to the minimum consumption level defined by equation (18).

7 Optimality of expansionary monetary policy

In this section we discuss the optimality of a finite expansionary policy under the invariant distribution of money holdings. To do this, we consider the limit $\rho \downarrow 0$, as typically done in similar contexts for reason of tractability.

From an ex ante perspective, the social planner chooses the optimal growth rate of money $\hat{\mu}$ in order to maximize

$$\begin{aligned} \mathcal{W}(\mu) &= \mathbb{E}_x \left\{ u\left(c^1(x,2;\mu)\right) + u\left(c^2(1-x,2;\mu)\right) - l^2(1-x,2;\mu) \middle|\, \mu \right\} \\ &= \int_0^1 f(x,2;\mu)\left[u\left(c^1(x,2;\mu)\right) + u\left(c^2(1-x,2;\mu)\right) - l^2(1-x,2;\mu)\right]\, dx \end{aligned} \tag{22}$$

where the notation emphasizes that the consumption paths and densities of money holdings depend on the choice of money growth μ. The expression measures the sum of the expected utility of each type of agent from an ex ante perspective. Types are given equal weights because the symmetric Markov process for the shocks implies that agent are productive 1/2 of the time. We will use this expression to show that the expected utility approaches $-\infty$ when $\mu \downarrow 0$ or when $\mu \uparrow \infty$, and also that it is finite for $0 < \mu < \infty$. Then, the optimal monetary policy $\hat{\mu}$ is expansionary at a finite rate.

First we show that a policy with $\mu \uparrow \infty$ implies that trade approaches zero and hence it cannot be optimal. From equation (11) we have that $\gamma(0,2) = 2/\mu$ and from Lemma 2 that $\gamma_x(x,2) < 0$ and $\gamma(x,1) < \gamma(x,2)$ for all x. As $\mu \uparrow \infty$, these imply that $\gamma(x,s) \downarrow 0$ for all x and s. In other words, the marginal value of a unit of money becomes zero because the monetary policy wipes out completely the current value of money holdings. Intuitively, the reason is that such a policy destroys all the incentives for a productive agent to accept money in exchange for goods. In fact, independently of the distribution of outstanding money holdings at the beginning of the period x, this policy is such that the distribution of money holdings across agents is constant at $x = 1/2$, with all the mass concentrated on this point (see Appendix G for a proof). Therefore, as $\mu \uparrow \infty$ the economy approaches autarky. Recall that Proposition 1 showed that any policy with $\mu < 0$ also implies that in the ergodic set the economy is in autarky. Hence, the expected utility $\mathcal{W}(\mu)$ diverges to $-\infty$ for both of these cases: the reason is that ex-ante agents are unproductive 1/2 of the times, and in this state their consumption is zero.

Next we show that ex-ante expected utility for finite money growth rates, $0 < \mu < \infty$, is bounded below. In order to do so, we want to analyze the limiting behavior of the expected utility for an unproductive agent as her money holdings approaches zero. Consider the following component of ex-ante expected utility

$$\underline{\mathcal{W}}(\mu) \equiv \lim_{x \downarrow 0} \int_x^{2x} f(y,2) \ln c^1(y,2) dy < 0 \quad . \tag{23}$$

This is a part of expected utility that stems from the histories in which an unproductive agent has "small" money holdings.[10] It is immediate that if $\underline{\mathcal{W}}(\mu)$ is finite then expected utility $\mathcal{W}(\mu)$ is finite, since consumption is continuous and increasing in x and the expected utility of the productive agent is always finite. We now construct a lower bound for $\underline{\mathcal{W}}(\mu)$.

From Lemma 4, we know that for a small $\varepsilon > 0$ and $x \in B_\varepsilon$ we have that $f(x,2) = Ge^{\left(\frac{Q_1}{2x} - Q_2 x\right)}$ and that the function $f(x,2)$ is locally decreasing and convex. This means that for $x \in B_\varepsilon$, $f(x,2)$ can be bounded below by a tangent plane. That is,

$$f(x,2) = Ge^{\left(\frac{Q_1}{2x} - Q_2 x\right)} > G\left(1 + \frac{Q_1}{2x} - Q_2 x\right) \tag{24}$$

[10] That $\underline{\mathcal{W}}(\mu)$ is negative comes from the fact that $c^1(0,2) < 1$ which can be seen since $c^1(0,2) = \frac{\gamma(1,1)}{\gamma(0,2)} < \frac{\gamma(1,2)}{\gamma(0,2)} < 1$. The continuity of $c^1(x,2)$ ensures this is true in a neighborhood of $x = 0$.

which implies that

$$\lim_{x \downarrow 0} \int_x^{2x} f(y,2)dy > \lim_{x \downarrow 0} G \int_x^{2x} \left(1 + \frac{Q_1}{2y} - Q_2 y\right) dy > 0$$

i.e. that the probability mass in a small interval of size x is non-zero. Using equation (23), and that the lowest attainable utility is $\ln c^1(0,2) = \ln\left(\frac{\mu}{2}\gamma(1,1)\right)$, it is straightforward to derive the following bound

$$\underline{\mathcal{W}}(\mu) > \ln\left(\frac{\mu}{2}\gamma(1,1)\right) \lim_{x \downarrow 0} \int_x^{2x} f(y,2)dy \equiv H(\mu)$$

where $H(\mu)$ denotes the lower bound as a function of the money growth parameter. Since the value of the integral term is finite, then $H(\mu)$ is finite and negative for every $\mu > 0$, continuous in μ, and $H(\mu) \downarrow -\infty$ as $\mu \downarrow 0$. Hence there exists a finite μ such that $\underline{\mathcal{W}}(\mu)$ is bounded below.

Next we show that expected utility $\mathcal{W}(\mu)$ diverges to $-\infty$ as μ becomes arbitrarily small. Intuitively, since the consumption of an unproductive agent with $x = 0$ becomes zero as μ becomes arbitrarily small (see equation (18)), it is immediate that the instantaneous utility will diverge at this point. But in order for this feature to imply a divergence of $\mathcal{W}(\mu)$ we must have that the probability mass of histories that end up with very small values of x must be "sufficiently large", namely that it should go to zero at a slower rate than the rate at which consumption becomes zero. We develop this argument formally using equation (23).

Recall that for $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$, Lemma 3 showed that $\ln c^1(x,2) \approx Q_0 + Q_1 \ln x$. Thus the limiting expected utility for an unproductive agent when her money holdings approaches zero and μ is small can be written as

$$\underline{\mathcal{W}}(\mu) \approx \lim_{x \downarrow 0} \int_x^{2x} f(y,2) \left[Q_0 + Q_1 \ln y\right] dy < 0$$

were we used that for $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$, $\ln c^1(x,2) \approx Q_0 + Q_1 \ln x < 0$ and therefore $\underline{\mathcal{W}}(\mu) <$

0 because $f(x,2) > 0 \ \ \forall x$. To prove that this is the case recall that $\ln c^1(0,2) = \ln \frac{\mu}{2}\gamma(1,1)$ with $\lim_{\mu\downarrow 0} \ln c^1(x,2) = -\infty$. From inspection it is trivial that $\ln c^1(x,2)$ continuous in both x and μ. Therefore by a continuity argument we have that for $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$, $\ln c^1(x,2) \approx Q_0 + Q_1 \ln x < 0$.

Using the lower bound for $f(x,2)$ in equation (24), gives that for $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$ the limiting expected utility for an unproductive agent has the following upper bound

$$\lim_{x\downarrow 0} \int_x^{2x} f(y,2)\left[Q_0 + Q_1 \ln y\right] dy \quad < \quad \lim_{x\downarrow 0} \int_x^{2x} G\left(1 + \frac{Q_1}{2y} - Q_2 y\right)\left[Q_0 + Q_1 \ln y\right] dy = -\infty \tag{25}$$

This shows that for the small values of x and μ, which ensure that the consumption approximation from Lemma 3 is accurate, the expected utility component $\underline{\mathcal{W}}(\mu)$ can be made arbitrarily small.

We formalize this point as follows. Fix a value $K > 0$. Equation (25) ensures that there exists an $\varepsilon > 0$ such that for $\mu \in B_\varepsilon$ we have $\underline{\mathcal{W}}(\mu) < -K$. Note that there is no bound on how large K can be. Suppose we choose $K' > K$. Again equation (25) guarantees that there exists a smaller ball $\varepsilon' < \varepsilon$ such that for $\mu \in B_{\varepsilon'}$ we have $\underline{\mathcal{W}}(\mu) < -K'$. This shows that it is always possible to find a set of small values for μ, such that the limiting expected utility for an unproductive agent when x approaches zero is below any arbitrarily finite negative value. We summarize this in the next statement:

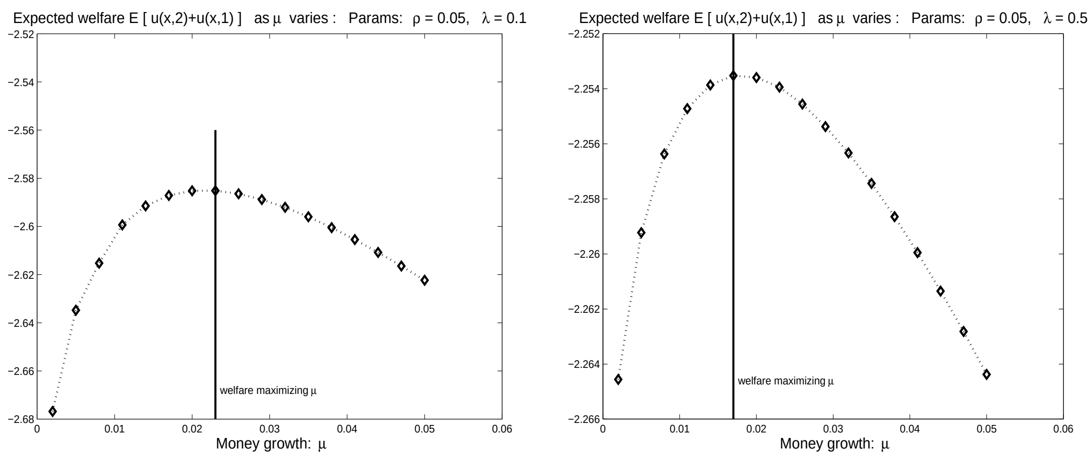
Corollary 1 *For any $0 < K < \infty$ there exists an $\varepsilon' > 0$ such that for any $\mu \in B_{\varepsilon'}$, $\underline{\mathcal{W}}(\mu) < -K$.*

This result, together with ex-ante expected utility going to $-\infty$ when μ goes to ∞, gives

Proposition 2 *The optimal monetary growth level $\hat{\mu}$ is strictly positive and finite. That is, $0 < \hat{\mu} < \infty$.*

The economics of this proposition are simple: in an economy with borrowing constraints agents will occasionally incur into histories, i.e. long spells of low-productivity, in which they run out of money. The positive money transfers the government distributes provide a floor to how bad consumption looks in these states. The formal analysis shows that the probability mass of such histories cannot be ignored in the welfare analysis. For this reason a strictly positive money growth rate is superior to constant money. On the other hand, positive money growth is costly because, by increasing inflation, it reduces the incentives for savings, and hence trading, for productive agents. The optimal finite money growth rate strikes a balance between these opposing forces.

$\lambda = 0.10$ $\lambda = 0.50$



8 Uncorrelated productivity shocks

In this section we show that the main result of Proposition 2 extends to a model where the productivity shocks are uncorrelated across agents. Assume a unit mass of agents, indexed by i, over the $[0, 1]$ interval. As before, the productivity state of each agent, s_i follows a Markov process, where λ denotes the rate at which the state switches.

Let m_t^i be the money holdings of agent i at time t, so that the total money supply is $m_t = \int_0^1 m_t^i \, di$. Let τ_t denote the per capita transfer from the government. The government budget constraint is

$$\tau_t = \tilde{q}_t \int_0^1 \dot{m}_t^i \, di = \mu \; m_t \; \tilde{q}_t = \mu \; q_t \tag{26}$$

where the last equality uses the homogeneity of $\tilde{q}$ with respect to m, a property that continues to hold in this model. In what follows we let $\mu \geq 0$. The same argument developed in Section 4 shows that no monetary equilibrium exists for $\mu < 0$.

Obviously $\int_0^1 x_t^i \, di = 1$ where $x_t^i = m_t^i / m_t$. Notice that in this model with a continuum of agents $x_i \in [0, +\infty)$, where $x_i = 1$ denotes the situation in which a single agent money balances equal the economy's average money, and $x_i \uparrow \infty$ denotes the situation in which one agent holds all of the money. Letting $h(x_t^i, s_t) = \dot{m}_t^i / m_t$ as above, gives $\dot{x}(x, s) = h(x, s) - x\mu$, where we omit the time and agent i indices for notation simplicity.

The agent first order conditions of this model are unchanged compared to the previous model (equation (8))). The unproductive agent budget constraint and Euler equation (i.e. when $s = 2$) give

$$\dot{x}(x, 2) = \mu \, (1 - x) - \frac{1}{\gamma(x, 2)} \tag{27}$$

This equation shows that money growth has no effect on the money share in the case where the agent's money holding equal the average money per capita in the economy, i.e. the ratio is $x_i = 1$.[11]

We assume the economy has a centralized competitive market where one unit of money buys $\tilde{q} = \frac{1}{m} q$ units of consumption. This implies that all productive agents are willing to produce in exchange for money as long as $\gamma(x, 1) > q$, and that there is a level of money holdings $\bar{x}$ where productive agents stop producing for the market: $\gamma(\bar{x}, 1) = q$. This implies that for a productive agent

$$\dot{x}(x, 1) = \mu \, (1 - x) + \bar{x} - x$$

Before moving on with solving the model we define an equilibrium for this economy.

Definition 2 *A monetary equilibrium is a price function $\tilde{q}(m) = \frac{1}{m} \, q$, with $q \in \mathbb{R}^+$, and a stochastic process $x(t, \omega)$ with values in $[0, 1]$, such that a consumer i maximizes expected discounted utility (equation (2)) subject to the constraints (equations (3), (4), and (5)) with*

[11] Note that the same property holds in the model with two agents, in which the total mass is 2, the index $x^i \in [0, 1]$ and an equal distribution of money holdings implies that the ratio of the agent money to the average (per capita) money holdings is $1/2$, so that μ does not affect $\dot{x}$ when $x = 1/2$.

$q(t, \omega) = q(s)$ and the government budget constraint (equation (26)) is balanced.

The lagrange multipliers $\gamma(x, 1), \gamma(x, 2)$ solve the system of differential equations that we determined before (equation (9) and (10)), which under the assumptions of this section (using equation (27)) simplifies to

$$\begin{aligned} \gamma_x(x,1)\left[\mu\,(1-x)+\bar{x}-x\right] &= (\rho+\mu)\gamma(x,1)-\lambda\left(\gamma(\bar{x},2)-q\right) \\ \gamma_x(x,2)\left[\mu\,(1-x)-\frac{1}{\gamma(x,2)}\right] &= (\rho+\mu)\gamma(x,2)-\lambda\left(\gamma(x,1)-\gamma(x,2)\right) \end{aligned}$$

where we used that $\gamma(\bar{x}, 1) = q$. The system decouples in two ODEs. The solution to the first one is

$$\gamma(x,1) = \lambda\,\frac{\gamma(\bar{x},2)-q}{\rho+\mu} \tag{28}$$

which indicates that the marginal utility of money for a productive agent is constant.[12] Since this equation also holds at $\bar{x}$, then $\gamma(x, 1) = q$. The ODE for $\gamma(x, 2)$ can be rewritten as

$$\gamma_x(x,2) = \frac{(\rho+\lambda+\mu)\gamma(x,2)^2-\lambda q\gamma(x,2)}{\gamma(x,2)\,(\mu)\,(1-x)-1} \tag{29}$$

with the following boundary conditions at $x = 0$ and $x = \bar{x}$:

$$\gamma(0,2) = \frac{1}{\mu} \qquad \text{and} \qquad \gamma(\bar{x},2) = q\left(\frac{\rho+\mu}{\lambda}+1\right)$$

where the first boundary is obtained from the unproductive agent's Euler equation and budget constraint at $x = 0$, while the second one is implied by equation (28). Notice that this problem has two unknowns, $q, \bar{x}$, and a constant of integration to be determined in the solution of equation (29). Three equations can be used to solve for the three unknowns: the two boundary conditions, and the condition that $\int_0^1 x_i\, di = 1$.[13]

[12]The assumption of linear disutility of labor is important for this result, as it implies that the productive agent immediately refills his money balances up to $\bar{x}$.

[13]The simplicity of the model with uncorrelated shocks allow us to show existence of equilibrium for the case with constant money (i.e. $\mu = 0$) in a very simple way. The characterization can be found in Appendix H.

Recall that consumption when unproductive obeys equation (16), which in this environment where $\gamma(x,1) = q$ amounts to

$$c^1(x,2) = \frac{q}{\gamma(x,2)} \quad \text{where, for } x = 0\text{:} \quad c^1(0,2) = \mu q$$

Note that this implies that consumption whe unproductive is increasing in x since

$$c^1_x(x,2) = -\frac{q}{\gamma(x,2)^2}\gamma_x(x,2) > 0$$

The highest consumption when unproductive obtains when $x = \bar{x}$, and is given by

$$c^1(\bar{x},2) = \frac{q}{\gamma(\bar{x},2)} = \frac{\lambda}{\lambda+\rho+\mu} \tag{30}$$

which shows that consumption when unproductive converges to zero as $\mu \to \infty$.

Let ε and ε' denote two arbitrarily small and positive numbers. Next we characterize the local behavior of consumption for an unproductive agent when her money holdings are close to 0 in a similar fashion to Lemma 3.

Lemma 5 $\lim_{x \downarrow 0} \frac{c^1_x(x,2)x}{c^1(x,2)} = Q_4$*, where* $Q_4 = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu} - \lambda q > 1$ *and finite* $\forall\ \mu \in (0,1)$*. Moreover, for* $x \in B_\varepsilon$ *and* $\mu \in B_{\varepsilon'}$, $\ln c^1(x,2) \approx Q_5 + Q_4 \ln x$*.*

The proof of the lemma is very similar to the proof of Lemma 3 and therefore is omitted. Finally, we discuss the invariant density of money holdings in this economy. Productive agents have money holdings $\bar{x}$, so that $F(x,1) = 0$ if $x < \bar{x}$, and $F(\bar{x},1) = 1$. The density function for unproductive agents can be derived from the usual Kolmogorov forward equation. Using $f(x,1) = 0$ and equation (27) to replace $\dot{x}(x,2)$ into equation (19) gives

$$\frac{f_x(x,2)}{f(x,2)} = \frac{(\lambda-\mu)\gamma(x,2) - \frac{(\rho+\lambda+\mu)\gamma(x,2)-\lambda q}{1+\mu(x-1)\gamma(x,2)}}{1+\mu\,(x-1)\,\gamma(x,2)} \tag{31}$$

(see Appendix E for more details).

Using this expression we can evaluate the local behavior of the density of money holdings close to 0.

Lemma 6 *For $x \in B_\varepsilon$ and $\mu \in (0,1)$: $f(x,2) = Q_6 e^{\frac{Q_4}{x}}$, $Q_6 > 0$, with (i) $\lim_{x\downarrow 0} f(x,2) = +\infty$, (ii) $\lim_{x\downarrow 0} f_x(x,2) = -\infty$, (iii) decreasing in x, (iv) convex in x.*

The proof of the lemma is very similar to the proof of Lemma 4 and therefore is omitted.

We now have all the necessary objects to discuss the optimal monetary policy in the economy with uncorrelated shocks. For $\mu < 0$ a similar argument to the one discussed in Section 4 shows that there is no monetary equilibrium and furthermore in the ergodic set no trade happens. Therefore, ex ante expected utility is $-\infty$. For $\mu \to \infty$, we had shown that the consumption function for an unproductive agent is increasing and her maximum consumption happens when $x = \bar{x}$. From equation (30) we have that $\lim_{\mu\uparrow\infty} c^1(\bar{x}, 2) = 0$ which further implies that ex ante expected utility is $-\infty$. Using Lemma 5 and Lemma 6 we can repeat the steps presented in Section 7 for the case with correlated shocks to show that for $x \in B_\varepsilon$ and $\mu \in B_{\varepsilon'}$ ex ante expected utility is $-\infty$ while finite for finite values of the monetary policy μ. Therefore, the optimal monetary policy is strictly positive and finite.

9 Concluding remarks and future work

This paper analyzed the optimal anticipated money growth rule in the Scheinkman and Weiss (1986) economy. The choice involves a tradeoff between insuring agents who are hit by long sequences of bad shocks vs. minimizing the distortion that the inflation tax induces on production decisions. We showed that a monetary policy involving deflation, such as the Friedman rule, or an infinite inflation, do not support a monetary equilibrium, and leads to autarky. With a constant money supply the government provides no insurance: we showed that this corner solution is not optimal; from an ex-ante perspective, agents are better off with strictly positive money growth. The presence of uninsurable risk is key to this result: in the non-stochastic of our economy the Friedman rule is optimal.

Two important assumptions behind the optimality of the expansionary monetary policy are that the marginal utility of consumption approaches infinity as consumption approaches zero (Inada conditions) and that unproductive agents can only consume if their money balances are not zero (e.g. no home production is allowed). Under these assumptions, the expansion of the money supply is good because the precautionary savings motive that follow from the unboundedness of the utility function dominates the cost of expansions (decrease in production). In other words, providing some extra cash to poor agents is always optimal even though it implies a decrease in aggregate production. If we relax either assumption, e.g. assume that the utility function is bounded or extend the model to allow for positive endowments, an expansionary monetary policy might not be optimal anymore as the precautionary savings motive can be dominated by the inflation effect on production. Kehoe, Levine, and Woodford (1990) explore a particular case of these economies, one where there are constant endowments, no production, and where agents differ in their marginal utility of consumption. Because agents can trade any fraction of their endowment, they can always overcome Inada conditions by simply keeping a fraction of their endowment for personal consumption. They show numerically that the optimality of an expansionary policy depends on parameter values and therefore is not guaranteed.

What would happen to optimality in our model if we modify either of the assumptions is not clear. But we conjecture that our results are robust to the introduction of small endowments. We argue that this is the case as our proof hinges on showing that an expansionary monetary policy dominates constant and contractionary monetary policies. Note that in our benchmark model endowments are zero. Because of continuity this implies that small endowments should not reverse the result, even though they provide some insurance

Another issue that we see as interesting for future research is to numerically investigate the properties of optimal state dependent rules. In this model we did not allow the money growth to vary with the business cycle. Intuitively, it would seem reasonable that a government wants to expand the money supply in a recession (e.g. when x is small), but may want to deflate when x is more evenly distributed. We leave this topic for future research.

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Appendix

A The Euler equation for the marginal utility of money

The maximum principle implies that the lagrange multiplier $\tilde{\gamma}$ follows[14]

$$
\begin{aligned}
&\mathbb{E}\left\{ e^{-\rho(t+dt)}\tilde{\gamma}(m(t+dt,\omega),x(t+dt,\omega),s(t+dt,\omega)) \,\middle|\, m(t,\omega)=m, x(t,\omega)=x, s(t,\omega)=1 \right\}\\
&\approx e^{-\rho t}\left[-\rho\tilde{\gamma}(m_t,x_t,1)dt+\tilde{\gamma}_x(m_t,x_t,1)\dot{x}_t dt+\tilde{\gamma}_m(m_t,x_t,1)\dot{m}_t dt+\tilde{\gamma}(m_t,x_t,1)(1-\lambda dt)\right]\\
&+e^{-\rho t}\tilde{\gamma}(m_t,x_t,2)\lambda dt\\
&=\frac{e^{-\rho t}}{m_t}\left[-\rho\gamma(x_t,1)dt+\gamma_x(x_t,1)\dot{x}_t dt-\gamma(x_t,1)\frac{\dot{m}_t}{m_t}dt+\gamma(x_t,1)(1-\lambda dt)+\gamma(x_t,2)\lambda dt\right]\\
&=\frac{e^{-\rho t}}{m_t}\left[\gamma(x_t,1)+\gamma_x(x_t,1)\dot{x}_t dt-\gamma(x_t,1)(\rho+\lambda+\mu)dt+\gamma(x_t,2)\lambda dt\right]
\end{aligned}
$$

Subtracting $e^{-\rho t}\gamma(x_t,1)/m_t$ from both sides, dividing by dt, taking the limit for $dt \downarrow 0$, and using equation (7) to replace $\dot{x}$ with its law of motion, gives the delay differential equation (9). An identical logic gives equation (10).

B Proof of Lemma 1

We will first prove by contradiction that $\bar{x} \notin [0, 1/2)$. Then we will show that $\bar{x} = 1/2$ it is enough to cover the lifetime tax obligations. Suppose that $\bar{x} < 1/2$. Without loss of generality assume that $x_t \in (\bar{x}, 1/2)$ and agent 1 is unproductive. Conditional on no reversal of the state, it follows that $x_{t+dt} < x_t$. Then for a given $\Delta \in \mathbb{R}^+$, $\Pr[x_{t+\Delta} < \bar{x}] > 0$ and therefore the agent will fail to comply with her tax obligations with positive probability. Then, $\bar{x} \notin [0, 1/2)$. Consider now the case where $x_t = \bar{x} = 1/2$. As the agent can decide not to trade she can always keep her share of outstanding money balances x above $1/2$ and therefore for any $\mu \in (0, 1)$ she will be able to cover her tax needs. That $x = 1/2$ is the ergodic set is trivial. If $x_0 < 1/2$ there is a positive probability that an agent fails to pay for her lifetime taxes. An unproductive agent with money holdings $x > 1/2$ is willing to buy goods (and the productive one with $x < 1/2$ willing to take the money) until x reaches $1/2$.

C Proof of Lemma 2

That $\gamma(x, s) > 0$, for $s = 1, 2$ is immediate for all internal solutions from the Khun-Tucker theorem. Next conjecture that $\dot{x}(x, 2) < 0$, as verified below. Notice that, since $\dot{x}(x, 2) + \dot{x}(1-x, 1) = 0$ it follows that $\dot{x}(x, 1) > 0$. Under these assumptions, the left side of equation (9) is smaller than zero. Using the right hand side of the equation we get $(\rho + \mu + \lambda)\gamma(x, 1) - \lambda\gamma(x, 2) < 0$ which gives

$$0 < \gamma(x,1) < \gamma(x,2) \quad . \tag{32}$$

[14] The variable $\tilde{\gamma}$ is a costate. This condition is equivalent to the first order condition $\dot{\tilde{\gamma}} - \rho\tilde{\gamma} = 0$ in the current value Hamiltonian program.

Next we show that this inequality implies $\dot{x}(x,2) < 0$. Differentiate the unproductive agent foc in equation (8), to write

$$\frac{\dot{\gamma}(x,2)}{\gamma(x,2)} = \frac{\dot{q}(x,2)}{q(x,2)} + \frac{u''\left(c^1(x,2)\right)}{u'\left(c^1(x,2)\right)}\, \dot{c}^1(x,2)$$

or, using that $q(x,2) = \gamma(1-x,1)$ and $\dot{c} = c_x\, \dot{x}$, we get

$$\frac{c_x^1(x,2)}{c^1(x,2)}\, \dot{x}(x,2) = \left[\frac{\dot{\gamma}(x,2)}{\gamma(x,2)} - \frac{\dot{\gamma}(1-x,1)}{\gamma(1-x,1)}\right] \frac{u'\left(c^1(x,2)\right)}{u''\left(c^1(x,2)\right)\, c^1(x,2)}$$

Assuming a CRRA utility such that $\frac{u'}{u''\,c} = -\theta < 0$, and using equation (9) and equation (10) to replace the terms in the square parenthesis gives

$$\frac{c_x^1(x,2)}{c^1(x,2)}\quad \dot{x}(x,2) = -\lambda\theta \left[\frac{\gamma(1-x,2)}{\gamma(1-x,1)} - \frac{\gamma(x,1)}{\gamma(x,2)}\right] < 0 \tag{33}$$

where the inequality follows since the term in the square parenthesis is positive (as implied by equation (32)). Since $c_x^1(x,2) > 0$, as from equation (17), it then follows that $\dot{x}(x,2) < 0$. The economic interpretation of this result is that the expected change in the value of money (the term $\frac{\dot{\gamma}(x,2)}{\gamma(x,2)}$) is larger than the expected return on money (the term $\frac{\dot{\gamma}(1-x,1)}{\gamma(1-x,1)}$), so that saving is not optimal. Equation (32)) and $\dot{x}(x,2) < 0$ imply, using the right hand side of equation (10), that $\gamma_x(x,2) < 0$.

Finally we show that $\lim_{x\downarrow 0} \dot{x}(x,2) = 0$. The right hand side of equation (33) is strictly negative at all x, including the boundary $x = 0$. Since x cannot be negative, this implies that $c_x^1(x,2) \uparrow +\infty$ and $\dot{x}(x,2) \uparrow 0$. Equation (34) in Appendix D can be used to verify that $\lim_{x\downarrow 0} c_x^1(x,2) = +\infty$.

D Proof of Lemma 3

That $Q_1 > 1$ follows since $\gamma(0,2) > \gamma(0,1)$. Using equation (16) and equation (17) we write

$$\frac{c_x^1(x,2)}{c^1(x,2)} = -\frac{\gamma_x(1-x,1)}{\gamma(1-x,1)} - \frac{\gamma_x(x,2)}{\gamma(x,2)}$$

From equation (14) (evaluated at $1-x$) and equation (15) (evaluated at x) we get

$$\frac{\gamma_x(1-x,1)}{\gamma(1-x,1)} = \frac{\rho + \lambda + \mu - \lambda\frac{\gamma(1-x,2)}{\gamma(1-x,1)}}{\mu\left(x - \frac{1}{2}\right) + \frac{1}{\gamma(x,2)}} \quad , \quad \frac{\gamma_x(x,2)}{\gamma(x,2)} = \frac{\rho + \lambda + \mu - \lambda\frac{\gamma(x,1)}{\gamma(x,2)}}{\mu\left(\frac{1}{2} - x\right) - \frac{1}{\gamma(x,2)}}\,.$$

Then, noting that $\gamma(0,2) = \frac{2}{\mu}$, some algebra gives that

$$\lim_{x\downarrow 0} \frac{c_x^1(x,2)x}{c^1(x,2)} = 1 + \frac{\rho}{\mu} + \frac{\lambda}{\mu}\left(1 - \frac{\gamma(0,1)}{\gamma(0,2)}\right) \equiv Q_1 \tag{34}$$

which proves the first part of the lemma.

We now turn to the approximation for small μ and x. Note that for $x \in B(\varepsilon)$, equation (34) implies the ordinary differential equation, $\frac{c^1_x(x,2)}{c^1(x,2)} = \frac{Q_1}{x}$ with solution $\tilde{c}^1(x,2) = \tilde{Q}_0 x^{Q_1}$, which approximates the true function $c^1(x,2)$ in the neighborhood of $x=0$. Notice that $\lim_{x\downarrow 0} \tilde{c}^1(x,2) = 0 \quad , \forall \mu > 0$. Hence the approximation error on the boundary condition $c^1(0,2) = \frac{\mu}{2}\gamma(1,1)$ converges to zero as $\mu \downarrow 0$. Taking logs and defining $Q_0 \equiv \ln(\tilde{Q}_0)$ provides $\ln(\tilde{c}^1(x,2)) = Q_0 + Q_1 \ln(x)$ which gives the approximation used in the text.

E Derivation of the invariant wealth distribution

The CDF for the money holdings, $F(x,s,t)$, with density $f(x,s,t)$ in states $s = 1,2$ follows

$$\begin{aligned} F(x,1,t+dt) &= (1-\lambda\ dt)F(x-\dot{x}(x,1)\ dt,1,t) + \lambda\ dt\ F(x-\dot{x}(x,2)\ dt,2,t) \\ F(x,2,t+dt) &= (1-\lambda\ dt)F(x-\dot{x}(x,2)\ dt,2,t) + \lambda\ dt\ F(x-\dot{x}(x,1)\ dt,1,t) \end{aligned}$$

Expanding $F(x,s,t)$ around x gives (we only report the one for $s=2$)

$$F(x,2,t+dt) = (1-\lambda\ dt)\left[F\left(x,2,t\right) - f\left(x,2,t\right)\dot{x}(x,2)dt\right] + \lambda dt\ \left[F\left(x,1,t\right) - f\left(x,1,t\right)\dot{x}(x,1)dt\right]$$

Subtracting $F(x,2,t)$ from both sides and dividing by dt and taking the limit for $dt \downarrow 0$

$$\lim_{dt\downarrow 0} \frac{F(x,2,t+dt)-F(x,2,t)}{dt} = \frac{\partial F(x,2,t)}{\partial t} = -f\left(x,2,t\right)\dot{x}(x,2) - \lambda\left(F\left(x,2,t\right) - F\left(x,1,t\right)\right)$$

Using this equation together with the corresponding one for state $s=1$ and imposing invariance give

$$0 = f\left(x,2\right)\dot{x}(x,2) + f\left(x,1\right)\dot{x}(x,1) \tag{35}$$

Taking the derivative w.r.t. x delivers the Kolmogorov forward equation

$$\frac{\partial\ \partial F(x,2,t)}{\partial x\ \partial t} = \frac{\partial f(x,2,t)}{\partial t} = \frac{\partial\left[-f\left(x,2,t\right)\dot{x}(x,2) - \lambda\left(F\left(x,2,t\right) - F\left(x,1,t\right)\right)\right]}{\partial x}$$

which, equated to zero (imposing invariance) gives equation (19).

E.1 Application to model with correlated shocks

Using the expressions in (13) to replace $\dot{x}(x,1)$ and $\dot{x}(x,2)$ into equation (19) gives

$$\frac{f_x(x,2)}{f(x,2)} = \frac{(\lambda-\mu)\gamma(x,2) + \frac{\gamma_x(x,2)}{\gamma(x,2)} - \lambda\gamma(x,2)\frac{f(x,1)}{f(x,2)}}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)}$$

Using equation (35) to replace the ratio $f(x,2)/f(x,1)$ in the above expression gives equation (20).

E.2 Application to model with uncorrelated shocks

For $x \in [0, \bar{x})$ we have $f(x,1) = 0$. Using equation (27) to replace $\dot{x}(x,2)$ in equation (19)

$$\frac{f_x(x,2)}{f(x,2)} = \frac{(\lambda-\mu)\gamma(x,2) + \frac{\gamma_x(x,2)}{\gamma(x,2)}}{1+\mu\,(x-1)\,\gamma(x,2)}$$

Using equation (29) to replace for $\frac{\gamma_x(x,2)}{\gamma(x,2)}$ in the above expression gives equation (31).

F Proof of Lemma 4

Consider the expression for $\frac{f_x(x,2)}{f(x,2)}$ in equation (20). When $x \downarrow 0$ the last term $\frac{\lambda}{\mu\left(\frac{1}{2}-x\right)+\frac{1}{\gamma(1-x,2)}}$ converges to $Q_2 \equiv \frac{\lambda}{\frac{\mu}{2}+\frac{1}{\gamma(1,2)}}$. Next let us analyze the behavior of the first term in the same equation as $x \downarrow 0$. Recall that $\gamma(0,2) = \frac{2}{\mu}$ so that

$$\lim_{x\downarrow 0} \frac{\gamma(x,2)\,(\lambda-\mu) - \frac{(\rho+\lambda+\mu)\gamma(x,2)-\lambda\gamma(x,1)}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)}}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)} = \lim_{x\downarrow 0} \frac{\lambda\gamma(x,1) - 2\left(\frac{\rho+\lambda}{\mu}+1\right)}{4x^2}$$

Because $\gamma(0,1) < \frac{2}{\mu}$ and finite for every μ as discussed before. Then the numerator of the last equation is finite and therefore the limit of the last equation is not finite. In fact, the limit of the numerator as x approaches 0 is $-2Q_1$. Therefore, $\lim_{x\downarrow 0} \frac{\lambda\gamma(0,1)-2\left(\frac{\rho+\lambda}{\mu}+1\right)}{4x^2} = \lim_{x\downarrow 0} \frac{-Q_1}{2x^2} < 0$ because $Q_1 > 1$ as established in Appendix D.

Note that then we have the following equation for x in the neighborhood of 0,

$$\lim_{x\downarrow 0} \frac{f_x(x,2)}{f(x,2)} = \lim_{x\downarrow 0} -\frac{Q_1}{2x^2} - Q_2$$

which is an ordinary differential equation. Solving the ODE and noting that the constant of integration is a constant $G > 0$ gives

$$\lim_{x\downarrow 0} f(x,2) = \lim_{x\downarrow 0} G e^{\left(\frac{Q_1}{2x} - Q_2 x\right)}$$

with $\lim_{x\downarrow} f(x,2) = +\infty$ and $\lim_{x\downarrow 0} f_x(x,2) = -\infty$. Moreover, note that

$$\begin{aligned} f_x(x,2) &= -\left(Q_2 + Q_1/2x^2\right) f(x,2) &< 0 \\ f_{xx}(x,2) &= \left[Q_1/2x^3 + \left(Q_2+Q_1/2x^2\right)^2\right] f(x,2) &> 0 \end{aligned}$$

Therefore, $f(x,2)$ is locally decreasing and convex.

We now turn to the last statement in the lemma. When $x \uparrow 1$ the first term of $\frac{f_x(x,2)}{f(x,2)}$ is a finite number provided that $\gamma(x,2) > 0 \;\; \forall \;\; x$. That is,

$$\lim_{x\uparrow 1} \frac{\gamma(x,2)\left(\lambda-\mu\right) - \frac{(\rho+\lambda+\mu)\gamma(x,2)-\lambda\gamma(x,1)}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)}}{1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)} \equiv Q_3$$

with $\lim_{\mu\downarrow 0} Q_3$ and $\lim_{\mu\uparrow\infty} Q_3$ are both finite. As Q_3 is continuous in μ, then Q_3 is finite for every admissible μ. Now we analyze the second term of $\frac{f_x(x,2)}{f(x,2)}$ as $x \uparrow 1$.

$$\lim_{x\uparrow 1}\left(-\frac{\lambda}{\mu\left(\frac{1}{2}-x\right)+\frac{1}{\gamma(1-x,2)}}\right) = -\frac{\lambda}{\mu} \;\; \lim_{x\uparrow 1}\left(\frac{1}{1-x}\right)$$

Then, when $x \uparrow 1$, $f(x,2)$ solves the following ODE,

$$f_x(x,2) = Q_3 f(x,2) - \left(\frac{\lambda}{\mu}\right)\left(\frac{1}{1-x}\right) f(x,2)$$

Therefore,

$$\lim_{x\uparrow 1} f(x,2) \propto \lim_{x\uparrow 1} e^{Q_3 x}(x-1)^{\frac{\lambda}{\mu}} = 0 \; .$$

G Mass point at $x = 1/2$ as $\mu \uparrow \infty$

From equation (20) we can rewrite $f_x(x,2)/f(x,2)$ as

$$\frac{f_x(x,2)}{f(x,2)} = \frac{\gamma(x,2)\left(\frac{\lambda}{\mu}-1\right)}{\frac{1}{\mu}+\left(x-\frac{1}{2}\right)\gamma(x,2)} - \frac{\gamma(x,2)\left(\frac{\rho+\lambda}{\mu}+1\right)}{\mu\left(\frac{1}{\mu}+\left(x-\frac{1}{2}\right)\gamma(x,2)\right)^2}$$
$$+\frac{\lambda\gamma(x,1)}{\left[1+\mu\left(x-\frac{1}{2}\right)\gamma(x,2)\right]^2} - \frac{\lambda}{\mu\left(\frac{1}{2}-x\right)+\frac{1}{\gamma(1-x,2)}}$$

taking the limit as $\mu \uparrow \infty$,

$$\lim_{\mu\to\infty}\frac{f_x(x,2)}{f(x,2)} = \frac{1}{\frac{1}{2}-x} \text{ for } x > 0$$

which is an ODE with solution $f(x,2) = \frac{C_0}{\frac{1}{2}-x}$. For $x > 0$ the only consistent solution is $C_0 = 0$ (otherwise $f(x,2) < 0$ for some values of x). This implies that $f(x,2)$ is discontinuous. Notice that $\lim_{\mu\to\infty}\frac{f_x(x,2)}{f(x,2)}$ is negative for $x > \frac{1}{2}$ and positive for $x < \frac{1}{2}$. In particular

$$\lim_{x\uparrow\frac{1}{2}}\left(\lim_{\mu\to\infty}\frac{f_x(x,2)}{f(x,2)}\right) = +\infty \quad , \quad \lim_{x\downarrow\frac{1}{2}}\left(\lim_{\mu\to\infty}\frac{f_x(x,2)}{f(x,2)}\right) = -\infty$$

which tells us that there is a mass point at $\frac{1}{2}$. Also note that $\lim_{\mu\to\infty}\frac{f_x(0,2)}{f(0,2)}=0$ so there cannot be a mass point at $x=0$. Then, there exists a unique mass point at $x=\frac{1}{2}$,

$$f(x,2)=\begin{cases}1 & \text{for } x=1/2\\ 0 & \text{otherwise}\end{cases}$$

Because $f(1/2,2)=f(1/2,1)=1$, a productive agent has no incentives to sell goods to an unproductive agent because she cannot affect the ex-post distribution of money holdings.

H Uncorrelated shocks and constant money ($\mu = 0$)

It is interesting to analyze the case with $\mu=0$ for the model with uncorrelated shocks because of its simplicity. We start by discussing existence of equilibria.

After setting $\mu=0$ equation (29) reduces to the following ODE: $\gamma_x(x,2)=\lambda q\gamma(x,2)-(\rho+\lambda)\gamma(x,2)^2$, with boundary conditions that reduce to $lim_{x\downarrow 0}\gamma(x,2)=\infty$ and $\gamma(\bar{x},2)=q\frac{\rho+\lambda}{\lambda}$. Using the ODE and the first boundary provides an expression for the marginal value of money for an unproductive agent as a function of the price level q,

$$\gamma(x,2)=\frac{q\lambda}{\rho+\lambda}\frac{e^{q\lambda x}}{e^{q\lambda x}-1}$$

which is strictly positive for every $q>0$.

Using the expression we just obtained for $\gamma(x,2)$ and the second boundary condition provides an equation that relates $\bar{x}$ and q,

$$\bar{x}=\frac{1}{q\lambda}\ln\left(\frac{\left(\frac{\rho+\lambda}{\lambda}\right)^2}{\left(\frac{\rho+\lambda}{\lambda}\right)^2-1}\right)>0$$

from where it can be seen that $\frac{\partial\bar{x}}{\partial q}=-\bar{x}/q<0$. Note that this also implies a unit elasticity between $\bar{x}$ and the price level q.

Now we turn to evaluate the density of money holdings $f(x,2)$. Setting $\mu=0$ in equation (31) reduces to

$$\frac{f_x(x,2)}{f(x,2)}=\lambda q\left(1-\frac{\rho}{\lambda+\rho}\frac{e^{q\lambda x}}{e^{q\lambda x}-1}\right)$$

where we used the expression we found for $\gamma(x,2)$. This expression is an ODE with solution $f(x,2)=Q_7 e^{q\lambda x}\left(e^{q\lambda x}-1\right)^{-\frac{\rho}{\rho+\lambda}}$, where Q_7 is a constant to be determined next. We have that

$$\int_0^{\bar{x}}f(y,2)dy=1-\frac{\lambda}{2}$$

because there is a mass point at $\bar{x}$ with mass $\lambda/2$. This is because of the half of the population that are productive in every instant with $x=\bar{x}$, a fraction λ of them switch states and become unproductive. We can use this equation to obtain Q_7. Therefore, the density $f(x,2)$ is

$$f(x,2) = q\lambda\frac{\lambda}{\rho+\lambda}\left(1-\frac{\lambda}{2}\right)\left(\left(\frac{\rho}{\rho+\lambda}\right)^2-1\right)^{\frac{\lambda}{\rho+\lambda}}\frac{e^{q\lambda x}}{(e^{q\lambda x}-1)^{\frac{\rho}{\rho+\lambda}}}$$

with $\lim_{x\downarrow 0} f(x,2) = \infty$ and

$$f_x(x,2) = q\lambda\left(1-\frac{\rho}{\rho+\lambda}\frac{e^{q\lambda x}}{e^{q\lambda x}-1}\right)f(x,2)$$

What remains is to provide existence and uniqueness of q.

Recall that $\int_0^1 x_i di = 1$. This expression reduces to

$$\frac{1}{2}\left(\int_0^{\bar{x}} yf(y,2)dy + \bar{x}\frac{\lambda}{2}\right) + \frac{\bar{x}}{2} = 1$$

or

$$\int_0^{\bar{x}} yf(y,2)dy = 2 - \bar{x}\frac{2+\lambda}{2}$$

Let $\Upsilon(q) \equiv \int_0^{\bar{x}} yf(y,2)dy$ and $\Psi(q) \equiv 2-\bar{x}\frac{2+\lambda}{2}$. It is easy to check that $\Psi'(q) > 0$, $\lim_{q\downarrow 0}\Psi(q) = -\infty$, and $\lim_{q\uparrow\infty}\Psi(q) = 2$.

Now we turn to characterize $\Upsilon(q)$. When q approaches 0 the density $f(x,2)$ degenerates into a single mass point at $\bar{x}$.[15] Moreover, $\bar{x}$ approaches ∞ as q approaches 0. Then, $\lim_{q\downarrow 0}\Upsilon(q) = \infty$. Also, $\Upsilon(q) > 0$ and $\lim_{q\uparrow\infty}\Upsilon(q) = 0$.

Therefore, $\Upsilon(q)$ and $\Psi(q)$ intersect at least once. Then, $\exists\ q > 0$ so that we verified existence of equilibrium.

[15] $\lim_{q\downarrow 0} f(x,2) = 0$.