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Optimal Price Setting with Observation and Menu Costs

by

Fernando Alvarez

(University of Chicago)

Francesco Lippi

(University of Sassari and EIEF)

Luigi Paciello

(EIEF)

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Fernando Alvarez
University of Chicago

Francesco Lippi
University of Sassari, EIEF

Luigi Paciello
EIEF

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Abstract

We study the price setting problem of a firm in the presence of both observation and menu costs. In this problem the firm optimally decides when to collect costly information on the adequacy of its price, an activity which we refer to as a price "review". Upon each review, the firm chooses whether to adjust its price, subject to a menu cost, and when to conduct the next price review. This behavior is consistent with recent survey evidence documenting that firms revise prices infrequently and that only a few price revisions yield a price adjustment. The goal of the paper is to study how the firm's choices map into several observable statistics, depending on the level and relative magnitude of the observation vs the menu cost. The observable statistics are: the frequency of price reviews, the frequency of price adjustments, the size-distribution of price adjustments, and the shape of the hazard rate of price adjustments. We provide an analytical characterization of the firm's decisions and a mapping from the structural parameters to the observable statistics. We compare these statistics with the ones obtained for the models with only one type of cost. The predictions of the model can, with suitable data, be used to quantify the importance of the menu cost vs. the information cost. We also consider a version of the model where several price adjustment are allowed between observations, a form of *price plans* or indexation. We find that no indexation is optimal for small inflation rates.

1 Introduction

We study the optimal control problem of an agent subject to two types of adjustment costs. One is a standard fixed cost of adjusting the state, the other is a fixed cost of observing the state. The effects of each of these adjustment costs have been thoroughly analyzed in the literature in a variety of contexts. An example of the fixed adjustment cost is the canonical sS problem. The analysis of the implications of costly observation of the relevant state is more recent, yet examples abound.[1] The model in this paper incorporates both costs, and provides an analytical characterization of the policy rules and the implications for several observable statistics.

We develop this model with several applications in mind, ranging from the consumption-savings, household portfolio choice, to the price-setting problem of a monopolist. Moreover, the availability of new data sets makes it possible to compare the theory with actual measures of observation and adjustment frequencies. For concreteness, and because of its importance in macroeconomics, in this paper we focus on the price-setting problem.[2] In this problem the firm optimally decides when to collect costly information on the adequacy of its price, an activity which we refer to as a price "review". In several papers Reis provides a broader interpretation of this activity, as well as the relationship to the rational inattention literature –see, for example, section 2.1 of Reis (2006a). Upon each review, the firm chooses whether to adjust its price, subject to a menu cost, and when to conduct the next price review. The goal of the paper is to study how the firm's choices concerning several observable statistics depend on the level and relative magnitude of the observation cost vs. the menu cost. Among the observable statistics we focus on are: the frequency of price reviews, the frequency of price adjustments, the size-distribution of price adjustments, and the shape of the hazard rate of

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[1]For examples with a fixed observation cost see Caballero (1989), Duffie and Sun (1990), Reis (2006b,a), Abel, Eberly, and Panageas (2007, 2009), and more generally the related "rational inattention" literature as in Moscarini (2004), Sims (2003).

[2]Alvarez, Guiso, and Lippi (2009) analyze a version of the consumption-savings and portfolio choice model studied by Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007). That paper uses a novel data set to measure the frequency of observation and adjustment of portfolios for Italian investors.

price adjustments.

We think that modeling the price setting of a firm as composing of two different activities is interesting for several reasons. First, survey data indicates that firms review the adequacy of their prices infrequently, and that not all price reviews yield a price adjustment. Such a pattern cannot be accounted for by existing menu cost models, where price reviews occur continuously, nor by costly observation models, where each price review is also a price adjustment. The model with both observation and menu costs naturally accounts for the observed patterns. Second, menu cost and observation cost models have different implications for the response to aggregate shocks. For instance models with only observation cost, such as Mankiw and Reis (2002, 2006, 2007), yield "time-dependent" rules, while models with only menu cost, as Golosov and Lucas (2007), yield "state-dependent" rules. Our theory makes a step towards understanding which of these frictions is more relevant: with suitable data the model can be used to quantify the magnitude of the menu vs. the observation cost. Third, the distinction between the two activities allows us to consider the economics behind "price plans" or "sticky plans", an assumption that has been used in the literature (e.g. Burstein (2006)) which has potentially important implications for monetary policy, as discussed below.

The model is a version of the tracking problem subject to two fixed costs. The starting point is the firm's static cost function, modeled as a quadratic of the difference between the choice variable, e.g. the log of the current price p, and the bliss point, e.g. the "target" log price p^* which is the price that maximizes current profits. We refer to the difference between p and p^* as the price gap, denoted by $\tilde{p} \equiv p - p^*$. We write the instantaneous firm's cost as $B(\tilde{p}(t))^2$. This gives the cost of having a price different from p^*, and is obtained as the second order expansion of the log of the profit function around p^* (with a minus sign). Hence B is a parameter that depends on the curvature of the profit function (for instance $B = \eta(\eta - 1)/2$ if the demand has constant elasticity η). We let p^* follow a random walk with drift, arising from innovations in the firm's marginal cost and benefit, where the drift is the inflation rate and the innovations are idiosyncratic shocks. We assume that the firm faces two fixed costs.

The first is a standard menu cost, ψ, that applies to any change in prices. The second is an observation cost, θ, that the firm bears to discover p^* and hence $\tilde{p}$.[3] The firm minimizes the expected present value of the quadratic losses plus the expected discounted sum of the fixed costs incurred. We consider two technologies for price adjustments: the first requires that an observation cost is paid every time that the price is adjusted, but that the firm can observe without adjusting the price (i.e. without paying the menu cost). The second allows for price changes to be done without observing the price gap, a form of indexation to inflation. We show that if the drift of the target (i.e. the inflation rate) is small enough, the decision rules are the same in both cases, i.e. there is no indexation.

Our model embeds the two "polar" cases of menu cost and observation cost only. We compare our analytical characterization of the mapping from the structural parameters to the firm decisions, and to several observable statistics, in each of three setups: menu cost only, observation cost only, and the case with both costs.[4] Analyzing the outcomes of the different setups we show the aspects for which the predictions of the model with two costs are similar to a "weighted average" of the two polar cases, and the ones where the interaction of the two cost yields novel predictions for e.g. the size distribution of price adjustments and the shape of the hazard-rate function.

A substantial part of the paper focuses on the characterization of the optimal decision rules and their implications for the case without drift, i.e. zero inflation. In this case, right after an observation of the price gap $\tilde{p}$, the firm chooses whether to pay the menu cost and adjust the price. Additionally the firm chooses the time until the next observation. We give an analytical characterization for the solution as we let the discount rate and the fixed menu cost to be small. The decision of whether to adjust the price follows an sS rule: the price is adjusted only if the price gap $\tilde{p}$ after an observation is outside the inaction interval $-\bar{p}, \bar{p}$, and (with zero inflation) $\tilde{p}$ is reset to zero. The optimal time until the next observation, $\tau(\tilde{p})$,

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[3]Both costs are measured as a fraction of current profits under the 2nd order expansion discussed above.

[4]Most, but not all, of the analysis in the case of one cost only is in the literature in different papers. We find it useful to collect the known result and to extend some of them in exactly the same framework so that the effect of the different frictions is easier to understand.

is also a function of $\tilde{p}$: its value is longest, equal to τ, in the range where price adjustment is optimal. In the inaction interval the function $\mathrm{T}(\cdot)$ has an inverted U-shape: it peaks at a zero price gap, attaining τ, and is otherwise decreasing in the size of the price gap, namely $\mathrm{T}(\tilde{p}) = \tau - (\tilde{p}/\sigma)^2$, where σ is the standard deviation of the idiosyncratic innovations of the target price. This is clear: as the price gap gets closer to $\bar{p}$ the firm realizes that it is more likely to cross that threshold, and hence decides to monitor sooner. To summarize, the optimal decision rules can be expressed as a function of two variables: $\bar{p}$ and τ.[5] We use this characterization of the decision rules to provide analytical expressions for several observable statistics: the expected number of price reviews per unit of time n_r, the expected number of price adjustment per unit of time n_a, the invariant distribution of price changes, and the hazard rate of price changes $\boldsymbol{h}(t)$. We find these statistics interesting because they can be computed using available survey and scanner micro data sets. We use these statistics to make several points:

First, the baseline model with two costs predicts that firms review their prices more often than they adjust them, i.e. that the ratio between the average frequency of review and adjustment is above one that: $n_r/n_a > 1$. This happens since the reviews where the price gap is in the inaction region do not produce a price adjustment. We find this prediction interesting because it seems to be typical in the data (see Section 3). Moreover, neither model with only one cost can produce it ($n_r/n_a = 1$ in the observation cost model, while $n_r/n_a = \infty$ in the observation cost model).

Second, we characterize the distribution of price changes and compare it to the data and the ones from the polar models. The distribution of price changes is normal, with the mode on zero, in the observation-cost model, and takes only two symmetric values $(-\bar{p}, \bar{p})$ in the menu-cost model. In the model with both costs the distribution has no mass between $-\bar{p}$ and $\bar{p}$, and outside these values it has a density with the same tails as a normal, but with more mass close to $-\bar{p}$ and $\bar{p}$. The presence of a positive menu cost $\psi > 0$ prevents too small price

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[5]We also show that, in turn, these two variables can be expressed in essentially closed form as functions of three structural parameters: the two normalized cost θ/B, ψ/B and the innovation variance σ^2.

changes from occurring. This seems consistent with the US evidence in Midrigan (2007) and the cross-country evidence in Cavallo (2009).

Third, we identify several "observable statistics" that provide direct information about the relative size of the menu and observation costs: $\alpha = \psi/\theta$. We show that n_r/n_a is a monotone function that depends only on α. This is quite intuitive if one considers the polar cases: n_r/n_a is 1 for the observation cost model where $\alpha = 0$, and infinity for the menu cost model where $\alpha = \infty$. In addition, we show that three statistics from the distribution of price changes are monotone functions only of α, namely: $std \; |\Delta p| \; / \; \mathbb{E}|\Delta p|$, $min \; |\Delta p| \; / \; \mathbb{E}|\Delta p|$, $mode \; |\Delta p| \; / \; \mathbb{E}|\Delta p|$. These results provide several tests to measure α using statistics that are, at least in principle, available. Our preliminary estimates based on this model indicate that $\alpha < 1$, and possibly much smaller.

Fourth, we derive the instantaneous hazard rate $\boldsymbol{h}(t)$ for the model with two costs. This function shares some properties with the observation cost model, like an initial value of zero for the hazard rate between times $t \in [0, \; \tau)$, and a spike in the hazard at $t = \tau$. But unlike that model, it has a finite continuos non-zero hazard rate for higher values of t. Loosely speaking the shape of the hazard rate function has some periodicity, in that it looks like a series of non-monotone functions around durations that are multiple of τ. The reason for this non-monotonicity comes from the fact that reviews happen at unequal length of time – given by the function $\textsc{t}(\cdot)$ – and that adjustment occurs depending on whether the price gap is larger than the threshold $\bar{p}$ at the time of observation. The non-monotonicity of the hazard rate is unique to the model with both costs. We find this feature appealing because most of the studies fail to find evidence for increasing hazard rates, which is the implication stemming from the polar cases with only one cost, as well as from most of the models in the literature.[6]

Most of our analysis assumes a zero inflation rate. The last part of the paper considers

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[6]Klenow and Kryvtsov (2008) estimate a flat hazard rate on U.S. CPI data. A downward sloping hazard is estimated by Nakamura and Steinsson (2008) on a similar U.S. dataset using a different methodology, and by Alvarez, Burriel, and Hernando (2005) for the Euro area. In contrast, Cavallo (2009) estimates upward sloping or hump shaped hazard functions for four Latin American countries.

the effect of inflation on the optimal decision rules under two alternative price-changing technologies. First we retain the assumption that each price change requires an observation. In this case the optimal return point is different from the static optimum (the zero price gap). Indeed, the firm anticipates the effect of inflation so that the optimal price gap upon adjustment is positive, and roughly equal to the inflation rate times half of the expected time until the next adjustment. Second, we relax the assumption that every price adjustment requires an observation. In this setting a firm can schedule the dates and the size of multiple price-changes after a single observation. Note that an adjustment without observation is a form of automatic indexation, or what has been called in the literature a *price path* or a *price plan*. We show that for $\psi > 0$ and a positive but sufficiently small inflation rate all adjustments will take place immediately after the observation. In other words, if inflation is small relative to the menu cost indexation is not optimal.

The paper is organized as follows. Section 2 gives a brief survey of the related literature. Section 3 discusses the recent survey evidence on the firms' decisions concerning the frequency of price reviews and price adjustments in a number of countries. Section 4 presents our model of the firm's price setting problem with observation and menu costs. The section characterizes the firm optimal policy for each of the two polar cases (menu or observation cost only) and for the general case. An analytical solution for the firm's optimal decision rules in the case of zero inflation is given in Section 5. The rules are used in Section 6 to characterize analytically the model predictions for the frequency of price reviews, the frequency of price adjustments, the size-distribution of price adjustments, and the shape of the hazard rate of price adjustments. A comparison between the results of our model with those produced by the polar cases is given. We discuss how these results match against the recent micro evidence on the distribution of price changes and the shape of the hazard rate of price adjustments. Section 7 explores the case of positive inflation. Finally, Section 8 summarizes the main findings and discusses some issues for future research. Proofs and documentation material is given in the Appendix. A set of Online Appendices provides additional documentation on

the models discussed in this paper and the findings in the related literature.

2   Related literature on optimal price setting

There is a large literature on price-setting decisions under imperfect information, starting with the seminal work by Phelps (1969) and Lucas (1972). In these models the speed with which prices respond to changes in the state relates to the speed with which information about the shocks is embedded in the price-setting decision. This literature has recently been revisited by several authors looking for alternative models of price setting in order to account for the sluggish response of prices to nominal shocks. In particular, Reis (2006b) and Mankiw and Reis (2002, 2006, 2007) generalize Caballero (1989) and model imperfect information as arising from a fixed cost of observing the state.[7] In these models, the firm reviews the state infrequently and, due to the absence of any adjustment cost, adjust prices anytime the state is reviewed. There is no arrival of new information in between times of reviews. Reis (2006b) shows that, in standard frameworks, the optimal rule is to review and, contemporaneously, adjust prices at a constant frequency. By including both observation and adjustment costs, our model contributes to this literature on several dimensions. First, consistently with the survey evidence, our model implies that not all price reviews yield an adjustment. Second, our model implies distribution and hazard rate of price adjustments that make progress in explaining the empirical evidence. In particular, our model avoids infinitesimally small price adjustments from occurring, and allows for richer shapes of the hazard rate of price adjustments than the model with observation cost only. Third, in our model the optimal time between reviews is state dependent. In addition, while in the framework considered by Reis (2006b) firms can choose a pricing plan at each observation date and freely adjust

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[7]Woodford (2001) and Mackowiak and Wiederholt (2009) model information flows as depending on a signal about the underlying state which realizes every period. In these models, what prevents the firm from perfectly observing the state in every period is limited information processing capabilities, as originally proposed by Sims (2003). Given that in these models new, although partial, information arrives every period, the firm changes its price continuously. See Matejka (2009) for generalizations of this framework to obtain infrequent adjustments.

prices at any moment between observation dates, in our model due to the adjustment cost this strategy is not optimal unless the inflation rate is relatively large. In fact, we show that if inflation is small relatively to the adjustment cost price adjustments only take place immediately after an observation. On this dimension, our results also differ from Burstein (2006) who studies the case where the firm reviews its state continuously, but can specify an entire sequence of future prices upon payment of a fixed cost. In fact, we assume that the firm has to pay a fixed cost to review the state, as in Burstein (2006), but also that the firm faces a fixed cost of adjustment anytime it changes its price.

The classic theory of infrequent price adjustment relies on fixed adjustment costs. This theory delivers the state dependence of individual decisions: the agent acts when the state crosses some critical threshold, balancing cost and benefits of adjustment. Within this literature, Dixit (1991) studies the problem of a firm that faces a state given by a random walk process.[8] These models deliver a simple and powerful explanation for infrequent adjustment, but several predictions of the basic theory are hard to reconcile with the data. For instance, under a fixed menu cost the distribution of price changes is concentrated on the 2 points (the barriers), and the hazard rate of price changes is an increasing function of the time elapsed since the adjustment.

A recent strand of literature studies price-setting decisions, and their aggregate consequences, with models that combine some form of infrequent information-review with infrequent price-adjustment.[9] A simplifying assumption in these studies is that at least one of these choices is exogenous. For instance Nimark (2008) assumes exogenous information structure and exogenous times of price adjustment. In Bonomo and Garcia (2004) the times of observation are exogenous and the firm choice is limited to the optimal pricing rule in the

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[8]A large literature analyzes the consequences of price adjustment costs for aggregate dynamics, see Dotsey, King, and Wolman (1999), Golosov and Lucas (2007) and Midrigan (2007) for a review. Also, see Sheshinski and Weiss (1977, 1979, 1983), Barro (1972) and Dixit (1991) for a review of early results on price setting under costly price adjustments.

[9]See e.g. Bonomo and Garcia (2004), Nimark (2008), Morris and Shin (2006), Woodford (2009), Angeletos and La'O (2009), Gertler and Leahy (2008) and Bonomo and Carvalho (2004) and the review in Mankiw and Reis (2010).

presence of a fixed adjustment cost, so that only the times of adjustment are endogenous.[10] In Woodford (2009) the realization of signals on the state determine the timing of reviews. The signal structure is chosen by the firm: the larger the precision of the signals, the larger the cost faced by the firm. Each review entails an additional fixed cost, which could alternatively be interpreted as an observation or a menu cost. In fact, differently from our paper, there are no separate fixed costs of adjustment and reviews, so that each review coincides with an adjustment. By studying both the review-times decision and the price-adjustment decision our model analyzes the interaction between these choices in the presence of two frictions. The closest frameworks to the one of our paper are the model by Gorodnichenko (2008) and the discussion of the work by Woodford (2009) done by Burstein.[11] Gorodnichenko studies a model where each firm has to pay a fixed cost to acquire information and a fixed cost to change the price. Differently from our paper, Gorodnichenko studies the price setting problem within a general equilibrium framework where firms observe the past realization of the aggregate price level at no cost. While this more general framework addresses some important questions, such as the response of inflation and output to nominal shocks, it implies that the average frequency of price adjustment is larger than the average frequency of information acquisition, a prediction that seems at odds with the survey evidence discussed in the next Section.

3   Evidence on price adjustments vs. price reviews

Several recent studies measure two distinct dimensions of the firm's price management: the frequency of price reviews, or the decision of assessing the appropriateness of the price currently charged, and the frequency of price changes, i.e. the decision to adjust the price. The typical survey question asks firms: "In general, how often do you review the price of your

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[10] A new Section of this paper, dated May 2010 (Bonomo, Carvalho, and Garcia (2010)), presents numerical results for a problem where both the optimal times and optimal price adjustment are solved for, alike to the theory developed in this paper (see the NBER WP 15852, March 2010) and the earlier numerical analysis in Alvarez, Guiso, and Lippi (2009).

[11]This is apparently based on work in preparation by Hellwig, Burstein, and Venki (2010).

main product (without necessarily changing it)?"; with possible choices yearly, semi-yearly, quarterly, monthly, weekly and daily. The same surveys contain questions on frequency of price changes too. Fabiani et al. (2007) survey evidence on frequencies of reviews and adjustments for different countries in the Euro area, and Blinder et al. (1998), Amirault, Kwan, and Wilkinson (2006), and Greenslade and Parker (2008) present similar evidence for US, Canada and UK. This section uses this survey data to document that the frequency of price reviews is larger than the frequency of price changes. We believe that the level of both frequencies, especially the one for reviews, are measured very imprecisely. Yet, importantly for the theory presented in this paper, we have found that in all countries, and in almost all industries in each country, and for almost all the firms in several countries, the frequency of review is consistently higher than the frequency of adjustment. We will argue that, given our understanding of the precision of the different surveys, the most accurate measures of the ratio of price reviews to price adjustments per year are between 1 and 2. In the rest of this section we document this fact, referring the interested reader to the Online Appendix AA-2 for more documentation.

Table 1: Price-reviews and price-changes per year

	AT	BE	FR	GE	IT	NL	PT	SP	EURO	CAN	UK	US
						Medians						
Review	4	1	4	3	1	4	2	1	2.7	12	4	2
Change	1	1	1	1	1	1	1	1	1	4	2	1.4
				Mass of firms (%) with at least 4 reviews/changes								
Review	54	12	53	47	43	56	28	14	43	78	52	40
Change	11	8	9	21	11	11	12	14	14	44	35	15

Number of price changes and reviews per year. The sources for the medians are Fabiani et al. (2007) 2003 Euro area survey, Amirault, Kwan, and Wilkinson (2006) 2003 Canadian survey, Greenslade and Parker (2008) 2008 UK survey. See the Online Appendix AA-2 for a discussion of the sources and the measurement issues involved.

The upper panel of Table 1 reports the median frequency of price reviews and the median frequency of price adjustments across all firms in surveys taken from various countries. The median firm in the Euro area reviews its price a bit less than three times a year, but changes

its price only about once a year, and similar for UK and US.[12]

These surveys collect a wealth of information on many dimensions of price setting, well beyond the ones studied in this paper. Yet, for the questions that we are interested in, the survey data from several countries have some drawbacks. We think that, mostly due to the design of these surveys, the level of the frequencies of price review and price adjustments are likely subject to a large amount of measurement error. One reason is that in most of the surveys firms were given the following choices for the frequency of price reviews: yearly, quarterly, monthly and weekly (in some also semi-yearly and less than a year). It turns out that these bins are too coarse for a precise measurement, given where the medians of the responses are. For example, consider the case where in the population the median number of price reviews is exactly one per year, but where the median number of price changes is strictly larger than one per year. Then, in a small sample, the median for reviews will be likely 1 or 2 reviews a year, with similar likelihood. Instead the sample median for number of adjustments per year is likely to be one. From this example we remark that the median for price reviews is imprecisely measured, as its estimates fluctuates between two values that are one hundred percent apart. The configuration described in this example is likely to describe several of the countries in our surveys.[13] Another reason is that in some cases the sample size is small. While most surveys are above one thousand firms, the surveys for Italy has less than 300 firms and the one for the US has about 200 firms. Yet another difficulty with these measures is that several surveys use different bins to classify the frequency of price reviews and that one price changes. For instance in France and Italy firms are asked the average number of changes, instead of being given a set of bins, as is the case for the frequency of

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[12]This evidence about the frequency of price adjustment is roughly consistent with previous studies at the retail level. See Alvarez et al. (2006) for more details.

[13]For example in Portugal the median frequency of review is 2, but the fraction that reviews at one year or less is 47%, while for price adjustment the median is one and the fraction of firms adjusting exactly once a year is 49.5%, see Martins (2005). In the UK for 1995 the median price review is 12 times a year, but the fraction that reviews at most 4 times a years is about 46%, while for price adjustment the median is 2 and the fraction of firms adjusting 2 times or less a year is 66%, see Hall, Walsh, and Yates (2000). Indeed, consistent with our hypothesis of measurement error, in a similar survey for the UK for year 2007-2008, the median price review is 4 and the median price adjustment is 4, see Greenslade and Parker (2008).

reviews.[14]

The bottom panel of Table 1 reports another statistic that is informative on the relative frequency of reviews and adjustments: the fraction of firms reviewing and changing respectively their price at least four times a year. We see this statistic as informative, and less subject to measurement error, because this frequency bin appears in the questionnaire for both the review and the adjustment decision for almost all countries. It shows that the mass of firms reviewing prices at least four times a year is substantially larger than the corresponding one for price changes.

Figure 1: Average industry frequency of price changes vs. adjustments
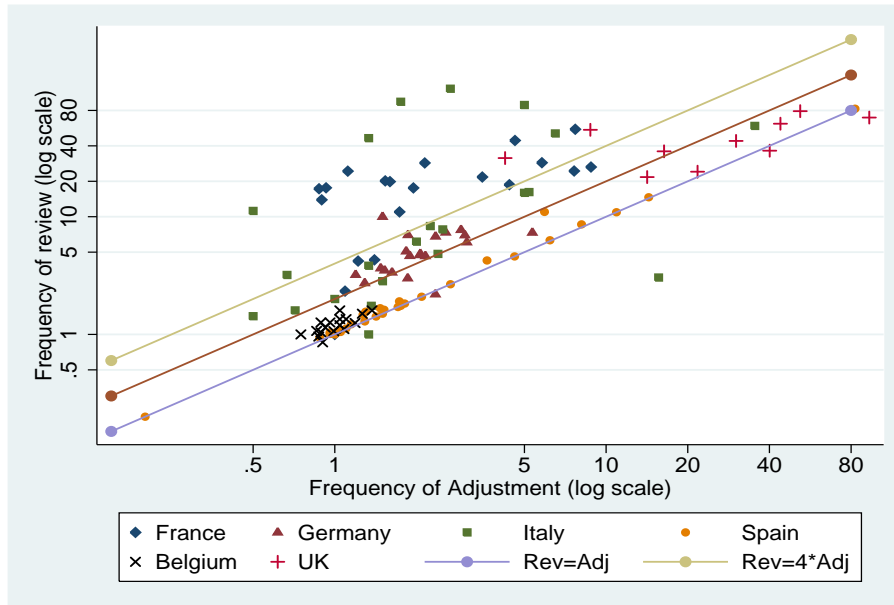
Note: data for each dot are the mean number of price changes and reviews in industry j in country i.

Figure 1 plots the average number of price reviews against the average number of price adjustments across a number of industries in six countries. This figure shows that in the six countries the vast majority of the industry observations lies above the 45 degree line, where the two frequencies coincide. Most of the industries for Belgium, Spain and the UK have a

[14]Furthermore for Germany firms where asked whether or not they adjusted the price in each of the preceding 12 months; this places an upper bound of 12 on the frequency of adjustments, while no such restriction applies to the number of reviews.

ratio of number of reviews per adjustment between 1 and 2 (i.e. lies between the two lower straight lines). The data for France, Italy and Germany has much higher dispersion in this ratio. We believe that the reason of the higher dispersion for Italy, Germany and France is due to the measurement error discussed above. Our belief is based on the fact that the questionnaire in the surveys for Belgium and Spain treat price reviews and price changes symmetrically and they record the average frequencies as an integer as opposed to a coarse bin.

For four countries Table 2 classifies the answers of each firm on three mutually exclusive categories: 1) those that change their prices more frequently than they review them, 2) those that change and review their prices at the same frequency, 3) and those that change their prices less frequently than they adjust them. Table 2 shows that most of the firms respond that they review their prices at frequencies greater or equal than the one in which change their prices. We conjecture that the percentage of firms in category 1, i.e. those changing the price more frequently than reviewing it, is actually even smaller than what is displayed in the table due to measurement error.

Table 2: Frequency of Price Changes and Reviews at Firm Level

	Belgium	France	Germany	Italy	Spain*
Percentage of Firms with:					
1) Change > Review	3	5	19	16	0
2) Change = Review	80	38	11	38	89
3) Change < Review	17	57	70	46	11
N of Observations (firms)	890	1126	835	141	194

* For Spain is only for firms that review four or more times a year. Sources: Table 17 in Aucremanne and Druant (2005) for Belgium, and our calculations based on the individual data described in Loupias and Ricart (2004), Stahl (2009), and Fabiani, Gattulli, and Sabbatini (2004) for France, Germany, and Italy. For Spain from section 4.4 of Alvarez and Hernando (2005). See Online Appendix AA-2 for details.

4 A price setting problem

We analyze the quadratic tracking problem of a firm facing instantaneous return function given by $-B\ (p\left(t\right) - p^{*}\left(t\right))^{2}$ where $p(t)$ is a decision for the agent and $p^{*}(t)$ is random target, i.e. the optimal value that she would set with full knowledge of the state of the problem. The target changes stochastically, and we assume that the agent must pay a fixed cost θ to observe the state $p^{*}(t)$, and that she maximizes expected discounted values. The constant B measures the cost elasticity to price deviations from the target. Moreover, it is assumed that the agent faces a physical cost ψ associated with resetting the price (a "menu cost").

The target price $p^{*}(t)$ follows a random walk with drift μ, with normal innovations with variance σ^{2} per unit of time, or

$$p^{*}(t_0 + t) = p^{*}(t_0)\ + \mu\ t\ + s\ \sigma\ \sqrt{t}\ , \tag{1}$$

where s is a standard normal, so that $\mathbb{E}_{t_0}\left[\ p^{*}(t_0 + t)\ \right] = p^{*}(t_0) + \mu\ t$, $Var_{t_0}\left[\ p^{*}(t_0 + t)\ \right] = \sigma^2\ t$, and $\mathbb{E}_{t_0}\left(p(t_0) - p^{*}(t_0 + t)\right)^{2} = \left(p\left(t_0\right) - p^{*}\left(t_0\right) - \mu t\right)^{2} + t\ \sigma^{2}$. In the case where the firm sets the price level $p(t_0)$ in nominal terms, the drift μ can be interpreted as the inflation rate, since $p^{*}(t)$ denotes a real variable.

We index the times at which the agent chooses to pay the cost θ and observe the state by T_i, with $0 = T_0 < T_1 < T_2 < \cdots$. After observing the value of the state at a date $t = T$, the agent decides whether to pay the cost ψ and adjust the value of its control variable $p(t)$ that will apply for the times $t \in (T_i, T_{i+1})$.

Online Appendix AA-1.1 discusses one case that is useful to interpret the units of B, θ, ψ, σ in the tracking problem described above. The instantaneous loss is the second-order term from a log approximation of the profit function, $p(t)$ is the log-price of a monopolistic firm, and $p^{*}(t)$ its corresponding static optimal level. Under the assumption of a constant demand elasticity $\eta > 1$, and constant marginal cost c, the parameter B in the flow return is given by half of the second derivative of the profit function, i.e. $B = \frac{1}{2}\ \eta\ (\eta - 1)$. The interpretation

for the costs θ and ψ is that they are measured as a proportion of profits per unit of time. The simplification of using a quadratic approximation to the profit function has been used in the seminal work on price setting problem with menu cost by Caplin and Spulber (1987) and Caplin and Leahy (1991, 1997), as discussed by Stokey (2008), and also by Caballero (1989) and Moscarini (2004) in the context of costly observation models.

We make two comments about the 'technology' to change prices. First, we assume that, after paying the adjustment cost, firms set a price that stays fixed until the next adjustment. In Section 4.1 we briefly comment on the possibility that the firm sets a path for the prices rather than a value. Second, for most of the paper we impose that observing is a necessary condition for an adjustment, i.e. that the agent must pay the observation cost for every price adjustment. Most of the paper treats the case where the target price $p^*(t)$ has no drift, i.e. the zero inflation case. In this case even if agents could adjust without observing they would not do so. Instead if the drift is large enough relative to the volatility, it may be optimal to adjust without observing. We discuss this issue in Section 7.3.

For a better understanding of the nature of the problem with both observation and menu costs, and to relate to the literature, the next two subsections discuss the polar cases in which there is an observation cost only ($\theta > 0$, $\psi = 0$) and a menu cost only ($\theta = 0$, $\psi > 0$). We then discuss the problem in the presence of both costs.

4.1 Price setting with observation cost

In the simpler case where the menu cost is zero ($\psi = 0$) the agent solves the following cost minimization problem:

$$V_0 \equiv \min_{\{T_i, p(T_i)\}_{i=0}^{\infty}} \mathbb{E}_0 \left[\sum_{i=0}^{\infty} e^{-\rho T_i} \left(\theta + B \int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \left(p(T_i) - p^*(t) \right)^2 dt \right) \right] \quad (2)$$

where, without loss of generality, we are starting at time $t = 0$ being an observation date, so that $T_0 = 0$.

The term $e^{-\rho T_i}\theta$ is the present value of the cost paid to observe the state $p^*(T_i)$ at time T_i. The term $\int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)}\mathbb{E}_{T_i}\left[B\left(p(T_i) - p^*(t)\right)^2\right] dt$ is the integral of the present value of the expected profits after observing the state at time T_i and setting the price to $p(T_i)$ to be maintained from time T_i until the new observation date v_{i+1}. The conditioning set is all the information available up to T_i. The expectation outside the sum of the maximization problem conditions on the information available at time zero. The notation of this problem assumes that a price adjustment can only happen at the time of a price review. This is the assumption we employ in most of the paper, except for Section 7.3 were we relax it.

Given two arbitrary T_i and T_{i+1}, define:

$$\hat{p} \equiv p(T_i) - p^*(T_i) \quad \text{and} \quad v(T_{i+1} - T_i) \equiv \min_{\frac{\hat{p}}{\mu}} \int_0^{T_{i+1}-T_i} e^{-\rho t}\left(\frac{\hat{p}}{\mu} - t\right)^2 dt \quad . \tag{3}$$

Two comments are in order. First, after observing the value of $p^*(T_i)$, the firm optimal pricing decision $\hat{p}$ concerns the new price in excess of $p^*(T_i)$ measured in units of the drift μ. Second, instead of setting a constant price we might consider letting the firm set a path for $p(t)$ for $t \in [T_i, T_{i+1})$. It is clear from the objective function that the firm would then choose $p(t) = p^*(T_i) + \mu t$, and hence the minimized objective function would be identically zero, i.e. $v(T_{i+1} - T_i) = 0$. Mechanically, this has the same effect on the solution of the problem in this section of setting the drift of the state to zero, i.e. setting the inflation rate to $\mu = 0$. Thus we can interpret the model with $\mu = 0$ as a problem in which the firm is allowed to set a path for $p(t)$. Alternatively, we can also consider the problem where the cost θ applies for a price review, where the firm finds out the value of $p^*(t)$, but that it has no cost of changing prices without gathering any new information. In this case the optimal policy is also to change the prices between reviews at the rate μ per unit of time. Hence, in either of these alternative scenarios the solution of the firm's problem is equivalent to the one we present in this section setting $\mu = 0$. We will revisit this issue in Section 7.3. For the moment we maintain the assumption that prices can be changes only after a review.

The first order condition with respect to $\hat{p}$ of the function $v(\tau)$, defined in equation (3), gives the optimal reset price

$$\frac{\hat{p}}{\mu} = \frac{\rho \int_0^\tau t e^{-\rho t} \, dt}{1 - e^{-\rho \tau}} \to \frac{\tau}{2} \text{ as } \rho \downarrow 0,$$

using the function v we write the firm problem as:

$$V_0 \equiv \min_{\{T_i\}_{i=0}^{\infty}} \left[\sum_{i=0}^{\infty} e^{-\rho T_i} \left(\theta + B \, \mu^2 \, v(T_{i+1} - T_i) + B \int_0^{T_{i+1} - T_i} e^{-\rho t} \, t \, \sigma^2 dt \right) \right] . \qquad (4)$$

Comparing equation (2) with equation (4) we notice that in the second expression we have solved for the expected values, and we have also subsumed the choice of the price into the function v. We can write this problem in a recursive way by letting $\tau \equiv T_{i+1} - T_i$ be the time between successive observation-adjustment dates:

$$V = \min_{\tau} \left[\theta + B \, \mu^2 \, v(\tau) + B \int_0^\tau e^{-\rho t} \, t \, \sigma^2 dt + e^{-\rho \tau} V \right] ,$$

where we use the result that the history of the shocks up to that time is irrelevant for the optimal choice of τ, given our assumptions on $p^*(t)$. The optimal time between observations (and adjustments) solves:

$$V = B \, \sigma^2 \, \min_{\tau} \frac{\tilde{\theta} + v(\tau) \frac{\mu^2}{\sigma^2} + \int_0^\tau e^{-\rho t} \, t \, dt}{1 - e^{-\rho \tau}} , \quad \text{where} \quad \tilde{\theta} \equiv \frac{\theta}{B \, \sigma^2} , \qquad (5)$$

a problem defined by three parameters: $\tilde{\theta}$, $(\mu/\sigma)^2$ and ρ.

Using this setup, the next proposition provides an analytical characterization of the optimal length of the inaction period for the case of small discounting ($\rho \to 0$) and for the case without drift ($\mu = 0$). Both cases provide very accurate approximation of the solution with non-zero drift and discounting provided these are small, as is likely the case in our data.

PROPOSITION 1.   The optimal decision rule τ for the time between observations when

$\rho \to 0$ is a function of two arguments, $\tau \left(\frac{\theta}{B \sigma^2} , \frac{\mu^2}{\sigma^2} \right)$, with the following properties:

1. τ is increasing in the normalized cost $\theta/(B\sigma^2)$, decreasing in the normalized drift μ/σ, and decreasing in the innovation variance σ^2

2. The elasticity of τ with respect to $\theta/(B\sigma^2)$ is: $1/2$ as $\theta/B \to 0$, or $\mu = 0$; $1/3$ for $\sigma = 0$.

3. For $\mu = 0$ and $\rho > 0$, then τ solves $\tilde{\theta} = \frac{1}{2}\tau^2 - \frac{1}{6}\rho\tau^3 + o(\rho^2\tau^3)$, so that τ is increasing in ρ provided that ρ or $\tilde{\theta}$ are small enough; for $\rho = 0$ we obtain: $\tau = \sqrt{2\,\tilde{\theta}}$

4. The derivative of τ with respect to μ is zero at zero inflation if $\sigma > 0$.

The proposition shows that for small values of the (normalized) observation cost $\tilde{\theta}$ the square root formula gives a good approximation, so that second order costs of observation gathering give rise to first order spells of inattention. Two special cases are worth mentioning. First the case with zero drift, i.e. $\mu = 0$, which gives a square root formula on the cost $\tilde{\theta}$ and with elasticity -1 on σ. As we discussed above, the $\mu = 0$ case can be interpreted as a setting where the firm is allowed to set a path for its price. The other special case of interest is when there is no uncertainty, i.e. $\sigma = 0$. We can think of this as a limiting case of the observation problem. When the uncertainty is tiny, the rule becomes even more inertial, switching from a square to a cubic root. Alternatively, this can be reinterpreted as a deterministic model with a physical menu cost of price setting (equal to θ), where the firm's price drifts away from the optimal level due to the inflation trend.

While the nature of our approximation is different, several of the conclusions of this proposition confirm previous findings in Reis (2006b). In particular, in his Proposition 4 the approximate optimal solution for the inaction interval follows a square root formula, just like we obtain under point 2. Also, as in his Proposition 5, the length of the inaction intervals is decreasing in the variance of the innovations, and increasing in the (normalized) cost of adjustment, as we find under point 1. Finally in this setup the length of the inaction spells is constant, as in the special case that Reis discusses in his Proposition 5. In both our and

his model the reason behind this result is that the state follows a brownian motion and that the level of the value function upon adjustment is independent of the state.

For future reference we note that in the case of no drift ($\mu = 0$) and no discount ($\rho \downarrow 0$), the time between observations/adjustments τ, the number of observations/adjustments per unit of time n, the average size of price adjustments $\mathbb{E}[\,|\Delta p|\,]$, and the instantaneous hazard rate of a price adjustment as a function of time $\boldsymbol{h}(t)$, are given by

$$n \equiv \frac{1}{\tau} \;=\; \sqrt{\frac{\sigma^2 B}{2\,\theta}} \quad, \quad \mathbb{E}[\,|\Delta p|\,] = \left[\frac{2\,\sigma^2\,\theta}{B}\right]^{\frac{1}{4}} \sqrt{2/\pi}\;, \tag{6}$$

$$\boldsymbol{h}(t) \;=\; 0 \text{ for } t \in [0, \tau), \text{ and } \boldsymbol{h}(\tau) = \infty\;.$$

where price changes are distributed as a normal with zero mean and standard deviation $\sigma\sqrt{\tau}$, and t measures the time elapsed since the last price change.

4.2 Price setting with menu cost

In this section we assume the firm observes the state p^* without cost, i.e. $\theta = 0$, but that it must pay a fixed cost ψ to adjust prices. This is the standard menu cost model. The firm observes the underlying target value $p^*(t)$ continuously but acts only when the current price, $p(t)$, is sufficiently different from it, i.e. when the deviation $\tilde{p}(t) \equiv p(t) - p^*(t)$, is sufficiently large. Thus optimal policy is characterized by a range of inaction. Using $\hat{p} \equiv p(T_i) - p^*(T_i)$ to denote the optimal reset price at the time of adjustment, and using the law of motion for p^* in equation (1), the evolution of the price deviation is $\tilde{p} = \hat{p} - \mu\,t - s\sigma\sqrt{t}$.

The Hamilton-Jacobi-Bellman equation in the range of inaction $\tilde{p} \in [\underline{p}\,,\,\bar{p}]$ is:

$$\rho V(\tilde{p}) = B\,\tilde{p}^2 - V'(\tilde{p})\,\mu + \frac{1}{2}V''(\tilde{p})\,\sigma^2\;. \tag{7}$$

The optimal return point is: $\hat{p} = \arg\,\min_{\tilde{p}} V(\tilde{p}) \implies V'(\hat{p}) = 0$, and the boundary conditions

are given by

$$V(\bar{p}) \;=\; V(\hat{p}) + \psi \;\; \text{and} \;\; V'(\bar{p}) = 0 \;, V(\underline{p}) = V(\hat{p}) + \psi \;\; \text{and} \;\; V'(\underline{p}) = 0 \;. \qquad (8)$$

While the standard way to solve this problem is to use the closed form solution of the ODE and use the boundary conditions to obtain an implicit equation for $\bar{p}$, we pursue an alternative strategy that will be useful to compare with the solution for the problem with both cost. Since for $\mu = 0$ the value function is symmetric and attains a minimum at $\tilde{p} = 0$, we use the following fourth-order approximation to $V(\cdot)$ around zero:

$$V(\tilde{p}) = V(0) + \frac{1}{2}V''(0) \; (\tilde{p})^2 + \frac{1}{4!}V''''(0) \; (\tilde{p})^4 \;\; \text{for all } \tilde{p} \in [-\bar{p} \;, \; \bar{p}], \qquad (9)$$

where the symmetry around zero implies $V'(0) = V'''(0) = 0$. Note that the optimality condition $V'(\hat{p}) = 0$ and the boundary conditions in equation (8) imply that $V(\cdot)$ is convex around $\tilde{p} = 0$ but concave around $-\bar{p}$ and $\bar{p}$. Thus a fourth order approximation is the smallest order that we can use to capture this, with $V''(0) > 0$ and $V''''(0) < 0$. The approximation will be accurate for small values of ψ, since in this case the range of inaction is small.

PROPOSITION 2. Given $\mu = 0$, the width of the range of inaction, $\bar{p}$, for small $\psi\sigma^2/B$ and ρ is approximately given by :

$$\bar{p} = \left(\frac{6 \; \sigma^2 \; \psi}{B} \right)^{1/4} . \qquad (10)$$

The result for the quartic root is essentially the one in Dixit (1991), who obtained it through a different argument. The approximation of equation (10) is very accurate for a large range of values of the cost ψ. In this case price adjustments take two values only, and hence the average size of price changes is $\mathbb{E}[\;|\Delta p|\;] = \bar{p}$. Instead, the number of adjustments per unit of time is more involved, but it can be computed using the function defining the expected time until adjustment, $\mathcal{T}_a(\tilde{p})$, i.e. the expected value of the time until $\tilde{p}$ first reaches $\bar{p}$ or $\underline{p}$. The average number of adjustments, denoted by n, is then $1/\mathcal{T}_a(\hat{p})$. The function

$\mathcal{T}_a(\tilde{p})$ satisfies the o.d.e.:

$$0 = 1 - \mu \ \mathcal{T}_a'(\tilde{p}) + \frac{\sigma^2}{2} \mathcal{T}_a''(\tilde{p}) \ \text{ and } \ \mathcal{T}_a(\underline{p}) = \mathcal{T}_a(\bar{p}) = 0 \ . \tag{11}$$

For the case of $\mu = 0$ the solution is $\mathcal{T}_a(\tilde{p}) = \frac{\bar{p}^2 - \tilde{p}^2}{\sigma^2}$. Hence the average number of adjustments per unit of time n satisfies $n \equiv \frac{1}{\mathcal{T}_a(0)} = \frac{\sigma^2}{\bar{p}^2}$. Indeed, the distribution of the first times between subsequent price adjustment is known in closed form, and hence one can use it to characterize $\boldsymbol{h}(t)$, the instantaneous hazard rate of a price change as a function of t the time elapsed since the last price change (see Online Appendix AA-1.3 for details). For future reference we have

$$n \ = \ \frac{\sigma^2}{\bar{p}^2} = \sqrt{\frac{\sigma^2 \ B}{6 \ \psi}} \ , \quad \mathbb{E}[\ |\Delta p| \ ] \ = \ \left[\frac{6 \ \sigma^2 \ \psi}{B} \right]^{\frac{1}{4}} \ \text{ and } \ \boldsymbol{h}(0) = 0 \ ,$$

$$\boldsymbol{h}'(t) \ \geq \ 0, \ \boldsymbol{h}(\cdot) \text{ is convex-concave with } \lim_{t \to \infty} \boldsymbol{h}(t) = \frac{\pi^2}{8} \frac{\sigma^2}{\bar{p}^2} = \frac{\pi^2}{8} \ n \ .$$

where we use the approximation for $\bar{p}$. We note that the hazard rate starts at zero, it is strictly increasing, it quickly attains its asymptote, which is approximately equal to n, the average number of adjustment (their ratio is $\frac{\pi^2}{8} \approx 1.23$). This means that the behavior is not that different from a constant hazard rate.

4.3   Price setting with observation and menu costs

In this section we consider the problem where the firm faces two costs: an observation cost θ that is paid to observe the state $p^*(t)$, and a menu cost ψ that is paid to change the price. We first write the corresponding sequence problem:

$$V(p_0) \ \equiv \tag{12}$$

$$\min_{\{T_i, p(T_i), \chi_{T_i}\}_{i=0}^{\infty}} \mathbb{E}_0 \left\{ \sum_{i=0}^{\infty} e^{-\rho T_i} \left[\theta \ + \ (1 - \chi_{T_i}) \ B \ \int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \left(p(T_{i-1}) - p^*(t) \right)^2 dt \right. \right.$$

$$\left. \left. + \ \chi_{T_i} \left(\psi + B \int_{T_i}^{T_{i+1}} e^{-\rho(t-T_i)} \mathbb{E}_{T_i} \left(p(T_i) - p^*(t) \right)^2 \ dt \right) \right] \right\}$$

where, as in Section 4.1, the sequence $\{T_i\}$ denotes the stopping times for the observation of the state $p^*(t)$, and where χ_{T_i} is an indicator that the agent will pay the menu cost ψ and adjust the price to $p(T_i)$ so that prices evolve according to:

$$p(t) = p(T_{i-1}) \text{ for } t \in [T_i, T_{i+1}) \text{ if } \chi_{T_i} = 0 , \tag{13}$$

$$p(t) = p(T_i) \quad \text{for } t \in [T_i, T_{i+1}) \text{ if } \chi_{T_i} = 1 .$$

where, without loss of generality, we are starting at time $t = 0$ being an observation date, so that $T_0 = 0$. As in the problem of Section 4.1, the notation $\mathbb{E}_{T_i}(\cdot)$ denotes expectations conditional on the history of $\{p^*(s)\}$ up to $t = T_i$. The process for $\{p^*(t)\}$ follows a random walk with drift as in equation (1), so the current value of the process is a sufficient statistics for the distribution of its future realizations. Finally notice that the objective function is homogeneous of degree one in (B, θ, ψ). Thus the optimal policy will be a function only of the parameters $(\rho, \mu, \theta/B, \psi/B, \sigma^2)$, so that we can normalize $B = 1$ without loss of generality where we find it convenient.

Following the same steps as in Section 4.1, we write the Bellman equation for the firm's problem excluding all the terms that the firm cannot affect and in terms of the deviations $\tilde{p}(T + t) \equiv p(T) - p^*(T + t) = \hat{p}(T) - \mu \ t - s\sigma\sqrt{t}$. We measure the value function just after paying the observation cost and observing the state $\tilde{p}$:

$$V(\tilde{p}) = \min \left\{ \bar{V}(\tilde{p}) \ , \ \hat{V} \right\} , \tag{14}$$

where $\bar{V}(\tilde{p})$ is the value function if the firm knows the state, but does not change the price, and $\hat{V}$ is the value function if the firm decides to set the optimal price $\hat{p}$. Thus

$$\hat{V} = \psi + \theta + \min_{\hat{p}, \tau} B \int_0^\tau e^{-\rho t} \left[(\hat{p} - \mu t)^2 + \sigma^2 t \right] dt + e^{-\rho \tau} \int_{-\infty}^\infty V \left(\hat{p} - \mu \tau - s\sigma\sqrt{\tau} \right) dN(s) \tag{15}$$

where $N(\cdot)$ is the CDF of a standard normal.[15] The value function conditional on the firm not changing the price is:

$$\bar{V}(\tilde{p}) = \theta + \min_{T} B \int_0^T e^{-\rho t} \left[(\tilde{p} - \mu t)^2 + \sigma^2 t \right] dt + e^{-\rho T} \int_{-\infty}^{\infty} V \left(\tilde{p} - \mu T - s\sigma\sqrt{T} \right) dN(s) \ .$$

(16)

One can show that the operator defined by the right side of equation (14), equation (15) and equation (16) is a contraction in a subset of the space of bounded and continuous functions. The argument is intuitive but slightly non-standard, since the length of the time period is a decision variable, potentially making the problem a continuous-time one. The Online Appendix AA-1.6 provides a formal proof. The value function is uniformly bounded and continuos on $\tilde{p}$ and if $\theta > 0$ then the optimal time between observations is uniformly bounded by $\underline{\tau} > 0$, and thus the operator defined by the right side of equation (14), equation (15) and equation (16) is a contraction of modulus $\exp(-\rho\underline{\tau})$.

We conjecture that the form of the optimal decision rule is that there will be two thresholds for $\tilde{p}$ that will define a range of inaction, $[\underline{p}, \ \bar{p}]$. These thresholds are such that

$$\bar{V}(\tilde{p}) \quad < \quad \hat{V} \ \text{ if } \ \tilde{p} \in (\underline{p}\,, \bar{p}) \ \text{ and that } \quad \bar{V}(\tilde{p}) > \hat{V} \ \text{ if } \ \tilde{p} < \underline{p} \ \text{ or } \ \tilde{p} > \bar{p} \ .$$

If upon paying the observation cost the firm discovers that the state $\tilde{p}$ is in the range of inaction, then the firm will decide not to pay the cost ψ and not to adjust the price; moreover the firm will observe again in $\tau(\tilde{p})$ units of time. If otherwise, the state is outside the range of inaction, the firm will pay the cost ψ, set the optimal price, and also set the new interval at which to observe the state to be τ.

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[15]Equations (15) and (16) use that the expected value: $\int_{-\infty}^{\infty} \left(p - \mu t - s\sigma\sqrt{t} \right)^2 dN(s) = (p - \mu t)^2 + \sigma^2 t.$

5 Optimal decision rules with zero inflation

In this section we study the case where p^* has no drift ($\mu = 0$) which simplifies the problem. We show that the value function is symmetric around zero, with a minimum at $\tilde{p} = 0$. The optimal choice of p upon paying the cost ψ is $\hat{p} = 0$, $\text{T}(0) = \tau$, and $\text{T}(\tilde{p})$ symmetric around zero. Moreover, $\text{T}(\cdot)$ has a maximum at $\tilde{p} = 0$, and has an inverted U-shape. The thresholds for the range of inaction satisfy: $\underline{p} = -\bar{p}$.

We begin by showing that $V(\cdot)$ is symmetric around $\tilde{p} = 0$ and increasing around it. The Bellman equations for $\tilde{p} \in (-\infty, \infty)$ are:

$$\bar{V}(\tilde{p}) = \theta + \min_T B \int_0^T e^{-\rho t} \left[\tilde{p}^2 + \sigma^2 t\right] dt + e^{-\rho T} \int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{T}\right) dN(s) \quad (17)$$

$$\hat{V} = \psi + \theta + \min_{\tau,\hat{p}} B \int_0^\tau e^{-\rho t} \left[\hat{p}^2 + \sigma^2 t\right] dt + e^{-\rho \tau} \int_{-\infty}^{\infty} V\left(\hat{p} - s\sigma\sqrt{\tau}\right) dN(s) \quad (18)$$

$$V(\tilde{p}) = \min\left\{\hat{V}, \bar{V}(\tilde{p})\right\} \quad (19)$$

PROPOSITION 3.   Let $\mu = 0$. The value function V is symmetric around $\tilde{p} = 0$, and V is strictly increasing in $\tilde{p}$ for $0 < \tilde{p} < \bar{p}$. The optimal price level conditional on adjustment is $\hat{p} = 0$. The derivative of $\bar{V}(\tilde{p})$ for $0 \leq \tilde{p}$ is given by

$$0 \leq \bar{V}'(\tilde{p}) = 2\, B\, \tilde{p}\, \frac{1 - e^{-\rho \text{T}(\tilde{p})}}{\rho} + e^{-\rho \text{T}(\tilde{p})} \int_0^\infty V'(z) \frac{e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\text{T}(\tilde{p})}}\right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\text{T}(\tilde{p})}}\right)^2}}{\sigma\,\sqrt{\text{T}(\tilde{p})\,2\,\pi}}\, dz$$

with strict inequality if $\text{T}(\tilde{p}) > 0$, where $\text{T}(\cdot)$ is the optimal decision rule for the time between observations. Thus $V'(0) = \bar{V}'(0) = 0$, $V''(0) > 0$, and $V'(\tilde{p}) = 0$, for $\tilde{p} > \bar{p}$ and hence V is not differentiable at $\tilde{p} = \bar{p}$.

Notice that in this case at the boundary of the range of inaction the value function has a kink, i.e. there is *no smooth pasting*. This differs from the model with menu cost only of Section 4.2, which featured the smooth pasting property, typical of continuous-time fixed-cost models, see e.g. Dixit (1993) and Stokey (2008).

The next proposition characterizes the function $\text{T}(\cdot)$ describing the time until the next review in the range of inaction. The proof of this proposition uses a third order expansion of $\text{T}(\cdot)$ around $\tilde{p} = 0$. It uses a second order expansion of the f.o.c., as well as the symmetry of functions $\text{T}(\cdot)$ and $V(\cdot)$.

PROPOSITION 4.   The limit as $\rho \downarrow 0$ of the optimal rule for the time to the next revision $\text{T}(\tilde{p})$ is given by:

$$\text{T}(\tilde{p}) = \tau \ - \ \left(\frac{\tilde{p}}{\sigma} \right)^2 \ + o(|\tilde{p}^3|) \quad \text{for } \tilde{p} \in (-\underline{p}, \ \bar{p}). \tag{20}$$

A few comments are in order. First, the shape of the optimal decision rule depends only on σ, and not on the rest of the parameters for the model, i.e. B, θ, and ψ. Second, if the agent finds herself after a review with a price gap $\tilde{p} = 0$, she will set $\text{T}(0) = \tau$, since the optimal adjustment would have implied a post adjustment price gap of zero. Third, the function $\text{T}(\tilde{p})$ is decreasing in the (absolute value of) $\tilde{p}$. If upon a review the agent finds the price gap close to the boundary of range of inaction, then she will plan for a review relatively soon, since it is likely that the target will cross the threshold $\bar{p}$. Fourth, the price gap is normalized by the standard deviation of the changes in the target price σ. This is also natural, since the interest of the decision maker is on the likelihood that the price target will deviate and hit the barriers, so that for a lower σ she is prepared to wait more for the same price gap $\tilde{p}$. Figure 2 plots the policy rule $\text{T}(\tilde{p})$ implied by Proposition 4 against the policy rule obtained from the numerical solution of the model, both evaluated at a set of structural parameters which we think are reasonable. The two vertical bars at $-\bar{p}$ and $\bar{p}$ denote the threshold values that delimit the inaction region. The vertical bar at $\hat{p}$, inside the inaction region, denotes the optimal return point after an adjustment. The approximation for $\text{T}(\cdot)$ is very accurate, we found this approximation to be precise for all the economically interesting parameters we have tried.

Next, we compute an analytical approximation to the value function and optimal policies. The approximation relies on the fact that $V(\cdot)$ is symmetric around $\tilde{p} = 0$, i.e. $V(\tilde{p}) = V(-\tilde{p})$,

Figure 2: Policy rule $\mathrm{T}(\tilde{p})$
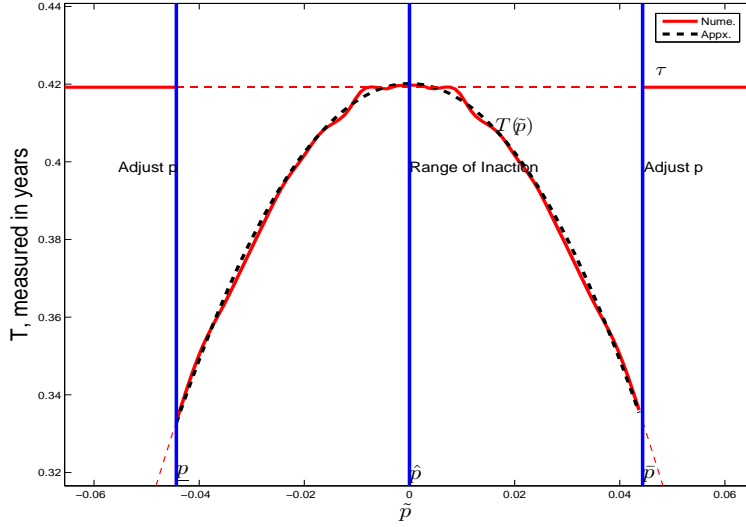
Note: parameter values are $B = 20$, $\rho = 0.02$, $\sigma = 0.15, \theta = 0.03$, and $\psi = 0.015$; the analytical approximation to $\mathrm{T}(\cdot)$ given in Proposition 4; the solid red line is the numerical solution to $\mathrm{T}(\cdot)$.

and hence all the derivatives of odd order are zero. Hence we approximate

$$\bar{V}(\tilde{p}) = V(0) + \frac{1}{2}V''(0)\ (\tilde{p})^2 + o(|\tilde{p}|^3) \approx V(0) + \frac{1}{2}V''(0)\ (\tilde{p})^2$$

since $V'(0) = V'''(0) = 0$ and $V''(0) > 0$. We refer to the LHS of this expression as the quadratic approximation, even though since $V'''(0) = 0$ the remainder is of order smaller than $|\tilde{p}^3|$. The other source of approximation, to simplify the analytical expressions, is that we let ρ converge to zero.[16] The quadratic approximation for the value function is globally accurate if the range of inaction, i.e. $[-\bar{p},\ \bar{p}]$, is small. Since $\bar{p}$ converges to zero as the menu cost ψ goes to zero, the approximation will be accurate for small values of ψ relatively to θ. We will discuss the accuracy of these approximations below. Proposition 5 (and Lemma 1 - 2 in Appendix A.3) uses these approximations to characterize the values of $\bar{p}$ and τ. To this end, it is convenient to define the variable $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$, which measures the minimum size of the innovation of a standard normal required to get out of the inaction region $[-\bar{p}, \bar{p}]$ after

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[16]This second approximation has negligible effects on the accuracy of the solution given the small discount rates that are appropriate for this problem.

resetting the price to $\hat{p} = 0$. We refer to the variable ϕ as the "normalized range of inaction".

PROPOSITION 5.   Define $\alpha \equiv \psi/\theta$ and $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$, and assume that $\theta > 0$ and $\psi > 0$ and $\alpha < (1/2 - 2(1 - N(1)))^{-1} \approx 5.5$. As $\rho \downarrow 0$, there exists a unique solution for $\bar{p}$ and τ, in that solution ϕ is a function of 2 arguments: the normalized costs $\left(\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B}\right)$. For small values of $\sigma^2 \frac{\psi}{B}$, the solution $\phi(\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B})$ is approximated by $\varphi(\alpha)$ which solves

$$1 = \varphi(\alpha)^2 \left(\frac{2}{\alpha} + 4\left[1 - N\left(\varphi(\alpha)\right)\right] \right) \text{ , with elasticity} \tag{21}$$

$$\frac{\partial \log \varphi}{\partial \log \alpha} = \frac{1}{2} \text{ at } \alpha = 0 \text{ and } \frac{\partial \log \varphi}{\partial \log \alpha} < \frac{1}{2} \text{ for } \alpha > 0 \ . \tag{22}$$

such that $\varphi(\alpha) - \phi(\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B}) = o\left(\left[\sigma^2 \frac{\psi}{B}\right]\right)$. The optimal values for the time until the next revision after an adjustment, τ and the width of the range of inaction, $\bar{p}$, are given by

$$\tau = \sqrt{\frac{\theta}{\sigma^2 B}} \ \frac{\sqrt{\alpha}}{\varphi(\alpha)} \ > \ \tau \mid_{\psi=0} \ = \ \sqrt{\frac{\theta}{\sigma^2 B}} \ 2 \tag{23}$$

$$\bar{p} = \left[\sigma^2 \frac{\psi}{B}\right]^{\frac{1}{4}} \ \sqrt{\varphi(\alpha)} \ < \ \bar{p} \mid_{\theta=0} \ = \ \left[\sigma^2 \frac{\psi}{B} 6\right]^{\frac{1}{4}} \ . \tag{24}$$

While the proof of Proposition 5 involves some algebra, the logic follows three simple steps. First, we develop a system of two equations in two unknowns, the equations are the f.o.c. for τ and the value matching condition at $\bar{p}$, i.e. $\bar{V}(\bar{p}) = \hat{V}$. These equations are simplified by using the quadratic approximation for $\bar{V}$ and by letting $\rho \downarrow 0$. Second, a bit of analysis of these equations shows that, under the conditions stated in the proposition, the solution is unique and well defined (i.e. it implies $\text{T}(\bar{p}) < 0$. Third, we obtain an approximation for ϕ, namely φ.

To understand the type of error produced by the quadratic approximation, recall that the nature of the approximation used in this section is that the value function $\bar{V}$ is assumed to be quadratic in $\tilde{p}$, and also that we let ρ converge to zero. Here we focus on the first feature, which turns out to be the most important. Since the function $\bar{V}$ is symmetric and has a minimum at $\tilde{p} = 0$, a quadratic approximation must be accurate around $\tilde{p} = 0$. Also,

recall that Proposition 3 shows that the function is increasing for all $\tilde{p} < \bar{p}$. Thus, since $\bar{p}$ tends to zero as ψ goes to zero, the relevant range of $\bar{V}$, given by $[-\bar{p}, \bar{p}]$, is very small and hence the approximation very accurate if ψ is small. On the other hand, when ψ is large relative to θ, the quality of the approximation deteriorates. In particular, as θ goes to zero, the problem converges to the menu cost model analyzed in Section 4.2. The value function $\bar{V}$ in the menu cost model is convex close to $\tilde{p} = 0$, but then it must be concave around $\bar{p}$, to satisfy smooth pasting. As explained in Section 4.2 above, this implies that $V''''(0) < 0$. Thus, as θ becomes small relative to ψ, the value function $\bar{V}$ becomes closer to the one of the menu cost, and hence our quadratic approximation of the value function becames worse, especially for values of $\tilde{p}$ away from zero. In particular, since our quadratic approximation has $V''''(0) = 0$, it tends to be higher for values of $\tilde{p}$ away from zero, and consequently the value of $\bar{p}$ that we obtain tends to be smaller.[17]
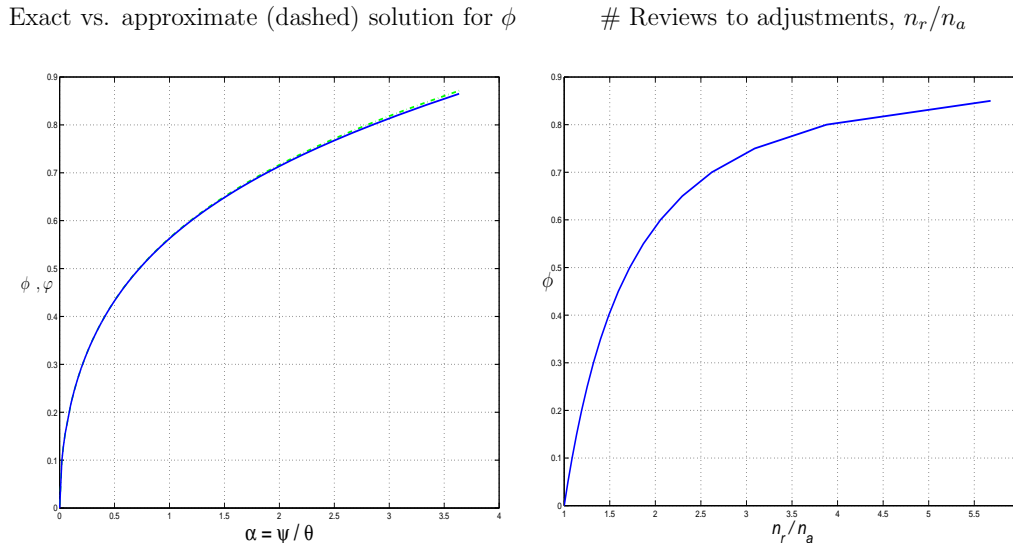
Proposition 5 shows that the expressions in equation (23) and equation (24) are the generalizations of the corresponding formulas for the case in which there is only an observation or a menu cost, respectively: for a given ratio of the cost α, they have the same exact functional form. The equations show that the length of time until the next revision, τ, is higher in the model with both costs than in the model with observation cost only, and that the width of the inaction band, $\bar{p}$, is smaller than in the menu cost model.[18] The reason why τ is higher is that the introduction of the menu cost increases the cost of one price adjustment (from θ to $\theta + \psi$) but not the benefit. As a consequence firms optimally economize on the number of times they pay the cost. The reason why $\bar{p}$ is smaller than in the menu cost case is more subtle. In the pure menu cost model observations are free, i.e. the firm can monitor when the state crosses the threshold at no cost. But with an observation cost this is not true, and when the firm discovers to be 'sufficiently close' to the barrier she prefers to adjust

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[17] In Online Appendix AA-3 we compare our analytical quadratic approximation for the value function and the thresholds with numerical solutions obtained of these objects. For large values of ψ/θ the quadratic approximation $\mathrm{T}(\cdot)$ is not accurate over the whole range. In this case the true value of τ is larger than the in our quadratic approximation.

[18] Also note that, from equation (23) and equation (22) it follows that τ is weakly increasing in the ratio of the cost α, with elasticity less than $1/2$.

rather than having to pay again for observing when exactly the barrier is crossed. In other words the barrier (both left and right) shifts inwards.

Figure 3: The relationship between ϕ and α, and between ϕ and n_r/n_a

Exact vs. approximate (dashed) solution for ϕ # Reviews to adjustments, n_r/n_a

Note: parameter values are $B = 20$, $\sigma = 0.15$, $\theta = 0.03$, and we let ψ to vary.

The left panel of Figure 3 plots the variable ϕ (the exact solution of the system (A-5)-(A-6)), and the variable φ (the solution of equation (21)), against the ratio of the two cost, $\alpha \equiv \frac{\psi}{\theta}$. The figure shows that φ and ϕ are almost indistinguishable, i.e. that the approximation is very precise for the parameters that are of conceivable interest to us.[19]

Finally, notice that the width of the inaction band has elasticity $1/4$ with respect to σ^2/B, as in the menu cost model, and that the time to the next review after a price adjustment has elasticity equal to $-1/2$ with respect to $B\sigma^2$. Using equation (22) into equation (23) and

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[19]We think of the value $\sigma = 0.15$ as the annual standard deviation of cost, as well as a value of $B = 20$, which in this context is implied by a markup of about 15%, as of the right order of magnitude. Multiplying this "reasonable" value of the σ^2/B ratio by a factor of 100 the approximation remains close to the one that is drawn in the figure. See Section 6.2 for further discussion on the calibration of the model.

equation (24) we obtain the following elasticities with respect to the cost:

$$0 \leq \frac{\partial \log \tau}{\partial \log \psi} = \frac{1}{2} - \frac{\partial \log \varphi}{\partial \log \alpha} \leq \frac{1}{2} \text{ and } 0 < \frac{\partial \log \tau}{\partial \log \theta} = \frac{\partial \log \varphi}{\partial \log \alpha} \leq \frac{1}{2} \; , \tag{25}$$

$$0 \leq \frac{\partial \log \bar{p}}{\partial \log \psi} = \frac{1}{2}\left(\frac{1}{2} + \frac{\partial \log \varphi}{\partial \log \alpha}\right) \leq \frac{1}{2} \text{ and } 0 > \frac{\partial \log \bar{p}}{\partial \log \theta} = -\frac{1}{2}\frac{\partial \log \varphi}{\partial \log \alpha} \geq -\frac{1}{2} \; . \tag{26}$$

Equation (25) says that the time to the next review after a price adjustment is increasing in both costs. Equation (26) says that the width of the inaction band is increasing in ψ, with an elasticity smaller than in the menu cost model, and *decreasing* in θ, since at the time of an observation an agent facing a higher cost minimizes the chances of paying further observation cost by narrowing the range of inaction.

6   Statistics for the case of zero inflation

In this section we characterize the implications for the following statistics of interest in the case of zero inflation (no drift): the frequency of price revisions, the frequency of price adjustment, the distribution of price adjustment, and the hazard rate for price changes.

6.1   Average frequencies of review and adjustment

First we turn to the development of expressions for statistics of interest implied by this model the frequency of price adjustments n_a, and the frequency of price reviews n_r. First, we turn to the characterization of the frequency of price adjustments. Let the function $\mathcal{T}_a(\tilde{p})$, describe the expected time needed for the price gap to get outside the range of inaction $[-\bar{p}\,,\,\bar{p}]$, conditional on the state (right after a revision) equal to $\tilde{p}$. This function solves the recursion:

$$\mathcal{T}_a(\tilde{p}) = \mathrm{T}(\tilde{p}) + \int_{\frac{\tilde{p}-\bar{p}}{\sigma\sqrt{\mathrm{T}(\tilde{p})}}}^{\frac{\tilde{p}+\bar{p}}{\sigma\sqrt{\mathrm{T}(\tilde{p})}}} \mathcal{T}_a\left(\tilde{p} - \sigma\sqrt{\mathrm{T}(\tilde{p})}\,s\right) dN(s) \tag{27}$$

for $\tilde{p} \in [-\bar{p}, \bar{p}]$. Since after a price adjustment the price gap $\tilde{p}$ is zero, then $\mathcal{T}_a(0)$ is the expected time between price adjustments. By the fundamental theorem of renewal theory, the average number of price adjustments per unit of time is given by $n_a = 1/\mathcal{T}_a(0)$.

In the next proposition we derive analytical approximation for the expected time between adjustments as function of only two arguments: $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$ and τ. The nature of the approximation is that we replace the function $\mathcal{T}_a(\tilde{p})$, that is symmetric around $\tilde{p} = 0$ and has zero odd derivatives at $\tilde{p} = 0$, by a quadratic function; moreover, based on Proposition 4, we use $\mathrm{T}(\tilde{p}) = \tau - (\tilde{p}/\sigma)^2$.

PROPOSITION 6. Let $\mathrm{T}(\tilde{p})$ be given by the quadratic approximation given in Proposition 4 and let $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$. The frequency of price adjustments $n_a = 1/\mathcal{T}_a(0)$, where $\mathcal{T}_a(0) = \tau \, \mathcal{A}(\phi)$, for some strictly increasing function $\mathcal{A}(\cdot)$ of one variable, with $\mathcal{A}(0) = 1$.

The proof of this proposition follows, essentially, from a change of variables from $\tilde{p}$ to $\tilde{p}/(\sigma\sqrt{\mathrm{T}(\tilde{p})})$. The function $\mathcal{A}(\phi)$ is the solution of a recursion which can be easily solved numerically, and can also be approximated analytically, see Online Appendix AA-5 for details. Clearly Proposition 6 implies that keeping ϕ fixed the expected time between price adjustments is increasing in τ. The higher ϕ, the larger the expected time between price adjustments for given τ.

Next we turn to the expected time between reviews. As an intermediate step we write a recursion for the distribution of the price gaps upon review (and before adjustment), for which we use density $g(\tilde{p})$. The recursion is akin to a Kolmogorov forward equation:

$$g(\tilde{p}) = \int_{-\bar{p}}^{\bar{p}} \frac{1}{\sigma\sqrt{\mathrm{T}(p)}} \, n\left(\frac{\tilde{p} - p}{\sigma\sqrt{\mathrm{T}(p)}}\right) \, g(p) \; dp + \frac{1}{\sigma\sqrt{\tau}} \, n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}}\right) \left[1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp\right] \quad (28)$$

for all $\tilde{p} \in [-\bar{p}, \bar{p}]$, where $n(\cdot)$ is the density of a standard normal.[20] The first term on the right side of this equation gives the mass of firms with values of the price gap p that in last review were in the inaction region, and drew shocks to transit from p to $\tilde{p} = p - s\sigma\sqrt{\mathrm{T}(p)}$

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[20] Notice that $\frac{1}{\sigma\sqrt{\mathrm{T}(p)}} \, n(\cdot)$ is the probability density of $\tilde{p}$ conditional on p, where $\tilde{p} = p - s \, \sigma\sqrt{\mathrm{T}(p)}$.

during $\text{T}(p)$ periods. The second term has the mass of firms that in the last review were outside the inaction region, and hence started with a price gap of zero, so that $1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp$ is the fraction of reviews that end up outside the range of inaction, and hence trigger an adjustment.[21] The expected time between price reviews is simply given by the expected value of $\text{T}(\cdot)$, the time until the next review, across the different price gaps, distributed according to $g(\cdot)$:

$$\mathcal{T}_r = \int_{-\bar{p}}^{\bar{p}} \text{T}(p) \, g(p)dp + \tau \left[1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp\right] \, . \tag{29}$$

Similarly to Proposition 6, we can derive an analytical approximation for the expected time between reviews as function of only $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$ and τ.[22] We establish the following:

PROPOSITION 7.   Let $\text{T}(\tilde{p})$ be given by the quadratic approximation given in Proposition 4 and let $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$. The average frequency of price reviews is $n_r = 1/\mathcal{T}_r$ where $\mathcal{T}_r = \tau \, \mathcal{R}(\phi)$, for some strictly decreasing function $\mathcal{R}(\cdot)$ of only with variable, with $\mathcal{R}(0) = 0$.

The proof follows, essentially, by using a change of variables from $\tilde{p}$ to $\tilde{p}/(\sigma\sqrt{\text{T}(\tilde{p})})$ and analogously for p on the Kolmogorov-like equation (28), and then using equation (29). The function $\mathcal{R}(\cdot)$ used the solution for g, that follows a recursion which can easily solved numerically, and can also be approximated analytically, see Online Appendix AA-6 for details. Note that Proposition 7 implies that keeping ϕ fixed the expected time between price reviews is increasing in τ. However, the higher ϕ, the smaller $\mathcal{R}(\phi)$, and the smaller the expected time between price reviews for given τ.

An important implication of Proposition 6 and Proposition 7 is that the ratio of the average number of price reviews to the average number of adjustments per unit of time, i.e. $\frac{n_r}{n_a} = \frac{\mathcal{A}(\phi)}{\mathcal{R}(\phi)}$, only depends on ϕ. This result is useful because it allows us to identify the value of ϕ using data on the ratio between the (mean) frequency of price adjustment and the (mean) frequency of price review. In addition, from Proposition 2, we know that ϕ is roughly

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[21]For future reference, notice that equation (28) does not use the values of $g(\cdot)$ outside the range of inaction.

[22] In particular, we replace the density $g(\tilde{p})$, that is symmetric around $\tilde{p} = 0$ and has zero odd derivatives at $\tilde{p} = 0$, by a quadratic function, and use $\text{T}(\tilde{p}) = \tau - (\tilde{p}/\sigma)^2$. See Online Appendix AA-5 for details.

determined by the relative cost of adjusting relative to reviewing prices, $\alpha = \frac{\psi}{\theta}$. Therefore the model delivers a tight relationship between $\frac{n_r}{n_a}$ and α, that can be brought to the data to estimate the relative cost of review to adjustment. The right panel in Figure 3 plots the mapping between $\frac{n_r}{n_a}$ and the $\varphi(\alpha)$, side by side to the function $\varphi(\alpha)$. For given value of $\frac{n_r}{n_a}$, the implied value for α given by our approximated solution is relatively larger than the value implied by the numerical solution of the model. The discrepancy between the approximated and numerical solution increases as α increases.

These results are useful to interpret the data of Figure 1. The slope of a line in the (n_a, n_r) plane is determined by the ratio of the costs of adjustment and review, α. Given the slope of the line, the position of the points along the line, i.e. the average level of activity of either review or adjustment, varies with 3 factors: the volatility of the state (σ), the curvature of the profit function (B), or an equal variation in the level of the costs ψ and θ (such that α remains constant).

Using the expressions for τ and $\bar{p}$ as functions of α from equation (23) and equation (24), and the results of Proposition 6 and Proposition 7, leads to a characterization of the elasticity of the frequencies of adjustment and review, summarized in:

PROPOSITION 8.   Assume that $\sigma^2 \, \psi/B$ is small so that approximating ϕ using φ is accurate. Then the elasticities of the number of price adjustments and revisions with respect to the cost satisfy:

$$\frac{\partial \log n_a(\alpha)}{\partial \log \theta} = -\frac{\partial \log \varphi(\alpha)}{\partial \log \alpha}\left(1 - \varphi(\alpha)\frac{\mathcal{A}'(\varphi(\alpha))}{\mathcal{A}(\varphi(\alpha))}\right) \geq -\frac{1}{2} \text{ and } = -\frac{1}{2} \text{ if } \alpha = 0 \,, \quad (30)$$

$$\frac{\partial \log n_r(\alpha)}{\partial \log \theta} = -\frac{\partial \log \varphi(\alpha)}{\partial \log \alpha}\left(1 - \varphi(\alpha)\frac{\mathcal{R}'(\varphi(\alpha))}{\mathcal{R}(\varphi(\alpha))}\right) \leq 0 \text{ and } = -\frac{1}{2} \text{ if } \alpha = 0 \,, \quad (31)$$

$$\frac{\partial \log n_r(\alpha)}{\partial \log \psi} = -\frac{1}{2} - \frac{\partial \log n_r(\alpha)}{\partial \log \theta}, \text{ and } \frac{\partial \log n_a(\alpha)}{\partial \log \psi} = -\frac{1}{2} - \frac{\partial \log n_a(\alpha)}{\partial \log \theta} \,. \quad (32)$$

Notice that equations (32) imply that an increase of θ and ψ in the same percentage, decreases the number of adjustments and the number of reviews n_a and n_r by half of that percentage. Note that this one half elasticity is the one present in the models described

in Section 4.1 and Section 4.2 that feature either information cost only or menu cost only. Furthermore, we remark that while an increase in the information cost θ decreases the ratio n_r/n_a, so each adjustment corresponds to fewer observations, the elasticity of n_a with respect to θ displayed in equation (30) is negative, because observations and adjustments are complementary activities.[23]

6.2 The distribution of price changes

Next we derive expressions for the average size of price change, $\mathbb{E}[\,|\Delta p|\,]$, and for the distribution of price changes. The average size of price adjustment equals the average price-gap upon review, conditional on that average gap being outside the range of inaction. Thus, to compute this average we extend $g(\cdot)$, defined in equation (28), to the values outside the range of inaction. Since the distribution of price changes has to be conditional on a price change, and the price change happens with probability $[1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp]$, then price adjustments have a density:

$$ w(\Delta p) \equiv \frac{g(\Delta p)}{1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp} = \frac{\int_{-\bar{p}}^{\bar{p}} \frac{1}{\sigma\sqrt{\mathrm{T}(p)}}\, n\left(\frac{\Delta p - p}{\sigma\sqrt{\mathrm{T}(p)}}\right) g(p)dp}{1 - \int_{-\bar{p}}^{\bar{p}} g(p)dp} + \frac{1}{\sigma\sqrt{\tau}}\, n\left(\frac{\Delta p}{\sigma\sqrt{\tau}}\right), \qquad (33) $$

for $|\Delta p| \in (\bar{p}, \infty)$. The average size of price change is then given by $\mathbb{E}[\,|\Delta p|\,] = 2\int_{\bar{p}}^{\infty} z\, w(z)dz$. The distribution of price changes $w(\cdot)$ is composed of two terms: the first term is positive and strictly decreasing in $|\Delta p|$; the second term is the density of a normal distribution with zero mean and standard deviation equal to $\sigma\sqrt{\tau}$. Therefore, conditional on adjusting prices, $w(\cdot)$ assigns relatively more weight to values of $|\Delta p|$ that are closer to $\bar{p}$ than the normal distribution. Moreover, $w(\cdot)$ is symmetric around zero, as both the normal distribution and the distribution of the price gaps upon review, $g(\cdot)$, are symmetric. The next proposition characterizes this distribution, and some related statistics, as a function of the parameters

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[23]The results in Proposition 8 use the approximations to the decision rules. In Online Appendix AA-4 we compute numerical solutions to n_r and n_a for a range of values of ψ and θ and verify that the pattern of elasticities of this proposition hold for several parameter configurations.

for the optimal decision rules.

PROPOSITION 9. Consider an optimal decision rule described by two parameters (τ, ϕ) and where, in the range of inaction, $\mathrm{T}(p)$ is given by the approximation described in Proposition 4, namely $\mathrm{T}(p) = \tau - (p/\sigma)^2$. Let x denote the normalized price changes: $x \equiv \Delta p/(\sigma\sqrt{\tau})$, and let $\bar{\phi} \equiv \frac{\phi}{\sqrt{1-\phi^2}}$. Then

$$\mathbb{E}[\,|\Delta p|\,] = \sigma\,\sqrt{\tau}\,\mathbb{E}[\,|x|\,;\,\phi]\,, \quad \mathbb{E}[\,|x|\,;\,\phi] = 2\int_{\bar{\phi}}^{\infty} x\,v(x\,;\,\phi)\,dx\,,$$

$$\frac{\min|\Delta p|}{\mathbb{E}[\,|\Delta p|\,]} = \frac{\mathrm{mode}\,|\Delta p|}{\mathbb{E}[\,|\Delta p|\,]} = \frac{\bar{\phi}}{2\int_{\bar{\phi}}^{\infty} x\,v(x\,;\,\phi)\,dx} \in [0,1]\,, \tag{34}$$

$$\frac{\mathrm{std}[|\Delta p|]}{\mathbb{E}[\,|\Delta p|\,]} = \sqrt{\frac{\int_{\bar{\phi}}^{\infty} x^2\,v(x\,;\,\phi)\,dx}{2\left(\int_{\bar{\phi}}^{\infty} x\,v(x\,;\,\phi)\,dx\right)^2} - 1} \in \left[0, \sqrt{\frac{\pi}{2} - 1}\right]\,. \tag{35}$$

where the density can be written as $v(x;\phi) = e(x;\phi) + n(x)$ for $|x| > \bar{\phi}$, and $= 0$, otherwise, for a strictly positive and strictly decreasing function $e(\cdot;\phi)$ for $x > 0$.

The proof uses the change of variables $\tilde{p}$ to $\tilde{p}/(\sigma\sqrt{\mathrm{T}(\tilde{p})})$. In Online Appendix AA-7 we give an analytical approximation to this density. This proposition, together with the previous characterization of (τ, ϕ) as functions of the parameters of the problem, completely characterizes the distribution of price changes. In particular, notice that using the approximations developed in Proposition 5, for ϕ in equation (21) and for τ in equation (23), we can write the average price change as:

$$\mathbb{E}[\,|\Delta p|\,] = \left(\frac{\theta\,\sigma^2}{B}\right)^{\frac{1}{4}}\mathcal{E}(\alpha)\,, \tag{36}$$

where $\mathcal{E}(\alpha) \equiv \varphi(\alpha)^{\frac{1}{2}}\,\mathbb{E}[\,|x|\,;\varphi(\alpha)]$. Notice that, holding constant the ratio of the two costs, α, this expression has the same comparative static with respect to a change in both cost (ψ, θ), and to a change in (B, σ^2) than the one for the models with observation cost only or with menu cost only. Finally, notice that equation (34) implies that the mode and the

Figure 4: The Distribution of Price Changes

The three theories   Theories vs. data
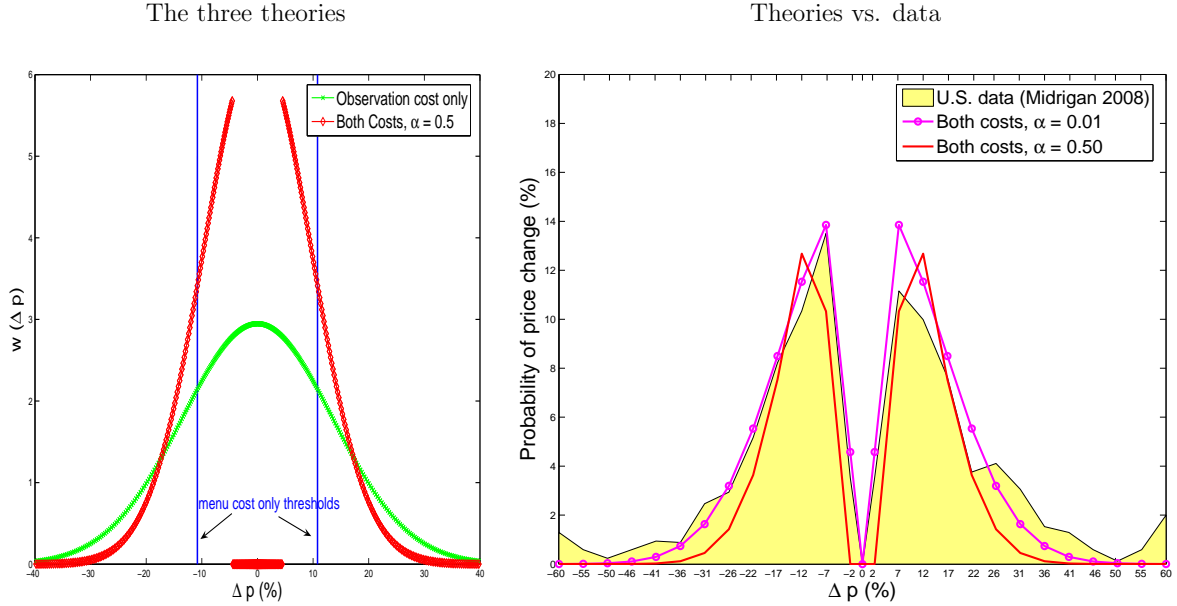
Note: Model's parameters as in Table 3. The distribution of price changes labeled U.S. data is from Midrigan (2007). The models' distributions in the right panel are computed over bins of price changes of size 0.05, as in the data, and linearly interpolated in between.

standard deviation of the absolute value of price changes, both relative to the expected value, are functions of α:

$$\frac{\min |\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} = \frac{\text{mode } |\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} = \mathcal{M}(\alpha) \ \ \text{and} \ \ \frac{\text{std}[|\Delta p|]}{\mathbb{E}[\ |\Delta p|\ ]} = \mathcal{S}(\alpha) \ , \tag{37}$$

where $\mathcal{M}(\alpha) \equiv \bar{\bar{\phi}}(\alpha) \ / \ 2 \int_{\bar{\bar{\phi}}(\alpha)}^{\infty} x \ v(x \ ; \varphi(\alpha)) \ dx$, and where $\bar{\bar{\phi}}(\alpha) \equiv \varphi(\alpha)/\sqrt{1 - \varphi(\alpha)^2}$. The function $\mathcal{S}(\cdot)$ is defined analogously. Thus the "shape" of the distribution depends on the ratio of the two costs. This result provides an additional identification scheme to measure α, using the distribution of price changes.

The left panel of Figure 4 plots the distribution of price changes implied by our model against the one predicted by the special cases of observation cost and menu cost only, under a parameterization of the model that we think of reasonable given the empirical evidence about average frequencies of adjustments and reviews. The right panel of Figure 4 plots the empirical distribution of price changes, measured at intervals of 5% size, corresponding to AC

Table 3: Moments from the distribution of price changes: theory vs. data

	$\mathbb{E}[|\Delta p|]$	$\frac{\sigma(|\Delta p|)}{\mathbb{E}[|\Delta p|]}$	1st quartile	third quartile	fraction $< \frac{\mathbb{E}[|\Delta p|]}{2}$	fraction $< \frac{\mathbb{E}[|\Delta p|]}{4}$
Menu Cost	0.11	0.00	0.11	0.11	0.00	0.00
Observation Cost	0.11	0.79	0.04	0.16	0.31	0.16
Both Costs, $\alpha = 0.50$	0.11	0.47	0.07	0.14	0.11	0.00
Both Costs, $\alpha = 0.01$	0.11	0.70	0.05	0.16	0.29	0.12
Data:						
Midrigan (2007)*	0.15	0.72	0.06	0.19	0.25	0.10
Klenow and Kryvtsov (2008)**	0.11	-	-	-	0.44	0.25

Parameters common to all models $B = 20$, $\rho = 0.02$; Model with 2 costs: for the case where $\alpha = 0.50$, we use $\sigma = 0.15$, $\psi = 0.015$, $\theta = 0.03$; For the case where $\alpha = 0.01$: $\sigma = 0.17$, $\psi = 0.001$, $\theta = 0.10$. Menu cost: $\psi = 0.15$, $\theta = 0$, $\sigma = 0.06$; Observation cost: $\psi = 0$, $\theta = 0.11, \sigma = 0.17$; These parametrizations generate approximately 1.6 adjustments per year and an average size of price change of 10.8% in all models, which is consistent with estimates reported by Klenow and Kryvtsov (2008) for regular price changes. * $\mathbb{E}[|\Delta p|]$ and quartiles from Table 1 in Midrigan (2007) , the others from normalized price changes of Table 2b, all from AC Nielsen data on non-sale prices. ** These numbers are obtained from Tables 3-4 in Klenow and Kryvtsov (2008) and refer to regular price changes.

Nielsen data from Midrigan (2007), against the corresponding distribution of price changes implied by the different theories. Likewise, Table 3 reports some key statistics related to the results of Proposition 9 from the distribution of the size of price changes in the three models in the top panel, and in the bottom panel the corresponding statistics for different data sets in the US. The parameters in Figure 4 and Table 3 are chosen such that the three models imply the same frequency of price changes, i.e. 1.6 adjustments per year. In Table 3 we include two parameterizations of the model with two costs, one with very low menu cost ($\alpha = 0.01$ case) and one with moderate menu cost ($\alpha = 0.50$). The histogram in the right panel corresponds to the high α case. In comparing the distribution of price changes from these models, several results are worth mentioning. First, our model assigns a zero probability to price changes smaller than $\bar{p}$ in absolute value, while the model with observation cost only assigns a positive probability mass also at arbitrary small price changes. Second, our model with both types of costs has a mode at $\pm\bar{p}$, while the model with observation cost only has a mode at zero. Finally, our model cannot generate price changes having fatter tails than innovations in p^*. Therefore, being innovations normally distributed, our model with both

costs implies a platykurtic distribution of price changes.

The distribution of price changes in our model is different from the one implied by the model with menu cost only, where price changes occur only at a size equal to $\bar{p}$ (in absolute value). Our model predicts a distribution of price changes that resembles the one from the menu cost model, as it puts a lot of weight to price changes close to $\bar{p}$ in absolute value, but also assigns some positive probability to price changes of size larger than $\bar{p}$. Hence, the average size of price changes in the model with both observation and menu costs is larger than $\bar{p}$.

The shape of the distribution of price changes implied by our model resembles in some aspects the distribution of price changes estimated by Midrigan (2007) on AC Nielsen data and by Cavallo (2009) for Latin American countries. In particular, these studies find distribution of price changes that are roughly bimodal, with relatively little mass close to zero when compared to a normal distribution.[24] The statistics in last two columns of Table 3 are informative about the mass of small price changes in the distribution. Our model implies less mass of price changes at values smaller than a half and a quarter of the average, than the mass implied by the normal distribution in the model with an observation costs only. At the same time, however, our model allows for smaller price changes than the menu cost model would imply. When comparing these statistics to the data, we see that our model implies too little mass of small price changes at $\alpha = 0.5$. Moreover, the ratio of the standard deviation to the average absolute price change is too small relatively to the data. However, this does not necessarily mean that our model cannot replicate these statistics, but just signals that menu cost will have to be quite small relative to the observation cost. In fact, at $\alpha = 0.01$ the model improves substantially on these statistics. Yet such a low value of α implies a value of the frequency of review to adjustment $n_r/n_a = 1.06$, that is clearly too low relative to what the data suggest. Based on these considerations we choose $\alpha = 0.5$ as our baseline parametrization, since we think that the measurement error affecting the size of small price

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[24]See Online Appendix AA-8 for more details on empirical evidence about the distribution of price changes.

changes is potentially larger that the one we discussed concerning the survey measures of $n_r/n_a \cong 1.5$. The level of the costs ψ, θ and the size of the fundamental innovation σ are chosen to match an average frequency of price adjustment $n_a \cong 1.6$, and an average frequency for the size of (absolute) price changes $|\Delta p| \cong 0.11$. As discussed above, $B = 20$ is consistent with a markup of around 15%.

6.3   The hazard-rate of price changes

In Section 4.1 and Section 4.2 we showed that in the models with only one type of cost the hazard rate of a price changes is monotone increasing on the time elapsed since the last change. We now show that with both costs the hazard rate function is *not* monotone.

Let t denote the time elapsed since the last price adjustment, and let $S(t)$ be the survival probability, i.e. the fraction of spells of unchanged prices that are of length t or longer. The instantaneous hazard rate is defined as $\boldsymbol{h}(t) = -S'(t)/S(t)$. We construct the hazard rate for the interval $t \in [0, \tau + \min\{\tau, 2\underline{\tau}\}]$, i.e. in this analytical characterization we restrict attention to the consequences of the first 2 reviews. First, notice that until τ units of time no firm will review its price, and hence no adjustments will take place, so that $S(t) = 1$ and the hazard rate is zero for $t \in [0, \tau)$. At τ all the firms review their prices, and a fraction of them adjusts. This fraction is $2(1 - N(\phi))$, i.e. the probability that after the review the target is outside the range of inaction. Thus, there is a jump down in the survivor function to $S(\tau) = 2\,N\,(\phi) - 1$, and thus the instantaneous hazard rate is infinite at this point. For the remaining firms the time of the next review depends on the current price gap $\tilde{p}$. The earliest next review among these firms occurs $\underline{\tau}$ periods after the first review, these are the firms that have a price gap inside the range of inaction but arbitrarily close to its boundary, i.e. very close to $\bar{p}$ or $-\bar{p}$. We describe the number of firms that change prices in their second review, between times $\tau + \tilde{t}$ and $\tau + \tilde{t} + \Delta$, as approximately $\partial S(\tau + \tilde{t})/\partial t \times \Delta$, satisfying:

$$\frac{\partial S(\tau + \tilde{t})}{\partial t} = \left[1 - N\left(\frac{\bar{p} - p(\tilde{t})}{\sigma\sqrt{\tilde{t}}} \right) + N\left(\frac{-\bar{p} - p(\tilde{t})}{\sigma\sqrt{\tilde{t}}} \right) \right] 2 \frac{\partial p(\tilde{t})}{\partial t} \frac{n\left(\frac{p(\tilde{t})}{\sigma\sqrt{\tau}} \right)}{\sigma\sqrt{\tau}} ,$$

for $\underline{\tau} < \tilde{t} < \tau$ where $p(\tilde{t}) \equiv \mathrm{T}^{-1}(\tilde{t})$ denotes the inverse of $\mathrm{T}(\cdot)$, so that $\mathrm{T}^{-1}(\tilde{t}) : [\underline{\tau}, \tau] \to [0, \bar{p}]$.[25]

We summarize these results in the next proposition:

PROPOSITION 10. The hazard rate of price adjustments starts flat at zero, it jumps to infinity at $t = \tau$, it returns flat to zero in the segment $t \in (\tau, \tau + \underline{\tau})$, it jumps to a positive value at $\tau + \underline{\tau}$. In the segment $[\tau + \underline{\tau}, \tau + \min\{\tau, 2\underline{\tau}\})$ it is strictly positive and, if $\underline{\tau} > (1/2)\tau$, it tends to infinity at the end of this segment and returns to zero:

$$\boldsymbol{h}(t) = 0, \text{ for } t \in [0, \tau), \quad \boldsymbol{h}(\tau) = \infty, \quad \boldsymbol{h}(t) = 0 \text{ for } t \in (\tau, \tau + \underline{\tau}),$$

$$\lim_{t \downarrow \tau + \underline{\tau}} \boldsymbol{h}(t) > 0, \quad 0 < \boldsymbol{h}(t) < \infty \text{ for } t \in [\tau + \underline{\tau}, \tau + \min\{\tau, 2\underline{\tau}\})$$

$$\text{if } \tau \leq 2\underline{\tau} : \lim_{t \uparrow 2\tau} \boldsymbol{h}(t) = \infty, \quad \text{and if } \tau < 2\underline{\tau} : \boldsymbol{h}(t) = 0 \text{ for } t \in (2\tau, \tau + 2\underline{\tau}).$$

The previous proposition does not characterize the hazard rate when durations are longer than $\tau + \min\{\tau, 2\underline{\tau}\}$. While an expression can be developed for larger durations it becomes increasingly complex because a price change can happen after several combinations of previous reviews. Indeed the larger the value of t, the larger the number of combinations of different duration of previous reviews that can happen. The effect of this feature is that the hazard rate for larger values of elapsed time t will tend to be smaller but without the "holes", i.e. stretches with zero values that we have identified for low values of t in Proposition 10. We illustrate these features in the left panel of Figure 5 that displays the *weekly* hazard rate based on a large number of simulations of the model's decision rules. The red histogram is the weekly hazard rate produced by the simulations. The black line is the analytical coun-

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[25]The first term in square brackets is the fraction of those firms that had price gap $\tilde{p} > 0$ at time τ and that after the second review are outside the range of inaction, and hence adjust their price.[26] The remaining term counts the number of firms that have a price gap $\tilde{p} = p(\tilde{t})$ so that they will adjust their price at $\tau + \tilde{t}$. This, in turn, is made of two terms. The second ratio is the density of innovations from time zero to time τ necessary to end up in the required value of the price gap $p(\tilde{t})$. The derivative, $\partial p(\tilde{t})/\partial t$, comes from a change of variables formula, to convert the density of prices into a density expressed with respect to times. If $\tau + \underline{\tau} > 2\tau$, the expression for $\partial S'(\tilde{t} + \tau)/\partial dt$ is valid for all $t \in [\tau + \underline{\tau}, 2\tau]$. In this case, since the symmetry of $\mathrm{T}(\cdot)$ implies that $\partial \mathrm{T}(0)/\partial p = 0$, then $\partial p(\tau)/\partial t = \infty$, and thus the hazard rate tends to infinity at the end of this interval, and reverts to zero afterwards. If this conditions is not satisfied, the expression for the derivative of S for values higher than $\tau + 2\underline{\tau}$ is more complex because a price change can occur at exactly the same time after two or three reviews.

terpart of the hazard developed in Proposition 10 rate. We plot dotted vertical lines to note that every τ periods there is a "wave" of price adjustments. At $t = \tau$ all the adjustments occur simultaneously, so the hazard rate has a spike. In the subsequent waves, i.e. for larger values of t, they are less concentrated around a single value, and hence the hazard rates have smaller spikes.We compare the shape of the hazard rate function with the one in the observation cost model, which is given by a single spike (in green), and one of the menu cost model (in blue). The latter is characterized by an increasing hazard rate in the first weeks, but quickly converging to a constant rate. The examples plotted for the three models have the same average number of price changes per year (equal to 1.6).

Figure 5: Hazard Rate of Price Changes, weekly and monthly rates

The three theories   Model with both costs vs. data
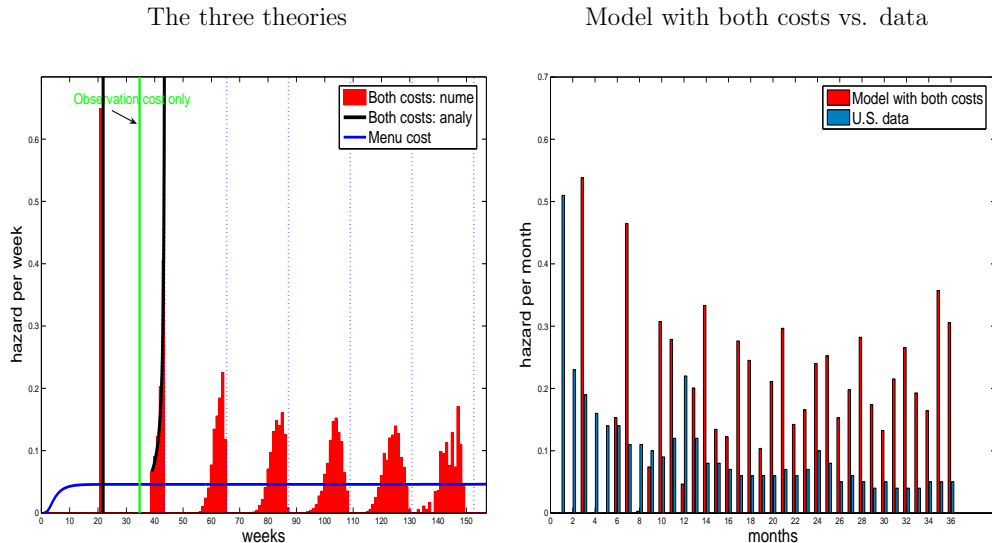
Note: Model with two costs: $B = 20$, $\rho = 0.02$, $\sigma = 0.15$, $\psi = 0.015$, $\theta = 0.03$. Menu cost only $\psi = 0.15$, $\theta = 0$, $\sigma = 0.06$; observation cost only $\psi = 0$, $\theta = 0.11, \sigma = 0.17$ (the other parameters unchanged), to generate 1.6 adjustments a year and an average size of price change of 10.8%, as in the model with both costs. The U.S. monthly hazard is from Klenow and Kryvtsov (2008).

There is no consensus in the literature about the shape of the hazard function. For instance, both Klenow and Kryvtsov (2008) and Nakamura and Steinsson (2008) show that the hazard rate computed on U.S. CPI data is downward sloping, being at odd with menu cost models of price adjustment. Similar evidence is reported by Alvarez, Burriel, and Hernando

(2005) for the Euro area. However, when adjusting for heterogeneity Klenow and Kryvtsov (2008) and Nakamura and Steinsson (2008) reach opposite conclusions. The former finds a flat hazard rate, while the latter finds a downward sloping hazard despite adjusting for heterogeneity. Finally, a recent study by Cavallo (2009) shows that the hazard rate is upward sloping in four Latin American countries.[27] As shown above, a model with menu cost only produces an upward sloping hazard. However, heterogeneity could account for the observed downward sloping hazard rate if data was generated by this model. Our model with both observation and menu costs predicts a non-monotone hazard. Among other things, this implies that an econometrician estimating the product level hazard rate on data generated from our model could obtain an upward sloping or a downward sloping hazard rate depending on the length of the sample, and on the frequency of observations within that sample, even in absence of heterogeneity. In the right panel of Figure 5 we plot the monthly hazard rate estimated by Klenow and Kryvtsov (2008) on U.S. CPI data, against the monthly hazard rate implied by our model with both costs, under a parametrization that we think reasonable. While at this parametrization our model is able to account for the downward-sloping shape of the hazard function, it is unable to generate the positive and high hazard rate at a horizon of 1 to 4 months. Reducing the size of the two costs would help reconciling the model with the empirical hazard rate, but at the cost of a higher frequency of price adjustment and observation than seen in the data. An alternative explanation could be found by allowing for heterogeneity in adjustment or observation costs across firms.

7   The case of positive inflation

In this section we discuss the effect of inflation, i.e. the drift of the (log of the) target price p^*, on the solution of the problem studied above. In the presence of inflation ($\mu > 0$), immediately after an adjustment the agent will set a price higher than value of the target at the time of the adjustment. This is quite intuitive, since the agent expects to adjust in

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[27]See the on-line appendix for more details on empirical evidence about the hazard rate.

the future and hence hedges against the forecastable part of the price gap deviation due to inflation.

It is useful to consider the case where price changes occur at deterministic intervals, as it is optimal to do when the menu cost is zero ($\psi = 0$) or when there is no uncertainty ($\sigma = 0$). In these cases, letting the discount factor $\rho \downarrow 0$, the optimal value for $\hat{p}$ is $\hat{p} = \mu \, \bar{T}/2$, where $\bar{T}$ denotes the time until the next price adjustment. This is quite intuitive: the optimal reset price implies a price gap equal to the value that the target will have exactly at half of the time until the next adjustment, so the first half of the time deviations are positive, and the second negative. A similar formula holds for the case where σ and ψ are not zero, from which one can develop the result that with positive inflation, $\hat{p} > 0$ for small σ or ψ (see the Online Appendix 15 for the exact expression and the corresponding result). Next we discuss the cases of low and high inflation, respectively.

7.1 "Low inflation"

In this section we give a partial characterization of the effect that a low inflation rate has in the decision rules. We focus on the effect of a low inflation rate on the value of τ, the optimal time until the next revision decided right after a price adjustment and show that this feature of the decision rule changes very little for small inflation, i.e. that its derivative with respect to the inflation rate is zero when evaluated at zero inflation rate. This is to be contrasted with $\hat{p}$, the value of the price gap right after a price adjustment, which has a strictly positive derivative with respect to the inflation rate. Thus, loosely speaking, the effect of a small inflation rate is concentrated on the size of the price changes rather than on the frequency of changes.

PROPOSITION 11.   Let $\tau(\mu)$ be optimal time until the next observation set right after a price adjustment, $n_a(\mu)$ the average number of adjustment per unit of time, $n_r(\mu)$ the average number of reviews per unit of time, and $\hat{p}(\mu)$ the optimal price gap set right after a price

adjustment, all as a function of the inflation rate. We have:

$$\left.\frac{\partial \hat{p}}{\partial \mu}\right|_{\mu=0} > 0 \text{ and } \left.\frac{\partial \tau}{\partial \mu}\right|_{\mu=0} = \left.\frac{\partial n_a}{\partial \mu}\right|_{\mu=0} = \left.\frac{\partial n_r}{\partial \mu}\right|_{\mu=0} = 0 \ . \tag{38}$$

The proof of this proposition follows from the symmetry of the instantaneous cost function (quadratic), law of motion of the price gap, and of the p.d.f. of a normal distribution. We note that the zero derivative of τ with respect to inflation at zero inflation is also a feature of the problem with observation cost only, as shown in Proposition 1. Below we illustrate this proposition by computing the decision rule for some numerical examples displayed in Figure 6 and Table 4 for different inflation rates.

7.2 "High" inflation

In this section we solve the problem assuming $\sigma \downarrow 0$ and $\mu > 0$. We study this case as an approximate solution of the problem where inflation (μ) is large relative to the volatility of the idiosyncratic shocks (σ).

When there is no uncertainty the optimal policy is to review *once* and, after the first price adjustment, to adjust prices every τ periods, by exactly the same amount. This is a version of the classical price adjustment model by Sheshinski and Weiss (1977). The optimal policy can also be written in terms of the price gap, following a sS band, adjusting the price when a lower barrier $\underline{p}$ is reached, to a value given by $\hat{p}$. Our interest in the function $\mathrm{T}(\cdot)$ and the thresholds $\bar{p}, \hat{p}$ and $\underline{p}$ in this limit is that it should be informative about the shape of $\mathrm{T}(\cdot)$ for very small, but strictly positive, values of σ, and in general for the forces that operates in the general case of μ and σ strictly positive.

PROPOSITION 12. Let $\sigma = 0$ and $\mu > 0$. As we let $\rho \downarrow 0$ the optimal decision rule satisfy:

$$n_a = \left(\frac{B \mu^2}{6 (\psi + \theta)}\right)^{1/3}, \quad \hat{p} = \frac{1}{2}\left(\frac{6 \mu (\psi + \theta)}{B}\right)^{1/3}, \quad \underline{p} = -\hat{p}, \quad \frac{\hat{p} - \underline{p}}{\bar{p} - \hat{p}} = \sqrt{3}$$

$$\mathrm{T}(\tilde{p}) = \frac{\tilde{p} - \underline{p}}{\mu} \text{ if } \tilde{p} \in [\underline{p}, \ \bar{p}] \text{ and } \mathrm{T}(\tilde{p}) = 1/n_a \text{ otherwise.}$$

This result follows from writing down the objective function at the limit of $\rho = 0$ and solving it explicitely. In the deterministic case with $\theta > 0$, the reviews will never be conducted unless there is an adjustment, and this is precisely the logic behind the linear decreasing shape of the function $\mathrm{T}(\cdot)$ in the proposition. In the range of inaction, $\mathrm{T}(\tilde{p})$ is exactly the time it takes until the next adjustment. This is in stark contrast with the symmetric shape of $\mathrm{T}(\cdot)$ in the case with no drift. In this deterministic case the cost of each adjustment is given by the sum $\theta + \psi$, due to our assumption that a review must be conducted at the time of an adjustment. In the limit case of Proposition 12 the frequency of adjustments has an elasticity of $1/3$ with respect to the cost $\theta + \psi$. This is different from the the case of zero drift, where the number of adjustments has an elasticity of $1/2$ with respect to an equal proportional change in the costs θ and ψ. The optimal return point $\hat{p}$ is strictly positive, and increasing in inflation, an application of the general result of Proposition 15. The optimal return point $\hat{p}$ and the lower bound of the range of inaction $\underline{p}$ are such that $\hat{p} = -\underline{p}$. This feature is very intuitive, since for $\rho = 0$ the agent gives the same weight to the deviations that occur just after adjusting as to those just before the next adjustment. This also differs from the optimal return of zero of the model with no drift. Finally, the boundaries of the range of inaction are asymmetric: $(\hat{p} - \underline{p})/(\bar{p} - \hat{p}) = \sqrt{3} > 1$. This asymmetry is due to the fact that $\hat{p} > 0$ already takes into account the effect of positive inflation, and hence at $\bar{p}$ the deviation from the static optimum value of $\tilde{p}$ are very large.

To evaluate the extent to which the features discussed in this limit case are present in the case of strictly positive μ and σ Figure 6 includes three panels where, for fixed values of all the parameters –including $\sigma = 0.15$–, we display the shape of the optimal decision rules for three strictly positive levels of the inflation rate. It is clear that for inflation rate of 5% the difference with the shape for zero inflation is very small, but we can already see that the range of inaction starts being asymmetric, that the optimal return point is positive, and that $\mathrm{T}(\cdot)$ is asymmetric, with a peak at the right of the optimal return point, and with $\mathrm{T}(\underline{p}) < \mathrm{T}(\bar{p})$. These features are more apparent for 20% inflation rate, and even more so for

Figure 6: Policy rule $T(\tilde{p})$ under different values of inflation: $\mu = 0.05, 0.2, 0.6$
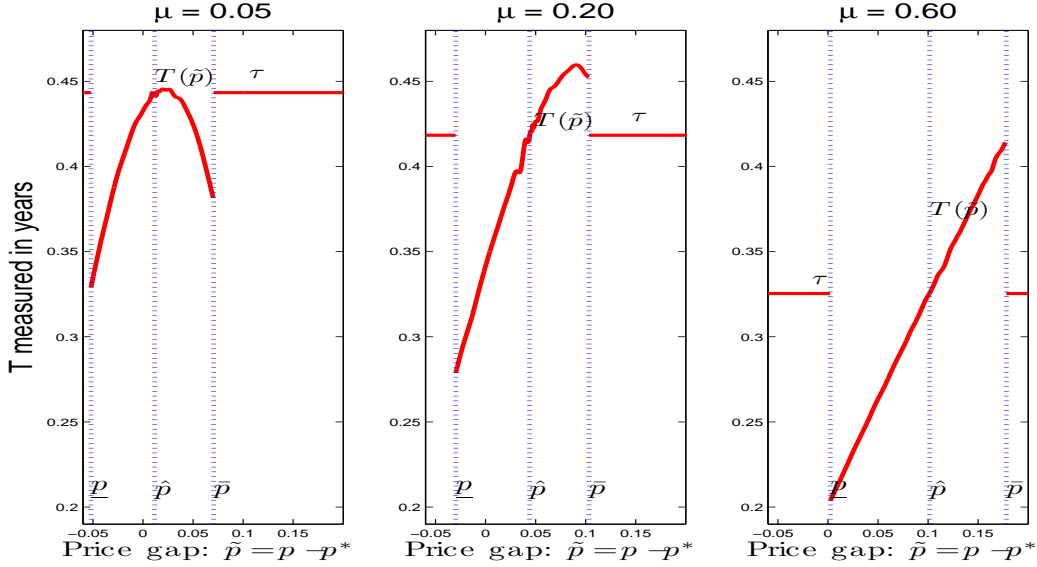
Baseline parameter values are: $B = 20,$, $\rho = 0.02$, $\sigma = 0.15$, $\psi = 0.015$, $\theta = 0.03$.

the 60% annual inflation rate.

7.3 Price adjustments with no observations

In this section we relax the assumption that every price adjustment requires an observation. Below we consider another technology to change prices that allows multiple price adjustments to be planned after a single review. We show that in the case where the state has no drift it is indeed optimal to observe every time a price is adjusted. On the other hand, as the volatility goes to zero relatively to the drift it is optimal to do some adjustments without observing the state.

We consider an extension of the problem where upon paying a cost θ, and finding the value of the price gap $\tilde{p}$, the firm decides whether to immediately change the price, paying the cost ψ or not. The firm can also decide a number n of future dates where the price will be adjusted without a new observation of the state. For each of these adjustments the firm pays only the cost ψ. In this case the Bellman equations for a firm just after finding the

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value of the price gap $\tilde{p}$ satisfy:

$$V_n(\tilde{p}) = \theta + \min_{\tau_0, \{\hat{p}_i, \tau_i\}_{i=1}^n} \int_0^{\tau_0} e^{-\rho t} B \left[(\tilde{p} - \mu t)^2 + \sigma^2 t \right] dt$$

$$+ \sum_{i=1}^n \left[e^{-\rho \tau_{i-1}} \psi + \int_{\tau_{i-1}}^{\tau_i} e^{-\rho t} B \left[(\hat{p}_i - \mu t)^2 + \sigma^2 t \right] dt \right]$$

$$+ e^{-\rho \tau_n} \int_{-\infty}^{\infty} V \left(\hat{p}_n - \mu \tau_n - s \sqrt{\tau_n} \sigma \right) dN(s) , \qquad (39)$$

$$V(\tilde{p}) = \min_{n \geq 0} V_n(\tilde{p}) \qquad (40)$$

where $0 \leq \tau_i \leq \tau_{i+1}$, and $n = 0, 1, 2, ...$, and where for $n = 0$ we denote $\hat{p}_0 = \tilde{p}$. The function $V_n(\tilde{p})$ gives the optimal value conditional on making n price adjustments before the next review, which will take place τ_n units of time after the current review. The first integral on the right hand side of $V_n(\tilde{p})$ corresponds to the losses incurred if there is no immediate price adjustment. Notice that if the firm chooses $\tau_0 = 0$, the time for which this loss is incurred is nil, and hence a price adjustment is conducted immediately. The next sum contains the losses corresponding to the n price adjustments. These n price adjustments are conducted using only the information of the initial price gap $\tilde{p}$. Notice that $V_0(\tilde{p})$ is the value of conducting a review and no price adjustment.[28] The value function V minimizes the cost by choosing whether to have an immediate price adjustment or not (choice of τ_0), and by minimizing on the number of price adjustments conducted between reviews (choice of n). If we restrict this problem to have $n = 1$ every time $\tau_0 = 0$, we will obtain exactly the same value functions of Section 5.

As a preliminary step we discuss the sense in which multiple price adjustments between price observations look like inflation indexation. Consider the case of equation (39) for $n \geq 3$. In particular, fixing the other choices for this problem, consider the first order condition for $\hat{p}_i$, $\hat{p}_{i+1}$ and τ_i for $0 < i < n$. The restriction that $0 < i < n$ means that the price adjustments $\hat{p}_i$ and $\hat{p}_{i+1}$ take place at times $\tau_{i-1} > 0$ and τ_i, and that there is still one more adjustment

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[28]In the previous section we refer to this function as $\bar{V}(\tilde{p})$.

before the next observation. After some algebra (letting $\rho \downarrow 0$ for simplicity), we get:

$$\hat{p}_i \;\; = \;\; \mu \, \frac{\tau_{i+1} + \tau_i}{2} \;\; , \;\; \hat{p}_{i+1} = \mu \, \frac{\tau_{i+2} + \tau_{i+1}}{2} \;\; \text{and} \;\; \tau_i = \frac{\tau_{i+1} + \tau_{i-1}}{2} \;\; ,$$

This shows the sense in which the optimal policy is an instance of indexation: prices increase with accumulated inflation, and the adjustments are equally spaced in time.

We discuss two extreme cases that imply infinitely many adjustments between observations. First, let the adjustment cost $\psi \downarrow 0$. In this case the firm will be adjusting infinitely often, so that the optimal number of adjustments between reviews will have $n_a \to \infty$ and the adjustment will be converging to $\hat{p}_i = \mu \tau_i$. In this case the value function converges to the one for the problem with observation cost only of Section 4.1 and no drift (i.e. $\mu = 0$). Thus, as the menu cost is small, and the drift is large, the optimal policy will involve more adjustments than reviews, contrary to what is found in the survey evidence discussed in Section 3. The second case is one where $\sigma \downarrow 0$. This case coincides with the model in Section 7.2, whose solution is characterized in Proposition 12, except that we now set $\theta = 0$. The intuition is clear: if there is no uncertainty, there is no need to pay the cost θ to observe the state.

The next proposition shows that if μ is sufficiently close to zero, there is at most one price adjustment between reviews, i.e. the firm will not find optimal to adjust without observing the price gap. Moreover, in such a case price adjustment will happen immediately upon a observation. The logic of this result is simplest when $\mu = 0$. In this case, it is immediate that the value function V is symmetric around zero. This symmetry implies that, if a price adjustment takes place, the optimal rest price is $\hat{p} = 0$.[29] Notice that in the absence of drift the *expected* price gap until the next observation remains zero. Thus it is not optimal to have any further price adjustment as long as $\psi > 0$, i.e. $V_n(\tilde{p}) > \min \{V_0(\tilde{p}), V_1(\tilde{p})\}$ for $n \geq 2$. By a continuity argument, along the lines of the Theorem of the maximum, the value function for $\mu > 0$, but small, is close to the one for zero inflation, and hence $\hat{p}$ is also close to zero. Again, having more than one adjustment between reviews will increase the cost by a discrete

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[29]This can be seen as a straightforward modification of the argument in the proof of Proposition 15.

amount ψ, unrelated to μ, and thus the argument extend to the case of small but positive inflation.[30] These arguments sketch the proof of the following proposition:

PROPOSITION 13.   Fix all the parameters, including $\psi > 0$ and $\sigma > 0$. Then there is a strictly positive inflation rate $\bar{\mu} > 0$ such that for all inflation rates $|\mu| < \bar{\mu}$ it is optimal to adjust prices at most once between successive reviews. Moreover, when it is optimal to adjust, the price change takes immediately after the price review.

In words, Proposition 13 shows that the decision rules computed in Section 7.2 are optimal for the problem defined in equations (39)-(40) for a small enough inflation rate. One interpretation of adjusting several times between observations is that the firm is indexing its price, or setting a plan, or a price path. Notice that in our set up, setting a plan or a path is costly, since it involves several price changes, each of them costing ψ. But also notice that menu cost are not the only cost, there are also observations cost θ. Thus we find this set up a useful benchmark to think about the benefit of indexation. In this sense, the previous proposition just says that if inflation is small relative to the menu cost, indexation is not optimal.

A simple calculation may help understand why relatively high inflation is required for more than one adjustment to take place between observations. Let V be the value function obtained in the problem where firms have to pay the observation cost at the time of an adjustment. Consider the benchmark case where the firm is paying the observation cost and changing the price for $\mu > 0$. We want to compare the value of the minimization of the Bellman equation when the firm is allowed to change the price several times until the new observation, paying the cost ψ each time. We will compute an upper bound on the savings of such policy. The upper bound is computed assuming that, until the next observation at τ, the firm adjusts the price continuously. We compare this with the cost that is incurred if

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[30] A more subtle point is to argue that for small but positive inflation optimal price adjustments occur immediately upon review. The argument against the optimality of a delayed adjustment is that during the first part of the period, of length τ_0 in the notation of this section, the firm is having a loss of the order of the price gap. Since this case is only relevant outside the range of inaction, the price gap is non trivial and thus the loss is large relative to μ.

the price gap is set to zero right after the observation and until the next observation at time τ. The cost-savings (i.e. the cost of the price not following trend inflation) is:

$$B \int_0^\tau e^{-\rho t}(t\mu)^2 dt \leq B \int_0^\tau (t\mu)^2 dt = \mu^2 \ B \ \frac{\tau^3}{3}$$

where the inequality comes from disregarding discounting (i.e. $\rho = 0$). For the baseline parameter values discussed in Section 6.2 we have that $\tau \approx .45$ (see Figure 6), which gives that the cost-savings from adjusting continuously will be about $\mu^2 \ B \ \frac{\tau^3}{3} \approx 0.006$ for a 10% inflation rate. In this example these savings are not even enough to pay for the cost of one extra adjustment $\psi = 0.015$. In fact, the cost-savings are most surely smaller, since $\hat{p}$ in the benchmark case will be set at a slightly positive value.

The top panel of Table 4 computes $\bar{\mu}$ for three set of parameter values, for which we also report statistics about the average frequencies of price changes and reviews, as well as the time to next review after an adjustment. The three cases are the baseline parameterization and two other ones with respectively a small and a large value of α. For each of these three cases we report the statistics for zero inflation, five percent, and $\bar{\mu}$, as defined in Proposition 13. The bottom panel displays the same statistics for two parameterizations that are chosen to match some average statistics corresponding to "Europe" and the "US" (frequency of review and adjustment from Table 1 and expected value of absolute price changes from Klenow and Kryvtsov (2008) and Alvarez et al. (2005)).

Table 4 shows that the statistics are not very sensitive to changes in the level of inflation, e.g. when comparing the results for zero inflation with the ones for 5% inflation. This low sensitivity is consistent with the zero derivative of τ, n_a and n_r at zero inflation shown in Proposition 11. The table also reports the level of inflation $\bar{\mu}$ that leaves the firm indifferent between following the policies outlined in Section 5 versus the one discussed in this section. In other words, $\bar{\mu}$ is the highest inflation rate for which indexation is not optimal: it takes a higher inflation rate for firms to be willing to adjust prices more than once between observations. For the baseline parameterization on the top panel the threshold inflation rate

Table 4: Statistics of the model as a function of μ

		$\mu = 0$			$\mu = 0.05$			$\mu = \bar{\mu}$			
		n_a	n_r	τ	n_a	n_r	τ	n_a	n_r	τ	$\bar{\mu}$
Baseline	$\alpha = 0.50$	1.59	2.44	0.42	1.61	2.45	0.42	1.74	2.62	0.39	0.16
Low α	$\alpha = 0.01$	1.60	1.70	0.59	-	-	-	1.60	1.70	0.59	0.00
High α	$\alpha = 2.50$	1.55	3.91	0.29	1.60	4.04	0.28	2.27	4.47	0.25	0.51
"Europe"	$\alpha = 2.33$	1.00	2.51	0.45	1.05	2.54	0.44	1.64	2.92	0.39	0.25
"US"	$\alpha = 0.40$	1.43	2.11	0.48	1.64	2.11	0.48	1.49	2.15	0.47	0.09

Parameters common to all models $B = 20, \rho = 0.02$; The parameter of the first three lines yield an average frequency of price adjustment of 1.6, and and average size of price change of 0.11. Baseline parameters: $\sigma = 0.15$, $\psi = 0.015$, $\theta = 0.03$; the 'Low α' case: $\sigma = 0.15$, $\psi = 0.001, \theta = 0.10$; the 'High α': $\sigma = 0.25$, $\psi = 0.063, \theta = 0.025$. The last two lines are parametrized to match approximately Euro-area and U.S. data about average frequencies of observation, adjustment and size of price changes; "Europe" uses $\sigma = 0.12$, $\psi = 0.035, \theta = 0.015$; "US" uses $\sigma = 0.14$, $\psi = 0.016$, $\theta = 0.04$.

$\bar{\mu}$ is 16%. In the bottom panel we also report the threshold inflation rate $\bar{\mu}$ for alternative parameters, which generates statistics similar to the ones reported for "Europe" or "US", and obtain values of $\bar{\mu}$ of 25 and 9% respectively.

8   Summary and concluding remarks

This paper developed a theoretical model to study price setting decisions in the presence of an observation cost θ and a menu cost ψ. Our model embeds the two polar cases in which one of the two cost is zero. A review of the results in each model is useful to understand what is learned by the joint analysis of the model with both costs in the case of zero inflation. In the observation cost model ($\theta > 0$, $\psi = 0$) the optimal policy is time-dependent: price reviews and price adjustments coincide and take place at equally spaced time intervals of length τ. Thus, the frequency of reviews and adjustment satisfies $n_a = n_r = 1/\tau$, which has elasticity 1/2 with respect to $\sigma^2 B/\theta$ (where σ^2 is the variance of idiosynchratic shocks and B relates to forgone profits from not charging the right price i.e. the curvature of the profit function. Price changes are normally distributed with variance $\sigma\sqrt{2\,\theta/B}$. In this case the mean (absolute value) of price changes, $\mathbb{E}[|\Delta p|]$, has an elasticity 1/4 with respect to $\sigma^2\theta/B$. The instantaneous hazard rate $\boldsymbol{h}(t)$ as a function of the time t between price changes, is

equal to zero until just before $t = 1/n_a$ at which time the review/adjustment takes place, and $\boldsymbol{h}(t) = \infty$. In the menu cost model ($\theta = 0$, $\psi > 0$) the optimal policy is state-dependent: price reviews occur continuously ($n_r = \infty$) while price adjustment happen only of the price gap crosses the threshold $\bar{p}$ of a symmetric range of inaction, which has elasticity $1/4$ with respect to $\sigma^2 \psi / B$. The average number of adjustments per unit of time n_a has elasticity $1/2$ with respect to $\sigma^2 B / \psi$, i.e. the same functional form than in the observation cost model. The instantaneous hazard rate $\boldsymbol{h}(t)$ starts at zero, is strictly increasing, and it asymptotes to a constant value. The time between price changes is random, but price changes take only two values: $\bar{p}$ and $-\bar{p}$, and hence $\mathbb{E}[|\Delta p|]$ has elasticity $1/4$ with respect to $\sigma^2 \psi / B$, as in the observation cost model.

The optimal decision rules for the model with two costs ($\theta > 0$, $\psi > 0$) combine several elements of these two models. As in the menu cost model, after observing the state the price is adjusted if the price gap $\tilde{p}$ is outside a range of inaction: $-\bar{p}, \bar{p}$. The optimal rule for price reviews is $\text{T}(\tilde{p}) = \tau - (\tilde{p}/\sigma)^2$ in the range of inaction, and otherwise it equals τ. Hence the optimal decision rules are characterized by two variables: τ and $\bar{p}$. We show that the presence of both costs makes τ smaller and $\bar{p}$ bigger compared to the models where one cost is zero. But, apart from the level, fixing a value of the ratio of the menu cost to the observation cost $\alpha \equiv \psi/\theta$, the variables τ and $\bar{p}$ have exactly the same functional than in the polar models: $1/\tau$ has elasticity $1/2$ with respect to $\sigma^2 B / \theta$, while $\bar{p}$ has elasticity $1/4$ with respect to $\sigma^2 \psi / B$. On the other hand, the model produces interactions between the review and adjustment decisions that are novel in the literature: the optimal times for review depend on the menu cost ψ, similarly the width of the inaction range depends on the observation cost θ. In particular, increasing α increases the threshold $\bar{p}$ and τ. The distribution of price changes in the model with two costs resembles a normal density with the mass chopped in the range of inaction. Such a bimodal distribution is interesting because a similar pattern for price changes seems typical of many micro datasets. Another original prediction of this model is that the ratio between the frequency of price adjustment an reviews is smaller than

one, as in the data of Section 3.

Table 5: Model based inference on α

	$\alpha = \mathcal{F}^{-1}(\frac{n_r}{n_a})$	$\alpha = \mathcal{S}^{-1}(\frac{\sigma(|\Delta p|)}{\mathbb{E}[|\Delta p|]})$	$\alpha = \mathcal{M}^{-1}(\frac{min\ (|\Delta p|)}{\mathbb{E}[|\Delta p|]})$	$\alpha = \mathcal{M}^{-1}(\frac{mode\ (|\Delta p|)}{\mathbb{E}[|\Delta p|]})$
U.S.	$\mathcal{F}^{-1}(1.4) = 0.38$	$\mathcal{S}^{-1}(0.72) = 0.01$	$\mathcal{M}^{-1}(0.07) = 0.01$	$\mathcal{M}^{-1}(0.33) = 0.26$
France	$\mathcal{F}^{-1}(4.0) = 2.91$	$\mathcal{S}^{-1}(1.21) \approx 0.00$	$\mathcal{M}^{-1}(0.13) = 0.03$	$\mathcal{M}^{-1}(0.25) = 0.12$

U.S. data is from Midrigan (2007) and Blinder et al. (1998); French data is from Loupias and Ricart (2004).

As argued above, in the models where each of the costs appears in isolation the resulting policy rules are either time-dependent or state-dependent. Understanding which of these rules underlies price-setting behavior is important because the rules have different implications for the economy's response to monetary shocks. A main result of our theoretical model is to establish an analytical mapping between several observable statistics and the fundamental model parameters. In particular we show that, provided the level of the normalized menu cost $\sigma^2 \frac{\psi}{B}$ and inflation μ are not too large, several observable statistics, such as the average frequency of price adjustment, or the average size of price changes, can be written as functions of the two normalized costs: $\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B}$. One application of this theoretical result identifies several observable statistics that can be used to measure the relative size of the menu cost vs the observation cost: $\alpha \equiv \psi/\theta$. For instance, the ratio between the frequency of price reviews and price adjustment n_r/n_a is a function only of α, and so are the following moments from the distribution of price changes: $std\ |\Delta p|\ /\ \mathbb{E}|\Delta p|$, $min\ |\Delta p|\ /\ \mathbb{E}|\Delta p|$, $mode\ |\Delta p|\ /\ \mathbb{E}|\Delta p|$. Some preliminary results of this application are reported in Table 5. For two countries for which we were able to gather some statistics that seem close to the ones of the model, we invert the model mapping and infer the magnitude of α. Two remarks are due. First, all cells but one suggest that α is smaller than 1, i.e. that the size of menu costs is smaller, typically much smaller, than the size of the observation cost. We find this result reasonable: the evidence in Zbaracki et al. (2004) supports the assumption that the information-costs associated with a price-review are much higher than the physical-costs of price changes. Second, the point "estimate" for α obtained for each country varies significantly, depending

on what statistics we use. This can be due to the measurement error problems that affect some of these statistics, some of which were discussed in Section 3, and also to other sources of heterogeneity that are overlooked by our analysis. For instance the theory assumes that level of the costs is identical for all firms, while the disaggregated data on price changes used in this table merge sectors with different levels (and ratios) of menu and observation costs, as suggested by the large variation in the level of the frequency of adjustment in Figure 1. We see this table only as an illustration of the potential application of our theory. We leave a more thorough empirical analysis for future work.

There are two interesting extensions/applications of the model we developed. The first one is to study the aggregation of sectors that are heterogenous on the parameters $(\sigma, \theta/B, \psi/B, \mu)$ in the steady state of an economy. This will provide a framework to evaluate wether using a representative sector is adequate to estimate the fundamental parameters of economies with sectors that differ dramatically in their observed behavior –see, for instance, the cross sector variations in Figure 1. The second one is the computation/characterization of an impulse response to an aggregate monetary shock of a simple economy with variable output. We think that evaluating an experiment analogous to Mankiw and Reis (2002) and Golosov and Lucas (2007) in the model with both costs will help to quantify the effects of monetary policy. We think these extensions/applications are interesting and hope that the analytical results provided by this paper will be helpful in carrying them through.

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A Proofs

This section provides details on several of the propositions in the paper. Those proofs that are straightforward extensions of existing results – or obtained through simple algebra– are given, for completeness, only in the Online appendix.

A.1   Proof of Proposition 3.

Proof. Under the conjecture that $\hat{p} = 0$ and that $V(\cdot)$ is symmetric around zero, and by the symmetry of the normal density, we can rewrite the Bellman equations (15) and (16) using only the positive range for $\tilde{p} \in [0 , \infty)$ as:[31]

$$\bar{V}(\tilde{p}) = \theta + \min_\tau B \int_0^\tau e^{-\rho t} \left[\tilde{p}^2 + \sigma^2 t \right] dt \ + $$

$$e^{-\rho\tau} \int_{-\tilde{p}/(\sigma\sqrt{\tau})}^\infty V\left(\tilde{p} + s\sigma\sqrt{\tau} \right) dN(s) \quad + e^{-\rho\tau} \int_{\tilde{p}/(\sigma\sqrt{\tau})}^\infty V\left(-\tilde{p} + s\sigma\sqrt{\tau} \right) dN(s) \quad \text{(A-1)}$$

$$\hat{V} = \psi + \theta + \min_\tau B \int_0^\tau e^{-\rho t} \left[\sigma^2 t \right] dt + e^{-\rho\tau} \, 2 \int_0^\infty V\left(s\sigma\sqrt{\tau} \right) dN(s) \quad \text{(A-2)}$$

We use the corollary of the contraction mapping theorem. First, notice that if the V in the right side of equation (18) is symmetric around $\tilde{p} = 0$, with a minimum at $\tilde{p} = 0$, then it is optimal to set $\hat{p} = 0$. Second, notice that if the function V in the right side of equation (17) is symmetric with a minimum at $\tilde{p} = 0$, then the value function in the left side of this equation is also symmetric, and hence V in equation (19) is symmetric. Third, using the symmetry, we show that if $V(\tilde{p})$ is weakly increasing, then the right side of equation (19) is weakly increasing. It suffices to show that $\bar{V}(\tilde{p})$ given by the right side of (A-1) is increasing in $\tilde{p}$ for a fixed arbitrary value of τ. We do this in two steps. The first step is to notice that the expression containing $\tilde{p}^2$ in (A-1) is obviously increasing in $\tilde{p}$. For the second step, without loss of generality, we assume that V is differentiable almost everywhere and compute the derivative with respect to $\tilde{p}$ of the remaining two terms involving the expectations of $V(\cdot)$ in (A-1). This derivative is:

$$\frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}} \left[V(0) \, dN\left(\frac{-\tilde{p}}{\sigma\sqrt{\tau}} \right) - V(0) \, dN\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}} \right) \right]$$

$$+ \ e^{-\rho\tau} \left[\int_{-\tilde{p}/(\sigma\sqrt{\tau})}^\infty V'(\tilde{p} + s(\sigma\sqrt{\tau})) dN(s) - \int_{\tilde{p}/(\sigma\sqrt{\tau})}^\infty V'(-\tilde{p} + s(\sigma\sqrt{\tau})) dN(s) \right]$$

$$= \ e^{-\rho\tau} \left[\int_0^\infty V'(z) \frac{1}{\sigma\sqrt{\tau}} \left(dN\left(\frac{z - \tilde{p}}{\sigma\sqrt{\tau}} \right) - dN\left(\frac{z + \tilde{p}}{\sigma\sqrt{\tau}} \right) \right) \right]$$

$$= \ e^{-\rho\tau} \left[\int_0^\infty V'(z) \frac{1}{\sigma\sqrt{\tau}\sqrt{2\pi}} \left(e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}} \right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}} \right)^2} \right) dz \right] \geq 0$$

where the term involving $V(0)$ is zero due symmetry of $dN(s)$, and where the inequality follows since $e^{-\frac{1}{2}\left(\frac{x-\tilde{p}}{\sigma\sqrt{\tau}} \right)^2} - e^{-\frac{1}{2}\left(\frac{x+\tilde{p}}{\sigma\sqrt{\tau}} \right)^2} > 0$ for $x > 0$ and $\tilde{p} > 0$. Notice that the inequality is

---

[31] The second line in equation (A-1) uses that $\int_{-\infty}^\infty V(p-s)dN(s) = \int_{-p}^\infty V(p+s)dN(s) + \int_p^\infty V(-p+s)dN(s)$.

strict if $\tilde{p} > 0$ and $V'(x) > 0$ in a segment of strictly positive length. If $\tilde{p} = 0$, then the slope is zero.

Finally, differentiating the value function twice, and evaluating at $\tilde{p} = 0$ we get

$$V''(0) = 2\ B\ \frac{1 - e^{-\rho\tau}}{\rho} + 2\ \frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}} \int_0^{\bar{p}} V'(z)\ z\ \frac{e^{-\frac{1}{2}\frac{z^2}{\sigma^2\tau}}}{\sigma\sqrt{\tau}\ 2\ \pi}\ dz > 0\ .$$

∎

A.2 Proof of Proposition 4.

Proof. The expression is based on a second order expansion of $\mathrm{T}(\cdot)$ around $\tilde{p} = 0$. The first order condition for τ can be written as:

$$F(\mathrm{T}; \tilde{p}) \equiv e^{-\rho\tau}\left(B\left(\tilde{p}^2 + \sigma^2\tau\right) - \rho \int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{\tau}\right) dN(s) + \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\right)\ .$$

At a minimum $F(\mathrm{T}(\tilde{p})\ ;\ \tilde{p}) = 0$ and $F_\tau(\mathrm{T}(\tilde{p})\ ;\ \tilde{p}) \geq 0$. We have $\left.\frac{\partial\mathrm{T}(\tilde{p})}{\partial\tilde{p}}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}}}{F_\tau} = 0$. That $\partial\mathrm{T}/\partial\tilde{p} = 0$ follows from the symmetry of $\mathrm{T}(\cdot)$ around $\tilde{p}$, which is verified directly by checking that $F_{\tilde{p}} = 0$ (see below). Totally differentiating $F_\tau\mathrm{T}' + F_{\tilde{p}}$ we obtain:

$$0 = F_{\tau\tau}(\mathrm{T}')^2 + F_{\tau\tilde{p}}\mathrm{T}' + F_\tau\mathrm{T}'' + F_{\tilde{p}\tau}\mathrm{T}' + F_{\tilde{p}\tilde{p}}\ ,$$

using that $\mathrm{T}' = 0$ we get the second derivative:

$$\left.\frac{\partial^2\mathrm{T}(\tilde{p})}{(\partial\tilde{p})^2}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}\tilde{p}}}{F_\tau} = 0$$

To compute this second derivative we first compute:

$$F_\tau(\mathrm{T}; \tilde{p}) = -\rho F(\tau; \tilde{p}) + e^{-\rho\tau}\left(B\sigma^2 - \rho \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right) \frac{-s\sigma}{2\ \sqrt{\tau}} dN(s)\right.$$

$$\left. - \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\tau^{-3/2}}{4}\ dN(s) + \int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s)\right)$$

Taking $\rho \downarrow 0$, using that at the optimum $F = 0$, that in the approximation $\bar{V}'(\tilde{p}) = V''(0)\ \tilde{p}$ and that $\bar{V}''(\tilde{p}) = V''(0)$ we obtain:

$$F_\tau(\mathrm{T}; 0) = B\ \sigma^2 - \int_{-\infty}^{\infty} V'\left(-s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\ \tau^{-3/2}}{4}\ dN(s) + \int_{-\infty}^{\infty} V''\left(-s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s)$$

$$= B\ \sigma^2 - \int_{-\infty}^{\infty} V''(0)\frac{s^2\sigma^2}{4\ \tau}\ dN(s) + \int_{-\infty}^{\infty} V''(0)\frac{s^2\sigma^2}{4\ \tau}\ dN(s)\ = B\ \sigma^2\ . \qquad\text{(A-3)}$$

We also have:

$$F_{\tilde{p}}(\text{T};\tilde{p}) = e^{-\rho\tau}\left(2B\tilde{p} - \rho\int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s) + \int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\frac{-s\,\sigma}{2\,\sqrt{\tau}}\,dN(s)\right)$$

$$F_{\tilde{p}\tilde{p}}(\tau;\tilde{p}) = e^{-\rho\tau}\left(2B - \rho\int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s) + \int_{-\infty}^{\infty} V'''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\frac{-s\,\sigma}{2\,\sqrt{\tau}}\,dN(s)\right)$$

Evaluating $F_{\tilde{p}\tilde{p}}$ at $\tilde{p} = 0$ for $\rho \downarrow 0$ and the approximation with $V'''(0) = 0$ gives:

$$F_{\tilde{p}\tilde{p}}(\text{T};0) = 2B .\tag{A-4}$$

Expanding $\text{T}(\cdot)$ around $\tilde{p} = 0$, using that its first derivative is zero, and that the second derivative is the negative of the ratio of the expressions in equation (A-3) and equation (A-4) we obtain:

$$\text{T}(\tilde{p}) = \text{T}(0) + \text{T}'(0)(\tilde{p}) + \frac{1}{2}\text{T}''(0)(\tilde{p})^2 = \tau - \frac{1}{2}\frac{F_{\tilde{p}\tilde{p}}}{F_\tau}\,(\tilde{p})^2 = \tau - \left(\frac{\tilde{p}}{\sigma}\right)^2 .$$

which appears in the proposition. ∎

A.3   Proof of Proposition 5.

Proof. We begin by establishing two lemmas that are useful to characterize the solution for $\bar{p}$ and τ. The proofs for this lemmas are given at the end of this Section.

LEMMA 1.   Let $\phi \equiv \frac{\bar{p}}{\sigma\sqrt{\tau}}$, then $V''(0)$, $\bar{p}$, and τ solve the recursive system:

$$\sigma^2\psi/B = f(\phi) , \quad \sigma^2\,\tau = h(\phi) , \quad\text{and}\quad V''(0) = 2\,\frac{\psi}{\bar{p}^2} ,$$

where $f(\cdot)$ and $h(\cdot)$ are the following known functions of ϕ and of two parameters $(\sigma^2\frac{\theta}{B}, \sigma^2\frac{\psi}{B})$:

$$\sigma^2\frac{\psi}{B} = f(\phi) \equiv \frac{\phi^2\,[h(\phi)]^2}{1 - 2\,\sqrt{h(\phi)}\,\int_0^\phi s^2\,dN(s)}\tag{A-5}$$

$$\tau = \frac{h(\phi)}{\sigma^2} \equiv \sqrt{2\,\frac{\theta + 2\,\psi(1 - N(\phi))}{\sigma^2 B}} .\tag{A-6}$$

Equation (A-5) and equation (A-6) can be thought of as the optimality conditions for $\bar{p}$ and τ. An immediate corollary of Lemma 1 is that the optimal values of ϕ and $\sigma^2\tau$ are only functions of two parameters $\sigma^2\theta/B$ and $\sigma^2\psi/B$. Notice also that the expression for τ in equation (A-6) is the same as the square root formula in Proposition 1 for the problem with observation cost only, except that the cost θ has been replaced by the "expected" cost $\theta + \psi\,2\,(1 - N(\phi))$.

Lemma 1 gives a recursive system of equations whose solution is the optimal value of $(\tau, \bar{p})$. The next Lemma gives a sufficient condition for the existence and uniqueness of the

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system, and provides some comparative statics. In fact, it turns out that the approximations used in this section can only be used globally – i.e. for all $\tilde{p}$ – if $\phi \in (0,1)$, as is clear from Proposition 4. Thus, the next Lemma restricts attention to parameter settings so that there is a unique solution in this range.

LEMMA 2.   Let $\phi \equiv \bar{p}/(\sigma\sqrt{\tau})$. Assume that $\psi/\theta \leq (1/2 - 2(1 - N(1)))^{-1} \approx 5.5$. Then there exists a unique value $\phi \in (0,1)$ that solves $\sigma^2 \frac{\psi}{B} = f(\phi)$ defined in equation (A-5). Also let τ be the solution of $\tau = h(\phi)/\sigma^2$ defined in equation (A-6). Then,

1. ϕ is decreasing in θ, and τ is increasing in θ,

2. ϕ is decreasing in $\frac{\sigma^2}{B}$, and $\sigma^2\tau$ is increasing in $\frac{\sigma^2}{B}$ with an elasticity $\geq 1/2$,

3. $\partial\phi/\partial\frac{\sigma^2}{B} = 0$ evaluated at $\frac{\sigma^2}{B} = 0$,

4. $\partial\phi/\partial\psi > 0$ if $\sigma^2\frac{\psi}{B}$ is small relative to θ.

The assumption in this Lemma is that the observation cost must be sufficiently large relative to the menu cost ($\psi/\theta < 5.5$) in order for the approximation to be globally valid –i.e. in order for $\phi < 1$– for arbitrary values of σ^2/B. The reason for this assumption is that the problem formulation presumes that after adjusting the price the firm waits for $\tau > 0$ periods before the next review since observation have a non-negligible cost relative to the adjustments. For instance when $\theta = 0$ the problem formulation is incorrect as the model becomes the menu cost of Section 4.2, where $\tau = 0$ and price reviews happen continuously.

We now use these lemmas to establish the result in Proposition 5. Write the solution as $\lambda(\psi\frac{\sigma^2}{B}, \alpha) \equiv \phi(\psi\frac{\sigma^2}{B}, \psi\frac{\sigma^2}{B}/\alpha)$. Then fixing α we can write $\lambda(\psi\frac{\sigma^2}{B}, \alpha) = \lambda(0,\alpha) + \lambda_1(0,\alpha)\psi\frac{\sigma^2}{B} + o(\psi\frac{\sigma^2}{B})$ where $\lambda(0,\alpha) = \varphi(\alpha)$ and where by Part 3 of Lemma 2: $\lambda_1(0,\alpha) = \partial\phi/\partial\frac{\sigma^2}{B} = 0$.
Some algebra, using the implicit definition of $\varphi(\alpha)$ by equation (21), gives

$$\frac{\partial \log \varphi}{\partial \log \alpha} = \frac{1 - \frac{4\alpha(1-N(\varphi))}{2+4\alpha(1-N(\varphi))}}{2 - \frac{4\alpha(1-N(\varphi))}{2+4\alpha(1-N(\varphi))}\frac{n(\varphi)\varphi}{(1-N(\varphi))}} \tag{A-7}$$

Since, $\varphi \to 0$ as $\alpha \to 0$, then $\frac{\partial \log \varphi}{\partial \log \alpha} \to 0$. For values of $\alpha > 0$, we have that

$$\frac{\partial \log \varphi}{\partial \log \alpha} < \frac{1}{2} \iff \frac{n(\varphi)\varphi}{(1-N(\varphi))} < 2, \tag{A-8}$$

which is a property of the normal distribution for values of $\varphi < 1$. Finally, inequality equation (23) follows from the definition of φ in equation (21) and because $2 \varphi/\alpha = [1 - \varphi^2 4(1 - N(\varphi))] < 1$. Inequality equation (24) follows because $\varphi < 1$ and equation (21).

A.3.1   Proof of Lemma 1.

First we notice that using the quadratic approximation into the definition of $\bar{p}$ given by $\bar{V}(\bar{p}) = \hat{V}$ implies

$$\psi = \frac{1}{2}V''(0)(\bar{p})^2 . \tag{A-9}$$

Second we derive equation (A-6) as the first order condition for τ. To this end, use the Bellman equation (18) for a fixed $\tau > 0$ evaluated at the optimal $\hat{p} = 0$, the symmetry of $V(\tilde{p})$, and the approximation

$$V(\tilde{p}) = \min\{\hat{V} , \ V(0) + \frac{1}{2}V''(0) \ (\tilde{p})^2\}$$

to write:

$$
\begin{aligned}
V(0) = \hat{V} - \psi &= \theta + B\sigma^2 \int_0^\tau e^{-\rho t} t \ dt + e^{-\rho\tau} \int_{-\infty}^\infty V\left(s\sigma\sqrt{\tau}\right) dN(s) \\
&= \theta + B\sigma^2 \int_0^\tau e^{-\rho t} t \ dt + e^{-\rho\tau} \ V(0) + \psi \ e^{-\rho\tau} \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right] \\
&\quad + e^{-\rho\tau} \ V''(0) \ \sigma^2 \tau \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2 \ dN(s)
\end{aligned}
$$

Thus

$$\rho V(0) = \frac{\theta + B\sigma^2 \int_0^\tau e^{-\rho t} t \ dt + \psi \ e^{-\rho\tau} \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right] + e^{-\rho\tau} \ V''(0) \ \sigma^2\tau \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2 \ dN(s)}{(1 - e^{-\rho\tau})/\rho}$$

letting $\rho \downarrow 0$ gives

$$\lim_{\rho\downarrow 0} \rho V(0) = \frac{\theta + \psi \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right]}{\tau} + B\sigma^2 \frac{\tau}{2} + V''(0) \ \sigma^2 \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2 \ dN(s)$$

Maximizing the right side of this expression gives

$$
\begin{aligned}
0 = &-\frac{\theta + \psi \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right]}{\tau^2} + \frac{B\sigma^2}{2} + \left(\psi \ 2 \ n\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right) \left(\frac{\bar{p}}{\sigma}\right)(\tau)^{-3/2}\right)\frac{1}{\tau} \\
&- V''(0) \ \sigma^2 \ \left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)^2 n\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\left(\frac{\bar{p}}{\sigma}\right)(\tau)^{-3/2}
\end{aligned}
$$

where we use $n(\cdot)$ for the density of the standard normal. Using that $V''(0) = 2 \ \psi \ / \ \bar{p}^2$, this expression simplifies to

$$0 = -\frac{\theta + \psi \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right]}{\tau^2} + \frac{B\sigma^2}{2}$$

rearranging and using the definition of ϕ gives $\sigma^2\tau = h(\phi)$ of equation (A-6).

Third, we obtain an expression for $V''(0)$. Differentiating the value function twice, and evaluating it at $\tilde{p} = 0$ we get

$$V''(0) = 2 \ B \ \frac{1 - e^{-\rho\tau}}{\rho} + 2 \ \frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}} \int_0^{\bar{p}} V'(z) \ z \ \frac{e^{-\frac{1}{2}\frac{z^2}{\sigma^2\tau}}}{\sigma\sqrt{\tau} \ 2 \ \pi} \ dz$$

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With a change in variable $s = z/(\sigma\sqrt{\tau})$ we have:

$$V''(0) = 2\ B\ \frac{1 - e^{-\rho\tau}}{\rho} + 2\ e^{-\rho\tau} \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} V'\left(\sigma\sqrt{\tau}\ s\right)\ s\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \pi}}\ ds\ .$$

Using the third order approximation $V(\tilde{p}) = V(0) + \frac{1}{2}V''(0)\ (\tilde{p})^2$ around $\tilde{p} = 0$ we obtain:

$$V''(0) = 2\ B\ \frac{1 - e^{-\rho\tau}}{\rho} + e^{-\rho\tau}\ V''(0)\ 2\ \sigma\sqrt{\tau} \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \pi}}\ ds$$

or collecting terms:

$$V''(0) = \frac{2\ B\ \frac{1-e^{-\rho\tau}}{\rho}}{1 - e^{-\rho\tau}\ 2\ \sigma\sqrt{\tau} \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \pi}}\ ds}$$

and letting $\rho \downarrow 0$, using the definition of ϕ and N for the CDF of a standard normal:

$$V''(0) = \frac{2\ B\ \tau}{1 - 2\ \sigma\sqrt{\tau} \int_0^\phi s^2\ dN(s)}\ . \tag{A-10}$$

Using equation (A-9) to replace $V''(0)$ into equation (A-10), using the definition of ϕ, and using $\sigma^2\tau = h(\phi)$ to replace τ and $\sqrt{\tau}$ we obtain equation (A-5).

A.3.2   Proof of Lemma 2.

Begin defining

$$\hat{f}(\phi) = \frac{\hat{h}(\phi)\ \phi^2}{1 - 2\sqrt{h(\phi)} \int_0^\phi s^2 dN(s)} \quad \text{where } \hat{h}(\phi) \equiv \frac{B}{\sigma^2}\ [h(\phi)]^2 = 2\left(\theta + 2\psi\ (1 - N(\phi))\right)\ , \text{ so that}$$

$$\hat{f}(\phi) = \frac{2\ \phi^2\ \left(\theta + 2\psi\ (1 - N(\phi))\right)}{1 - 2\left[2\sigma^2\frac{\theta}{B} + 4\sigma^2\frac{\psi}{B}\ (1 - N(\phi))\right]^{1/4} \int_0^\phi s^2 dN(s)}\ ,$$

and noting that $\psi = \hat{f}(\phi)$ is the same as the solution of equation (A-5) and equation (A-6).

First we turn to the existence and uniqueness of the solution. We show that it follows from an application of the intermediate function theorem, together with monotonicity. We show that if $\frac{\theta}{\psi} > 1/2 - 2(1 - N(1)) \approx 0.1827$ then: there is a value $0 < \phi' \leq 1$ so that: i) the function $\hat{f}$ is continuous and increasing in $\phi \in [0, \phi')$, ii) $\hat{f}(0) = 0$, iii) $\hat{f}(\phi') > \psi$, iv) $\hat{f}(\phi) < 0$ for $\phi \in (\phi', 1]$.

The value of ϕ' is given by the minimum of 1 or the solution to

$$1 = 2\left[2\sigma^2\frac{\theta}{B} + 4\sigma^2\frac{\psi}{B}\ (1 - N(\phi'))\right]^{1/4} \int_0^{\phi'} s^2 dN(s)\ , \tag{A-11}$$

so that if $\phi' < 1$, the function $\hat{f}$ has a discontinuity going from being positive and tending to $+\infty$ to being negative and tending to $-\infty$.

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The rest of the proof fills in the details: Step (1): Show that $\hat{h}(\phi)^2 \cdot (\phi)^2$ is increasing in ϕ if $\theta/\psi > 0.1667$ for $\phi < 1$. Step (2): Show that $\sqrt{h(\phi)} \cdot \int_0^\phi s^2 dN(s)$ is increasing in ϕ if $\phi < 1$. Step (3): Using (1) and (2) the function $\hat{f}$ is increasing in ϕ for values of ϕ that are smaller than 1, provided that its denominator is positive.

Step (1) follows from totally differentiating $h(\phi)^2 \cdot (\phi)^2$ with respect to ϕ. Collecting terms we obtain that the derivative is proportional to $\theta + 2 \cdot \psi(1 - N(\phi) - \phi \cdot N'(\phi))$. Since the function $1 - N(\phi) - \phi \cdot N'(\phi)$ is positive for small values of ϕ and negative for large values, we evaluate it at its upper bound for the relevant region, obtaining: $\theta + 2\psi(1 - N(1) - N'(1)) > 0$ or $\theta > \psi 2[N(1) + N'(1) - 1] \approx \psi \ 0.1667$. But notice that this condition is implied by the assumption: $\theta > \psi[1/2 - 2(1 - N(1))] \approx \psi \ 0.1827$.

Step (2) follows from totally differentiating $\sqrt{h(\phi)} \cdot \int_0^\phi s^2 dN(s)$ with respect to ϕ. Collecting terms we obtain that the derivative is proportional to $\phi^2 - \int_0^\phi s^2 dN(s) \ \psi/2(\theta + \psi \ (1 - N(\phi)))$. This expression is greater than $\phi^2 - \int_0^\phi s^2 dN(s) \ /2((1 - N(\phi)))$, which is obtained by setting θ to zero. This integral is positive for the values of ϕ in $(0, 1)$.

Now we turn to the comparative statics results. That ϕ is decreasing in θ follows since $\hat{f}$ is increasing in θ. That $\sigma^2 \tau$ is increasing it follows from the previous result and inspection of h. That ϕ is decreasing in σ^2/B follows since $\hat{f}$ is increasing in σ^2/B. That $\sigma^2 \tau$ is increasing follows from the previous result and inspection of h. That $\partial\phi/\partial\frac{\sigma^2}{B} = 0$ at $\sigma^2/B = 0$ follows from differentiating $\hat{f}$ with respect to σ^2/B and verifying that that derivative is zero when evaluated at at $\sigma^2/B = 0$. That ϕ is strictly increasing in ψ when σ^2/B is small relative to θ it follows from differentiating $\hat{f}/\psi$ with respect to ψ. That derivative is strictly negative and continuous on the parameters, when evaluated at $\theta > 0$ and $\sigma^2/B = 0$. ∎

A.4 Proof of Proposition 6.

Proof. We show that, given equation (20), the average frequency of price adjustment can be written as $n_a = 1/\mathcal{T}_a(0)$ where $\mathcal{T}_a(0) = \tau \ \mathcal{A}(\phi)$. Rewrite equation (27) in terms of $\mathrm{P}(p) = \frac{p}{\sigma\sqrt{\mathrm{T}(p)}}$,

$$\mathcal{T}_a(\phi^{-1}(\tilde{\phi})) \equiv \tilde{\mathcal{T}}(\tilde{\phi}) = \mathrm{T}(\tilde{\phi}) + \int_{-\bar{\phi}}^{\bar{\phi}} \tilde{\mathcal{T}}(v) \ n\left(v\frac{\sqrt{\mathrm{T}(v)}}{\sqrt{\mathrm{T}(\tilde{\phi})}} - \tilde{\phi}\right)\frac{\sqrt{\mathrm{T}(v)}}{\sqrt{\mathrm{T}(\tilde{\phi})}}dv,$$

$$= \frac{\tau}{1 + \tilde{\phi}^2} + \int_{-\bar{\phi}}^{\bar{\phi}} \tilde{\mathcal{T}}(v) \ n\left(v\frac{\sqrt{1 + \tilde{\phi}^2}}{\sqrt{1 + v^2}} - \tilde{\phi}\right)\frac{\sqrt{1 + \tilde{\phi}^2}}{\sqrt{1 + v^2}}dv,$$

where the first equality follows from strict monotonicity of $\mathrm{P}(p)$. Then we can write $\mathcal{T}_a(0) = \tilde{\mathcal{T}}(0) = \tau\tilde{\mathcal{A}}(0; \phi) \equiv \tau\mathcal{A}(\phi)$, where

$$\tilde{\mathcal{A}}(\tilde{\phi}; \phi) = \frac{1}{1 + \tilde{\phi}^2} + \int_{-\phi/\sqrt{1-\phi^2}}^{\phi/\sqrt{1-\phi^2}} \tilde{\mathcal{A}}(v; \bar{\phi}) \ n\left(v\frac{\sqrt{1 + \tilde{\phi}^2}}{\sqrt{1 + v^2}} - \tilde{\phi}\right)\frac{\sqrt{1 + \tilde{\phi}^2}}{\sqrt{1 + v^2}}dv.$$

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where we have used that $\bar{\phi} = \phi/\sqrt{1-\phi^2}$. ∎

A.5 Proof of Proposition 7.

Proof. Let $q(\phi)$ and $Q(\phi)$ be respectively the density and CDF of $\phi = \mathrm{P}\,(p) \equiv \frac{p}{\sigma\sqrt{\mathrm{T}(p)}}$.
Notice that $\mathrm{P}\,(p)$ is a monotonic transformation of p, $\frac{d\mathrm{P}}{dp} = \frac{1}{\sigma\sqrt{\mathrm{T}(p)}} - \mathrm{P}(p)\frac{\mathrm{T}'(p)}{2\mathrm{T}(p)} > 0$. Notice
that using $p(\phi)$ to denote the inverse function of P, we compute $\mathrm{T}(p) = \tau - \left(\frac{p}{\sigma}\right)^2 = \frac{\tau}{1+(\mathrm{P}(p))^2}$
which, abusing notation, defines the new function $\mathrm{T}(\phi) = \frac{\tau}{1+\phi^2}$.

The monotonicity of the transformation also gives that $Q(\mathrm{P}(p)) = G(p)$ at all p, implying

$$g(p)\frac{dp}{d\phi}d\phi = q(\mathrm{P}(p))d\phi \tag{A-12}$$

Using equation (A-12) and the change of variables from p to ϕ in equation (28) we write:

$$g(\tilde{p}) = \int_{-\bar{\phi}}^{\bar{\phi}} g(p(\phi))\, n\left(\frac{\tilde{p} - p(\phi)}{\sigma\sqrt{\mathrm{T}(p(\phi))}}\right) \frac{1}{\sigma\sqrt{\mathrm{T}(p(\phi))}}\frac{dp}{d\phi}\, d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} g(p(\phi))\frac{dp}{d\phi}d\phi\right] n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}}\right)\frac{1}{\sigma\sqrt{\tau}}$$

$$= \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)\, n\left(\frac{\tilde{p}}{\sigma\sqrt{\mathrm{T}(p(\phi))}} - \phi\right) \frac{1}{\sigma\sqrt{\mathrm{T}(p(\phi))}}d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi\right] n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}}\right)\frac{1}{\sigma\sqrt{\tau}}.$$

For clarity, rewrite the previous equation using the density of $\tilde{p}$ conditional on ϕ:

$$f(\tilde{p}|\phi) \equiv n\left(\frac{\tilde{p}}{\sigma\sqrt{\mathrm{T}(p(\phi))}} - \phi\right)\frac{1}{\sigma\sqrt{\mathrm{T}(p(\phi))}}$$

This gives:

$$g(\tilde{p}) = \int_{-\bar{\phi}}^{\bar{\phi}} f(\tilde{p}|\phi)\, q(\phi)d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi\right] f(\tilde{p}|0). \tag{A-13}$$

Now consider the following monotone transformation of the random variable $\tilde{\phi}$: $\Theta(\tilde{\phi}, \phi) \equiv \tilde{\phi}\frac{\sqrt{\mathrm{T}(\tilde{\phi})}}{\sqrt{\mathrm{T}(\phi)}}$, where the function $\mathrm{T}(\phi)$ is given in equation (A.5).[32] Using the definition of ϕ and
the law of motion for $\tilde{p}$ it follows that $\Theta(\tilde{\phi}, \phi) - \phi$ is a random variable with the standard
normal distribution: $n\left(\Theta(\tilde{\phi}, \phi) - \phi\right)$.

By doing the change in variables from $\tilde{p}$ to $\tilde{\phi}$ on the left-hand side of equation (A-13),

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[32]The monotonicity holds since θ is increasing in $\tilde{\phi}$.

and from $\tilde{p}$ to $\Theta(\tilde{\phi}, \phi)$ on the right-hand side of equation (A-13), we obtain

$$q(\tilde{\phi})\frac{d\tilde{\phi}}{d\tilde{p}} = \int_{-\bar{\phi}}^{\bar{\phi}} n\left(\Theta(\tilde{\phi}, \phi) - \phi\right) \frac{d\Theta(\tilde{\phi}, \phi)}{d\tilde{\phi}} \frac{d\tilde{\phi}}{d\tilde{p}} \, q(\phi) \, d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi\right] n\left(\tilde{\phi}\right) \frac{d\Theta(\tilde{\phi}, \tilde{\phi})}{d\tilde{\phi}} \frac{d\tilde{\phi}}{d\tilde{p}},$$

$$q(\tilde{\phi}) = \int_{-\bar{\phi}}^{\bar{\phi}} n\left(\Theta(\tilde{\phi}, \phi) - \phi\right) \frac{d\Theta(\tilde{\phi}, \phi)}{d\tilde{\phi}} \, q(\phi) \, d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi\right] n\left(\tilde{\phi}\right) \quad .$$

Replacing these expressions into equation (29) gives

$$\mathcal{T}_r(0) = 2 \int_0^{\bar{\phi}} \mathrm{T}(\tilde{\phi})q(\tilde{\phi})d\tilde{\phi} + \left[1 - 2\int_0^{\bar{\phi}} q(\tilde{\phi})d\tilde{\phi}\right]\tau \qquad (\text{A-14})$$

$$= \tau \left[1 + 2\int_0^{\phi/\sqrt{1-\phi^2}} \left(\frac{1}{1+\tilde{\phi}^2} - 1\right) q(\tilde{\phi})d\tilde{\phi}\right] \equiv \tau \, \mathcal{R}(\phi)$$

where we use the definitions of $q(\cdot)$ and $\mathrm{P}(p)$, and the quadratic approximation of $\mathrm{T}(p)$. ∎

A.6 Proof of Proposition 9.

Proof. Rewrite equation (33) as

$$w(\Delta p) = \frac{1}{\sigma\sqrt{\tau}} \frac{\int_{-\bar{\phi}}^{\bar{\phi}} \sqrt{1+\phi^2}\, n\left(\frac{\Delta p \sqrt{1+\phi^2}}{\sigma\sqrt{\tau}} - \phi\right) q(\phi)d\phi}{1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi} + \frac{1}{\sigma\sqrt{\tau}}\, n\left(\frac{\Delta p}{\sigma\sqrt{\tau}}\right),$$

where $\bar{\phi} \equiv \frac{\phi}{\sqrt{1-\phi^2}}$. Using the change of variable $\phi = \tilde{p}/(\sigma\sqrt{\mathrm{T}(\tilde{p})})$, using the approximation for the optimal policy: $\mathrm{T}(p) = \tau - (p/\sigma)^2 = \frac{\tau}{1+\phi^2}$, which implies that $p(\phi) = \sigma\sqrt{\tau}\,\phi/\sqrt{1+\phi^2}$, and hence this change of variables gives the density of the normalized prices: $q(\phi) = g(p(\phi))dp(\phi)/d\phi$. Thus letting the normalized price changes be: $x \equiv \frac{\Delta p}{\sigma\sqrt{\tau}}$, we define the density of the normalized price changes x as $v(\cdot)$, satisfying $v(x)/(\sigma\sqrt{\tau})$ and then the distribution of normalized price adjustment have density given by the change of variable formula: $v(x) = w\left(\frac{\Delta p}{\sigma\sqrt{\tau}}\right)\sigma\sqrt{\tau}$

$$v(x) = \frac{\int_{-\bar{\phi}}^{\bar{\phi}} \sqrt{1+\phi^2}\, n\left(x\sqrt{1+\phi^2} - \phi\right) q(\phi)d\phi}{1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi} + n(x) \quad \text{for } |x| > \bar{\phi} . \qquad (\text{A-15})$$

Thus we have that

$$e(x, \phi) \equiv \frac{\int_{-\bar{\phi}}^{\bar{\phi}} \sqrt{1+\phi^2}\, n\left(x\sqrt{1+\phi^2} - \phi\right) q(\phi)d\phi}{1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi)d\phi} \quad .$$

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Using the change of variable and the density q, the ratio of the moments in the proposition follows from direct computation. The limits of the range of equation (34) and equation (35) follow since $|\Delta p|$ takes only one value for the menu cost model, and it is the absolute value of a normal with zero mean for the observation cost model. ∎

A.7 Proof of Proposition 11.

Proof. To show that $\frac{\partial \hat{p}}{\partial \mu}|_{\mu=0} > 0$ we note that $\hat{p} = 0$ for zero inflation, and -by equation (AA-11) in Proposition 15- strictly positive for positive inflation and strictly negative for negative inflation. To show that $\frac{\partial \tau}{\partial \mu}|_{\mu=0} = \frac{\partial n_a}{\partial \mu}|_{\mu=0} = \frac{\partial n_r}{\partial \mu}|_{\mu=0} = 0$, we show that these three functions are symmetric around $\mu = 0$. To show this symmetry we use a guess and verify strategy on the value function and associated polices described by equation (16). In particular denote $V(\tilde{p}; \mu)$, $\tau(\mu), \text{т}(\tilde{p}; \mu), \hat{p}(\mu), \bar{p}(\mu), \underline{p}(\mu)$ where we include the parameter μ as an argument of these functions. Assume that $V(x + \hat{p}(\mu); \mu) = V(-x - \hat{p}(\mu); -\mu)$ for all $x \in \mathbb{R}$ and $\mu \geq 0$. Then, $\tau(\mu) = \tau(-\mu)$, $\text{т}(x + \hat{p}(\mu); \mu) = \text{т}(-x - \hat{p}(\mu); -\mu)$, $-\hat{p}(-\mu) = \hat{p}(\mu)$, $\hat{p}(-\mu) - \underline{p}(-\mu) = \bar{p}(\mu) - \hat{p}(\mu)$, and $\bar{p}(-\mu) - \hat{p}(-\mu) = \hat{p}(\mu) - \underline{p}(\mu)$ for all $x \in \mathbb{R}$ and $\mu \geq 0$. The symmetry follows because the instantaneous return function for both the $\hat{V}$ and $\bar{V}$ value functions, $B \int_0^T (\hat{p}(\mu) - \mu t)^2 dt$ and $B \int_0^T (x + \hat{p}(\mu) - \mu t)^2 dt$ respectively are symmetric, as well as from the symmetry of the pdf from the innovations $dN(s)$. A similar argument follows for the distribution of times until next adjustment, the invariant distribution of price gaps, and the times between reviews. ∎

A.8 Proof of Proposition 12.

Proof. We solve the menu cost model in the steady state, i.e. for $\rho = 0$. This problem corresponds to the the limit case when $\rho \downarrow 0$. When $\mu > 0$ and $\sigma = 0$ we can write the objective function as:

$$\min_{\hat{p},\tau} \frac{1}{\tau} \left[\int_0^\tau B (\mu t - \hat{p})^2 \, dt + (\psi + \theta) \right] = \min_{\hat{p},\tau} \frac{1}{\tau} \left[B \left(\frac{\mu^2 \tau^3}{3} - \frac{2\hat{p}\mu\tau^2}{2} + \hat{p}^2 \tau \right) + (\psi + \theta) \right]$$
$$= \min_\tau \frac{1}{\tau} \left[B \frac{\mu^2 \tau^3}{12} + (\psi + \theta) \right] = \min_\tau \left[B \frac{\mu^2 \tau^2}{12} + \frac{(\psi + \theta)}{\tau} \right]$$

where we used that $\frac{\tau\mu}{2} = \arg\min_{\hat{p}} -\frac{2\hat{p}\mu\tau^2}{2} + \hat{p}^2\tau$. The first order condition for τ gives

$$\frac{2 \, B \, \mu^2 \, \tau}{12} = \frac{(\psi + \theta)}{\tau^2} \, .$$

This gives the optimal rules in the proposition.

The value of n_a is obtained as $n_a = 1/\tau$. The values for $\text{т}(\tilde{p})$ are obtained by requiring that the review happens exactly at the time of an adjustment: $\text{т}(\tilde{p})\mu = \tilde{p} - \underline{p}$ in the range of inaction. The optimal policy has this form because, due to the deterministic evolution of $\tilde{p}(t)$, if $\theta > 0$ it is optimal to review only at the time of an adjustment.

To show the asymmetry of the range of inaction, i.e. that $\bar{p} - \hat{p} < \hat{p} - \underline{p}$, we use the following strategy. We solve the menu cost version for $\rho > 0$, $\sigma = 0$ and $\mu > 0$. Start

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by solving the value function in the range of inaction $\tilde{p} \in (\underline{p}, \bar{p})$, which satisfies the ODE: $\rho V(\tilde{p}) = B\tilde{p}^2 - \mu V'(\tilde{p})$. Then using the 2 value matching conditions at the boundary (for $\underline{p}$ and $\bar{p}$) and the optimality condition for $\hat{p}$, gives 3 equations for these 3 variables. Comparing $\bar{p} - \hat{p}$ with $\hat{p} - \underline{p}$ shows that $(\bar{p} - \hat{p})/(\hat{p} - \underline{p}) = 1/\sqrt{3}$. ∎

Online Appendices - NOT FOR PUBLICATION

Optimal price setting with observation and menu costs

Fernando Alvarez (U. Chicago)
Francesco Lippi (U. Sassari, EIEF)
Luigi Paciello (EIEF)

AA-1   Details on manuscript proofs

AA-1.1   Log approximation of monopolist profit function

Let $\Pi(P)$ be the profit function of the monopolist as a function of the price P. Let P^* be the optimal price, satisfying $\Pi_P(P^*) = 0$. Consider a second order approximation of the log of Π around $P = P^*$ obtaining:

$$\log \Pi(P) = \log \Pi(P^*) + \frac{1}{2} \frac{\partial^2 \log \Pi(P)}{\partial (\log P)^2}\bigg|_{P=P^*} (\log P - \log P^*)^2 + o\left((\log P - \log P^*)^2\right) .$$

A useful example for this approximation is the case with a constant elasticity of demand equal to $\eta > 1$ where: $q(P) = E\ P^{-\eta}$ where E is a demand shifter and where the monopolist faces a constant marginal cost C, where:

$$P^* = \frac{\eta}{\eta - 1}\ C \quad \text{and} \quad \frac{\partial^2 \log \Pi(P)}{\partial (\log P)^2}\bigg|_{P=P^*} = -\eta\ (\eta - 1)$$

So, letting $B = -\eta\ (\eta - 1)$, $p = \log P$ and $p^* = \log P^*$ we obtain the problem in the body of the paper.

AA-1.2   Non-linear Price setting model

In this section we describe a fully non-linear price setting model. Let the instantaneous profit of a monopolist be as in Appendix AA-1.1, where the cost C and the relative price is P. We solve this model numerically and compare its predictions with the simple tracking problem of the paper. [TO BE REPORTED IN FUTURE DRAFTS]. This model also helps to interpret ψ and θ in the tracking problem as cost in proportion to the period profit.

At the time of the observation we let the general price level be one. This price level increases at the rate μ per unit of time. The log of constant margnal cost in real terms evolve as a random walk with innovation variance σ^2 and with drift μ. The demand has constant elasticity η with respect to the price P relative to the general price level. Thus the real demand t periods afger observing a real cost C with a price P is given by $A\left(\hat{P}e^{-\mu t}\right)^{-\eta}$ where A is a constant the determined the level of demand. The nominal mark-up t period after is then $\left(\hat{P} - Ce^{\mu t + \mu t + s(t)\sigma\sqrt{t}}\right)$, where $s(t)$ is a standard normal random variable. Thus the real profitst are given by $A\left(\hat{P}e^{-\mu t}\right)^{-\eta}\left(\hat{P}e^{-\mu t} - Ce^{\mu t + s(t)\sigma\sqrt{t}}\right)$. The profit level, if prices are chosen to maximize the instantaneous profit when real cost are C and the general price level is one are given by $\Pi^*(C) \equiv A\ C^{1-\eta}\left(\eta/(\eta - 1)\right)^{-\eta}\left(1/(\eta - 1)\right)$. The corresponding

Bellman equation is:

$$
\begin{aligned}
v(P,C) &= \max\left\{\hat{v}(C)\,,\,\bar{v}(P,C)\right\}\\
\hat{v}(C) &= -(\theta+\psi)\Pi^*(C)+\\
&\quad \max_{\tau,\hat{P}}\int_0^\tau e^{-\rho t}\int_{-\infty}^\infty A\left(\hat{P}e^{-\mu t}\right)^{-\eta}\left(\hat{P}e^{-\mu t}-Ce^{\mu t+s(t)\sigma\sqrt{t}}\right)dN(s(t))\ dt+\\
&\quad +\ e^{-\rho\tau}\int_{-\infty}^\infty v\left(\hat{P}e^{-\mu\tau},Ce^{\mu\tau+s\sigma\sqrt{\tau}}\right)dN(s)\\
\bar{v}(P,C) &= -\theta\Pi^*(C)+\max_\tau\int_0^\tau e^{-\rho t}\int_{-\infty}^\infty A\left(Pe^{-\mu t}\right)^{-\eta}\left(Pe^{-\mu t}-Ce^{\mu t+s(t)\sigma\sqrt{t}}\right)dN(s(t))\ dt+\\
&\quad +\ e^{-\rho\tau}\int_{-\infty}^\infty v\left(Pe^{-\mu\tau},Ce^{\mu\tau+s\sigma\sqrt{\tau}}\right)dN(s)
\end{aligned}
$$

This Bellman equation is very similar to the one we solve numerically in Alvarez, Guiso, and Lippi (2009) for a saving and portfolio problem for households. As in the problem of that paper, it is easy to show that the value function is homogenous of degree $1-\eta$. In this case we can simplify the problem considering only one state, say P/C. We can develop the expectations and collect terms to obtain:

$$
\begin{aligned}
\hat{v}(C) &= -(\theta+\psi)\Pi^*(C)+\\
&\quad \max_{\tau,\hat{P}}\int_0^\tau e^{-\rho t+(1-\eta)(\mu+\sigma^2/2)t}A\ C^{1-\eta}\left(\frac{\hat{P}e^{-\mu t}}{Ce^{(\mu+\sigma^2/2)\ t}}\right)^{-\eta}\left(\frac{\hat{P}e^{-\mu t}}{Ce^{(\mu+\sigma^2/2)\ t}}-1\right)dt\\
&\quad +\ e^{-\rho\tau+(1-\eta)(\mu+\sigma^2/2)\tau}\int_{-\infty}^\infty e^{-(1-\eta)(\mu+\sigma^2/2)\tau}v\left(\hat{P}e^{-\mu\tau},Ce^{\mu\tau+s\sigma\sqrt{\tau}}\right)dN(s)
\end{aligned}
$$

and likewise for $\bar{v}$. Letting a modified discount factor to be $\tilde{\rho}\equiv\rho-(1-\eta)(\mu+\sigma^2/2)$ and using that v is homogeneous of degree $1-\eta$:

$$
\begin{aligned}
\frac{\hat{v}(1)}{\Pi^*(1)} &= -(\theta+\psi)+\max_{\tau,\hat{P}/C}\int_0^\tau e^{-\tilde{\rho}t}\frac{A}{\Pi^*(1)}\left(\frac{(\hat{P}/C)\ e^{-\mu t}}{e^{(\mu+\sigma^2/2)\ t}}\right)^{-\eta}\left(\frac{(\hat{P}/C)e^{-\mu t}}{e^{(\mu+\sigma^2/2)\ t}}-1\right)dt\\
&\quad +\ e^{-\tilde{\rho}\tau}\int_{-\infty}^\infty e^{(1-\eta)(s\sigma\sqrt{\tau}-(\sigma^2/2)\tau)}v\left(\frac{(\hat{P}/C)e^{-\mu\tau}}{e^{\mu\tau+s\sigma\sqrt{\tau}}},1\right)\frac{1}{\Pi^*(1)}dN(s)\\
\frac{\bar{v}(P/C,\ 1)}{\Pi^*(1)} &= -(\theta+\psi)+\max_\tau\int_0^\tau e^{-\tilde{\rho}t}\frac{A}{\Pi^*(1)}\left(\frac{(P/C)e^{-\mu t}}{e^{(\mu+\sigma^2/2)\ t}}\right)^{-\eta}\left(\frac{(P/C)e^{-\mu t}}{e^{(\mu+\sigma^2/2)\ t}}-1\right)dt\\
&\quad +\ e^{-\tilde{\rho}\tau}\int_{-\infty}^\infty e^{(1-\eta)(s\sigma\sqrt{\tau}-(\sigma^2/2)\tau)}v\left(\frac{(P/C)e^{-\mu\tau}}{e^{\mu\tau+s\sigma\sqrt{\tau}}},1\right)\frac{1}{\Pi^*(1)}dN(s)\\
v(P/C,1) &= \max\left\{\hat{v}(1)\,,\bar{v}(P/C,1)\right\}.
\end{aligned}
$$

AA-1.3   Hazard rate of menu cost model

In this appendix we described the details for the characterization of the hazard rate of price adjustments of the menu cost model of Section 4.2. Section 2.8.C formula (8.24) of Karatzas

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and Shreve (1991) displays the density of the distribution for the first time that a brownian motion hit either of two barriers, starting from an arbitrary point inside the barriers. In our case, the initial value is the price gap after adjustment, namely zero, and the barriers are symmetric, given by $-\bar{p}$ and $\bar{p}$. We found more useful for the characterization of the hazard rate to use a transformation of this density, obtained in Kolkiewicz (2002), section 3.3, as the sum of expressions (15) and (16). In our case we set the initial condition $x_0 = 0$ and the barriers $a < x_0 < b$ are thus given by $a = -\bar{p}$ and $b = \bar{p}$, thus obtaining the density $\boldsymbol{f}(t)$:

$$\boldsymbol{f}(t) = \frac{\pi}{2\,(\bar{p}/\sigma)^2} \sum_{j=0}^{\infty} (2j+1)(-1)^j \, \exp\left(-\frac{(2j+1)^2\pi^2}{8\,(\bar{p}/\sigma)^2}\,t\right). \qquad \text{(AA-1)}$$

The hazard rate $\boldsymbol{h}(t)$ is then defined as:

$$\boldsymbol{h}(t) = \frac{\boldsymbol{f}(t)}{\int_t^{\infty} \boldsymbol{f}(s)ds}. \qquad \text{(AA-2)}$$

Notice that since equation (AA-1) is a sum of exponentials evaluated at the product of $-t$ times a positive quantity, each of them larger. Thus, for large values of t the first term in the sum dominates, and hence the expression for $\boldsymbol{f}(t)$ becomes

$$\boldsymbol{f}(t) \approx \frac{\pi}{2\,(\bar{p}/\sigma)^2} \exp\left(-\frac{\pi^2}{8\,(\bar{p}/\sigma)^2}\,t\right) \quad \text{for large } t. \qquad \text{(AA-3)}$$

and hence $\lim_{t\to\infty} \boldsymbol{h}(t) = \frac{\pi^2}{8\,(\bar{p}/\sigma)^2}$. Indeed, the shape of this hazard rate is independent of $\bar{p}/\sigma$, this value only scales it up and down. Moreover the asymptote is approximately attained well before the expected value of the time.

AA-1.4 Proof of Proposition 1.

Proof. We let $x^* = \hat{p}/\mu$. Note that as $\rho \to 0$ we have:

$$\frac{\rho e^{-\rho t}}{1 - e^{-\rho \tau}} \;\to\; \frac{1}{\tau} \quad \text{so} \quad x^* \to \frac{1}{\tau}\int_0^{\tau} t\,dt = \frac{\tau}{2}, \quad \text{and} \quad \int_0^{\tau} e^{-\rho t}\,t\,dt \to \tau\int_0^{\tau} t\,\frac{1}{\tau}dt = \frac{\tau^2}{2}.$$

$$v(\tau) \;\to\; \int_0^{\tau}\left(\frac{\tau}{2}-t\right)^2 dt = \tau\int_0^{\tau}\left(\frac{\tau}{2}-t\right)^2\frac{1}{\tau}dt = \tau\frac{\tau^2}{12} = \frac{\tau^3}{12}.$$

Thus

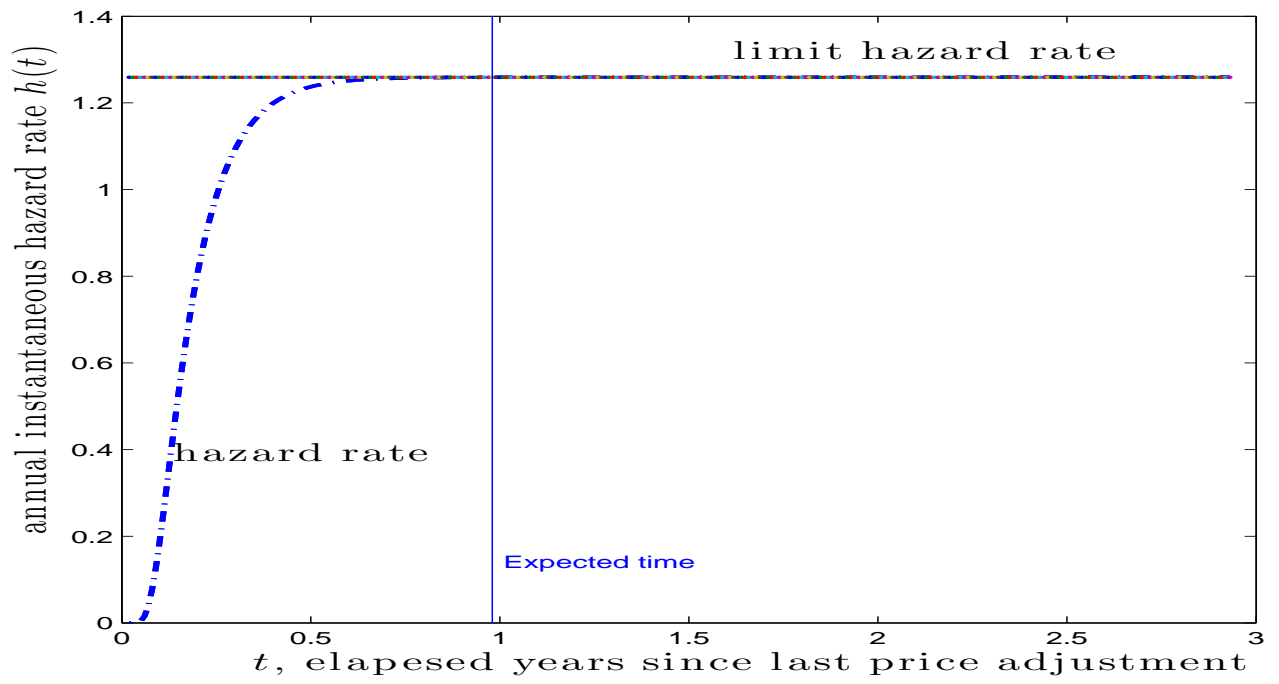$$\lim_{\rho\to 0}\rho V(\tau) \;=\; \lim_{\rho\to 0}\left(\frac{\mu^2}{\sigma^2}\frac{\rho\tau^3}{12\,(1-e^{-\rho\tau})} + \frac{\rho\,\tau^2}{2\,(1-e^{-\rho\tau})} + \frac{\tilde{\theta}\rho}{(1-e^{-\rho\tau})}\right) = \frac{\mu^2}{\sigma^2}\frac{\tau^2}{12} + \frac{\tau}{2} + \frac{\tilde{\theta}}{\tau}$$

The f.o.c. for τ is:

$$\frac{\tilde{\theta}}{(\tau)^2} = \frac{\mu^2}{\sigma^2}\frac{\tau}{6} + \frac{1}{2} \quad \text{or} \quad \tilde{\theta} = \frac{\mu^2}{\sigma^2}\frac{(\tau)^3}{6} + \frac{(\tau)^2}{2}$$

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Figure AA-7: Hazard Rate of Menu Cost Model

Note: $B = 25, \sigma = 0.1$ and $\psi = 0.04$.

From here we see that the optimal inaction interval τ is a function of 2 arguments, it is increasing in the normalized cost $\tilde{\theta}$, and decreasing in the normalized drift $(\mu/\sigma)^2$. Keeping the parameters B, θ and μ constant we can write:

$$\frac{\theta}{B} = \mu^2 \frac{(\tau)^3}{6} + \sigma^2 \frac{(\tau)^2}{2}$$

which implies that τ is decreasing in σ.

Note that for $\mu = 0$ we obtain a square root formula on the cost θ and with elasticity minus one on σ:

$$\tau = \sqrt{2\,\tilde{\theta}} = \sqrt{2\,\frac{\theta}{B\,\sigma^2}} = \sqrt{\frac{2}{B}}\,\theta^{1/2}\,\frac{1}{\sigma}\ .$$

The total differential of the foc for τ gives:

$$\tau(\tilde{\theta})\left(\frac{\mu^2}{\sigma^2}\frac{\tau(\tilde{\theta})}{2} + 1\right)\left(\frac{\partial \tau(\tilde{\theta})}{\partial \tilde{\theta}}\right) = 1$$

since $\lim_{\tilde{\theta}\to 0}\tau(\tilde{\theta}) = 0$, then one obtains the same expression than in the case of $\mu = 0$, and thus the elasticity is $1/2$, or: $\lim_{\tilde{\theta}\to 0}\frac{\tilde{\theta}}{\tau(\tilde{\theta})}\frac{\partial \tau(\tilde{\theta})}{\partial \tilde{\theta}} = \frac{1}{2}$. To see the result for $\sigma = 0$, let us write:

$$\tilde{\theta}\,\sigma^2 \equiv \frac{\theta}{B} = \mu^2 \frac{(\tau)^3}{6} + \sigma\frac{(\tau)^2}{2}\ ,$$

then we let $\sigma^2 \to 0$ to get $\frac{\theta}{B} = \mu^2 \frac{\tau^3}{6}$ which implies a cubic root formula on the cost θ and with elasticity $-2/3$ on μ:

$$\tau = \left(\frac{6\,\theta}{B\,\mu^2}\right)^{1/3} = \left(\frac{6}{B}\right)^{1/3}\theta^{1/3}\,\mu^{-2/3}.$$

That $\partial\tau/\partial\mu = 0$ evaluated at $\mu = 0$ follows from totally differentiating $\frac{\theta}{B} = \mu^2 \frac{(\tau)^3}{6} + \sigma^2\frac{(\tau)^2}{2}$.

Now consider the case when $\rho > 0$ and $\mu = 0$ then $\rho V(\tau)$ equals

$$\rho V(\tau) = \frac{\rho\tilde{\theta}}{1 - e^{-\rho\tau}} - \frac{\tau e^{-\rho\tau}}{1 - e^{-\rho\tau}} + \frac{1}{\rho}\ .$$

The first order condition with respect to τ implies that the optimal choice satisfies:

$$\tilde{\theta} = \frac{\rho\tau - 1 + e^{-\rho\tau}}{\rho^2}. \tag{AA-4}$$

A third order expansion of the right hand side of equation (AA-4), gives:

$$\tilde{\theta} = \frac{1}{2}\tau^2 - \frac{1}{6}\rho\tau^3 + o(\rho^2\tau^3).$$

The expression shows that if $\rho = 0$ we obtain a square root formula: $\tau = \sqrt{2\,\tilde{\theta}}$, and that the optimal τ is increasing in ρ provided ρ or $\tilde{\theta}$ are small enough. $\blacksquare$

AA-1.5 Proof of Proposition 2.

Proof. Differentiating the Bellman equation and evaluating it at zero we obtain:

$$\rho V''(0) = 2B + \sigma^2/2 V''''(0) \tag{AA-5}$$

and evaluating this expression for $\rho = 0$ we have

$$V''''(0) = -\frac{2B}{\sigma^2/2} \; . \tag{AA-6}$$

Differentiating the quartic approximation equation (9), evaluating at $\bar{p}$ and imposing the smooth pasting equation (8) we obtain:

$$0 = V''(0)\bar{p} + \frac{1}{3\,2}V''''(0)\,\bar{p}^3 \; . \tag{AA-7}$$

Replacing into this equation the expression for $V''''(0)$ in equation (AA-6) and solving for $V''(0)$ we obtain

$$V''(0) = -\frac{1}{3\,2}V''''(0)\bar{p}^2 = \frac{1}{3\,2}\frac{2B}{\sigma^2/2}\bar{p}^2 \; . \tag{AA-8}$$

Using the quartic approximation into the (levels) of equation (8) we obtain:

$$\psi = \frac{1}{2}V''(0)\bar{p}^2 + \frac{1}{4!}V''''(0)\bar{p}^4 \; , \tag{AA-9}$$

replacing into this equation $V''(0)$ from equation (AA-8) and $V''''(0)$ from equation (AA-6) we obtain

$$\psi = \left(\frac{1}{2\,3\,2} - \frac{1}{4\,3\,2}\right)\frac{2B}{\sigma^2/2}\,\bar{p}^4 = \frac{B}{6\,\sigma^2}\,\bar{p}^4 \; , \tag{AA-10}$$

thus solving for $\bar{p}$ we obtain the desired expression. $\blacksquare$

AA-1.6 Proof of Proposition 14.

TO BE REVISED

The next proposition states that the operator defined by the right side of equation (14), equation (15) and equation (16) is a contraction in the space of bounded and continuous functions. We will use this for further characterizations. The argument is intuitive but non-standard, since the length of the time period is a decision variable, potentially making the problem a continuous time one. Since we assume that $\theta > 0$ revisions should be optimally spread out. We then have:

PROPOSITION 14.   The value function is uniformly bounded and continuos on $\tilde{p}$. If $\theta > 0$ then the optimal time between observations is uniformly bounded by $\underline{\tau} > 0$, and thus the

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operator defined by the right side of equation (14), equation (15) and equation (16) is a contraction of modulus $\exp(-\rho\underline{\tau})$.

Proof. We start with a simple preliminary result to set up the analysis showing that the fixed point of V coincides with the solution of the sequence problem. Denote by $V_{\mathrm{info},\theta+\psi}$ the solution of the problem with observation cost only of Section 4.1, but where the observation cost has been set to $\theta + \psi$, and denote the optimal value of the time between observations and price changes as τ_i. We interpret this as the value of following the feasible policy of observing and adjusting $(\chi_{T_i} = 1$ all $T_i)$ every τ periods, and hence $V(\tilde{p}) \leq V_{\mathrm{info},\theta+\psi}$. Also denote by $V_{\mathrm{menu},\psi}(\cdot)$ the solution of the problem with menu cost ψ and no observation cost analyzed in Section 4.2. Since the observation cost is set to zero, this provides a lower bound for the value function: $V(\tilde{p}) \geq V_{\mathrm{menu},\psi}(\tilde{p})$. Letting $\mathbf{T}$ the operator defined by the right side of equation (14), equation (15) and equation (16), and by $\mathbf{T}^n V_0$ the outcome of n successive applications of $\mathbf{T}$ to an initial function V_0, we have that for each $\tilde{p}$:

$$V_{\mathrm{menu},\psi}(\tilde{p}) \leq \mathbf{T}^n V_{\mathrm{menu},\psi}(\tilde{p}) \leq \mathbf{T}^n V_{\mathrm{info},\theta+\psi}(\tilde{p}) \leq V_{\mathrm{info},\theta+\psi}(\tilde{p}) \ ,$$

and the two sequence of functions converge pointwise. Since they converge to a finite value, their limit must be the same, by an adaptation of Theorem 4.14 in Stokey and Lucas (1989). Thus:

$$V(\tilde{p}) = \lim_{n\to\infty} \mathbf{T}^n V_{\mathrm{info},\theta+\psi}(\tilde{p}) = \lim_{n\to\infty} \mathbf{T}^n V_{\mathrm{menu},\psi}(\tilde{p}) \ .$$

pointwise. Furthermore, since $V_{\mathrm{info},\theta+\psi}(\tilde{p})$ is a constant function, i.e. independent of $\tilde{p}$, and since $V_{\mathrm{menu},\psi}(\tilde{p}) > 0$ for all $\tilde{p}$, then we have that the value function $V(\tilde{p})$ is uniformly bounded.

We sketch the argument to show that the value function V is continuous on $\tilde{p}$. Suppose not, that there is jump down at $\bar{p}$ so that $V(\bar{p}) > \lim_{p\downarrow\bar{p}} V(\bar{p})$. Then, fixed the policies that correspond to a value of $p > \bar{p}$ in terms of stopping times and prices $\hat{p}(\cdot)$ as defined in the sequence formulation of equation (12). Thus, the agent will observe, starting with $\bar{p}$, after the same cumulated value of the innovations on p^*, and it will adjust to the value Let $\epsilon = p - \bar{p}$ the difference between these two prices at time zero, and denote by $\{p(t)\}, \{\bar{p}(t)\}$ the stochastic process for the price that follow from the two initial prices. Notice that $\bar{p}(t) - p(t) = \epsilon$ for all $t > 0$. By following that policy when the initial price is $\bar{p}$, the expected discounted value of fixed cost paid are exactly the same for the two initial prices. Thus, the difference of the value function at $p(0)$ and at $\bar{p}(0)$ is given by the

$$\frac{\epsilon^2}{\rho} + \epsilon\, \mathbb{E}_0\Big[\int_0^\infty e^{-\rho t}\, (p(T_i) - p^*(t))\, dt \Big] \ .$$

Since this difference is clearly continuous on ϵ and goes to zero as $\epsilon \downarrow 0$, then the value function is continuous.

Now we use the fact that V is bounded above uniformly and continuous to show that for $\theta > 0$, then the optimal policy for $\mathrm{T}(\tilde{p})$, is uniformly bounded away from zero. Here is a sketch of the argument. Suppose, as a way of contradiction, that for any $\epsilon > 0$, we can find a $\tilde{p}$ for which $\mathrm{T}(\tilde{p}_\epsilon) < \epsilon$. We will argue that for small enough ϵ it is cheaper to double $\mathrm{T}(\tilde{p}_\epsilon)$. The main idea is that this decreases the fixed by $e^{-\rho\epsilon}\theta$, and increases the cost due to different information gathering in a quantity that is a continuous function of ϵ. The reason why the

second part is continuous as function of ϵ is that the distribution of value of $p^*(t + \epsilon)$ and $p^*(t + 2\epsilon)$ have most of the mass concentrated in a neighborhood of $p^*(t)$. The effect due to the increase in cost due to evaluation of the period return objective function is small, since it is the expected value of the integral of a bounded function between 0 and ϵ or between 0 and 2ϵ. Thus, for small ϵ this difference is small. The effect on the value function of the mass $p^*(t)$ is small, because the value function is uniformly bounded and this probability is small, i.e. goest to zero as $\epsilon \downarrow 0$. For the mass that is in the neighborhood of $p^*(t)$, the effect in the value function is small because the value function is continuous.

Using that $\inf \mathrm{T}(\tilde{p}) \equiv \underline{\tau} > 0$, by Blackwell's sufficient conditions, we obtain that $\mathbf{T}$ is a contraction of modulus $\exp(-\underline{\tau}\rho)$ in the space of continuous and bounded functions. ∎

AA-1.7 Proof of Proposition 15.

For the next proposition set, without loss of generality, the current time at $t = 0$, and assume that the agent will adjust the price at this time. Also, without loss of generality we set $p^* = 0$. Let $\bar{T}$ be a stopping time indicating the time of the next price adjustment. Notice that $\bar{T}$ is the sum of the time elapsed between consecutive reviews with no adjustment plus the time until review and adjustment.

PROPOSITION 15.   Let $\hat{p}$ be optimal price set after an adjustment exceeds the value of the target p^*, and let $\bar{T}$ the time elapsed between price adjustments. As ρ goes to zero, the optimal value of the initial price gap satisfies:

$$\hat{p} = \frac{\mu \, \mathbb{E}\left(\frac{\bar{T}^2}{2}\right) + \sigma \, \mathbb{E}\left[\int_0^{\bar{T}} W(t) \, dt\right]}{\mathbb{E}\left(\bar{T}\right)}, \tag{AA-11}$$

where $W(t)$ is a standard Brownian motion, with $W(0) = 0$. Thus, fixing $\theta > 0$, for sufficiently small ψ, the value of $\hat{p}$ is strictly positive.

The terms in equation (AA-11) are intuitive if you first consider the case where the time to the next adjustment is deterministic, so that $\bar{T}$ has a degenerate distribution. In this case we obtain: $\hat{p} = \mu \, \bar{T}/2$, so that the initial gap is equal to the value that the target will have exactly at half of the time until the next adjustment, so the first half of the time deviations are positive, and the second negative. To understand the expression for $\hat{p}$ in the case in which $\bar{T}$ is random *and unrelated* with $\{W(t)\}$, consider the following simple example. Let $\bar{T} = \hat{T}$ with probability q and otherwise $\bar{T} = \epsilon$ where we let ϵ to go to zero. In this case there is an immediate adjustment with probability $1 - q$ and otherwise the next adjustment happen in exactly $\hat{T}$ periods. Notice that the value of $\hat{p}$ is irrelevant in the case that the adjustment is immediate, so its determination should be governed by $\hat{T}$. But in this case the same logic than the one in the purely deterministic case applies, and hence we should set $\hat{p} = \hat{T}/2$. Finally, notice that the general expression in equation (AA-11) gives the correct answer in this particular case, since $\mathbb{E}[\bar{T}] = q\hat{T}$ and $\mathbb{E}[\bar{T}^2] = q\hat{T}^2$, and hence we obtain again that $\hat{p} = \hat{T}/2$. In our model the stopping time for adjustment $\bar{T}$ is a function of the path of $\{W(t)\}$. Thus, the condition that ψ is small, reduces the depedence of $\bar{T}$ on its path, ensuring that the first term in equation (AA-11) dominates the expression of $\hat{p}$. Notice that

if $\psi = 0$, the value of $\bar{T}$ is indeed deterministic, and hence the second term is exactly zero.

Proof. Consider

$$G(\hat{p} \; ; \bar{T}, \mu) \equiv \mathbb{E} \left(\int_0^{\bar{T}} e^{-\rho t} \left[(\mu t - \hat{p}) + \sigma W(t) \right]^2 dt \right) \tag{AA-12}$$

where $W(t)$ is a standard brownian motion. If $\hat{p}$ and the $\bar{T}$ are optimal, then $G(\cdot; \bar{T}, \mu)$ should be maximized at $\hat{p}$. We will show that, provided that the stopping time is positive and finite,

$$\frac{\partial G(0 \; ; \bar{T}, \mu)}{\partial \hat{p}} < 0 \text{ if } \mu > 0 \; . \tag{AA-13}$$

We write equation (AA-12) as

$$G(\hat{p} \; ; \bar{T}, \mu) \equiv \mathbb{E} \left(\int_0^{\bar{T}} e^{-\rho t} \left[(\mu t - \hat{p})^2 + 2\sigma W(t) (\mu t - \hat{p}) + \sigma^2 W(t)^2 \right] dt \right)$$

and thus

$$
\begin{aligned}
\frac{\partial G(\hat{p} \; ; \bar{T}, \mu)}{\partial \hat{p}} &= \mathbb{E} \left(\int_0^{\bar{T}} e^{-\rho t} \left[-2 (\mu t - \hat{p}) - 2\sigma W(t) \right] dt \right) \\
&= -\mathbb{E} \left(\int_0^{\bar{T}} e^{-\rho t} 2 (\mu t - \hat{p}) \, dt \right) - \sigma \, 2\mathbb{E} \left[\int_0^{\bar{T}} e^{-\rho t} W(t) dt \right]
\end{aligned}
$$

Equating this first order condition to zero, rearranging, and taking ρ to zero:

$$\hat{p} = \frac{\mu \, \mathbb{E} \left(\frac{\bar{T}^2}{2} \right) + \sigma \, \mathbb{E} \left[\int_0^{\bar{T}} W(t) dt \right]}{\mathbb{E} \left(\bar{T} \right)}$$

AA-2   More detailed information on country surveys

Figure AA-8 plots the CDF for the frequencies of review and adjustment. The source of the data are Stahl (2009) for Germany, Loupias and Ricart (2004) for France, Fabiani, Gattulli, and Sabbatini (2004) for Italy and Greenslade and Parker (2008) for UK. We tossed those observation that were either missing or reporting an irregular frequency of review or adjustment.

AA-2.0.1   Computation of the means

Austria: The source of the data is Table 2 and Table 3 in Kwapil, Baumgartner, and Scharler (2005). In order to compute the means, we assigned a yearly frequency of 0.5 to frequency smaller than a year, and took the midpoint of all intervals; we assigned a value of 75 to the

Figure AA-8: Cumulative distribution of frequency of price adjustment and review
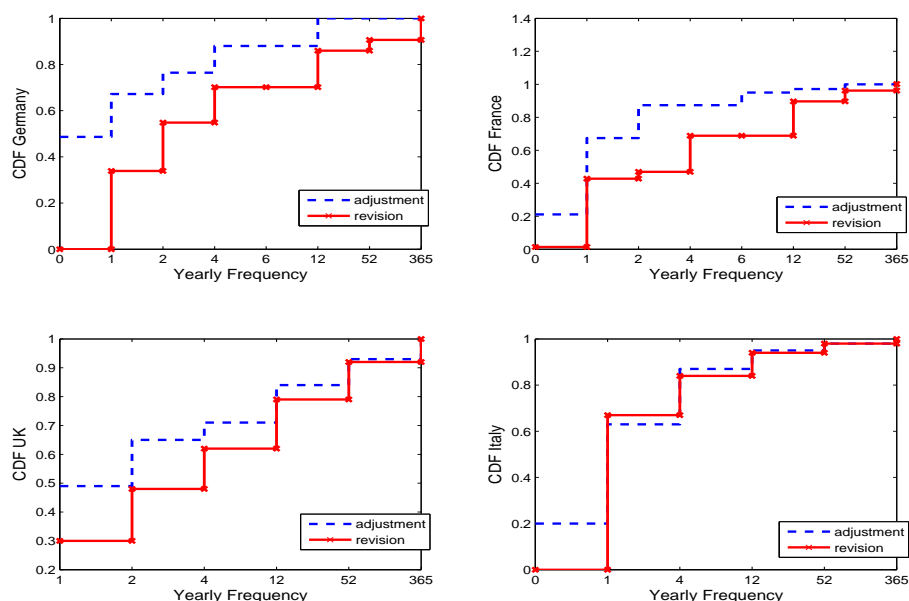
Note: Frequencies are measured on a per-year basis.

Table AA-6: Price-reviews and price-changes per year: medians and means

	AT	BE	FR	GE	IT	NL	PT	SP	EURO	CAN	UK	US
					Medians							
Review	4	1	4	3	1 2	4	2	1	2.7	12	4	2
Change	1	1	1	1	1 1	1	1	1	1	4	2	1.4
					Means							
Review	12.3	1.2	23.2	4.9	27.7	52.4	3.6	1.89	16.9	99.7	39.2	30
Change	2.6	0.9	3.5	2.1	5.1	2.3	1.9	1.85	3.3	61.3	33.5	27

Number of changes and reviews per year. The sources for the medians are Fabiani et al. (2007) 2003 Euro area survey, Amirault, Kwan, and Wilkinson (2006) 2003 Canadian survey, Greenslade and Parker (2008) 2008 UK survey. The sources for the means are discussed below.

group of firms reporting a frequency of price adjustment higher than 50. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Belgium: The source of the data is Section IV in Aucremanne and Druant (2005). From Aucremanne and Druant (2005), we have information on the average time between consecutive price reviews to be 13 months and the average number of consecutive price changes to be about 10 months. From section IV.1.2 "Overall, the average duration between two consecutive price reviews is 10 months." From section IV.2, "...This implies that the average duration between two consecutive price changes is almost 13 months..."The following table is from section IV.3. counts the number of firms in the sample that review and adjust prices in a given pair of durations. Below we copy Table 17 - Duration of prices from these

authors:

Table AA-7: Belgium: Duration of prices (number of firms in each bin)

	Price change			
	<= 1	> 1 and < 12	12	> 12
Price review				
<= 1	**31**	12	8	1
> 1 and < 12	1	**197**	72	21
12	2	15	**436**	37
> 12	0	1	5	**51**

Source: NBB, Aucremanne and Druant (2005). duration <= 1: price is changed/reviewed monthly or more frequently.duration > 1 and < 12 : price is changed/reviewed with a frequency from one month up to one year.duration = 12: price is changed /reviewed once a year.duration > 12 : price is changed/reviewed less than once a year.

France and Italy: The source is the raw data from Loupias and Ricart (2004) and Fabiani, Gattulli, and Sabbatini (2004). We removed missing observations from both series of the frequencies of adjustment and review and averaged across the remaining observations. Notice that we are keeping those firms for which we only have observation of one of the two frequencies.

Germany: The source is the raw data from Stahl (2009). We removed missing observations from both series of the frequencies of adjustment and review. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight. Notice that we are keeping those firms for which we only have observation of one of the two frequencies. In addition, the highest frequency of observation for price adjustment is monthly, while the highest frequency of price review is daily. In order to make the data comparable, we assigned a monthly frequency to all observations at a frequency higher than monthly.

Netherlands: The source of the data is Tables 4A-B in Hoeberichts and Stokman (2006). In order to compute the means, we assigned a yearly frequency of 0.5 to firms reporting of adjusting occasionally. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Portugal: The source of the data is Char 15-16 in Martins (2005). In order to compute the means, we assigned a yearly frequency of 0.5 to firms reporting of adjusting/reviewing less than once a year, and a yearly frequency of 18 to firm reporting of adjusting/reviewing more than twelve times a year. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Spain: I order to compute the mean, we used data in Tables A10-A11 in Alvarez and Hernando (2005). We assigned a frequency of 6, 2.5 and 0.5 to firms reviewing/adjusting more than four times, between two and three times and less than once a year respectively. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

Moreover, Alvarez and Hernando (2005) show that for Spain, quoting from their section 4.4:

When we compare the frequencies of price reviews and of changes, restricting the comparison to those firms that responded to both questions we observe that price changes occur only slightly less frequently than price reviews. The correlation between both frequencies is very high. For instance, among those firms reviewing their prices four or more times a year, 89% declare changing their prices at least four times a year, 4% change them two or three times a year, 6% once a year and 1% less than once a year.

The following table constructed from the first row of tables A10 and A11 in Alvarez and Hernando (2005)

Table AA-8: Spain: Frequency of price reviews and price changes (% of firms in each bin)

	At least 4 times per year	2 or 3 times per year	once a year	< once a year
Price change	13.9	15.1	56.8	14.3
Price review	14.0	15.6	63.1	7.4

Source: Alvarez and Hernando (2005)

Euro: We used the 2003 nominal GDP to compute the weights and averaged across the countries.

Canada: The source of the data is Figure 1 and Table 14 in Amirault, Kwan, and Wilkinson (2006). We assigned a frequency of 0.5 to firms reporting to review sporadically. We took the midpoint in each closed interval for the frequency of price changes (e.g. 3 for firms reporting between 2 and 4), and assigned 547.5=365*1.5 frequency of price changes to firms reporting to adjust prices more than 365 times a year. Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

UK: The source of the data is Table C on page 406 and chart 5 on page 407 in Greenslade and Parker (2008). In computing the mean, we excluded firms reporting "irregularly" and "other". Then we averaged across the different frequencies, using the fraction of firms at each frequency as a weight.

US: Source data from Blinder et al. (1998) 1992 US survey. To compute the means we use Table 4.1 and Table 4.7, interpolating the bins. Both means and medians are based on a small number of responses (186 and 121), and both are sensitive to details used for the interpolation.

AA-3   Numerical evaluation of Proposition 1.

In this section we evaluate the accuracy of the approximated solution in Proposition 1. To do so, we solve the model numerically on a grid for $\tilde{p}$ and obtain the numerical counterparts to the policy rule derived in Proposition 1. In doing so we approximate $\bar{V}(\cdot)$ through either a cubic spline or a sixth order polynomial. Results are invariant to the latter.

As Figure AA-9 shows, the solution for $\bar{p}$ (and as a consequence for ϕ) in Proposition 1 diverges from its numerical counterpart the more, the larger the ratio $\alpha \equiv \frac{\psi}{\theta}$ is. In particular, the approximated solution tend to understate the value of $\bar{p}$ relatively to the numerical

Figure AA-9: Numerical and approximated ϕ, τ, $\bar{p}$ as a function of $\alpha \equiv \frac{\psi}{\theta}$
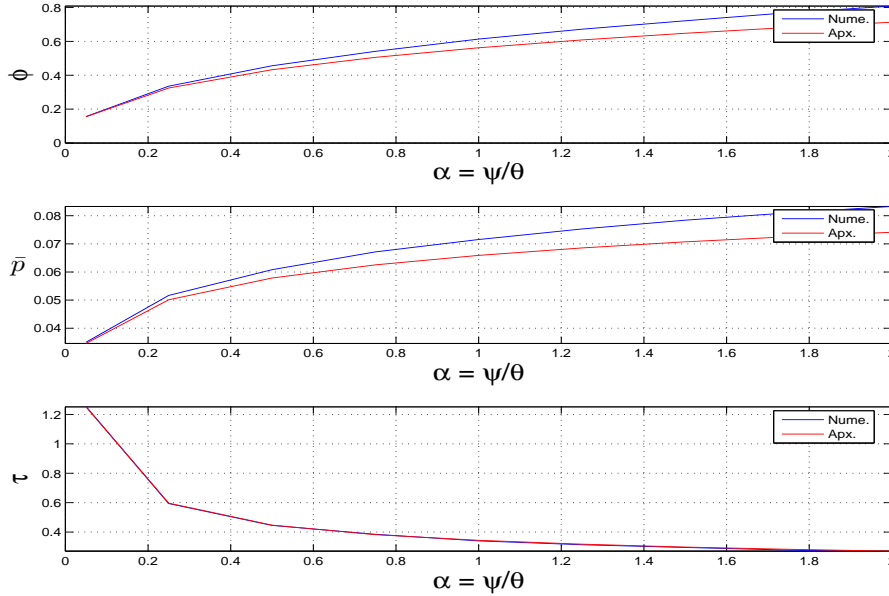
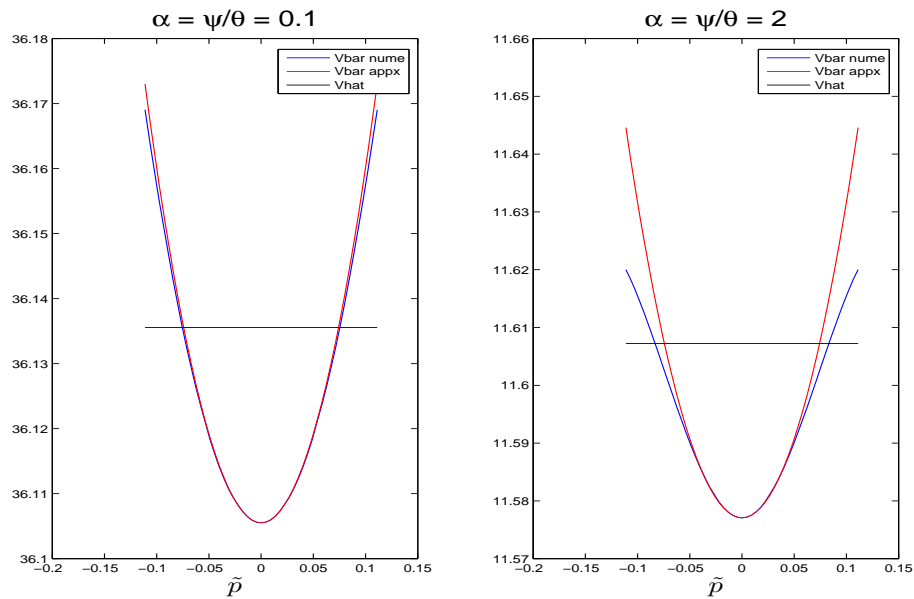Note: parameter values are $B = 20$, $\rho = 0.02$, $\sigma = 0.2$, $\psi = 0.03$. We let θ to vary.

solution. As a consequence, also ϕ is understated. In fact, the approximation of τ works pretty well.

This discrepancy is due to the nature of our approximation which relies on a second order approximation of $\bar{V}(\cdot)$, while higher orders (the fourth one in particular) become more relevant as ψ/θ increases, causing the inaction range to widen. To document this effect, we show the following computations. We solved the model numerically, assuming a polynomial of order sixth for $\bar{V}(\cdot)$, on a grid of values for $\tilde{p}$ for values of $\alpha = 0.1$ and $\alpha = 2$. We used the symmetry property of $\bar{V}(\cdot)$ to set the value of all the odd derivatives evaluated at $\tilde{p} = 0$ equal to zero. We then compared the numerical solution for $\bar{V}(\cdot)$ with the approximated one given by Proposition 1, but having an intercept (i.e. $\bar{V}(0)$) equal to the constant term in the sixth order polynomial. As Figure AA-10 shows, the quadratic approximation for $\bar{V}(\cdot)$ works better for low values of $\tilde{p}$, and more generally for low values of α. The second order approximation for $\bar{V}(\cdot)$ tends to overstate the value of the function for values away from zero, as it ignores the fourth derivative $V''''(0)$, which is negative. While the difference in the approximation will also affect the value of $\hat{V}$, we find this effect much smaller in our computations. Therefore, the quadratic approximation tends to understate the inaction range, i.e. to produce values of $\bar{p}$ that are smaller. This is consistent with the values displayed in Figure AA-9.

## AA-4   Numerical evaluation of elasticities in Proposition 8.

In this section we report the numerical solution of the model to the following experiments: (i) a change in ψ holding θ fixed; (ii) a change in θ holding ψ fixed. These experiments are meant to capture the elasticities of n_a and n_r with respect to ψ and θ.

Figure AA-10: Numerical and approximated $\bar{V}$ as a function of $\alpha \equiv \frac{\psi}{\theta}$

Note: parameter values are $B = 20$, $\rho = 0.02$, $\sigma = 0.2$, $\psi = 0.03$. We let θ to vary.

Figure AA-11: Frequency of adjustment and review as function of costs θ, ψ
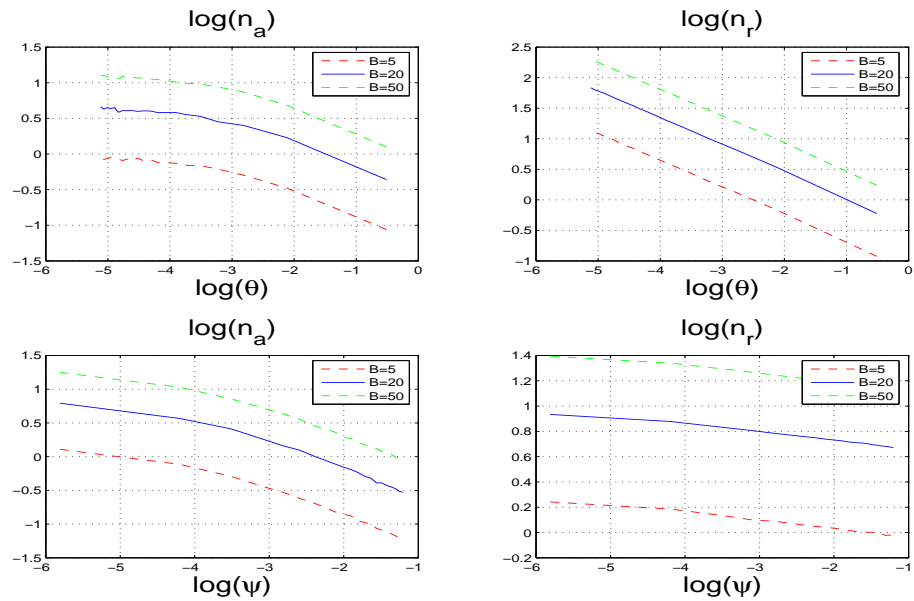
Note: fixed parameter values are $\rho = 0.02$, $\sigma = 0.2$; $\psi = 0.03$ when θ is allowed to change; $\theta = 0.06$ when ψ is allowed to change.

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We show results for different parameterizations of $B = 5, 20, 50$. The first row in Figure AA-11 display results for $\log(n_a), \log(n_r)$ to changes in $\log(\theta)$ holding $\psi = 0.03$ as in our benchmark calibration. The larger θ, the closer the value of α to zero. As we can see, the elasticity of n_r with respect to θ is roughly equal to - 1/2, independently of the level of B and the level of α. Similarly, the elasticity of n_a with respect to θ is not changing much to changes in the level of B, however it is sensitive to the level of α, being roughly equal to - 1/2 for large values of θ, i.e. for α closer to zero, and smaller at smaller values of θ.

The second row in Figure AA-11 display results for $\log(n_a), \log(n_r)$ to changes in $\log(\psi)$ holding $\theta = 0.06$ as in our benchmark calibration. The smaller ψ, the closer the value of α to zero. The elasticities of n_a and n_r with respect to ψ are smaller at smaller values of ψ.

AA-5   Recursion and Numerical evaluation of $\mathcal{A}(\cdot)$ from Proposition 6.

We give an analytical approximation for the function $\mathcal{A}$. Let $\bar{\phi} \equiv \phi/\sqrt{1-\phi^2}$

$$\mathcal{A}(\phi) \approx \frac{(1-\phi^2)}{1.5 - N\left(2\bar{\phi}\right)} \left(1 - \frac{1/2 \left(\int_0^{2\bar{\phi}} \left(\bar{\phi} - s\right)^2 dN(s) - \bar{\phi}^2 \right)}{1 - N(\phi) + \int_0^{\phi} s^2 dN(s) + \phi n(\phi)} \right) \tag{AA-14}$$

where $\mathcal{A}(0) = 1$, the approximation for $\mathcal{A}(\phi)$ is strictly increasing for $0 \leq \phi \leq \phi_{sup}$, $\phi_{sup} \approx 0.75$, and where $\mathcal{A}(\phi_{sup}) \approx 1.8$. The value ϕ_{sup} delimits the range over which the approximation is accurate.

Here we describe the derivation of the analytical approximation. We start by computing the second derivative of $\mathcal{T}_a(p)$ at $p = 0$ (for notation simplicity we use p in the place of $\tilde{p}$). To further simplify notation rename the extreme of integration as:

$$s_1 \equiv \frac{p - \bar{p}}{\sigma\sqrt{\mathrm{T}(p)}} \quad , \quad s_2 \equiv \frac{p + \bar{p}}{\sigma\sqrt{\mathrm{T}(p)}}$$

which depend on p with derivatives

$$\frac{\partial s_1}{\partial p} = \frac{\sqrt{\mathrm{T}(p)} - (p - \bar{p})\frac{\mathrm{T}'(p)}{2\sqrt{\mathrm{T}(p)}}}{\sigma \mathrm{T}(p)} \quad , \quad \frac{\partial s_2}{\partial p} = \frac{\sqrt{\mathrm{T}(p)} - (p + \bar{p})\frac{\mathrm{T}'(p)}{2\sqrt{\mathrm{T}(p)}}}{\sigma \mathrm{T}(p)} \quad .$$

The first order derivative is:

$$\mathcal{T}_a'(p) = \mathrm{T}'(p) + \int_{s_1}^{s_2} \mathcal{T}_a'\left(p - s\sigma\sqrt{\mathrm{T}(p)}\right)\left(1 - \frac{\sigma \mathrm{T}'(p)}{2\sqrt{\mathrm{T}(p)}} s\right) dN(s)$$

$$- \mathcal{T}_a(\bar{p})\, n\left(s_1\right) \frac{\partial s_1}{\partial p} + \mathcal{T}_a(-\bar{p})\, n\left(s_2\right) \frac{\partial s_2}{\partial p}$$

where $n(s)$ denotes the density of the standard normal. The second order derivative is:

$$\mathcal{T}_a''(p) = \mathrm{T}''(p) + \int_{s_1}^{s_2} \mathcal{T}_a'' \left(p - s\sigma\sqrt{\mathrm{T}(p)}\right) \left(1 - \frac{\sigma\mathrm{T}'(p)}{2\sqrt{\mathrm{T}(p)}} s\right)^2 dN(s)$$

$$+ \int_{s_1}^{s_2} \mathcal{T}_a' \left(p - s\sigma\sqrt{\mathrm{T}(p)}\right) \frac{s\sigma}{2\mathrm{T}(p)} \left(-\mathrm{T}''(p)\sqrt{\mathrm{T}(p)} + \frac{(\mathrm{T}'(p))^2}{2\sqrt{\mathrm{T}(p)}}\right) dN(s)$$

$$- \mathcal{T}_a'(\bar{p}) \left(1 - \frac{\sigma\mathrm{T}'(p)}{2\sqrt{\mathrm{T}(p)}} s_1\right) n(s_1) \frac{\partial s_1}{\partial p} + \mathcal{T}_a'(-\bar{p}) \left(1 - \frac{\sigma\mathrm{T}'(p)}{2\sqrt{\mathrm{T}(p)}} s_2\right) n(s_2) \frac{\partial s_2}{\partial p}$$

$$- \mathcal{T}_a(\bar{p}) \left(n'(s_1) \left(\frac{\partial s_1}{\partial p}\right)^2 + n(s_1) \frac{\partial^2 s_1}{(\partial p)^2}\right) + \mathcal{T}_a(-\bar{p}) \left(n'(s_2) \left(\frac{\partial s_2}{\partial p}\right)^2 + n(s_2) \frac{\partial^2 s_2}{(\partial p)^2}\right) \quad .$$

To evaluate this expression note that at $p = 0$ we have $\mathrm{T}'(0) = 0$, $\mathrm{T}(0) = \tau$, $-s_1 = s_2 = \phi$, $\frac{\partial s_1}{\partial p} = \frac{\partial s_2}{\partial p} = \frac{1}{\sigma\sqrt{\tau}}$ (recall the notation already used above $\phi \equiv \frac{\bar{p}}{\sigma\sqrt{\tau}}$). Hence the second to last line in the previous formula is $-2\mathcal{T}_a'(\bar{p}) n(s_1) \frac{\partial s_1}{\partial p}$ by the symmetry of $\mathcal{T}_a(p)$. Note moreover that at $p = 0$

$$\frac{\partial^2 s_1}{(\partial p)^2} = -\frac{\partial^2 s_2}{(\partial p)^2} = \frac{\bar{p}\,\mathrm{T}''(0)\sqrt{\tau}}{2\sigma\,\tau^2}$$

Thus we get:

$$\mathcal{T}_a''(0) = \mathrm{T}''(0) + \int_{-\phi}^{\phi} \mathcal{T}_a'' \left(-s\sigma\sqrt{\tau}\right) dN(s) - \frac{\sigma\mathrm{T}''(0)}{2\sqrt{\tau}} \int_{-\phi}^{\phi} \mathcal{T}_a' \left(-s\sigma\sqrt{\tau}\right) s \, dN(s)$$

$$- 2\mathcal{T}_a'(\bar{p}) \frac{n(\phi)}{\sigma\sqrt{\tau}} - 2\mathcal{T}_a(\bar{p}) \left(-\frac{n'(\phi)}{\sigma^2\,\tau} + n(\phi)\frac{\bar{p}\,\mathrm{T}''(0)\sqrt{\tau}}{2\sigma\,\tau^2}\right) \quad \text{(AA-15)}$$

Using that $\mathrm{T}''(0) = -2/\sigma^2$ (from Proposition 4), the last term in the previous equation can be rewritten as $-2\frac{\mathcal{T}_a(\bar{p})}{\sigma^2\tau}\left(-n'(\phi) - \phi\,n(\phi)\right)$, which is zero since $n'(x) + x\,n(x) = 0$ for a standard normal density.

Given that $\mathcal{T}_a'(0) = \mathcal{T}_a'''(0) = 0$, and $\mathcal{T}_a''(0) < 0$, we approximate $\mathcal{T}_a(p)$ with a quadratic function on the interval $[-\bar{p}\,,\,\bar{p}]$:

$$\mathcal{T}_a(p) = \mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\,(p)^2 \,.$$

Using the first and the second derivative of this quadratic approximation into the right hand side of equation (AA-15), and $\mathrm{T}''(0) = -2/\sigma^2$, gives:

$$\mathcal{T}_a''(0) = -2/\sigma^2 + \mathcal{T}_a''(0)\,(2\,N(\phi) - 1) - 2\mathcal{T}_a''(0) \int_0^{\phi} s^2 dN(s) - 2\,\mathcal{T}_a''(0)\,\phi\,n(\phi)$$

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or

$$\mathcal{T}_a''(0) = \frac{-1/\sigma^2}{1 - N(\phi) + \int_0^\phi s^2 dN(s) + \phi n(\phi)} \quad . \tag{AA-16}$$

To solve for $\mathcal{T}_a(0)$ let us evaluate $\mathcal{T}_a(p)$ at $\bar{p}$ obtaining:

$$\mathcal{T}_a(\bar{p}) = \mathrm{T}(\bar{p}) + \int_0^{2\bar{\phi}} \mathcal{T}_a\left(\bar{p} - s\sigma\sqrt{\mathrm{T}(\bar{p})}\right) dN(s) \quad , \text{ where } \quad \bar{\phi} \equiv \frac{\bar{p}}{\sigma\sqrt{\mathrm{T}(\bar{p})}} = \frac{\phi}{\sqrt{1-\phi^2}} \quad .$$

Using the quadratic approximation for $\mathcal{T}_a$ in the previous equation we get

$$\mathcal{T}_a(\bar{p}) = \mathrm{T}(\bar{p}) + \int_0^{2\bar{\phi}} \left(\mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\left(\bar{p} - s\sigma\sqrt{\mathrm{T}(\bar{p})}\right)^2\right) dN(s)$$

$$= \mathrm{T}(\bar{p}) + \left(N\left(2\bar{\phi}\right) - \frac{1}{2}\right)\mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\int_0^{2\bar{\phi}}\left(\bar{p} - s\sigma\sqrt{\mathrm{T}(\bar{p})}\right)^2 dN(s)$$

Replacing $\mathcal{T}_a(\bar{p}) = \mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\,\bar{p}^2$ on the left hand side, and collecting terms gives

$$\mathcal{T}_a(0) = \frac{\mathrm{T}(\bar{p}) + \frac{1}{2}\mathcal{T}_a''(0)\left(\int_0^{2\bar{\phi}}\left(\bar{p} - s\sigma\sqrt{\mathrm{T}(\bar{p})}\right)^2 dN(s) - \bar{p}^2\right)}{1.5 - N\left(2\bar{\phi}\right)}$$

$$= \mathrm{T}(\bar{p}) \frac{1 + \frac{\sigma^2}{2}\,\bar{\phi}^2\,\mathcal{T}_a''(0)\left(\int_0^{2\bar{\phi}}\left(1 - \frac{s}{\phi}\right)^2 dN(s) - 1\right)}{1.5 - N\left(2\bar{\phi}\right)}$$

$$= \tau\left(1 - \phi^2\right) \frac{1 + \frac{\sigma^2}{2}\,\bar{\phi}^2\,\mathcal{T}_a''(0)\left(\int_0^{2\bar{\phi}}\left(1 - \frac{s}{\phi}\right)^2 dN(s) - 1\right)}{1.5 - N\left(2\bar{\phi}\right)} \tag{AA-17}$$

where the last line uses the equality $\mathrm{T}(\bar{p}) = \tau - \left(\frac{\bar{p}}{\sigma}\right)^2 = \tau\left(1 - \phi^2\right)$. Substituting equation (AA-16) into equation (AA-17) gives

$$\mathcal{T}_a(0) = \tau\frac{(1 - \phi^2)}{1.5 - N\left(2\bar{\phi}\right)}\left(1 - \frac{1/2\left(\int_0^{2\bar{\phi}}\left(\bar{\phi} - s\right)^2 dN(s) - \bar{\phi}^2\right)}{1 - N(\phi) + \int_0^\phi s^2 dN(s) + \phi n(\phi)}\right) \tag{AA-18}$$

which gives the approximation for the expression $\mathcal{T}_a(0) = \tau \cdot \mathcal{A}(\phi)$. A numerical study of the function $\mathcal{A}(\phi)$ shows that $\mathcal{A}(0) = 1$, and that the function approximation is accurate and increasing for $\phi \in (0, 0.75)$, that $\mathcal{A}(0.75) \cong 1.78$ and decreasing thereafter.

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AA-6 Recursion and Numerical evaluation of $\mathcal{R}(\cdot)$ from Proposition 7.

Let $\bar{\phi} \equiv \phi/\sqrt{1-\phi^2}$ and

$$\mathcal{R}(\phi) \approx 1 - 2 \int_0^{\bar{\phi}} \left(q(0) + \frac{1}{2} q''(0) \ \phi^2 \right) \frac{\phi^2}{1+\phi^2} d\phi, \qquad \text{(AA-19)}$$

where $q(0)$ and $q''(0)$ are known functions of $\bar{\phi}$ derived below, and where $\mathcal{R}(0) = 1$, the approximation for $\mathcal{R}(\phi)$ strictly decreasing for $0 \le \phi \le \phi_{sup}$ where $\phi_{sup} \approx 0.65$, and $\mathcal{R}(\phi_{sup}) \approx 0.96$.

To derive the quadratic approximation for q we notice that $q(\cdot)$ attains its maximum at $\tilde{\phi} = 0$, and that it is symmetric, so that $q'(0) = q'''(0) = 0$ and $q''(0) < 0$. Furthermore, $q(\bar{\phi}) > 0$. Then, for small $\bar{\phi} = \frac{\bar{p}}{\sigma\sqrt{\tau}}$, this function can be approximated by a quadratic function with

$$q(\phi) = q(0) + \frac{1}{2} q''(0) \ \phi^2 \ .$$

The value of $q(\cdot)$ and its first and second derivatives with respect to $\tilde{\phi}$, evaluated at $\tilde{\phi} = 0$, are given by

$$q(0) = \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) n(-\phi) \frac{d\Theta(0,\phi)}{d\tilde{\phi}} d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) d\phi \right] n(0),$$

$$q'(0) = \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) n'(-\phi) \left(\frac{d\Theta(0,\phi)}{d\tilde{\phi}} \right)^2 d\phi + \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) n(-\phi) \frac{d^2\Theta(0,\phi)}{(d\tilde{\phi})^2} d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) d\phi \right] n'(0),$$

$$q''(0) = \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) n''(-\phi) \left(\frac{d\Theta(0,\phi)}{d\tilde{\phi}} \right)^3 d\phi + \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) n'(-\phi) 3 \frac{d\Theta(0,\phi)}{d\tilde{\phi}} \frac{d^2\Theta(0,\phi)}{(d\tilde{\phi})^2} d\phi$$

$$+ \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) n(-\phi) \frac{d^3\Theta(0,\phi)}{(d\tilde{\phi})^3} d\phi + \left[1 - \int_{-\bar{\phi}}^{\bar{\phi}} q(\phi) d\phi \right] n''(0).$$

Using equation (A.5) gives

$$\frac{d\Theta(0,\phi)}{d\tilde{\phi}} = \sqrt{1+\phi^2} \ , \quad \frac{d^2\Theta(0,\phi)}{(d\tilde{\phi})^2} = 0 \ , \quad \frac{d^3\Theta(0,\phi)}{(d\tilde{\phi})^3} = -3\sqrt{1+\phi^2}.$$

Using that $n''(-\phi) = n(-\phi)(\phi^2 - 1)$, we rewrite $q(0)$ and $q''(0)$ as

$$q(0) = 2 \int_0^{\bar{\phi}} q(\phi) n(\phi) \sqrt{1+\phi^2} d\phi + \left(1 - 2 \int_0^{\bar{\phi}} q(\phi) d\phi \right) n(0),$$

$$q''(0) = 2 \int_0^{\bar{\phi}} q(\phi) n(\phi) \sqrt{1+\phi^2} (\phi^4 - 4) d\phi - \left(1 - 2 \int_0^{\bar{\phi}} q(\phi) d\phi \right) n(0).$$

These two equations and the quadratic approximation for $q(\cdot)$ give a system of 2 equations

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in 2 unknowns: $q(0)$ and $q''(0)$:

$$q(0) + q''(0) = 2 \int_0^{\bar{\phi}} \left(q(0) + \frac{1}{2} q''(0)\, \phi^2 \right) n(\phi) \sqrt{1 + \phi^2}(\phi^4 - 3) d\phi$$

$$q(0) = 2 \int_0^{\bar{\phi}} \left(q(0) + \frac{1}{2} q''(0)\, \phi^2 \right) n(\phi) \sqrt{1 + \phi^2} d\phi + \left(1 - 2 \int_0^{\bar{\phi}} \left(q(0) + \frac{1}{2} q''(0)\, \phi^2 \right) d\phi \right) n(0) \quad .$$

The equations above imply

$$\frac{q''(0)}{q(0)} = -\frac{1 - 2 \int_0^{\bar{\phi}} n(\phi) \sqrt{1 + \phi^2}(\phi^4 - 3) d\phi}{1 - \int_0^{\bar{\phi}} n(\phi) \phi^2 \sqrt{1 + \phi^2}(\phi^4 - 3) d\phi} \equiv \kappa(\bar{\phi})$$

$$q(0) = \frac{n(0)}{1 + 2 \int_0^{\bar{\phi}} \left(n(0) - n(\phi) \sqrt{1 + \phi^2} \right) d\phi + \kappa(\bar{\phi}) \int_0^{\bar{\phi}} \left(n(0) - n(\phi) \sqrt{1 + \phi^2} \right) \phi^2 d\phi} \quad .$$

$$\mathcal{T}_r(0) = 2 \int_0^{\bar{\phi}} \mathrm{T}(\phi) q(\phi) d\phi + \left[1 - 2 \int_0^{\bar{\phi}} q(\phi) d\phi \right] \tau \tag{AA-20}$$

$$\approx \tau - 2\tau \left[-\int_0^{\bar{\phi}} (1 + \phi^2)^{-1} \left(q(0) + \frac{1}{2} q''(0)\phi^2 \right) d\phi + \int_0^{\bar{\phi}} \left(q(0) + \frac{1}{2} q''(0)\phi^2 \right) d\phi \right] \tag{AA-21}$$

$$\approx \tau - 2\tau \left[\int_0^{\bar{\phi}} \left(q(0) + \frac{1}{2} q''(0)\phi^2 \right) \frac{\phi^2}{(1 + \phi^2)} d\phi \right] \tag{AA-22}$$

where we use the quadratic approximation of $q(\cdot)$, the definition of $\mathrm{P}(p)$, and the quadratic approximation of $\mathrm{T}(p)$.

Notice that, given equation (20), the average frequency of price review can be always written as $n_r = 1/\mathcal{T}_r(0)$ where $\mathcal{T}_r(0) = \tau \mathcal{R}(\phi)$. This result follows directly from substituting equation (20) into equation (AA-21).

AA-7 Approximation for the density of normalized price changes

We use the approximation for $q(\phi) \approx q(0) + \frac{1}{2} q''(0)\phi^2$ and the formulas for $q(0)$ and $q(0)''$ developed in Appendix AA-6. The expressions obtained there for $q(0)$ and $q''(0)$ are a function of $\bar{\phi}$. Thus we can write:

$$v(x; \bar{\phi}) = \frac{\int_{-\bar{\phi}}^{\bar{\phi}} \left[q(0) + \frac{1}{2} q''(0)\phi^2 \right] \sqrt{1 + \phi^2}\, n\left(x \sqrt{1 + \phi^2} - \phi \right) d\phi}{1 - \int_{-\bar{\phi}}^{\bar{\phi}} \left[q(0) + \frac{1}{2} q''(0)\phi^2 \right] d\phi} \tag{AA-23}$$

$$+ \quad n(x) \quad \text{for } |x| > \bar{\phi} \;.$$

where $\bar{\phi} \equiv \phi/\sqrt{1 - \phi^2}$.

AA-8   Evidence about the distribution and hazard rate of price changes

Figure AA-12: Distribution of price changes in Cavallo (2009)

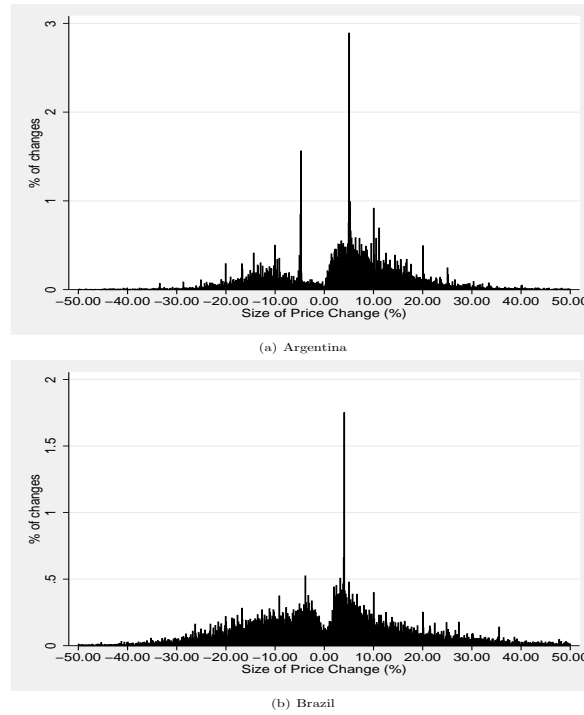

(a) Argentina

(b) Brazil

Figure 3: Size of Price Changes

Notes: Bin size is 0.1%. Brazil shown without changes on 15/12/07
and 29/12/07 (see Appendix for full distribution).

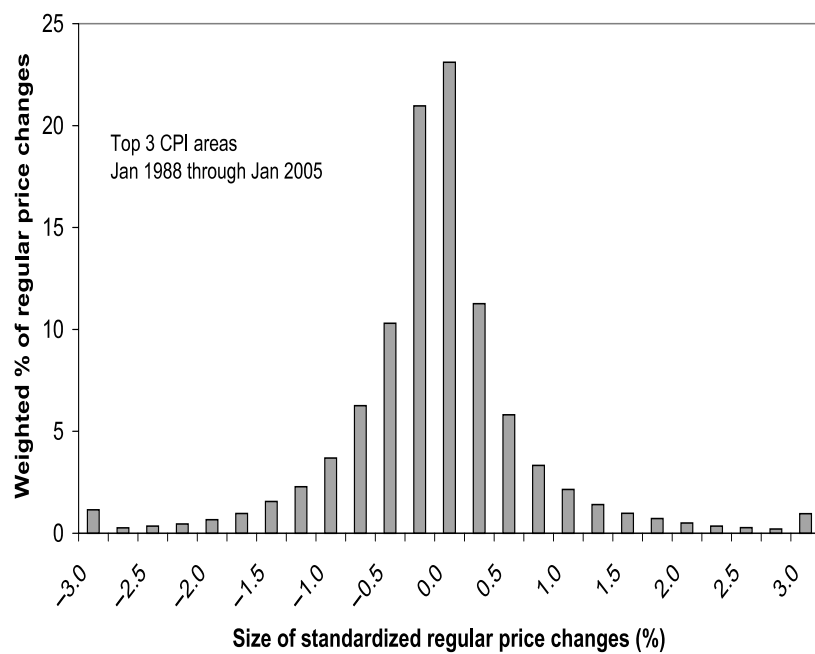Figure AA-13: Distribution of price changes in Klenow and Kryvtsov (2008)

FIGURE III
Weighted Distribution of Standardized Regular Price Changes

Figure AA-14: Hazard rate in Cavallo (2009)

(a) Argentina

(b) Brazil
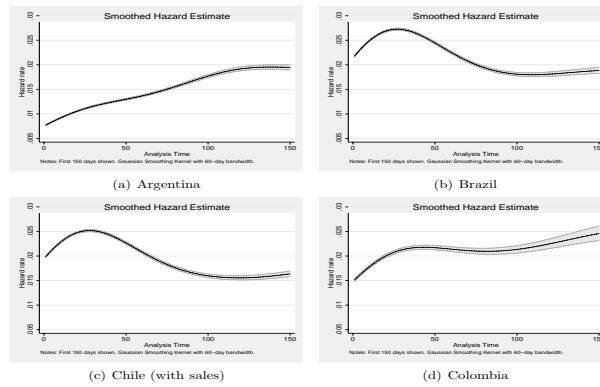
(c) Chile (with sales)

(d) Colombia

Figure 5: Smoothed Hazard Functions

Notes: All uncensored spells are included. Sale events are excluded, with the exception of Chile where sales
information is not available. Smoothing function with a Gaussian Kernel and 60-day bandwidth. First 150
days are shown.

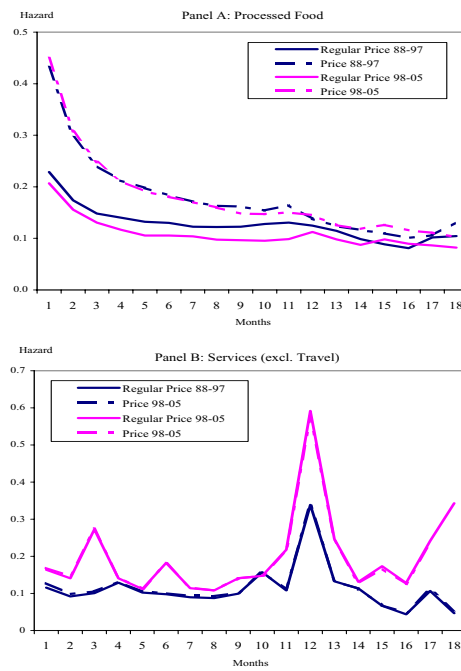Figure AA-15: Hazard rate in Nakamura and Steinsson (2008)

Figure VIII
Hazard Function for Consumer Prices

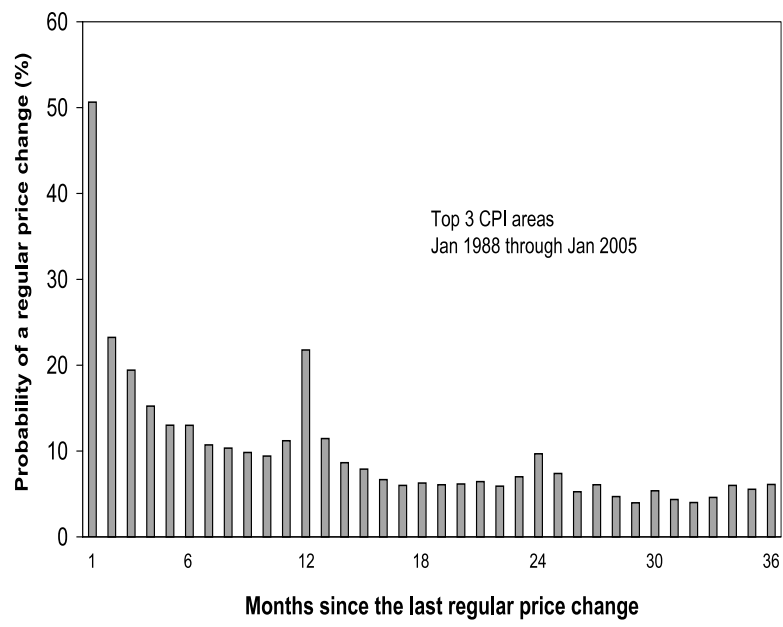Figure AA-16: Hazard rate in Klenow and Kryvtsov (2008)

FIGURE V
Weighted Hazard Rates for Regular Prices

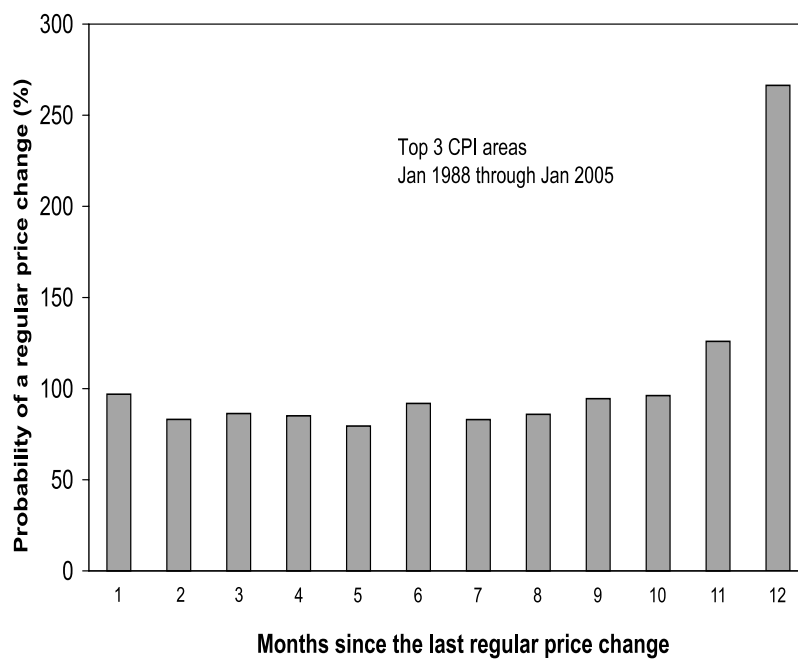Figure AA-17: Hazard rate in Klenow and Kryvtsov (2008)

FIGURE VI
Weighted Hazards for Regular Prices vs. Decile Fixed Effects

Figure AA-18: Hazard rate in Alvarez, Burriel, and Hernando (2005)

Figure 1: International evidence on decreasing hazard functions for price changes

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