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Persistent Liquidity Effect and Long Run Money Demand

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Abstract

We present a monetary model in the presence of segmented asset markets that implies a persistent fall in interest rates after a once and for all increase in liquidity. The gradual propagation mechanism produced by our model is novel in the literature. We provide an analytical characterization of this mechanism, showing that the magnitude of the liquidity effect on impact, and its persistence, depend on the ratio of two parameters: the long-run interest rate elasticity of money demand and the intertemporal substitution elasticity. At the same time, the model has completely classical long-run predictions, featuring quantity theoretic and Fisherian properties. The model simultaneously explains the short-run "instability" of money demand estimates as-well-as the stability of long-run interest-elastic money demand.

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1 Introduction

We develop a model based on the ideas on the liquidity effect of monetary shocks proposed by Lucas (1990), Fuerst (1992), Christiano and Eichenbaum (1992), and others. To keep the analysis tractable the previous literature is constructed to ignore the lingering distributional effects that follow an injection of liquidity. We develop a tractable model of these effects, and analytically characterize the determinants of the persistence of the interest changes on interest rate that follow an open market operation. For instance we show that including this distributional effect, a once-and-for-all increase in the money supply necessarily produces a persistent drop of the nominal interest rate, whose half life is determined by two type of factors: those determine the adjustment of consumption in permanent income theory, as well as the long-run interest rate elasticity of money demand. We use the model to show that the persistent liquidity effects imply a behavior of velocity and interest rates that is consistent with the short run "instability" of money demand in the presence of transitory monetary shocks. At the same time, and also novel in the segmented market models applied to monetary shocks, we show that if the changes in money growth are nearly permanent, the model delivers a standard money demand, with a positive relationship between velocity and interest rates.

The model in this paper is a descendent of the monetary models with segmented markets of Grossman and Weiss (1983) and Rotemberg (1984). These models are motivated by the hypothesis that, as described in Friedman's presidential address, and in contrast with the working of simpler neoclassical monetary models, an open market operation that increases the quantity of money once-and-for-all produces a protracted decrease of the nominal interest rate.[1] In the models in these two papers agents are subject to a cash in advance constraint,

[1] For instance, pages 5-6 of Friedman's 1968 presidential address: "Let the Fed set out to keep interest rates down. How will it try to do so? By buying securities. This raises their prices and lowers their yields. In the process, it also increases the quantity of reserves available to banks, hence the amount of bank credit, and, ultimately the total quantity of money. That is why central bankers in particular, and the financial community more broadly, generally believe that an increase in the quantity of money tends to lower interest rates. [...] The initial impact of increasing the quantity of money at a faster rate than it has been increasing is to make interest rates lower for a time than they would otherwise have been. But this is only the beginning

but access to asset markets where open market operation takes place is restricted to a fraction of the agents. This fraction of agents, who holds only half of the money stock, have to absorb the entire increase on the money supply associated with the open market operation, which will be absorbed only with a lower real interest rate. Moreover, if this effect is large enough, the nominal interest rate decreases also. Mostly for tractability, these two papers assume that agents have access to the asset markets every-other period. Not surprisingly, with this pattern of visits to the asset market, the effect of a once and for all increase in money supply on interest rates are short lived: the largest effect is in the first two periods, after which there are small lingering echo effects.

Several monetary models of segmented asset markets have been written to analyze a variety of related questions since the seminal work of Grossman and Weiss, and Rotemberg. In all of them, some carefully chosen assumptions are used to avoid having to keep tract of the lingering effects on the cross section distribution of asset holdings produced by an open market operation. The simplifications have the advantage of allowing a sharper analytical characterization of the equilibrium.[2] For instance, in Lucas (1990) asset and good markets are separated within the period, but at the end of the period all agents pool their resources. Similar assumptions are made in the base-line models in Fuerst (1992) and Christiano and Eichenbaum (1992), who use set-ups closely related to the one in Lucas (1990) to study the effects on output of a monetary shock. In Alvarez and Atkeson (1997) agents visit the asset market at exogenously randomly distributed times, at which they effectively −via the complete market assumption− pool resources. In Alvarez, Atkeson, and Kehoe (2002) agents

of the process not the end. [...] These three effects will reverse the initial downward pressure on interest rates fairly promptly, say, in something less than a year. Together they will tend, after a somewhat longer interval, say, a year or two, to return interest rates to the level they would otherwise have had."

[2]See, for instance Lucas (1990): "In [2] and [13], (referring to Grossman and Weiss, and Rotemberg) an open market operation that induces a liquidity effect will also alter the distribution of wealth, since agents who participate in the trade will have different post-trade portfolios than those who were absent. These distributional effects linger on indefinitely (as they no doubt do in reality), a fact that vastly complicates the analysis, effectively limiting both papers to the study of a one-time, unanticipated bond issue in an otherwise deterministic setting. This paper studies this same liquidity effect using a simple device that abstracts from these distributional effects." Fuerst (1992) uses the same assumption and comments about it: "This methodology is not without cost. By entirely eliminating these wealth effects, the model loses the persistent and lingering effects of a monetary injection captured, for example, in Grossman and Weiss (1983)."

face a fixed cost of accessing the asset market, making participation endogenous and time-varying. Yet the parameters considered and the nature of the equilibrium is such that, at the end of the period the distribution of asset is degenerate. Finally, the closest environments to the one consider in this paper are the ones in Alvarez, Lucas, and Weber (2001) and in Occhino (2004). In these set ups a fraction λ of the agents have permanent access to asset markets, the remaining fraction is excluded from the asset markets permanently. Apart from these differences in the way that agents are segmented in their participation in asset markets, these models have a binding cash-in-advance with exogenous velocity.

All these models have in common that an open market operation that produces a once and for all increase in the money supply decreases interest rate on impact, but then the interest rate returns immediately to its previous level. These models also have in common that they produce a version of the quantity theory in which different permanent values of the growth rate of money supply are associated with different inflation rates, and hence -via a Fisher equation- different nominal interest rates, but with the same level of velocity. In this sense, these models have an interest rate inelastic long run money demand. In this paper, we introduce a modification of the set-up in which a once and for all increase in the money supply has a persistent liquidity effect, due to persistent redistribution effects of the open market operation, and additionally it implies a long run interest elastic money demand.

There are examples in the literature of models where a once and for all increase in the money supply produce a transitory, but persistent, decline in interest rates. These examples feature a different mechanism to generate persistence. An early example is the model in Christiano and Eichenbaum (1992). In their basic set up firms face a CIA constraint and, given the assumption on when the household have to decide their cash holdings, asset market are segmented. In this set-up, as well in the closely related set-up of Fuerst (1992), liquidity effect are short lived.[3] Christiano and Eichenbaum (1992) add to this basic-set up, a convex

[3] Fuerst explains the lack of propagation in terms of his model very clearly: “As for the failures, the most glaring is that the model is lacking a strong propagation mechanism. The real effects of monetary injections are a result of (serially uncorrelated) forecast errors. These effects will therefore be strongest during the initial period of the shock.” …“This failure to achieve persistence is more a criticism of my

adjustment cost on households cash holdings, which produces a persistent liquidity effect. A recent contribution by Curdia and Woodford (2008) uses a modeling device close to the one in Alvarez and Atkeson (1997), and adds another channel that produces a persistent liquidity effect: slow price adjustment of firms a la Calvo.

Our model is a version of the segmented market model in Alvarez, Lucas, and Weber (2001) where instead of using a binding cash in advance constraint, we use a Sidrauski money in the utility function set-up. In this setup we obtain the standard result that velocity, is endogenous and forward looking, it is a function of an appropriately expected discounted value of future monetary expansions. This implies that the price level and inflation are also functions of future expected monetary expansions. Surprisingly, we show that this relationship between inflation and money is, up to first order, exactly the same regardless of the degree of segmentation, and hence it is the same as in an otherwise standard monetary model. Instead, the degree of segmentation on the asset markets determines the effect of monetary expansions on interest rates, i.e. whether or not there is a liquidity effect. To understand the behavior of interest rate, we consider two extreme cases that differ in terms of the persistence of monetary innovations.

The first case is a once-and-for-all increase of the money supply. In this case there is a jump on impact in the price level, and no effect on expected inflation. The model with segmented markets necessarily produces a persistent decrease in nominal interest rates. The magnitude of this effect at impact is inversely proportional to the fraction of agents participating in the open market operation. The persistence of the liquidity effect depends on the ratio of two elasticities, that govern how long it takes to return to the steady state distribution of liquidity across agents: the half-life of the liquidity effect is shorter, the higher is the ratio of the interest-elasticity of the money demand relative to the intertemporal substitution elasticity. In the second extreme case, a persistent increase in the growth rate of

particular modeling strategy than of this class of models as a whole. If it takes more than one period for the economy to re-balance its portfolio and 'undo' the monetary injection, then the effects of monetary shocks will of course persist. While this assumption may be the most natural, it is also somewhat intractable for the reasons outlined in section 2."

money supply, that immediately increases expected inflation, produces a persistent increase in nominal interest rates and no liquidity effect.

Our model displays liquidity effect for both unanticipated and anticipated monetary shocks, although the effects of anticipated monetary shocks may differ from anticipated shocks. In the model in Lucas (1990) and in versions of Christiano and Eichenbaum (1992) and Fuerst (1992) only unanticipated shocks display liquidity effects. On the other hand, in Alvarez, Lucas, and Weber (2001), Occhino (2004), Alvarez, Atkeson, and Kehoe (2002), there is no distinction between the effect of expected and unexpected monetary shocks on interest rates. The fact that both anticipated and unanticipated monetary shocks have an effect on interest rates, and that these effects differ, can be used in future applied work to estimate the impulse responses of interest rates by applying the identification strategy proposed by Cochrane (1998). The model is also related to Scheinkman and Weiss (1986), who study an economy where production possibilities shift stochastically between two agents. Absent complete markets, the productive agent has an incentive to accumulate money balances to finance future consumption in the times when he will be unproductive. When unproductive, the agent will use money balances to finance consumption until he becomes productive again. The dynamics of money balances in this model are similar to the dynamic transition towards the steady state for the non-trader agent in our model.

Combining the results for the dynamics of monetary expansions, inflation and interest rates we obtain the following behavior for money demand. In the long run, i.e. comparing across steady states with different constant monetary growth rates, the model displays an interest-elastic money demand. In the short run, i.e. comparing mean reverting shocks on the growth rate of money, the model features persistent departures of the relationship between real balances and interest rates, due to the persistent liquidity effects. We view this property of the model as a useful step to reconcile low frequency US data on interest rate and velocity, which is consistent with an interest elastic money demand, and the high frequency data, which features persistent deviation from it, and with lower interest rate elasticities. We illustrate the quantitative impact of this feature by simulating the model economy with a process for the growth rate of money supply with both low and high frequency components, and comparing it with the US economy.

2 The Environment

Let $U(c, m)$ be the period utility function, where c^i is consumption m^i are beginning-of-period real balances. We assume that U is strictly increasing and concave. Let $i = T, N$ be the type of agents, trader, or which there are λ, and non-traders, of which there are $(1-\lambda)$, respectively. While the focus of the paper is on the analysis of the behavior of interest rates due to the unequal access to asset market at the time of open market operations, we first describe the equilibrium in a model without a bond market, and hence without open market operations. We do so because it is easier, and it highlights the logic of the determination of different equilibrium variables.

Time is discrete and starts at $t = 0$. The timing within a period is as follows: agent start with nominal beginning-of-period cash balances M_t^i, they receive nominal income $P_t\ y^i$, choose real consumption c_t^i and end-of-period nominal balances N_t^i. Next period nominal cash balances M_{t+1}^i are given this period nominal balances plus $P_{t+1}\ \tau_{t+1}^i$ the nominal lump-sum transfer from the government. Their budget constraint at $t \geq 0$ are:

$$N_t^i + P_t c_t^i = P_t y^i + M_t^i \quad , \quad M_{t+1}^i = N_t^i + P_{t+1}\tau_{t+1}^i \tag{1}$$

Our choice of the timing convention for this problem is consistent with an interpretation of U as a cash-credit good, as in Lucas and Stokey (1987). We use a Sidrauski money-in-the-utility function specification because, relative to a cash-in-advance model, it allows more flexibility to accommodate a stock m and a flow c, albeit in a mechanical way. We will return to the discussion of the specification, and the relation between stocks and flows, below.

The problem of an agent of type i is

$$\max_{\{N_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U \left(c_t^i \ , \ M_t^i / P_t \right) \right]$$

subject to (1) given M_0. Notice that while we labeled the agents traders and non-traders, the budget constraint in (1) indicates that neither type of agent is allowed to trade in bonds or any other security. Their only intertemporal choice is the accumulation of cash balances. Yet in Section 2.1 we show that the equilibrium of the model will be the same as one in which traders and the government participate in a market for nominal bonds.

Market clearing of goods and money is given by

$$\begin{aligned} \lambda c_t^T + (1-\lambda) \, c_t^N &= \lambda y^T + (1-\lambda) y^N, \\ \lambda M_t^T + (1-\lambda) \, M_t^N &= M_t \end{aligned}$$

for all $t \geq 0$. The government budget constraint is given by

$$M_t - M_{t-1} = P_t \left[\lambda \tau_t^T + (1-\lambda) \, \tau_t^N \right] \ ,$$

for all $t \geq 1$. Notice that the government budget constraint does not apply for $t = 0$, since our timing convention is that we start the period with the cash after transfers. To simplify the notation, in this section we assume that the government does not trade in bonds, an assumption that we remove in Section 2.1. We note for future reference that the budget constraint of the agents and market clearing imply that aggregate beginning-of period and end-of-period money balances are the same:

$$M_t = N_t \equiv \lambda N_t^T + (1-\lambda) \, N_t^N$$

for all $t \geq 0$. Notice that using the definition of N_t the budget constraint of the government follows from the budget constraint of the households:

$$M_{t+1} = M_t + P_{t+1}\left[\lambda\tau^T_{t+1} + (1-\lambda)\,\tau^N_{t+1}\right] = N_t + P_{t+1}\left[\lambda\tau^T_{t+1} + (1-\lambda)\,\tau^N_{t+1}\right] \ .$$

We define inflation π_t, growth rate of money, μ_t, beginning-of-period real balances, m_t, and end-of-period real balances, n_t as:

$$\pi_{t+1} = \frac{P_{t+1}}{P_t}, \mu_{t+1} = \frac{M_{t+1}}{M_t},\ m_t = \frac{M_t}{P_t},\ m^i_t = \frac{M^i_t}{P_t},\ n^i_t = \frac{N^i_t}{P_t},$$

for $i = T, N$ and $t \geq 0$. With these definitions we write the budget constraints as

$$c^i_t + n^i_t = y^i + m^i_t \quad \text{and} \quad m^i_{t+1} = n^i_t/\pi_{t+1} + \tau^i_{t+1}, \quad t \geq 0 \tag{2}$$

market clearing as

$$\lambda m^T_t + (1-\lambda)\, m^N_t = m_t, \quad , \quad \lambda c^T_t + (1-\lambda)\, c^N_t = \lambda y^T + (1-\lambda)\, y^N, \quad t \geq 0 \tag{3}$$

the money-growth identity

$$\pi_t = \mu_t\ \frac{m_{t-1}}{m_t},\ t \geq 0 \tag{4}$$

and the government budget constraint as

$$m_t - \frac{m_{t-1}}{\pi_t} = \lambda\tau^T_t + (1-\lambda)\,\tau^N_t \quad , \quad t \geq 1. \tag{5}$$

The exogenous random processes for this economy are given by $s_t \equiv (\mu_t, \tau^T_t, \tau^N_t)$. We use s^t for the histories of such shocks, but we avoid this notation when it is clear from its context. The f.o.c. for the agent problem w.r.t. n^i_t , $i = T, N$ are:

$$U_1\left(c^i_t, m^i_t\right) = \mathbb{E}_t\left\{\frac{\beta}{\pi_{t+1}}\left[U_1\left(c^i_{t+1}, m^i_{t+1}\right) + U_2\left(c^i_{t+1}, m^i_{t+1}\right)\right]\right\}. \tag{6}$$

for $t \geq 0$, where the expectation is with respect to the realization of inflation and the lump sum subsidy τ_{t+1}^i. We can now define an equilibrium:

Definition 1. Given initial conditions $\{M_0^i\}$ and a monetary and fiscal policy described by stochastic processes $\left\{\mu_{t+1}, \tau_{t+1}^T, \tau_{t+1}^N\right\}_{t=0}^{\infty}$ an *Equilibrium* is an initial price level P_0, inflation rates $\{\pi_{t+1}\}_{t=0}^{\infty}$, and stochastic processes $\{n_t^i, m_t^i, c_t^i, m_t\}_{t=0}^{\infty}$ for $i = T, N$ such that: the budget constraints (2), market clearing (3), identity (4), the government budget constraint (5), and the Euler equations (6) hold.

We conclude this section with two comments on the set-up. The first is that our convention for the timing of the model is one where the initial conditions M_0^i contain the period zero monetary injection, and thus τ_0^i and μ_0 are not part of the set-up. For instance, our convention entails that $\{\mu_1, \tau_1^i\}$ are random variables, whose realization is not known as of time $t = 0$. Second, in the monetary-fiscal policy described by (5), the resources obtained by monetary expansions can be used for redistribution across agents, since the lump sum transfers τ_t^T and τ_t^N are allowed to differ across types.

2.1 Interest Rates in an Equilibrium with an Active Bond Market

In this section we introduce a bond market where traders participate and where the open market operation takes place. We show an equivalence result for the equilibrium with and without an active bond market, and analyze how the interest rate depends on the equilibrium allocation.

We assume that traders at time t have access to a bond market and a set of Arrow securities that pay contingent on the realization of s_{t+1} which opens at the beginning of the period, before consumption takes place, at the same time of the money transfer τ_t^T. Thus, if a trader buys one nominal zero coupon bond in period t, he reduces his money holdings M_t^T by Q_t dollars (the bond price), and increases next period holding of money in all states by one dollar. Let W_t be the beginning-of-period money stock, after the current period lump-sum transfer from the government but before participating in the bond market. Let B_t^T be the number of nominal bonds purchased, and $A_t(s_{t+1})$ the quantity of Arrow securities that pay one dollar contingent on the realization of s_{t+1}. The price of each of these securities at the beginning of period t is denoted by $q_t(s_{t+1})$. In this case we can write the budget constraint of the trader at time t and history s^t —which we omit to simplify the notation—

$$\begin{aligned}
\sum_{s_{t+1}} A_t^T(s_{t+1})q_t(s_{t+1}) + Q_t B_t^T + M_t^T &= W_t, \\
N_t^T + P_t c_t^T &= M_t^T + P_t y^T, \\
W_{t+1}(s_{t+1}) &= N_t^T + B_t^T + A_t^T(s_{t+1}) + P_{t+1}(s_{t+1})\ \tau_{t+1}^T(s_{t+1})
\end{aligned}$$

for $t \geq 0$. The interpretation is that during the period the agent chooses bond holdings B_t^T and consumption c_t^T, and given the budget constraint, this gives next period cash balances before transfers N_t^T. The last line shows the beginning-of-next-period cash balances, which include the cash from the bonds purchased this period and the cash transfer from the government. Equivalently, we can write the budget constraint in real terms as:

$$\begin{aligned}
\sum_{s_{t+1}} a_t^T(s_{t+1})q_t(s_{t+1}) + Q_t b_t^T + m_t^T &= w_t, \\
n_t^T + c_t^T &= m_t^T + y^T, \\
w_{t+1}(s_{t+1}) &= \frac{n_t^T + b_t^T + a_t^T(s_{t+1})}{\pi_{t+1}(s_{t+1})} + \tau_{t+1}^T(s_{t+1})\ ,
\end{aligned} \tag{7}$$

where $w_t = W_t/P_t$, $a_t = A_t/P_t$, $b_t = B_t/P_t$ and $t \geq 0$. The government budget constraint is:

$$\sum_{s_{t+1}} A_t(s_{t+1})q_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} + P_t\left[\lambda\tau_t^T + (1-\lambda)\,\tau_t^N\right], \tag{8}$$

for $t \geq 1$ or in real terms:

$$\sum_{s_{t+1}} a_t(s_{t+1})q_t(s_{t+1}) + Q_t b_t + m_t = \frac{b_{t-1} + a_{t-1} + m_{t-1}}{\pi_t} + \lambda\tau_t^T + (1-\lambda)\,\tau_t^N\ \ , \tag{9}$$

for $t \geq 1$. The trader's problem is to maximize utility by choice of $\left\{n_t^T, b_t^T\right\}_{t=0}^{\infty}$ subject to (7), and given an initial condition w_0.

The first order conditions for the choice of nominal bond holdings B_t^T and Arrow securities $A_t^T(s_{t+1})$ are:

$$\begin{aligned} Q_t \; \left[U_1\left(c_t^T, m_t^T\right) + U_2\left(c_t^T, m_t^T\right)\right] &= \mathbb{E}_t\left[\frac{\beta}{\pi_{t+1}}\left[U_1\left(c_{t+1}^T, m_{t+1}^T\right) + U_2\left(c_{t+1}^T, m_{t+1}^T\right)\right]\right] \quad (10) \\ q_t(s_{t+1})\left[U_1\left(c_t^T, m_t^T\right) + U_2\left(c_t^T, m_t^T\right)\right] &= \frac{\beta \; \Pr(s_{t+1}|s^t)}{\pi_{t+1}}\left[U_1\left(c_{t+1}^T, m_{t+1}^T\right) + U_2\left(c_{t+1}^T, m_{t+1}^T\right)\right] \end{aligned}$$

where $\Pr(s_{t+1}|s^t)$ is the probability of state s_{t+1} conditional on the history s^t. Combining this first order condition with the Euler equation for n_t^T we obtain:

$$Q_t = \frac{U_1\left(c_t^T, m_t^T\right)}{U_1\left(c_t^T, m_t^T\right) + U_2\left(c_t^T, m_t^T\right)}, \tag{11}$$

or, by letting r_t be the nominal interest rate, $1 + r_t \equiv Q_t^{-1}$, we can write:

$$r_t = \frac{U_2\left(c_t^T, m_t^T\right)}{U_1\left(c_t^T, m_t^T\right)} \; .$$

Notice that, given the timing assumptions for the bond market, interest rates are functions of the time t allocation, i.e. they do not involve any expected future values. For instance consider the the utility function

$$U\left(c, m\right) = \frac{h\left(c, m\right)^{1-\gamma} - 1}{1-\gamma}, \quad \text{where } h\left(c, m\right) = \left[c^{-\rho} + \mathcal{A}^{-1} m^{-\rho}\right]^{-\frac{1}{\rho}}, \tag{12}$$

which has a constant elasticity of substitution $1/\left(1+\rho\right)$ between c and m, a constant intertemporal substitution elasticity $1/\gamma$ between the consumption-money bundles, and $\mathcal{A}$ is a parameter. This case yields the constant elasticity, unitary income, money demand:

$$r_t = \frac{U_2\left(c_t^T, m_t^T\right)}{U_1\left(c_t^T, m_t^T\right)} = \frac{1}{\mathcal{A}}\left(\frac{m_t^T}{c_t^T}\right)^{-(1+\rho)} . \tag{13}$$

Finally, market clearing for Arrow securities and bonds, under the assumption that only traders participate in these markets, is:

$$b_t = \lambda \, b_t^T \quad \text{and} \quad a_t(s_{t+1}) = \lambda \, a_t^T(s_{t+1}) \quad , \; \forall s_{t+1} \tag{14}$$

for all $t \geq 0$. Next we give an equilibrium definition for the model with an active nominal bond market:

Definition 2. Given initial conditions $\left\{\tilde{M}_0^N, W_0\right\}$ and a monetary and fiscal policy described by stochastic processes $\left\{\tilde{\mu}_{t+1}, \tilde{\tau}_{t+1}^T, \tilde{\tau}_{t+1}^N, \tilde{b}_t, \tilde{a}_t\right\}_{t=0}^{\infty}$, *an Equilibrium with an active bond market* is an initial price level $\tilde{P}_0$, inflation rate process $\{\tilde{\pi}_{t+1}\}_{t=0}^{\infty}$, stochastic processes $\{\tilde{n}_t^i, \tilde{m}_t^i, \tilde{c}_t^i, \tilde{m}_t\}_{t=0}^{\infty}$ for $i = T, N$, and stochastic processes $\left\{w_t, b_t^T, a_t^T, q_t, Q_t\right\}_{t=0}^{\infty}$ that satisfy: the budget constraints for non-traders (2) and traders (7), identity (4), Euler equations for end-of-period cash balances (6), the government budget constraint (9), the f.o.c. for bonds and Arrow securities (10), and market clearing (3) and (14).

As in the equilibrium described in Definition 1, our convention for the initial conditions W_0, M_0^N include the time zero money injection, and so neither τ_0^i nor μ_0 are part of the definition. Instead B_0^T and M_0^T are choices for the traders, and hence bond prices Q_t are determined starting from period $t = 0$ on. The following proposition shows the sense in which the equilibrium with and without an active bond market are equivalent.

Proposition 1. Consider an equilibrium in the model without bond market: $\langle P_0, \{\pi_{t+1}\}_{t=0}^{\infty},$ $\{n_t^i, \; c_t^i, \; m_t^i, \; m_t\}_{t=0}^{\infty}\rangle$, for initial conditions $\{M_0^i\}$ and policy $\left\{\mu_{t+1}, \tau_{t+1}^T, \tau_{t+1}^N\right\}_{t=0}^{\infty}$ for $i = T, N$. Then, for any stochastic process of transfers to traders, $\{\tilde{\tau}_{t+1}^T\}_{t=0}^{\infty}$, there is an equilibrium with an active bond market that satisfies:

$$\left\langle \tilde{P}_0, \{\tilde{\pi}_{t+1}\}_{t=0}^{\infty}, \{\tilde{n}_t^i, \; \tilde{c}_t^i, \; \tilde{m}_t^i, \; \tilde{m}_t\}_{t=0}^{\infty} \right\rangle = \left\langle P_0, \{\pi_{t+1}\}_{t=0}^{\infty}, \{n_t^i, \; c_t^i, \; m_t^i, \; m_t\}_{t=0}^{\infty} \right\rangle \; ,$$

with bond and Arrow prices $\{Q_t, q_t\}_{t=0}^{\infty}$ given by equation (10) for all $t \geq 0$, the fiscal and monetary policy given by $\left\{\tilde{\tau}_t^N, \tilde{\mu}_t\right\}_{t=1}^{\infty} = \left\{\tau_t^N, \mu_t\right\}_{t=1}^{\infty}$, and $\{a_t, b_t\}_{t=1}^{\infty}$ satisfying (14) and:

$$\frac{a_{t-1} + b_{t-1}}{\tilde{\pi}_t} - Q_t b_t - \sum_{s_{t+1}} q_t(s_{t+1}) a_t(s_{t+1}) = \lambda \left(\tau_t^T - \tilde{\tau}_t^T\right) \ , \quad t \geq 1.$$

with initial conditions $\tilde{M}_0^N = M_0^N, \quad W_0 = M_0^T - Q_0(a_0 + b_0)/\lambda$, where $a_0 + b_0$ satisfies an appropriately chosen present value.

The proposition shows that it is only the combination of monetary and fiscal policy that matters. We use this proposition to analyze an equilibrium where all monetary injections are carried out through open market operations. To see this, first consider an equilibrium without an active bond market and where $\tau_t^N = 0$. In this case the budget constraint of the government is:

$$M_t - M_{t-1} = \lambda P_t \tau_t^T \quad .$$

Then, using the previous proposition, we can construct an equilibrium where $\tilde{\tau}_t^T = 0$ for all $t \geq 0$. The government budget constraint is:

$$\sum_{s_{t+1}} q_t(s_{t+1}) A_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} \ ,$$

which in the case of a time varying but deterministic policy is:

$$Q_t B_t + M_t - M_{t-1} = B_{t-1} \ , \ t \geq 1 \quad \text{or} \quad \sum_{t=1}^{\infty} \left(M_t - M_{t-1}\right) \left(\Pi_{i=1}^{t} Q_i\right) = B_0 \quad .$$

In this equilibrium traders start with an initial value of government bonds that enables them to buy the present value of the future seigniorage. The equivalence of real allocations, in the equilibrium where transfers differ across agents and in the one where money injections are carried out through open market operations, illustrates the sense in which open market operations with segmented asset markets have redistributive effects.

In Section 5 we will consider a fiscal - monetary policy that is similar to the one described above. We will assume $\tau_t^N = \tilde{\tau}_t^N = \bar{\tau}$ and $\tilde{\tau}_t^T = 0$. In this case non-traders receive a transfer with constant real value $\bar{\tau}$ and the government budget constraint is:

$$\sum_{s_{t+1}} q_t(s_{t+1})A_t(s_{t+1}) + Q_t B_t + M_t - M_{t-1} = B_{t-1} + A_{t-1} + P_t (1-\lambda)\bar{\tau}$$

We will choose the value of $\bar{\tau}$ so that it coincides with the "average" seignorage in the economy, as described in Section 5. By doing so we make traders and non-traders completely symmetric, at least in some average long-run sense.[4]

3 Approximate aggregation with Segmented Markets

This section studies the determination of inflation in the model with segmentation, i.e. when $\lambda \in (0,1)$. Given the equivalence established by Proposition 1, the argument is developed using the simpler framework without an active bond market. In particular, we allow traders and non-traders to be subject to different arbitrary processes for $\left\{\tau_t^N\right\}_{t=1}^{\infty}$ and $\left\{\tau_t^T\right\}_{t=1}^{\infty}$. We show that, somewhat surprisingly, up to a linear approximation the relation between aggregate inverse velocity m_t/y and money growth rates $\{\mu_t\}_{t=1}^{\infty}$ is independent of the fraction of traders (λ), and hence that it is the same one obtained in a model with a representative agent where $\lambda = 1$.

We assume that both types of agents have the same real income, so that $y^T = y^N = y$. We define an approximate equilibrium by replacing the Euler Equations of each agent and the budget constraints by linear approximations around the values that correspond to the steady state of an aggregate model with $\bar{m}^i = \bar{m}$, $\bar{c}^i = \bar{c} = y$ for $i = T, N$, and constant money supply growth $\bar{\mu} = \bar{\pi}$, satisfying

$$\bar{\tau} = \bar{m}\left(\bar{\mu} - 1\right) \ , \ U_1\left(y, \bar{m}\right) = \frac{\beta}{\bar{\mu}}\left[U_1\left(y, \bar{m}\right) + U_2\left(y, \bar{m}\right)\right] \quad . \tag{15}$$

[4] We can preserve the symmetry in other ways too. For instance, we can let non-traders have a higher real income, i.e. we can let y^N be higher than y^T and assume that non-traders receive zero real lump sum transfer.

Let $\hat{x}_t \equiv x - \bar{x}$ denote the deviation of the variable x from its steady state value $\bar{x}$. The linearization of the Euler equation (6) gives:

$$\bar{U}_{11}\hat{c}_t^i + \bar{U}_{12}\hat{m}_t^i = \frac{\beta}{\bar{\pi}}\mathbb{E}_t\left[\left(\bar{U}_{11} + \bar{U}_{21}\right)\hat{c}_{t+1}^i + \left(\bar{U}_{12} + \bar{U}_{22}\right)\hat{m}_{t+1}^i\right] - \frac{\beta}{\bar{\pi}^2}\mathbb{E}_t\left[\bar{U}_1 + \bar{U}_2\right]\hat{\pi}_{t+1} \quad (16)$$

The linearization of the identity in (4), and the budget constraints in (2) and (5) gives:

$$\hat{\pi}_t = \hat{\mu}_t - \frac{\bar{\mu}}{\bar{m}}\left(\hat{m}_t - \hat{m}_{t-1}\right), \quad \hat{m}_t^i = \frac{\hat{n}_{t-1}^i}{\bar{\pi}} - \frac{\bar{n}}{\bar{\pi}^2}\hat{\pi}_t + \hat{\tau}_t^i, \quad \hat{m}_t - \frac{\hat{m}_{t-1}}{\bar{\pi}} + \frac{\bar{m}}{\bar{\pi}^2}\hat{\pi}_t = \lambda\hat{\tau}_t^T + (1-\lambda)\hat{\tau}_t^N, \quad (17)$$

for $i = N, T$. We are now ready to define an approximate equilibrium.

Definition 3. Given initial conditions $\{M_0^i\}$ and a fiscal and monetary policy described by $\left\{\mu_{t+1}, \tau_{t+1}^i\right\}_{t=0}^{\infty}$ an *Approximate Equilibrium* is given by $\{n_t^i, m_t^i, c_t^i, m_t\}_{t=0}^{\infty}$ for $i = T, N$, and $P_0, \{\pi_{t+1}\}_{t=0}^{\infty}$ that satisfy: market clearing (3), the linearized Euler equation (16), and the linearized constraints (17).

Using this definition we state our main result on aggregation:

Proposition 2. In an approximate equilibrium the processes for aggregate real balances and inflation $\{m_t, \pi_{t+1}\}_{t=0}^{\infty}$ and the initial price level P_0 are the same for any $\lambda \in [0, 1]$.

The proof of this result is simple: in the equilibrium without active bond markets traders and non-traders optimal decisions are characterized by the *same* Euler equation, evaluated at different shocks for $\{\tau_t^i\}$, and the same inflation process. Linearizing these equations, and using market clearing, one obtains the aggregation result. This result is important for two reasons. First, substantively, it says that the relation between inflation and money growth, to a first order approximation, is independent of the fraction of traders λ. Second, it shows that the equilibrium has a recursive nature. One can determine first the path of aggregate real balances obtaining the process for inflation, as we do in Section 4, and then solve for the decision problem of the non-trader, obtaining the process for the non-trader consumption and real balances. Using feasibility and the process for aggregate real balances, one can finally solve for the traders' real balances and consumption, which in turns gives us the interest rate from equation (11). Since the problem of the non-trader is a key intermediate step to determine the behavior of interest rates, Section 5 analyzes it in detail.

4 Velocity and Money Growth

In this section we consider a model with one type of agent, or $\lambda = 1$, to obtain a description of inverse velocity and inflation as functions of future expected money growth rates, as in the representative-agent model of Sidrauski, or in Cagan's.[5] Our interest in the setup with $\lambda = 1$ comes from Proposition 2, which shows that the equilibrium path for aggregate inverse velocity and inflation is the same irrespective of $\lambda \in [0,\ 1]$.

Using market clearing ($c_t = y$) into the f.o.c. for m, and the inflation identity $\pi_{t+1} = \mu_{t+1}\ m_t/m_{t+1}$ we can write

$$U_1\left(y, m_t\right) m_t = \mathbb{E}_t \left\{ \frac{\beta}{\mu_{t+1}} \left[U_1\left(y, m_{t+1}\right) + U_2\left(y, m_{t+1}\right)\right] m_{t+1} \right\} . \tag{18}$$

Our next task is to analyze the behavior of this system. We first consider the steady state, the case where money supply grows at a constant rate $\bar{\mu}$ and $\bar{r}$ is the net interest rate that corresponds to a constant money growth rate and inflation $\bar{\mu}$:

$$\frac{U_2\left(y, \bar{m}\right)}{U_1\left(y, \bar{m}\right)} = \bar{r} = \frac{\bar{\mu}}{\beta} - 1 . \tag{19}$$

As in Lucas (2000) we interpret the function $\bar{m}$ of $\bar{r} = \bar{\mu}/\beta - 1$, solving equation (19), as the "long run" money demand. For the case where U is given by equation (12), this money demand has a constant interest rate elasticity $1/(1 + \rho)$.

In what follows we analyze a linearized version of the difference equation (18), expanded around a constant $\mu = \bar{\mu}$ and $m = \bar{m}$. We seek a solution for real balances as a function of

[5] Equivalently, we consider a set-up where traders and non-traders are identical, i.e. where $y^T = y^N = y,\ \tau_t^N = \tau_t^T$ for all $t \geq 0$ with initial conditions $M_0^T = M_0^N = M_0$ for any λ.

the future expected growth of the money supply.

Proposition 3. Let $\hat{m}_t \equiv m_t - \bar{m}$, $\hat{\mu}_t \equiv \mu_t - \bar{\mu}$. Linearizing (18) around (19) we have:

$$\hat{m}_t = \alpha \mathbb{E}_t\left[\hat{\mu}_{t+1}\right] + \phi \mathbb{E}_t\left[\hat{m}_{t+1}\right] \qquad \text{where} \tag{20}$$

$$\alpha \equiv -\frac{\bar{m}}{\bar{\mu}}\,\frac{\bar{U}_1}{\bar{U}_1 + \bar{m}\bar{U}_{12}} \quad \text{and} \quad \phi \equiv \frac{\beta}{\bar{\mu}}\left[1 + \frac{\bar{U}_2 + \bar{m}\bar{U}_{22}}{\bar{U}_1 + \bar{m}\bar{U}_{12}}\right] \tag{21}$$

and where $\bar{U}_i$, $\bar{U}_{ij}$ are the derivatives of $U(\cdot)$ evaluated at $(y, \bar{m})$. With $0 < \phi < 1$ we can express its unique bounded solution as

$$\hat{m}_t = \alpha \sum_{i=1}^{\infty} \phi^{i-1} \mathbb{E}_t\left[\hat{\mu}_{t+i}\right] \quad t \geq 0\ . \tag{22}$$

Thus, if $0 < \phi < 1$ and $\alpha < 0$, future expected money growth reduces current real money balances. We briefly discuss sufficient conditions for this configuration. Equation (21) shows that the condition for $\alpha < 0$ requires that $U_1 + mU_{12} > 0$, which is always the case if $U_{12} > 0$. The assumption of $U_{12} > 0$ has the interpretation that the durable good is a complement to the consumption of non-durables, and will be maintained for the rest of the paper. Notice that $-\alpha(\bar{\mu}/\bar{m})$ is decreasing in $U_{12} > 0$, starting from a value of 1 at $U_{12} = 0$.

When U is given by equation (12) the requirements for ϕ can be written in terms of conditions on: γ, ρ, $\bar{r}$ and $\bar{m}/y$. Lemma 1 in Appendix A.4 shows that when $\rho > -1$, a sufficient condition for $\phi < 1$ and $\alpha < 0$ is

$$\gamma < 2 + \rho + \frac{y}{\bar{r} \cdot \bar{m}} \tag{23}$$

(otherwise if $\rho = -1$, then $\phi = 1$). Notice that condition (23) holds for a wide range of parameters of interest.[6] For instance, with an annual nominal interest rate of 4 percent,

[6]Note that when U is given by (12) a sufficient condition for $U_{12} > 0$ is that the intertemporal elasticity of substitution, $1/\gamma$, is higher than the intratemporal elasticity of substitution between c and m, given by

annual money-income ratio of 1/4, and an elasticity of substitution between consumption and real balances of 1/2, so that $\bar{r} = 0.04$, $\bar{m}/y = 1/4$ and $\rho = 1$, then γ has to be smaller than 103, a condition that is easily satisfied by any reasonable estimate of the risk aversion parameter γ.

Lemma 1 also shows that the condition $0 < \phi$, which ensures monotone dynamics, is:

$$\gamma < \frac{1}{1+\bar{r}} \left(2 + \rho + \frac{y}{\bar{r}\ \bar{m}} + \bar{r} - \frac{\rho\ y}{\bar{m}} \right) \quad .$$

This inequality is implied by equation (23) as long as the length of a time period is sufficiently small.[7]

4.1 Linear State Space Representation for Velocity and Inflation

We specify a linear time series process for $\{\mu_t\}$ and rewrite the initial conditions exclusively in terms of real variables. We start with a representation for inflation as a function of future money growth rates and the initial aggregate money balances. Using (17) and (22) we obtain

$$\hat{\pi}_t = \hat{\mu}_t + \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t-1} - \frac{\bar{\mu}}{\bar{m}} \alpha \sum_{i=1}^{\infty} \phi^{i-1} \mathbb{E}_t \left[\hat{\mu}_{t+i} \right] \ , \quad t \geq 0 \ . \tag{24}$$

Notice that equation (17) is defined for $\pi_0 \equiv P_0/P_{-1}$ and μ_0. This representation avoids us to carry a nominal level variable, such as M_0, as the initial state. Instead, the initial state is the real level of $\hat{m}_{-1}$.

We will consider the case where we can write the detrended growth of money supply $\hat{\mu}_t$ as a function of an exogenous state z_t :

$$\hat{\mu}_{t+1} = \nu \ z_{t+1} \quad , \quad z_{t+1} = \Theta z_t + \epsilon_{t+1} \tag{25}$$

$1/(1+\rho)$.

[7]Note, by contrast, that inequality (23) is independent of the model's time period since the term $y/(\bar{r}\ \bar{m})$ contains the ratio of two flows $(\bar{r}/y)$, and the rest of the terms are parameters independent of time. We return to this issue in the discussion of Proposition 5.

for $t \geq 0$, where z_0 is given, ν is a $k \times 1$ vector, Θ is a $k \times k$ matrix with k stables eigenvalues, and ϵ_{t+1} is a $k \times 1$ vector of innovations. In this case

$$\hat{m}_t \;\; = \;\; \alpha \sum_{i=1}^{\infty} \phi^{i-1} \mathbb{E}_t \left[\hat{\mu}_{t+i} \right] = \alpha \; \nu \; \Theta \left[I - \phi \Theta \right]^{-1} \; z_t \; , \; t \geq 0 \; .$$

Replacing (25) into (24) we obtain:

$$\hat{\pi}_t = \nu \; z_t + \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t-1} - \alpha \; \frac{\bar{\mu}}{\bar{m}} \nu \; \Theta \left[I - \phi \Theta \right]^{-1} z_t \; , \quad t \geq 0 \; .$$

For example if z_t follows the AR(1) process $z_{t+1} = \theta z_t + \epsilon_{t+1}$ and $\nu = 1$, then $k = 1$, $\hat{\mu}_t = z_t$, and $\Theta = \theta$, so that for $t \geq 0$:

$$\hat{m}_t = \alpha \; \frac{\theta}{1 - \phi \theta} \; \hat{\mu}_t \quad , \quad \hat{\pi}_t = \frac{\bar{\mu}}{\bar{m}} \hat{m}_{t-1} + \left[1 - \alpha \; \frac{\bar{\mu}}{\bar{m}} \frac{\theta}{1 - \phi \theta} \right] \hat{\mu}_t \; ,$$

given $\hat{m}_{-1}$. Recall that $\alpha < 0$ if (23) holds, so that real balances are decreasing in $\hat{\mu}_t$ and hence inflation increases more than one for one with $\hat{\mu}_t$. This is a well known feature of the standard variable-velocity model.

We summarize the linear equilibrium representation for the aggregate economy as a function of the innovations $\{\epsilon_t\}_{t=1}^{\infty}$, the parameters $\{\bar{\mu}/\bar{m}, \; \alpha, \; \phi, \; \nu, \; \Theta\}$, and initial conditions z_0 and $\hat{m}_{-1}$ as follows:

$$\hat{m}_t = \kappa \; z_t \quad , \quad \hat{\pi}_t = \left(\frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta \; z_t \quad , \quad z_{t+1} = \Theta \; z_t + \epsilon_{t+1} \; , \tag{26}$$

for all $t \geq 0$, where the vectors ζ and κ are given by:

$$\zeta \equiv \nu \; \left(I - \alpha \frac{\bar{\mu}}{\bar{m}} \; \Theta \left[I - \phi \Theta \right]^{-1} \right) \quad , \quad \kappa \equiv \; \alpha \; \nu \; \Theta \left[I - \phi \Theta \right]^{-1} \; . \tag{27}$$

For future reference we also produce a formula for expected inflation:

$$\mathbb{E}_t\left[\hat{\pi}_{t+j}\right] = \bar{\Pi}\ \Theta^{j-1}\ z_t, \quad \text{where } \bar{\Pi} \equiv \frac{\bar{\mu}}{\bar{m}}\ \kappa + \zeta\Theta. \tag{28}$$

As a special case notice that if $\nu = 1$ and $\Theta = 0$, so that $\hat{\mu}_t$ is i.i.d., we have that real money balances are constant ($\kappa = 0$), inflation is equal to money growth ($\zeta = 1$), and hence expected inflation is constant ($\bar{\Pi} = 0$), or:

$$\kappa = \bar{\Pi} = 0, \text{ and } \zeta = 1\ . \tag{29}$$

Finally we examine the behavior of interest rates in the case of $\lambda = 1$ for the utility function in (12). Denoting $\hat{r}_t = r_t - \bar{r}$ and linearizing equation (13) we have: $\hat{r}_t = -\left(1+\rho\right)\left(\bar{r}/\bar{m}\right)\hat{m}_t$. Replacing $\hat{m}_t$ by (26) we have:

$$\hat{r}_t = -\left(1+\rho\right)\frac{\bar{r}}{\bar{m}}\ \kappa\ z_t\ , \quad t \geq 0\ . \tag{30}$$

For instance, in the case where $z_{t+1} = \theta z_t + \epsilon_{t+1}$ and $\nu = 1$ we have

$$\hat{r}_t = \left(1+\rho\right)\left(-\alpha\frac{\bar{r}}{\bar{m}}\right)\frac{\theta}{1-\phi\theta}\ \hat{\mu}_t\ . \tag{31}$$

Since $\alpha < 0$ under our maintained assumptions, interest rates move in the same direction than $\hat{\mu}_t$, and hence there is *no liquidity effect in the standard model.* Additionally, nominal interest rates inherit the persistence of $\hat{\mu}_t$: in the case where $\theta = 0$, so that $\hat{\mu}_t$ is i.i.d., then the nominal interest rate is constant.

5 Interest Rates with Segmented Markets

This section analyzes interest rates for the following fiscal-monetary policy. We consider a steady state, i.e a value of $\bar{m}$ for the aggregate balances that corresponds to a constant money growth rate $\bar{\mu}$ (the unconditional mean of the process for money growth). We set the fiscal policy as follows: $\tau_t^N = \bar{\tau}$, and $\tau_t^T = 0$, and endow the traders with an initial bond position that allows them to buy the seignorage not allocated to the non-traders, as outlined in Section 2.1. In the absence of shocks −"in a steady state"− traders and non-traders are symmetric. Yet when there are shocks, traders must absorb all innovations to the money supply.

While our focus is on interest rates, for simplicity we analyze the equilibrium where there is no active bond markets, which in Section 2.1 was shown to be equivalent to one where all the money injections are carried out through open market operations. In particular, as explained above, the nature of the equilibrium is recursive: we first solve for the process for inflation in the aggregate model, then we solve for the non-trader's problem −which is done in the next subsection− and finally, using the equivalence of the allocations with and without an active bond market, we solve for interest rates.

5.1 The Non-Trader problem

In this section we consider the problem of a non-trader choosing $\{n_t\}_{t=0}^{\infty}$, facing a constant real lump sum transfer τ^N, a given process for inflation $\{\pi_{t+1}\}_{t=0}^{\infty}$ and given initial condition $m_0 = n_{-1}/\pi_0 + \tau^N$. We simplify the notation in this section by dropping the superindex N from n, c, and m, whenever it is clear from the context.

We start by studying the non-trader problem assuming inflation is constant at $\bar{\pi} \geq 1$. In this case the state of the problem is given simply by n_{t-1}. We solve for the optimal decision rule $g(\cdot)$, that gives $n_t = g(n_{t-1})$, and find conditions under which it has a unique steady state $\bar{n} = g(\bar{n})$ that is globally stable. Furthermore we characterize the local dynamics of this problem, i.e. the value of $g'(\bar{n})$. In the second part of this section we use these results to characterize the solution of the linearized Euler equation when inflation follows an arbitrary process.

The next proposition uses the Bellman equation for the non-trader problem with $\bar{\pi} > 1$:

$$V(n) = \max_{0 \leq \tilde{n} \leq y + \tau^N + n/\bar{\pi}} \left\{ U\left(y + \tau^N + n/\bar{\pi} - \tilde{n},\ n/\bar{\pi} + \tau^N\right) + \beta V(\tilde{n}) \right\}. \tag{32}$$

to characterize the policy function $\tilde{n} = g(n)$, and the uniqueness and stability of the steady state. The proposition also characterizes the value of $g'(\bar{n})$, which is important to determine the speed of convergence to steady state for the non-trader problem.

Proposition 4. Assume $\bar{\pi} > 1$, $\tau^N > 0$, that U is strictly concave and bounded above, and $0 < U_{12} < -U_{11}$. Then the function $g(n)$ is strictly increasing, it has a unique interior steady state $\bar{n} = g(\bar{n})$ that is globally stable, with $0 < g'(\bar{n}) < 1$.

Using the the decision rule $g(\cdot)$ and the budget constraint, we can define the optimal consumption rule. For future reference, it turns out to be more convenient to use m (real cash balances after the transfer) as the state. The budget constraint of the agent is given by $m = n/\pi + \tau^N$ and $m + y = c + g(n)$. Thus $c(m) \equiv m + y - g\left(\left[m - \tau^N\right]\pi\right)$ and $c'(m)$ is given by $1 - \pi\ g'(n)$. The elasticity of the ratio $m/c(m)$ with respect to m is:

$$\chi(m) \equiv \frac{m}{m/c(m)} \frac{\partial(m/c(m))}{\partial m} = 1 - \frac{m}{c(m)} \frac{\partial c(m)}{\partial m} = 1 - \frac{m}{c(m)} \left(1 - \pi\ g'(n)\right) . \tag{33}$$

We are interested in this elasticity because the interest rate response to money shocks depends on the changes of the m/c ratio. In particular, χ determines the value of the impact effect and $g'(\bar{n})$ the half-life of the impulse response of interest rates to money shocks. As the nominal interest rate is proportional to the m/c ratio, see e.g. equation (13), a zero value of χ implies no liquidity effect, positive values imply a liquidity effect.

In the next proposition we specialize the utility function to U CRRA and h CES, as described in equation (12), and characterize the slope $g'(\bar{n})$ and the elasticity $\chi(\bar{m})$.

Proposition 5. Assume U is given by equation (12), $0 < U_{12} < -U_{11}$ and $\bar{\pi} > 1$. For any values of the triplet ρ, $r = \bar{\pi}/\beta - 1$ and $\bar{m}/\bar{c} > 1$, let $\mathcal{A}$ be such that $r = U_2/U_1$ evaluated at

$\bar{m}/\bar{c}$. Then $g'(\bar{n})$ and $\chi(\bar{m})$ depend only on $\bar{m}/\bar{c}$, $\gamma/(1+\rho)$, $\bar{\pi}$, β. Moreover:

$$0 < g'(\bar{n}) < 1 \quad , \quad 0 \leq \chi(\bar{m})$$

and $g'(\bar{n})$ and $\chi(\bar{m})$ are increasing in the ratio $\gamma/(1+\rho)$.

Recall that $1/\gamma$ is the intertemporal elasticity of substitution of the bundle h, and that $1/(1+\rho)$ is the intratemporal elasticity of substitution between cash balances and consumption. Proposition 5 shows that, somewhat surprisingly, the values of $g'(\bar{n})$ and $\chi(\bar{m})$ depend on the ratio between the elasticities, $\gamma/(1+\rho)$, as opposed to the values of γ and $1+\rho$ separately. The proposition establishes that χ, the elasticity of m/c, is increasing in the ratio: $\gamma/(1+\rho)$. Intuitively, a smaller value of χ indicates that m/c is stickier to adjust, and hence the liquidity effect is larger. For instance, $\chi = 0$ is zero when $\gamma = 0$: in this case the ratio m/c remains constant, i.e. the adjustment occurs along the inter-temporal margin, which has an infinite substitution elasticity. Analogously $\chi = 0$ when $1/(1+\rho) = 0$, since in this case the intra-temporal substitution elasticity between m and c is nil, and no liquidity effect arises.

This result can be understood by considering the behavior of an agent that starts with cash balances below the steady state $\bar{m}$. To reach the steady state the agent must reduce consumption. If the reduction in consumption is large, then convergence to the steady state is fast, and the liquidity effect is short lived. To see why this decision depends on $\gamma/(1+\rho)$, let us consider a case where the convergence to steady state is fast, i.e. $g' \approx 0$. According to Proposition 5 this can happen either because of a small γ or a high $1+\rho$. Let us consider each of these possibilities in turn.

Consider a case where $\gamma > 0$, but cash and consumption are poor substitutes, so that $1/(1+\rho) \approx 0$ and $\gamma/(1+\rho) \approx 0$. In this case, the agent would like to keep m/c almost constant, which implies that $\chi \cong 0$, and since the initial decrease in consumption is large then g' must be small. Alternatively, a similar behavior is produced if the intertemporal substitution elasticity is high. Let $1/(1+\rho) > 0$ and assume that the bundles are very close substitutes intertemporally, so that $1/\gamma \approx \infty$ and $\gamma/(1+\rho) \approx 0$. Since the agent substitutes intertemporally very easily then the speed of convergence is high, or g' is small, and thus it must be that the initial drop in consumption is large, or $\chi \cong 0$.

These examples illustrate why only the ratio matters. To see why g' (hence χ) is increasing in $\gamma/(1+\rho)$ consider now the case where γ is large, so the agent dislikes large variations in the bundle h. In this case, the initial drop in m will be partly compensated by not decreasing c too much, and hence $\chi > 0$, and it can be close to one. This also implies that the adjustment to steady state will take more time, or that g' is high.

We conclude this part with two comments on the role of the steady state value of $\bar{m}/\bar{c}$ in Proposition 5. First, as a technical comment notice that the condition that $\bar{m}/\bar{c} > 1$ is only sufficient, but not necessary, for $0 < g'(\bar{n}) < 1$.[8] Second, and more importantly, the result on the dependence of the speed of convergence on the ratio m/c is less standard, but it should be clear in this context. If the stock of money is very small relative to consumption, the effect of starting with a value of this stock below steady state can be quickly corrected: if $\bar{m}/\bar{c} = 1$ then $g'(\bar{n}) = 0$, so the steady state is attained immediately. In other words, if the stock m is small relative to the flow c, it must be that the length of the model period is so big that it makes the analysis of convergence uninteresting. While this property was derived for a money-in-the-utility function, we think that inventory theoretical models on the lines of Baumol-Tobin, Miller and Orr, Alvarez and Lippi (2009), must display similar behavior. Indeed, in Appendix AA-1 we analyze the continuous time version of the non-trader problem, that deals more naturally with the stock/flow distinction. In this case the corresponding results of Proposition 5 holds for any value of $\bar{m}/\bar{c}$.

So far we have analyzed the problem for a non-trader when inflation is constant. Now we move to the problem of the non-trader facing the (linearized) equilibrium process for

[8]The following case obtains $0 < g'(\bar{n}) < 1$, without requiring $\bar{m}/\bar{c} > 1$. Notice that Proposition 4 holds if $\gamma > 1$, since in this case the utility is bounded above, and it only requires that in addition $U_{11} + U_{12} < 0$, which is weaker than $\bar{m}/\bar{c} > 1$.

inflation. We use the steady state $\bar{n}$ to define $\hat{n}_t$, the deviations of the end-of-period real cash balances $\hat{n}_t \equiv n_t - \bar{n}$. Notice that a bounded process $\{n_t\}_{t=0}^{\infty}$ satisfying the Euler equation (6) is a solution to the non-trader's problem. Now we are ready to state a characterization of the linearized solution to the non-traders' Euler equation. Replacing the budget constraint (2) into the Euler equation (6) for non-traders with constant lump sum transfers $\tau_t^N = \tau^N$, and linearizing with respect to (c, m, π) around the values $(\bar{c}, \bar{m}, \bar{\pi})$ we obtain:

$$\mathbb{E}_t\left[\hat{n}_{t+1}\right] = \xi_0 \hat{n}_t - \frac{1}{\beta}\hat{n}_{t-1} + \xi_1 \hat{\pi}_t + \xi_2 \mathbb{E}_t\left[\hat{\pi}_{t+1}\right]. \tag{34}$$

where the coefficients ξ_i are functions of the second derivatives of U evaluated at the steady state as well as β and $\bar{\pi}$, given by equation (A-10) in Appendix A.7. The next proposition assumes that inflation is governed by the linearized equilibrium described in Section 4:

Assumption 1. The deviation of inflation and real balances from their steady state value $\{\hat{\pi}_t, \hat{m}_t\}_{t=0}^{\infty}$ are given by the linear stochastic difference equation with exogenous driving shocks $\{z_{t+1}\}_{t=0}^{\infty}$ described by the matrix and vectors $\{\Theta, \zeta, \kappa\}$ as detailed in equation (26). They imply that $\mathbb{E}_t\left[\hat{\pi}_{t+1}\right] = \bar{\Pi}\ z_t$ as given in equation (28).

The state for the dynamic program of the non-trader problem is given by the last period real balances $\hat{n}_{t-1}$, and the variables needed to forecast inflation, which given the linear representation in Section 4, are $(z_t, \hat{m}_{t-1})$. The next proposition characterizes the solution $\hat{n}_t = \hat{g}\left(\hat{n}_{t-1}, z_t, \hat{m}_{t-1}\right)$ for the linearized Euler equation.

Proposition 6. Assume that U is bounded from above, that $0 < U_{12} < -U_{11}$, and that the deviation of inflation and real balances from their steady state value $\{\hat{\pi}_t, \hat{m}_t\}_{t=0}^{\infty}$ are given by Assumption 1. The unique bounded solution of the linearized Euler equation is given by:

$$\hat{n}_t = \hat{g}\left(\hat{n}_{t-1},\ z_t,\ \hat{m}_{t-1}\right) = \varphi_0\ \hat{n}_{t-1} + \varphi_1\ z_t + \varphi_2 \hat{m}_{t-1} \tag{35}$$

where the φ_0 coefficient satisfies: $0 < \varphi_0 = g'\left(\bar{n}\right) < 1$, and where φ_i are functions of the coefficients ξ_i of the linearization of the Euler equation, the parameters $\beta, \bar{\pi}, \ \bar{m}$, and the coefficients describing the (linearized) equilibrium law of motion $\kappa, \ \Theta, \ \zeta$ and $\bar{\Pi}$ are given by equation (A-10) in Appendix A.7.

The result that the slope of the linear optimal policy $\hat{g}(\cdot)$ that solves the linearized Euler $\partial \hat{g}/\partial \hat{n} \equiv \varphi_0$ is the same as the slope at the steady state of the optimal decision rule for the non-linear problem $g'(\bar{n})$, is a standard one. As it is standard, φ_0 is the (stable) solution of a quadratic equation with coefficients defined by ξ_0 and β. Recall that if the growth rate of money $\hat{\mu}_t$ is i.i.d. real balances are constant (see equation (29)), inflation is i.i.d., and hence expected inflation is constant or: $\Theta = \kappa = \bar{\Pi} = 0, \quad \zeta = 1$. In this case, using our notation for $\hat{g}(\cdot)$ and the expressions (A-10) for the coefficients φ_i in Appendix A.7 we obtain that

$$\varphi_1 = -\beta \varphi_0 \xi_1 = -\varphi_0 \frac{\bar{n}}{\bar{\pi}}$$

so that for all $t \geq 0$, we have $\hat{m}_t = 0$, and thus $\hat{n}_t = \varphi_0 \ \left(\hat{n}_{t-1} - \frac{\bar{n}}{\bar{\pi}} \hat{\mu}_t\right) \ + \varphi_2 \hat{m}_{t-1}$. If the economy starts with $\hat{m}_{-1} = 0$:

$$\hat{n}_t = g'(\bar{n}) \ \hat{n}_{t-1} - g'(\bar{n}) \frac{\bar{n}}{\bar{\pi}} \hat{\mu}_t$$

so $\hat{n}_t$ follows an autoregressive process of order one, with parameter $g'(\bar{n})$, and innovations that are proportional to inflation $\hat{\pi}_t = \hat{\mu}_t$ with a (negative) coefficient given by $-g'(\bar{n})\, \bar{n}/\bar{\pi}$.

5.2 Interest Rates in a Linearized Equilibrium

In this section we use the aggregation result of Section 3, the inflation dynamics of Section 4 and the characterization of the non-trader's dynamic problem of Section 5.1 to solve for the effect of open market operations on interest rates.

We are interested in the following particular monetary-fiscal policy. Non-traders receive a constant real transfer per period, equal to the steady state value of seigniorage. Traders receive the remaining part of the seigniorage. The deviation from the steady state growth of money supply evolves according to $\hat{\mu}_t = \nu\, z_t$, for an exogenous process z_t as described in (25). Equivalently, as shown in Section 2.1, we can regard this equilibrium as one in which non-traders receive a constant real tax rebate τ_t^N and traders receive no lump sum rebate but participate in open market operations. Returning to the equilibrium without an active bond market, the values of τ_t^N and τ_t^T are given as follows. Let $\bar{m}\,(\bar{\mu}-1)/\bar{\mu}$ be the average seigniorage,

$$\tau_t^N = \frac{\bar{m}\,(\bar{\mu}-1)}{\bar{\mu}} \quad \text{and} \quad \tau_t^T = \frac{1}{\lambda}\left[\frac{M_t - M_{t-1}}{P_t}\right] - \frac{(1-\lambda)}{\lambda}\frac{\bar{m}\,(\bar{\mu}-1)}{\bar{\mu}}.$$

where $\bar{m}$ solves $\frac{U_2(y,\bar{m})}{U_1(y,\bar{m})} = \bar{r} \equiv \frac{\bar{\mu}}{\beta} - 1$. In steady state (i.e. when $\mu_t = \bar{\mu}$ all t), the value of τ_t^T is also constant, and hence $\tau_t^N = \tau_t^T = \bar{m}\,(\bar{\mu}-1)\,/\bar{\mu}$. It is straightforward to verify that these choices satisfy the government budget constraint (5). Also it is easy to verify that with these choices for τ_t^N and τ_t^T if M_t grows at a constant rate $\bar{\mu}$, then traders and non-traders will have the same consumption and money holdings, $\bar{m}^T = \bar{m}^N = \bar{m}$ and $\bar{c}^T = \bar{c}^N = y$.

Now we turn to the determination of the path of interest rates. To do so, let's use a first order approximation around $\mu_t = \pi_t = \bar{\mu}$ and $m_t = m_t^T = m_t^N = \bar{m}$, $c_t^T = c_t^N = y$ where $\hat{r}_t = r_t - \bar{r}$. Linearizing the f.o.c. of the traders (11) with respect to c^T and m^T, replacing $\hat{c}_t^T$, $\hat{m}_t^T$ using market clearing for goods and money (3) to write the resulting expression in terms of $\hat{c}_t^N$, $\hat{m}_t^N$, and using that the elasticity of substitution between m and c is given by $1/\,(1+\rho)$, we obtain

$$\frac{\hat{r}_t}{\bar{r}} = -\,(1+\rho)\left(\frac{\hat{m}_t^T}{\bar{m}} - \frac{\hat{c}_t^T}{\bar{c}}\right) = -\frac{(1+\rho)}{\lambda}\left(\frac{\hat{m}_t}{\bar{m}} - (1-\lambda)\left(\frac{\hat{m}_t^N}{\bar{m}} - \frac{\hat{c}_t^N}{\bar{c}}\right)\right)$$

Finally we use this equation to solve for interest rates as follows. The term $\hat{m}_t$ is determined by the equilibrium in the aggregate economy, i.e. $\hat{m}_t = \kappa\ z_t$. The terms $\hat{m}_t^N$, $\hat{c}_t^N$ are determined by the solution of the non-trader problem. Using the budget constraint of the non-trader we have $m_t^N = n_{t-1}^N/\pi_t + \tau^N$ and $c_t^N = y + n_{t-1}^N/\pi_t + \tau^N - n_t^N$. Linearizing these expressions gives

$$\frac{\hat{m}_t^N}{\bar{m}} - \frac{\hat{c}_t^N}{\bar{c}} = \hat{n}_{t-1}\frac{1}{\bar{\pi}}\left(\frac{1}{\bar{m}} - \frac{1}{y}\right) + \frac{\bar{n}}{\bar{\pi}^2}\left(\frac{1}{y} - \frac{1}{\bar{m}}\right)\hat{\pi}_t - \frac{1}{y}\hat{n}_t \ .$$

Using the decision rule of non-traders: $\hat{n}_t = \varphi_0 \ \hat{n}_{t-1} + \varphi_1 \ z_t + \varphi_2 \ \hat{m}_{t-1}$, to replace $\hat{n}_t$, and that inflation dynamics are given by $\hat{\pi}_t = \left(\frac{\bar{\mu}}{\bar{m}}\right)\hat{m}_{t-1} + \zeta \ z_t$ we obtain:

$$\begin{aligned} \frac{\hat{r}_t}{\bar{r}} \quad = \quad & \frac{(1+\rho)}{\bar{\pi}}\frac{(1-\lambda)}{\lambda}\left[\frac{1}{\bar{m}} - \frac{1}{y} + \frac{\bar{\pi}\varphi_0}{y}\right]\hat{n}_{t-1} \\ & + \frac{(1+\rho)}{\lambda}\left(\frac{(1-\lambda)}{\bar{\pi}}\left[\frac{\bar{n}}{\bar{\pi}}\left(\frac{1}{y} - \frac{1}{\bar{m}}\right)\zeta + \frac{\bar{\pi}\varphi_1}{y}\right] - \frac{\kappa}{\bar{m}}\right) \ z_t \\ & - (1+\rho)\frac{(1-\lambda)}{\lambda}\left[\frac{\bar{n}}{\bar{\pi}^2}\left(\frac{1}{y} - \frac{1}{\bar{m}}\right)\left(\frac{\bar{\mu}}{\bar{m}}\right) + \frac{\varphi_2}{y}\right]\hat{m}_{t-1} \ . \end{aligned} \tag{36}$$

The interest rate is a function of $(\hat{n}_{t-1}, \ z_t, \ \hat{m}_{t-1})$. The variables $\hat{m}_{t-1}$ and $\hat{n}_{t-1}$ encode the distributional effects between traders and non-traders. Indeed, as $\lambda \to 1$, the expression for the interest rate converges to equation (30) and, as already noticed, there is no liquidity effect.

5.2.1 Unexpected once-and-for-all increase in the money supply

This section studies the impulse-response of nominal interest rates when the growth rate of money supply follows an iid process. Equivalently, we analyze the effect of starting the system at the steady state corresponding to $\bar{\mu}$ and then shock it with an unexpected transitory one time increase in the growth rate of the money supply at t, i.e., $\mu_t > \bar{\mu}$ and $\mu_{t+s} = \bar{\mu}$ for all $s \geq 1$, i.e. a once and for all permanent increase in the level of the money supply.

Let the initial conditions $\hat{m}_{t-1} = \hat{n}_{t-1}^N = 0$, if $\hat{\mu}_t$ is iid we have that, as shown in equation (29), $\kappa = \Theta = 0, \ \nu = \zeta = 1$. This gives

$$\hat{\pi}_t = \hat{\mu}_t > 0, \quad \hat{\pi}_{t+s} = \hat{\mu}_{t+s} = 0 \ \text{ all } s \geq 1, \text{ and } \hat{m}_{t+s} = 0, \, s \geq 0.$$

For the non-traders we have that for all $s \geq 0$:

$$\hat{n}_{t+s} = \varphi_0 \ \hat{n}_{t+s-1} + \varphi_1 \hat{z}_{t+s} + \varphi_2 \hat{m}_{t+s-1} = \varphi_0 \ \hat{n}_{t+s-1} = [\varphi_0]^s \ \hat{n}_t.$$

Using $\bar{\pi} = \bar{\mu}$ and that for iid shocks $\varphi_1 = -\varphi_0 \frac{\bar{n}}{\bar{\pi}}$ we derive the following proposition, which assumes that $m/c > 1$ (a condition related to the choice of time units discussed in the comment to Proposition 5).

Proposition 7. The effect of an unexpected once and for all increase in the money supply at time t of size $(\mu_t - \bar{\mu}) / \bar{\mu}$ is to decrease interest rates on impact, and gradually return to the steady state value $\bar{r}$, according to

$$\frac{\hat{r}_{t+s}}{\bar{r}} = -\frac{(1-\lambda)\ (1+\rho)\ \bar{n}}{\lambda\ \bar{m}\bar{\pi}}\ \chi(\bar{m})\ [\varphi_0]^s \left(\frac{\mu_t - \bar{\mu}}{\bar{\mu}}\right) \tag{37}$$

for all $s = 0, 1, 2, ...$, where $\chi(\bar{m}) = 1 - \frac{\bar{m}}{y}(1 - \bar{\pi}\ \varphi_0)$.

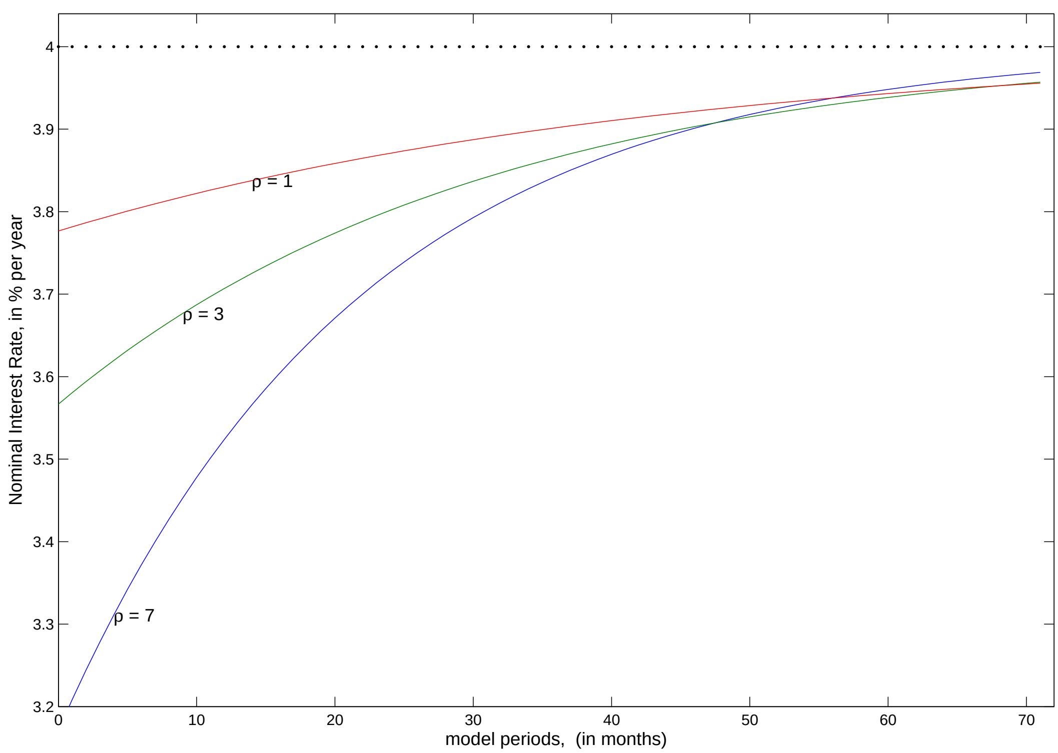
Equation (37) shows that the sign and persistence of the liquidity effect depend on the magnitude of φ_0. The impact effect is negative, i.e. the nominal interest rate decreases when money increases, if $\chi(\bar{m}) > 0$ a condition established in Proposition 5.

To understand the mechanics of the liquidity effect, note that the effect of an iid shock to money supply to the non-trader is to increase the price level, thus decreasing the post-transfer real money balances m of the non-trader. If the consumption elasticity is smaller than one then the ratio of money to consumption for the non-traders decreases, i.e. $\chi(\bar{m}) > 0$. Since with an iid shock aggregate real balances remain the same, this implies that the ratio of money to consumption must increase for traders. In turn a higher m/c ratio for traders implies, by equation (11), that the nominal interest rate must decrease.

The size of the impact decrease in interest rate after a once and for all increase in money increases with $\chi(\bar{m}) > 0$. Recall that Proposition 5 establishes that χ is an increasing function of $\gamma / (1 + \rho)$. The persistence of the liquidity effect also depends on the magnitude of $\varphi_0 = g'(\bar{n})$, which is also increasing in $\gamma/(1+\rho)$. The closer the value of φ_0 is to one, the more persistent the liquidity effect is. Finally, note that the more segmented markets are (the smaller is λ), the larger the (absolute value of the) liquidity effect at all horizons.

To illustrate this proposition we compute the impulse response to a once and for all shock to the money supply for different parameter values. In all the impulse responses we use annual inflation and real rate of 2 percent, U given by (12) and choose the value of the parameter $\mathcal{A}$ to obtain a steady-state value of $\bar{m}/\bar{c}$ equal to 0.25 at annual frequency. We let the model period to be a month. In the first figure we plot impulse responses for a once and for all shock to the money supply which implies a 1 percent increase in the price level on impact, so $(\mu_0 - \bar{\mu})/\bar{\mu}$ is 0.01. In the figures we let $\lambda = 0.25$, $\gamma = 4$ and we vary the value of ρ. As it is clear from Proposition 7, different values of λ scale the distance to steady state by the same proportion at all horizons.

Figure 1: Response to a once-and-for-all money supply shock



The shock is a 1% unanticipated increase in the money supply (same effect on the price level). The other parameters are: $\gamma = 4$, $\lambda = 0.25$.

Figure 1 shows that for larger values of ρ, and hence lower elasticity of the long-run money demand, there are larger liquidity effects at impact, with shorter life-times. The value of ρ has two opposite effects on the impulse response of interest rates, as can be seen from equation (37). The first is a direct effect of the preferences: a large value of $1 + \rho$ raises the sensitivity of the marginal rate of substitution of traders, increasing the effect of a given change in the money-consumption ratio m/c onto the interest rate. The second effect operates through the equilibrium determination of the elasticity χ, that was discussed in Proposition 5. For a fixed value of γ larger values of ρ decrease χ and hence imply a smaller decrease at impact on the ratio m/c. The impulse responses in Figure 1 show that the first effect almost completely dominates the second one, since the vertical distance between the impulse responses at $t = 0$, is almost proportional to the change in the value of $1 + \rho$. Additionally, different values of ρ correspond to different persistence of the liquidity effect, through changes in $g'(\bar{n})$. Higher values of ρ, as shown in Proposition 5, imply faster convergence, as the figure shows.

Table 1 complements Figure 1 by computing two of the determinants of the impulses response of interest rates after a once and for all change in the money supply for different combinations of intertemporal elasticity of substitution $1/\gamma$, and intratemporal elasticity of substitution $1/(1 + \rho)$. One of the determinants, the half-life of the shock, is a simple transformation of φ_0, and is given by $\tau \equiv [\log(1/2) / \log(\varphi_0)]$, expressed in years. The other determinant, denoted by χ, is the impact elasticity of m/c with respect to a change in m, which is also a simple function of φ_0.

The values for ρ and γ for Table 1 are chosen so that the ratio $\gamma/(1 + \rho)$ is constant on the diagonal. The values of τ and χ across the diagonal of abreftab-iid-mu-shocks are the same, which follows from Proposition 5, where it is shown that the decision rules $c(\cdot)$ and $g(\cdot)$ are functions of the ratio of the elasticities $\gamma/(1 + \rho)$. The value of χ has the interpretation of the elasticity of the ratio m/c with respect to an unanticipated once and for all shock to the price level (hence to m). The values for this elasticity varies between 0.82 and 0.95 across

Table 1: Once and for all shock to μ: half-life (τ, in years), and elasticity of m/c (χ)

γ: \ ρ:	1	3	7
2	$\chi = 0.91$ $\tau = 1.77$	$\chi = 0.87$ $\tau = 1.25$	$\chi = 0.82$ $\tau = 0.89$
4	$\chi = 0.94$ $\tau = 2.53$	$\chi = 0.91$ $\tau = 1.77$	$\chi = 0.87$ $\tau = 1.25$
8	$\chi = 0.96$ $\tau = 3.63$	$\chi = 0.94$ $\tau = 2.53$	$\chi = 0.91$ $\tau = 1.77$

Note: $\tau = [\log(1/2) / \log(\varphi_0)]$ half-life of the interest shock , $\chi(\bar{m}) = 1 - \frac{\bar{m}}{\bar{c}} \frac{dc}{dm}$ elasticity of m/c w.r.t m on impact.

the values of $\gamma / (1 + \rho)$ reported in the table. The range of half-lives across the values of $\gamma / (1 + \rho)$ reported in Table 1 is between a bit less than a year, to more than 3 and a half years.

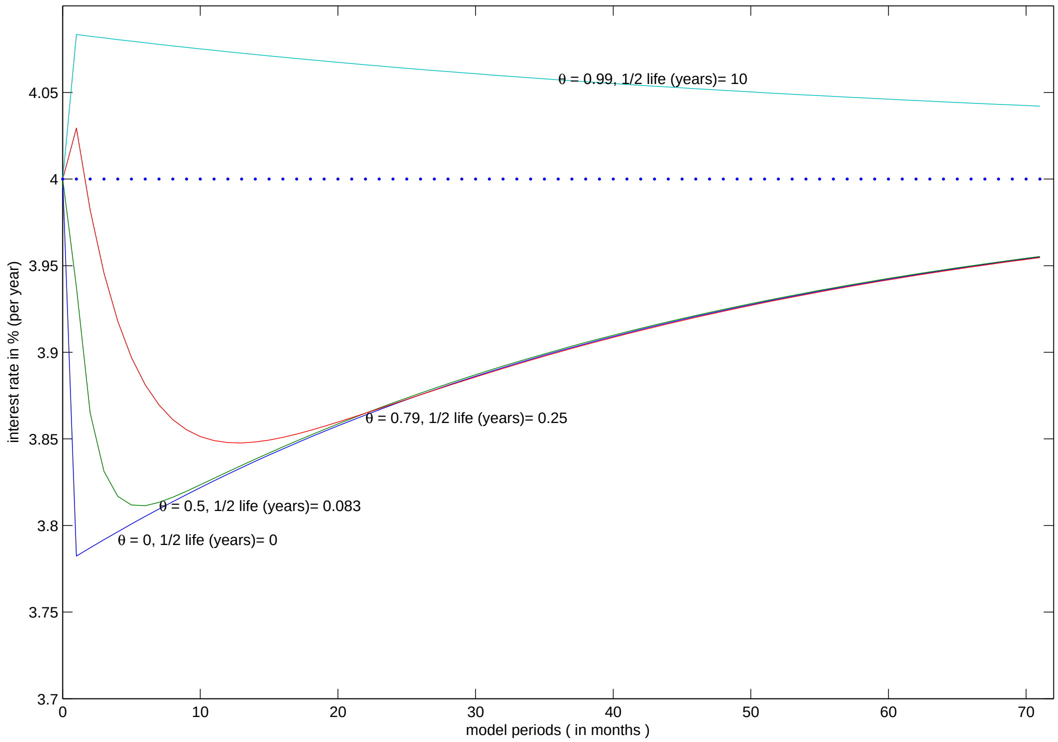
5.2.2 Persistent increase in the growth rate of the money supply

This section analyzes the effect of a persistent increase in the growth rate of the money supply on interest rates. We use the general expression for r_t in (36), the evolution of the state m_t given by (26), and the evolution of n_t given by (35).

Figure 2 plots the impulse responses of interest rates to money shocks under the assumption that the growth rate of the money supply is an AR(1) with autocorrelation θ, as opposed to iid. Otherwise the parameters are the ones used in Figure 1 in the case where $\rho = 1$ (so the long run elasticity of the money demand is 1/2). We plot the impulse response for four values of θ, corresponding to a half life of zero months, one month, three months, and ten years. The zero half life coincides with the iid case of Figure 1, and is included to help in the comparisons. The size of the initial shock to money is chosen so that the effect on the price level on impact is an increase of 1 percent, as in the case of iid money growth.

As can be seen from the impulse responses in Figure 2, the monetary shocks with a shorter half-life produce a liquidity effect. If the monetary shock is very persistent, instead,

Figure 2: Response to a persistent money supply shock

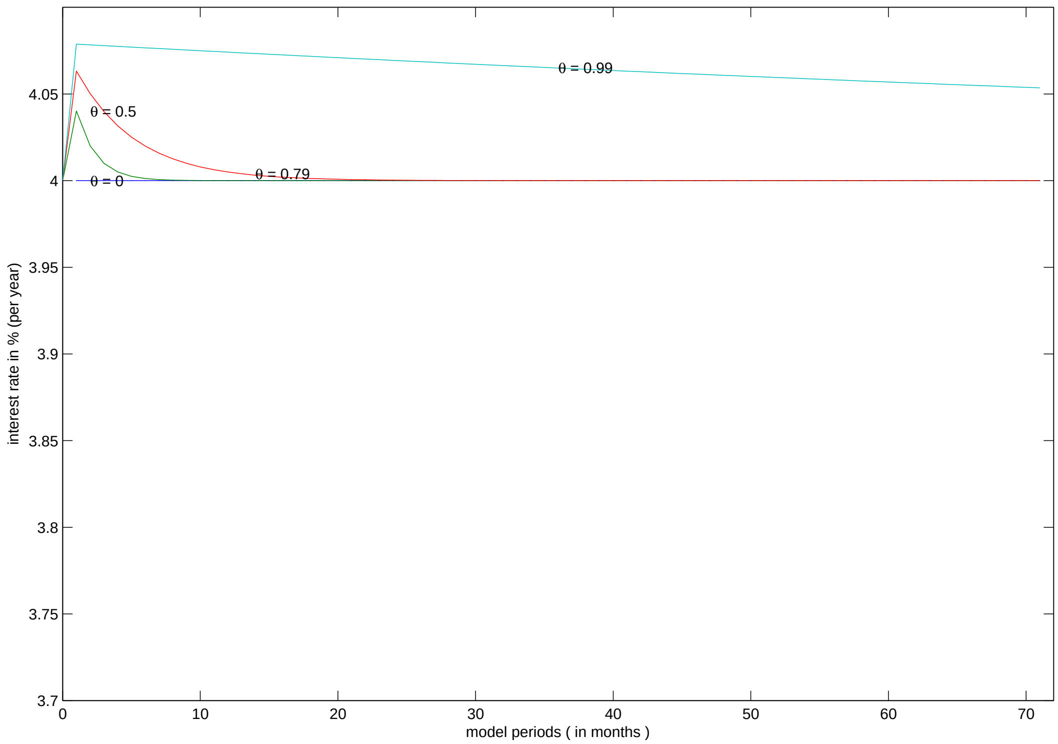


The shock is an unanticipated increase money causing a 1% increase of the price level. The other parameters are: $\gamma = 4$, $\lambda = 0.25$.

the Fisherian aspects of the model take over, expected inflation rises considerably on impact, and there is no liquidity effect. Notice that for intermediate values of θ the impulse response has an inverted hump shape, attaining the minimum some periods after the impact effect. The hump shape of the impulse response is due to the fact that the dynamic system has two eigenvalues: θ, governing aggregate real balances and inflation, and φ_0, governing the non-traders adjustment of their real balances. Note that for the three smallest values of θ, the impulse response converge to the same line, since the short run behavior is dominated by θ and the long run by φ_0. Instead, for the case where θ is much larger, the fisherian effect dominates and the impulse response is almost identical to the one where markets are not segmented.

To identify the effect of segmented markets on interest rates, Figure 3 displays the impulse response (to the same shocks) for the model with $\lambda = 1$, which has no liquidity effects.[9] When shocks are short lived, so that there are no movements in the expected growth rate of money, interest rates remain almost constant at the steady state level. Comparing Figure 2 and 3 shows that when monetary shocks are very persistent the behavior of interest rates in the model with segmented markets ($\lambda = 0.25$) is similar to the one in model with homogenous agents ($\lambda = 1$).

Figure 3: Response with no segmentation ($\lambda = 1$)



The shock is an unanticipated increase money causing a 1% increase of the price level. The other parameters are: $\gamma = 4$, $\lambda = 1.0$ (i.e. no segmentation).

5.2.3 Short vs Long Run Money Demand Elasticities and the Liquidity Effect

We conclude the section with a comment on the relation between the liquidity effect and the interest elasticity of money demand. The thought experiment that reveals a liquidity effect

[9] As shown in Section 4, in the model with $\lambda = 1$ interest rates move in the same direction than expected changes in money supply.

on interest rate is an open market operation, i.e. an increase of the money supply. Instead, the slope of the money demand is a relationship between real money balances −or velocity− and interest rates. As explained, in this model the "long run" interest-elasticity of the money demand is $-1/(1+\rho)$. The liquidity effect of an increase in the (nominal) money supply, too, depends on ρ, among other parameters.

As done in the literature −see e.g. Christiano, Eichenbaum, and Evans (1999)−, we define as the "short-run money demand elasticity" the ratio of the impact effect on real balances relative to the impact effect on interest rates following a monetary shock. We argue that there is no "constant" short-run elasticity of the money demand in the model. We emphasize that this is consistent with the unstable estimates of the interest-elasticity of money demand equations that are obtained using high-frequency data.

To fix ideas consider the case where the growth rate of the money supply follows an AR(1) process with parameter θ. From our previous analysis we have that the decrease on impact of aggregate real balances after a shock to the money supply is given by equation (26):

$$\left.\frac{1}{m}\frac{dm}{d\hat{\mu}}\right|_{m=\bar{m},\ \hat{\mu}=0} = \frac{\kappa}{\bar{m}} = \frac{\alpha}{\bar{m}}\ \frac{\theta}{1-\phi\theta},$$

where $\alpha < 0$ and $0 < \phi < 1$. Hence real balances decrease after a money growth shock, the more so the more persistent is the shock. From our analysis of the impact on interest rates of a monetary shock, equation (36), we have that:

$$\left.\frac{1}{r}\frac{dr}{d\hat{\mu}}\right|_{m=\bar{m},\ \hat{\mu}=0,\ n=\bar{n}} = \frac{(1+\rho)}{\lambda}\left(\frac{(1-\lambda)}{\bar{\pi}}\left[\frac{\bar{n}}{\bar{\pi}}\left(\frac{1}{y}-\frac{1}{\bar{m}}\right)\zeta + \frac{\bar{\pi}\varphi_1}{y}\right] - \frac{\kappa}{\bar{m}}\right)$$

where ζ is given in (27) and the expression for φ_1 is given in Appendix A.7 by (A-10). Thus we define the short run elasticity of the money demand as the ratio:

$$\eta \equiv \frac{r}{m}\frac{dm}{dr} = \left.\frac{\frac{1}{m}\frac{dm}{d\hat{\mu}}}{\frac{1}{r}\frac{dr}{d\hat{\mu}}}\right|_{m=\bar{m},\ \hat{\mu}=0,\ n=\bar{n}}.$$

To sign this expression notice that $(1/m)(dm/d\mu) < 0$, so the sign of the elasticity η depends on whether there is a liquidity effect or not. If there is a liquidity effect then the elasticity is positive on impact, otherwise it is negative. We consider two interesting special cases:

$$\eta \equiv \frac{r}{m}\frac{dm}{dr} = 0 \ \text{ if } \theta = 0 \quad , \quad \eta \equiv \frac{r}{m}\frac{dm}{dr} = -\frac{1}{1+\rho} \ \text{ if } \lambda = 1,$$

In the case case of a once-and-for-all increase in the money supply ($\theta = 0$) expected inflation is constant and thus aggregate real balances remain constant ($\kappa = 0$, and $\hat{m}_t = 0$). Thus the impulse response of a purely transitory shock on the growth rate of the money supply μ_t will display a short-run interest elasticity of the money demand equal to zero. Instead, in the case where $\lambda = 1$, i.e. when markets are not segmented, the short-run and long-run elasticities are the same since the standard interest elastic money demand equation holds at all frequencies. To illustrate the effect of other parameters' values on this elasticity, Table 2 reports the value of η using three values of θ used in Figure 2, denoted by the corresponding half-life of the shocks $\tau(\theta)$, two values of ρ, determining the long-run elasticity $-1/(1+\rho)$, and two values of λ, determining the degree of market segmentation.

Table 2: Interest elasticity of money demand at various horizons

shock half-life: $\tau(\theta)$	$\rho = 1\ ,\ \lambda = 0.2$	$\rho = 1\ ,\ \lambda = 0.1$	$\rho = 3\ ,\ \lambda = 0.1$
1-month	$+\ 0.34$	$+\ 0.08$	$+\ 0.19$
4-months	$+\ 0.93$	$+\ 1.25$	$-\ 0.40$
10-years	$-\ 0.47$	$-\ 0.43$	-0.23

As predicted by our analysis the last row of Table 2 shows that when the monetary shock is persistent the short-run elasticity is closer to the long-run elasticity $-1/(1+\rho)$, although the limit as $\theta \to 1$ depends on λ and does not converge exactly to $-1/(1+\rho)$. Instead, when the monetary shock are short lived, the liquidity effect makes the sign of the short-run elasticity the opposite of the one on the long-run elasticity.

5.3 The effect of anticipated monetary policy shocks

The analysis until now focussed on the effect of an unanticipated shock to the money supply. This section uses the model to explore the effect of anticipated monetary injections.

For simplicity we focus on the case of a once and for all increase of the money supply. To be concrete suppose that agents learn at the beginning of period t that the money supply will jump in period $t+1$, i.e. $\hat{\mu}_{t+1}$, and $\hat{\mu}_{t+1+j} = 0, \; i = 0, 1, 2, ...$

Using equation (22) and equation (24) shows that inflation and real money balances (i.e. the price level) adjust immediately upon the announcement at t, with

$$\hat{\pi}_t = -\alpha \frac{\bar{\mu}}{\bar{m}} \, \hat{\mu}_{t+1} \quad , \quad \hat{m}_t = \alpha \hat{\mu}_{t+1} \ .$$

Therefore, upon the announcement the price level increases reducing real money balances (remember that our assumption are such that $\alpha < 0$).

The adjustment process is completed in period $t+1$ with

$$\hat{\pi}_{t+1} = \left(1 + \alpha \frac{\bar{\mu}}{\bar{m}}\right) \hat{\mu}_{t+1} \quad , \quad \hat{m}_{t+1} = 0$$

TO BE COMPLETED....

6 A calibration on US data

In this section we compute the linearized equilibrium for a calibration based on US data on M1 growth and velocity. As discussed in Appendix AA-2, estimates of long run interest elasticities of money demand range between 1/4 and 1/2. In this section we use 1/2, the value estimated by Lucas (2000). We specify $U(c, m)$ to be CRRA and h CES as in equation (12). The 1/2 elasticity corresponds to a value of $\rho = 1$. We use a monthly model, with an money to income ratio equal to 0.25 at annual frequencies, consistent with the US data for M1. This corresponds to a value of $\mathcal{A} = 33.33$ for the monthly model . We use a value of the intertemporal substitution elasticity of 1/2, or $\gamma = 2$, as is common in the literature and within the range of estimates by Ogaki and Reinhart (1998) and others. The values β and $\bar{\mu}$ are chosen so that the annual real interest rates and inflation rates are two percent (the values for our monthly model are $\beta = 0.9983$, and $\bar{\mu} = 1.0017$). We set the fraction of traders to $\lambda = 0.25$.

We model the money supply process $\hat{\mu}_t/\bar{\mu}$ as the sum of two independent AR(1) processes. Appendix B discusses this statistical model and displays some estimates of a parameteric representation. The estimates typically capture a low and a high frequency component. The calibration below uses an estimate where one component is highly persistent, with a monthly autocorrelation equal to 0.995, which corresponds to a half-life of about 12 years. The other component is less persistent, with a monthly autocorrelation 0.5, i.e. a half-life of a month. The (annualized) unconditional standard deviation of these processes are 0.006 and 0.065 respectively. It appears that the largest part of year to year variation is explained by high-frequency innovations.

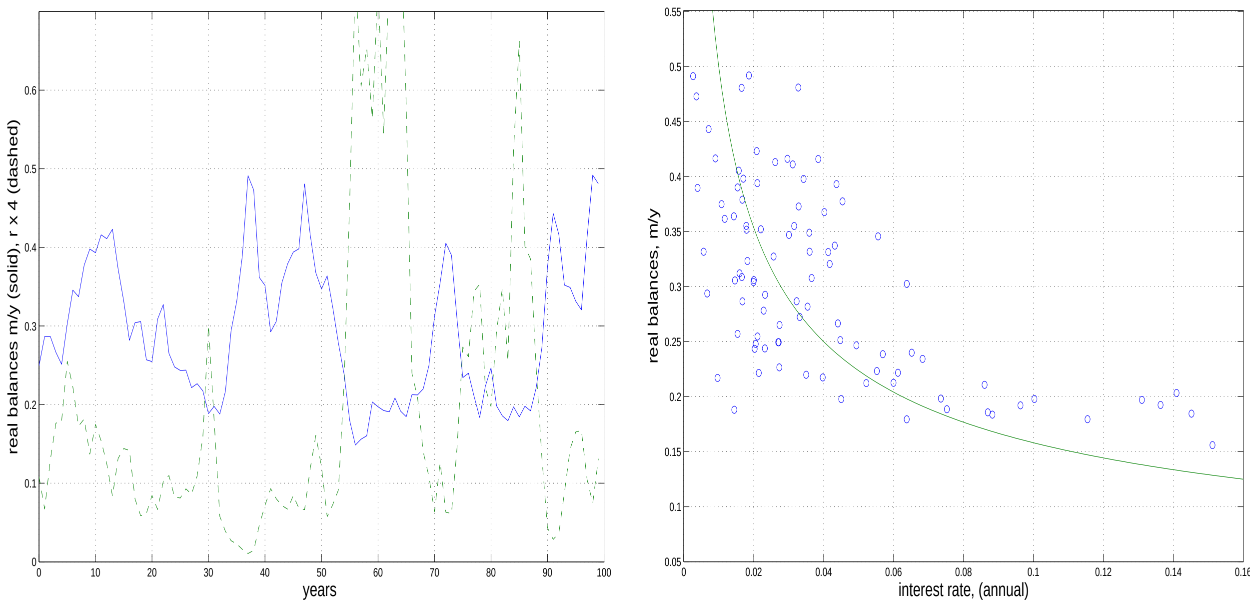
Figure 4 reports the result of an exercise where we simulate the model and display a 100 year time series of interest rate and velocity in the left panel using annual data simulated data (by averaging twelve consecutive non-overlapping periods). The right panel reports a scatter plot of the annual data: the solid line displays the log-log money demand with elasticity 1/2 which corresponds to permanent changes in the growth rate of money. These figures are meant to be compared with the scatter plot and time series graph from annual US data for the 20-th century in Lucas (2000).

The mechanics behind the figure is as follows. The effect of the persistent shock is quantitatively similar to the effect of a permanent change in the money growth rate. The effect of the transitory shock is quantitatively similar to the one of an iid shock of the money growth rate, which has no effect on real balances, and produces a liquidity effect. To better understand the mechanism that generates this time series recall the impulse response of Figure 2: a persistent money growth shock does not cause a liquidity effect, i.e. the interest

Figure 4: Simulated real balances and interest rates

time series

scatter plot



The simulation uses the following parameters: $\gamma = 4$, $\rho = 1$, $\lambda = 0.25$, $\bar{m}/y = 0.25$. Money growth is modelled as the sum of two AR(1) processes.

rate and money growth move in the same direction. Instead, the response of an innovation to a transitory shock displays a liquidity effect.

7 Concluding remarks

In this paper we presented a theoretical mechanism that gives rise to a persistent liquidity effect, and characterized its relationship with a long-run interest rate-elastic money demand. Our interest in monetary models that feature liquidity effects based on segmented asset markets is to study monetary policy questions as in Lahiri, Singh, and Vegh (2007), Nakajima (2006), Lama and Medina (2007), Khan and Thomas (2007), King and Thomas (2008), Bilbiie (2008), Curdia and Woodford (2008), Gust and Lopez-Salido (2009), and Zervou (2008). Comparing with this literature, we have deliberately kept the model as simple as possible. The simplicity allowed us to give a relatively sharp characterization of the theoretical results, such as the relationship between money and prices in the presence of segmentation, and the impact effect and persistence of the liquidity effect as a function of simple elasticities. The disadvantage of this simplicity is that this version of the model lacks many important features of so that many question of interest cannot be addressed. For instance, the model has exogenous endowment, and flexible prices. In this regard we view the result of the model as applying to the aggregate nominal demand.

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A Proofs

A.1 Proof of Proposition 1.

The tilde allocation satisfies market clearing for cash balances and consumption, since they are satisfied in the original equilibrium. Market clearing of bonds is satisfied trivially by construction. The budget constraint of the government is satisfied by construction too, given that the budget constraint for the original equilibrium is satisfied. The tilde allocation solve the problem of the non-traders since their budget constraint is identical. Finally, for traders, at the tilde equilibrium prices and interest rates and chosen initial conditions, the allocation $\left\{c_t^T, m_t^T\right\}_{t=0}^{\infty}$ is budget feasible, and the f.o.c. for the bond holdings is satisfied. Hence, $\left\{c_t^T, m_t^T\right\}_{t=0}^{\infty}$ solve the trader's problem. *Q.E.D.*

A.2 Proof of Proposition 2.

By adding the Euler equation of the traders times λ to the Euler equation of the non-traders times $(1-\lambda)$ we obtain the following Euler equation for the aggregate real balances:

$$\bar{U}_{11}\hat{c}_t + \bar{U}_{12}\hat{m}_t \;=\; \mathbb{E}_t\frac{\beta}{\bar{\pi}}\left[\left(\bar{U}_{11}+\bar{U}_{21}\right)\hat{c}_{t+1} + \left(\bar{U}_{12}+\bar{U}_{22}\right)\hat{m}_{t+1}\right] - \mathbb{E}_t\frac{\beta}{\bar{\pi}^2}\left[\bar{U}_1+\bar{U}_2\right]\hat{\pi}_{t+1}.$$

Feasibility implies that $\hat{c}_t = 0$, hence the Euler equation becomes

$$\bar{U}_{12}\hat{m}_t = \mathbb{E}_t\frac{\beta}{\bar{\pi}}\left[\left(\bar{U}_{12}+\bar{U}_{22}\right)\hat{m}_{t+1}\right] - \mathbb{E}_t\frac{\beta}{\bar{\pi}^2}\left[\bar{U}_1+\bar{U}_2\right]\hat{\pi}_{t+1}$$

Using the linearized money growth identity in equation (17) to replace $\hat{\pi}_{t+1}$ into the Euler equation we get

$$\bar{U}_{12}\hat{m}_t = \mathbb{E}_t\frac{\beta}{\bar{\pi}}\left[\left(\bar{U}_{12}+\bar{U}_{22}\right)\hat{m}_{t+1}\right] - \mathbb{E}_t\frac{\beta}{\bar{\pi}^2}\left[\bar{U}_1+\bar{U}_2\right]\left(\hat{\mu}_{t+1} + \frac{\bar{\mu}}{\bar{m}}\hat{m}_t - \frac{\bar{\mu}}{\bar{m}}\hat{m}_{t+1}\right)$$

or

$$\bar{U}_{12}\hat{m}_t + \frac{\beta}{\bar{\pi}^2}\left[\bar{U}_1+\bar{U}_2\right]\frac{\bar{\mu}}{\bar{m}}\hat{m}_t = \mathbb{E}_t\frac{\beta}{\bar{\pi}}\left[\left(\bar{U}_{12}+\bar{U}_{22}\right)\hat{m}_{t+1}\right] - \mathbb{E}_t\frac{\beta}{\bar{\pi}^2}\left[\bar{U}_1+\bar{U}_2\right]\left(\hat{\mu}_{t+1} - \frac{\bar{\mu}}{\bar{m}}\hat{m}_{t+1}\right)$$

multiplying by $\bar{m}$ and using that in the steady state $\bar{\pi} = \bar{\mu}$ and $\bar{U}_1 = \beta/\bar{\mu}\left(\bar{U}_1+\bar{U}_2\right)$:

$$\left[\bar{U}_{12}\bar{m} + \bar{U}_1\right]\hat{m}_t = -\mathbb{E}_t\frac{\beta}{\bar{\mu}^2}\left[\bar{U}_1+\bar{U}_2\right]\bar{m}\hat{\mu}_{t+1} + \mathbb{E}_t\frac{\beta}{\bar{\pi}}\left[\bar{m}\left(\bar{U}_{12}+\bar{U}_{22}\right) + \bar{U}_1+\bar{U}_2\right]\hat{m}_{t+1} \quad \text{(A-1)}$$

which is identical to the Euler Equation for the aggregate model that is obtained by linearizing (18) around the steady state. *Q.E.D.*

A.3 Proof of Proposition 3.

Linearizing (18) around the steady state gives equation (A-1). Using the definitions for α and ϕ in equation (21) gives (20). *Q.E.D.*

A.4 Sufficient conditions for $0 < \phi < 1$ and $\alpha < 0$

LEMMA 1. Let the utility be given by equation (12). Then $\phi < 1$ and $\alpha < 0$ if

$$\gamma \;<\; 2+\rho+\frac{1}{r\,(m/c)} \quad \text{and} \quad \rho > -1$$

If $\rho = -1$ then $\phi = 1$. Finally $\phi > 0$ if

$$\gamma \;<\; \frac{1}{1+r}\left(2+\rho+\frac{1}{r\,(m/c)}+r-\frac{\rho}{(m/c)}\right) \quad .$$

Proof. Simple algebra using equation (12) gives

$$\begin{aligned}
U_2 &= \left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-\frac{1-\gamma}{\rho}-1}\frac{1}{\mathcal{A}}m^{-\rho-1} \\
mU_{22} &= \left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-\frac{1-\gamma}{\rho}-1}\frac{1}{\mathcal{A}}m^{-\rho-1}\,(1-\gamma+\rho)\,\frac{\frac{1}{\mathcal{A}}m^{-\rho}}{\left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]} \\
&\quad -\left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-\frac{1-\gamma}{\rho}-1}\frac{1}{\mathcal{A}}m^{-\rho-1}\,(\rho+1)
\end{aligned}$$

and

$$\begin{aligned}
U_1 &= \left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-\frac{1-\gamma}{\rho}-1}c^{-\rho-1} \\
mU_{12} &= \left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-\frac{1-\gamma}{\rho}-1}c^{-\rho-1}\,(1-\gamma+\rho)\,\frac{\frac{1}{\mathcal{A}}m^{-\rho}}{\left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]}
\end{aligned}$$

thus

$$\phi \;\equiv\; (\beta/\bar{\mu})\left[1+\frac{\bar{U}_2+\bar{m}\bar{U}_{22}}{\bar{U}_1+\bar{m}\bar{U}_{12}}\right]=\frac{1}{1+r}\left[1+r\,\frac{(1-\gamma)\;r\,(m/c)-\rho}{(2-\gamma+\rho)\,r\,(m/c)+1}\right] \; .$$

Sufficient conditions for $0 < \phi < 1$ are the following. The condition for $\phi < 1$ requires:

$$\frac{(1-\gamma)\;r\,(m/c)-\rho}{(2-\gamma+\rho)\,r\,(m/c)+1} < 1$$

If $(2-\gamma+\rho)\,r\,(m/c)+1 > 0$, this inequality is:

$$0 < (1+\rho)\,(1+rm/c)$$

which holds if $\rho > -1$. If $\rho = -1$, then $\phi = 1$. This establishes the condition for $\phi < 1$:

$$\gamma \;<\; 2 + \rho + \frac{1}{r\,(m/c)} \quad \text{and } \rho > -1 \;\;.$$

The condition for $\phi > 0$ requires

$$-1 < r \frac{(1-\gamma)\;r\,(m/c) - \rho}{(2-\gamma+\rho)\,r\,(m/c) + 1}$$

Since the denominator is positive (whenever $\phi < 1$), this inequality implies

$$-\left(2-\gamma+\rho\right) r\,(m/c) - 1 < r\left[(1-\gamma)\;r\,(m/c) - \rho\right]$$

which, after rearranging terms, gives the second inequality in Lemma 1 .

Finally the condition for $\alpha < 0$ if and only if $\bar{U}_1 + \bar{m}\bar{U}_{12} > 0$. Using equation (13) and the expressions computed above for the utility function equation (12) we have that

$$\frac{U_1 + mU_{12}}{\left[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}\right]^{-\frac{1-\gamma}{\rho}-1} c^{-\rho-1}} = \frac{(2-\gamma+\rho)\,r\,(m/c) + 1}{1 + r\,(m/c)}$$

so $U_1 + mU_{12} > 0$ holds if $(2-\gamma+\rho)\,r\,(m/c) + 1 > 0$ or, as is immediate to verify, the condition for $\phi < 1$ holds.

Using the steady state equilibrium condition in equation (13) and the definition for the nominal interest rate in equation (19) we can rewrite the condition for $\alpha < 0$ and $\phi < 1$ in terms of exogenous parameters as

$$\gamma \;<\; 2 + \rho + \mathcal{A}^{\frac{1}{1+\rho}} \left(\frac{\bar{\mu}}{\beta} - 1\right)^{-\frac{\rho}{1+\rho}} \quad \text{and } \rho > -1 \;\;.$$

Q.E.D.

A.5 Proof of Proposition 4.

Under the stated conditions the value function V is strictly concave and differentiable. The f.o.c. for this problem is

$$U_1\left(y + \tau^N + \frac{n}{\bar{\pi}} - g\,(n)\,, \frac{n}{\bar{\pi}} + \tau^N\right) = \beta V'\left(g\,(n)\right) \tag{A-2}$$

for all $n \geq 0$. Since V is concave, V' is decreasing and since $U_{11} + U_{12} < 0$ the LHS of the f.o.c. (A-2), for a fixed value $n' = g\,(n)$, is decreasing in n, and hence $g\,(\cdot)$ is increasing. The envelope gives:

$$V'\,(n) \;=\; \frac{1}{\bar{\pi}} U_1\left(y + \tau^N + \frac{n}{\bar{\pi}} - g\,(n)\,, \frac{n}{\bar{\pi}} + \tau^N\right) + \frac{1}{\bar{\pi}} U_2\left(y + \tau^N + \frac{n}{\bar{\pi}} - g\,(n)\,, \frac{n}{\bar{\pi}} + \tau^N\right).$$

Using the f.o.c. and the envelope evaluated at steady state we obtain:

$$U_1\left(y+\tau^N-\bar{n}\left(1-\frac{1}{\bar{\pi}}\right),\ \frac{\bar{n}}{\bar{\pi}}+\tau^N\right)\left(1-\frac{\beta}{\bar{\pi}}\right)=\frac{\beta}{\bar{\pi}}U_2\left(y+\tau^N-\bar{n}\left(1-\frac{1}{\bar{\pi}}\right),\ \frac{\bar{n}}{\bar{\pi}}+\tau^N\right)$$

Under the assumption that $U_{12}\geq 0$ and $\bar{\pi}>1$ it is easy to see that there is a unique steady state $\bar{n}$ satisfying this equation.

We now show that this steady state is globally stable. Suppose not, i.e. that $g'(n)>1$, and assume that $n_0>\bar{n}$, then $\lim n_t=\infty$. But notice that V is bounded below, since $V(n)\geq V(0)\geq U\left(y+\tau^N,\tau^N\right)/(1-\beta)$. Additionally we assume that U is bounded above. In this case, since V is concave,

$$V(n_t)\geq V(0)+V'(n_t)\,n_t$$

and hence as $n_t\to\infty$ it must be that

$$V'(n_t)=U_1\left(n_t/\bar{\pi}+y+\tau^N-n_{t+1},n_t/\bar{\pi}+\tau^N\right)+U_2\left(n_t/\bar{\pi}+y+\tau^N-n_{t+1},n_t/\bar{\pi}+\tau^N\right)\to 0.$$

and since $U_2\geq 0$:

$$\lim_{t\to\infty}U_1\left(n_t/\bar{\pi}+y+\tau^N-n_{t+1},n_t/\bar{\pi}+\tau^N\right)=0$$

Since $U_{12}\geq 0$ then

$$U_1\left(n_t/\bar{\pi}+y+\tau^N-n_{t+1},n_t/\bar{\pi}+\tau^N\right)\geq U_1\left(n_t/\bar{\pi}+y+\tau^N-n_{t+1},\tau^N\right)$$

but if $n_{t+1}>n_t$, for $\bar{\pi}>1$

$$U_1\left(n_t/\bar{\pi}+y+\tau^N-n_{t+1},\tau^N\right)\geq U_1\left(y+\tau^N,\tau^N\right)>0$$

a contradiction. Hence $\lim_t n_t$ must be bounded, and thus $g'(\bar{n})<1$. *Q.E.D.*

A.6 Proof of Proposition 5.

-Part I. Using $\hat{n}_t=g'\cdot\hat{n}_{t-1}$ for the policy rule gives $\hat{m}_t=\hat{n}_{t-1}/\bar{\pi}$, $\hat{c}_t=(1/\bar{\pi}-\ g')\,\hat{n}_{t-1}$ and $\hat{m}_{t+1}=g'\hat{n}_{t-1}/\bar{\pi}$, $\hat{c}_{t+1}=(1/\bar{\pi}-\ g')\,g'\hat{n}_{t-1}$. Totally differentiating the Euler equation (6) and using the above policy functions gives a second order ODE with characteristic equation

$$0=\beta\left(\varphi_0\right)^2+b\varphi_0+1,$$

that has $\varphi_0\equiv g'(\bar{n})$ as its smallest root, where the coefficient b is:

$$-b\equiv\frac{\bar{\pi}+\frac{\beta}{\bar{\pi}}\left(1+2\,\frac{\bar{U}_{12}}{\bar{U}_{11}}+\frac{\bar{U}_{22}}{\bar{U}_{11}}\right)}{1+\frac{\bar{U}_{12}}{\bar{U}_{11}}}\ >\ 0. \qquad \text{(A-3)}$$

Note that $-b>0$ under the assumption $U_{11}+U_{12}<0$ and $\pi>1$. As an intermediate step for the proof, the next lemma gives the properties of φ_0 as a function of $-b$.

Lemma 2. The expression for the root that is smaller in absolute value is:

$$\varphi_0 = \frac{-b - \sqrt{b^2 - 4\beta}}{2\beta} \tag{A-4}$$

with $0 < \varphi_0 \leq 1$, provided that $-b \geq 1 + \beta$. Moreover, φ_0 is decreasing in $-b$.

Proof of Lemma 2. A real solution requires $b^2 - 4\beta \geq 0$. If $-b \geq 1 + \beta$, then $b^2 \geq (1+\beta)^2 = 1 + \beta^2 + 2\beta$, and $b^2 - 4\beta \geq (1-\beta)^2 > 0$. If $-b = 1 + \beta$ then $\varphi_0 = 1$. To see that φ_0 is decreasing in $-b$:

$$\frac{\partial \varphi_0}{\partial (-b)} = \frac{\partial}{\partial (-b)} \left(\frac{-b - \sqrt{(-b)^2 - 4\beta}}{2\beta} \right) = \frac{1}{2\beta} \left(1 - \frac{1}{\sqrt{b^2 - 4\beta}} \right) \leq \frac{1}{2\beta} \left(1 - \frac{1}{1-\beta} \right) < 0.$$

Q.E.D.

\- Part II. We show that the coefficients of the equation that defines $\varphi_0 \equiv g'(\bar{n})$ are a function of $\gamma/(1+\rho)$, β, π and m/c. Using $U(c,m) = \left(h(c,m)^{1-\gamma} - 1\right)/(1-\gamma)$ gives:

$$\frac{U_{22}}{U_{11}} = \frac{h_{22}/h_{11} + \gamma r^2 (h_1 h_1)/(-h h_{11})}{1 + \gamma (h_1 h_1)/(-h h_{11})} \quad , \quad \frac{U_{12}}{U_{11}} = \frac{h_{12}/h_{11} + \gamma r (h_1 h_1)/(-h h_{11})}{1 + \gamma (h_1 h_1)/(-h h_{11})}.$$

Using that h is CES we have

$$\begin{aligned}
h_{11} &= (1+\rho) \left[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho - 1} c^{-\rho-2} \left\{ \frac{c^{-\rho}}{\left[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}\right]} - 1 \right\}, \\
h_{22} &= (1+\rho) \left[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho - 1} \frac{1}{\mathcal{A}} m^{-\rho-2} \left\{ \frac{\frac{1}{\mathcal{A}} m^{-\rho}}{\left[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho}\right]} - 1 \right\}, \\
h_{12} &= (1+\rho) \left[c^{-\rho} + \frac{1}{\mathcal{A}} m^{-\rho} \right]^{-1/\rho - 2} c^{-\rho-1} m^{-\rho-1}/\mathcal{A}.
\end{aligned}$$

Thus

$$\frac{h_{22}}{h_{11}} = \left(\frac{c}{m}\right)^2 \quad \text{and} \quad \frac{h_{12}}{h_{11}} = -\left(\frac{c}{m}\right).$$

And using that, as from equation (13), $\left(\frac{m}{c}\right)^{-\rho}/\mathcal{A} = r\ m/c$

$$\frac{h_1 h_1}{-h h_{11}} = \frac{1}{(1+\rho)\, r\, (m/c)}.$$

Plugging these expressions into the ones for U_{22}/U_{11} and U_{12}/U_{11} gives

$$\frac{U_{22}}{U_{11}} = \frac{\left(\frac{c}{m}\right)^2 + r\left(\frac{c}{m}\right)\left(\frac{\gamma}{1+\rho}\right)}{1 + \frac{1}{r}\left(\frac{c}{m}\right)\left(\frac{\gamma}{1+\rho}\right)} \quad , \quad \frac{U_{12}}{U_{11}} = \frac{-\left(\frac{c}{m}\right) + \left(\frac{c}{m}\right)\left(\frac{\gamma}{1+\rho}\right)}{1 + \frac{1}{r}\left(\frac{c}{m}\right)\left(\frac{\gamma}{1+\rho}\right)} \tag{A-5}$$

which are functions of the ratio $\gamma/\left(1+\rho\right)$, the number r, and the ratio m/c.

Replacing (A-5) into (A-3) gives:

$$-b=\frac{\bar{\pi}+\frac{\beta}{\bar{\pi}}(1-x)^{2}+x\frac{\tilde{\gamma}}{r}(\bar{\pi}+\frac{\beta}{\bar{\pi}}(1+r)^{2})}{1-x+x\frac{\tilde{\gamma}}{r}(1+r)} \tag{A-6}$$

where $x\equiv c/m\in(0,1)$ and $\tilde{\gamma}\equiv\gamma/\left(1+\rho\right)$.

The partial derivative of (A-6) with respect to $\tilde{\gamma}$ gives

$$\frac{\partial(-b)}{\partial\tilde{\gamma}}=\frac{-\frac{x}{r}(x+r)}{(1-x+x\frac{\tilde{\gamma}}{r}(1+r))^{2}}\left(\bar{\pi}-\frac{\beta}{\bar{\pi}}(1+r)(1-x)\right) \tag{A-7}$$

Noting that at the steady state $\frac{\beta}{\bar{\pi}}(1+r)=1$ (by equation (13)) establishes that $-b$ is decreasing in $\gamma/(1+\rho)$. This, by Lemma 2, implies that the root φ_0 is increasing in $\gamma/(1+\rho)$.

-Part III. We conclude the proof by showing that $0<\chi(\bar{m})$ for $\bar{\pi}>1$ and $m/c>1$. The proof is in two parts. We first analyze the case of $\gamma=0$. Then we extend the results for the case of $\gamma>0$.

Assume $\gamma=0$. We show that the elasticity of $c\left(m\right)$ with respect to m, evaluated at $m=\bar{m}$, is smaller than one, which implies that $\chi(\bar{m})>0$. Note by equation (33) that $\frac{m}{c}\frac{\partial c}{\partial m}<1$ requires $\left(1-\frac{c}{m}\right)\frac{1}{\pi}\leq\varphi_0$. Using equation (A-4) this inequality becomes

$$-b-2\beta\left(1-\frac{c}{m}\right)\frac{1}{\pi}-\sqrt{b^{2}-4\beta}\geq 0 \tag{A-8}$$

Using equation (A-6) for $\gamma/\left(1+\rho\right)=0$ gives

$$-b=\frac{\pi}{1-x}+\frac{\beta}{\pi}\left(1-x\right) \tag{A-9}$$

where $x\equiv c/m$. We want to show that inequality (A-8) holds for $x\in(0,1)$, i.e. that

$$a(x)\equiv-b\left(x\right)-2\beta\left(1-x\right)\frac{1}{\pi}-\sqrt{b\left(x\right)^{2}-4\beta}>0 \quad .$$

This follows because: $a\left(0\right)>0$ and $a'\left(x\right)>0$ for $x\in(0,1)$. These two inequalities follow from:

$$a\left(0\right)=\pi-\frac{\beta}{\pi}-\sqrt{\pi^{2}+\left(\frac{\beta}{\pi}\right)^{2}+\frac{2\beta}{\pi}-4\beta}$$

Using that $\pi>1$, we have

$$a\left(0\right)>\pi-\frac{\beta}{\pi}-\sqrt{\pi^{2}+\left(\frac{\beta}{\pi}\right)^{2}+\frac{2\beta}{\pi}-4\frac{\beta}{\pi}}=\pi-\frac{\beta}{\pi}-\sqrt{\left(\pi-\frac{\beta}{\pi}\right)^{2}}=0 \quad .$$

Finally for $a'(x) > 0$ we have

$$a' = -b'\left[1 + \frac{b}{\sqrt{b(x)^2 - 4\beta}}\right] + 2\frac{\beta}{\pi} = \left(\frac{\pi}{(1-x)^2} - \frac{\beta}{\pi}\right)\left[1 + \frac{b}{\sqrt{b^2 - 4\beta}}\right] + 2\frac{\beta}{\pi} > 0 \quad .$$

We conclude the proof for the $\gamma = 0$ case by showing that $-b > 1 + \beta$ (an assumption in Lemma 2). From equation (A-6) with $\tilde{\gamma} = 0$, simple algebra shows that the inequality $-b > 1 + \beta$ holds if $\pi > 1$ and $m/c > 1$.

These results extend to the case where $\gamma > 0$. As above, the inequality $0 < \chi$ requires $\left(1 - \frac{c}{m}\right)\frac{1}{\pi} \leq \varphi_0$. This inequality was shown to hold for $\gamma = 0$. Since φ_0 is increasing in $\tilde{\gamma}$, then it holds *a fortiori* for $\gamma > 0$. The inequality $-b \geq 1 + \beta$ holds, since $-b$ is decreasing in $\tilde{\gamma}$ and $\lim_{\tilde{\gamma}\to\infty}(-b) = 1 + \beta$.

A.7 Expressions for the linearized Non-Trader problem

Coefficients for the linearization of the Euler equation of the Non-Trader's problem.

$$\begin{aligned}
\xi_0 &= \frac{\left[\bar{\pi}\bar{U}_{11} + \frac{\beta}{\bar{\pi}}\left(\bar{U}_{11} + \bar{U}_{12} + \bar{U}_{21} + \bar{U}_{22}\right)\right]}{\beta\left[\bar{U}_{11} + \bar{U}_{21}\right]} \quad , \quad \xi_1 = \frac{\bar{n}}{\beta\bar{\pi}}, \\
\xi_2 &= -\left[\frac{\frac{\beta}{\bar{n}}\left(\bar{U}_1 + \bar{U}_2\right) + \frac{\beta}{\bar{\pi}}\left(\bar{U}_{11} + \bar{U}_{12} + \bar{U}_{21} + \bar{U}_{22}\right)}{\left[\bar{U}_{11} + \bar{U}_{21}\right]}\right]\frac{\bar{n}}{\beta\bar{\pi}}.
\end{aligned}$$

Coefficients for the equilibrium solution of the Non-Trader's problem:

$$\begin{aligned}
(\varphi_0 - \xi_0)\,\varphi_0 + \frac{1}{\beta} &= 0, \quad , \quad (\varphi_0 - \xi_0)\,\varphi_2 = \xi_1\left(\frac{\bar{\mu}}{\bar{m}}\right), \\
(\varphi_0 - \xi_0)\,\varphi_1 + \varphi_1\Theta + \varphi_2\kappa &= \xi_1\zeta \; + \xi_2\bar{\Pi}
\end{aligned}$$

A.8 Proof of Proposition 6.

We try a solution of the form

$$n_t = \varphi_0\ \hat{n}_{t-1} + \varphi_1\ z_t + \varphi_2\hat{m}_{t-1}$$

with coefficients φ_0, φ_2 and φ_2 to be determined. Replacing the hypothesis for inflation and expected inflation the Euler equation:

$$\mathbb{E}_t\left[\hat{n}_{t+1}\right] = \xi_0\hat{n}_t - \frac{1}{\beta}\hat{n}_{t-1} + \xi_1\left[\left(\frac{\bar{\mu}}{\bar{m}}\right)\hat{m}_{t-1} + \zeta\ z_t\right] + \xi_2\bar{\Pi}\ z_t\ .$$

Taking expected values on the guess for the solution of the policy:

$$\mathbb{E}_t\left[\hat{n}_{t+1}\right] = \varphi_0\ \hat{n}_t + \varphi_1\ \mathbb{E}_t\left[z_{t+1}\right] + \varphi_2\hat{m}_t \ = \varphi_0\ \hat{n}_t + \varphi_1\Theta\ z_t + \varphi_2\kappa\ z_t$$

Equating the two terms:

$$\varphi_0\ \hat{n}_t + \varphi_1 \Theta\ z_t + \varphi_2 \kappa\ z_t \ \ = \ \ \xi_0 \hat{n}_t - \frac{1}{\beta}\hat{n}_{t-1} + \xi_1 \left[\left(\frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta\ z_t \right] + \xi_2 \bar{\Pi}\ z_t$$

rearranging

$$(\varphi_0 - \xi_0)\ \ \hat{n}_t + \varphi_1 \Theta\ z_t + \varphi_2 \kappa\ z_t \ \ = \ \ -\frac{1}{\beta}\hat{n}_{t-1} + \xi_1 \left[\left(\frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta\ z_t \right] + \xi_2 \bar{\Pi}\ z_t$$

Replacing again the guess for the optimal policy in $\hat{n}_t$:

$$(\varphi_0 - \xi_0) \left[\varphi_0 \hat{n}_{t-1} + \varphi_1 z_t + \varphi_2 \hat{m}_{t-1} \right] + \varphi_1 \Theta z_t + \varphi_2 \kappa z_t \ \ = \ \ -\frac{1}{\beta}\hat{n}_{t-1} + \xi_1 \left[\left(\frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} + \zeta z_t \right] + \xi_2 \bar{\Pi} z_t$$

rearranging:

$$\begin{aligned} & (\varphi_0 - \xi_0)\, \varphi_0\ \hat{n}_{t-1} + \left[(\varphi_0 - \xi_0)\, \varphi_1 + \varphi_1 \Theta + \varphi_2 \kappa \right] z_t + (\varphi_0 - \xi_0)\, \varphi_2 \hat{m}_{t-1} \\ = \ \ & -\frac{1}{\beta}\hat{n}_{t-1} + \left(\xi_1 \zeta + \xi_2 \bar{\Pi} \right) z_t + \xi_1 \left(\frac{\bar{\mu}}{\bar{m}} \right) \hat{m}_{t-1} \end{aligned}$$

matching coefficients:

$$(\varphi_0 - \xi_0)\, \varphi_0 = -\frac{1}{\beta} \ \ , \ \ (\varphi_0 - \xi_0)\, \varphi_2 = \xi_1 \left(\frac{\bar{\mu}}{\bar{m}} \right) \ \ , \ \ (\varphi_0 - \xi_0)\, \varphi_1 + \varphi_1 \Theta + \varphi_2 \kappa \ \ = \ \ \xi_1 \zeta \ + \xi_2 \bar{\Pi}$$

That $\varphi_0 = g'(\bar{n})$ can be verified immediately by linearizing the f.o.c. of the problem for the non-trader with constant inflation $\bar{\pi}$. Then, using Proposition 4, we then have $0 < \varphi_0 = g'(\bar{n}) < 1.Q.E.D.$

B Estimating a two-component process for μ_t

Here we set up a simple two unobservable components process for money growth. Let $\{\mu_t\}_{t=1,...,T}$ be the sample of observed money growth rates, i.e. the log of the consecutive levels of the nominal money stock. We assume that it is the sum of two unobserved, independent processes, each of them of the form:

$$\mu_{i,t+1} - \bar{\mu} = \rho_i \left(\mu_{i,t} - \bar{\mu} \right) + \epsilon_{i,t+1}$$

where the innovations $\{\epsilon_{i,t}\}$ are i.i.d., through time and independent of each other, normally distributed with mean zero and variance ${\sigma_i}^2$. We have then $\mu_t = \mu_{1,t} + \mu_{2,t}$ for $t = 1, ..., T$. We conjecture that the process was in place for a long time before the beginning of our sample, so that we assume that the initial observation was drawn out of the invariant distribution, which is normal with mean μ and variance $\sum_{i=1,2} {\sigma_i}^2/(1 - {\rho_i}^2)$.

The theoretical autocovariance function for μ is given by:

$$\hat{\Omega}(k) = \sigma_1^2\ \frac{\rho_1^k}{1-\rho_1^2} + \sigma_2^2\ \frac{\rho_2^k}{1-\rho_2^2}$$

Table A: Estimates for money growth (M1) process and implied half-life of shocks (τ)

	ρ_1	σ_1	τ_1	ρ_2	σ_2	τ_2
$J = 12$	0.94	0.0011	10.7	0.05	0.0055	0.2
$J = 12 \times 2$	0.91	0.0014	7.4	0.007	0.0054	0.1
$J = 12 \times 5$	0.83	0.0024	3.6	0.26	0.00001	0.5
$J = 12 \times 10$	0.83	0.0024	3.6	0.37	0.00001	0.7
$J = [1:3], [57:60], [117:120]$	0.995	0.00005	154	0.47	0.0048	0.9

Note: Based on M1 monthly data (not seasonally adjusted) over the sample period 1959:1-2009:9. The time unit is 1 month. Data source: Board of Governors of the Federal Reserve System. The estimates reported in the last line uses only focuses on fitting only a subset of moments in the ACF, i.e. the cross-correlation at long ($J = 117:120$), medium ($J = 57:60$) and low ($J = 1:3$) displacement.

for $k = 0, 1, 2, 3, ...$ To estimate the 4 unknown parameters $(\rho_1, \rho_2, \sigma_1, \sigma_2)$, we use a GMM approach that minimizes the distance between the first J values of the ACF function for M1 (or M2) and this theoretical counterpart (where identification requires that $J \geq 4$), giving an equal weight to each of these moments. Let $\Omega(k)$ be the sample measure of the autocovariance function. Then

$$\rho_1^*, \rho_2^*, \sigma_1^*, \sigma_2^* = \ argmin_{\rho_1,\rho_2,\sigma_1,\sigma_2} \sum_{j=1}^{J} \left(\Omega(j-1) - \hat{\Omega}(j-1) \right)^2$$

The Table below reports some of these estimates, based on monthly data, as we vary the value of J. Larger values of J are better at capturing the low frequency properties of the series. We consider horizons of 1 year ($J = 12$), 2 years ($J = 24$), and 5 years ($J = 120$)

Online Appendices

Persistent Liquidity Effect and Long Run Money Demand

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AA-1 The Non-Trader problem in continuous time

Since in our discrete time analysis, the speed of convergence and the elasticity of m/c depend on the value of a stock/flow, namely the steady state value of m/c, we analyze the continuous time version of the non-trader problem. The continuous time version has two advantages. One is that it deals more naturally with the stock/flow distinction. The other one is that it simplifies some expression, which are key for the impact of money on interest rate shocks.

Consider the problem of a non-trader maximizing

$$\max \int_0^\infty e^{-\delta t} U\left(c\left(t\right), m\left(t\right)\right) dt$$

subject to

$$\dot{m}\left(t\right) + c\left(t\right) = y + \tau - \pi m\left(t\right)$$

and $m\left(0\right) > 0$ given. Here π is the continuously compounded inflation rate and δ is the discount rate. The steady state of this problem is the same as the discrete time version, given by

$$\frac{U_m\left(\bar{c}, \bar{m}\right)}{U_c\left(\bar{c}, \bar{m}\right)} = \delta + \pi \equiv r.$$

In the case where U is given by (12), recall that $\bar{c}$ and $\bar{m}$ are independent of γ and depend only on the properties of h, in particular we obtain $\bar{m}/\bar{c} = \left(1/\mathcal{A}\right) r^{-1/(1+\rho)}$. We let $c = \psi\left(m\right)$ the optimal decision rule for consumption. From the budget constraint the dynamics of $m\left(t\right)$ for values close to the steady state $\bar{m}$, is governed by $\partial \dot{m}\left(t\right)/\partial m = -\pi - \psi'\left(\bar{m}\right)$. In the next proposition we keep fixed the value of δ, π, and regard γ, ρ and the steady state value of $\bar{m}/\bar{c}$ as parameters. The interpretation is that the value of the parameter $\mathcal{A}$ is changed as a function of ρ and $\bar{m}/\bar{c}$.

Proposition 8. Assume that $\pi \geq 0$, and that U is given by (12). Keeping the steady state value $\bar{m}/\bar{c}$ fixed, the slope of the optimal consumption function evaluated at steady state is a function of $\gamma/\left(1+\rho\right)$ and satisfy

$$\frac{c}{m}\frac{\partial c\left(m\right)}{\partial m}|_{m=\bar{m}} \equiv \frac{\bar{m}}{\bar{c}}\psi'\left(\bar{m}\right) = \begin{cases} 1 & \text{for } \gamma/\left(1+\rho\right) = 0 \\ < 1 & \text{for } \frac{\gamma}{1+\rho} > 0 \end{cases}$$

and $\frac{\bar{m}}{\bar{c}}\psi'\left(\bar{m}\right)$ is decreasing in $\frac{\gamma}{1+\rho}$.

As anticipated in the continuous time limit the particular threshold value of $m/c = 1$ plays no role.

Proof of Proposition 8. Using λ for the co-state, the Hamiltonian H is

$$H\left(c, m\right) = U\left(c, m\right) + \lambda\left[-m\pi + y + \tau - c\right]$$

The f.o.c. are:

$$\dot{\lambda} = -\rho\lambda - H_m : \dot{\lambda} = \lambda\left(\delta + \pi\right) - U_m\left(c, m\right), \text{ and } H_c = 0 : U_c\left(c, m\right) = \lambda.$$

The steady state is then:

$$\frac{U_m}{U_c}(\bar{c},\bar{m}) \;\; = \;\; \delta+\pi \quad , \quad \pi\bar{m}+\bar{c}=y+\tau.$$

From $U_c(c(t),m(t))=\lambda(t)$ we obtain:

$$\dot{c}U_{cc}+\dot{m}U_{cm}=\dot{\lambda},$$

replacing into the $\dot{\lambda}$ expression:

$$\dot{c}U_{cc}(c,m)+\dot{m}U_{cm}(c,m)=U_c(c,m)(\pi+\delta)-U_m$$

and using the budget constraint for $\dot{m}$:

$$\dot{c}U_{cc}(c,m)=U_c(c,m)(\pi+\delta)-[y+\tau-m\pi-c]\,U_{cm}(c,m)-U_m(c,m)$$

Linearizing this ODE around $(\dot{c},\dot{m},c,m)=(0,0,\bar{c},\bar{m})$ we obtain

$$\begin{aligned}\dot{c} \;&=\; (\pi+\delta)\frac{U_{cc}}{U_{cc}}(c-\bar{c})+(\pi+\delta)\frac{U_{cm}}{U_{cc}}(m-\bar{m})+\frac{U_{cm}}{U_{cc}}(c-\bar{c})\\ &\quad+\frac{\pi U_{cm}}{U_{cc}}(m-\bar{m})-\frac{U_{mm}}{U_{cc}}(m-\bar{m})-\frac{U_{mc}}{U_{cc}}(c-\bar{c})\end{aligned}$$

where all the second derivatives are evaluated at the steady state values. Summarizing we have the linear system:

$$\begin{aligned}\dot{c} \;&=\; a(c,m)\equiv[\pi+\delta](c-\bar{c})+\left[\frac{U_{cm}(2\pi+\delta)-U_{mm}}{U_{cc}}\right](m-\bar{m})\\ \dot{m} \;&=\; b(c,m)\equiv y+\tau-m\pi-c\end{aligned}$$

We are looking for a solution of the form

$$c=\psi(m)=\bar{c}+\psi'(\bar{m})(m-\bar{m}).$$

We use the method of undetermined coefficients:

$$\frac{\partial\psi(\bar{m})}{\partial m} \;\equiv\; \psi'(\bar{m})=\frac{dc/dt}{dm/dt}=\frac{a(\psi(\bar{m}),\bar{m})}{b(\psi(\bar{m}),\bar{m})}=\frac{0}{0} \;=\; \frac{(\pi+\delta)\psi'(\bar{m})+\left[\frac{U_{cm}(2\pi+\delta)-U_{mm}}{U_{cc}}\right]}{-\pi-\psi'(\bar{m})},$$

where the last equality uses L'Hopital rule. The quadratic equation for ψ' is then

$$-\psi'(\pi+\psi')+(\pi+\delta)\psi'+\left[\frac{U_{cm}(2\pi+\delta)-U_{mm}}{-U_{cc}}\right]=0.$$

The stable solution is given by

$$\psi' = \frac{-\left[r+\pi\right] + \sqrt{\left[r+\pi\right]^2 + 4\frac{U_{cm}(r+\pi)-U_{mm}}{(-U_{cc})}}}{2}.$$

Now we specialize the utility function U to (12). First we show that if $\gamma = 0$, then $\psi'(\bar{m}) = m/c$. Using that h is a CES we obtain that

$$\frac{h_{22}}{h_{11}} = \frac{1}{(m/c)^2} \quad , \quad h_{12}/h_{11} = -\frac{1}{m/c}$$

we have

$$\psi' = \frac{-\left[r+\pi\right] + \sqrt{\left[r+\pi\right]^2 + 4\frac{h_{12}(r+\pi)-h_{22}}{-h_{11}}}}{2}$$

so

$$\frac{h_{12}(r+\pi)-h_{22}}{-h_{11}} = \frac{(\pi+r)}{m/c} + \left(\frac{1}{m/c}\right)^2$$

Thus

$$\left[r+\pi\right]^2 + 4\frac{h_{12}(r+\pi)-h_{22}}{-h_{11}} = \left[r+\pi\right]^2 + 2(\pi+r)\left(\frac{2}{m/c}\right) + \left(\frac{2}{m/c}\right)^2 = \left(\left[r+\pi\right] + \left(\frac{2}{m/c}\right)\right)^2$$

and hence

$$\psi' = \frac{-\left[r+\pi\right] + \sqrt{\left(\left[r+\pi\right] + \left(\frac{2}{m/c}\right)\right)^2}}{2} = \frac{-\left[r+\pi\right] + \left[r+\pi\right] + \left(\frac{2}{m/c}\right)}{2} = \frac{1}{m/c}.$$

In the case of $\gamma > 0$ we have $U = f(h)$:

$$\begin{aligned} U_{11} &= f'h_{11} + f''h_1h_1 = f'\left\{h_{11} + \frac{f''}{f'}h_1h_1\right\} \quad , \quad U_{22} = f'\left\{h_{22} + \frac{f''}{f'}h_2h_2\right\} \\ U_{12} &= f'h_{12} + f''h_1h_2 = f'\left\{h_{12} + \frac{f''}{f'}h_1h_2\right\} \end{aligned}$$

so

$$\frac{U_{22}}{U_{11}} = \frac{h_{22} + \frac{f''}{f'}h_2h_2}{h_{11} + \frac{f''}{f'}h_1h_1} = \frac{h_{22}/h_{11} + \frac{f''}{f'}h_2h_2/h_{11}}{1 + \frac{f''}{f'}h_1h_1/h_{11}} = \frac{h_{22}/h_{11} + \gamma r^2 (h_1h_1)/(-hh_{11})}{1 + \gamma (h_1h_1)/(-hh_{11})}$$

or, since

$$h_1 = \left[c^{-\rho} + \frac{1}{\mathcal{A}}m^{-\rho}\right]^{-1/\rho-1} c^{-\rho-1} \quad , \quad h_{11} = (1+\rho)\left[c^{-\rho} + \frac{1}{\mathcal{A}}m^{-\rho}\right]^{-1/\rho-1} c^{-\rho-2}\left\{\frac{c^{-\rho}}{\left[c^{-\rho} + \frac{1}{\mathcal{A}}m^{-\rho}\right]} - 1\right\}$$

$$\frac{h_1 h_1}{-h h_{11}} = -\frac{\left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-2/\rho-2} c^{2(-\rho-1)}}{(1+\rho)\left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]^{-2/\rho-1} c^{-\rho-2}\left\{\frac{c^{-\rho}}{\left[c^{-\rho}+\frac{1}{\mathcal{A}}m^{-\rho}\right]}-1\right\}} = \frac{1}{(1+\rho)\,r\,(m/c)}$$

Thus

$$\frac{U_{22}}{U_{11}} = \frac{h_{22}/h_{11}+\gamma r^2 (h_1 h_1)/(-h h_{11})}{1+\gamma (h_1 h_1)/(-h h_{11})} = \frac{\left(\frac{1}{m/c}\right)^2+\frac{\gamma r^2}{(1+\rho) r (m/c)}}{1+\frac{\gamma}{(1+\rho) r (m/c)}}$$

and

$$\frac{U_{12}}{U_{11}} = \frac{h_{12}/h_{11}+\gamma r (h_1 h_1)/(-h h_{11})}{1+\gamma (h_1 h_1)/(-h h_{11})}$$

so

$$\begin{aligned}\Delta(\gamma) &\equiv \frac{U_{cm}(2\pi+\delta)-U_{mm}}{(-U_{cc})} = \frac{\left(\frac{1}{m/c}-\frac{\gamma}{(1+\rho)(m/c)}\right)[\pi+r]+\left(\frac{1}{m/c}\right)^2+\frac{\gamma r}{(1+\rho)(m/c)}}{1+\frac{\gamma}{(1+\rho) r (m/c)}} \\ &= \frac{\frac{1}{m/c}[\pi+r]-\frac{\gamma}{(1+\rho)(m/c)}\pi+\left(\frac{1}{m/c}\right)^2}{1+\frac{\gamma}{(1+\rho) r (m/c)}} = \frac{\frac{1}{m/c}[\pi+r]+\left(\frac{1}{m/c}\right)^2}{1+\frac{\gamma}{1+\rho}\frac{1}{r(m/c)}} - \frac{\frac{\pi}{(m/c)}}{\frac{(1+\rho)}{\gamma}+\frac{1}{r(m/c)}}\end{aligned}$$

Thus $\Delta(\gamma/(1+\rho))$ is decreasing in γ provided that $\pi > 0$. Since

$$\psi'(\bar{m},\gamma) = \frac{-[r+\pi]+\sqrt{[r+\pi]^2+4\Delta(\gamma)}}{2}$$

then $\psi'(\gamma)$ is decreasing in γ. $Q.E.D.$

AA-2 Literature Review: interest elasticity, Liquidity Effects and the Price Puzzle

Here we give a brief account of estimates of parameters that quantify two key experiments for the model: the long run interest rate elasticity of the money demand, and the size and persistence of the liquidity effect.

There are many estimates of the interest rate elasticity of money demand. In our model, the parameter $1/(1+\rho)$ correspond to the long run elasticity of the money demand, so we refer to the studies that better much this concept. Lucas (2000) and Stock and Watson (1993) identify this elasticity using the long-run behavior of interest rates and velocity using almost a century of annual data. Lucas preferred estimate is 0.5 for the elasticity of M1 velocity with respect to interest rates, or equivalently a semi-elasticity of velocity with respect to interest rates of 0.8, while Stock and Watson's estimate of the M1 semi-elasticity is 1.0. Hoffman, Rasche, and Tieslau (1995) find long run elasticities similar to the ones estimated by Lucas for five industrialized countries.[10] There is a widespread view that the short run money demand elasticity is substantially smaller than the long run elasticity, see, for instance, Goodfriend 1991). Lucas (2000) and Stock and Watson (1993) interpret the lack of stability as evidence that it takes a very long-span on data to have enough low frequency variation to uncover the elasticities.

Papers that incorporate more recent data, such as Ball (2001) and Ireland (2009), find evidence of instability on the long-run money demand using M1, and also evidence of smaller elasticities. Ireland (2009) finds evidence of a downward shift on money demand after 1980. Ireland estimates elasticities of M1 velocity with respect to interest rates of about 0.05 and semi-elasticities about 1.5 using data after 1980. Ireland, as well as others, interpret the shift on money demand as due to financial innovation and deregulation. One approach suggested by this finding is to use a different monetary aggregate constructed to better reflect the changes in regulatory environment, such as MZM, or "money of zero maturity", or M1S (M1 plus balances "swept", see Dutkowsky, Cynamon, and Jones (2006) for details). Studies using MZM, such as Teles and Zhou (2005) and Carlson et al. (2000), find a more stable money demand. Teles and Zhou (2005) report elasticities of the MZM velocity with respect to interest rates of about 0.25, and Carlson and Keen (1996) find a interest rate semi-elasticity of around 4.0 (See table 10).

Summarizing, estimates of long run interest rate elasticities for money demand range between 0.25 and 0.5, depending on the time periods and monetary aggregates. These estimates are based on low frequency, decade to decade, changes on velocity and interest rates.

While the liquidity effect is present in most description of how monetary shocks affect interest rates, innovations on interest rates and monetary growth do not display the negative correlation suggested by the liquidity effect –see, for instance, Leeper and Gordon (1992). The most common interpretation of this feature of the data is that there are lots of changes in the money supply that are accommodating high frequency changes on money demand. Following this interpretation the literature has used a variety of identifying assumptions to isolate the changes in money supply that accommodate high frequency money demand shocks.

There is a large literature on the effect of "monetary policy shocks" using VARs. The estimates in this literature differ according to the assumptions used to identify the monetary policy shock, the time period and the variables included in the VARs. Yet, almost all of the estimates are based on the assumption that there is a liquidity effect. While the focus of this literature is on the effect of monetary shocks on output, the estimated impulse responses can be used to quantify the size of the impact effect, as well as the persistence of interest rates and monetary aggregates after a monetary shock.

Estimates using VARs typically display persistent effects on interest rates of an innovation on monetary policy, or a monetary policy shock. See for instance the survey of Christiano, Eichenbaum, and Evans (1999) where the impulse response of interest rates seem to be well described as an AR process, with a half life of about half a year. The VARs also estimate the path of monetary aggregates that will follow a monetary shock. For the purpose of

[10]The difference between the interest rate elasticities around 0.5 found by Lucas (2000) and Hoffman, Rasche, and Tieslau (1995), and the one in Stock and Watson (1993) around 0.1 is due to the treatment of the income elasticity of money demand. In Lucas (2000) and Hoffman, Rasche, and Tieslau (1995) a unitary elasticity is imposed, while in Stock and Watson (1993) income elasticities are estimated.

the theory developed in this paper, the persistence of the liquidity effect, as well as its magnitude depends on the persistence of the path of the relevant monetary aggregate that follows a monetary shock. Depending on the assumption on identification of a "monetary shock", and the choices of monetary aggregate, different behavior of monetary aggregates is found.

For instance, one of the identification schemes used in the literature focuses on the behavior of Non-borrowed reserves (or the ratio of borrowed to non-borrowed reserves), as opposed to a broader monetary aggregate such as M1 or M2. This assumption is meant to reflect the operating procedure of the Fed, which accommodates high frequency shocks to reserves. While this assumption may be a good description of the Fed operating procedures, it is not part of the theoretical model, which complicates the mapping between the estimated impulse response and the theoretical path described in the models. In the survey by Christiano, Eichenbaum, and Evans (1999) they compare different identification schemes, and in particular the behavior of Non-Borrowed reserves and broader monetary aggregates. Indeed the patterns of adjustment seem to differ across monetary aggregates. Non-borrowed reserves, M0, and M1 seem to display "u" shaped patterns after a contractionary shock, while M2 seem to display a persistent decrease.

Another statistic of interest that appears in the literature is the difference between the long and short run elasticities of the money demand. For instance Christiano, Eichenbaum, and Evans (1999) compute the short run money demand elasticity, by the ratio of the impact effects on interest rates and money on the impulse response of the estimated VAR. They find that the semi-elasticity of the money demand is smaller than what other estimate based on long run data, say -1.0, to be compared with values between -4.0 and -8.0 discussed above.

Finally, we comment briefly on the behavior of the price level in the model and in the estimated impulse responses. A common specification in the literature is that, by construction, the price level cannot jump contemporaneously with the monetary contraction. The estimated subsequent behavior of the price level after a monetary contraction, depends on the variables included in the system, and to a lesser extent, the period considered. Indeed, the simpler system tend to find an increase in the price level after a contractionary shock, a feature that is called "the price puzzle", since it is against the hypothesis of nominal rigidities.[11] Specifications where the price puzzle is not observed, which typically consist on including variables such as the price of commodities, display a slow decrease in the price level after a contractionary shock. This pattern is consistent with model with nominal rigidities.

The model in this paper has no rigidities on the price adjustment, so the price level can jump on impact after a monetary shock. The importance of nominal rigidities is not a central topic of this paper, indeed one can consider the same model with sticky prices. In the current version with flexible prices, the reaction of inflation diminishes the strength of the liquidity effect. Recall that in the case of i.i.d. shocks to the growth rate of money μ_t, interest rates decrease after an open market operation because the price level increase proportionally with the amount of money, but traders receive a more than proportional amount. Instead, in a sticky price version of the model, the increase on traders' real balances will be even higher.

Christiano, Eichenbaum, and Evans (1999) use M2 as a monetary aggregate, and in a

[11]Notice that an alternative explanation of this puzzle is that the shock identify the economy behaves according to the Fisherian fundamentals, i.e. higher interest rates are associated with higher inflation.

quarterly model find that after a monetary expansion, interest rates decrease in a persistent fashion. The point estimates indicate that the response of interest rates is a bit more persistent than the response of M2, which can be well approximated by an AR(2) with autocorrelation 0.5 and hence a half-life of 3 months. Their estimates display the "price puzzle", i.e. inflation has a small protracted initial decrease. The estimates in Christiano, Eichenbaum, and Evans (2005) also show similar patterns, where the point estimates of the interest rate impulse response is more persistent than the one for money growth. Altig et al. (2005) estimate the effect of a monetary policy shock, using MZM as a measure of money. The result are roughly similar to those in Christiano, Eichenbaum and Evans (2001, 2005) but they also feature velocity. They estimate that the impulse response of velocity and interest rates comoves, as if were generated by a money demand with a small interest rate semi-elasticity.

In summary, the literature present plenty of estimates of the liquidity effect after a monetary shock, but the emphasis and interest of the papers is not on measuring monetary aggregates or the relative persistence. As a consequence, future version of this paper will include estimated VARs that are tailored to estimate the relative persistence of interest rates and velocity of monetary aggregates. In particular the monetary aggregates will be chosen to be consistent with the ones that produce stable long run money demands.

AA-3 Linearizing the model for a generic μ_t process

Uhlig (1998) develops a convenient matlab code to solve linear models. A linear model is defined by the following equations,

$$0 = A\, x_t + B\, x_{t-1} + C\, y_t + D\, z_t \tag{AA-1}$$

$$0 = E_t \left[F\, x_{t+1} + G\, x_t + H\, x_{t-1} + J\, y_{t+1} + K\, y_t + L\, z_{t+1} + M\, z_t \right] \tag{AA-2}$$

$$z_{t+1} = N\, z_t + \varepsilon_{t+1} \quad \text{with } \mathbb{E}_t(\varepsilon_{t+1}) = 0 \tag{AA-3}$$

where A, B, C, D, F, G, H, J, K, L. M, N are matrices and x_t, y_t and z_t are vectors. One should think of x_t as endogenous state-type variables, y_t as control-type variables and z_t as exogenous state-type variables. The timing of the model is that the beginning of a period x_{t-1}, and z_t are given by history and x_t and y_t should be determined so that the equations above hold. Harald's code is a sophisticated version of the method of undetermined coefficients, i.e. it gives a linear solution as

$$\begin{aligned} x_t &= P\, x_{t-1} + Q\, z_t \\ y_t &= R\, x_{t-1} + S\, z_t \end{aligned}$$

i.e. the code gives the matrices P, Q, R and S.

In our case the vector x, y and z are

$$x_t = \left(\hat{m}_t, \hat{n}_t^N\right), \; z_t = \left(\hat{\mu}_{1,t}, \hat{\mu}_{2,t}\right) \text{ and } y_t = \left(\hat{c}_t^T, \hat{c}_t^N, \hat{m}_t^T, \hat{m}_t^N, \hat{\pi}_t, \hat{r}_t\right)$$

where $\mu_{1,t}$ and $\mu_{2,t}$ are two AR(1) process that compose money growth according to $\mu_t = \mu_{1,t} + \mu_{2,t}$. The equations in (AA-1) are the market clearing for consumption, market clearing for money, definition of $\hat{n}_t^N$ in terms of $\hat{m}_t^N$, the definition of $\hat{\pi}_t$ in terms of $\hat{\mu}_t, \hat{m}_t$ and $\hat{m}_{t-1}$, the definition of the interest rate, and the budget constraint of the non-traders:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -\bar{\mu}/\bar{m} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{m}_t \\ \hat{n}_t^N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1/\bar{\pi} \\ \bar{\mu}/\bar{m} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{m}_{t-1} \\ \hat{n}_{t-1}^N \end{bmatrix}$$
$$+ \begin{bmatrix} \lambda & 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & -1 & -\bar{n}/\bar{\pi}^2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ (1+\rho)/\bar{c} & 0 & -(1+\rho)/\bar{m} & 0 & 0 & -1/\bar{r} \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t^T \\ \hat{c}_t^N \\ \hat{m}_t^T \\ \hat{m}_t^N \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix}$$

the equations in (AA-2) are the aggregate money demand, and the Euler equation for the non-traders

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & -\beta\left[\bar{U}_{11}+\bar{U}_{21}\right]/\bar{\pi} \end{bmatrix} \begin{bmatrix} \hat{m}_{t+1} \\ \hat{n}_{t+1}^N \end{bmatrix}$$
$$+ \begin{bmatrix} -1 & 0 \\ 0 & \bar{U}_{11} + \frac{\beta}{\bar{\pi}^2}\left(\bar{U}_{11}+\bar{U}_{12}+\bar{U}_{21}+\bar{U}_{22}\right) \end{bmatrix} \begin{bmatrix} \hat{m}_t \\ \hat{n}_t^N \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\left(\bar{U}_{11}+\bar{U}_{12}\right)/\bar{\pi} \end{bmatrix} \begin{bmatrix} \hat{m}_{t-1} \\ \hat{n}_{t-1}^N \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\left[\frac{\beta}{\bar{\pi}^2}\left(\bar{U}_1+\bar{U}_2\right) + \frac{\beta\bar{n}}{\bar{\pi}^3}\left(\bar{U}_{11}+U_{12}+U_{21}+U_{22}\right)\right] & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_{t+1}^T \\ \hat{c}_{t+1}^N \\ \hat{m}_{t+1}^T \\ \hat{m}_{t+1}^N \\ \hat{\pi}_{t+1} \\ \hat{r}_{t+1} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left[\bar{U}_{11}+\bar{U}_{12}\right]\frac{\bar{n}}{\bar{\pi}^2} & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t^T \\ \hat{c}_t^N \\ \hat{m}_t^T \\ \hat{m}_t^N \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} + \begin{bmatrix} \alpha & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t+1} \\ \hat{\mu}_{2,t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix}$$

and the equation in (AA-3) is the law of motion for the growth rate of the money supply.

$$\begin{bmatrix} \hat{\mu}_{1,t+1} \\ \hat{\mu}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{1,t} \\ \hat{\mu}_{2,t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix}.$$