Open Market Operations and Money Supply at Zero Nominal Interest Rates

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Abstract

What are the paths of money supply that are consistent with zero nominal interest rates and that can be implemented through open market operations? To answer this question, I present an Irrelevance Proposition for some open market operations that exchange money and one-period bonds at the zero lower bound. The result has implications for the conduct of monetary policy (non-irrelevant open market operations modify the equilibrium) and for the implementation of the Friedman rule (any growth rate of money supply above the negative of the rate of time preferences can implement zero nominal interest rate forever, even a positive growth rate, provided that the government collects "large enough" surpluses).

The result holds under several specifications, including models with heterogeneity, non-separable utility, borrowing constraints, incomplete and segmented markets and sticky prices; and generalizes to any situation in which the cost of holding money vs. bonds is zero, such as the payment of interest on money.

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1 Introduction

At zero nominal interest rates, money is perfect substitutes for bonds from the point of view of the private sector. Money demand is thus not uniquely determined, therefore the equilibrium in the money market depends on the supply of money. In this paper, I want to answer the following question: what are the paths of money supply that are consistent with zero nominal interest rates and that are implemented by open market operations? To answer the question, I present an equivalence result that has the interpretation of an irrelevance proposition for (some) open market operations that exchange money and one-period bonds at zero nominal interest rates. The proposition identifies a precise class of open market operations that are irrelevant, in the sense that they change the supply of money and one-period bonds but they do not affect the equilibrium outcome.

There are two main implications of the result. First, for the conduct of monetary policy at the zero lower bound, if the nominal interest rate is expected to be positive at some future dates, there exist non-irrelevant open market operations that modify the equilibrium outcome if implemented. Second, any growth rate of money supply greater or equal then the rate of time preferences is consistent with the Friedman rule (zero nominal interest rates forever), even a positive growth rate of money, provided that the fiscal authority collects "sufficiently large" surpluses. I also generalizes the results to the case in which the opportunity cost of holding money versus bonds is zero, which is relevant in practice because the Federal Reserve and other central banks can pay interests on reserves.

I focus on open market operations, rather than helicopter drops of money, because such tools are the primary means used by many central banks to implement monetary policy decision. In normal times, open market operations are used to maintain a target for the nominal interest rate, while in "unconventional times", when the zero lower bound is binding, open market operations have been used with the aim of stimulating the economy. The economic literature has provided some propositions showing that open market operations are irrelevant in some special cases, in the same sense in which alternative corporate liabilities structures are irrelevant in the Modigliani-Miller theorem. An irrelevant open market operation replaces a given government policy by another equivalent policy, and the original equilibrium outcome is still an equilibrium under the new policy. This type of result is similar also to the Ricardian equivalence in which, under some assumptions, changing the timing of taxation is irrelevant for the equilibrium, so different paths of taxes are equivalent. The basic idea of these equivalence results is the following: if the (real) budget sets of all the agents in the economy are the same under two different policies, then the choice of all agents and thus the equilibrium outcome must be the same, so the two policies are equivalent. Wallace (1981) is one of the first examples of irrelevance theorem for open market operations[1]. More recently, there has been a resurgence of interest in such literature because the zero lower bound on interest rates has become a binding constraint: at zero nominal interest rates, some assets are perfect substitute with money from the point of view of private agents, so the effects and effectiveness of open market operations become non-trivial questions.

To prove the proposition, I show that the budget sets of all the agents are the same under two equivalent policies, as emphasized before. Using this approach, the result holds in a large variety of frameworks, including in models with heterogeneity, non-separable utility, some type of borrowing constraints, incomplete and segmented markets, and sticky prices. The key assumption is that there must exist a satiation threshold for money balances: if this is not the case (for instance, in a model with money in the utility function and log utility from real money balances) then the demand for money at zero nominal interest rates is infinite, so it does not even make sense to talk about open market operation at the zero lower bound in this setting. In practice, the satiation threshold does not seems to be a bad assumption, because recent experiences in Japan and the US have shown that money demand at zero nominal interest rate is high but nonetheless bounded. Thus, the satiation assumption does not look strong or empirically unreasonable.

Loosely speaking, at the zero lower bound, an open market operation that exchanges money with short-term bonds is irrelevant if and only if:

- the effects on the supply of money and bonds are "undone" by some other open market operation(s) before the economy switches back to positive nominal rate, or

[1] See also Peled (1985), Sargent and Smith (1987), Sargent (1987), Wallace (1989) and Lucas (1984).

- the nominal interest rate is zero forever.

Short-term bonds and money are perfect substitute from the point of view of each private agent. But from an equilibrium perspective, it's the infinite-horizon sequences of money and bond supply that matters in determining whether two policies are equivalent, rather than just the time-t supply decisions. For instance, the result of Eggertsson and Woodford (2003) fits the first bullet point above, because the class of open market operations that they consider have effects on bonds and money supply that are reversed while the nominal interest rate is still at the zero lower bound.

By a contraposition argument, any open market operation (involving short-term bonds) that is not irrelevant modifies the equilibrium outcome. For instance, a permanent once-and-for-all increase in money supply (implemented by an open market operation), in an economy with positive future nominal rates, modifies the equilibrium outcome; thus, I generalize and formalize a similar result already noted by Auerbach and Obstfeld (2005), Bernanke, Reinhart, and Sack (2004), Clouse et al. (2003) and Krugman (1998). Or, a change in money and bond supply that is offset after the zero lower bound is not anymore binding is another example of non-irrelevant open market operations. The general result that I derive, however, comes at the cost of being unable to predict the effect of a non-irrelevant open market operations on the equilibrium: I can only say that there will be *some* consequence. On the contrary, a specific model (such as in Auerbach and Obstfeld (2005)) can give a precise description of the effects, but in this case the result applies only to a particular economy and to a particular type of open market operation.

With respect to the implementation of the Friedman rule, which is the first best in many monetary models, previous literature such as Cole and Kocherlakota (1998) and Wilson (1979) concludes that any average growth rate of money between $-\rho$ (where ρ is the rate of time preferences) and zero can implement zero nominal interest rates forever. This result is derived in a model where the government transfers money directly to households through lump-sum transfer. When the desired path of money supply is instead implemented through open market operations, any growth rate of money at least as large as $-\rho$ can implement the Friedman rule, even a positive growth rate. Thus, a path of money supply where money grows a rate larger then $-\rho$ is associated to at least two equilibria: an equilibrium with zero nominal interest rate, and an equilibrium where inflation is equal to the growth rate of money supply, as predicted by the quantity theory. Crucially, fiscal policy is different in this two equilibria, and in the simplified version of the model I show that the fiscal authority must collect "sufficiently large" surpluses in order to achieve zero nominal interest rates.

I first present the results in a simple stylized model with a one-period bond, a representative household, no uncertainty and perfect foresight, in Section 2. I use this framework to provide intuition of the results and to discuss the implications for the conduct of monetary policy (Section 3.1) and for the implementation of the Friedman rule (Sections 3.2 and 3.3), and to generalize the analysis to the case in which the central bank pays interest on money (Section 3.4). Then, I extend the results to a more general class of models in Section 4.

2 A simple example

2.1 Preferences and household choices

Consider a frictionless, perfect-foresight, pure-endowment economy. The economy is composed by a mass 1 of identical households. Let c_t and M_t denote consumption and nominal money holding per household, and P_t the aggregate price level. The utility function of the representative household is given by:

$$\sum_{t=0}^{\infty} \beta^t \left[\log c_t + v\left(m_t\right)\right] \tag{1}$$

where $m_t = \frac{M_t}{P_t}$ are real money balances. This "Sidrauski" specification of money-in-the-utility function is similar to the one in Lucas (2000): agents hold money (cash, checkable accounts, etc) rather than interest-bearing bonds because they need money for transactions, and money-in-the-utility is a simple shortcut to represent such need. With this formulation, money is just like any other good, so money demand depends on its price: the interest rate. Differently from other goods, it is reasonable to impose an assumption of satiation, so that the demand for money is finite when the interest rate converges to zero: when money is above a certain threshold, any extra dollar does not help with transactions, so it does not provide additional utility. I assume the following functional form for the function $v(\cdot)$:

$$v(m) = \begin{cases} -\frac{1}{2}(K-m)^2 & \text{if } 0 \leq m < K \\ 0 & \text{if } m \geq K \end{cases} \tag{2}$$

where $K > 1$ is a parameter. With this specification, $v(m)$ is strictly increasing and strictly concave for $0 < m < K$ and flat for $m \geq K$. Let:

$$-\rho \equiv \log \beta \tag{3}$$

so $-\rho$ is the rate of time preference.

Households maximize (1) subject to:

$$B_{t-1} + M_{t-1} + P_t Y - P_t T_t \geq P_t c_t + \frac{1}{1+i_t} B_t + M_t \tag{4}$$

so the timing for the household is the following:

- the household enters period t with a given amount of money M_{t-1}, bonds B_{t-1} and endowment $Y = 1$;
- the household observes (takes as given) the aggregate price P_t, real lump-sum taxes from the government[2] T_t and the nominal interest rate i_t;
- the household chooses consumption c_t, the nominal amount of bonds B_t to be bought at price $\frac{1}{1+i_t}$ and the nominal amount of money holding M_t to be carried to tomorrow.

I also require that each household satisfy the solvency constraint:

$$\lim_{t \to \infty} Q_{0,t+1} (B_t + M_t) \geq 0 \tag{5}$$

[2] T_t can be negative, so I allow for the possibility of lump-sum transfers.

where:

$$Q_{t,t+s} \equiv (1+i_t)^{-1}(1+i_{t+1})^{-1}...(1+i_{t+s-1})^{-1}. \tag{6}$$

Equation (5) considers the sum of money and bonds, and it is independent of the composition of the wealth of the household. There is also a non-negativity constraint on money and consumption:

$$\begin{aligned} M_t &\geq 0, \quad (7) \\ c_t &\geq 0, \quad (8) \end{aligned}$$

for all $t \geq 0$. Defining inflation $1+\pi_{t+1} = P_{t+1}/P_t$, the optimality conditions of the household are given by a standard Euler equation:

$$\frac{1}{c_t} = \beta \frac{1+i_t}{1+\pi_{t+1}} \frac{1}{c_{t+1}} \tag{9}$$

and by an intra-temporal condition for optimal money holding:

$$v'(m_t) = \frac{i_t}{1+i_t} \frac{1}{c_t}. \tag{10}$$

The transversality condition is satisfied if:

$$\lim_{t\to\infty} \beta^{t+1} \frac{1}{c_{t+1}} \frac{1}{P_{t+1}} (B_t + M_t) = 0. \tag{11}$$

I want to emphasize a key implication of the satiation assumption. When the nominal interest rate is zero, $i_t = 0$, the intra-temporal optimality condition that relates money and the marginal utility of consumption, (10), becomes:

$$v'(m_t) = 0 \tag{12}$$

so the demand of real money balances must be high enough to satiate the agent. Given the specification of $v(\cdot)$ in (2), then any level of money balances above the satiation threshold satisfies this condition, thus:

$$m_t \geq K. \tag{13}$$

In this simple model, $i_t = 0$ allows to reach the first-best because it maximizes the utility of agents, even though consumption is independent of i_t because this is an endowment economy. The Pareto optimality of $i_t = 0$ holds also in other type of monetary models. For instance, assume that there two consumption goods at each point in time: a "cash" good and a "credit" good, and only the "cash" good is subject to a cash-in-advance constraint, as in Lucas and Stokey (1987), and the sum of the consumption of the cash good and of the credit good must be equal to the endowment in the economy. Even in this case, zero nominal interest rates allows to reach the first best, because there is no opportunity cost of holding money, so there is no distortion in the choice between cash good and credit good.

2.2 Government

I use the "upper bar" on nominal money ($\overline{M}_t$) and nominal bonds ($\overline{B}_t$) to denote the supply of money and bonds by the government, in order to distinguish it from the amount demanded by the households (even though in equilibrium they must be the same by market clearing). Also, $\overline{m}_t = \frac{\overline{M}_t}{P_t}$ is real money supply.

I refer to the "government" as the authority in charge of the following policies:

- fiscal policy, which is described by a sequence of real taxes $\{T_t\}_{t=0}^{\infty}$;
- interest rate policy, which is described by a sequence of nominal interest rates $\{i_t\}_{t=0}^{\infty}$;
- (government-issued) assets supply policy, which is described by a sequence of money supply and bond supply $\left\{\overline{M}_t, \overline{B}_t\right\}_{t=0}^{\infty}$.

A government policy is thus described by taxes, nominal interest rate, money and bond supply. In particular, the supply of money and bonds can be altered by the government through open market operations. In the literature, there are two approaches to define open market operations. One approach is used in e.g. Sargent (1987), where an open market operation is defined by an exchange of money and bonds, together with a change in taxes that exactly offset the change in seigniorage arising from the open market operation. The other approach, that I follow, is used in e.g. Wallace (1989) and consists of exchanging money and bonds and keeping real net-of-interest surpluses unchanged.

With respect to money supply, one can think that the monetary authority sets either the level of money $\overline{M}_t$ or the growth rate of money, that I denote as μ_t, since the two are equivalent given $\overline{M}_{t-1}$:

$$\overline{M}_t = (1+\mu_t)\,\overline{M}_{t-1}. \tag{14}$$

The government has to satisfy the following constraint:

$$\overline{B}_{t-1} = P_t T_t + \frac{1}{1+i_t}\overline{B}_t + \mu_t \overline{M}_{t-1} \qquad \text{for all } t \geq 0. \tag{15}$$

In each period, the government has to repay back the bonds $\overline{B}_{t-1}$ issued in the previous periods, and it finances such expenditures imposing lump sum taxes $P_t T_t$, issuing new one-period bond $Q_{t,t+1}\overline{B}_t$ and printing money $\mu_t \overline{M}_{t-1}$. For simplicity, there is no public expenditure in this Section.

You can rewrite equation (15) using a present-value formulation[3]:

$$\overline{B}_{-1} + \overline{M}_{-1} = \sum_{t=0}^{\infty} Q_{0,t} P_t T_t + \sum_{t=0}^{\infty}\left(Q_{0,t}\frac{i_t}{1+i_t}\overline{M}_t\right) + \lim_{s\to\infty} Q_{0,s+1}\left(\overline{B}_s + \overline{M}_s\right) \tag{16}$$

and, in equilibrium, the last term on the right-hand side is zero:

$$\overline{B}_{-1} + \overline{M}_{-1} = \sum_{t=0}^{\infty} Q_{0,t} P_t T_t + \sum_{t=0}^{\infty}\left(Q_{0,t}\frac{i_t}{1+i_t}\overline{M}_t\right). \tag{17}$$

2.3 Equilibrium: definition

I consider the following definition of equilibrium.

[3] From (15), add $\overline{M}_{t-1}$ on both sides and replace $\overline{M}_{t-1} + \mu_t \overline{M}_{t-1}$ with $\overline{M}_t$ on the right-hand side using (14); also add and subtract $\frac{1}{1+i_t}\overline{M}_t$ on the right-hand side getting:

$$\overline{B}_{t-1} + \overline{M}_{t-1} = Q_{t,t+1}\left(\overline{B}_t + \overline{M}_t\right) + P_t T_t + \left(1 - \frac{1}{1+i_t}\right)\overline{M}_t\,, \quad t = 0, 1, 2, \ldots$$

and then combine all these constraints from $t = 0$ to $t = \tau$ and finally take the limit as $\tau \to \infty$.

Definition 2.1. *An equilibrium:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

is a collection of:

- *initial conditions for money* $\overline{M}_{-1}$ *and bonds* $\overline{B}_{-1}$
- *a sequence of consumption* $\{c_t\}_{t=0}^{\infty}$
- *a sequence of prices* $\{P_t\}_{t=0}^{\infty}$
- *fiscal policy and interest rate policy* $\{T_t,\, i_t\}_{t=0}^{\infty}$
- *assets supply policy* $\left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}$

such that:

- *households maximize utility:*
  - *the optimality conditions* (9) *and* (10) *holds for all* $t \geq 0$ *at* $m_t = \frac{\overline{M}_t}{P_t}$*;*
  - *the period-by-period budget constraint* (4) *holds for all* $t \geq 0$ *at* $B_t = \overline{B}_t$ *and* $M_t = \overline{M}_t$*;*
  - *the transversality conditions* (11) *holds at* $M_t = \overline{M}_t$ *and* $B_t = \overline{B}_t$*;*
  - *the solvency constraint* (5) *holds at* $M_t = \overline{M}_t$ *and* $B_t = \overline{B}_t$*;*
  - *the non-negativity constraint for money and consumption* (7) *and* (8) *holds, at* $M_t = \overline{M}_t$*;*
- *the present-value relation of the government* (17) *holds;*
- *the goods market clear:* $c_t = Y = 1$*.*

This notion of equilibrium allows me to abstract from the fiscal-monetary regime which is actually in place. Definition 2.1 can thus fit both the approach of the fiscal theory of the price level and a monetarist approach[4].

[4] See e.g. Cochrane (2005) and Kocherlakota and Phelan (1999) for a discussion of the two regimes, and Bassetto (2002) for an analysis of the role of the budget equation (17) at out-of-equilibrium paths.

2.4 Equivalence Proposition in the simple economy

The next Proposition takes an equilibrium as given, and construct an equivalent class of money and bond supply that leaves the equilibrium unchanged. Also, the equilibrium is unchanged only if money and bond supply are within such class, therefore this is an "if and only if" statement.

The supply of money and bonds can be changed through open market operations (in this simple model, one-period bond is the only type of asset, so an open market operation is implemented by exchanging money with such bonds). So this result can have the interpretation of an Irrelevance Proposition for some open market operations, though in some circumstances there are *combinations* of current and future open market operations that are irrelevant, so one should be careful with this interpretation. Indeed, as I stress in Section 3.1, the entire infinite sequence of money and bonds supply $\left\{\overline{M}_t,\, \overline{B}_t\right\}_{t=0}^{\infty}$ matters, rather then just their time-t value.

Since the initial equilibrium is taken as given, the Proposition is non-vacuous if there exist an equilibrium to start with: the existence of an equilibrium in this simple economy is proved in Appendix A.

Proposition 2.2. *Let:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

be an equilibrium. Then:

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t^*,\, \overline{M}_t^*\right\}_{t=0}^{\infty}\right\}$$

is an equilibrium if and only if:

$$\left.\begin{aligned} \overline{M}_t^* &= \overline{M}_t \\ \overline{B}_t^* &= \overline{B}_t \end{aligned}\right\} \textit{when } i_t > 0$$

$$\left.\begin{aligned} \overline{M}_t^* + \overline{B}_t^* &= \overline{M}_t + \overline{B}_t \\ \overline{M}_t^* &\geq P_t K \end{aligned}\right\} \textit{when } i_t = 0$$

To understand why the Proposition holds, let me use and emphasize the approach of the "equivalence of budget sets". When $i_t = 0$, the period-by-period budget constraint of the household, (4), can be written as:

$$\underbrace{\left(\overline{M}_{t-1} + \overline{B}_{t-1}\right) + P_t Y - P_t T_t}_{\text{given}} - \underbrace{\left(\overline{B}_t + \overline{M}_t\right)}_{=\overline{B}_t^* + \overline{M}_t^*} = P_t c_t \tag{18}$$

where I have used the conditions $M_t = \overline{M}_t$ and $B_t = \overline{B}_t$ since the Proposition assumes that the initial allocation is an equilibrium.

Households take as given their own initial wealth $\overline{M}_{t-1} + \overline{B}_{t-1}$, the endowment and taxes. Such resources are used to buy money and bonds $\overline{B}_t + \overline{M}_t$ and for consumption. Replacing money and bonds with $\overline{B}_t^* + \overline{M}_t^*$, the end-of-period wealth of the agent is unchanged by assumption. Thus, the budget set for consumption choices is not affected. As a result, the same path of prices $\{P_t\}_{t=0}^{\infty}$ still supports the equilibrium, given the goods market clearing condition $c_t = Y$. Since this is an infinite horizon problem, looking at the period-by-period budget constraint is not enough. But the borrowing limit (5) and the transversality condition (11) depend just on the sum of money bonds, so they are not affected by assumption of the Proposition (the sum of money and bonds is always unchanged, no matter whether $i = 0$ or $i > 0$). Finally, the budget constraint of the government is unchanged as well because of market clearing and Walras law. The formal proof of the Proposition is provided in Appendix E, as a special case of the result of Section 4.

3 Discussion

3.1 Irrelevance of open market operations and quantitative easing

Money and short-term bonds are perfect substitutes from the point of view of of the private sector. But this perfect substitutability for private agents does not imply an equivalence of equilibria in general. From a policy perspective, it is important to understand whether two different policies of money and bonds supply are equivalent from an equilibrium perspective, namely if a change in the supply of money and bonds is consistent or not with a given equilibrium. And, a policy is the (infinite) sequence of money and bonds that are supplied, not just their time-t value.

Loosely speaking, at the zero lower bound, an open market operation that exchanges money with short-term bonds is irrelevant if and only if[5]:

- the effects on the supply of money and bonds are "undone" by some other open market operation(s) before the economy switches back to positive nominal rate, or
- the nominal interest rate is zero forever.

By a contraposition argument, any non-equivalent policy results in a different equilibrium; for instance, examples of non-irrelevant open market operations are:

- a one-time exchange of money and short-term bonds
- an open market operations whose effects on the supply of money and bonds are undone after the nominal interest rate has switched to positive.

The case of permanent versus transitory open market operations at the zero lower bound is an idea already discussed in Auerbach and Obstfeld (2005), Bernanke, Reinhart, and Sack (2004) and

[5] An open market operation that can be defined as "irrelevant" according to Proposition 2.2 does not necessarily result in an unchanged equilibrium. For instance, if there are feedback from monetary to fiscal policy, a temporary change in money supply, even if reversed while i_t is still zero, might nonetheless affect government expenditure or taxation; as an example, this could be the case if the time-t policies of the fiscal authority depend on the quantity of bonds held by the private sector. Eggertsson and Woodford (2003) and Bernanke, Reinhart, and Sack (2004) emphasize the importance of ruling out effects of monetary policy on the government budget constraint to get an irrelevance result.

Clouse et al. (2003); Krugman (1998) has a similar discussion about permanent versus transitory changes in money supply. However, this is not the only possibility to modify the equilibrium in a liquidity trap. What matters is that the effects of an open market operation are not reversed as long as $i_t = 0$, but there are still effects on the equilibrium if these effects are reversed later.

Though I formalize and generalize this result already discussed in the literature, I am able to do so at a cost: I only define the class of open market operations involving short-term bonds that are irrelevant, while a fully specified model such as Auerbach and Obstfeld (2005) can also derive precise implications for non-irrelevant open market operations on the equilibrium outcome. However, the strength of my analysis is to show that irrelevant and non-irrelevant open market operations can be identified similarly in a large class of monetary models, as discussed in Section 4.

Also, the Irrelevance Proposition of Eggertsson and Woodford (2003) represents a special case of my analysis. Eggertsson and Woodford (2003) assume that the central bank follows a policy rule that generates exactly a class of equivalent asset supply policies, because the effects of any open market operation at the zero lower bound is undone by the time in which the nominal interest becomes positive, according to equation (11) in their model, and there is no feedback from monetary to fiscal policy.

Finally, notice that quantitative easing can be seen as the combination of two actions: an exchange of money and short-term bonds, together with the swap of short-term bonds and long-term securities[6]. In this perspective, if the exchange of money and short-term bonds will not be reversed before the nominal interest rate switches to positive, then the effects of quantitative easing on the economy must be analyzed as the combination of the consequences of the two actions, and an analysis limited to the swap of of short-term money with long-term bonds would give an incomplete result.

[6] Hamilton and Wu (2012) analyze this second effect.

3.2 Money supply, Friedman rule and quantity theory

What are the paths of money supply that can implement the Friedman rule (zero nominal interest rates forever)? At $i_t = 0$ forever, Proposition 2.2 implies that any process of money and bond is consistent with the original equilibrium, provided that the sum of money and bonds is unchanged and there is enough money to satiate the demand by the private sector. Thus, the traditional quantity-theory result, that links one-to-one the growth rate of money and inflation, is not the unique equilibrium outcome for the process of money supply. Define the average growth rate of money $\{\bar{\mu}_t\}_{t=1}^{\infty}$ as:

$$\left\{\bar{\mu}_t = \frac{1}{t}\sum_{s=1}^{t}\mu_s\right\}_{t=1}^{\infty}$$

then any average growth rate at least as large as $-\rho$ can arise in equilibrium, as shown by the next Corollary[7], which is proved in Appendix B.

Corollary 3.1. *Let:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t = 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

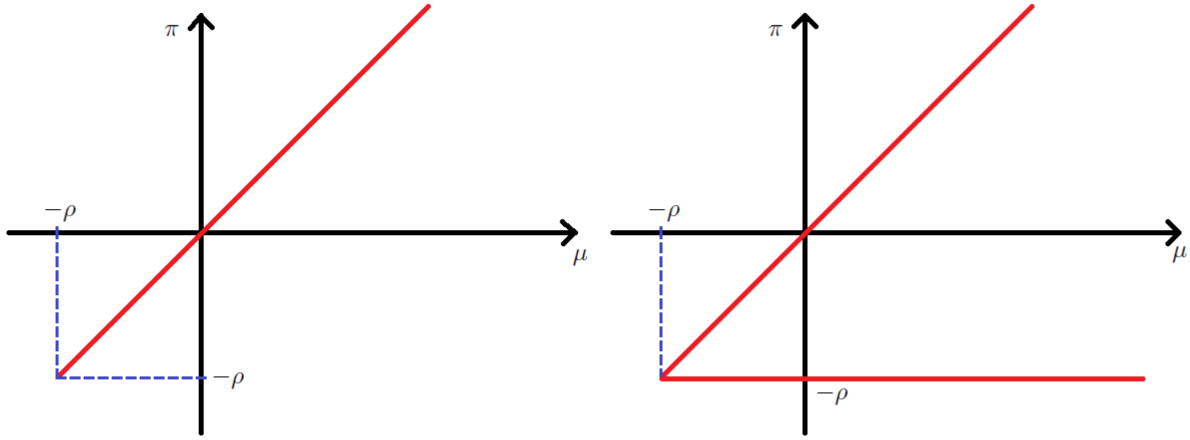
be an equilibrium, with $\frac{\overline{M}_0}{P_0} = K$. *Then* $-\rho \leq \inf\{\bar{\mu}_t\}_{t=1}^{\infty}$.

If the Central Bank implements open market operations such that the average growth rate of money falls below $-\rho$, then agents are not satiated and thus equilibrium condition (10) is not satisfied. On the other hand, any growth rate of money $\geq -\rho$ is an equilibrium outcome, and the path of bonds $\left\{\overline{B}_t\right\}_{t=0}^{\infty}$ adjusts accordingly. Notice that the growth rate of money can even be permanently strictly positive, with the consequence that eventually, at some point in time t, $\overline{B}_{t+s} < 0$ for all $s \geq 0$, so the fiscal authority accumulates positive assets (Section 3.3 analyze the role of fiscal policy in the implementation of the Friedman rule).

Figure 1 is a simple graphical representation of the results that I have derived. The left panel is the representation of the standard view about the quantity theory in this simple economy without

[7] For simplicity, Corollary 3.1 assumes that $\overline{m}_0 = K$. This assumption simplifies the analysis without altering the idea of the result, see the discussion in Appendix B.

Figure 1: The growth rate of money supply: quantity theory and open market operations



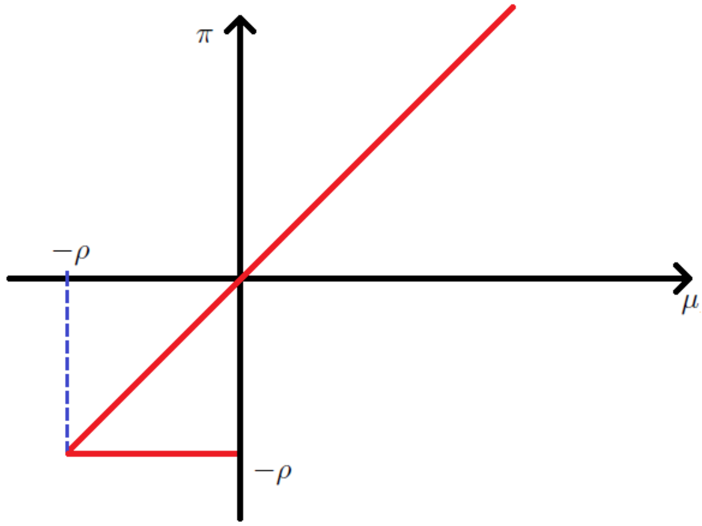
growth[8]: there is one-to-one link between money growth and inflation. The right panel includes the implications of Proposition 2.2: when $i_t = 0$ forever, the inflation rate is $\pi \approxeq -\rho$ and any growth rate $\mu \geq -\rho$ can arise in equilibrium.

The implications of $i_t = 0$ forever have been analyzed by some other studies, especially because the first-best in many monetary models can be achieved at zero nominal interest rates. Wilson (1979) recognizes that there are many paths of money supply that are consistent with $i_t = 0$ forever, but he emphasizes the case when borrowing by the fiscal authority is zero in every period. That's equivalent to imposing the restrictions $\overline{B}_t = 0$ for all $t \geq 0$ in my model, therefore $\overline{M}_t \to 0$ eventually. This is also the result of Cole and Kocherlakota (1998), who derive it as an implication of the transversality condition. Indeed, with no bonds, the transversality condition at $i = 0$ requires that money holdings by household goes to zero in the limit, so the agent has no wealth in the limit. Since money must go to zero in their frameworks (and must be high enough to satiate households), any constant growth rate of money $\mu \in [-\rho, 0)$ is an equilibrium outcome: the result of Wilson (1979) and Cole and Kocherlakota (1998) is depicted in Figure 2.

Woodford (1994) recognizes an indeterminacy in the process of money and bonds at $i = 0$ forever: Proposition 11 in his paper emphasizes that any level of money above a satiation threshold is an equilibrium outcome, and that bonds adjust accordingly. However, he does so in a particular model (a standard model of cash-credit goods) and he does not emphasize the implications for the

[8]The plots refer to equilibria with constant inflation and constant growth rate of money supply.

Figure 2: The growth rate of money supply in Wilson (1979) and Cole and Kocherlakota (1998)



implementation of the Friedman rule.

3.3 Friedman rule and fiscal policy

Figure 3 emphasizes that there exists (at least) two equilibria associated with a given growth rate of money supply $> -\rho$: an equilibrium with zero nominal interest rates, and an equilibrium where inflation is equal to the growth rate of money supply, as predicted by the quantity theory. How are this two equilibria related? It turns out that fiscal policy is crucially different in the two equilibria. Therefore fiscal policy is crucial both to implement the Friedman rule or to avoid zero nominal interest rates forever.

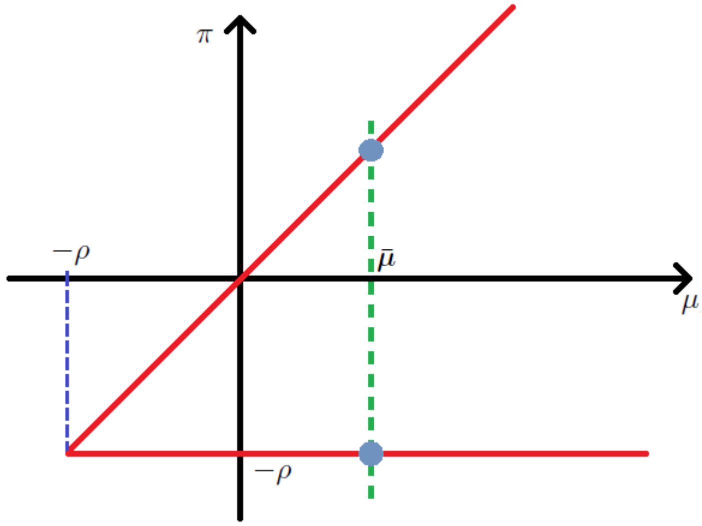
Let me start by stating the following corollary, that follows directly from the Equivalence Proposition 2.2, to formally confirm the idea that there exist more then one equilibrium with a given path of money supply.

Corollary 3.2. *Let:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t \geq 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\} \tag{19}$$

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t^*,\, P_t^*,\, T_t^*,\, i_t^* = 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t^*,\, \overline{M}_t^*\right\}_{t=0}^{\infty}\right\} \tag{20}$$

Figure 3: Examples of equilibria associated with a growth rate of money supply $> -\rho$



be two equilibria such that:

- $i_t > 0$ *for some* t *and* $i_t^* = 0$ *for all* $t \geq 0$*;*
- $\overline{M}_t \geq \overline{M}_t^*$ *for all* $t \geq 0$.

Then there exists a $\left\{\overline{B}_t^{**}\right\}_{t=0}^{\infty}$ *such that:*

$$\left\{\overline{M}_{-1}, \overline{B}_{-1}, \{c_t^*, P_t^*, T_t^*, i_t^* = 0\}_{t=0}^{\infty}, \left\{\overline{B}_t^{**}, \overline{M}_t\right\}_{t=0}^{\infty}\right\} \tag{21}$$

is an equilibrium.

Given the equilibrium in (20), it is possible to implement open market operations such that (21) has the same path of money supply as (19). The next result shows how fiscal policy differs in the two equilibria: real taxes must be higher with zero nominal interest rate forever. I can only prove Proposition 3.3 and the results that follows in the framework of the simple model of this Section. In the general formulation of Section 4, I argue that a similar result must holds, but I can't formally prove it. Since I do not generalize the result anyway, I focus on the simple case in which both equilibria have constant interest rates to emphasize the intuition, but the result holds also if the equilibrium that does not implement the Friedman rule, in (22) has non-constant interest rates.

Proposition 3.3. *Let:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t = \bar{i} > 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\} \tag{22}$$

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t^*,\, P_t^*,\, T_t^*,\, i_t^* = 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t^{**},\, \overline{M}_t\right\}_{t=0}^{\infty}\right\} \tag{23}$$

be two equilibria (based on the result of Corollary 3.2). Then:

$$\sum_{t=0}^{\infty} \beta^t T_t < \sum_{t=0}^{\infty} \beta^t T_t^*$$

Proof. Since (22) is an equilibrium, the present-discounted budget equation of the government is given by (see Appendix A):

$$\frac{\overline{B}_{-1} + \overline{M}_{-1}}{P_0} = \sum_{t=0}^{\infty} \beta^t T_t + S_0$$

where S_0 is the present-discounted value of seigniorage (see equation (29) in Appendix A for the definition of S_0) and $S_0 > 0$ because by assumption $i_t = \bar{i} > 0$ for all $t \geq 0$. Rearranging:

$$P_0 = \frac{\overline{B}_{-1} + \overline{M}_{-1}}{\sum_{t=0}^{\infty} \beta^t T_t + S_0}.$$

Similarly, for equilibrium (23):

$$P_0^* = \frac{\overline{B}_{-1} + \overline{M}_{-1}}{\sum_{t=0}^{\infty} \beta^t T_t^*}$$

and there is no seigniorage in the last expression because the nominal interest rate in (23) is zero forever.

The result is driven by two effects:

1. fixing $P_0 = P_0^*$, then $S_0 > 0$ thus $\sum_{t=0}^{\infty} \beta^t T_t < \sum_{t=0}^{\infty} \beta^t T_t^*$;

2. because $i_t = \bar{i} > 0 = i_t^*$ for all t, including $t = 0$, then:

$$\frac{\overline{M}_0}{P_0^*} \geq K > \frac{\overline{M}_0}{P_0}$$

   thus it is actually the case that $P_0 > P_0^*$; if it were the case that $S_0 = 0$, then $\sum_{t=0}^{\infty} \beta^t T_t <$

$\sum_{t=0}^{\infty} \beta^t T_t^*$;

so the two effects go in the same direction and thus $\sum_{t=0}^{\infty} \beta^t T_t < \sum_{t=0}^{\infty} \beta^t T_t^*$. $\square$

From the perspective of the fiscal theory of the price level, the outcome looks like a "fiscal selection" of equilibria. However, the result is more general because it does not rely on any specific fiscal-monetary interaction mechanism, but only on the existence of the equilibria.

I can now summarize the fiscal and monetary policy stances that allow to implement the Friedman rule.

Corollary 3.4. *Let:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t = 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

be an equilibrium with $\frac{\overline{M}_0}{P_0} = K$. *Then:*

- $-\rho \leq \inf \{\bar{\mu}_t\}_{t=1}^{\infty}$
- $\sum_{t=0}^{\infty} \beta^t T_t \geq \frac{K\left(\overline{M}_{-1}+\overline{B}_{-1}\right)}{\overline{M}_0}$

The first condition derives from Corollary 3.1, while the second condition derives from the requirements $\frac{\overline{M}_0}{P_0} \geq K$ and from the expression for the initial price level $P_0 = \frac{\overline{M}_{-1}+\overline{B}_{-1}}{\sum_{t=0}^{\infty} \beta^t T_t}$. Let me thus emphasize that the implementation of the Friedman rule requires higher taxes for a *given* path of money supply: the second condition of Corollary 3.4 depends not only on taxes, but also on the money supply $\overline{M}_0$ at $t = 0$.

Finally, I also summarize the conditions that allow to avoid zero nominal interest rates forever. The outcome can be achieved either by setting "low enough" taxes, or by setting a positive growth rate of money supply together with a path of bonds supply that does not diverge to minus infinity.

Corollary 3.5. *The set of prices and quantities:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t = 0\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

is not an equilibrium if:

- $\sum_{t=0}^{\infty} \beta^t T_t < \frac{K\left(\overline{M}_{-1}+\overline{B}_{-1}\right)}{\overline{M}_0}$

or:

- $\inf \{\bar{\mu}_t\}_{t=0}^{\infty} > 0$
- $\lim_{t\to\infty} \overline{B}_t > -\infty$

3.4 Interests on money

All the previous results can be extended to the situation in which the central bank pays interests on money, with some remarks. Let r_t denote the interest which is paid on money, so an interest rate policy is now defined by a sequence $\{i_t,\, r_t\}_{t=0}^{\infty}$, and an equilibrium is a collection:

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t,\, r_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

such that household maximizes, the government budget equation holds and markets clear. The equivalence proposition can thus be stated as follow.

Proposition 3.6. *Let:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t,\, r_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

be an equilibrium. Then:

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t,\, r_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t^*,\, \overline{M}_t^*\right\}_{t=0}^{\infty}\right\}$$

is an equilibrium if and only if:

$$\left.\begin{aligned} \overline{M}_t^* &= \overline{M}_t \\ \overline{B}_t^* &= \overline{B}_t \end{aligned}\right\} \textit{when } i_t > r_t$$

$$\left.\begin{aligned} \overline{M}_t^* + \overline{B}_t^* &= \overline{M}_t + \overline{B}_t \\ \overline{M}_t^* &\geq P_t K \end{aligned}\right\} \textit{when } i_t = r_t$$

Therefore, having interest on money $r_t = i_t$ is equivalent to having zero nominal interest rates. With respect to the irrelevance of open market operations, when $i_t = r_t$ or $i_t = 0$, an open market operation that exchanges money with short-term bonds is irrelevant if and only if:

- the effects on the supply of money and bonds are "undone" by some other open market operation(s) before the opportunity costs of holding money switches to positive, or
- the opportunity cost of holding money is zero forever.

Having interests on money $r_t = i_t$ also allows to achieve the Friedman rule. Indeed, with a zero opportunity cost of holding money, the money demand of an household is $\frac{\overline{M}_t}{P_t} \geq K$, no matter what the interest rate is. Restricting for simplicity the analysis to an environment with constant inflation π and constant interest rates $\bar{i}$, since:

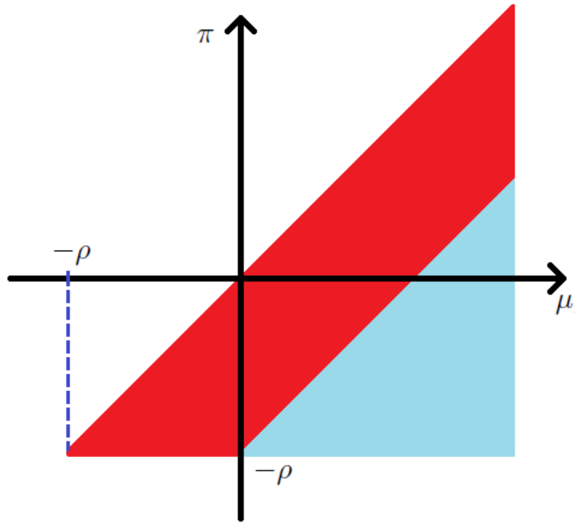
$$\frac{\overline{M}_t}{P_t} = \frac{\overline{M}_{t-1}}{P_{t-1}} \frac{(1+\mu_t)}{(1+\pi)} \geq K$$

the result of Corollary 3.1 can be generalized to obtain:

$$\inf \{\bar{\mu}_t\}_{t=1}^{\infty} \geq \pi$$

Figure 4 depicts the growth rates of money supply that can arise from open market operations when the central bank pays interest on money, which is a generalization of the right panel of Figure 1. For any level of inflation π, any corresponding growth rate of money supply above π can be

Figure 4: The growth rate of money supply with interests on money



Red area: equilibria such that $\lim_{t\to\infty} Q_{0,t+1}\overline{B}_t = 0$; light blue area: equilibria such that $\lim_{t\to\infty} Q_{0,t+1}\overline{B}_t = -\infty$.

an equilibrium. Moreover, using the solvency constraint (5) and the transversality condition (11), you get:

$$\lim_{t\to\infty} Q_{0,t+1}\left(\overline{B}_t + \overline{M}_t\right) = 0$$

Therefore, in an economy with a constant interest rate $\bar{i}$ and constant growth rate of money μ:

$$\lim_{t\to\infty}\left(\frac{1}{1+\bar{i}}\right)^t \overline{B}_t = -\overline{M}_0 \lim_{t\to\infty}\left(\frac{1+\mu}{1+\bar{i}}\right)^t$$

Thus, if $\mu < i \approxeq \rho + \pi$ then the present-discounted value of bonds is zero, while if $\mu > i \approxeq \rho + \pi$ then the present-discounted value of bonds is $-\infty$. The red (darker) area in Figure 4 represents the equilibria such that $\lim_{t\to\infty} Q_{0,t+1}\overline{B}_t = 0$ while the light blue area represents the equilibria such that $\lim_{t\to\infty} Q_{0,t+1}\overline{B}_t = -\infty$.

With respect to the analysis of fiscal policy, implementing the Friedman rule requires “sufficiently large” taxes not only with $i_t = 0$ forever, but also with $i_t = r_t$ forever. Indeed, in this case there is no seigniorage so the initial price level is given by:

$$P_0 = \frac{\overline{B}_{-1} + \overline{M}_{-1}}{\sum_{t=0}^{\infty}\beta^t T_t}$$

and since $\frac{\overline{M}_0}{P_0} \geq K$, then:

$$\sum_{t=0}^{\infty} \beta^t T_t \geq \frac{K\left(\overline{M}_{-1} + \overline{B}_{-1}\right)}{\overline{M}_0}$$

which is the same condition of Corollary 3.4.

4 A general formulation

The notation that I introduce in this Section is meant to formalize a model which is as general as possible, in order to state and prove a general formulation of Proposition 2.2.

The economy can be characterized by uncertainty about some exogenous variables (e.g. productivity,...). I denote $\omega^t = \{\omega^{-1}, \omega_0, ..., \omega_t\}$ to be a history of realization of exogenous variables up to time t; if such realization has occurred, I will say that the economy is in *node* ω^t. Also, let Ω be the set of all nodes that can be reached conditional on ω^{-1}:

$$\Omega = \left\{\omega^s,\, s \geq 0 | \omega^{t+s} = \left\{\omega^{-1}, \omega_0, ..., \omega_s\right\}\right\}.$$

To state the Equivalence Proposition, it turns out that I can look only at the aggregate outcome of an equilibrium, without reference to the choice of each single agent. Next, I describe the government in Section 4.1, then define an "aggregate equilibrium outcome" in Section 4.2, and I finally state and explain the general formulation of the Equivalence Proposition in Section 4.3.

However, to prove the Proposition, I need to check that the budget set of all the agents in the economy are unchanged. The full description of the economy (agents budget sets and choices, and definition of equilibrium) and the proof of the Equivalence Proposition are deferred to Appendix C. Let me just summarize the most important features described there:

- the economy can be characterized by heterogeneity: each agent is indexed by j, and $\mathcal{J}$ is the set of all agents;
- there can be borrowing constraint, provided that such constraint are imposed on the value of money plus bond; in equilibrium, there is no default;

- there can be incomplete markets: it's enough to have just one asset in the economy, namely a one-period bond; but there can be also long-term bonds and other assets;

- the result holds in a pure-endowment economy, as well as in an economy with a production side; I denote with $\overline{\boldsymbol{Z}}(\omega^t)$ a vector of variables that are taken as given by households: this vector can include either the endowment of the household, or the wage rate and variables arising from the profit maximization of firms, together with other variables taken as given by households;

- a crucial assumption is the existence of a satiation threshold for real money balances, for each j: when the nominal interest rate is zero, any level of money above the satiation threshold of agent j satisfies its demand; in contrast to Section 2, the satiation threshold doesn't have to be constant across agents, over time and across states; and it can be due to a formulation of money in the utility function, or to some type of transaction constraints (such as a cash-in-advance constraint); I denote with $\overline{K}(\omega^t)$ the aggregate satiation threshold: if real money supply is above this level, there is enough money to satiate all agents in the economy[9];

I want to stress that the general model of this Section includes a new Keynesian model or any model with nominal rigidities as a special case, provided that the crucial assumption of the existence of a satiation threshold is satisfied[10].

In addition, Section 4.5 discusses how the above assumptions can be relaxed to allow for borrowing constraints only on the value of bonds (excluding money) and for fixed costs to adjust the portfolio of bonds, in the spirit of Baumol-Tobin and of the literature on segmented asset market such as Alvarez, Atkeson, and Kehoe (2002), thought I do not formally prove the equivalence result in these cases.

[9] $\overline{K}(\omega^t)$ denotes the value of the aggregate satiation threshold in node ω^t, which is just the aggregation of the values of the thresholds for each j; as discussed in the Appendix, the individual threshold can be a function of the endogenous choices of the agent, such as consumption.

[10] For instance, Golosov and Lucas (2007) describe a monetary model with money in the utility function, and with log felicity from real money; this specification does not allow for a satiation threshold, therefore my analysis does not apply to their model.

4.1 Government

Let $P(\omega^t)$ and $i(\omega^t)$ denote the price level and the nominal interest rate in node ω^t. In node ω^t, the money supply in node is given by $\overline{M}(\omega^t)$ and the supply of one-period bonds maturing in $t+1$ is given by $\overline{B}(\omega^t)$. The price of such bonds is given by $p(\omega^t)$.

A government policy is given by:

- fiscal policy: the fiscal authority can now choose any type of instrument, not just lump-sum taxes and transfers; the specification of fiscal policy is included in $\left\{\overline{\boldsymbol{Z}}(\omega^t)\right\}_{\omega^t\in\Omega}$;
- interest-rate policy: $\{i(\omega^t)\}_{\omega^t\in\Omega}$;
- (government-issued) asset supply policy: $\left\{\overline{M}(\omega^t), \overline{B}(\omega^t)\right\}_{\omega^t\in\Omega}$ and possibly other assets (such as long-term securities) that are included in $\left\{\overline{\boldsymbol{Z}}(\omega^t)\right\}_{\omega^t\in\Omega}$.

In equilibrium, the following budget constraint holds:

$$\overline{B}\left(\omega^{t-1}\right) = \left[\overline{M}\left(\omega^t\right) - \overline{M}\left(\omega^{t-1}\right)\right] + p\left(\omega^t\right)\overline{B}\left(\omega^t\right) + G\left[\overline{\boldsymbol{Z}}\left(\omega^{t-1}\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), P\left(\omega^t\right)\right] \tag{24}$$

for all $\omega^t = \{\omega^{t-1}, \omega_t\} \in \Omega$. Equation (24), which is similar to equation (15) of Section 2, says that one-period bonds $\overline{B}(\{\omega^{t-1}, \omega_t\})$ must be repaid by printing money, issuing new bonds, or with other revenues captured by the term $G\left[\overline{\boldsymbol{Z}}(\omega^{t-1}), \overline{\boldsymbol{Z}}(\{\omega^{t-1}, \omega_t\}), P(\omega^t)\right]$ that includes primary surpluses and adjustments of the supply of other assets such as long-term securities.

4.2 Aggregate equilibrium outcome

An aggregate equilibrium outcome is a collection of: money and bonds supply, nominal interest rate, price level, price of bonds, outstanding value of government debt, end-of-period government debt, government real primary surplus, aggregate satiation threshold and other variables taken as given by the household, for each node $\omega^t \in \Omega$:

$$\mathcal{E}^{(\mathcal{A})} = \left\{\overline{M}\left(\omega^t\right), \overline{B}\left(\omega^t\right), i\left(\omega^t\right), P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), \overline{K}\left(\omega^t\right) \middle| \omega^t \in \Omega\right\}$$

such that there exists an equilibrium that supports $\mathcal{E}^{(\mathcal{A})}$. An aggregate equilibrium outcome is thus a collection of prices and quantities that that can be observed when looking only at aggregate variables, provided that this aggregate outcome is supported by some equilibrium in which the allocation for each agent $j \in \mathcal{J}$ is optimal given her budget set. Appendix C.4 gives a definition of equilibrium (including also the requirement that each agent maximize utility given her budget set) and a more formal definition of aggregate equilibrium outcome.

Similarly to Section 2.3, I do not take any stand on the fiscal-monetary regime in the economy, because the Equivalence Proposition depends only on the existence of an aggregate equilibrium outcome.

4.3 Equivalence Proposition

I can now state the general formulation of the Equivalence Proposition.

Proposition 4.1. *Let:*

$$\mathcal{E}^{(\mathcal{A})} = \left\{ \overline{M}\left(\omega^t\right), \overline{B}\left(\omega^t\right), i\left(\omega^t\right), P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), \overline{K}\left(\omega^t\right) \middle| \omega^t \in \Omega \right\} \tag{25}$$

be an aggregate equilibrium outcome. Then:

$$\left(\mathcal{E}^{(\mathcal{A})}\right)^* = \left\{ \overline{M}^*\left(\omega^t\right), \overline{B}^*\left(\omega^t\right), i\left(\omega^t\right), P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), \overline{K}\left(\omega^t\right) \middle| \omega^t \in \Omega \right\} \tag{26}$$

is an aggregate equilibrium outcome if and only if, for all $\omega^t \in \Omega$:

$$\left.\begin{array}{rcl} \overline{M}^*\left(\omega^t\right) & = & \overline{M}\left(\omega^t\right) \\ \overline{B}^*\left(\omega^t\right) & = & \overline{B}\left(\omega^t\right) \end{array}\right\} \text{ when } i\left(\omega^t\right) > 0 \tag{27}$$

$$\left.\begin{array}{rcl} \overline{M}^*\left(\omega^t\right) + \overline{B}^*\left(\omega^t\right) & = & \overline{M}\left(\omega^t\right) + \overline{B}\left(\omega^t\right) \\ \overline{M}^*\left(\omega^t\right) & \geq & P\left(\omega^t\right) \overline{K}\left(\omega^t\right) \end{array}\right\} \text{ when } i\left(\omega^t\right) = 0. \tag{28}$$

In node ω^t such that $i(\omega^t) = 0$, the price of a one-period bond is $p(\omega^t) = 1$ because the nominal interest rate is zero in node ω^t, and there is no default either for the government or for private agents.

I now provide an intuition for the proof, while the details are presented in Appendix D. The first part of the proof (conditions (27) and (28) imply that (26) is an aggregate equilibrium outcome) works as follow. Money and one-period bonds are perfect substitutes, therefore I can assign to each agent a portfolio of money and bonds such that the sum of the two is unchanged. This portfolio is feasible, because the sum of the supply of money and bonds is unchanged by assumption. Therefore, fixing the prices in the aggregate equilibrium outcome $\mathcal{E}^{(\mathcal{A})}$, which are the same as the prices of $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$, the budget set of all agents for choices other than money and bonds is unchanged. Thus, the choices that they were making in the original equilibrium must still be optimal. Appendix D shows that there exist a "well-specified" equilibrium that supports the aggregate equilibrium outcome $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$.

To show the reverse (i.e. (26) is an aggregate equilibrium outcome implies that conditions (27) and (28) must hold) I use the government budget set equation, (24): to keep it unchanged, the sum of money and bonds when $i(\omega^t) = 0$ must be unchanged, and both the supply of money and bonds must be unchanged when $i(\omega^t) > 0$. The last condition in (28) must be satisfied otherwise some agent is not satiated.

4.4 Implications and discussion

The implications for the conduct of monetary policy and for the analysis of quantitative easing are the same as in Section (3.1).

With respect to the implementation of the Friedman rule, the precise definition of "zero nominal interest rates forever", in this more general framework, is "zero nominal interest rates in all time periods and in all states", or $i(\omega^t) = 0$ for all $\omega^t \in \Omega$. In the general framework of this Section, it is not possible to derive a simple expression for the lower bound of the growth rate of money, because such lower bound depends on how the aggregate satiation threshold for real money balances $\overline{K}(\omega^t)$

evolves over time and across states. Anyway, the results that there exists some finite lower bound on the growth rate of money, and that there is no upper bound, still holds in the more general framework.

The role of fiscal policy and the implications for the implementation of the Friedman rule cannot be discussed in this general formulation. Corollary 4.2 below generalizes Corollary 3.2, so the result that there exist more than one equilibrium associated with a given path of money supply still holds. However, I cannot formally analyze the role of fiscal policy in this more general framework, even though the intuitions discussed in Section 3.3 and the proof of Proposition 3.3 suggest that the result is more general.

Corollary 4.2. *Let:*

$$\left\{\overline{M}\left(\omega^t\right), \overline{B}\left(\omega^t\right), i\left(\omega^t\right) \geq 0,\ P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), \overline{K}\left(\omega^t\right) \middle| \omega^t \in \Omega\right\}$$

$$\left\{\overline{M}^*\left(\omega^t\right), \overline{B}^*\left(\omega^t\right), i^*\left(\omega^t\right) = 0,\ P^*\left(\omega^t\right), p^*\left(\omega^t\right), \overline{\boldsymbol{Z}}^*\left(\omega^t\right), \overline{K}^*\left(\omega^t\right) \middle| \omega^t \in \Omega\right\}$$

be aggregate equilibrium outcomes such that:

- $i\left(\omega^t\right) > 0$ *for some* $\omega^t \in \Omega$ *and* $i^*\left(\omega^t\right) = 0$ *for all* $\omega^t \in \Omega$*;*
- $\overline{M}\left(\omega^t\right) \geq \overline{M}^*\left(\omega^t\right)$ *for all* $\omega^t \in \Omega$*.*

Then there exists a $\left\{\overline{B}^{**}\left(\omega^t\right)\right\}_{\omega^t \in \Omega}$ *such that:*

$$\left\{\overline{M}\left(\omega^t\right), \overline{B}^{**}\left(\omega^t\right), i^*\left(\omega^t\right) = 0,\ P^*\left(\omega^t\right), p^*\left(\omega^t\right), \overline{\boldsymbol{Z}}^*\left(\omega^t\right), \overline{K}^*\left(\omega^t\right) \middle| \omega^t \in \Omega\right\}$$

is an aggregate equilibrium outcome.

4.5 Extensions: borrowing constraints on bonds only and segmented asset markets

The assumptions of the model of Section 4 can be relaxed even further, but this may requires some changes in the statement and the implications of the equivalence proposition.

First, the model can be enriched to allow for borrowing constraints on the value of bonds (excluding money). If only a subset of agents have borrowing constraints on the value of bonds, and another subset of agents have only borrowing constraints on the total value of money and bonds, then the result of the previous analysis is unchanged[11]. On the other hand, if all agents on the economy have borrowing constraints on the value of bonds (excluding money), then the equivalence proposition must impose a lower bound on the value of government bonds. For instance, if all agents have a constraint $B^j(\omega^t) \geq 0$, then it must be the case that $\overline{B}(\omega^t) \geq 0$, otherwise the market clearing condition cannot hold. If the private sector holds a non-negative amount of Treasury securities and open market operations are small compared to the size of government debt (as it is the case in practice), the implications for the conduct of monetary policy are unchanged. With respect to the Friedman rule, zero nominal interest rates forever can only be implemented with a growth rate of money below a certain threshold[12].

Second, I can introduce fixed costs to adjust the portfolio of money and bonds, in the spirit of Baumol-Tobin and of the literature on segmented asset market such as Alvarez, Atkeson, and Kehoe (2002). In this case, the result is unchanged, provided that, in every period, there is a subset of agents that adjust their own portfolio. The result, however, is not robust to the introduction of proportional transaction costs, because such costs would alter the budget sets of agents.

[11] The subset of agent that has only borrowing constraints on the total value of money and bonds must be "large enough": for instance, if $\mathcal{J} = [0, 1]$, then there must be at least a continuum of agents that has borrowing constraints on the total value of money and bonds, so open market operations can be implemented by the central bank with these agents.

[12] In the simple model with $\overline{B}_t \geq 0$, the Friedman rule can be implemented with a growth rate $-\rho \leq \mu < 0$.

5 Conclusions

I have presented an equivalence result that has the interpretation of an irrelevance proposition for some combination open market operations that exchange money and short-term bonds at zero nominal interest rates. The result is based on the equivalence of budget sets and thus it holds for a large class of monetary models, and it can be generalizes to any situation in which the opportunity cost of holding money versus bonds is zero (such has having interests on money).

The analysis has two main implications. First, non-irrelevant open market operations that changes the supply of short-term bonds do affect the equilibrium outcome, a result that has consequences for the conduct of monetary policy. For instance, since quantitative easing can be decomposed in an exchange of money and short-term bonds together with a swap of short-term and long-term securities, then analyses of quantitative easing that look only at the latter part may be incomplete.

Second, the Friedman rule can be implemented with any growth rate of money (provided that money does not shrink "too fast"); in particular, a positive constant growth rate of money supply is still consistent with the implementation of the Friedman rule. As a result of this second implication, the one-to-one link between money growth and inflation emphasized by the quantity theory does not hold at zero nominal interest rate. For a given path of money supply, there exists at least two equilibria: an equilibrium with zero nominal interest rates forever, and an equilibrium where inflation is equal to the growth rate of money supply; fiscal policy is different in these two equilibria, and the fiscal authority must collect "sufficiently large" surpluses in order to implement the Friedman rule.

The lack of any relationship between money growth and inflation holds at zero nominal interest rate, but as a topic for future research it would be interesting to analyze if a weak link between money growth and inflation can be established when the nominal interest rate is "close" to zero, but not necessarily equal to zero.

References

Alvarez, F., A. Atkeson, and P.J. Kehoe. 2002. "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets." *Journal of Political Economy* 110 (1).

Auerbach, A. and M. Obstfeld. 2005. "The Case for Open-Market Purchases in a Liquidity Trap." *American Economic Review* 95 (1):110–137.

Bassetto, M. 2002. "A Game–Theoretic View of the Fiscal Theory of the Price Level." *Econometrica* 70 (6):2167–2195.

Bernanke, B., V. Reinhart, and B. Sack. 2004. "Monetary Policy Alternatives at the Zero Bound: An Empirical Assessment." *Brookings Papers on Economic Activity* 2004 (2):1–100.

Clouse, J., D.W. Henderson, A. Orphanides, D.H. Small, and P. Tinsley. 2003. "Monetary Policy When the Nominal Short-Term Interest Rate is Zero." *The BE Journal of Macroeconomics* (1).

Cochrane, J.H. 2005. "Money as stock." *Journal of Monetary Economics* 52 (3):501–528.

Cole, H.L. and N. Kocherlakota. 1998. "Zero nominal interest rates: why they're good and how to get them." *Federal Reserve Bank of Minneapolis Quarterly Review* 22:2–10.

Eggertsson, G.B. and M. Woodford. 2003. "Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity* 2003 (1):139–233.

Golosov, M. and R.E. Lucas. 2007. "Menu Costs and Phillips Curves." *Journal of Political Economy* 115:171–199.

Hamilton, J.D. and J.C. Wu. 2012. "The effectiveness of alternative monetary policy tools in a zero lower bound environment." *Journal of Money, Credit and Banking* 44:3–46.

Kocherlakota, N. and C. Phelan. 1999. "Explaining the fiscal theory of the price level." *Federal Reserve Bank of Minneapolis Quarterly Review* 23 (4):14–23.

Krugman, P. 1998. "It's Baaack: Japan's Slump and the Return of the Liquidity Trap." *Brookings Papers on Economic Activity* 29 (2):137–206.

Lucas, R.E. 1984. "Money in a theory of finance." In *Carnegie-Rochester Conference Series on Public Policy*, vol. 21. Elsevier, 9–46.

———. 2000. "Inflation and welfare." *Econometrica* 68 (2):247–274.

Lucas, R.E. and N.L. Stokey. 1987. "Money and Interest in a Cash-in-Advance Economy." *Econometrica: Journal of the Econometric Society* :491–513.

Peled, D. 1985. "Stochastic inflation and government provision of indexed bonds." *Journal of Monetary Economics* 15 (3):291–308.

Sargent, T.J. 1987. *Dynamic macroeconomic theory*. Harvard Univ Pr.

Sargent, T.J. and B.D. Smith. 1987. "Irrelevance of open market operations in some economies with government currency being dominated in rate of return." *The American Economic Review* :78–92.

Wallace, N. 1981. "A Modigliani-Miller theorem for open-market operations." *The American Economic Review* 71 (3):267–274.

———. 1989. "Some Alternative Monetary Models and Their Implications for the Role of Open-Market Policy." *Modern Business Cycle Theory, Harvard University Press, Cambridge, Massachusetts* .

Wilson, C. 1979. "An infinite horizon model with money." In *General equilibrium, growth, and trade*, edited by J. R. Green and J. A. Scheinkman. New York: Academic Press.

Woodford, M. 1994. "Monetary policy and price level determinacy in a cash-in-advance economy." *Economic theory* 4 (3):345–380.

A Equilibrium in the Economy in Section 2: Existence

For convenience, define S_0 to be the real present-discounted value of seigniorage that the government collects:

$$S_0 \equiv \sum_{t=0}^{\infty} \beta^t \left(\frac{i_t}{1+i_t}\right) \left[K - \frac{i_t}{1+i_t}\right]. \tag{29}$$

To get an intuition for equation (29), note that the term $K - \frac{i_t}{1+i_t}$ is time-t real money demand in the equilibrium of Proposition A.2, so the term $\frac{i_t}{1+i_t}\left(K - \frac{i_t}{1+i_t}\right)$ is the real seigniorage collected in period t by the government[13]. The next Lemma shows that S_0 is bounded.

Lemma A.1. *The seigniorage S_0 defined in equation* (29) *is bounded:* $0 \leq S_0 < \infty$.

Proof. Notice that:

$$i \to \infty \;\Rightarrow\; \frac{i}{1+i} \to 1;$$

$$i \to 0 \;\Rightarrow\; \frac{i}{1+i} \to 0;$$

$$\frac{\partial}{\partial i}\left(\frac{i}{1+i}\right) = \frac{1}{(1+i)^2} > 0 \quad \text{for all } i \geq 0. \tag{30}$$

Then define the continuous function $f : [0,1] \to \mathbb{R}$:

$$f(y) = yK - y^2 \quad , \quad 0 \leq y \leq 1 \quad , \quad K > 1$$

which is bounded (by the boundedness theorem). Define:

$$y(i) = \frac{i}{1+i}$$

Therefore, letting $y^* \in [0,1]$ be the maximizer of $f(y)$, then $\exists i^* \geq 0$ such that $y(i^*) = y^*$ because

[13] To understand this expression, consider that nominal money demand is $M_t = P_t\left[K - \frac{i_t}{1+i_t}\right]$ in the Equilibrium of Proposition A.2. Holding bonds instead of money, households could get an extra $i_t M_t$ tomorrow, or dividing it by $\frac{1}{1+i_t}$ to discount it to today:

$$\frac{i_t}{1+i_t} M_t = \frac{i_t}{1+i_t} P_t \left(K - \frac{i_t}{1+i_t}\right)$$

which is the nominal seigniorage collected by the government in period t. Dividing by P_t you get the expression for real seigniorage $\frac{i_t}{1+i_t}\left(K - \frac{i_t}{1+i_t}\right)$.

$y(i)$ is strictly increasing by (30) and thus invertible. As a consequence:

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{i^*}{1+i^*}\right)\left[K - \frac{i^*}{1+i^*}\right] = \sum_{t=0}^{\infty} \beta^t f\left(y\left(i^*\right)\right) < \infty$$

since $0 < \beta < 1$. Thus, using the Assumption $K > 1$ (stated in Section 2.1), $0 \leq S_0 < \infty$. $\square$

I can now state and prove the following existence Proposition.

Proposition A.2. *There exists an equilibrium:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

such that:

- $c_t = Y = 1$ *for all* $t \geq 0$*;*
- $\{T_t,\, i_t\}_{t=0}^{\infty}$, $\overline{M}_{-1}$ *and* $\overline{B}_{-1}$ *can take any value, provided that they satisfy*[14]*:*
    - $\overline{M}_{-1} > 0$ *and* $\overline{M}_{-1} + \overline{B}_{-1} > 0$
    - $0 < \sum_{t=0}^{\infty} \beta^t T_t < \infty$
    - $0 \leq i_t < \infty$ *for all* $t \geq 0$*;*
- *prices are given by:*

$$P_0 = \frac{\overline{B}_{-1} + \overline{M}_{-1}}{S_0 + \sum_{t=0}^{\infty} \beta^t T_t} = \frac{\overline{B}_{-1} + \overline{M}_{-1}}{\sum_{t=0}^{\infty} \beta^t \left\{T_t + \left(\frac{i_t}{1+i_t}\right)\left[K - \frac{i_t}{1+i_t}\right]\right\}} = \frac{\textit{government liabilities}}{\textit{real taxes+real seigniorage}}, \quad (31)$$

$$P_t = P_0 \beta^t \prod_{j=0}^{t-1} (1 + i_j) \quad , t \geq 1; \quad (32)$$

- *nominal money supply is given by:*

$$\overline{M}_t = P_t \left(K - \frac{i_t}{1+i_t}\right) \qquad \textit{for all } t \geq 0; \quad (33)$$

[14]These assumptions are sufficient but not necessary for the existence of the equilibrium. I have chosen these restrictive assumptions to simplify the proof, because the objective of this Proposition is just to show that there exist an equilibrium so the Equivalence Proposition 2.2 is non-vacuous.

- *the path of bond supply satisfies:*

$$\overline{B}_t = (1 + i_t)\,\overline{B}_{t-1} - (1 + i_t)\left[\left(\overline{M}_t - \overline{M}_{t-1}\right) + P_t T_t\right] \tag{34}$$

for all $t \geq 0$.

Proof. Market clearing is trivially satisfied.

By assumption, $\overline{M}_{-1} + \overline{B}_{-1} > 0$ and $0 < \sum_{t=0}^{\infty} \beta^t T_t < \infty$. Using also Lemma A.1, then the initial price level (31) is bounded: $0 < P_0 < \infty$.

Using $Q_{t,t+1} = \frac{1}{1+i_t}$ from equation (6) and $c_t = Y = 1$ for all $t \geq 0$, the Euler equation (9) and the money FOC (10) become:

$$\frac{1}{1 + i_t} = \beta \frac{1}{1 + \pi_{t+1}}, \tag{35}$$

$$v'(m_t) = \frac{i_t}{1 + i_t}. \tag{36}$$

Re-arranging the Euler equation (35) you get:

$$1 + \pi_{t+1} = \beta\,(1 + i_t)$$

The path of prices is:

$$P_t = P_0 \beta^t \prod_{j=0}^{t-1} (1 + i_j)\,.$$

Since $P_0 > 0$, you can take the ratio of P_{t+1} and P_t:

$$\frac{P_{t+1}}{P_t} = \beta\,(1 + i_t)$$

Since $1 + \pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$, then the Euler equation is satisfied.

In order to show that the money FOC (36) is satisfied, define a function $\tilde{v}(m) : [0, K] \to \mathbb{R}$ such that:

$$\tilde{v}(m) \equiv v(m) \quad \text{for } 0 \leq m \leq K$$

so $\tilde{v}(m)$ is strictly monotone on its domain and thus invertible. Notice that if m^* solves $\tilde{v}'(m^*) = \frac{i_t}{1+i_t}$ then the same m^* must solve $v'(m^*) = \frac{i_t}{1+i_t}$. Since $\tilde{v}(\cdot)$ is strictly concave, then $\tilde{v}'(\cdot)$ is invertible, and:

$$m_t = (\tilde{v}')^{-1}\left(\frac{i_t}{1+i_t}\right) = K - \frac{i_t}{1+i_t}$$

(where the second inequality uses the functional form $\tilde{v}(m) = -\frac{1}{2}(K-m)^2$ so $\tilde{v}'(m) = K - m$) which is solved by:

$$\overline{m}_t = \frac{\overline{M}_t}{P_t} = K - \frac{i_t}{1+i_t} \tag{37}$$

using the equilibrium requirement $m_t = \overline{m}_t$ and the definition $\overline{m}_t = \frac{\overline{M}_t}{P_t}$. Note that $\overline{m}_t = \frac{\overline{M}_t}{P_t} > 0$ for all $t \geq 0$ since, by assumption, $K > 1$, so the non-negativity constraint for money holding (7) is never binding, since $P_0 > 0$, $i_t \geq 0$ and $\beta > 0$ imply $P_t > 0$.

From the government present-value equation (16):

$$\overline{B}_{-1} + \overline{M}_{-1} = \sum_{t=0}^{\infty} Q_{0,t} P_t T_t + \sum_{t=0}^{\infty} Q_{0,t}\left(1 - Q_{t,t+1}\right)\overline{M}_t + \lim_{t\to\infty} Q_{0,t+1}\left(\overline{B}_t + \overline{M}_t\right), \tag{38}$$

using the definition of $Q_{0,t}$ in (6) and the path of prices in (32), the second series on the right-hand side becomes:

$$\begin{aligned}
\sum_{t=0}^{\infty}\left(Q_{0,t} - Q_{0,t+1}\right)\overline{M}_t &= \overline{M}_0\left(1 - Q_{0,1}\right) + \sum_{t=1}^{\infty}\left(1 - Q_{t,t+1}\right) Q_{0,t} P_t \frac{\overline{M}_t}{P_t} \\
&= P_0 \frac{\overline{M}_0}{P_0}\frac{i_0}{1+i_0} + P_0 \sum_{t=1}^{\infty}\left(\frac{i_t}{1+i_t}\right)\frac{1}{\prod_{j=0}^{t-1}(1+i_j)}\beta^t \prod_{j=0}^{t-1}(1+i_j)\left[K - \frac{i_t}{1+i_t}\right] \\
&= P_0 S_0
\end{aligned}$$

where:

$$S_0 \equiv \sum_{t=0}^{\infty}\beta^t\left(\frac{i_t}{1+i_t}\right)\left[K - \frac{i_t}{1+i_t}\right]$$

is the (present-discounted value of) real seigniorage that the government obtains. The present-discounted value of taxes in (16) can be written as:

$$\begin{aligned}\sum_{t=0}^{\infty} Q_{0,t} P_t T_t =& P_0 \sum_{t=0}^{\infty} \left[\frac{1}{\prod_{j=0}^{t-1} (1+i_j)} \beta^t \left\{ \prod_{j=0}^{t-1} (1+i_j) \right\} T_t \right] \\ =& P_0 \sum_{t=0}^{\infty} \beta^t T_t\end{aligned}$$

Thus equation (16) becomes:

$$\overline{B}_{-1} + \overline{M}_{-1} = P_0 \left[\sum_{t=0}^{\infty} \beta^t T_t + S_0 \right] + \lim_{t\to\infty} Q_{0,t+1} \left(\overline{B}_t + \overline{M}_t \right).$$

Re-arranging:

$$P_0 = \frac{\overline{B}_{-1} + \overline{M}_{-1} - \lim_{t\to\infty} Q_{0,t+1} \left(\overline{B}_t + \overline{M}_t \right)}{S_0 + \sum_{t=0}^{\infty} \beta^t T_t} = \frac{\overline{B}_{-1} + \overline{M}_{-1} - \lim_{t\to\infty} Q_{0,t+1} \left(\overline{B}_t + \overline{M}_t \right)}{\sum_{t=0}^{\infty} \beta^t \left\{ T_t + \left(\frac{i_t}{1+i_t} \right) \left[K - \frac{i_t}{1+i_t} \right] \right\}}$$

and using (31), you get:

$$\lim_{t\to\infty} Q_{0,t+1} \left(\overline{B}_t + \overline{M}_t \right) = 0 \tag{39}$$

Thus, the solvency constraint of the agent (5), evaluated at $M_t = \overline{M}_t$ and $B_t = \overline{B}_t$ is satisfied. Also, the transversality condition (11) becomes:

$$\begin{aligned}0 = \lim_{t\to\infty} \beta^{t+1} \frac{1}{c_{t+1}} \frac{1}{P_{t+1}} (B_t + M_t) =& \lim_{t\to\infty} \beta^{t+1} \frac{1}{P_0 \beta^{t+1} \prod_{j=0}^{t} (1+i_j)} \left(\overline{B}_t + \overline{M}_t \right) = \\ =& \frac{1}{P_0} \lim_{t\to\infty} Q_{0,t+1} \left(\overline{B}_t + \overline{M}_t \right),\end{aligned}$$

which again holds because $0 < P_0 < \infty$ and because of equation (39).

Finally, the period-by-period budget constraint (4), evaluated at $c_t = Y$, $B_t = \overline{B}_t$ and $M_t = \overline{M}_t$ becomes:

$$(1+i_t) \overline{B}_{t-1} - \left[\left(\overline{M}_t - \overline{M}_{t-1} \right) + P_t T_t \right] \geq \overline{B}_t$$

which is satisfied by (34).

$\square$

Since Section 3.2 analyzes open market operation when $i_t = 0$ forever, I want to emphasize that the Equivalence Proposition is non-vacuous also for this case. The next Corollary states the result for $i_t = 0$ forever; the proof is omitted since it is identical to the proof of Proposition A.2.

Corollary A.3. *There exists an equilibrium:*

$$\left\{\overline{M}_{-1},\, \overline{B}_{-1},\, \{c_t,\, P_t,\, T_t,\, i_t\}_{t=0}^{\infty},\, \left\{\overline{B}_t,\, \overline{M}_t\right\}_{t=0}^{\infty}\right\}$$

such that:

- $i_t = 0$ *for all* $t \geq 0$*;*
- $c_t = Y = 1$ *for all* $t \geq 0$*;*
- $\{T_t\}_{t=0}^{\infty}$, $\overline{M}_{-1}$ *and* $\overline{B}_{-1}$ *can take any value, provided that they satisfy:*
    - $\overline{M}_{-1} > 0$ *and* $\overline{M}_{-1} + \overline{B}_{-1} > 0$
    - $0 < \sum_{t=0}^{\infty} \beta^t T_t < \infty$
- *prices are given by:*

$$P_0 = \frac{\overline{B}_{-1} + \overline{M}_{-1}}{\sum_{t=0}^{\infty} \beta^t T_t} = \frac{\textit{government liabilities}}{\textit{real taxes}},$$

$$P_t = \beta P_{t-1} \quad , \, t \geq 1;$$

- *nominal money supply is given by:*

$$\overline{M}_t = P_t K \qquad \textit{for all } t \geq 0;$$

- *the path of bond supply satisfies:*

$$\overline{B}_t = \overline{B}_{t-1} - \left[\left(\overline{M}_t - \overline{M}_{t-1}\right) + P_t T_t\right]$$

  for all $t \geq 0$.

B Proof of Corollary 3.1

Proof. By assumption, there exists an equilibrium with $i_t = 0$ forever. By Proposition 2.2, such equilibrium must display $\overline{m}_t = \frac{\overline{M}_t}{P_t} \geq K$ for all t. Thus:

$$K \leq \overline{m}_t = \overline{m}_0 \frac{\prod_{j=1}^{t} (1+\mu_j)}{(1+\pi)^t} = \overline{m}_0 \exp\left\{ t\,\frac{1}{t} \sum_{j=1}^{t} \underbrace{\log(1+\mu_j)}_{<\mu_j} - t\underbrace{\log \beta}_{\equiv -\rho} \right\} < \underbrace{\overline{m}_0}_{=K} \exp\left\{ t\left[\bar{\mu}_t + \rho\right]\right\}$$

then $\bar{\mu}_t > -\rho$ or $\inf \{\bar{\mu}_t\}_{t=1}^{\infty} \geq -\rho$. □

If $\overline{m}_0 > K$, then the average growth rate of money supply can be temporarily smaller then $-\rho$. However, as t gets large, the condition $\bar{\mu}_t > -\rho$ must be satisfied eventually. Formally, one could state that there exists a $\bar{t}$ satisfying $0 \leq \bar{t} < \infty$ such that for all $t \geq \bar{t}$ the condition $\bar{\mu}_t > -\rho$ must hold, so $\inf \{\bar{\mu}_t^*\}_{t=\bar{t}}^{\infty} \geq -\rho$. Thus, the statement that money cannot shrink faster then the rate of time preferences must be eventually true.

C General formulation: agents choices and equilibrium definition

C.1 Private agents

In each node $\omega^t = \{\omega^{t-1}, \omega_t\}$, agent $j \in \mathcal{J}$ chooses how much money $M^j(\omega^t)$ and one-period bonds $B^j(\omega^t)$ to hold, given her initial wealth $W^j(\omega^{t-1}, \omega^t)$: the value of the initial wealth depends on quantities decided in ω^{t-1} (first argument of W^j) and on prices in $\omega^t = \{\omega^{t-1}, \omega_t\}$ (second argument of W^j).

Define the following objects:

- let $A^j(\omega^t)$ be the value of money and short-term bonds held by agent j in node ω^t:

$$A^j\left(\omega^t\right) = M^j\left(\omega^t\right) + p\left(\omega^t\right) B^j\left(\omega^t\right); \tag{40}$$

- let $c^j(\omega^t)$ denote consumption of some good by agent j; the unit price of this consumption good is $P(\omega^t)$;
- let $\boldsymbol{x}^j(\omega^t)$ denote a vector of choice variables other than money, one-period bonds and $c^j(\omega^t)$, that are under the control of agent j in node ω^t; for instance, $\boldsymbol{x}^j(\omega^t)$ can include labor supply, long-term bonds, other assets and consumption of goods other than $c^j(\omega^t)$;
- let $m^j(\omega^t)$ be the real money demand of agent j in node ω^t, $m^j(\omega^t) = \frac{M^j\left(\omega^t\right)}{P(\omega^t)}$;
- let $\overline{\boldsymbol{Z}}(\omega^t)$ denote a vector of variables that are taken as given by the agents:
  - if the economy has endogenous production, then $\overline{\boldsymbol{Z}}(\omega^t)$ can include wages and variables arising from the maximization of profits by firms, such as labor demand, price settings by monopolistic firms and profits;
  - if the economy has an exogenous endowment, $\overline{\boldsymbol{Z}}(\omega^t)$ includes the endowment for each agent $j \in \mathcal{J}$;
  - $\overline{\boldsymbol{Z}}(\omega^t)$ includes policy instruments other than nominal interest rates, money and bonds (e.g. lump-sum taxes, tax rates for proportional labor and consumption taxes, ...);
  - $\overline{\boldsymbol{Z}}(\omega^t)$ includes the aggregate values of $c^j(\omega^t)$ and of $\boldsymbol{x}^j(\omega^t)$ denoted by the aggregator $X\left(\{\boldsymbol{x}^j(\omega^t)\}_{j\in\mathcal{J}}\right)$ and $C\left(\{c^j(\omega^t)\}_{j\in\mathcal{J}}\right)$;
  - $\overline{\boldsymbol{Z}}(\omega^t)$ includes also debt limit(s) discussed in Section C.2.

C.2 Borrowing limits

Let $\mathcal{H}(\omega^t, s)$ be the set of all the s-ahead histories that can be reached conditional on the realization of history ω^t:

$$\mathcal{H}\left(\omega^t, s\right) = \left\{\omega^{t+s} | \omega^{t+s} = \left\{\omega^t, \omega_{t+1}, ..., \omega_{t+s}\right\}\right\}, \quad s \geq 0$$

In node ω^t, the choice of agent j is subject to some borrowing constraints. Let $\Theta^j(\omega^t) < \infty$ denote the natural debt limit, namely the maximal value that agent j can repay starting from node ω^t, assuming that consumption $c^j(\omega^t)$ and other endogenous expenditure (if any) are zero forever[15]. Also, let:

$$0 \leq \tilde{\Upsilon}^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right) \leq +\infty$$

be an exogenous state-contingent debt limit that must be satisfied in node ω^t for all $\{\omega^t, \omega_{t+1}\} \in \mathcal{H}(\omega^t, 1)$. Then, in node ω^t, agent j has to satisfy the constraint:

$$W^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right) \geq -\Upsilon^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right) \quad \text{for all } \left\{\omega^t, \omega_{t+1}\right\} \in \mathcal{H}\left(\omega^t, 1\right) \tag{41}$$

where:

$$\Upsilon^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right) = \min\left\{\Theta^j\left(\left\{\omega^t, \omega_{t+1}\right\}\right), \tilde{\Upsilon}^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right)\right\}.$$

If you set $\tilde{\Upsilon}^j(\omega^t, \{\omega^t, \omega_{t+1}\}) = +\infty$, then the choice of agent j is only subject to a state-contingent natural debt limit. The debt limits described so far are included in the vector $\overline{\boldsymbol{Z}}(\omega^t)$:

$$\left\{\Theta^j\left(\omega^t\right), \left\{\tilde{\Upsilon}^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right), \Upsilon^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right)\right\}_{\left\{\omega^t, \omega_{t+1}\right\} \in \mathcal{H}\left(\omega^t, 1\right)}\right\}_{j \in \mathcal{J}} \in \overline{\boldsymbol{Z}}\left(\omega^t\right)$$

because they are taken as given by the agents.

I also allow for the possibility of a debt limit on the value of money and debt at the end of node ω^t, therefore, I write the borrowing constraint:

$$\boldsymbol{\chi}^j\left[W^j\left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right), A^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right)\right] \geq \mathbf{0} \tag{42}$$

for some (possibly) vector-valued function $\boldsymbol{\chi}^j$. As an example, this specification can include an ad-hoc constraint of the form $A^j(\omega^t) \geq 0$; or, as an other example, if $\boldsymbol{x}^j(\omega^t)$ includes other assets whose prices are in the vector $\overline{\boldsymbol{Z}}(\omega^t)$, then (42) can represent a borrowing constraint on the total

[15] Also, if there is any endogenous income decision (e.g. labor supply), this must be set at the lower bound, because it is not possible to force people to work to repay their debt. The value of the natural debt limit is taken as given by the agent, but it depends upon the equilibrium values of other endogenous variables.

value of wealth.

C.3 Budget sets and optimal choices

To describe the choice of an agent, I now define a plan and a feasible plan. I then summarize the problem of agent $j \in \mathcal{J}$ and I provide a definition of an optimal plan.

Definition C.1. *(Plan) Given ω^0, a plan for agent $j \in \mathcal{J}$ is a mapping that assigns a choice of real money $m^j(\omega^t)$, bonds $B^j(\omega^t)$, consumption $c^j(\omega^t)$ and other variables $\boldsymbol{x}^j(\omega^t)$, for each node $\omega^t \in \Omega$:*

$$\left\{m^j\left(\omega^t\right), B^j\left(\omega^t\right), c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right)\right\}_{\omega^t \in \Omega}.$$

Definition C.2. *(Feasible plan) Given:*

- $W^j\left(\omega^{-1}, \{\omega^{-1}, \omega_0\}\right)$;
- *prices* $\{P(\omega^t)\}_{\omega^t \in \Omega}$ *and* $\{p(\omega^t)\}_{\omega^t \in \Omega}$;
- $\left\{\overline{\boldsymbol{Z}}(\omega^t)\right\}_{\omega^t \in \Omega}$;

a feasible plan for agent $j \in \mathcal{J}$ is a plan:

$$\left\{m^j\left(\omega^t\right), B^j\left(\omega^t\right), c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right)\right\}_{\omega^t \in \Omega}$$

that satisfies, for all $\omega^t = \{\omega^{t-1}, \omega_t\} \in \Omega$:

- *a non negativity constraint for nominal money:*

$$M^j\left(\omega^t\right) \equiv m^j\left(\omega^t\right) P\left(\omega^t\right) \geq 0; \tag{43}$$

- *a budget constraint, given initial wealth $W^j(\omega^{t-1}, \omega^t)$:*

$$g^j\left[\boldsymbol{x}^j\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right)\right] + P\left(\omega^t\right) c^j\left(\omega^t\right) + A^j\left(\omega^t\right) \leq W^j\left(\omega^{t-1}, \omega^t\right) \tag{44}$$

where $\omega^t = \{\omega^{t-1}, \omega_t\}$ *and* g^j *is a single-valued function*[16]*;*

- *the borrowing constraints defined in Section C.2:*

$$\boldsymbol{\chi}^j \left[W^j \left(\omega^t, \omega^{t+1}\right), A^j \left(\omega^t\right), \boldsymbol{x}^j \left(\omega^t\right), \overline{\boldsymbol{Z}} \left(\omega^t\right)\right] \geq \mathbf{0} \tag{45}$$

for all $\omega^{t+1} = \{\omega^t, \omega_{t+1}\} \in \mathcal{H}(\omega^t, 1)$*;*

- *a transaction constraint:*

$$\gamma^j \left[c^j \left(\omega^t\right) P \left(\omega^t\right), \boldsymbol{x}^j \left(\omega^t\right), \overline{\boldsymbol{Z}} \left(\omega^t\right)\right] \leq m^j \left(\omega^t\right) P \left(\omega^t\right); \tag{46}$$

where the evolution of wealth is given by:

$$\begin{aligned} W^j \left(\omega^t, \left\{\omega^t, \omega_{t+1}\right\}\right) = \left(M^j \left(\omega^t\right) - \gamma^j \left[c^j \left(\omega^t\right) P \left(\omega^t\right), \boldsymbol{x}^j \left(\omega^t\right), \omega^t\right]\right) + \\ + B^j \left(\omega^t\right) + P \left(\omega^t\right) \Xi^j \left[\boldsymbol{x}^j \left(\omega^t\right), \overline{\boldsymbol{Z}} \left(\omega^t\right), \overline{\boldsymbol{Z}} \left(\left\{\omega^t, \omega_{t+1}\right\}\right)\right] \end{aligned} \tag{47}$$

for all $\omega^t \in \Omega$ *and for all* $\{\omega^t, \omega_{t+1}\} \in \mathcal{H}(\omega^t, 1)$.

The value of initial wealth in node $\omega^{t+1} = \{\omega^t, \omega_{t+1}\}$ includes unspent money, bonds $B^j(\omega^t)$ that reach maturity in $t+1$ and an extra term (the function Ξ^j) that includes other sources of wealth; for instance, it can include the value of the endowment sold in node ω^t, or the value of the endowment in ω^{t+1}, or the total wage earned in ω^t, or the value of long-term bonds and other assets.

I denote the objective function of agent $j \in \mathcal{J}$ as[17]:

$$V^j \left(\left\{c^j \left(\omega^t\right), \boldsymbol{x}^j \left(\omega^t\right), m^j \left(\omega^t\right)\right\}_{\Omega}\right). \tag{48}$$

This notation captures the possibility that the objective function depends directly on real money

[16]The function g^j can be formulated as $g^j \left[\boldsymbol{x}^j \left(\omega^{t-1}\right), \boldsymbol{x}^j \left(\omega^t\right), \overline{\boldsymbol{Z}} \left(\omega^t\right)\right]$ to allow for the possibility of a fixed and variable transaction cost to adjust the portfolio of other assets, if any.

[17]The expression in (48) should be defined with respect to a probability space, according to the exogenous process for ω_t (and possibly according to beliefs of agent j about the exogenous process). To simplify the notation, I only emphasize the dependence of V^j on the plan chosen by agent j.

holdings (such as in Section 2). But it can also represent a different formulation in which money does not affect directly utility: in this case, you can express V^j to be independent from the entry $m^j(\omega^t)$ for all $\omega^t \in \Omega$.

I can now define an optimal plan:

Definition C.3. *(Optimal plan) Given:*

- $W^j(\omega^{-1}, \{\omega^{-1}, \omega_0\})$;
- *prices* $\{P(\omega^t)\}_{\omega^t \in \Omega}$ *and* $\{p(\omega^t)\}_{\omega^t \in \Omega}$;
- $\{\overline{\boldsymbol{Z}}(\omega^t)\}_{\omega^t \in \Omega}$;

a feasible plan:

$$\left\{m^j\left(\omega^t\right),\, B^j\left(\omega^t\right),\, c^j\left(\omega^t\right),\, \boldsymbol{x}^j\left(\omega^t\right)\right\}_{\omega^t \in \Omega}$$

is optimal if, for any other feasible plan $\left\{\hat{m}^j(\omega^t),\, \hat{B}^j(\omega^t),\, \hat{c}^j(\omega^t),\, \hat{\boldsymbol{x}}^j(\omega^t)\right\}_{\omega^t \in \Omega}$*:*

$$V^j\left(\left\{c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), m^j\left(\omega^t\right)\right\}_{\omega^t \in \Omega}\right) \geq V^j\left(\left\{\hat{c}^j\left(\omega^t\right), \hat{\boldsymbol{x}}^j\left(\omega^t\right), \hat{m}^j\left(\omega^t\right)\right\}_{\omega^t \in \Omega}\right).$$

I now impose a standard assumptions of monotonicity in consumption, and then I assume that there exist a satiation threshold for real money balances.

Assumption C.4. *For all $j \in \mathcal{J}$, the objective function* (48) *is strictly increasing in $c^j(\omega^t)$ for all $\omega^t \in \Omega$.*

Assumption C.5. *For each $j \in \mathcal{J}$, and for any given $\{c^j(\omega^t), \boldsymbol{x}^j(\omega^t)\}_{\omega^t \in \Omega}$, there exist satiation thresholds defined by (with some abuse of notation):*

$$K^j\left(\omega^t\right) = K^j\left(c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), \omega^t\right) \quad \textit{for all } \omega^t \in \Omega$$

such that:

$$0 \leq K^j\left(c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), \omega^t\right) < \infty \quad \textit{for all } \omega^t \in \Omega$$

and exactly one of the following statement holds:

1. V^j *has the property "money in the utility function":*

   (a) V^j *is strictly increasing in real money balances* $m^j(\omega^t)$*, provided that* $0 \leq m^j(\omega^t) < K^j(\omega^t)$*;*

   (b) V^j *is constant in real money balances* $m^j(\omega^t)$*, provided that* $m^j(\omega^t) \geq K^j(\omega^t)$*;*

   and the transaction constraint (46) *satisfies, for any* P *and for any* $\overline{\boldsymbol{Z}}$*:*

$$\gamma^j\left[c^j\left(\omega^t\right)P, \boldsymbol{x}^j\left(\omega^t\right), \overline{\boldsymbol{Z}}\right] \equiv 0;$$

2. *the objective function* V^j *is independent of* $m^j(\omega^t)$ *for all* $\omega^t \in \mathcal{N}(\omega^0)$ *and the transaction constraint* (46) *satisfies the following: given* $W^j(\omega^{-1}, \{\omega^{-1}, \omega_0\})$*, prices* $\{P(\omega^t)\}_{\omega^t\in\Omega}$ *and* $\{p(\omega^t)\}_{\omega^t\in\Omega}$ *and* $\left\{\overline{\boldsymbol{Z}}(\omega^t)\right\}_{\omega^t\in\Omega}$*, if*

$$\left\{m^j\left(\omega^t\right), B^j\left(\omega^t\right), c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right)\right\}_{\omega^t\in\Omega}$$

   is an optimal plan given, then for all $\hat{\omega}^t$ *such that* $i(\hat{\omega}^t) = 0$*:*

$$\gamma^j\left[c^j\left(\hat{\omega}^t\right)P\left(\hat{\omega}^t\right), \boldsymbol{x}^j\left(\hat{\omega}^t\right), \overline{\boldsymbol{Z}}\left(\hat{\omega}^t\right)\right] = P\left(\hat{\omega}^t\right)K\left(c^j\left(\hat{\omega}^t\right), \boldsymbol{x}^j\left(\hat{\omega}^t\right), \hat{\omega}^t\right).$$

Assumption C.5 is sufficient but not necessary to prove the result in Section 4.3. Some conditions can be relaxed (for instance, utility from money balances doesn't have to be strictly increasing for all level of money balances below the satiation threshold), but they are anyway usually met by standard monetary models, and this formulation simplifies the proof of the results.

If Case 2 of Assumption C.5 holds, then equation (46) implies that, for all $\hat{\omega}^t$ such that $i(\hat{\omega}^t) = 0$:

$$m^j\left(\hat{\omega}^t\right) \geq K\left(c^j\left(\hat{\omega}^t\right), \boldsymbol{x}^j\left(\hat{\omega}^t\right), \hat{\omega}^t\right). \tag{49}$$

C.4 Equilibrium and Aggregate Equilibrium Outcome

Definition C.6. *Given debt limits*

$$L\left(\omega^t\right)=\left\{\tilde{\Upsilon}^j\left(\omega^t,\left\{\omega^t,\omega_{t+1}\right\}\right)\middle| j\in\mathcal{J},\,\left\{\omega^t,\omega_{t+1}\right\}\in\mathcal{H}\left(\omega^t,1\right)\right\}$$

for all $\omega^t\in\Omega$, an equilibrium $\mathcal{E}=\left\{\mathcal{E}^{(\mathcal{J})},\,\mathcal{E}^{(\mathcal{A})}\right\}$, where:

$$\mathcal{E}^{(\mathcal{J})}=\Bigg\{M^j\left(\omega^t\right),\,m^j\left(\omega^t\right),\,B^j\left(\omega^t\right),\,c^j\left(\omega^t\right),\,\boldsymbol{x}^j\left(\omega^t\right),$$
$$W^j\left(\omega^{t-1},\omega^t\right),\,A^j\left(\omega^t\right),\,K^j\left(\omega^t\right)\Bigg|j\in\mathcal{J},\,\omega^t=\left\{\omega^{t-1},\omega_t\right\}\in\mathcal{N}\left(\omega^0\right)\Bigg\}$$

$$\mathcal{E}^{(\mathcal{A})}=\left\{\overline{M}\left(\omega^t\right),\,\overline{B}\left(\omega^t\right),\,i\left(\omega^t\right),\,P\left(\omega^t\right),\,p\left(\omega^t\right),\,\overline{\boldsymbol{Z}}\left(\omega^t\right),\,\overline{K}\left(\omega^t\right)\middle|\omega^t\in\Omega\right\}$$

is given by:

- *a feasible plan $\{m^j(\omega^t),\,B^j(\omega^t),\,c^j(\omega^t),\,\boldsymbol{x}^j(\omega^t)\}_{\omega^t\in\Omega}$ for each $j\in\mathcal{J}$;*
- *nominal money holding $M^j(\omega^t)$, initial wealth $W^j(\omega^{t-1},\{\omega^{t-1},\omega^t\})$, the value of money and bonds $A^j(\omega^t)$ and a value for the satiation threshold $K^j(\omega^t)$ for all $\omega^t=\{\omega^{t-1},\omega_t\}\in\Omega$ and for all $j\in\mathcal{J}$;*
- *an asset supply policy and an interest rate policy $\left\{\overline{M}(\omega^t),\,\overline{B}(\omega^t),\,i(\omega^t)\right\}_{\omega^t\in\Omega}$;*
- *price level $P(\omega^t)$ and price of bonds $p(\omega^t)$ for all $\omega^t\in\Omega$;*
- *a vector $\overline{\boldsymbol{Z}}(\omega^t)$ of other variables, for all $\omega^t\in\Omega$;*
- *an aggregate satiation threshold $\overline{K}(\omega^t)$, for all $\omega^t\in\Omega$;*

such that:

- *household optimality holds: the plan*

$$\left\{m^j\left(\omega^t\right),\,B^j\left(\omega^t\right),\,c^j\left(\omega^t\right),\,\boldsymbol{x}^j\left(\omega^t\right)\right\}_{\omega^t\in\Omega}$$

satisfies Definitions C.2 (feasibility) and C.3 (optimality), for all $j \in \mathcal{J}$ *and for all* $\omega^t \in \Omega$*;*

- *the satiation threshold* $K^j(\omega^t)$ *satisfies Assumption C.5, for all* $j \in \mathcal{J}$ *and for all* $\omega^t \in \Omega$*;*

- *aggregation holds:* $X\left(\{\boldsymbol{x}^j(\omega^t)\}_{j\in\mathcal{J}}\right) \in \overline{\boldsymbol{Z}}(\omega^t)$, $C\left(\{c^j(\omega^t)\}_{j\in\mathcal{J}}\right) \in \overline{\boldsymbol{Z}}(\omega^t)$ *and* $\int K^j(\omega^t)\,dj = \overline{K}(\omega^t)$ *for all* $\omega^t \in \mathcal{N}(\omega^0)$*;*

- *market clearing for money and bonds market holds:* $\overline{M}(\omega^t) = \int M^j(\omega^t)\,dj$ *and* $\overline{B}(\omega^t) = \int B^j(\omega^t)\,dj$ *for all* $\omega^t \in \Omega$*;*

- *household holding of money and one-period bond equal the value of government debt:*

$$\overline{M}\left(\omega^t\right) + p\left(\omega^t\right)\overline{B}\left(\omega^t\right) = \int A^j\left(\omega^t\right) dj \text{for all } \omega^t \in \Omega; \tag{50}$$

- *the government period-by-period budget equation* (24) *is satisfied for all* $\{\omega^{t-1}, \omega_t\} \in \Omega$*;*

- *other equilibrium conditions hold, described by the function:*

$$\boldsymbol{h}\left(\left\{\overline{M}\left(\omega^t\right), \overline{B}\left(\omega^t\right), i\left(\omega^t\right), P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), L\left(\omega^t\right)\right\}_{\omega^t\in\Omega}\right) = \mathbf{0} \tag{51}$$

where $\boldsymbol{h}$ *is (possibly) a vector-valued function.*

Definition C.7. *An Aggregate Equilibrium Outcome is:*

$$\mathcal{E}^{(\mathcal{A})} = \left\{\overline{M}\left(\omega^t\right), \overline{B}\left(\omega^t\right), i\left(\omega^t\right), P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right), \overline{K}\left(\omega^t\right) \middle| \omega^t \in \Omega\right\}$$

such that $\mathcal{E} = \left\{\mathcal{E}^{(\mathcal{J})}, \mathcal{E}^{(\mathcal{A})}\right\}$ *is an equilibrium for some*

$$\mathcal{E}^{(\mathcal{J})} = \left\{M^j\left(\omega^t\right), m^j\left(\omega^t\right), B^j\left(\omega^t\right), c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right),\right.$$
$$\left. W^j\left(\omega^{t-1}, \omega^t\right), A^j\left(\omega^t\right), K^j\left(\omega^t\right) \middle| j \in \mathcal{J}, \omega^t = \left\{\omega^{t-1}, \omega_t\right\} \in \Omega\right\}.$$

D Proof of Proposition 4.1

Proof. By assumption, $\mathcal{E}^{(\mathcal{A})}$ is an aggregate equilibrium outcome, so according to Definition C.7, there exists an equilibrium $\mathcal{E} = \left\{\mathcal{E}^{(\mathcal{J})},\, \mathcal{E}^{(\mathcal{A})}\right\}$ for some:

$$\mathcal{E}^{(\mathcal{J})} = \Bigg\{ M^j\left(\omega^t\right),\, m^j\left(\omega^t\right),\, B^j\left(\omega^t\right),\, c^j\left(\omega^t\right),\, \boldsymbol{x}^j\left(\omega^t\right),\, W^j\left(\omega^{t-1}, \omega^t\right),\, A^j\left(\omega^t\right),\, K^j\left(\omega^t\right) \Bigg| j \in \mathcal{J},\, \omega^t = \left\{\omega^{t-1}, \omega_t\right\} \in \Omega \Bigg\}.$$

Notice that:

$$i\left(\omega^t\right) = 0 \;\Rightarrow\; p\left(\omega^t\right) = 1 \tag{52}$$

because: there is no default risk, since all agents have to satisfy a natural debt limit (described in Section C.2) and the government always repays its debt (equation (24) must always hold); and there is no transaction costs for money and bonds in the term $A^j\left(\omega^t\right)$, defined in equation (40), that enters the budget constraint (44).

I split the proof of the Proposition in two parts.

PART 1: conditions (27) and (28) hold $\Rightarrow$ (26) is an aggregate equilibrium outcome.

I have to show that there exists:

$$\left(\mathcal{E}^{(\mathcal{J})}\right)^* = \Bigg\{ M^{j*}\left(\omega^t\right),\, m^{j*}\left(\omega^t\right),\, B^{j*}\left(\omega^t\right),\, c^{j*}\left(\omega^t\right),\, \boldsymbol{x}^{j*}\left(\omega^t\right),\, W^{j*}\left(\omega^{t-1}, \omega^t\right),\, A^{j*}\left(\omega^t\right),\, K^{j*}\left(\omega^t\right) \Bigg| j \in \mathcal{J},\, \omega^t = \left\{\omega^{t-1}, \omega_t\right\} \in \Omega \Bigg\}$$

such that $\mathcal{E}^* = \left\{\left(\mathcal{E}^{(\mathcal{J})}\right)^*,\, \left(\mathcal{E}^{(\mathcal{A})}\right)^*\right\}$ is an equilibrium, so $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$ is an aggregate equilibrium outcome. To do so, set $c^{j*}\left(\omega^t\right) = c^j\left(\omega^t\right)$ and $\boldsymbol{x}^{j*}\left(\omega^t\right) = \boldsymbol{x}^j\left(\omega^t\right)$, therefore:

$$K^{j*}\left(\omega^t\right) = K^j\left(c^{j*}\left(\omega^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \omega^t\right) = K^j\left(c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), \omega^t\right) = K^j\left(\omega^t\right);$$

I claim that there exists a portfolio of money and bonds such that, for all $j \in \mathcal{J}$ and for all

$\omega^t = \{\omega^{t-1}, \omega_t\} \in \Omega$:

$$\begin{aligned} W^{j*}\left(\omega^{t-1}, \omega^t\right) &= W^j\left(\omega^{t-1}, \omega^t\right), \\ A^{j*}\left(\omega^t\right) &= A^j\left(\omega^t\right). \end{aligned}$$

To show this, define an allocation for money $\{M^{j*}(\omega^t)\}_{\omega^t \in \Omega}$ that satisfies:

$$M^{j*}\left(\omega^t\right) \begin{cases} = M^j(\omega^t) & \text{if } i(\omega^t) > 0 \\ \geq P(\omega^t) K^j(\omega^t) & \text{if } i(\omega^t) = 0 \end{cases} \tag{53}$$

and:

$$\int M^{j*}\left(\omega^t\right) dj = \overline{M}^*\left(\omega^t\right) \quad \text{if } i\left(\omega^t\right) = 0 \tag{54}$$

for all $j \in \mathcal{J}$. Under the money supply policy $\left\{\overline{M}^*(\omega^t)\right\}_{\omega^t \in \Omega}$, the market clearing condition for money is thus satisfied for all ω^t s.t. $i(\omega^t) = 0$ by (54), and for all ω^t such that $i(\omega^t) > 0$ because:

$$\int M^{j*}\left(\omega^t\right) dj = \int M^j\left(\omega^t\right) dj = \overline{M}\left(\omega^t\right) = \overline{M}^*\left(\omega^t\right),$$

where the first equality follows by (53), the second equality follows by the assumption that $\mathcal{E}$ is an equilibrium and the last equality is given by (27). So the market clearing condition is satisfied for all ω^t. Moreover, there exists at least one allocation of money that satisfies (53) and (54): indeed, if you assign, in node ω^t such that $i(\omega^t) = 0$:

$$M^{j*}\left(\omega^t\right) = P\left(\omega^t\right) K^j\left(\omega^t\right)$$

then, using (28)

$$\overline{M}^*\left(\omega^t\right) \geq P\left(\omega^t\right) \int K^j\left(\omega^t\right) dj = \int M^{j*}\left(\omega^t\right) dj$$

and, if this last relationship holds with strict inequality, you can just assign the difference between $\overline{M}^*(\omega^t)$ and $\int M^{j*}(\omega^t)\, dj$ to any subset of agent, in order to satisfy (54).

Then, assign to each agent $j \in \mathcal{J}$ bonds $B^{j*}\left(\omega^t\right)_{\omega^t \in \Omega}$ such that:

- if $i\left(\omega^t\right) > 0$:

$$B^{j*}\left(\omega^t\right) = B^j\left(\omega^t\right);$$

- if $i\left(\omega^t\right) = 0$:

$$M^{j*}\left(\omega^t\right) + B^{j*}\left(\omega^t\right) = M^j\left(\omega^t\right) + B^j\left(\omega^t\right), \tag{55}$$

  provided that:

$$\int B^{j*}\left(\omega^t\right) dj = \overline{B}^*\left(\omega^t\right). \tag{56}$$

There exists an allocation that is feasible, since, using the previous assumptions together with (27) and (28):

- if $i\left(\omega^t\right) > 0$:

$$\int B^{j*}\left(\omega^t\right) dj = \int B^j\left(\omega^t\right) = \overline{B}\left(\omega^t\right) = \overline{B}^*\left(\omega^t\right);$$

- if $i\left(\omega^t\right) = 0$:

$$\begin{aligned}\int \left[M^{j*}\left(\omega^t\right) + B^{j*}\left(\omega^t\right)\right] dj &= \int \left[M^j\left(\omega^t\right) + B^j\left(\omega^t\right)\right] dj \\ &= \overline{M}\left(\omega^t\right) + \overline{B}\left(\omega^t\right) \\ &= \overline{M}^*\left(\omega^t\right) + \overline{B}^*\left(\omega^t\right).\end{aligned}$$

Now, I want to show that the plan:

$$\begin{aligned}&\left\{c^{j*}\left(\omega^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), m^{j*}\left(\omega^t\right), B^{j*}\left(\omega^t\right)\right\}_{\omega^t \in \Omega} = \\ &\qquad = \left\{c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), m^{j*}\left(\omega^t\right), B^{j*}\left(\omega^t\right)\right\}_{\omega^t \in \Omega}\end{aligned} \tag{57}$$

is optimal given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$, for all $j \in \mathcal{J}$. To do so, let me first state and prove this intermediate result:

Lemma D.1. *For each agent $j \in \mathcal{J}$, a plan:*

$$\left\{\check{c}^j\left(\omega^t\right), \check{\boldsymbol{x}}^j\left(\omega^t\right), \check{m}^j\left(\omega^t\right), \check{B}^j\left(\omega^t\right)\right\}_{\omega^t \in \Omega}$$

is feasible given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$ if and only if the same plan is feasible given $\mathcal{E}^{(\mathcal{A})}$.

Proof. By definition C.2, a plan is feasible if it satisfies equations (43), (44), (42), (46) and (47). These equations depends on conditions that are taken as given by the agent:

$$\left\{W^j\left(\omega^{-1}, \omega_0\right), \left\{P\left(\omega^t\right), p\left(\omega^t\right), \overline{\boldsymbol{Z}}\left(\omega^t\right)\right\}_{\omega^t \in \Omega}\right\}.$$

Since this list in $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$ is the same as in $\mathcal{E}^{(\mathcal{A})}$, then a plan is feasible given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$ if and only if it is feasible given $\mathcal{E}^{(\mathcal{A})}$. $\square$

Then, I show that the plan (57) is feasible given $\mathcal{E}^{(\mathcal{A})}$:

- (non-negativity constraint on money) the non-negativity constraint $M^j\left(\omega^t\right) \geq 0$ is satisfied by (53), since the aggregate price level must be non-negative in equilibrium, because V^j is strictly increasing in $c^j\left(\omega^t\right)$, for all $\omega^t \in \Omega$;
- (budget constraint) the sum of money and bonds is unchanged, when evaluated at prices $p\left(\omega^t\right)$ in the original equilibrium: the result is trivial for ω^t such that $i\left(\omega^t\right) > 0$ (since the portfolio of money and bonds is unchanged), and for ω^t such that $i\left(\omega^t\right) = 0$ you get

$$\begin{aligned} A^{j*}\left(\omega^t\right) &= \left[M^{j*}\left(\omega^t\right) + B^{j*}\left(\omega^t\right)\right] = \\ &= \left[M^j\left(\omega^t\right) + B^j\left(\omega^t\right)\right] = A^j\left(\omega^t\right) \end{aligned}$$

then, given $W^j\left(\omega^{t-1},\{\omega^{t-1},\omega_t\}\right)$, the budget constraint is still satisfied:

$$\begin{aligned}
&g^j\left[\boldsymbol{x}^{j*}\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]+P\left(\omega^t\right)c^{j*}\left(\omega^t\right)+A^{j*}\left(\omega^t\right)= \\
&\qquad\qquad =g^j\left[\boldsymbol{x}^{j}\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]+P\left(\omega^t\right)c^{j}\left(\omega^t\right)+A^{j}\left(\omega^t\right)\leq \\
&\qquad\qquad\qquad\qquad \leq W^j\left(\omega^{t-1},\left\{\omega^{t-1},\omega_t\right\}\right)
\end{aligned}$$

for all $\omega^t=\{\omega^{t-1},\omega_t\}\in\Omega$;

- (transaction constraint) the constraint (46) is trivially satisfied in nodes ω^t such that $i\left(\omega^t\right)>0$ since $m^{j*}\left(\omega^t\right)=m^j\left(\omega^t\right)$, $c^{j*}\left(\omega^t\right)=c^j\left(\omega^t\right)$ and $\boldsymbol{x}^{j*}\left(\omega^t\right)=\boldsymbol{x}^j\left(\omega^t\right)$; in nodes such that $i\left(\omega^t\right)=0$, then Assumption C.5 implies that, given $\mathcal{E}^{(\mathcal{A})}$, either:

$$\gamma^j\left[c^j\left(\omega^t\right)P\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]=0$$

or:

$$\gamma^j\left[c^j\left(\omega^t\right)P\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]=P\left(\omega^t\right)K^j\left(\omega^t\right),$$

therefore, by (53), the constraint is satisfied given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$, since:

$$m^{j*}\left(\omega^t\right)\geq K^j\left(\omega^t\right)=K^{j*}\left(\omega^t\right);$$

- (initial wealth) from equation (47), the evolution of initial wealth in the original equilibrium can be written:

$$\begin{aligned}
&W^j\left(\omega^t,\left\{\omega^t,\omega_{t+1}\right\}\right)=M^j\left(\omega^t\right)+B^j\left(\omega^t\right)+ \\
&\quad-\gamma^j\left[c^j\left(\omega^t\right)P\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]+P\left(\omega^t\right)\Xi^j\left[\boldsymbol{x}^j\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\left\{\omega^t,\omega_{t+1}\right\}\right)\right] \\
&\qquad\qquad ,
\end{aligned}$$

and the evolution of initial wealth in the "*" equilibrium is:

$$W^j\left(\omega^t,\left\{\omega^t,\omega_{t+1}\right\}\right)=M^{j*}\left(\omega^t\right)+B^{j*}\left(\omega^t\right)+$$
$$-\gamma^j\left[c^j\left(\omega^t\right)P\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]+P\left(\omega^t\right)\Xi^j\left[\boldsymbol{x}^j\left(\omega^t\right),\,\overline{\boldsymbol{Z}}\left(\omega^t\right),\,\overline{\boldsymbol{Z}}\left(\left\{\omega^t,\omega_{t+1}\right\}\right)\right],$$

and using equations (55):

$$W^{j*}\left(\omega^{t-1},\left\{\omega^{t-1},\omega_t\right\}\right)=W^j\left(\omega^{t-1},\left\{\omega^{t-1},\omega_t\right\}\right) \tag{58}$$

for all $\omega^t=\{\omega^{t-1},\omega_t\}\in\Omega$.

- (borrowing constraint) using the result in (58), then the borrowing constraint (42) is still satisfied:

$$\boldsymbol{\chi}^j\left[W^{j*}\left(\omega^t,\left\{\omega^t,\omega_{t+1}\right\}\right),A^{j*}\left(\omega^t\right),\boldsymbol{x}^{j*}\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]=$$
$$=\boldsymbol{\chi}^j\left[W^j\left(\omega^t,\left\{\omega^t,\omega_{t+1}\right\}\right),A^j\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),\overline{\boldsymbol{Z}}\left(\omega^t\right)\right]\geq 0$$

for all $\omega^t\in\Omega$ and for all $\{\omega^t,\omega_{t+1}\}\in\mathcal{H}(\omega^t,1)$.

Therefore, by Lemma D.1, the plan is feasible given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$.

To show that the plan (57) is optimal given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$, it is enough to check that the plan is optimal given $\mathcal{E}^{(\mathcal{A})}$: as a Corollary of Lemma D.1, a plan which is optimal given $\mathcal{E}^{(\mathcal{A})}$ is also optimal given $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$, according to Definition C.3. Thus, it is enough to show that:

$$V^j\left(\left\{c^j\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),m^{j*}\left(\omega^t\right)\right\}_{\omega^t\in\Omega}\right)=V^j\left(\left\{c^j\left(\omega^t\right),\boldsymbol{x}^j\left(\omega^t\right),m^j\left(\omega^t\right)\right\}_{\omega^t\in\Omega}\right).$$

To do so, I consider two possibilities, based on Assumption C.5:

1. if V^j has the "money in the utility function" property (Case 1 in Assumption C.5), then real

money balances in the original equilibrium $\mathcal{E}$ must satisfy:

$$i\left(\omega^t\right)=0 \quad \Rightarrow \quad m^j\left(\omega^t\right) \geq K^j\left(c^{j*}\left(\omega^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \omega^t\right)=K^{j*}\left(\omega^t\right)$$

To see that this condition must hold, assume by contradiction that it is not verified. But then there exist a feasible plan with money above the satiation threshold when $i\left(\omega^t\right)=0$: the plan in (57) is feasible as shown before, and it allows to achieve a higher value of the objective function, according to Case 1 in Assumption C.5.

Therefore real money in the original equilibrium satisfies $m^j\left(\omega^t\right) \geq K^j\left(\omega^t\right)$ and thus, by Case 1 in Assumption C.5:

$$V^j\left(\left\{c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), m^{j*}\left(\omega^t\right)\right\}_{\omega^t \in \mathcal{N}(\omega^0)}\right)=V^j\left(\left\{c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), m^j\left(\omega^t\right)\right\}_{\omega^t \in \mathcal{N}(\omega^0)}\right);$$

2. if Case 2 in Assumption C.5 holds, then the objective function is simply independent of real money balances; thus:

$$V^j\left(\left\{c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), m^{j*}\left(\omega^t\right)\right\}_{\omega^t \in \mathcal{N}(\omega^0)}\right)=V^j\left(\left\{c^j\left(\omega^t\right), \boldsymbol{x}^j\left(\omega^t\right), m^j\left(\omega^t\right)\right\}_{\omega^t \in \mathcal{N}(\omega^0)}\right).$$

To complete this part of the proof, I still have to check the remaining conditions of the Definition of equilibrium C.6:

- the equilibrium condition "aggregation" holds trivially in $\mathcal{E}^*$, since $c^{j*}\left(\omega^t\right)=c^j\left(\omega^t\right)$ and $\boldsymbol{x}^{j*}\left(\omega^t\right)=\boldsymbol{x}^j\left(\omega^t\right)$ for all j and for all ω^t, and the vector of other variables $\overline{\boldsymbol{Z}}\left(\omega^t\right)$ is unchanged in the two equilibria;
- the market clearing conditions in bonds and money market hold, as discussed before;
- given $\overline{\boldsymbol{Z}}\left(\omega^t\right)$ and $\overline{\boldsymbol{Z}}\left(\omega^{t-1}\right)$, the period-by-period government budget set (24) is unchanged;
- equation (51) is unchanged, so all the other equilibrium conditions still hold.

Therefore, $\mathcal{E}^* = \left\{ \left(\mathcal{E}^{(\mathcal{J})}\right)^*, \left(\mathcal{E}^{(\mathcal{A})}\right)^* \right\}$ is an equilibrium and thus $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$ is an aggregate equilibrium outcome.

PART 2: (26) is an aggregate equilibrium outcome $\Rightarrow$ conditions (27) and (28) hold.

Since $\left(\mathcal{E}^{(\mathcal{A})}\right)^*$ is an aggregate equilibrium outcome, there exists:

$$\left(\mathcal{E}^{(\mathcal{J})}\right)^* = \Bigg\{ M^{j*}\left(\omega^t\right), m^{j*}\left(\omega^t\right), B^{j*}\left(\omega^t\right), c^{j*}\left(\omega^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \\ W^{j*}\left(\omega^{t-1}, \omega^t\right), A^{j*}\left(\omega^t\right), K^{j*}\left(\omega^t\right) \Bigg| j \in \mathcal{J}, \omega^t = \left\{\omega^{t-1}, \omega_t\right\} \in \Omega \Bigg\} \quad (59)$$

such that $\mathcal{E}^* = \left\{ \left(\mathcal{E}^{(\mathcal{J})}\right)^*, \left(\mathcal{E}^{(\mathcal{A})}\right)^* \right\}$ is an equilibrium.

Rearranging the government budget constraint (24), when $i\left(\omega^t\right) = 0$:

$$\overline{B}\left(\omega^{t-1}\right) + \overline{M}\left(\omega^{t-1}\right) = \overline{M}\left(\omega^t\right) + \overline{B}\left(\omega^t\right) + G\left[\overline{\boldsymbol{Z}}\left(\omega^{t-1}\right), \overline{\boldsymbol{Z}}\left(\left\{\omega^{t-1}, \omega_t\right\}\right), P\left(\omega^t\right)\right]$$

thus it must be the case that $\overline{M}^*\left(\omega^t\right) + \overline{B}^*\left(\omega^t\right) = \overline{M}\left(\omega^t\right) + \overline{B}\left(\omega^t\right)$ to keep this equation unchanged;

When $i\left(\omega^t\right) > 0$, to have an aggregate equilibrium outcome such that the path of nominal interest rates, prices, prices of bonds, satiation threshold and other aggregate variables are unchanged, it must be the case that the budget sets of all the agents in the economy are unchanged. Thus, the supply of money must be unchanged, $\overline{M}^*\left(\omega^t\right) = \overline{M}\left(\omega^t\right)$, and the supply of one-period bonds must be unchanged, $\overline{B}^*\left(\omega^t\right) = \overline{B}\left(\omega^t\right)$.

To show that the condition:

$$i\left(\omega^t\right) = 0 \Rightarrow \overline{M}^*\left(\omega^t\right) \geq P\left(\omega^t\right) \int K^j\left(c^{j*}\left(\hat{\omega}^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \hat{\omega}^t\right) dj$$

must hold, assume that, by contradiction, it is not satisfied in node $\hat{\omega}^t \in \Omega$ such that $i\left(\hat{\omega}^t\right) = 0$. Then, for some $j \in \mathcal{J}$:

$$M^{j*}\left(\hat{\omega}^t\right) < P\left(\hat{\omega}^t\right) K^j\left(c^{j*}\left(\hat{\omega}^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \hat{\omega}^t\right)$$

or:

$$m^{j*}\left(\hat{\omega}^t\right) < K^j\left(c^{j*}\left(\hat{\omega}^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \hat{\omega}^t\right) \tag{60}$$

Then, using Assumption C.5:

1. if V^j has the property "money in the utility function" (Case 1 of Assumption C.5), define the plan:

$$\left\{c^{j*}\left(\omega^t\right), \boldsymbol{x}^{j*}\left(\omega^t\right), \tilde{m}^{j*}\left(\omega^t\right), \tilde{B}^{j*}\left(\omega^t\right)\right\}_{\omega^t\in\Omega} \tag{61}$$

where:

$$\tilde{m}^{j*}\left(\omega^t\right) = \begin{cases} K^j\left(c^j\left(\hat{\omega}^t\right), \boldsymbol{x}^j\left(\omega^t\right), \hat{\omega}^t\right) & \text{if } \omega^t = \hat{\omega}^t \quad (\text{recall } i\left(\hat{\omega}^t\right) = 0) \\ m^{j*}\left(\omega^t\right) & \text{otherwise} \end{cases}$$

therefore $\tilde{m}^{j*}\left(\hat{\omega}^t\right) > m^{j*}\left(\hat{\omega}^t\right)$, and:

$$\tilde{B}^{j*}\left(\omega^t\right) = \begin{cases} B^{j*}\left(\hat{\omega}^t\right) - P\left(\hat{\omega}^t\right)\left[K^j\left(c^j\left(\hat{\omega}^t\right), \boldsymbol{x}^j\left(\omega^t\right), \hat{\omega}^t\right) - m^{j*}\left(\hat{\omega}^t\right)\right] & \text{if } \omega^t = \hat{\omega}^t \\ B^{j*}\left(\omega^t\right) & \text{otherwise} \end{cases}$$

The plan (61) is feasible, since, in node $\omega^t \neq \hat{\omega}^t$ it is the same as the plan in (59), and for node $\hat{\omega}^t$:

- the non-negativity constraint for money is satisfied since $\tilde{m}^{j*}\left(\hat{\omega}^t\right) > m^{j*}\left(\hat{\omega}^t\right)$ and $m^{j*}\left(\hat{\omega}^t\right)$ satisfies by assumption the constraint;
- the budget constraint is satisfied because:

$$i\left(\hat{\omega}^t\right) = 0 \;\Rightarrow\; \tau\left(\hat{\omega}^t\right) \geq 1$$

  and therefore:

$$\begin{aligned}\tilde{M}^{j*}\left(\hat{\omega}^t\right) + \tilde{B}^{j*}\left(\hat{\omega}^t\right) = P\left(\hat{\omega}^t\right) K^j\left[\hat{\omega}^t, \boldsymbol{x}^j\left(\hat{\omega}^t\right)\right] + B^{j*}\left(\hat{\omega}^t\right) + \\ -\left(P\left(\omega^t\right) K^j\left(c^j\left(\hat{\omega}^t\right), \boldsymbol{x}^j\left(\omega^t\right), \hat{\omega}^t\right) - M^{j*}\left(\hat{\omega}^t\right)\right) = B^{j*}\left(\hat{\omega}^t\right) + M^{j*}\left(\hat{\omega}^t\right)\end{aligned} \tag{62}$$

and using $p(\hat{\omega}^t) = 1$ (that follows from (52)):

$$\begin{aligned}
\tilde{A}^{j*}\left(\hat{\omega}^t\right) &= \tilde{B}^{j*}\left(\hat{\omega}^t\right) + \tilde{M}^{j*}\left(\hat{\omega}^t\right) \\
&= M^j\left(\hat{\omega}^t\right) + B^j\left(\hat{\omega}^t, t+1\right) \\
&= A^{j*}\left(\omega^t\right)
\end{aligned}$$

so, for $\hat{\omega}^t = \{\hat{\omega}^{t-1}, \hat{\omega}_t\}$:

$$\begin{aligned}
& g^j\left[\boldsymbol{x}^j\left(\hat{\omega}^t\right), \overline{\boldsymbol{Z}}\left(\hat{\omega}^t\right)\right] + P\left(\hat{\omega}^t\right) c^j\left(\hat{\omega}^t\right) + \tilde{A}^{j*}\left(\hat{\omega}^t\right) = \\
& \qquad = g^j\left[\boldsymbol{x}^j\left(\hat{\omega}^t\right), \overline{\boldsymbol{Z}}\left(\hat{\omega}^t\right)\right] + P\left(\hat{\omega}^t\right) c^j\left(\hat{\omega}^t\right) + A^{j*}\left(\hat{\omega}^t\right) \leq \\
& \qquad\qquad \leq W^j\left(\hat{\omega}^{t-1}, \left\{\hat{\omega}^{t-1}, \hat{\omega}_t\right\}\right)
\end{aligned}$$

- the function γ^j is identically equal to zero since I am analyzing case (1) of Assumption C.5;
- initial wealth $W^{j*}(\hat{\omega}^t, \{\hat{\omega}^t, \hat{\omega}_{t+1}\})$ is unchanged using (61), and (62);
- the borrowing constraint (42) is satisfied because $W^{j*}(\hat{\omega}^t, \{\hat{\omega}^t, \hat{\omega}_{t+1}\})$, $A^{j*}(\omega^t)$ and $\boldsymbol{x}^{j*}(\omega^t)$ are unchanged.

By the assumption of money in the utility function, then the plan (61) is strictly preferred by agent j to its plan in $\left(\mathcal{E}^{(\mathcal{J})}\right)^*$, because $\tilde{m}^{j*}(\omega^t) \geq m^{j*}(\omega^t)$, with strict inequality for $\omega^t = \hat{\omega}^t$. But this is a contradiction, because by Assumption $\mathcal{E}^* = \left\{\left(\mathcal{E}^{(\mathcal{J})}\right)^*, \left(\mathcal{E}^{(\mathcal{A})}\right)^*\right\}$ is an equilibrium, so the plan of agent j in $\left(\mathcal{E}^{(\mathcal{J})}\right)^*$ must be optimal.

2. if case 2 in Assumption C.5 holds, then (60) implies that the necessary condition $m^{j*}(\omega^t) \geq K(c^{j*}(\omega^t), \boldsymbol{x}^{j*}(\omega^t), \omega^t)$ stated in (49) is not satisfied, which is a contradiction.

$\square$

E Proof of Proposition 2.2

To prove Proposition 2.2, I map the economy of Section 2 in the general formulation of Section 4, and then I invoke Proposition 4.1.

In the economy of Section 2 there is no uncertainty, therefore $\omega^t = t$ and:

$$\mathcal{H}\left(\omega^t, s\right) = t + s$$

$$\Omega = \{0, 1, 2, \dots\}$$

Also, there is no heterogeneity, therefore I'll drop the index j in what follows. The economy of Section 2 can be mapped in the formulation of Section 4 as follow:

- bonds: $B\left(\omega^t\right) = B_t$;
- money: $M\left(\omega^t\right) = M_t$ and $m\left(\omega^t\right) = m_t$;
- price level: $P\left(\omega^t\right) = P_t$;
- value of money and bonds: $A\left(t\right) = M_t + Q_{t,t+1}B_t$;
- consumption: $c\left(\omega^t\right) = c_t$;
- other endogenous choices: $\boldsymbol{x}^j\left(\omega^t\right) = \{\varnothing\}$;
- exogenous debt limits: $\tilde{\Upsilon}\left(\omega^t, \{\omega^t, \omega_{t+1}\}\right) = \tilde{\Upsilon}_{t+1} = +\infty$;
- $\overline{Z}\left(\omega^t\right) = \{Y, \Upsilon_t, c_t, T_t\}$, where Υ_t is the debt limit (see below the description of the equilibrium conditions $\boldsymbol{h}$);
- plan: $\{m_t, B_t, c_t\}_{t=0}^{\infty}$;
- budget constraint:

$$g\left[\boldsymbol{x}\left(\omega^t\right), \overline{Z}\left(\omega^t\right)\right] = 0 \qquad \text{for all } \boldsymbol{x}\left(\omega^t\right), \overline{Z}\left(\omega^t\right);$$

- transaction constraint:

$$\gamma\left[c\left(\omega^{t}\right)P\left(\omega^{t}\right),\boldsymbol{x}\left(\omega^{t}\right),\overline{Z}\left(\omega^{t}\right)\right]=0\qquad\text{for all }c\left(\omega^{t}\right)P\left(\omega^{t}\right),\boldsymbol{x}\left(\omega^{t}\right),\overline{Z}\left(\omega^{t}\right);$$

- evolution of wealth:

$$\begin{aligned}W\left(\omega^{t},\left\{\omega^{t},\omega_{t+1}\right\}\right) &= M_{t}+B_{t}+P\left(\omega^{t}\right)\Xi\left[\boldsymbol{x}\left(\omega^{t}\right),\overline{Z}\left(\omega^{t}\right),\overline{Z}\left(\left\{\omega^{t},\omega_{t+1}\right\}\right)\right]=\\ &= M_{t}+B_{t}+P_{t}\left[\overline{Z}\left(\left\{\omega^{t},\omega_{t+1}\right\}\right)(1)-\overline{Z}\left(\left\{\omega^{t},\omega_{t+1}\right\}\right)(4)\right]=\\ &= M_{t}+B_{t}+P_{t}\left(Y-T_{t}\right)\end{aligned}$$

where $\overline{Z}\left(\{\omega^{t},\omega_{t+1}\}\right)(s)$ is the s-th entry of the vector $\overline{Z}\left(\{\omega^{t},\omega_{t+1}\}\right)$

- objective function:

$$V\left(\left\{c\left(\omega^{t}\right),\boldsymbol{x}\left(\omega^{t}\right),m\left(\omega^{t}\right)\right\}_{\omega^{t}\in\mathcal{N}\left(\omega^{0}\right)}\right)=V\left(\{c_{t},m_{t}\}_{t=0}^{\infty}\right)=\sum_{t=0}^{\infty}\beta^{t}\left[\log c_{t}+v\left(m_{t}\right)\right]$$

so Assumption C.4 is satisfied (V strictly increasing in consumption);

- feasible and optimal plan: the maximization of (1) subject to (4), (5), (7) and (8) is equivalent to the choice of the optimal plan described in Definition C.3;

- Assumption C.5 is satisfied (Case 1 holds);

- government fiscal policy: $T_{t}\in\overline{Z}\left(\omega^{t}\right)$;

- government budget set, equation (24):

$$\overline{M}_{t-1}+\overline{B}_{t-1}=P_{t}T_{t}+\overline{M}_{t}+\frac{1}{1+i_{t}}\overline{B}_{t};$$

$$\Rightarrow\overline{B}_{t-1}=P_{t}T_{t}+\frac{1}{1+i_{t}}\overline{B}_{t}+\mu_{t}\overline{M}_{t-1};$$

that corresponds to equation (15);

- other equilibrium conditions $\boldsymbol{h}$:

- $Q_{t,t+1} = p\left(\omega^t, t+1\right) = \frac{1}{1+i_t}$;
- natural debt limit:

$$\Theta\left(\omega^t\right) = \Theta_t = P_t\left(Y - T_t\right) + \sum_{s=1}^{\infty} \frac{P_{t+s}\left(Y - T_{t+s}\right)}{\prod_{k=0}^{s-1}\left(1 + i_{t+k}\right)}$$

  where endowment Y and taxes T_{t+s}, $s \geq 0$, are element of the vectors $\left\{\overline{\boldsymbol{Z}}\left(\omega^t\right)\right\}_{\omega^t \in \mathcal{N}(\omega^0)}$; the debt limit is derived using the budget constraint (4), the non-negativity constraint on money and consumption (7) and (8) and the solvency constraint (5);
- debt limit:

$$\Upsilon\left(\omega^{t-1}, \left\{\omega^{t-1}, \omega_t\right\}\right) = \Upsilon_t = \min\left\{\Theta_t, \tilde{\Upsilon}_t\right\} = \min\left\{\Theta_t, +\infty\right\} = \Theta_t;$$

- aggregate equilibrium outcome: all agents are alike, so the equilibrium described in Definition 2.1 is identical to an aggregate equilibrium outcome.

Therefore, since the economy of Section 2 is a special case of the general formulation of Section 4, Proposition 2.2 is just a special case of Proposition 4.1.