Delayed Capital Reallocation

Job Market Paper

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Abstract

How do firms adjust their balance sheets and reallocate capital stock in response to recurrent productivity or profitability shocks? Why does capital reallocation fluctuate procyclically, while the potential benefits to reallocate appear to be countercyclical? To answer these questions, this paper develops a tractable dynamic general equilibrium model. In the model, firms face idiosyncratic productivity shocks while at the same time are restricted by the illiquidity of capital stock and financing constraints. The model shows that asset illiquidity and financing constraints interact and generate capital reallocation delays. These delays result in cross-sectional productivity dispersion and losses of total factor productivity (TFP), which become more severe during recessions.

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1 Introduction

How do firms adjust their balance sheets of productive assets and liabilities in response to recurrent shocks to productivity or profitability? How does the existing capital stock reallocate across firms and interact with new investment? Two empirical observations provide some guidance of the balance sheet adjustment of U.S. corporate firms.

The first is aggregate capital stock reallocation. Figure 1 plots cyclical components of aggregate capital reallocation (solid line) and GDP (dashed line), which suggests that reallocation of existing productive capital is highly procyclical. Following Eisfeldt and Rampini (2006), capital reallocation includes sales of property, plants, and equipment and acquisitions from the COMPUSTAT database.[1] This observation is in contrast with creative destruction theory in which more capital stock should be liquidated in recessions. Moreover, existing literature shows that firm-level total factor productivity (TFP) become more dispersed in recessions.[2] Therefore, in recessions, there are highest potential benefits[3] to reallocate which should imply the most capital reallocation.

The second looks at firms that liquidate assets. Figure 2 plots debt-to-asset ratios of firms over time during which they do not sell assets until time 0.[4] We learn that most of the firms are reluctant to sell assets quickly. In addition, they shrink their debt burdens before selling: their liabilities are reduced relative to their assets.

Figure 1 is puzzling as Eisfeldt and Rampini (2006) point out: why is there less reallocation in recessions (especially when there are larger potential benefits to reallocate)? This paper asks what reason(s) can delay reallocation and generate larger TFP dispersion in recessions *endogenously*. Figure 2 suggests that the outside financing condition should be important in firms' liquidation. The changes of the condition may affect the timing of reallocation and may explain why there is less reallocation but larger TFP dispersion in recessions.

To examine outside financing's impact on capital reallocation, I construct a tractable dynamic general equilibrium model in which firms face idiosyncratic and aggregate shocks while being restricted by two frictions: asset illiquidity and financing constraints. The two frictions interact and generate capital reallocation delays from unproductive firms. In response to credit crunches, the delays are prolonged and the TFP dispersion thus expands.

[1] Jovanovic and Rousseau (2002) also use this measure for studying the purchase of used assets. To give a sense of reallocation market size, in 2011, the reallocation from COMPUSTAT is about \$0.65 trillion whereas the total U.S. fixed investment is about \$1.6 trillion. Non-listed firms probably buy more used assets according to Eisfeldt and Rampini (2007). In sum, capital reallocation is comparable to new investment.

[2] For example, Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012) shows that the dispersion of plant level total factor productivity increases in recessions, replicated in Figure 10 in the Appendix. Since the plant-level productivity dispersion measures potential gains from reallocation, the figure suggests that the gains are countercyclical. Other measures of TFP dispersion are also larger in recessions as in Table 8 in the Appendix.

[3] Mergers and acquisitions (M&A) sometimes occur for market power motives; but in firm-level data, M&A generally increase efficiency as shown in Maksimovic and Phillips (2001). (A transaction that does not increase efficiency is "a minority of transactions".)

[4] Except for firms with very high leverage ratios, see Figure 11 in the Appendix. Covered firms are with asset sells in the years 2000-2012 in the SDC Platinum database and have corresponding information in the COMPUSTAT database. Those who sell multiple times are excluded. See the data description in the Appendix.

Figure 1: **Capital reallocation over cycles**

The series plotted are cyclical components of HP-filtered log data normalized by standard deviations. Solid line represents seasonally adjusted reallocation, i.e., the sum of sales of property, plant, and equipment (SPPE) and acquisition (AQC) in 2005 dollars. Dashed line represents real GDP in 2005 dollars. Shaded regions denote NBER recessions. For the separate cyclical patterns of SPPE and AQC, see Figure 12 in the Appendix. See also Tabel 6 and Table 7 in the Appendix for summary statistics and more statistics of cyclical patterns.

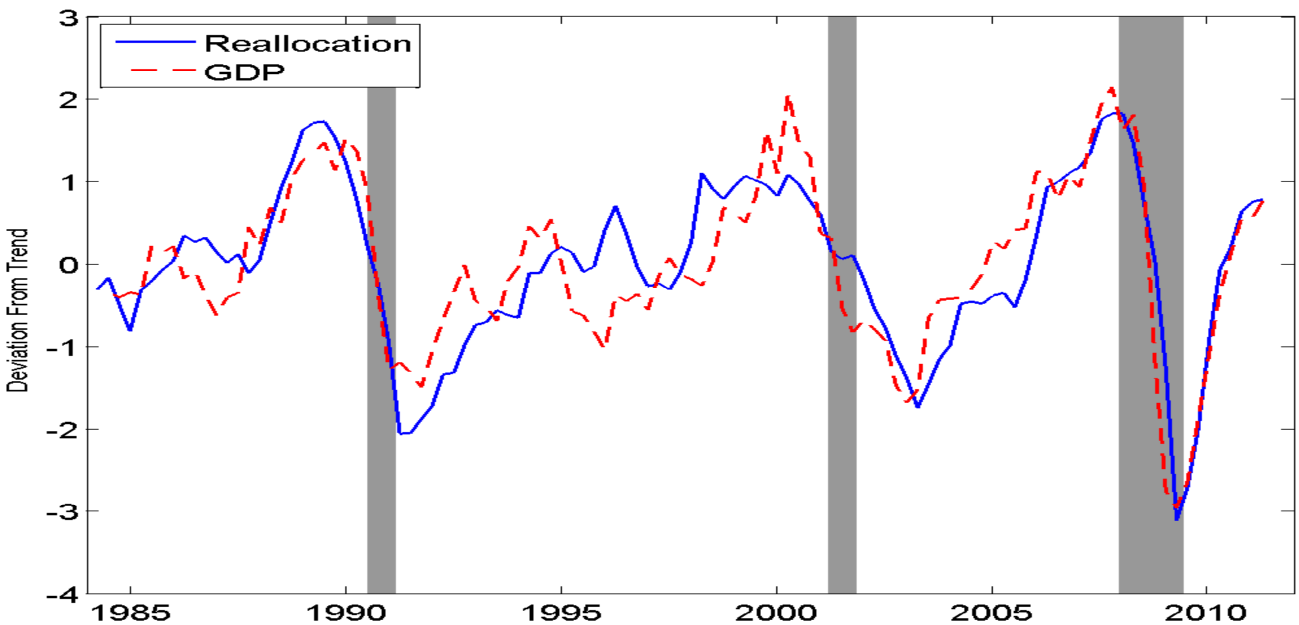
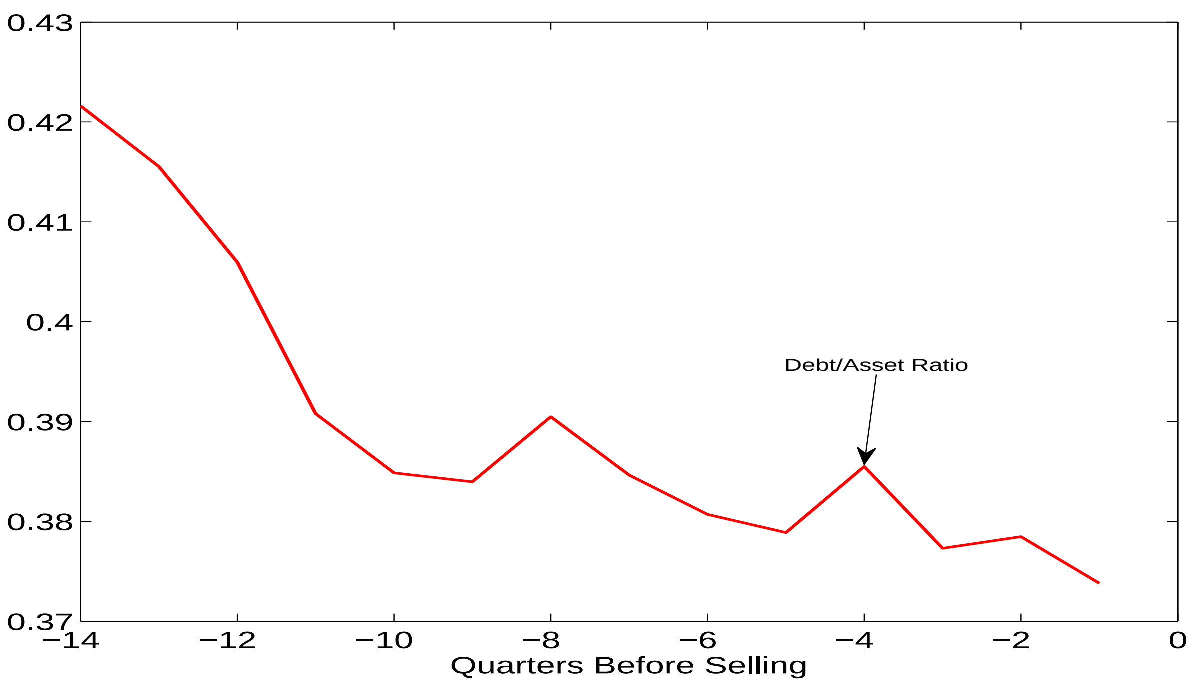


Figure 2: **Debt-to-asset ratio before liquidation**

Debt-to-asset ratios before selling assets of all firms who sold at least 50% of the assets in 2000-2012 (2071 such firms in total). Time 0 denotes the time when firms sell assets and time t denotes t quarters before selling assets. By construction, there is no assets selling at time t. For a more cross-sectional detail, see Figure 11 in the Appendix.



To be more specific, the key features are: (1) collateralized borrowing constraints, (2) capital resale discount[5,6] (assets will be sold at discount in liquidation), and (3) fixed costs in running firms. In this economy, idiosyncratic productivity shocks create the benefits to reallocate capital stock. Productive firms expand by borrowing, but collateral constraints restrict the expansion so that not every capital stock can be reallocated. For example, not every production line of electric cars can be transferred to productive car companies.

In contrast, firms whose productivity falls are hesitant to sell assets because of the resale discount, gambling on the hopes that they might regain productivity soon. Meanwhile, these firms have accumulated a large amount of debt. The interest rate on the debt is higher than the rate of return on capital stock. They let the capital depreciate while pay down existing debt by shrinking dividends (modeled as consumption). If they persist in this unproductive way, profitability stays low and they gradually shrink. But they will eventually give up their capital when the option value of maintaining the depreciated capital is not enough to compensate for the fixed costs of operation. Thus, the model is able to generate balance sheet dynamics as in Figure 2 (see Figure 5 later).

The main result is that aggregate adverse shocks to borrowing constraints prolong the selling delay through the general equilibrium. Consider a credit crunch that further limits efficient firms from expanding. These firms' purchase of existing capital stock decreases. More importantly, economy-wide hiring drops and wage rates decrease such that the labor costs to run firms decrease. In response to lower input costs, the more inefficient firms postpone liquidation and less capital is sold. At the same time, these inefficient firms slowly pay down debt to reduce interest payments and to increase future borrowing capacity. In summary, the interaction is a result of the general equilibrium effect: when a financing problem restrict productive firms to expand, reduce demand for labor, and therefore lowers input costs, keeping assets and slowly deleveraging are more attractive to inefficient firms.

Because capital reallocation slows down during recessions, the idiosyncratic TFP dispersion across firms expands and the aggregate TFP declines with the tightened financing constraints, leading to a deepening recession. Thus, aggregate shocks to financing constraints interact with asset illiquidity, which helps explain why capital reallocation slows down in spite of larger potential benefits to reallocate during recessions. A major credit crunch after a banking crisis, such as the one in the U.S. in 2008, exemplifies these interactions.[7]

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[5] Shleifer and Vishny (1992) summarize two usual reasons for resale costs. First, when firms are liquidating, the potential buyers with the highest valuation are often those in the same industry who generally also have financial troubles. Assets may not go to the highest valuation users. Second, because of antitrust reasons, assets may need to be sold to industry outsiders, causing lower values for assets.

[6] Ramey and Shapiro (2001) provide empirical evidence of investment specificity and selling costs. They estimate the wedge between purchase price and resale price for different types of capital. Machine tools are sold at about a 69% discount off the purchase value, and structural equipment is sold at a 95% discount. These estimates suggest a large degree of specificity. Other evidence includes Holland (1990), in which a 50% to 70% discount is associated with the liquidation of the assets of a machine-tool manufacturer.

[7] U.S. economy after 2008 experiences similar massive deleveraging in Japan after 1990, summarized in Shirakawa (2012). Koo (2011) calls this type of recessions "balance sheet recessions". Meanwhile, Japanese corporate sector

Aggregate TFP shocks, however, generate different dynamics. When adverse aggregate TFP shocks hit, the profit rate is lower because of a lower productivity. Keeping capital is less profitable and inefficient firms have higher incentives to liquidate. Therefore, more reallocation and smaller TFP dispersion should be seen during recessions.[8] Meanwhile, deleveraging is much smaller and more short-lived compared to responses after a credit crunch in which inefficient firms slowly pay down debt.

Finally, I estimate the two aggregate shocks (aggregate shocks to financing constraints and aggregate TFP shocks) using Bayesian estimation methods and simulate the economy with only aggregate TFP shocks and only financial shocks. I confirm that aggregate TFP shocks alone cannot generate both observed procyclical capital reallocation and countercyclical TFP dispersion. Financial shocks are necessary to capture both dynamics. The joint dynamics thus offer some natural identification of the source(s) of business cycles.

The contribution of this paper is to consider the interaction of the two frictions. Without asset illiquidity, there will not be selling delay. Without financing constraints, productive firms can borrow as much as they want, pushing up the wage rate and interest rate. Thus, unproductive firms have small incentives of keeping assets, leaving a very short delays of selling assets.

The technical innovation of this paper is to propose a tractable method for firm dynamics with asset illiquidity and for the distribution of firms. Solutions to such model are usually complex[9] and sometimes infeasible with aggregate shocks (not to mention estimations of the shocks). To maintain tractability[10], I simplify the problem by solving portfolio choices between bonds and capital stock with (real) "option values", using finance portfolio choice theory, e.g., in Campbell and Viceira (2002). Therefore, the option value of capital depends on the portfolio weight (or leverage ratio) which is a new endogenous state variable, similar in Miao and Wang (2010).

Using the closed-form portfolio choice, individuals' decision rules are easily aggregated. Note that finite moments are not enough to characterize the firm distribution. But the tractability of the distribution still leads to exact aggregation and avoids the approximation method as in Krusell and Smith (1998). Therefore, system dynamics can be analyzed by solving simple simultaneous non-linear difference equations.

Literature Review. Real option is the salient feature of this paper. Dixit and Pindyck (1994) and Caballero and Engel (1999) focus on the timing of irreversible investment. This paper focuses on asset selling. Since assets may turn to be productive, running unproductive firms has an option value which may exceed the resale value. I show how to directly quantify the option value which is history dependent and summarized in firms' leverage ratios. The history dependent option value is

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has substantial less restructuring found by Hoshi, Koibuchi, and Schaede (2011). Thus, linking deleveraging and capital reallocation sheds some light on corporate balance sheet adjustments.

[8]Note that this is the standard creative destruction theory, but the opposite phenomena occur in data.

[9]See, for example, Bloom, Bond, and Reenen (2007), Bloom (2009), and Khan and Thomas (2011), who use piece-wise functions to approximate individual value functions.

[10]I follow and extend previous works by Angeletos (2007), Kiyotaki and Moore (2011), and Buera and Moll (2012). Under the class of CRRA preferences, if individual production functions feature constant returns to scale, the wealth spent on capital and bonds is simplified to a portfolio choice between the two.

similarly to that in Philippon and Sannikov (2007) where the value is from the history dependent contract. Additionally, the option value of keeping illiquid assets partially explains why firms tend to sell more liquid assets initially as in Duffie and Ziegler (2003).

The real option is linked to the delayed capital reallocation which generates larger dispersion during recessions. Implication of shocks to the dispersion of firm-specific conditions can be found, for example, in Bloom (2009), Arellano, Bai, and Kehoe (2012), Gilchrist, Sim, and Zakrajsek (2010), Panousi and Papanikolaou (2012), and Vavra (2012). But Bachmann and Bayer (2012a,b) show that large dispersion shocks are difficult to reconcile with other observations such as the investment rate dispersion. This paper shows how standard credit crunches can increase the dispersion endogenously through general equilibrium. Similarly, Bachmann and Moscarini (2011) study endogenous dispersion through the risk-taking behaviors of firms during recessions.

Further literature of macroeconomic implications of asset illiquidity and implications of financing constraints can be found in surveys by Caballero (1999) for capital illiquidity,[11] and Bernanke, Gertler, and Gilchrist (1999) and more recently Brunnermeier, Eisenbach, and Yuliy (2012) for financing constraints. Whether asset illiquidity or financing constraints can quantitatively amplify TFP and output losses is a matter of some debate.[12,13]

Innovation of this paper is to consider the interactions between asset illiquidity and financing constraints. The calibration shows that the aggregate TFP gap between the model economy and an economy without illiquidty of capital or without financing constraints is significant in the steady state and expands during recessions caused by credit crunches. In this sense, the closest papers are perhaps Kurlat (2011) and Khan and Thomas (2011). Kurlat (2011) shows analytically why the secondary market for existing capital may shut down and its macroeconomic implications through adverse selection. He focuses on the resale prices by simplifying outside financing: entrepreneurs are not allowed to borrow. Instead, I focus on different degrees of borrowing constraints and the impact on the portfolio choices among capital and bonds. Khan and Thomas (2011) quantitatively examine reallocation efficiency for given degrees of resale costs and financing frictions, focusing mainly on numerical aspects. I extensively use analytical methods (by focusing on more specific process of idiosyncratic shocks) to better explain the interaction of the two frictions on the capital

[11] Partial irreversibility is important for the interaction in the model. Previous work on investment irreversibility focuses on zero resale value, or completely irreversible investment, such as in Abel and Eberly (1996, 1999) and Thomas (2002). With zero resale value, firms only consider when to buy instead of when to sell.

[12] Thomas (2002) and Veracierto (2002) argue that irreversibility is not important in general equilibrium since idiosyncratic adjustments will be smoothed out. However, Kashyap and Gourio (2007) show that whether lumpy investment is important in aggregate depends on production function of firms and the distribution of fixed costs. Recently, Kiyotaki and Moore (2011) study the illiquidity shocks and the amplification. Eisfeldt (2004) and Kurlat (2011) model the illiquidity through asymmetric information.

[13] See financial constraints' impact on long-run output and TFP losses in Buera, Kaboski, and Shin (2011), Moll (2010), and Midrigan and Xu (2012). For example, Midrigan and Xu (2012) argue that financing frictions cannot generate the misallocation observed in Hsieh and Klenow (2009). Moll (2010) suggests that as firms have persistent idiosyncratic productivity shocks, they save enough to undo financial frictions. See also financing constraints' effect on short-run output and TFP fluctuations in Kocherlakota (2000), Cordoba and Ripoll (2004) and more recently Chen and Song (2012).

reallocation delays through general equilibrium,[14] before calibration and estimation. More importantly, in contrast to both Kurlat (2011) and Khan and Thomas (2011), I look at the deleveraging behaviors of firms before liquidations.[15] The deleveraging occurs because of risk-averse agents who try to smooth consumption. Thus, there are firms that borrow but are not constrained.

2 The Model

2.1 Preferences, Technology, and Information

Time is discrete and the horizon is infinite. There are two types of agents: households (with measure L) and entrepreneurs (with measure 1). Households are hand-to-mouth and supply labor inelastically. Entrepreneurs own production technology and some of them run firms.

Preferences. At time t, a typical entrepreneur j has preferences over the consumption stream $c_{jt}, c_{jt+1}, c_{jt+2}...$, and leisure stream $(1-h_{jt}), (1-h_{jt+1}), (1-h_{jt+2})...$, given by

$$E_t \sum_{s=t}^{\infty} \beta^{s-t}[u(c_{js}) + \eta(1-h_{js})] \tag{1}$$

where $\beta \in (0,1)$ is the discount factor, E_t is the conditional expectation operator, and $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. σ is the relative risk-aversion parameter. To simplify, I use $\sigma = 1$, i.e., $u(c) = log(c)$, leaving the general case in the Appendix. If j runs the firm, $h_{jt} = 1$; if j does not run the firm, $h_{jt} = 0$, and there is η extra leisure utility. η represent the fixed costs in running the business and will be important in the exit decisions later.[16]

Production. In the beginning of time t, j's firm uses capital k_{jt} (installed in $t-1$) and hire labor l_{jt} at a competitive wage rate w_t, to produce output:

$$y_{jt} = A_t \tilde{z}_{jt} k_{jt}^{\alpha} l_{jt}^{1-\alpha} = A_t (z_{jt} k_{jt})^{\alpha} l_{jt}^{1-\alpha}$$

where $\alpha \in (0,1)$, z_{jt} is the idiosyncratic productivity, and A_t is aggregate productivity. Aggregate productivities A_t are realized at the beginning of t, while idiosyncratic productivities z_{jt} are known at time $t-1$. Similarly, entrepreneur j learns z_{jt+1} at time t. Let $a_t = (z_{jt}, z_{jt+1})$ denote the productivity pair at time t. Some entrepreneurs are productive at time t ($z_{jt} = z^h$) while others

[14] The interactions in the model occur through general equilibrium. Credit crunches reduce wage rates because of a frictionless labor market. Empirically, despite wage rigidities, real wage rates decline during recessions, as found by Solon, Barsky, and Parker (1994) and Haefke, Sonntag, and Van Rens (2012). The decline of real wages is a consequence of lower wages of newly hired workers, in spite of moderate wage rigidity for longer term employees. Caggese and Cunat (2008) show firms can substitute flexible employment contracts for permanent employment contracts to reduce efficiency wages. Berger (2012) takes this a step further: firms hire more unproductive workers in expansions, but quickly fire them during recessions to reduce costs.

[15] Recently, Guerrieri and Lorenzoni (2011) look at deleveraging after credit crunches in households who face durable consumption goods illiquidity and financing constraints.

[16] Alternatively, an entrepreneur's engagement in running the firm produce output that are the fixed costs required for production. Modeling fixed costs as η utility will give rise to closed form solution later.

are unproductive ($z_{jt} = z^l$), with $z^h > z^l > 0$. For convenience, $\tilde{z}^h = (z^h)^\alpha$ and $\tilde{z}^h = (z^h)^\alpha$ denote the "measured" idiosyncratic productivity levels. The idiosyncratic productivity follows a two state Markov process[17] where the transition probabilities are

$$\text{Prob}(z_{jt+2} = z^l \mid z_{jt+1} = z^h) = p^{hl}$$

$$\text{Prob}(z_{jt+2} = z^h \mid z_{jt+1} = z^l) = p^{lh}$$

Capital Accumulation. Capital depreciates at a rate δ. Firms can invest in new capital stock, buy existing assets from the secondary market, or sell existing assets to the market. Inactive investment decisions are also allowed, i.e., j can choose to neither buy nor sell capital. One unit of efficient used assets, after being installed, is the same as one unit of new assets. Thus, the entrepreneur j's capital stock evolves according to

$$k_{jt+1} = (1-\delta)k_{jt} + i_{jt}$$

where $i_{jt} > 0$, $i_{jt} < 0$ and $i_{jt} = 0$ denote buying, selling, and inaction in investment, respectively.

As in neo-classical growth model, a buyer pays one unit of consumption goods for investment goods. Thus, amplification from asset price channel is switched off. For each unit of used assets sold, only $(1-d)$ fraction is useful for other buyers which implies that sellers receive a payment of $(1-d)$ for each unit of asset sold from them.

In sum, it costs 1 to invest (new or old capital) and $(1-d)$ to retire a unit of old capital. If the firm changes its quantity of capital from k to k', the cost of doing so is

$$\psi(k', k) = \begin{cases} k' - (1-\delta)k, & \text{if } k' > (1-\delta)k \\ 0, & \text{if } k' = (1-\delta)k \\ -(1-d)[(1-\delta)k - k'], & \text{if } k' < (1-\delta)k \end{cases}$$

Budget and Collateral Constraints. Entrepreneur j has access to the credit market. Denote the bond position as b_{jt} at the beginning of t and the interest rate from $t-1$ to t as R_t. The budget constraint of j can be written as

$$c_{jt} + b_{jt+1} + \psi(k_{jt+1}, k_{jt}) = y_{jt} - w_t l_{jt} + R_t b_{jt}.$$

j earns profits and interests, which are spent on consumption, new bonds, and paying the capital adjustment costs. Note that one can simplify profits further. Because firm j has a constant return

[17] Note that, $0 < p^{hl} < 1$, $0 < p^{lh} < 1$, and $p^{hl} + p^{lh} < 1$.

to scale (CRS) production technology, the instantaneous profits of j are linear in k_{jt}[18,19]

$$\Pi(z_{jt}, k_{jt}; w_t) = \max_{l_{jt}}\{(A_t z_{jt} k_{jt})^{\alpha} l_{jt}^{1-\alpha} - w_t l_{jt}\} = (z_{jt}\pi_t)k_{jt}$$

where $\pi_t = \alpha A_t^{\frac{1}{\alpha}} (\frac{1-\alpha}{w_t})^{(1-\alpha)/\alpha}$. Thus, the budget constraint can be simplified to

$$c_{jt} + b_{jt+1} + \psi(k_{jt+1}, k_{jt}) = z_{jt}\pi_t k_{jt} + R_t b_{jt}. \tag{2}$$

Entrepreneur j can short bonds (borrow), but not capital stock. Borrowing is bounded because j faces collateral constraints similar to those in Kiyotaki and Moore (1997) and Hart and Moore (1994).[20] The collateral constraint here includes the resale friction and an extra degree of financing friction θ_t:

$$R_{t+1} b_{jt+1} \geq -\theta_t (1-d)(1-\delta) k_{jt+1} \tag{3}$$

where $1 - \theta_t$ is the "haircut". Collateral constraint (3) says that debt value cannot exceed θ_t fraction of the resale value of the residual capital at $t + 1$. Also, for one unit of capital stock, the investing entrepreneur only needs to pay $1 - \theta(1 - d)(1 - \delta)/R_{t+1}$ as down payment. θ_t fluctuates and measures the financial market development, reflecting the external financing difficulties. For example, a permanently higher θ_t represents a better financial development, whereas a temporary decline in θ_t represents a sudden banking problem.

θ_t of (3) constrains capital stock allocation efficiency. Without (3), z^h owners can obtain any funds needed to invest in capital stock. The economy would reach the efficient production frontier, and as many entrepreneurs as possible can enjoy leisure.

A Summary. Each entrepreneur j maximizes (1) subject to (2) and (3), by choosing consumption c_{jt}, leisure h_{jt}, labor input l_{jt}, capital k_{jt+1}, and bonds b_{jt+1}, while taking the wage rate w_t and the interest rate R_{t+1} as given.

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[18] To see this, the first-order condition for labor is $(A_t^{\frac{1}{\alpha}} z_{jt} k_{jt})^{\alpha} (1-\alpha) l_{jt}^{-\alpha} = w_t$, so that the optimal labor demand is $l_{jt}^{*} = A_t^{\frac{1}{\alpha}} z_{jt} k_{jt} \left[\frac{1-\alpha}{w_t}\right]^{1/\alpha}$, from which profits are

$$\begin{aligned}\Pi(z_{jt}, k_{jt}; w_t) = A_t (z_{jt} k_{jt})^{\alpha} l_{jt}^{1-\alpha} - w_t l_{jt} &= A_t^{\frac{1}{\alpha}} z_{jt} k_{jt} \left[\left(\frac{1-\alpha}{w_t}\right)^{(1-\alpha)/\alpha} - w_t \left(\frac{1-\alpha}{w_t}\right)^{1/\alpha}\right] \\ &= A_t^{\frac{1}{\alpha}} z_{jt} k_{jt} \left(\frac{1-\alpha}{w_t}\right)^{1/\alpha} [\frac{w_t}{1-\alpha} - w_t] = z_{jt}\pi_t k_{jt}.\end{aligned}$$

[19] After substitution, labor demand is $l_{jt}^{*} = (\frac{\pi_t}{\alpha A_t})^{1/(1-\alpha)} z_{jt} k_{jt}$. Thus, total output produced by entrepreneur j can be written as $y_{jt} = \frac{z_{jt}\pi_t k_{jt}}{\alpha}$. To interpret this result, α fraction of the output becomes j's profits while the $1-\alpha$ fraction is paid through wages.

[20] This is a consequence of the fact that the human capital of the agent who is raising outside funds is inalienable. To ensure no "run away" default, the lender should be able to seize the tangible assets.

2.2 Recursive Equilibrium

I rewrite the entrepreneur's problem recursively and then define recursive equilibrium. Denote aggregate state as $X = (\Gamma(k, b, a), \theta, A)$ where $\Gamma(k, b, a)$ is the distribution of individuals' capital stock, bonds, and productivity pair at the beginning of each period. To emphasize, θ and A are the primitive shocks, i.e., financial disturbances and aggregate productivity fluctuations are exogenous shocks. Let V be the optimal value of an entrepreneur with k, b, and a, given the aggregate state variable X. The value function $V(k, b, a; X)$ satisfies the Bellman equation:

$$V(k, b, a; X) = \max\{W^1(k, b, a; X), W^0(k, b, a; X)\} \tag{4}$$

$$W^1(k, b, a; X) = \max_{\substack{k'>0 \\ R'b' \geq -\theta(1-d)(1-\delta)k'}} \{u(z\pi k + Rb - \psi(k', k) - b') + \beta E[V(k', b', a'; X')|a, X]\}$$

$$W^0(k, b, a; X) = \max_{b'}\{u(z\pi k + Rb + (1-\delta)(1-d)k - b') + \eta + \beta E[V(0, b', a'; X')|a, X]\}$$

The first step maximization is over the two actions: (1) to run the firm and get W^1 and (2) not to run the firm and get W^0. The second step is to choose the optimal consumption and savings (in capital stock and in bonds). Note that W^0 has the leisure utility η today, as an entrepreneur who gets W^0 does not run the firm today and there is no output tomorrow. The existence and uniqueness of the value function are standard by contraction mapping, as in Chapter 9 of Stokey, Lucas, and Prescott (1989).

Finally, I define the recursive equilibrium to close the model:

Definition 1 (The First Recursive Equilibrium Definition)**:**
The equilibrium is a law of motion H, policy functions $l = g^l(k, b, a; X)$, $k' = g^k(k, b, a; X)$, $b' = g^b(k, b, a; X)$, and pricing functions $\pi(X)$ and $R'(X)$ such that:
(i) l, k' and b' solve the entrepreneur's problem in (4) given the wage and the interest rate.
(ii) Markets for labor and bonds clear

$$\int l_{jt} dj = L, \int b_{jt+1} dj = 0.$$

(iii) The distribution evolution H is consistent with policy functions.

3 Decision Rules

The challenge of equilibrium characterization is to track the distribution of firms. Fortunately, the economy turns out to be highly tractable. I begin by describing the general solution under the economy with an active secondary market, leaving the mathematical details for later. Doing so will give readers an idea of where the argument flow is and allow them to skip the details.

In the details, I first show some general properties of entrepreneurs' recursive problems, regardless of the parameters. Then I shift the focus to certain parameters under which the equilibrium has both an active credit market and an active secondary asset market, since my focus is on the imperfect secondary market and delayed capital reallocation. In the next section, I show how the distribution can be easily handled.

3.1 A Quick Preview

It turns out that an individual entrepreneur's policy depends only on the leverage ratio, i.e., capital stock over equity $k/(k+b)$. Under certain parameters, z^h owners buy capital while z^l owners hold on to it before liquidation, in the steady state and the neighborhood around the steady state. I focus on equilibrium of such because it has imperfect capital reallocation and possible binding financing constraints for productive firms. In numerical analysis, I confirm such equilibrium.

In steady state, the optimal policy functions can be shown in two ways. One is to examine tomorrow's leverage given the leverage today (Figure 3a). When drawing z^h, entrepreneurs always lever up to some leverage ratio $\bar{\lambda}$, denoted as $z' = z^h$ line. When drawing z^l, entrepreneurs let the capital depreciate and pay back existing debt by consuming less. To see this, leverage tomorrow can be found through $z' = z^l$ and 45-degree lines. Leverage today can be mapped into leverage tomorrow by the following procedure. First, cut horizontally the z^l line in which intersection point G_1 has $k/(k+b)$ as today's leverage. Then, cut vertically the z^l line in which the intersection G_2 has the same $k/(k+b)$ as $k'/(k'+b')$ of G_1, which is tomorrow's leverage. Tomorrow's leverage keeps decreasing if an entrepreneur keeps drawing z^l until leverage reaches some threshold $\underline{\lambda}$. Then, the firm is liquidated since the capital stock will be very small and the fixed costs (the loss of leisure utility) will force the entrepreneur to do so.

Alternatively, one can examine the dynamics of k and b (Figure 3b). z^h owners always expand through the $z' = z^h$ line so that the leverage remains as $\bar{\lambda}$ and k/b is kept as $\frac{\bar{\lambda}}{1-\bar{\lambda}}$. For example, when $\bar{\lambda}$ is the leverage under the borrowing constraint, z^h owners are constrained by the credit limit. z^l owners, on the other hand, shrink their debt while letting the capital depreciate until they reach leverage $\underline{\lambda}$ (i.e., k/b ratio is $\frac{\underline{\lambda}}{1-\underline{\lambda}}$) when their firms are liquidated. The region characterized by the two lines with slope $\frac{\bar{\lambda}}{1-\bar{\lambda}}$ and $\frac{\underline{\lambda}}{1-\underline{\lambda}}$ denotes the inaction region. Inside the region, the reward for changing capital stock is insufficient. From outside the region (to the right of the $\frac{\underline{\lambda}}{1-\underline{\lambda}}$ slope line), the optimal policies are such as to proceed instantly to the $k = 0$ line, that is, to liquidate. Finally, $\bar{\lambda}$ and $\underline{\lambda}$ will change in response to aggregate shocks.

3.2 General Properties

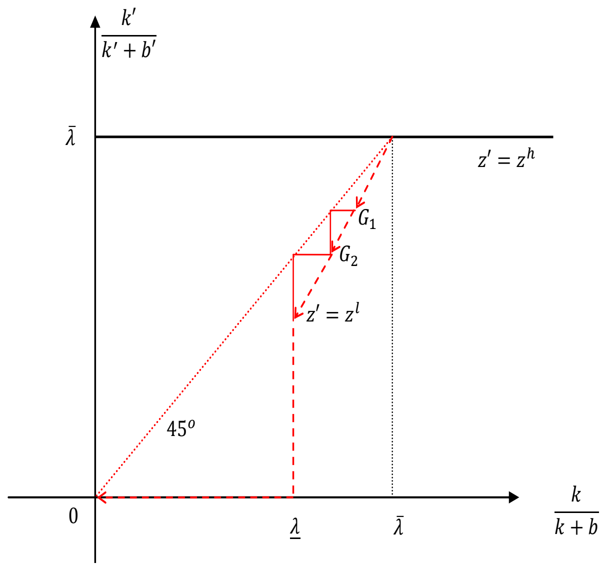
3.2.1 Properties of the Value Function

I establish some useful properties of the value function which will be used later. The value function behaves normally and has the "scale-invariant" property:

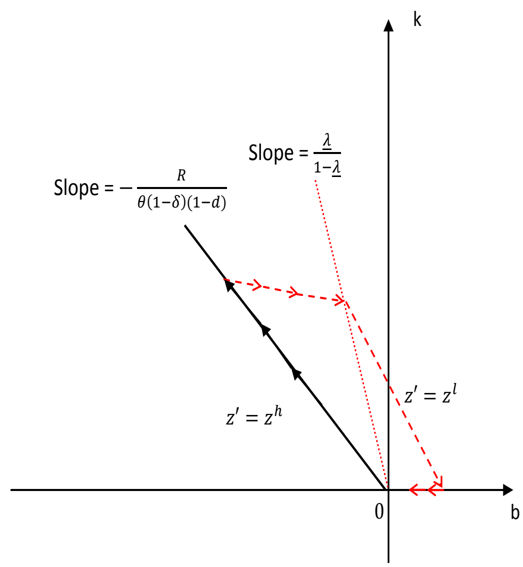
Figure 3: **Policy function illustration**

(a) Policy function mapping leverage today to leverage tomorrow. The $z' = z^h$ line denotes the target leverage when entrepreneurs draw z^h. They target at $\bar{\lambda}$ independent of their leverage today. The $z' = z^l$ line (which is below the 45-degree line) denotes the target leverage when drawing z^l. The target leverage is lower than today's leverage. When today's leverage reaches $\underline{\lambda}$ and the entrepreneur still draws z^l, the entire firm will be liquidated and leverage will be 0. (b) Dynamics of k and b. When entrepreneurs draw z^h, their firms expand (increase k while decrease b) along the solid line. Whenever entrepreneurs draw z^l, they step on the dashed line (one specific path): let k depreciate while paying back existing debt (increase b) until $k/b = \frac{\underline{\lambda}}{1-\underline{\lambda}}$ when they liquidate the firm.

(a) Policy function 1



(b) Policy function 2



Lemma 1 (Properties of the Value Function):
The value function V has the following properties

i $V(k, b, a; X)$ is increasing in k, b, and a, and concave in (k, b).

ii V satisfies

$$V(\gamma k, \gamma b, a; X) = V(k, b, a; X) + \frac{log\gamma}{1-\beta}. \tag{5}$$

Proof. See the Appendix. □

One can prove Lemma 1 by contraction mapping, which maps the space of functions with properties (i) and (ii) to itself. Let leverage of a firm defined as $k/(k + b)$. (ii) of Lemma 1 says that value functions of entrepreneurs with the same leverage ratio and a are affine transformations of each other. More importantly, target leverage of these entrepreneurs will be the same.[21]

Lemma 1 also suggests that fixed costs do not affect the difference of values of two entrepreneurs with the same leverage and a. Later, this property is important in deriving the liquidation strategy (i.e., when should an entrepreneur liquidates the firm). Intuitively from Lemma 1, the liquidation strategy depends only on the leverage ratio $k/(k + b)$. In the appendix, I prove that this property still holds under general CRRA utility.

[21]Their policies are (k', b') and $(\gamma k', \gamma b')$ so the target leverages are $k'/(k' + b')$ and $\gamma k'/(\gamma k' + \gamma b')$.

To derive policy functions, one needs derivatives of the value function. A potential problem is that $\psi(k', k)$ has no derivative when $k' = (1-\delta)k$. The left derivative is strictly smaller than the right one. Such functions are called sub-differentiable (at $k' = (1-\delta)k$). If ψ has a kink, so will V.[22] Therefore, V is also sub-differentiable at $k' = (1-\delta)k$. Fortunately, the value function is an upper envelope and it will thus be super-differentiable (the opposite of sub-differentiable). The function that is both sub-differentiable and super-differentiable is differentiable.[23]

Lemma 2 (Differentiability)**:**
$V(k, b, a; X)$ is differentiable for $k > 0$ and satisfies the envelope condition.

Proof. See the Appendix. □

3.2.2 Closed-form Policy Functions for $k' > 0$

Let $z(a)$ and $z'(a)$ denote today's and tomorrow's productivity. Because of potential inaction investment decisions, it is useful to work with "shadow value" of capital, i.e., $q(k, b, a; X)$ that satisfies the envelope condition:

$$V_k(k, b, a; X) = u'(c(k, b, a; X))[z(a)\pi + q(k, b, a; X)(1-\delta)], \tag{6}$$

for $k > 0$. q measures the value of capital in consumption goods unit. It shows how much entrepreneurs value their capital internally, particularly when the investment decision is inaction. Later, it turns out to be useful in solving policy functions.

q is equivalent to the marginal reward to adjust capital. When the marginal reward to increase capital reaches 1, a firm buys capital. When the marginal reward to decrease capital reaches $1-d$, the firm sells it. When there are no active purchases or sales, the marginal reward to increase capital is q, which should be less than 1; the marginal reward to decrease capital is q, which should be greater than $1-d$. Therefore, it is not optimal to adjust capital stock when

$$1 - d < q(k, b, a; X) = \frac{V_k/u'(c) - z\pi}{1-\delta} < 1.$$

Inside the inaction region, q is the option value of staying. Such characterization is similar to that in Dixit (1997). Moreover, q depends only on leverage, keeping everything else fixed:

Lemma 3 (Scale Invariance and Shadow Prices)**:**
The value function V and the shadow value q have the following properties

i V_k is homogeneous with degree -1.

ii For given a and X, $V_k/u'(c)$ depends only on $k/(k+b)$, but not on k or b level.

[22] As shown in Mordukhovich, Nam, and Yen (2006).
[23] The proof closely follows recent work by Clausen and Strub (2012).

iii $q(k, b, a; X)$ *can be simplified to* $q(\frac{k}{k+b}, a; X)$.

Proof. See the Appendix. □

One may also interpret $q(\frac{k}{k+b}, a; X)$ as the "stock price" of each share of a firm with leverage $k/(k+b)$ if the firm can be traded among entrepreneurs. When the firm is investing, each share of the stock is priced at 1. When sold, each share of the stock is priced at $1-d$. When firms are inactive in investment, each share of the stock is $q \in (1-d, 1)$. Having established the "competitiveness" of the q, we can express the first-order conditions as:

Proposition 1 (First-order Conditions)**:**
Define $\mu(k, b, a; X)$ *as the Lagrangian multiplier to the borrowing constraint. The first-order condition for* $k' > 0$ *is*

$$u'(c)q\left(\frac{k}{k+b}, a; X\right) = \beta E[V_k(k', b', a'; X')|a, X] + \mu(k, b, z; X)\theta(1-\delta)(1-d),$$

where $q(\frac{k}{k+b}, a; X)$ *is defined in equation (6). The first-order condition for* b' *is*

$$u'(c)R = \beta E[V_b(k', b', a'; X')|a, X] + \mu(k, b, a; X)R,$$

where V_b *is* $V_b(k, b, a; X) = u'(c)R$*. Finally,* $\mu(k, b, a; X) > 0$ *when the borrowing constraint binds, and* $\mu(k, b, a; X) = 0$ *otherwise.*

Proof. See the Appendix. □

When $\mu(k, b, z; X) = 0$, the first-order condition and the envelop condition yield:

$$E\left[\beta \frac{u'(c')}{u'(c)} \frac{z'(a)\pi' + (1-\delta)q(\frac{k'}{k'+b'}, a'; X)}{q(\frac{k}{k+b}, a; X)} \middle| a, X\right] = 1$$

which exemplifies classic asset pricing formula "$E_t[\Lambda_{t+1} r_{t+1}] = 1$" or "$E_t[\Lambda_{t+1}(r_{t+1} - R_{t+1})] = 0$", where "$\Lambda$" is the stochastic discount factor and r is the return from an asset. Here, the return on capital is $\frac{z'(a)\pi' + (1-\delta)q(\frac{k'}{k'+b'}, a'; X)}{q(\frac{k}{k+b}, a; X)}$, where $q(\frac{k}{k+b}, a; X)$ takes different values depending on the decision of buying, selling, or being inactive.

The asset pricing formula also sheds some light on solving portfolio choices between capital stock and bonds. To see this, first define the rate of return of having capital ($k' > 0$) as

$$r'(k', b', a'; X'|k, b, a; X) = \frac{z'(a)\pi' + (1-\delta)q(\frac{k'}{k'+b'}, a'; X')}{q(\frac{k}{k+b}, a; X)},$$

define the net worth of an entrepreneur using the shadow value of capital as

$$n(k,b,a;X) = z(a)\pi k + q(\frac{k}{k+b},a;X)(1-\delta)k + Rb,$$

and let ϕ denote the fraction of net worth spent on capital. We have the closed-form solution as:

Proposition 2 (Closed-form Policy Functions)**:**
The policy function on consumption $c = c(k,b,a;X)$, capital $k' = k'(k,b,a;X) > 0$, and bonds $b' = b'(k,b,a;X)$ can be expressed as

$$c = (1-\beta)n(k,b,a;X), \quad k' = \frac{\phi}{q(\frac{k}{k+b},a;X)}\beta n(k,b,a;X), \quad b' = (1-\phi)\beta n(k,b,a;X).$$

where ϕ satisfies

$$\begin{cases} E\left[\frac{r'-R'}{\phi r'+(1-\phi)R'}\middle| a,X\right] = 0, & \text{if } E\left[\frac{r'}{\phi r'+(1-\phi)R'}\middle| a,X\right] = 1 \\ \phi = \frac{1}{1-\theta(1-\delta)(1-d)/qR'}, & \text{if } E\left[\frac{r'}{\phi r'+(1-\phi)R'}\middle| a,X\right] < 1 \end{cases}$$

Finally, k' is consistent with $q(\frac{k}{k+b},a;X)$, so that $k' > (1-\delta)k$ for $q(\frac{k}{k+b},a;X) = 1$ and $k' < (1-\delta)k$ for $q(\frac{k}{k+b},a;X) = 1-d$. Otherwise, ϕ should be such that $k' = (1-\delta)k$.

Proof. See the special case $\sigma = 1$ of the proof under general CRRA utility in the Appendix. □

Notice that the stochastic discount factor here is $\Lambda' = \frac{1}{\phi r'+(1-\phi)R'}$ such that asset pricing formula $E[\Lambda'(r'-R')] = 0$ holds. A typical entrepreneur consumes $(1-\beta)$ fraction and saves the other β fraction of the net worth. She uses the savings to invest in a portfolio. The portfolio consists of risky assets (capital stock) and risk-free assets (bonds), allowing shorting on risk-free assets but not on risky ones. If she invests ϕ fraction of a dollar in risky assets and the other $1-\phi$ fraction in risk-free assets, the next period's rate of return is $\phi r' + (1-\phi)R'$. The goal of portfolio choice is to maximize the expected log rate of return (i.e., the solution of ϕ).[24]

Even though the saving rate is a constant (β) under log utility, different entrepreneurs save different fractions of the "accounting" net worth which is either $z\pi k + (1-\delta)k + Rb$ or $z\pi k + (1-\delta)(1-d)k + Rb$. Unlike the accounting net worth, the "economic" net worth evaluates capital at shadow prices, which varies across entrepreneurs when the investment decisions opt for inaction.

[24]Policy functions have closed-form expressions for any σ (see the Appendix). But under general CRRA utility, the saving rate (not necessarily β) and portfolio weight ϕ intertwine with each other. The reason is that with general CRRA utility the income and substitution effect do not offset each other, for example illustrated in Campbell and Viceira (2002). The combination of the two effects are so-called "hedging demand" in the asset pricing literature. Depending on the investment opportunities in the long time frame, agents put different weights on capital and consume differently.

3.3 The Inaction Regions and Liquidation Choices

I confine my attention to the equilibrium with an active credit market and an active secondary market. There may or may not be inaction in investment. When there is, there exists at least a q that is between $1-d$ and 1. To characterize the inaction region, one only needs to check how the shadow price $q(k/(k+b), a; X)$ varies as $k/(k+b)$ and a change (for a given X). The inaction region is the set of $k/(k+b)$ and a such that the shadow price is between $1-d$ and 1.

In such equilibrium, z^h owners should always invest and z^l owners should not because:

$$z^h\pi' + (1-\delta) > R' \geq z^l\pi' + (1-\delta)$$

The first inequality should hold; otherwise no entrepreneurs will invest. The second inequality should hold. If $R' < z^l\pi' + (1-\delta)$, z^l owners always find a higher return from investing than the return from holding bonds regardless of drawing z^h or z^l tomorrow. They strictly prefer to invest and borrow to the credit limit. In that economy, everyone is a borrower, which is inconsistent with equilibrium definition since the bond market cannot clear.

Therefore, some or all z^h owners invest and borrow. Because of the linear rate of return in individual level, they have the same target leverage $k'/(k'+b')$ tomorrow regardless of their leverage today (Proposition 3). $k'/(k'+b')$ may or may not reach the leverage under credit limits.

For z^l owners, profits from capital stock are low. Their investment decision is either to hold or to sell. It turns out that an entrepreneur j who persistently draws z^l hold capital for finite periods. The shadow price during the process of holding capital is monotonically decreasing until it reaches $1-d$ when j liquidates assets. Additionally, the leverage decreases before liquidation.

Proposition 3 (Leverage and Deleverage)**:**
In equilibrium with an active secondary market

i z^h *owners borrow and invest. Moreover, they have the same target leverage* $\frac{k'}{k'+b'} = \bar{\lambda}$.

ii Denote today's shadow price as q *and tomorrow's shadow price as* q'*. Then,*

$$q' \begin{cases} = 1 & \text{if } z' = z^h \\ < q & \text{if } z' = z^l \end{cases} \quad \text{and} \quad \frac{k'}{k'+b'} \begin{cases} = & \bar{\lambda} \text{ if } z' = z^h \\ < & \frac{k}{k+b} \text{ if } z' = z^l \end{cases}$$

Proof. See the Appendix. □

The delveraging behavior during the inaction process are intuitive. For z^l entrepreneurs, running business is not profitable compared to risk-free rate. Without resale costs, they will liquidate and repay all the debt. But with resale costs, those who just turn from z^h to z^l hold capital initially. They can still shrink interest payment in order to smooth consumption. Not surprisingly, capital is less and less valued.

After some periods of inaction, capital stock gradually shrinks to a very small amount. The fixed costs of running a business eventually force the z^l owners to liquidate. To see this, z^l owners compare the value of liquidating and holding. Once the value after liquidation is the same as the value of holding strategy, z^l owners start to liquidate, that is, there exists a stopping time:

Proposition 4 (Optimal Stopping Time):
For z^l owners, there exists an optimal capital liquidation rule (stopping-time rule or exit rule). Let $n = z^l\pi + (1-\delta)(1-d) + R\frac{1-\underline{\lambda}}{\underline{\lambda}}$ and suppose a finite $\underline{\lambda} \in [0, \bar{\lambda}]$ is a root of

$$\begin{aligned} \eta &= \frac{\beta}{1-\beta} p^{lh} E\left[log\left(1 + (1-\delta)\frac{z^l\pi' + (1-\delta) - (1-d)R'}{\beta n R'}\right)\Big| X\right] \\ &+ \frac{\beta}{1-\beta} p^{ll} E\left[log\left(1 + (1-\delta)\frac{z^l\pi' + (1-\delta)(1-d) - (1-d)R'}{\beta n R'}\right)\Big| X\right] \end{aligned} \quad (7)$$

i When $\frac{k}{k+b} > \underline{\lambda}$, z^l owners are inactive in adjusting capital. When $\frac{k}{k+b} < \underline{\lambda}$, they liquidate the whole firm. When $\frac{k}{k+b} = \underline{\lambda}$, they are indifferent between holding or liquidating capital.

ii If no $\underline{\lambda}$ satisfies equation (7), then no z^l entrepreneur sells capital.

Proof. See the Appendix. □

The indifference condition (7) is intuitive. Entrepreneurs are indifferent between liquidation and holding when the gains of liquidation (extra η utility) equals the expected discounted cost-s of not doing so (the right hand side, extra value of holding capital stock one more period). In calculating the costs of not liquidating, for each unit of net worth saved in capital stock and bonds, the excess return is $\left(1 + (1-\delta)\frac{z^l\pi' + (1-\delta) - (1-d)R'}{\beta n R'}\right)$ when drawing z^h tomorrow and $\left(1 + (1-\delta)\frac{z^l\pi' + (1-\delta)(1-d) - (1-d)R'}{\beta n R'}\right)$ when drawing z^l tomorrow.

So far, I have shown the steps to establish the decision rules in Figures 3a and 3b. Now, the inaction region can be easily expressed by the set of leverage ratios and productivities

$$\{(\frac{k}{k+b}, a) : \underline{\lambda} \leq \frac{k}{k+b} \leq \bar{\lambda} \quad and \quad z'(a) = z^l\}$$

where $\underline{\lambda}$ is the lower bound while $\bar{\lambda}$ is the upper bound. To understand the changes of the inaction region, I show what leads to $(\bar{\lambda} - \underline{\lambda})$ changes. If $(\bar{\lambda} - \underline{\lambda})$ is larger, the inaction region expands. I focus on the borrowing constrained economy, i.e, $\bar{\lambda} = 1/(1 - \theta(1-\delta)(1-d)/R')$. Intuitively, z^l owners have more incentive to hold capital if (1) they are more patient (a larger β), (2) the fixed costs are smaller (a smaller leisure utility η), and (3) the selling discount d is higher. All of these increase the net benefits of holding capital and expand the inaction region.

Corollary 1 (Changes of Inaction Region: Partial Equilibrium Effect):
If borrowing is constrained, the inaction region expands when

i β *is higher, that is,* $\partial(\bar{\lambda} - \underline{\lambda})/\partial\beta > 0$

ii η *is smaller, that is,* $\partial(\bar{\lambda} - \underline{\lambda})/\partial\eta < 0$

iii d *is higher, that is,* $\partial(\bar{\lambda} - \underline{\lambda})/\partial d < 0$

Proof. Define m to be the right hand side of equation (7). From the proof of Proposition 4, $\partial m/\partial\underline{\lambda} < 0$. Notice that $\partial m/\partial\beta < 0$ and $\bar{\lambda}$ does not depend on β. Then using the implicit function theorem, we know that $\partial(\bar{\lambda} - \underline{\lambda})/\partial\beta > 0$ which proves (i). (ii) can be proved by similar steps. (iii) can be proved by similar steps and by taking into account $\frac{\partial\bar{\lambda}}{\partial d} = -(\bar{\lambda})^2\theta(1-\delta)/R'$. □

A larger degree of asset illiquidity directly expands the inaction region. In contrast, a larger degree of financing frictions (a lower θ) does not have a direct effect, from equation (7). Moreover, when θ goes down, the highest leverage become smaller ($\bar{\lambda}$ is smaller) and $(\bar{\lambda} - \underline{\lambda})$ decreases. But a lower θ has a general equilibrium effect. If financing frictions limit the expansion of productive firms so that aggregate demand shrinks and labor input costs are lower, π' will be higher and z^l owners will wait until a even lower leverage before liquidation ($\underline{\lambda}$ is much smaller). In that case, $(\bar{\lambda} - \underline{\lambda})$ increases in response to a lower θ.

Corollary 2 (Changes of Inaction Region: General Equilibrium Effect)**:**
In borrowing constrained equilibrium, $\pi'(X')$ *depends on* θ*. The inaction region expands when*

i profits rate is higher, that is, $\partial(\bar{\lambda} - \underline{\lambda})/\partial\pi'(X') > 0$

ii financing constraints are tighter and $\partial\pi'(X')/\partial\theta$ *is negative and sufficiently small, that is,* $\partial(\bar{\lambda} - \underline{\lambda})/\partial\theta < 0$

Proof. By similar steps in Corollary 1. □

The changes of profits rate and subsequent impacts on liquidation choices are essential for understanding the system dynamics in response to aggregate shocks. The Corollary provides us some intuition. For example, a credit crunch (a lower θ) will reduce investment and employment, which leads to lower input costs and a higher profits rate. Therefore, after a credit crunch, inefficient firms have higher incentives to hold assets. Later, the intuition from the Corollary will be confirmed in the numerical analysis.

4 Recursive Equilibrium Revisit

So far, we know that entrepreneurs with the same leverage $k/(k+b)$ and productivity put the same portfolio weights on k and b. Thus, I can define aggregate capital stock and aggregate bonds for a specific $k/(k+b)$ ratio, given a productivity pair a, i.e.,

$$K(x,a) = \int_{\{(k,b):\frac{k}{k+b}=x\}} k\Gamma(dk,db,a), \quad B(x,a) = \int_{\{(k,b):\frac{k}{k+b}=x\}} b\Gamma(dk,db,a)$$

Equilibrium can be redefined as a mapping $(K(x,a), B(x,a), \theta, A) \rightarrow (K'(x,a), B'(x,a), \theta', A')$. I apply this idea to characterize the evolution of the firm distribution. Subsection 4.1 shows the details and Subsection 4.2 redefines the equilibrium using the distribution in Subsection 4.1.

4.1 The Distribution of Firms

Since drawing z^h always means investing, keeping track of the firm distribution is equivalent to keeping track of firms with the time length of having been drawing z^l. Thanks to Proposition 4, there is a stopping rule for entrepreneurs who run the firms but always draw z^l.

At the beginning of time t, let $s = 1, 2, ...$ denote the *vintage* of entrepreneurs, who have been drawing s times of z^l. These firms did not invest in $t-1$. Clearly, $s = 0$ denotes the state in which the entrepreneur just finished investing (or drew z^h as time t productivity in time $t-1$). Drawing z^h means entrepreneurs will go to vintage $s = 0$, whereas drawing z^l means going to the next vintage, i.e., the vintage whose number equal current vintage number plus 1. Inside each vintage, the $k/(k+b)$ ratio is the same, which allows me to replace $q(\frac{k}{k+b}, a; X)$ by vintage-specific price. When entrepreneurs with $k/(k+b)$ decide to go from vintage s to s', the shadow price of capital will be $q(s'; X)$, which is vintage-specific and corresponds to a specific $k'/(k'+b')$.

When the secondary market is active, there exists an integer $N_t < +\infty$ at time t, such that entrepreneurs who are from vintage $N_t + 1$, $N_t + 2$,... and draw z^l hold no capital stock; while those who are from vintage 0,1,...,N_t and draw z^l will be inactive in capital.

For simplicity, I focus on small exogenous shocks around the steady state such that the equilibrium vintages do not change, i.e., $N_t = N$ where N is an endogenous constant integer. (Note that N itself varies in different steady states). In numerical exercises, I verify that the shocks do not change vintage numbers N. When N stays the same, entrepreneurs from vintage N who draw z^l again are indifferent between liquidating and keeping capital. They play a mixed strategy between staying and liquidating.

Corollary 3:

In the equilibrium with capital reallocation, there exists an integer N such that:

i Entrepreneurs go to vintage 0 once they draw z^h.

ii For those entrepreneurs who draw z^l, they go to the next vintage.

- *Those in vintage* 0 *to* $N-1$ *hold on to capital.*
- *Those in vintage N are indifferent between being inactive in capital or liquidating.*

- *Those in vintage $N+1$ liquidate the firm. Those in $N+2$, ... do not run the firm.*

Now, I describe the features of the vintages. One can group vintages after $N+2$ together to be one vintage, since entrepreneurs in vintages $N+2$, $N+3$,... only hold bonds, thanks to Corollary 3. To simplify, the probability of drawing z^h and z^l in each vintage is

$$\tilde{P} = \begin{bmatrix} p^{hh} & p^{lh} & \dots & p^{lh} \\ p^{hl} & p^{ll} & \dots & p^{ll} \end{bmatrix}^T_{(N+2)\times 2}$$

where $\tilde{P}_{i1}$ and $\tilde{P}_{i2}$ are the probability of drawing z^h and z^l in vintage i. The associated vintage specific productivity vector is

$$Z = \begin{bmatrix} z^h & z^l & z^l & \dots & z^l & z^l \end{bmatrix}'_{(N+2)\times 1}$$

With slight abuse of notation, let $p^{ih} = \tilde{P}_{i1}$ and $p^{il} = \tilde{P}_{i2}$ be the probability of drawing z^h and z^l as the new productivity respectively; let z^i as the ith element of Z which is the current productivity of an entrepreneur in vintage i.

A fraction of the entrepreneurs from vintage N who draw z^l goes to vintage $N+1$ and the other fraction liquidates capital and goes to vintage $N+2$. They are indifferent between holding or selling capital. As in Proposition 4, the optimal indifference leverage $\underline{\lambda}$ solves[25]

$$\begin{aligned}\eta &= \frac{\beta p^{(N+1)h}}{1-\beta} E\left[log\left(1+(1-\delta)\frac{z^{N+1}\pi_{t+1}+(1-\delta)-(1-d)R_{t+1}}{\beta R_{t+1}(z^N\pi_t+(1-\delta)(1-d)+R_t(1-\underline{\lambda}_t)/\underline{\lambda}_t)}\right)\middle| X_t\right] \\ &+ \frac{\beta p^{(N+1)l}}{1-\beta} E\left[log\left(1+(1-\delta)\frac{z^{N+1}\pi_{t+1}+(1-\delta)(1-d)-(1-d)R_{t+1}}{\beta R_{t+1}(z^N\pi_t+(1-\delta)(1-d)+R_t(1-\underline{\lambda}_t)/\underline{\lambda}_t)}\right)\middle| X_t\right]\end{aligned}$$

Finally, let f_t^i $(i = 1, 2, ..., N+2)$ be the fraction of entrepreneurs who go to vintage i out of all entrepreneurs who draw vintage i productivity z^i, and $1-f_t^i$ be the other fraction of entrepreneurs who liquidate capital. Notice that $f_t^i = 1$ for $i = 0, 1, 2, ..., N$: those who draw z^h will always invest and go to vintage 0, and those who draw z^l in vintage $i-1$ will be inactive in investment and go to vintage i. Also, $f_t^{N+2} = 1$ because entrepreneurs who are from vintage $N+1$ or $N+2$ and draw z^l will always hold only bonds and go to vintage $N+2$. Finally, $f_t^{N+1} \in [0, 1)$ because entrepreneurs who are from vintage N and who draw z^l play mixed strategy: f_t^{N+1} fraction of them go to vintage $N+1$ and $1-f_t^{N+1}$ fraction of them go to vintage $N+2$.

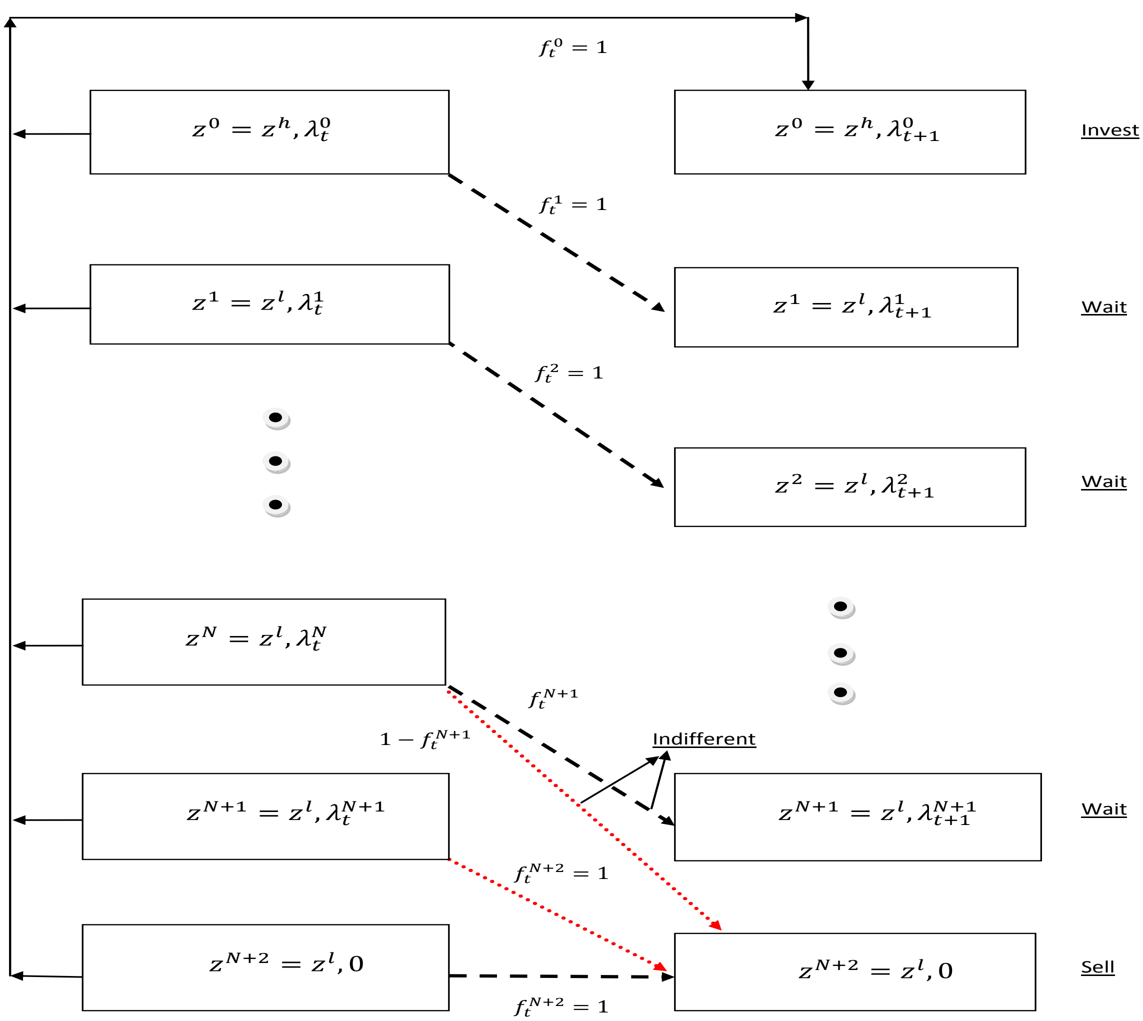
Now, we can fully characterize the firm distribution evolution from t to $t+1$ in Figure 4.[26]

[25] In equilibrium, $(1-\underline{\lambda}_t)/\underline{\lambda}_t$ is equal to the ratio of b_t^N/k_t^N in vintage N. Those from vintages $N+1$ and $N+2$ who also draw z^l go to vintage $N+2$ by holding only bonds.

[26] If we expand the state space of idiosyncratic producibility and reclassify each vintage as a state, the matrix

Figure 4: **Evolution of the distribution**

Each box represents a vintage in which firms have the same $\lambda = \frac{k}{k+b}$ leverage ratio. The vintage number is identical to how many periods an entrepreneur has been drawing z^l. Entrepreneurs who draw z^h invest and move to vintage 0. Entrepreneurs who are from vintage 0 to $N-1$ and draw z^l are inactive. Entrepreneurs who are from vintage N and draw z^l are indifferent between liquidating or continuing production. Entrepreneurs in vintage $N+1$ or the last vintage $N+2$ hold only bonds if drawing z^l (liquidate the firm or continuing holding only bonds). f_t^i denotes the fraction of entrepreneurs who go to vintage i out of all entrepreneurs who draw vintage i productivity z^i. $1 - f_t^i$ then denotes the other fraction of entrepreneurs who do not go to vintage i but liquidate their firms.



P below is the transition probability matrix

$$P = \begin{bmatrix} p^{hh} & p^{hl} & & & \\ p^{lh} & & p^{ll} & & \\ \dots & \dots & \dots & \dots & \dots \\ p^{lh} & \dots & \dots & p^{ll} f_t^{N+1} & p^{ll}(1 - f_t^{N+1}) \\ p^{lh} & \dots & \dots & 0 & p^{ll} \\ p^{lh} & \dots & \dots & 0 & p^{ll} \end{bmatrix}_{(N+2)\times(N+2)}$$

where P_{ij} denotes the probability from vintage i to vintage j. The right eigenvector of P^T associated with eigenvalue one is the population of entrepreneurs in each vintage in the steady state. See Chapter 2 of Ljungqvist and Sargent (2004) for details. In calibration, I use this property to calculate the cross-section standard deviation of TFP of the existing firms.

4.2 Recursive Equilibrium Revisit

Thanks to the vintage distribution, I can leave aggregate state X out and denote variables with vintage superscript and time subscript t, $t+1$,.... For example at time t, the capital shadow price of entrepreneurs who are going to vintage i is q_t^i. For consistency, let $q_t^0 = 1$ denote the buying price.[27] Define the (risky) rate of return on capital from time t to time $t+1$ of entrepreneurs who are going to vintage i as r_{t+1}^{ij}, where $i = 0, 1, ..., N+1$ and $j \in h, l$ indicates drawing (time $t+2$ productivity) z^h or z^l at time $t+1$.[28] Specifically, the vintage i specific rate of return on capital when z^h or z^l is realized can be written as

$$r_{t+1}^{ih} = \frac{z^i \pi_{t+1} + (1-\delta) q_{t+1}^0}{q_t^i}, r_{t+1}^{il} = \frac{z^i \pi_{t+1} + (1-\delta) q_{t+1}^{i+1}}{q_t^i}, \quad \text{for } i = 1, 2, ..., N+1.$$

For convenience, denote $\bar{r}_{t+1}^i$ as the average return, i.e., for $i = 0, 1, 2, ..., N+1$,

$$\bar{r}_{t+1}^i = p^{ih} E[r_{t+1}^{ih} | X_t] + p^{il} E[r_{t+1}^{il} | X_t].$$

Then, according to Proposition 2, the portfolio weight ϕ on capital can be simplified as:

Corollary 4 (Vintage-specific Portfolio Choices)**:**
The capital weight ϕ_t^i ($i = 0, 1, 2..N$) for entrepreneurs who are going to vintage i solves

$$\min\{\frac{1}{1-\theta_t(1-\delta)(1-d)/R_{t+1}}, \phi_t^i = -\frac{R_{t+1}(\bar{r}_{t+1}^i - R_{t+1})}{(r_{t+1}^{ih} - R_{t+1})(r_{t+1}^{il} - R_{t+1})}\}$$

Now, we are ready to redefine the equilibrium. Denote K_t^i and B_t^i as the aggregate capital stock and bonds in vintage i. Thanks to the closed-form decision rules in Proposition 2, the transition dynamics is highly tractable as in the following non-linear equations. Capital transition can be characterized by aggregate capital in vintage 0

$$q_t^0 K_{t+1}^0 = f_t^0 \phi_t^0 \sum_{i=0}^{N+2} p^{ih} \beta [z^i \pi_t K_t^i + (1-\delta) q_t^0 K_t^i + R_t B_t^i], \tag{8}$$

by aggregate capital in vintage $i = 1, 2, ..., N+1$

$$q_t^i K_{t+1}^i = f_t^i \phi_t^i p^{(i-1)l} \beta [z^{i-1} \pi_t K_t^{i-1} + (1-\delta) q_t^i K_t^{i-1} + R_t B_t^{i-1}], \tag{9}$$

and by aggregate capital in vintage $N+2$

$$K_{t+1}^{N+2} = 0. \tag{10}$$

[27] "Shadow price" of capital of entrepreneurs who are going to invest and go to vintage 0.

[28] For example, at time $t+1$, an entrepreneur in vintage 3 draws (time $t+2$ productivity) z^h, her rate of return on capital from t to $t+1$ is r_{t+1}^{3h}.

The transition of bonds can be characterized by aggregate bonds in vintage 0

$$B_{t+1}^0 = f_t^0(1-\phi_t^0)\sum_{i=0}^{N+2} p^{ih}\beta[z^i\pi_t K_t^i + (1-\delta)q_t^0 K_t^i + R_t B_t^i], \tag{11}$$

by aggregate bonds in vintage $i = 1, 2, ..., N+1$,

$$B_{t+1}^i = f_t^i(1-\phi_t^i)p^{(i-1)l}\beta[z^{i-1}\pi_t K_t^{i-1} + (1-\delta)q_t^i K_t^{i-1} + R_t B_t^{i-1}], \tag{12}$$

and finally by aggregate bonds in vintage $N+2$

$$\begin{aligned} B_{t+1}^{N+2} &= \sum_{i=1}^{N}(1-f_t^{i+1})p^{il}\beta[z^i\pi_t K_t^i + (1-\delta)q_t^{N+2}K_t^i + R_t B_t^i] \\ &+ \sum_{i=N+1}^{N+2} f_t^{N+2}p^{il}\beta[z^i\pi_t K_t^i + (1-\delta)q_t^{N+2}K_t^i + R_t B_t^i]. \end{aligned} \tag{13}$$

The aggregate capital in vintages $i = 1, 2, ..., N+1$ satisfies

$$K_{t+1}^i = p^{(i-1)l} f_t^i (1-\delta) K_t^{i-1}, \tag{14}$$

together with consistent f_t^i

$$f_t^i \begin{cases} = & 1, & \text{if } i = 0, 1, ..., N \\ \in & [0,1), & \text{if } i = N+1 \\ = & 1, & \text{if } i = N+2. \end{cases} \tag{15}$$

The labor market and bond market clearing conditions are

$$\left(\frac{\pi_t}{\alpha A_t}\right)^{\frac{1}{1-\alpha}}\left(\sum_{i=0}^{N+2} z^i K_t^i\right) = L, \quad \sum_{i=0}^{N+2} B_{t+1}^i = 0. \tag{16}$$

Finally, the stopping condition of an entrepreneur from vintage N who draws z^l again is

$$\begin{aligned} \eta &= \frac{\beta p^{(N+1)h}}{1-\beta}E\left[log\left(1+(1-\delta)\frac{z^{N+1}\pi_{t+1} + (1-\delta)q_{t+1}^0 - q_t^0(1-d)R_{t+1}}{\beta R_{t+1}(z^N\pi_t + (1-\delta)(1-d)q_t^0 + R_t B_t^N/K_t^N)}\right)\middle| X_t\right] \\ &+ \frac{\beta p^{(N+1)l}}{1-\beta}E\left[log\left(1+(1-\delta)\frac{z^{N+1}\pi_{t+1} + (1-\delta)q_{t+1}^0(1-d) - q_t^0(1-d)R_{t+1}}{\beta R_{t+1}(z^N\pi_t + (1-\delta)(1-d)q_t^0 + R_t B_t^N/K_t^N)}\right)\middle| X_t\right]. \end{aligned} \tag{17}$$

Definition 2 (The Second Recursive Equilibrium Definition)**:**
The recursive competitive equilibrium is functions ($\{\phi_t^i\}_{i=0}^{N+2}$, $\{f_t^i\}_{i=1}^{N+2}$, $\{q_t^i\}_{i=0}^{N+2}$, $\{K_{t+1}^i\}_{i=0}^{N+2}$, $\{B_{t+1}^i\}_{i=0}^{N+2}$, π_t,R_{t+1}) *of state variables* ($\{K_t^i\}_{i=0}^{N+2}, \{B_t^i\}_{i=0}^{N+2}, \theta_t, A_t$) *and a given initial condition* ($\{K_0^i\}_{i=0}^{N+2}$, $\{B_0^i\}_{i=0}^{N+2}$, θ_0,A_0)*, such that:*

i equations (8) to (17) are satisfied

ii $\{\phi_t^i\}_{i=0}^N$ *solve the portfolio problems in Corollary 4*

iii $q_t^0 = 1$ *and* $q_t^{N+1} = q_t^{N+2} = 1 - d$

iv together with the law of motion of (θ_t, A_t)

The capital market clearing is embedded in the capital transition dynamics, and one can easily verify that the goods market clearing condition is satisfied (i.e., Walras' Law holds).

4.3 Efficiency and Delayed Reallocation in the Steady State

The longer the waiting periods, the more capital reallocation is delayed and the less efficient is the economy (i.e., the lower is the aggregate TFP). In steady state, aggregate productivity $A_t = 1$ and the aggregate TFP is defined as

$$TFP = \frac{Y}{K^\alpha L^{1-\alpha}}$$

where Y is the total output and K is the total capital stock. Note that α fraction of the output is entrepreneurs' profits. Output can be written as $Y = \frac{\pi}{\alpha}(z^h K^h + z^l K^l)$, where $K^h = K^0$ and $K^l = \sum_{i=1}^{N+1} K^i$ denote the capital stock under z^h and z^l technology respectively. Together with the labor market clearing condition $(\frac{\pi}{\alpha})^{\frac{1}{1-\alpha}}(z^h K^h + z^l K^l) = L$, TFP can be simplified to

$$TFP = \frac{(z^h K^h + z^l K^l)^\alpha}{(K^h + K^l)^\alpha} = \frac{(z^h K^h / K^l + z^l)^\alpha}{(K^h / K^l + 1)^\alpha}. \tag{18}$$

When $K^l \to 0$, all capital is installed under z^h technology, and the TFP reaches the upper bound $\tilde{z}^h = (z^h)^\alpha$. When $K^l > 0$, we know that the relative capital stock ratio K^h/K^l determines the economy efficiency. Intuitively, the longer the waiting period, the smaller K^h/K^l ratio and thus a lower TFP in the economy. The quantitative effects of delayed reallocation and aggregate TFP losses are the main targets in the next section.

What determines the delayed reallocation and thus the aggregate TFP of the economy? Intuitively, the essential parameters that affect the trade-off between liquidation and continued production are the relative productivity gap, the persistence of the transition matrix, the outside option utility η, the resale costs d, and the degree of financing frictions θ. For example, if the relative productivity gap is larger, holding capital has a higher benefit so waiting periods tend to be longer. But also the interest rate in the steady state is higher because z^h entrepreneurs can accumulate more capital and collateralized borrowing is easier. In that case, liquidation is more preferred. The net effects are unclear and further numerical examinations are needed.

However, the next proposition shows that labor supply and capital share do not have any impact on the trade-offs in the steady state, so is the absolute levels of z^h and z^l (as long as the relative gap remains the same, the vintage number does not change).

Proposition 5 (Waiting Time)**:**
Changing the following parameters does not change the steady state waiting periods N:

i Inelastic labor supply unit L

ii Capital share α in the production function

iii z^h and z^l as long as the ratio of $\frac{z^h}{z^l}$ stays the same.

Proof. (i) Suppose we have the solution for a given L. Consider changing L to $(1 + \Delta)L$. The steady state equations are still satisfied by varying only K^i and B^i to be $(1+\Delta)K^i$ and $(1+\Delta)B^i$, while keeping other variables the same. Similar results hold for (ii) and (iii). □

From now on, I will turn to quantitative exercises of the model. The above proposition shows that we should give more consideration to parameters other than labor supply, capital share, and the level of z^h or z^l (but z^h/z^l is important).

5 Numerical Examples

5.1 Calibration and Estimation

While the model is stylized, I bring it to data as close as possible. I match the steady state result to several U.S. long-run economy characteristics. Further, I estimate the shocks to financing constraints and aggregate productivity for short-run analysis. Each period represents a quarter and the full calibrated parameters are in Table 1.

Following Veracierto (2002), the capital abstracts from components such as land, residential structure, and consumer durables. Thus, the capital corresponds to non-residential structures, plant, and equipment while the investment corresponds to the non-residential investment in the National Income and Product Accounts (NIPA). Meanwhile, the empirical counterpart for consumption should be non-durable goods and services consumption. Output is then defined as the sum of the consumption and the investment. The investment-to-output ratio is found to be 0.165, which translates into the capital share in the production function as $\alpha = 0.258$. The capital to annual output ratio is 1.5 which translates into a depreciation rate $\delta = 2.58\%$. $\beta = 0.9847$ targets at the risk-free interest rate. The interest rate is low in equity premium puzzle literature (e.g. Mehra and Prescott (1985)) and is commonly chosen to be 3% to 4% annually. Here, I chose 3.5%.

The employment (labor measure) is set to be $L = 4$, so that roughly 80% of the working age population is employed. I normalize the low productivity $\tilde{z}^l = 1$. As shown by Proposition 5, L and the level z^l do not affect the waiting periods in the steady state.

For the productivity transition matrix, one only needs p^{hl} and p^{lh}. The primary targets are (1) the fraction of firms that are constrained, and (2) the turn-over of capital reallocation over empirical relevant capital stock. The target of (1) is from the studies of Almeida, Campello, and

Table 1: **Calibrated Parameters**

Parameters calibrated to the long-run U.S. economy. α is the capital share, β is the discount factor, δ is the capital stock depreciation rate, $\tilde{z}^h$ and $\tilde{z}^l$ are the high and low idiosyncratic productivities, p^{hl} and p^{lh} are the transition probabilities in the transition matrix, θ measures the tightness of financing constraints, d is the proportional resale costs, and finally η is the leisure utility that captures the fixed costs in running firms.

	Value	Target	Source/Note
α	0.2580	Investment/Output Ratio: 0.165	NIPA data
β	0.9847	Quarterly Discount Rate	Common discount rate
δ	0.0258	Capital to Output Ratio:1.5	NIPA data
$\tilde{z}^h$	1.1307	Standard Deviation of TFP: 5.7%	Basu, Fernald, and Kimball (2006)
$\tilde{z}^l$	1.0000	Normalization	Does not change N
L	4.0000	80% of the working-age population is employed	Does not change N
p^{hl}	0.0665	Constrained firms: 64%	Almeida, Campello, and Weisbach (2004)
p^{lh}	0.0400	Reallocation/capital stock: 1.44%	COMPUSTAT and SDC data
$\bar{\theta}$	0.4000	Average debt/asset ratio: 0.325	Flow of funds data
d	0.1000	Reallocation/capital expenditure ratio: 0.40	COMPUSTAT data
η	0.3000	Annual real interest rate 3.5%	Mehra and Prescott (1985)

Weisbach (2004), who identify the number of constrained firms to be 64% from COMPUSTAT data (which I average across from all alternative ways of measurement in their studies). Also, the turn-over of capital reallocation over total property, plant, and equipment is 5.7% annually and 1.4% quarterly in the COMPUSTAT data.

Once we have the transition matrix, we can determine $\tilde{z}^h$. I follow the cross-sectional standard deviation of productivity (5.7%) in Basu, Fernald, and Kimball (2006). Note that the measure is the standard deviation of TFP of existing firms in the model, excluding the TFP of entrepreneurs who exited before. Then, $\tilde{z}^h$ turns out to be 1.1307.

The parameters left are θ, d, and η. These three affect decisions on leverage, investment, and liquidation. The haircut θ targets at leverage where empirically, the debt-to-asset ratio is averaged to be 0.325 from flow of funds data. The degree of asset irreversibility d targets at reallocation. The fraction of capital reallocation over total capital purchase (roughly 35% quarterly) is stable. However, COMPUSTAT data only include publicly traded firms that are relatively large. Smaller firms, according to Eisfeldt and Rampini (2007), use more used capital. Therefore, 35% is naturally the lower bound and I chose 40% for benchmark calibration. Finally, the leisure utility η measures "fixed costs" and controls how long a persistently unproductive firm will hold the assets and deleveraging. I chose $\eta = 0.30$ such that there will be 12 quarters of waiting periods, roughly the same as the number of quarters of deleveraging before selling found in the introduction.

The calibration is to capture the long-run steady state. For estimating the shocks and their persistence, I use output and capital reallocation (both after HP-filtered) as the observations. The unobservable shocks are financial shocks and aggregate productivity shocks. Therefore, the model linked the observations to the shocks. I use Bayesian methods to back out the information of the shocks, conditional on observations. Specifically, I assume $\theta_t = \theta e^{\hat{\theta}_t}$ and $A_t = e^{\hat{A}_t}$, where θ_t and

Table 2: **Steady State: Calibrated Benchmark**

Each row represents a particular vintage. q: shadow prices. $\frac{k}{k+b}$: leverage. K: total capital stock. B: total bond assets. f: the number of entrepreneurs who go to vintage i over the total number of entrepreneurs who draw z^i (vintage i productivity). "Binding" indicates whether the borrowing constraint is binding for entrepreneurs who are going to a specific vintage.

Vintage	q	$\frac{k}{k+b}$	Binding?	K	B	f
0	1.0000	1.5330	Yes	33.419	-11.620	100%
1	0.9228	1.4680	No	2.165	-0.690	100%
2	0.9211	1.4454	No	2.025	-0.624	100%
3	0.9194	1.4230	No	1.894	-0.563	100%
4	0.9175	1.4009	No	1.771	-0.507	100%
5	0.9155	1.3789	No	1.656	-0.455	100%
6	0.9133	1.3572	No	1.549	-0.408	100%
7	0.9110	1.3357	No	1.449	-0.364	100%
8	0.9085	1.3143	No	1.355	-0.324	100%
9	0.9059	1.2932	No	1.267	-0.287	100%
10	0.9030	1.2723	No	1.185	-0.254	100%
11	0.9000	1.2516	No	0.607	-0.122	54.79%
12	0.9000	0.0000	No	0.000	16.218	0%

A_t follow AR(1) processes:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_t^\theta,$$

$$\hat{A}_t = \rho_A \hat{A}_{t-1} + \epsilon_t^A.$$

Innovation process $\epsilon_t = [\epsilon_t^\theta, \epsilon_t^A]^T$ is Gaussian with $E[\epsilon_t] = 0$, $E[\epsilon_t \epsilon_s'] = 0$, $E[\epsilon_t \epsilon_t'] = \Sigma_\epsilon$ and

$$\Sigma_\epsilon = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_A^2 \end{bmatrix}.$$

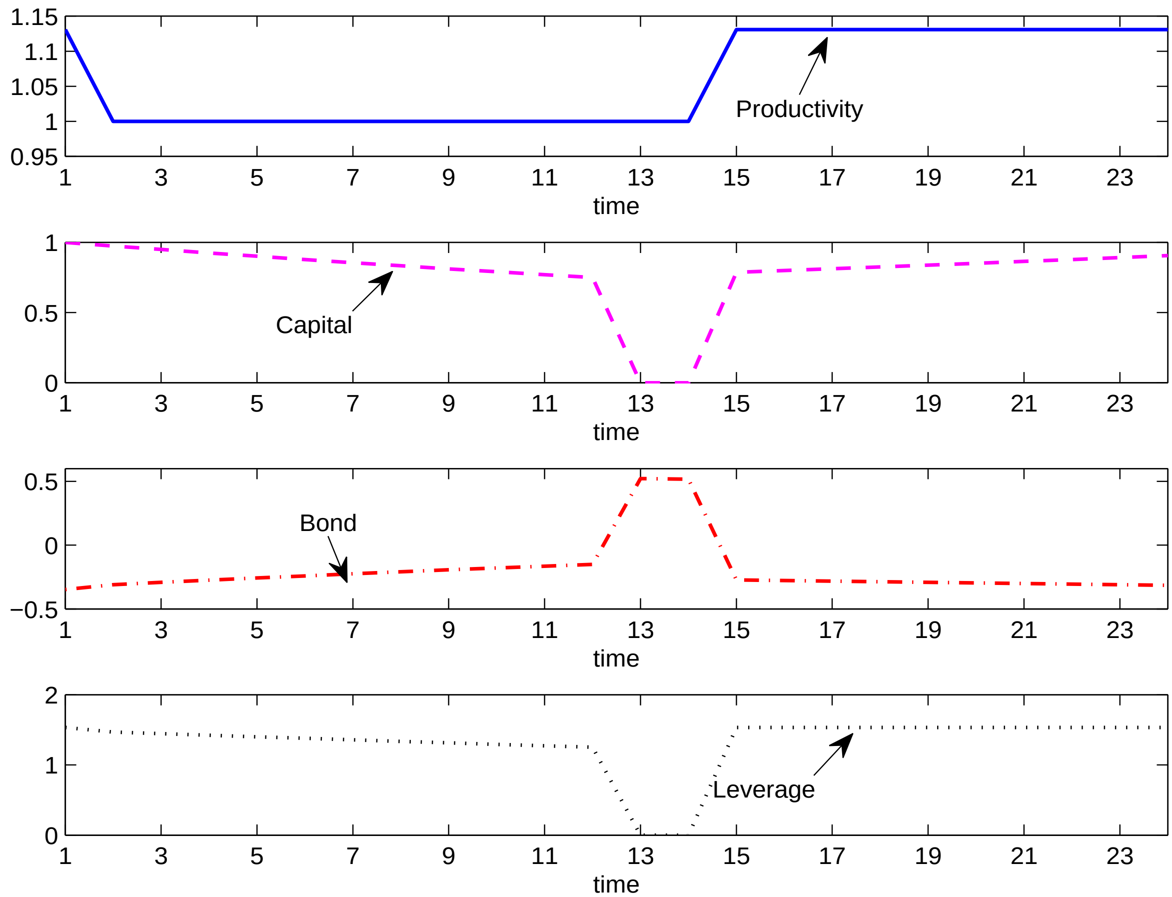
The estimation exercise is to back out σ_A, σ_θ, ρ_A and ρ_θ, using Bayesian methods. The detail on estimation will become clear in the business cycle analysis.

5.2 Interactions in the Steady State

5.2.1 The Calibrated Steady State

Under the calibrated parameters, there are 10 to 11 inactive quarters in the steady state. That is, entrepreneurs who turn from z^h to z^l and draw z^l for 10 quarters in a row neither buy nor sell capital during those 10 quarters (Table 2). When they unfortunately draws the 11th z^l, one fraction of them sells the firm and saves in bonds while the other fraction decides to be inactive for another quarter. For those who still run firms but draw a 12th z^l, they liquidate the entire firm and save the revenue in bonds until they become productive again.

Figure 5: **Capital, bond and leverage dynamics of a firm**
The firm's physical capital is normalized to be 1. Solid line: productivity draws. Dash line: physical capital. Dash dotted line: bond. Dotted line: leverage ratio defined as $k/(k+b)$.



As predicted, the real option value of capital decreases as the vintage number increases, which shows directly the reduced incentives to maintain the capital as a firm keeps drawing z^l and waiting. Meanwhile, the borrowing constraint only binds when firms invest. Once a firm draws z^l, the financial constraint is slack since the firm pays down existing debt.

To illustrate, suppose entrepreneur j has one unit of capital and was investing and borrowing before. Then her bond position is $-\theta(1-\delta)(1-d)/R$. Unfortunately, j draws 11 quarters of z^l in a row from time $t=1$ on. In the 12th quarter ($t=12$), j draws z^l again and decides to liquidate the entire firm. After that, j keeps drawing two z^l for quarters 13 and 14 but draws z^h afterwards.

j lets the capital depreciate in the first 11 quarters and liquidates it in the 12th quarter (firm dynamics in Figure 5), i.e., capital at the beginning of the 13th quarter is 0. During the inactive investment process, debt is being paid and leverage decreases. After liquidation, j saves only in bonds and consume $(1-\beta)$ of the bond value. Importantly, the leverage evolution before selling is similar to Figure 2.

j continues to hold bonds until drawing z^h again in the 15th quarter. Then, she uses her net worth as a down payment to borrow and invest. Though she borrows to the limit, capital stock after investing is less than one, the amount j started with. The firm size is not as large as before because j does not have enough resources to expand. Her business was not profitable under z^l technology and capital was sold at a discount before. If j keeps drawing z^h, she can continue investing and capital stock can gradually go back to one.

5.2.2 Delayed Reallocation and Aggregate TFP

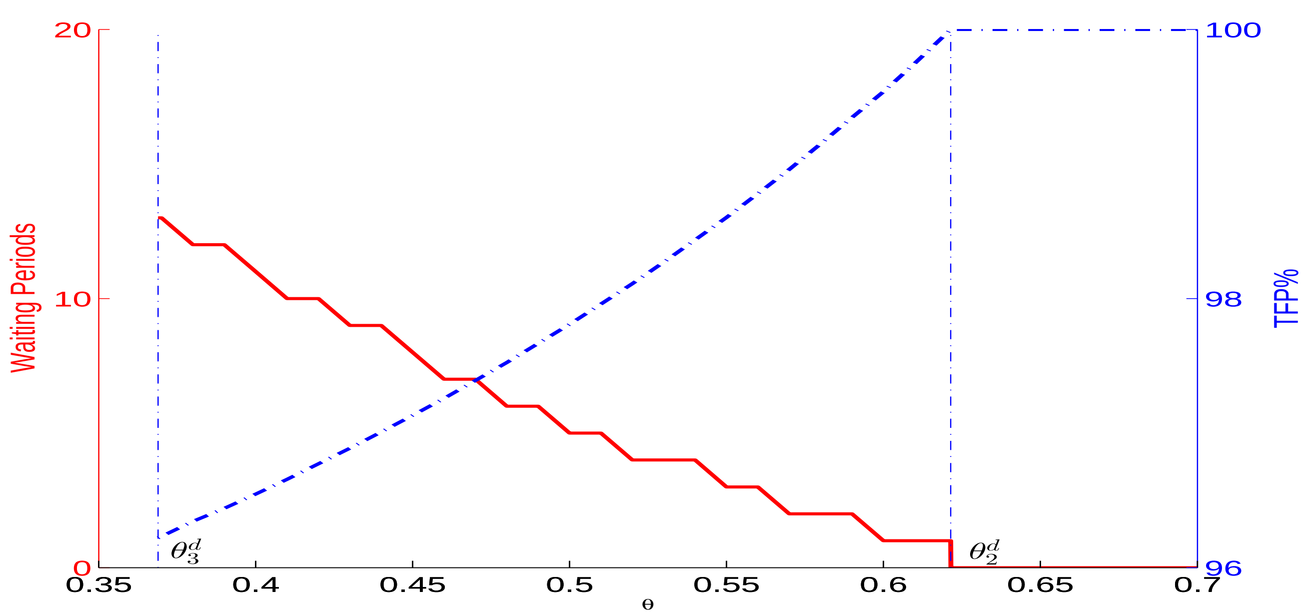
To examine the interactions of asset illiquidity and financing frictions, I vary θ in the $d > 0$ economy to see the changes of aggregate total factor productivity (TFP). The exercises can be thought of as comparing aggregate TFP across countries with different financing frictions but with the same degree of asset illiquidity. Then, I redo the exercise for the $d = 0$ economy. After that, I can compare how much the asset illiquidity can contribute to aggregate TFP losses. As θ becomes smaller, one can see how a tightening funding liquidity (a smaller θ) has different impacts on the two economies. Such a comparison reveals the interaction of the two frictions in the steady state.

The $d > 0$ Economy. Let θ decrease from $+\infty$ to 0. The d economy features no borrowing constraint when $\theta = \theta > \theta_1^d = 0.6540$. z^h firms have enough credit to reallocate all available capital from z^l firms. For the calibrated benchmark d, every z^l owners liquidate their firms when θ is above θ_1^d. Capital stock is fully under z^h technology and thus aggregate TFP equals $\tilde{z}^h$. Notice that if d is large enough, z^l owners may not sell their capital, even if θ is very large.

When θ reaches $\theta_2^d = 0.6214$, some previous z^h entrepreneurs who just drew z^l start to hold capital for one period (Figure 6). When $\theta_2^d = 0.6214$, the inaction region is the line that is the same as the borrowing constraint line in the z^l plain (recall Figure 3b). z^h owners invest and borrow to the limit; when turned into z^l owners, their leverage ratio is the one under the borrowing constraint. As θ becomes even smaller, the inaction region starts from a line to a fan as

Figure 6: **Aggregate TFP Losses and Waiting Periods**

Steady state TFP and waiting periods as only θ changes and when the steady state has capital reallocation. The red solid line denotes the waiting periods $N + 1$. The blue dashed line denotes aggregate TFP. Aggregate TFP is the percentage of $\tilde{z}^h$.



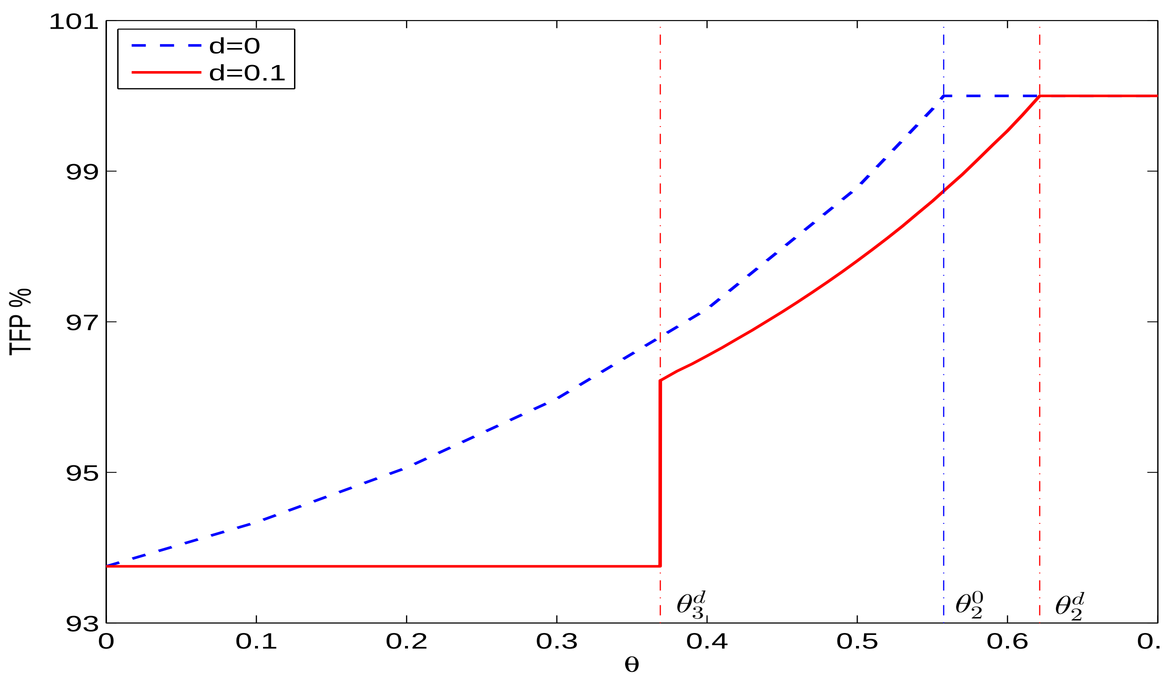
in Figure 3b. Persistently unlucky z^l owners wait longer and longer before selling capital. Capital reallocation is thus less and aggregate TFP is smaller (Figure 6).

When $\theta = \theta_3^d = 0.3689$, the secondary market shuts down so that no single z^l owner sells capital (Figure 7). In addition, all entrepreneurs save through running firms regardless of their productivity. The reason is that a larger degree of financial frictions further limit reallocation and borrowing. Therefore, both wage rate and risk-free interest rate will be low, i.e., π will tend to be large and R will tend to be small. When $\theta < \theta_3^d$, the condition $R' \geq z^l\pi' + (1-\delta)$ under which z^l owners do not invest is no longer satisfied. Therefore, z^l owners always find investing in capital stock better than saving in bonds. The economy thus is characterized by autarky allocation (Figure 7) and no productivity risk-sharing exists through the financial market.

$\theta \in [0, \theta_3^d]$ is an extreme interaction between asset illiquidity and financial constraints. Asset illiquidity delays liquidation. Tighter borrowing constraints prolong the delay. Once the profit rate is high enough and the interest rate is low enough due to large financial frictions, no liquidation takes place and the credit market effectively shuts down. Therefore, the important message is that both markets can shut down together if the two frictions interact. Then the economy is the same as the $d = 0$ economy with $\theta = 0$, even though $d > 0$ and $\theta > 0$ (θ_3^d is still far from zero, which exemplifies the interaction).

To summarize, when $\theta \in [\theta_2^d, \theta_1^d)$, the economy is inefficient only because investment from z^h firms is constrained by financial frictions. When $\theta \in [\theta_3^d, \theta_2^d)$, z^l owners delay selling and the economy is inefficient because of insufficient investment from z^h firms and insufficient reallocation from z^l firms. Finally, when $\theta \in [0, \theta_1^d)$, both secondary market and credit market shut down.

Figure 7: **Aggregate TFP Losses in $d > 0$ and $d = 0$ economy**
Aggregate TFP as percentage of $\tilde{z}^h$ in the steady state, when only θ changes. The red solid line: $d > 0$ economy. The blue dashed line: $d = 0$ economy.



The $d = 0$ Economy. As a comparison, there is no inactive investment decisions in the $d = 0$ economy. The stationary economy can be characterized by two cut-offs, $0.5575 = \theta_2^0 < \theta_1^0 = 0.7121$. The financial constraint is slack when $\theta \geq \theta_1^0$. Because of constant return to scale technology, only a zero measure of firms operate. The rest of entrepreneurs enjoy returns on bonds and leisure utility. When θ decreases in the region $[\theta_2^0, \theta_1^0]$, more and more z^h firms produce.

When $\theta \in [0, \theta_2^0]$, a fraction of z^l owners produces and TFP is less than $\tilde{z}^h$. TFP is lower when θ decreases in this region because more and more z^l firms produce. Notice that there is no asset illiquidity ($d = 0$) so that the return on capital stock is risk-free. The return must be higher than interest rate, otherwise z^l owners will not operate and enjoy extra leisure. Therefore, these existing z^l firms will borrow to the credit limit.

To summarize, there is no delay of selling in $d = 0$ economy. When $\theta \in [\theta_2^0, \theta_1^0)$, the economy is inefficient only because investment from z^h firms is constrained by financial frictions. When $\theta \in [0, \theta_2^0)$, the economy is inefficient in two ways: not enough investment from z^h firms and not enough reallocation from z^l firms.

How much are the TFP losses from steady state when $d = 0$ changes to $d = 0.10$? The answer obviously depends on what θ the economy has (Figure 7). In the calibrated $d = 0.1$ economy, the TFP losses increase by almost 25%. In percentage terms of $\tilde{z}^h$, the largest TFP losses are the following two cases. First, about 1.5% of $\tilde{z}^h$ more losses when $\theta = \theta_2^0$. z^l owners produce in the $d > 0$ economy, but not in the $d = 0$ economy. Second, about 2.5% of $\tilde{z}^h$ more losses when $\theta = \theta_3^d$. The secondary market shuts down in the $d > 0$ economy but not in the $d = 0$ economy. TFP losses in other regions are typically from 0.5% to 1.5% of $\tilde{z}^h$.

Such TFP losses are large and significant compared to the literature on financial frictions' impact on capital misallocation.[29] Given a degree of financial frictions, asset illiquidity can add losses of 0.5% to 1.5% of the efficient economy aggregate TFP ($\tilde{z}^h$). In the extreme case, there is about 2.5% more losses when borrowing is allowed but no lending is available (when both credit market and secondary market are effectively shut down). The studies in the literature are thus sensitive to the introduction of asset illiquidity, a common phenomenon in the secondary market.

5.3 Interactions During Business Cycles

Returning to the cycle properties of capital reallocation, I experiment with standard aggregate TFP shocks and credit crunch shocks. With large aggregate shocks, the model becomes intractable because the number of vintages changes after large shocks, leaving complex dynamics to solve. Instead, I focus on small aggregate shocks such that the equilibrium vintages do not change. I solve the dynamics around the steady state using first-order perturbation methods. Then I verify that the shocks are small enough through the response of the fraction (f_t^{N+1}) of entrepreneurs

[29] For example, Midrigan and Xu (2012) found that misallocation results in TFP losses of only about 0.3% in the benchmark calibrated economy and at most 5% when the credit market completely shuts down. Similarly, in Moll (2010) the magnitude of TFP losses depends on the persistence of idiosyncratic productivity shocks.

Table 3: **Priors and Posteriors**
"Prior s.d." denotes the standard deviation of the prior. "Post mean" denotes the posterior mean. "5%" and "95%" denote the 5 and 95 percentile. Posteriors are drawn using Markov Chain Monte Carlo (MCMC) methods such as in An and Schorfheide (2007).

	Prior Distribution	Prior mean	Prior s.d	Post mean	5 %	95 %
σ_A	Inverse Gamma	0.01	1	0.0045	0.0041	0.0049
σ_θ	Inverse Gamma	0.01	1	0.0115	0.0103	0.0129
ρ_A	Beta	0.9	0.05	0.8721	0.8297	0.9227
ρ_θ	Beta	0.9	0.05	0.9701	0.9472	0.9873

that stay in vintage $N = 10$. If f_t^{N+1} is still less than 1, the vintages do not change.

5.3.1 Estimation Results

I use the HP-filtered cyclical components of real reallocation and real GDP data from 1984Q1 to 2011Q4 to estimate the standard deviation and the persistence parameters ρ_θ and ρ_A. I apply Bayesian methods to estimate the standard deviation and the persistence of the shocks, as standard in the DSGE model estimation.[30] Prior and posterior information is in Table 3 and Figure 13.

I use the mean estimator for cycle analysis. Using the mode estimator will not change the result much since the mean and the mode are close to each other (Figure 13). There are several features of the mean estimators. The standard deviation of aggregate TFP shocks (shocks to A) is 0.45%, which is close to the estimation results found in the literature such as in Thomas (2002) (with 0.53%). Second, the size of the credit shocks (about 1.15%) is even larger than aggregate TFP shocks (0.45%). Finally, credit shocks ($\rho_\theta = 0.9701$) are more persistent than TFP shocks ($\rho_A = 0.8721$).

Even though I only use the two observed series (output and reallocation) for estimation (to avoid stochastic singularity issues because I focus on two shocks), the estimated aggregate TFP shocks and financing constraints shocks generate key business cycle statistics that are close to the data (Table 9).

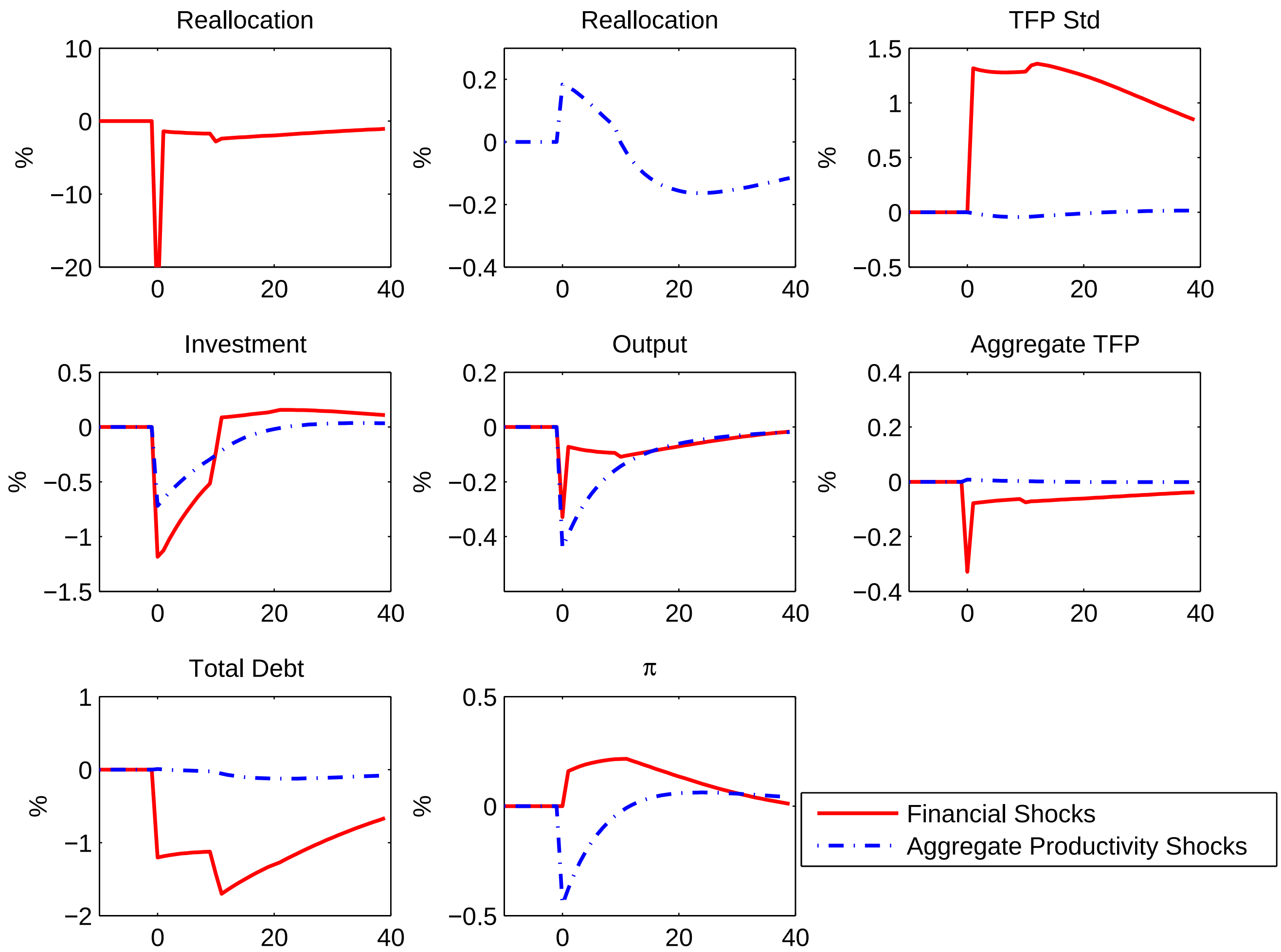
5.3.2 Financial Shocks and Aggregate Productivity Shocks

Figure 8 show the impulses to a one standard deviation (1.15%) credit shocks and a one standard deviation (0.45%) aggregate productivity shocks.

In response to credit shocks, tightened financing constraints largely reduce the investment from z^h firms. Demand for labor shrinks and real wage rate decreases in equilibrium. Running firms now has lower labor input costs (therefore π increases). In response to lower input costs, more z^l firms delay selling assets. More selling delays lead to less reallocation and thus a larger TFP dispersion across firms. The direct consequence is that aggregate TFP is smaller and total output

[30] See, for example Smets and Wouters (2003) and An and Schorfheide (2007).

Figure 8: **Experiment: Responses to two types of shocks**
Responses to one standard deviation of negative financial shocks (shocks to θ) and negative aggregate productivity shocks (shocks to A). Reallocation: capital reallocation. TFP Std: standard deviation of firm-level TFP employed. Aggregate TFP: the Solow residuals after adjusted by A changes. The solid line denotes the response to financial shocks while the dashed line denotes the response to aggregate productivity shocks.



drops. As for the debt, there is persistent and sizable deleveraging. Though z^h firms can no longer raise as much debt as before, inactive z^l owners pay back more debt by shrinking consumption. After financial shocks, the reduced reallocation and the increased dispersion of TFP across firms are in line with the data.

Output and TFP responses are sizable given the small credit crunch that does not change the number of equilibrium vintages. This result is under the assumption that vintage number N does not change. Since the correlation between reallocation and output is lower in the model than in the data (Table 9), if we can estimate under an endogenous N, the standard deviation of credit shocks should be larger than the mean estimator because it will increase N (also because adverse A shocks will reduce reallocation as will be clear soon). Therefore, the credit crunch in reality should be larger. In a larger credit crunch, the number of vintages can suddenly increase (capital reallocation suddenly disappears) and production efficiency is suddenly reduced.

In response to aggregate productivity shocks, debt level changes little (i.e., a magnitude of

0.1%) compared to credit shocks. More importantly, responses have at least two aspects that are not observed. First, capital reallocation is more initially. Since aggregate productivity drops, the profit rate of investing in capital is down (π_t responses). The z^l owners thus have less incentive to hold capital, and more capital is liquidated. Second, compared to the economy before the shocks, fewer z^l owners stay to operate firms such that the measured TFP dispersion is slightly smaller in recessions.

I have shown responses to a one-time financial shock and aggregate TFP shock. Financial shocks increases the return from running firms, which induces less capital reallocation from inefficient firms. However, aggregate TFP shocks generate the opposite dynamics. Though these exercises are impulses, they shed light on why aggregate TFP shocks might not be able to capture capital reallocation dynamics. In what follows, I confirm the intuition learned from impulse responses.

5.3.3 Simulations

The key for less reallocation in recessions is whether shocks can delay capital selling from z^l firms. To examine more thoroughly the reallocation-output co-movement and TFP dispersion-output co-movement, I simulate the model (i.e., financial shocks or aggregate productivity shocks repeatedly hit the economy), using parameters from the estimation. Table 4 shows the correlation of the key variables and output, using one type of shocks each time.

First, reallocation is more volatile in the economy with only financial shocks. From the impulse responses, aggregate TFP shocks have the opposite effects on reallocation. That is why we should observe a more volatile reallocation in responses to only financial shocks.

Second, aggregate TFP shocks generate a positive correlation between reallocation and output. After one-time aggregate TFP shock, eventually capital available for reallocation will be less, as in the impulse responses in Figure 8. Nevertheless, TFP dispersion shrinks in recessions from aggregate TFP shocks since more firms are liquidating, as in Figure 8.

Table 4: **Only One Type of Shocks**

	Volatility		Co-movement	
	Standard deviation	Standard deviation to that of output	Correlation with output	
	Output	Reallocation	Reallocation	TFP dispersion
Data:	1.42%	10.91	0.85	-0.44
Model:				
Only financial shocks	1.38%	11.03	0.83	-0.67
Only aggregate TFP shocks	1.31%	9.11	0.18	0.53

Table 5: **Variance Decomposition**

	Output	Reallocation	Investment	TFP dispersion
Financial shocks	23.88%	99.29%	74.38%	99.23%
Aggregate TFP shocks	76.12%	0.71%	25.62%	0.77%

To further decompose the effects from financial shocks and aggregate productivity shocks, I decompose the variance of reallocation and output explained by each type of shocks, using the mean estimators from the Bayesian exercise. As in Table 5, almost all the reallocation and TFP dispersions fluctuations are caused by financial shocks. In addition, financial shocks can also explain a large portion of the variation in investment and output. This result is because: (1) financial shocks lead to changes of "measured" aggregate TFP; (2) aggregate TFP shocks, similar as a neoclassical growth model, explain most of the output and a large portion of the investment. Therefore, it is not surprising that financial shocks can also explain a large portion of the variation in investment and output.

Finally, I apply Kalman smoother to reconstruct the implied financial shocks and aggregate productivity shocks conditional on the whole sample (Figure 9). The adverse financial shocks are particularly important during the 2008 recession. In addition, the shocks are relatively large during the 1990 recessions but quite mild during the 2000 recessions. If we relate the financial shocks to the capital reallocation time series in Figure 1, the drops of reallocation are large during 1990 and 2008 recessions but small during the 2000 one.

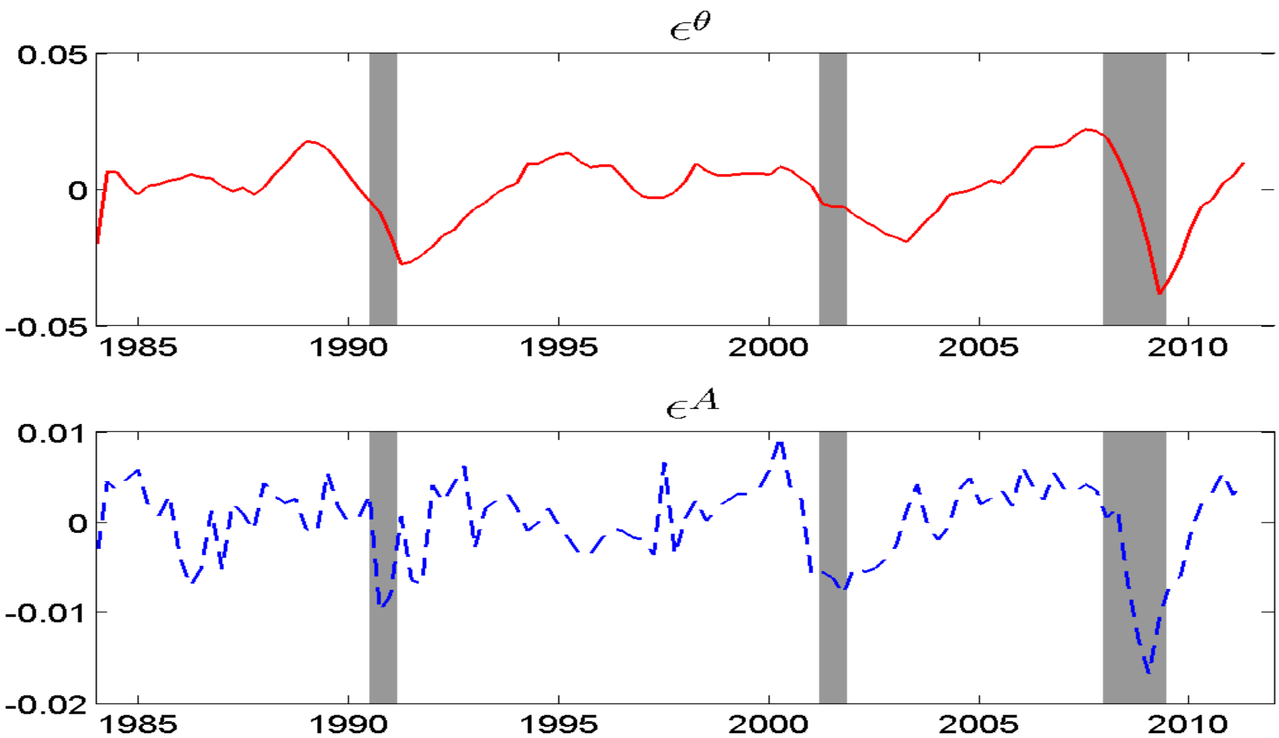
In summary, one needs both aggregate TFP shocks and credit crunch shocks to generate consumption, investment, and output dynamics as in Table 9; however, to capture both procyclical reallocation and countercyclical TFP dispersion, financial shocks are necessary. Therefore, dynamics of capital reallocation and the TFP dispersion in the data provide us some useful identification of the source(s) of business cycles.

6 Discussion

The interactions between asset illiquidity and financial frictions can be directly seen from the waiting periods in the steady state. Without asset illiquidity, there is no inactive investment decisions so that there is no waiting periods. Without borrowing constraints, z^h firms can borrow as much as possible to reallocate assets. The number of waiting periods is small and equal zero in our calibrated d economy. Thus, in order to generate prolonged capital reallocation delay during recessions, the interactions between the two frictions are the key ingredients.

In reality, recessions might originate from both aggregate TFP shocks and financial shocks. The relative importance, however, changes from recession to recession. The recent credit crunch

Figure 9: **Shocks back out from data**
Unobserved shocks computed from Kalman Smoother over the entire 1984Q1-2011Q4 sample using all the information in the sample.



since 2008 exemplifies a huge drop in θ. Less capital reallocation and slow deleveraging[31] are more significant than in past recessions. It is therefore reasonable to believe that financial shocks are essential in 2008 recessions and also important in previous recessions. Policy targeted at secondary market illiquidity should be able to help reverse the adverse shocks.

Importantly, the exercise does *not* imply that financial shocks are the only primitive shocks for recessions. Both aggregate TFP shocks and financial shocks are needed to generate business cycle statistics as in Table 9. Instead, this paper shows that if the economy features asset illiquidity, financial shocks are necessary to generate less capital reallocation and larger TFP dispersion during recessions.

Finally, this paper does not model changes of illiquidity. The first reason is that if illiquidity comes from asymmetric information, some good quality assets might be forced to be liquidated in recessions and mitigate the information problem as in Eisfeldt (2004). The second reason is that, if the increases of illiquidity are all because of fire-sale of real assets as in Shleifer and Vishny (1992), the larger TFP dispersion during recessions is hard to be justified. Fire-sale theories suggest the most efficient firms of using the assets are also in financial troubles, which should lead to a smaller TFP dispersion. The last and probably the most important one is that if the illiquidity can be amplified, then this paper proposes one cause for the *initial* drop of asset liquidity: a credit crunch can reduce the number of buyers and sellers simultaneously.

[31]See Shirakawa (2012) and Koo (2011) for the evidence.

7 Final Remark

This paper begins with two empirical facts: (1) capital reallocation is procyclical, but the benefits to reallocate are countercylical; (2) firms without surviving problems shrink liabilities relative to assets before selling assets. These two observations can be generated in the model with asset illiquidity and financing constraints in response to shocks to financing constraints, instead of aggregate TFP shocks. When negative financial shocks hit, inefficient firms are more willing to hold assets because of a lower input costs (a lower wage rate) and because of a lower interest rate. Therefore, financial shocks are important not only during the 2008 recession but also during previous ones.

The challenge to link individual firm's asset liquidation and aggregate capital reallocation is the complex distribution of firms. I model the selling decision as a stopping-time problem that turns out to simplify the aggregate distribution dramatically. Meanwhile, the real option value of capital stock before liquidation shed some light on how firms price their assets internally.

One future prospect is how the resale costs endogenously interact with the depth of asset markets. The asset specificity costs, in that case, come from matching between buyers and sellers. Sellers may find it costly to search potential buyers, especially during downturns. In contrast, asset markets are generally deeper in economic booms. The resale discounts are smaller in boom times and delayed selling by inefficient firms is reduced. A better allocation of assets will deepen asset markets further, and labor market conditions will improve too. Therefore, policy targeted at the resale market depth may have a large effect by improving the efficiency of asset allocation and labor market. This channel may also shed light on unemployment issues and labor input costs for firms.

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Appendices

A Data Description

For capital reallocation, the quarterly COMPUSTAT contains useful information for ownership changes of productive assets from 1984Q1. Following Eisfeldt and Rampini (2006), who use annual COMPUSTAT data from 1971, I measure capital reallocation by sales of property, plant and equipment (SPPE, data item 107 with combined data code entries excluded), plus acquisitions (AQC, data item 129 with combined data code entries excluded). The measure captures transactions after which the capital is used by a new firm and new productivity thus applied. The advantage of using quarterly data compared to annual data is more observations. However, quarterly data is shown in the "cash flow statement" and there is a substantial seasonal pattern. Therefore, I apply seasonal adjustment to the data.

For debt-to-asset ratio of companies before selling assets, I merge quarterly COMPUSTAT and SDC file. SDC file contains merger and acquisition of all U.S. firms. I get information of all the companies who sold at least 50 % of their assets in the SDC file after 2000 until the most recent available date (currently April 2012), then keep firms with information in COMPUSTAT and delete those who sell multiple times in the sample periods. The merging of COMPUSTAT and SDC allows me to trace back the leverage of companies before they sell assets.

For aggregate consumption, investment, and GDP, I obtain the data from FRED, a macroeconomic dataset managed by Federal Reserve Bank at St. Louis. Note that I exclude residential investment, consumer durables, government expenditure, and net export because the model abstract from these components.

B Proofs

B.1 Lemma 1

I prove a general result under general CRRA utility. First, define the Bellman operator $\mathcal{T}$:

$$\mathcal{T}V(k,b,a;X) = \max\left\{W^1(k,b,a;X), W^0(k,b,a;X)\right\}$$

$$W^1(k,b,a;X) = \max_{k'>0, R'b'\geq -\theta(1-d)(1-\delta)k'} u(z\pi k + Rb - \psi(k',k) - b') + \beta E[V(k',b',a';X')|a,X]$$

$$W^0(k,b,a;X) = \max_{b'}\left\{u\left(z\pi k + Rb + (1-\delta)(1-d)k - b'\right) + \eta + \beta E\left[V(0,b',a';X'|a,X\right]\right\}$$

The value function is the fixed point of the contraction mapping in the space $\mathcal{V}_1$ of well defined functions as in Stokey, Lucas, and Prescott (1989). Further, the Bellman operator $\mathcal{T}$ is closed on the class of functions $\mathcal{V}_1$ satisfying the properties in the Lemma. I simplify notation by

$$w^1(k,b,k',b',a;X) = u(k,b,k',b',a;X) + \beta E[V(k',b',a';X')|a,X]$$

$$w^0(k,b,k',b',a;X) = u(k,b,0,b',a;X) + \eta + \beta E[V(0,b',a';X')|a,X]$$

with slight abuse of notation of utility function $u(.)$.

(i) *Increasing in a, k and b* and concavity

Standard as in Stokey, Lucas, and Prescott (1989).

(ii) $V(\gamma k,\gamma b,a;X) = \gamma^{1-\sigma}V(k,b,a;X) + \frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta} + (1-\gamma^{1-\sigma})\eta(1-h)$

I will prove $\mathcal{TV}$ has the same property. Consider an agent with state (k,b,a) and (k',b') is the optimal policy. For any $\gamma>0$, when the state is $(\gamma k,\gamma b,a)$, the policy $(\gamma k',\gamma b')$ are feasible, i.e., it satisfies budget and borrowing constraints. Therefore, given an consistent choice $h \in \{0,1\}$,

$$\begin{aligned}
\mathcal{T}V(\gamma k, \gamma b, a; X) &\geq w^h(\gamma k, \gamma b, \gamma k', \gamma b', a; X) \\
&= \frac{(z\pi k + Rb - \psi(k', k) - b')^{1-\sigma}\gamma^{1-\sigma} - 1}{1-\sigma} + \eta(1-h) \\
&+ \beta\gamma^{1-\sigma}E[V(k', b', a'; X')|a, X] + \beta\frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta} \\
&= \gamma^{1-\sigma}\left[u(k, b, k', b'; X) + \eta(1-h)\right] + \frac{\gamma^{1-\sigma}-1}{1-\sigma} + \left(1-\gamma^{1-\sigma}\right)\eta(1-h) \\
&+ \beta\gamma^{1-\sigma}E[V(k', b', z'; X')|a, X] + \beta\frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta},
\end{aligned}$$

and thus

$$\mathcal{T}V(\gamma k, \gamma b, z; X) \geq \gamma^{1-\sigma}\mathcal{T}V(k, b, z; X) + \frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta} + \left(1-\gamma^{1-\sigma}\right)\eta\left(1-h\right).$$

Conversely, starting at $(\gamma k, \gamma b, a)$, scaling by $1/\gamma$, and following similar procedure above, one has

$$\mathcal{T}V(k, b, a; X) \geq (1/\gamma)^{1-\sigma}\mathcal{T}V(\gamma k, \gamma b, a; X) + \frac{\frac{(1/\gamma)^{1-\sigma}-1}{1-\sigma}}{1-\beta} + \left(1-(1/\gamma)^{1-\sigma}\right)\eta\left(1-h\right).$$

Combining the two gives

$$\mathcal{T}V(\gamma k, \gamma b, a; X) = \gamma^{1-\sigma}\mathcal{T}V(k, b, a; X) + \frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta} + \left(1-\gamma^{1-\sigma}\right)\eta\left(1-h\right).$$

Note that when $\sigma = 1$, $\mathcal{T}V(\gamma k, \gamma b, a; X) = \mathcal{T}V(k, b, a; X) + \frac{log\gamma}{1-\beta}$. Finally, the difference between $V(\gamma k, \gamma b, a; X)$ and $V(k, b, a; X)$ does not depend on the fixed costs because

$$V(\gamma k, \gamma b, a; X) - V(k, b, a; X) = (\gamma^{1-\sigma}-1)(V - \eta(1-h)) + \frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta}.$$

(Noticing that $V - \eta(1-h)$ does not depend on η.)

B.2 Lemma 2 and Proposition 1

The differentiability of $V(k, b, a; X)$ when $k' \gtrless (1-\delta)k$ is trivial, which relies on the differentiability of standard dynamic programming problem as proved by Benveniste and Scheinkman (1979) or Stokey, Lucas, and Prescott (1989). Next, I prove the differentiability of $V(k, b, a; X)$ when $k' = (1-\delta)k$.

I follow methods from Clausen and Strub (2012) in Banach space (the space of k and b) and adjust to the dynamic programming problem in this paper. The general idea is that the value function is the upper envelop of value function of buying, inactive and selling. It is therefore super-differentiable. At the same time, it has potential downward kink (sub-differentiable) because of $\psi(k', k)$ function. Therefore, the value function will be both super-differentiable and sub-differentiable, and therefore differentiable. First, we need the definition of Fréchet sub-differential of a function (since we have both k and b as state variables, while the analogy for function of one variable is sub-derivative of a function).

Definition 3:

The Fréchet sub-differential of $f : X \to R$ is a set

$$\partial_F f(x) = \left\{ m^* \in X : \lim_{\Delta x \to 0} \inf \frac{f(x+\Delta x) - f(x) - m^* \Delta x}{\|\Delta x\|} \geq 0 \right\}$$

and the Fréchet super-differential of f is a set

$$\partial^F f(x) = \left\{ m^* \in X : \lim_{\Delta x \to 0} \inf \frac{f(x+\Delta x) - f(x) - m^* \Delta x}{\|\Delta x\|} \leq 0 \right\}$$

Definition 4:
f is Fréchet sub-differentiable (or super-differentiable) if $\partial_F f(x)$ (or $\partial^F f(x)$) is non-empty

Remark f is (Fréchet) differentiable at x, if and only if f is both sub-differntiable and super-differentiable at x.

- ***Sub-differentiability of utility function***

For notation convenience, I rewrite the utility function as

$$U\left(k, b, k', b', 1; a\right) = u(z(a)\pi k + (1-\delta)k + Rb - (1-\delta)k' - b')$$

for $k' > (1-\delta)k$,

$$U\left(k, b, k', b', 0; a\right) = u(z(a)\pi k + Rb - b')$$

for $k' = (1-\delta)k$,

$$U\left(k, b, k', b', -1; a\right) = u(z(a)\pi k + (1-\delta)(1-d)k + Rb - (1-d)k' - b')$$

for $k' < (1-\delta)k$. Suppose $x = (k, b)$ denotes the state of capital and bond, the choice is x' with the optimal choice as x'_*, let the consistent capital buying, inactive or selling decision as e' with the optimal choice as e'_*. Then the utility can be written as $U(x, x', e'(x, x'); a)$ and the value function can be written as $V(x, a; X)$.

$U(x, x', e'; a)$ is sub-differentiable at x. To see this, the sub-differentiability is trivial when $k' \neq (1-\delta)k$ because U is differentiable. When $k' = (1-\delta)k$, one wants to check if there exist a m^* such that

$$\lim_{\Delta x \to 0} \inf \frac{U(x+\Delta x, x'; a) - U(x, x'; a) - m^*[\Delta k, \Delta b]^T}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} \geq 0$$

Notice that when $\Delta k < 0$

$$\begin{aligned} \frac{U(x+\Delta x, x'; a) - U(x, x'; a)}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} &= \frac{U(x+\Delta x, x'; a) - U(x+[0, \Delta b]^T, x'; a)}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} \\ &+ \frac{U(x+[0, \Delta b]^T, x'; a) - U(x, x'; a)}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} \end{aligned}$$

As $\Delta x = [\Delta k, \Delta b]^T \to 0$, the second term goes to $u'(z(a)\pi k + Rb - b')R$ and the first term goes to $u'((z(a)\pi + (1-\delta)(1-d))k + Rb - b'))(z\pi + (1-\delta)(1-d)) = o^{1-d} > 0$. When $\Delta k > 0$

$$\begin{aligned} \frac{U(x+\Delta x, x'; a) - U(x, x'; a)}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} &= \frac{U(x+\Delta x, x'; a) - U(x+[0, \Delta b]^T, x'; a)}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} \\ &+ \frac{U(x+[0, \Delta b]^T, x'; a) - U(x, x'; a)}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} \end{aligned}$$

As $\Delta x = [\Delta k, \Delta b]^T \to 0$, the second term goes to $u'\left(z(a)\pi k + Rb - b'\right)R$ and the first term goes to $u'\left((z(a)\pi + (1-\delta))k + Rb - b'\right)\left(z\pi + (1-\delta)\right) = o^1 > 0$. Therefore, if $o^{1-d} < o^1$, one can let $m^* = [p, q]$ where

$$p \in [o^{1-d}, o^1], \quad q = u'\left(z(a)\pi k + Rb - b'\right)R$$

so that if $\Delta x \to 0^-$

$$\lim_{\Delta x \to 0} \inf \frac{U\left(x + \Delta x, x'; a\right) - U\left(x, x'; a\right) - m^*[\Delta k, \Delta b]^T}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} = o^{1-d} + q - (-p + q) > 0,$$

and if $\Delta x \to 0^+$

$$\lim_{\Delta x \to 0} \inf \frac{U\left(x + \Delta x, x'; a\right) - U\left(x, x'; a\right) - m^*[\Delta k, \Delta b]^T}{\sqrt{(\Delta k)^2 + (\Delta b)^2}} = o^1 + q - (p + q) \geq 0.$$

When $o^{1-d} \geq o^1$, one can let $m^* = \left[o^1, u'\left(z(a)\pi k + Rb - b'\right)R\right]$ and have the same result. Thus, $U\left(x, x'; a\right)$ is sub-differentiable at x.

- ***Sub-differentiability of*** $V(k, b, a; X)$

Now we are ready to prove the sub-differentiability of V because U is sub-differentiable. For a more detail chain rule of sub-derivative, see Mordukhovich, Nam, and Yen (2006).

Lemma B1:
If x' is in the interior of the feasible set, then the value function $V(x, a)$ is sub-differentiable at x with $u_x\left(x, x', e'; a\right) \in \partial_F V\left(x, a; X\right)$

Proof. Notice that in the interior of the feasible set

$$\begin{aligned} V(x + \Delta x, a; X) - V(x, a; X) &\geq \left[U(x + \Delta x, x'_*, e'_*; a) + \beta E\left[V(x'_*, a', X')|a, X\right]\right] \\ &\quad - \left[U(x, x'_*, e'_*; a) + \beta E\left[V(x'_*, a'; X')|a, X\right]\right] \\ &= U\left(x + \Delta x, x'_*, e'_*; a\right) - U(x, x'_*, e'_*; a) \end{aligned}$$

Notice that $U(., x'_*, e'_*; a)$ is sub-differentiable at x, one can subtract $m\Delta x$, divide by $\|\Delta x\|$, and take limits on both sides:

$$\lim_{\Delta s \to 0} \inf \frac{V(x + \Delta x, a; X) - V(x, a; X) - m\Delta x}{\|\Delta x\|} \geq 0$$

since the right hand side has Fréchet derivative as $m = u_x(x, x', e')$. Because this is true for any small open ball around x, $V(x, a; X)$ is thus sub-differentiable at x and $u_x\left(x, x'_*, e'_*\right) \in \partial_F V(x, a; X)$.

□

- ***Super-differentiability***

The bellman equation can be rewritten in two stages as

$$v(x, x'; a; X) = u(x, x', e'(x', x); a) + \beta E\left[V(x', a'; X')|a, X\right] \tag{19}$$

$$V(x, a, X) = \sup_{x'} v(x, x'; a, X).$$

I will prove that $v(x, x'; a)$ is super-differntiable at x'_*. At the same time, 0 is one super-derivative of $v(x, x'; a)$.

Lemma B2:
If x'_* is an interior point and maximizes $v(x, x'_*; a, X)$, then $v(x, x'; a, X)$ is super-differentiable at x'_* with $0 \in \partial^F v(x, x'_*; a, X)$.

Proof. Notice that $v(x, x'_* + \Delta x'; a, X) \leq v(x, x'_*; a, X)$, for small $\Delta x'$. Dividing by $\|\Delta x'\|$ and take the limits gives

$$\lim_{\Delta x' \to 0} \frac{v(x, x'_* + \Delta x'; a, X) - v(x, x'_*; a, X)}{\|\Delta x'\|} \leq 0$$

Therefore, $v(x, x'; a, X)$ is super-differentiable and 0 must be an element of $\partial^F v(x, x'_*; a, X)$.

□

- ***Lemma 2* and Proposition 1**

Lemma B2 shows $v(x, x'; a, X)$ is super-differentiable at x'_*. Notice that v can be expressed by instantaneous utility and expectation of discounted future value, both of which is sub-differentiable from Lemma B1. Therefore $v(x, x'; a, X)$ is both sub-differentiable and super-differentiable at interior point x'_*. One can thus differentiate $v(x, x'; a, X)$ at $x' = x'_*$. Lemma B2 shows that 0 is an element of the super-derivative and the derivative of v at x'_* is 0. Namely, each term on the right hand side of (19) is differentiable, and once we take differentiation:

$$u_{x'}(x, x'_*; e'_*) + \beta E\left[V_s\left(x'_*, a'; X'\right)|a, X\right] = 0$$

Therefore, value function has derivative on x'_*. This result implies that V has partial derivative on the k that is chosen as the optimal policy before and the envelop condition is satisfied. In equilibrium, V has derivative $w.r.t$ k when $k > 0$. I thus conclude that the (Fréchet) derivative of $V(k, b, a; X)$ $w.r.t$ $k > 0$ exists, and one can use V_k for $k > 0$.

B.3 Lemma 3

(i) To save notation, I abstract from aggregate state variable X. From the proof in Lemma 1,

$$V(\gamma(k+e), \gamma b, a) = \gamma^{1-\sigma} V(k+e, b, a) + \frac{\frac{\gamma^{1-\sigma}-1}{1-\sigma}}{1-\beta} + (1-\gamma^{1-\sigma})\eta(1-h)$$

Take a derivative with respect to e and evaluate it at $e = 0$; one has $\gamma V_k(\gamma k, \gamma b) = \gamma^{1-\sigma} V_k(k, b)$. Divide γ on both sides and one can prove that V_k is homogeneous with degree $-\sigma$. When $\sigma = 1$, V is homogenous with degree -1 as in the main text.

(ii) Consider two entrepreneurs with (k_0, b_0, a) and $(\gamma k_0, \gamma b_0, a)$. Using equation (5) of Lemma 1, the targeted capital stock and bonds are scaled up by γ and thus the optimal consumption choices are c_0 and γc_0 from the budget constraints. Therefore, using property (1) of this Lemma, $V_k/u'(c)$ is the same for the two entrepreneurs. More generally, $V_k/u'(c)$ depends only on $k/(k+b)$.

(iii) By definition, $q(k, b, a; X) = (V_k/u'(c) - z\pi)(1-\delta)^{-1}$. Using (2), we know that $q(k, b, a; X)$ can be written as $q(\frac{k}{k+b}, a; X)$.

B.4 Proposition 2

I prove the policy functions for general CRRA utility. Using the net worth definition, I propose the following solution:

$$V(k, b, a; X) = J(a; X) + \frac{\frac{(g(a;X)n(k,b,a;X))^{1-\sigma}-1}{1-\sigma}}{1-\beta}, \tag{20}$$

and the associated policy functions

$$
\begin{aligned}
c(k,b,a;X) &= (1-s(k,b,a;X))n(k,b,a;X) \\
qk'(k,b,a;X) &= \phi(k,b,a;X)s(k,b,a;X)n(k,b,a;X) \\
b'(k,b,a;X) &= (1-\phi(k,b,a;X))s(k,b,a;X)n(k,b,a;X)
\end{aligned} \tag{21}
$$

where J, g, ϕ and s are to be determined. Note that, s is the saving rate and ϕ is the portfolio weight on capital. Notice that

$$
\frac{V_k(k,b,a;X)}{u'(c)} = z\pi + q(1-\delta), \quad \frac{V_b(k,b,a;X)}{u'(c)} = R.
$$

The first-order conditions with respect to k' and b' give:

$$
qc^{-\sigma} = \beta E\left[\frac{(g')^{1-\sigma}(n')^{-\sigma}}{1-\beta}\left(z'\pi' + (1-\delta)q'\right)|a,X\right] + \mu\theta(1-\delta)(1-d) \tag{22}
$$

$$
c^{-\sigma} = \beta E\left[\frac{(g')^{1-\sigma}(n')^{-\sigma}}{1-\beta}R'|a,X\right] + \mu R' \tag{23}
$$

where μ is the Lagrangian multipliers attached to the borrowing constraint. When $\mu = 0$, multiply (22) by $\frac{\phi}{q}$ and (23) by $(1-\phi)$, and then sum them up, we have

$$
c^{-\sigma} = \beta E\left[\frac{(g')^{1-\sigma}(n')^{-\sigma}}{1-\beta}\left(\phi r' + (1-\phi)R'\right)|a,X\right], \tag{24}
$$

where $r' = \frac{z'\pi' + (1-\delta)q'}{q}$. When $\mu > 0$, we know that $\phi = \frac{1}{1-\theta(1-d)(1-\delta)/qR'}$ from the borrowing constraint $R'b' = -\theta(1-\delta)(1-d)k'$. Again multiply (22) by $\frac{\phi}{q}$ and (23) by $(1-\phi)$, and then sum them up, we still have equation (24) because the part that has μ is cancelled out.

Next, notice that the envelope condition under the proposed value function is

$$
V_k = \frac{g^{1-\sigma}n^{-\sigma}}{1-\beta}(z\pi + (1-\delta)q),
$$

from which one has $\frac{g^{1-\sigma}n^{-\sigma}}{1-\beta} = c^{-\sigma}$. Together with

$$
n' = z'\pi'k' + q'(1-\delta)k' + R'b' = \left[\phi r' + (1-\phi)R'\right]sn \equiv \rho' sn, \tag{25}
$$

equation (24) can be rewritten as

$$
(1-s)^{-\sigma} = \beta E[(1-s')^{-\sigma}s^{-\sigma}(\rho')^{1-\sigma}|a,X], \tag{26}
$$

- $\sigma = 1$, i.e., log utility

Equation (26) is simplified to be

$$
\frac{s}{1-s} = E\left[\frac{\beta}{1-s'}|a,X\right].
$$

For convenience, let me temporarily get the time subscript back. After recursive substitution:

$$
\frac{s_t}{1-s_t} = \beta + \beta^2 + ... + \beta^j E_t\left[\frac{s_{t+j}}{1-s_{t+j}}\right]
$$

Notice that $s \in (0,1)$, i.e., it is not optimal to save everything ($s = 1$) or consume everything ($s = 0$). Otherwise, the marginal utility of today or tomorrow will go to infinity because of CRRA utility assumptions. Then $E_t\left[\frac{s_{t+j}}{1-s_{t+j}}\right]$ is bounded by some positive numbers. Let $j \to \infty$ and the solution is $s_t = \beta$. Once the consumption choice is fixed, i.e., $s = \beta$, ϕ should be picked accordingly to solve equation (22) and (23), i.e.,

$$\begin{cases} E[\frac{r'-R'}{\phi r'+(1-\phi)R'}|a, X] = 0, & \text{if } E[\frac{r'}{\phi r'+(1-\phi)R'}|a, X] = 1 \\ \phi = \frac{1}{1-\theta(1-\delta)(1-d)/qR'}, & \text{if } E[\frac{r}{\phi r+(1-\phi)R}|a, X] < 1 \end{cases}$$

- $\sigma \neq 1$

From equation (26), and the difference of equation (22) and equation (23), ϕ and s jointly solve the recursive simultaneous equations:

$$E\left[\beta\left(\frac{1-s}{(1-s')s\rho'}\right)^{\sigma}\rho'\middle| a, X\right] = 1$$

$$\begin{cases} E\left[\beta\left(\frac{1-s}{(1-s')s\rho'}\right)^{\sigma}(r'-R')\middle| a, X\right] = 0, & \text{if } E\left[\beta\left(\frac{1-s}{(1-s')s\rho'}\right)^{\sigma} r|a, X\right] = 1 \\ \phi = \frac{1}{1-\theta(1-\delta)(1-d)/qR'}, & \text{if } E\left[\beta\left(\frac{1-s}{(1-s')s\rho'}\right)^{\sigma} r|a, X\right] < 1 \end{cases}$$

Notice that, $\beta\left(\frac{1-s}{(1-s')s\rho'}\right)^{\sigma}$ is the stochastic discount factor.

- $k' = 0$

When the optimal choice is $k' = 0$, there is only one first order condition for b',

$$c^{-\sigma} = \beta E\left[(c')^{-\sigma}R'\right] \tag{27}$$

where I use the fact that the leisure utility is a constant term and will not be shown in the first order condition for b', once (20) is plugged into the Bellman equation. Notice that,

$$n' = z'\pi'k' + q'(1-\delta)k' + R'b' = R'sn$$

and consumption choice in (21), one has

$$(1-s)^{-\sigma} = \beta E[(1-s')^{-\sigma}s^{-\sigma}(R')^{1-\sigma}],$$

which is the same as that in $k' = 0$. Everything else goes through the same way by replacing $\rho' = R'$.

- Verification

Finally, I verify the proposed value function (21) and policy functions (21) solve the Bellman equation. When $k' \neq 0$, substitute (20) back into the Bellman equation (4).

$$J_t + \frac{\frac{(g_t n_t)^{1-\sigma}-1}{1-\sigma}}{1-\beta} = \frac{((1-s_t)n_t)^{1-\sigma}-1}{1-\sigma} + \beta E_t[J_{t+1} + \frac{\frac{(g_{t+1}n_{t+1})^{1-\sigma}-1}{1-\sigma}}{1-\beta}]$$

Plug in the envelop conditions $\frac{(g')^{1-\sigma}}{1-\beta} = (1-s')^{-\sigma}$, one have

$$J_t + \frac{(1-s_t)^{-\sigma}n_t^{1-\sigma}}{1-\sigma} = \frac{((1-s_t)n_t)^{1-\sigma}}{1-\sigma} + \beta E_t[J_{t+1} + \frac{(1-s_{t+1})^{-\sigma}n_{t+1}^{1-\sigma}}{1-\sigma}]$$

Using (25),

$$J_t + \frac{(1-s_t)^{-\sigma}n_t^{1-\sigma}}{1-\sigma} = \frac{((1-s_t)n_t)^{1-\sigma}}{1-\sigma} + \beta E_t[J_{t+1} + \frac{(1-s_{t+1})^{-\sigma}\rho_{t+1}^{1-\sigma}s_t^{1-\sigma}n_t^{1-\sigma}}{1-\sigma}]$$

Then use (26), one can simplify the above equation to be

$$J_t = \beta E_t \left[J_{t+1} \right]$$

Therefore, J_t does not depend on the net-worth n_t. When $k' = 0$, substitute (20) back into the Bellman equation (4) by noticing that an extra leisure utility

$$J_t + \frac{\frac{(g_t n_t)^{1-\sigma}-1}{1-\sigma}}{1-\beta} = \frac{((1-s_t)n_t)^{1-\sigma}-1}{1-\sigma} + \eta + \beta E_t[J_{t+1} + \frac{\frac{(g_{t+1}n_{t+1})^{1-\sigma}-1}{1-\sigma}}{1-\beta}]$$

Then following the similar steps and one have

$$J_t = \beta E_t \left[J_{t+1} \right] + \eta$$

Again, J_t does not depend on the net-worth n_t. Then, I verify that the guessed value function is correct and the policy functions proposed solve the Bellman equation.

B.5 Proposition 3 : Leverage and Deleverage

(i) I prove that in equilibrium, entrepreneurs who draw $z_{t+1} = z^h$ at time t invest and borrow to a common target leverage $\bar{\lambda}_t$. First, because idiosyncratic productivity follows the two state Markov process, it is straightforwad to show that these entrepreneurs will invest. Second, I show that entrepreneurs will borrow to a common leverage if they invest. For notation simplicity, I use vintage specfic shadow prices and rate of return. Moreover, instead of using leverage, I use the vintage specific portfolio weight on capital ϕ_t^0. Suppose an entrepreneur have net-worth $n_t = (z_t\pi_t + (1-\delta)q_t^0)k_t + R_t b_t$, where $q_t^0 = 1$ denotes the buying price. When the entrepreneur decides to invest, the rate of return on capital is $\frac{z^h\pi_{t+1}+(1-\delta)q_{t+1}^0}{q_t^0}$ and $\frac{z^h\pi_{t+1}+(1-\delta)q_{t+1}^1}{q_t^0}$. The value from investing is

$$\begin{aligned} V^{buy} &= log\left((1-\beta)n_t\right) + \beta p^{hh} E_t \left[J_{t+1}^0 + \frac{log\left(\frac{z^h\pi_{t+1}+(1-\delta)q_{t+1}^0}{q_t^0}\phi_t^0\beta n_t + R_{t+1}(1-\phi_t^0)\beta n_t\right)}{1-\beta} \right] \\ &\quad + \beta p^{hl} E_t \left[J_{t+1}^1 + \frac{log\left(\frac{z^h\pi_{t+1}+(1-\delta)q_{t+1}^1}{q_t^0}\phi_t^0\beta n_t + R_{t+1}(1-\phi_t^0)\beta n_t\right)}{1-\beta} \right] \end{aligned}$$

Now consider one-shot deviation this period by taking a different portfolio weight on capital as $\phi_t^{0'}$. The value of such one-shot deviation is

$$\begin{aligned} V^{in} &= log\left((1-\beta)n_t\right) + \beta p^{hh} E_t \left[J_{t+1}^0 + \frac{log\left(\frac{z^h\pi_{t+1}+(1-\delta)q_{t+1}^0}{q_t^0}\phi_t^{0'}\beta n_t + R_{t+1}(1-\phi_t^{0'})\beta n_t\right)}{1-\beta} \middle| X_t \right] \\ &\quad + \beta p^{hl} E_t \left[J_{t+1}^m + \frac{log\left(\frac{z^h\pi_{t+1}+(1-\delta)q_{t+1}^m}{q_t^0}\phi_t^{0'}\beta n_t + R_{t+1}(1-\phi_t^{0'})\beta n_t\right)}{1-\beta} \middle| X_t \right] \end{aligned}$$

for some time-varying constant J^m_{t+1} and shadow value q^m_{t+1}. Therefore, the difference between these two values is

$$
\begin{aligned}
V^{buy} - V^{in} \quad = \quad & \frac{\beta}{1-\beta} p^{hh} E\left[log \frac{\phi^0_t (r^{0h}_{t+1} - R_{t+1}) + R_{t+1}}{\phi^{0\prime}_t (\frac{z^h \pi_{t+1} + (1-\delta) q^m_{t+1}}{q^0_t} - R_{t+1}) + R_{t+1}} \middle| X_t \right] \\
& \frac{\beta}{1-\beta} p^{hl} E\left[log \frac{\phi^0_t (r^{0l}_{t+1} - R_{t+1}) + R_{t+1}}{\phi^{0\prime}_t (\frac{z^h \pi_{t+1} + (1-\delta) q^m_{t+1}}{q^0_t} - R_{t+1}) + R_{t+1}} \middle| X_t \right] + \beta p^{hl} E\left[J^1_{t+1} - J^m_{t+1} | X_t \right]
\end{aligned}
$$

which does not depend on n_t. So if there exist one shot deviation for some entrepreneurs who draw z^h, then similar one-shot deviation always exist for any entrepreneurs who draw z^h so that no one will invest. Therefore, entrepreners who draw z^h will borrow to the same target leverage.

(ii) First, the option value decreases when drawing $z_{t+1} = z^l$. Suppose not, then the rate of return on capital from t to $t+1$ are

$$
\frac{z^l \pi_{t+1} + (1-\delta)}{q_t}, \quad \frac{z^l \pi_{t+1} + (1-\delta) q_{t+1}}{q_t}
$$

with $q_{t+1} > q_t$. Notice that, the rate of return for an investing entrepreneur is

$$
\frac{z^l \pi_{t+1} + (1-\delta)}{1}, \quad \frac{z^l \pi_{t+1} + (1-\delta) q^1_{t+1}}{1}
$$

where $q^1_{t+1} < 1$. Therefore, the rate of return of capital for entrepreneurs who draw z^l is higher than that of an investing entrepreneur, state by state because $q_t < q_{t+1} < 1$. This result suggest that z^l entrepreneurs should invest rather than holding capital stock, a contradction.

Second, I prove that entrepreneurs who draw z^l and who hold capital will deleverage. Without loss of generality, consider an entrepreneur with (k_t, b_t) and $k_t = 1$ who draws $z_{t+1} = z^l$ and lets the capital depreciate to $k_{t+1} = 1 - \delta$. It is straightforward to show that borrowing to the credit constraint limit is not optimal because productivity is low.

Suppose $b_{t+1} \leq (1-\delta) b_t$. Then, the Euler equation (or the asset pricing formula) can be written as

$$
\begin{aligned}
& \beta p^{lh} E\left[\frac{z^l \pi_t + (1-\delta) q_t + R_t b_t}{(z^l \pi_{t+1} + (1-\delta))(1-\delta) + R_{t+1} b_{t+1}} \middle| X_t \right] \frac{z^l \pi_{t+1} + (1-\delta)}{q_t} \\
+ & \beta p^{ll} E\left[\frac{z^l \pi_t + (1-\delta) q_t + R_t b_t}{(z^l \pi_{t+1} + (1-\delta) q_{t+1})(1-\delta) + R_{t+1} b_{t+1}} \middle| X_t \right] \frac{z^l \pi_{t+1} + (1-\delta) q_{t+1}}{q_t} \quad = \quad 1
\end{aligned}
$$

Notice that $b_{t+1} \leq (1-\delta) b < 0$, the left hand side

$$
\begin{aligned}
LHS \quad \leq \quad & \frac{\beta (z^l \pi_t + (1-\delta) q_t + R_t b_t)}{(1-\delta) q_t} \left[p^{lh} E\left[\frac{z^l \pi_{t+1} + (1-\delta)}{z^l \pi_{t+1} + (1-\delta) + R_{t+1} b_t} \middle| X_t \right] + p^{lh} E\left[\frac{z^l \pi_{t+1} + (1-\delta) q_{t+1}}{z^l \pi_{t+1} + (1-\delta) + R_{t+1} b_t} \middle| X_t \right] \right] \\
= \quad & \frac{\beta (z^l \pi_t + (1-\delta) q_t + R_t b_t)}{(1-\delta) q_t} \left[p^{lh} E\left[\frac{1}{1 + \frac{R_{t+1} b_t}{z^l \pi_{t+1} + (1-\delta)}} \middle| X_t \right] + p^{lh} E\left[\frac{1}{1 + \frac{R_{t+1} b_t}{z^l \pi_{t+1} + (1-\delta) q_{t+1}}} \middle| X_t \right] \right] \\
< \quad & \frac{\beta (z^l \pi_t + (1-\delta) q_t + R_t b_t)}{(1-\delta) q_t} \frac{1}{1 + b_t}
\end{aligned}
$$

where the last inequality uses the condition in equilibrium $z^l \pi_{t+1} + (1-\delta) \leq R_{t+1}$, $b_t < 0$, and $q_{t+1} < 1$.

Further,

$$\begin{aligned} q_t(1-\delta)(1+b_t) &= q_t(1-\delta)+q_t(1-\delta)b_t \\ &> q_t(1-\delta)+b_{t+1} \\ &= \beta(z^l\pi_t+(1-\delta)q_t+R_tb_t) \end{aligned}$$

so that $LHS < 1$, which contradict the Eular equation. Therefore, $b' > (1-\delta)b$ and because $k_{t+1} = (1-\delta)k_t$, we know that

$$\frac{k_{t+1}}{k_{t+1}+b_{t+1}} < \frac{k_t}{k_t+b_t}$$

B.6 Proposition 4: Existence of Stopping Time

Using Proposition 2, there is vintage specific (time varying) constant J in the value function of entrepreneurs in that specific vintage. Denote J^i $(i = 0, 1, ..., N+2)$ as the constant in the value function.

(1) I prove that it is never optimal to sell part of the capital stock using principle of unimprovability (to check one-shot deviation). Consider an agent with state $(1, \tilde{b}, a)$, where $z'(a) = z^l$, i.e., she draws z^l. The net worth is $n_t = z_t\pi_t + (1-\delta)(1-d) + R_t\tilde{b}$. Suppose the partial selling strategy is optimal, and $0 < \tilde{k} < 1-\delta$ is left and bonds are $\beta n_t - (1-d)\,\tilde{k}$. Such partial selling strategy gives value

$$\begin{aligned} V^{part}(\tilde{k}) &= log((1-\beta)n_t) + \beta p^{lh} E\left[J^0_{t+1} + \frac{log\left((z^l\pi_{t+1}+(1-\delta))\tilde{k} + R_{t+1}\left(\beta n_t - (1-d)\,\tilde{k}\right)\right)}{1-\beta}|X_t\right] \\ &\quad + \beta p^{ll} E\left[J^K_{t+1} + \frac{log\left((z^l\pi_{t+1}+(1-\delta)\,q_{t+1})\tilde{k} + R_{t+1}\left(\beta n_t - (1-d)\,\tilde{k}\right)\right)}{1-\beta}|X_t\right] \end{aligned}$$

where J^K_{t+1} is some time varying constant and q_{t+1} is some consistent shadow price for the action tomorrow. However, there always exists an one shot deviation by inaction today in which the shadow value of capital is $q_t \geq 1-d$. To see this, the one shot deviation gives value

$$\begin{aligned} V^{in}(1-\delta) &= log((1-\beta)\tilde{n}_t) + \beta p^{lh} E\left[J^0_{t+1} + \frac{log\left((z^l\pi_{t+1}+(1-\delta))(1-\delta) + R_{t+1}\left(\beta\tilde{n}_t - (1-d)\,(1-\delta)\right)\right)}{1-\beta}|X_t\right] \\ &\quad + \beta p^{ll} E\left[J^K_{t+1} + \frac{log\left((z^l\pi_{t+1}+(1-\delta)\,q_{t+1})(1-\delta) + R_{t+1}\left(\beta\tilde{n}_t - (1-d)\,(1-\delta)\right)\right)}{1-\beta}|X_t\right] \end{aligned}$$

where $\tilde{n}_t = z_t\pi_t + (1-\delta)q_t + R_t\tilde{b}_t \geq n_t$. Notice that $V^{part}(\tilde{k}) \leq V^{part}(1-\delta) \leq V^{in}(1-\delta)$, where the first inequality uses the monotonicity of V^{part} and the second uses both monotonicity and $\tilde{n}_t \geq n_t$. Therefore, partial selling strategy is never optimal.

(2) We are left to prove when inaction strategy dominates full liquidation strategy, and vice versa. It is sufficient to look at the region where liquidation strategy dominates. Since I can always normalize capital stock by 1, the region is the set of b/k (or leverage $k/(k+b)$) . Inside the region, I need to prove no one-shot deviation exists. Again, consider the same agent with state $(1, \tilde{b}, a)$, The net-worth is $n_t = z_t\pi_t + (1-\delta)(1-d) + R_t\tilde{b}$. Liquidation strategy gives value

$$V^{out} = log((1-\beta)n_t) + \eta + \beta p^{lh} E\left[J^0_{t+1} + \frac{log\,(\beta n_t R_{t+1})}{1-\beta}|X_t\right] + \beta p^{ll} E\left[J^{N+2}_{t+1} + \frac{log\,(\beta n_t R_{t+1})}{1-\beta}|X_t\right].$$

One shot inaction deviation strategy gives value

$$
\begin{aligned}
V^{in} &= log((1-\beta)n_t) + \beta p^{lh} E\left[J^0_{t+1} + \frac{log\left((z^l\pi_{t+1}+(1-\delta))(1-\delta)+R_{t+1}\left(\beta n_t-(1-d)\,\tilde{k}\right)\right)}{1-\beta}|X_t\right] \\
&\quad +\beta p^{ll} E\left[J^{N+2}_{t+1} + \frac{log\left((z^l\pi_{t+1}+(1-\delta)\,(1-d))(1-\delta)+R_{t+1}\left(\beta n_t-(1-d)\,(1-\delta)\right)\right)}{1-\beta}|X_t\right]
\end{aligned}
$$

where terms in the bracket are continuation values for drawing z^h and z^l tomorrow, respectively. The value difference between the two strategies is

$$
\begin{aligned}
V^{in}-V^{out} &= \frac{\beta}{1-\beta}p^{lh}E\left[log\left(1+(1-\delta)\frac{z^l\pi_{t+1}+(1-\delta)-(1-d)R_{t+1}}{\beta n_t R_{t+1}}\right)|X_t\right] \\
&\quad +\frac{\beta}{1-\beta}p^{lh}E\left[log\left(1+(1-\delta)\frac{z^l\pi_{t+1}+(1-\delta)(1-d)-(1-d)R_{t+1}}{\beta n_t R_{t+1}}\right)|X_t\right]-\eta
\end{aligned}
$$

Notice that $\frac{\partial(V^{in}-V^{out})}{\partial n_t}$ is equal to

$$
-\frac{\beta}{1-\beta}\left[p^{lh}E\left[\frac{\frac{z^l\pi_{t+1}+(1-\delta)-(1-d)R_{t+1}}{\beta n_t^2 R_{t+1}}}{1+(1-\delta)\frac{z^l\pi_{t+1}+(1-\delta)-(1-d)R_{t+1}}{\beta n_t R_{t+1}}}|X_t\right]+p^{ll}E_t\left[\frac{\frac{z^l\pi_{t+1}+(1-\delta)(1-d)-(1-d)R_{t+1}}{\beta n_t^2 R_{t+1}}}{1+(1-\delta)\frac{z^l\pi_{t+1}+(1-\delta)(1-d)-(1-d)R_{t+1}}{\beta n_t R_{t+1}}}|X_t\right]\right]
$$

which must be less than 0 in equilibrium. To see this, I only need to prove that

$$
p^{lh}E\left[z^l\pi_{t+1}+(1-\delta)-(1-d)R_{t+1}|X_t\right]+p^{ll}E\left[z^l\pi_{t+1}+(1-\delta)(1-d)-(1-d)R_{t+1}|X_t\right]>0
$$

or

$$
p^{lh}E\left[\frac{z^l\pi_{t+1}+(1-\delta)}{1-d}|X_t\right]+p^{ll}E\left[\frac{z^l\pi_{t+1}+(1-\delta)(1-d)}{1-d}|X_t\right]>R_{t+1}
$$

so that the expected rate of return of capital stock (with price as the liquidation price) must be greater than the risk-free rate. Suppose not, then the rate of return of inactive entrepreneurs will be less than interest rate because

$$
p^{lh}E_t\frac{z^l\pi_{t+1}+(1-\delta)}{q_t}+p^{ll}E_t\frac{z^l\pi_{t+1}+(1-\delta)q_{t+1}}{q_t}<p^{lh}E_t\frac{z^l\pi_{t+1}+(1-\delta)}{1-d}+p^{ll}E_t\frac{z^l\pi_{t+1}+(1-\delta)(1-d)}{1-d}\leq R_{t+1}
$$

by noticing that $1-d<q_t<1$ and $q_t>q_{t+1}$ (shadow price of capital decreases as it is held longer). This inequality says that inactive entrepreneurs' strategy is not consistent. They earn a lower expected rate of return on capital than risk-free rate which implies that they should liquidate.

Therefore, $\frac{\partial(V^{in}-V^{out})}{\partial n_t}<0$ and $\frac{\partial(V^{in}-V^{out})}{\partial\tilde{b}}<0$. Notice that, $V^{in}-V^{out}\to-\eta$ as $\tilde{b}\to+\infty$ (so that $n_t\to+\infty$). If $V^{in}-V^{out}$ will ever cross 0 at some $\tilde{b}=\frac{1-\underline{\lambda}_t}{\underline{\lambda}_t}$, then $V^{in}-V^{out}>0$ when $\tilde{b}<\frac{1-\underline{\lambda}_t}{\underline{\lambda}_t}$, and $V^{in}-V^{out}\leq 0$ when $\tilde{b}\geq\frac{1-\underline{\lambda}_t}{\underline{\lambda}_t}$. Equivalently, entrepreneurs liquidate the capital stock when $\frac{k}{k+b}<\underline{\lambda}_t$ and there is no one-shot deviation. In sum, if there is liquidation, the cut-off leverage $\underline{\lambda}_t$ solves

$$
\begin{aligned}
\eta &= \frac{\beta}{1-\beta}p^{lh}E\left[log\left(1+(1-\delta)\frac{z^l\pi_{t+1}+(1-\delta)-(1-d)R_{t+1}}{\beta n_t R_{t+1}}\right)|X_t\right] \\
&\quad +\frac{\beta}{1-\beta}p^{lh}E\left[log\left(1+(1-\delta)\frac{z^l\pi_{t+1}+(1-\delta)(1-d)-(1-d)R_{t+1}}{\beta n_t R_{t+1}}\right)|X_t\right]
\end{aligned}
$$

where $n_t=z^l\pi_t+(1-\delta)(1-d)+R_t\frac{1-\underline{\lambda}_t}{\underline{\lambda}_t}$.

C Extra Tables

Table 6: **Summary Statistics for COMPUSTAT Capital Reallocation**
Level variables are in millions of 2005 dollars for a given calendar quarter. "PP&E" stands for property, plant and equipment, "CapEx" for capital expenditures, "Reallocation" is the sum of acquisitions plus sales of PP&E, and "Investment" is defined as the capital expenditure plus acquisition. Total Reallocation/Total Previous PP&E ratio is computed as the sample mean of the numerator over the sample mean of the denominator to avoid the problem of firms with extremely large assets.

Variable	Mean	Median	Std. Dev.
Assets	2435.11	129.94	15712.73
PP&E	602.24	17.16	3851.315
CapEx	20.12	1.23	101.23
Acquisitions	6.12	0.00	45.67
Sales of PP&E	3.51	0.00	18.50
Total Sales of PP&E/Total Reallocation	30.71%		
Total Reallocation/Total Investment	32.1%		
Total Reallocation/Total Previous PP&E	1.44%		

Table 7: **Capital reallocation**
Correlation of real GDP and the various definitions of capital reallocation, after taking natural log and then HP filtered. Numbers in the bracket are the standard deviation after correcting heteroscedasticity and autocorrelation. Acquisition: COMPUSTAT data items 129. SPPE: sales of property, plant and equipment, COMPUSTAT data item 107. AQC turnover: acquisition divided by total asset (item 6) last period. SPPE turnover: SPPE divided by total property, plant and equipment (item 8) last period. Total Reallocation is the sum of acquisition and SPPE. GDP is real GDP in 2005 dollars. All series are seasonal adjusted and "***" denotes 1% signifance level.

Corrrelation	Acquisition	SPPE	Reallocation	SPPE turnover	AQC turnover
Corr with GDP	0.840***	0.430***	0.854***	0.411***	0.786***
	(0.064)	(0.148)	(0.057)	(0.128)	(0.071)

Table 8: **Benefits to reallocation**

Correlation of standard deviation of various measure of productivity changes, reproduction from Eisfeldt and Rampini (2006) Table 3. TFP growth: TFP growth rate of durable and non-durable manufacturing industries from Bureau of Labor Statistics and NBER-CES Manufacturing Industry database. Productivity Changes: productivity changes in 2 SIC digit manufacturing and 1 SIC digit outside manufacturing adjusted by variation in capacity utilization and value of sectoral value-added from Basu, Fernald, and Kimball (2006)

Standard Deviation of	TFP growth (2 SIC digit)	TFP growth (4 SIC digit)	Productivity Changes
Corr with GDP	-0.465^{***}	-0.384^{***}	-0.437^{***}

Table 9: **Key statistics in the data and in the model**

Data are cyclical components of HP filtered series from 1984Q1 to 2011Q4. Standard deviations denote the standard deviations of percentage deviations from trends.

	Volatility				Co-movement			
	Standard deviation	Standard deviation to that of output			Correlation with Output			
	Output	Consumption	Investment	Reallocation	Consumption	Investment	Reallocation	TFP dispersion
Data:	1.42%	0.55	3.86	10.91	0.91	0.96	0.85	-0.42
Model:	1.35%	0.61	4.01	11.05	0.88	0.91	0.61	-0.37

D Extra Graphs

Figure 10: **The potential benefits to capital reallocation**
Solid line (left scale) is the interquartile range (the gap between the 75% level and 25% level) of establishment level idiosyncratic TFP shocks (annual frequency), constructed by Bloom, Floetotto, Jaimovich, Saporta-Eksten, and Terry (2012) through Annual Manufacturing Survey and Census of Manufacturing. Dashed line (right scale) is the cyclical component of HP-filtered log of real GDP (annual frequency) normalized by its standard deviation. Shaded regions denote NBER recessions.

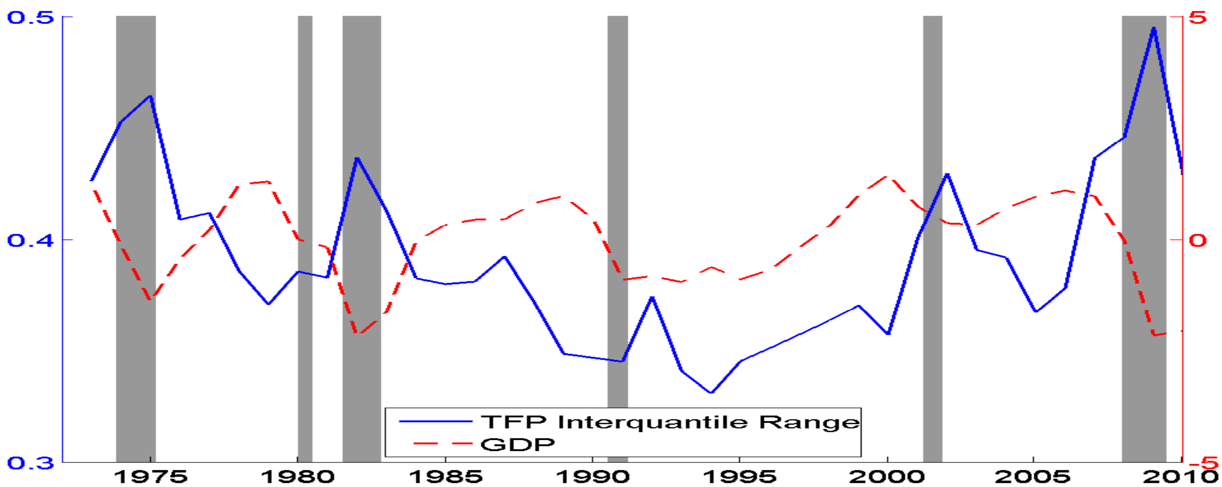

Figure 11: **Debt-to-Asset Ratio before liquidation in different groups.**
Plotted series are debt-to-asset ratios before selling assets in each quantile group. Time 0 denotes the time when firms sell assets. Each firm is classified by their positions of debt-to-asset ratios quantile at time 0. Each plot traces back average debt-to-asset ratios in each quarter before time 0, in each quantile group. For example, the debt/asset ratio at time -10 in "50% - 75% quantile" plot, means the average debt/asset ratio of companies 10 quarters before selling assets in the 50% to 75% quantile group. This figure generally shows that firms that sell assets deleverage before they sell, in addition to firms that probably have surviving problems (the 75-100% quantile group).

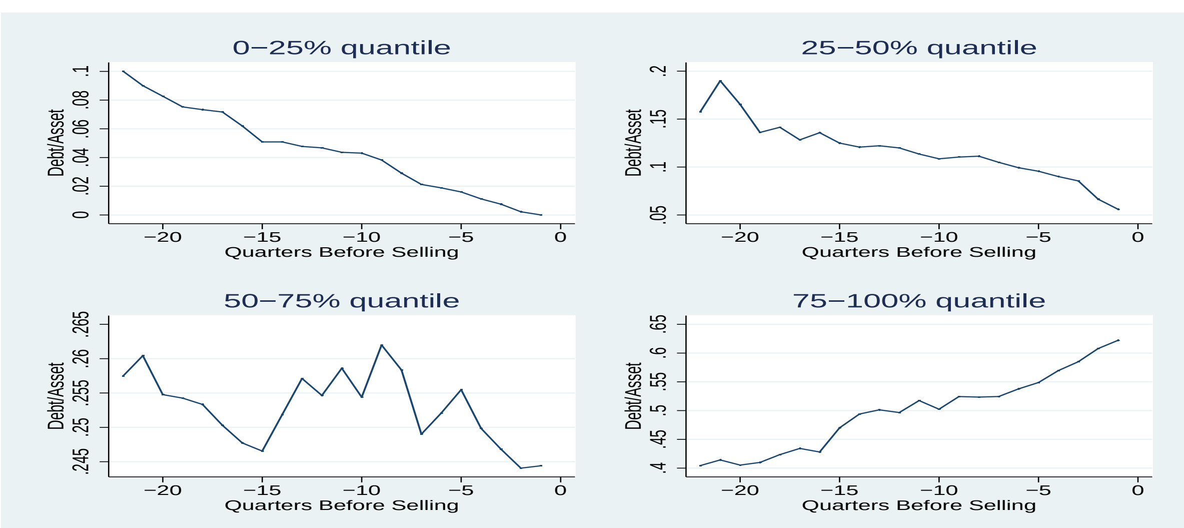

Figure 12: **Capital reallocation over cycles**

Cyclical components of HP filtered log data normalized by standard deviations. Solid lines: real GDP in 2005 dollars. Dashed lines: seasonally adjusted sales of property, plant and equipment in 2005 dollars. Dashed dotted lines: seasonally adjusted acquisitions in 2005 dollars. Shaded regions denote NBER recessions.

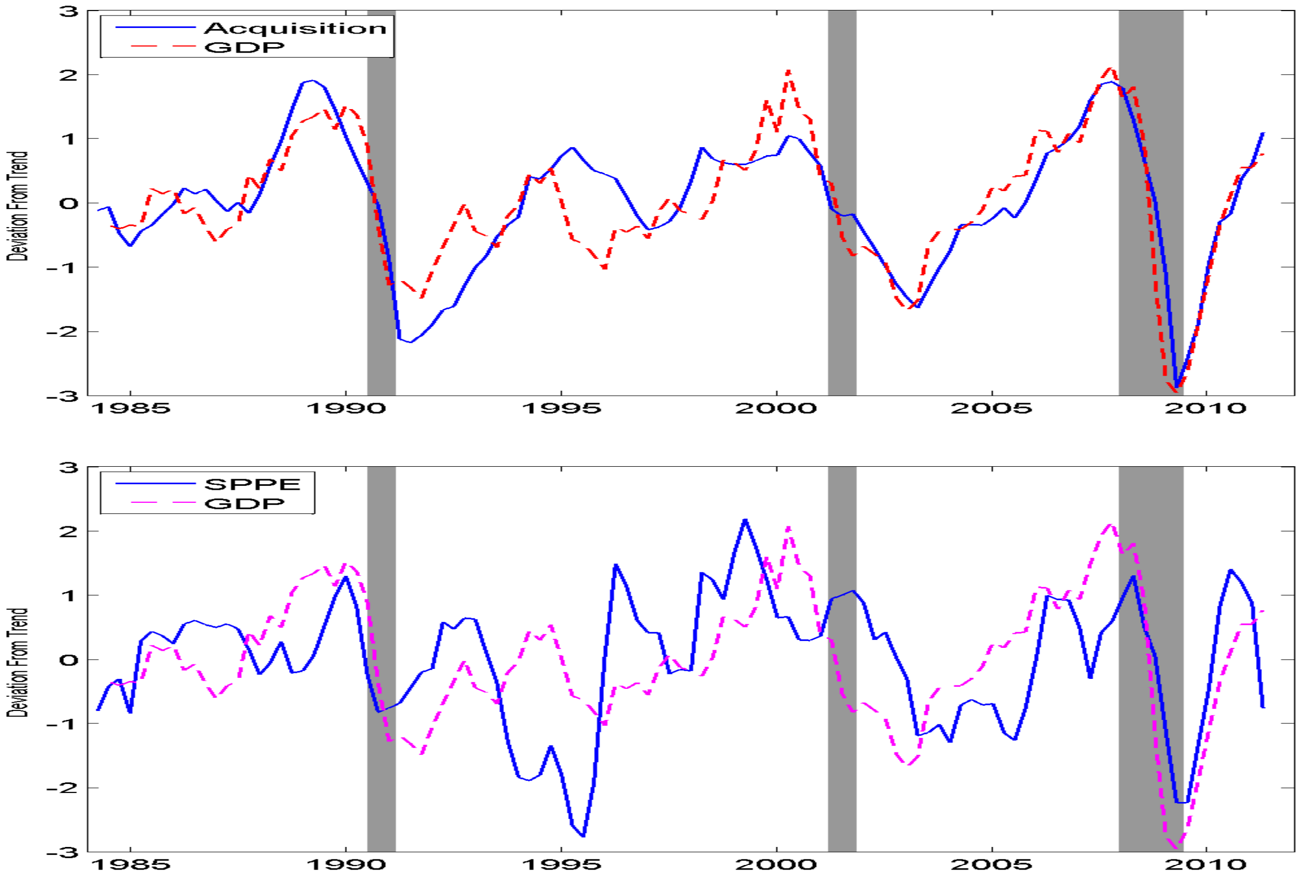

Figure 13: **Priors and Posteriors**

Priors and posteriors in graphs. Blue dashed lines denote priors. Red solid lines denote posteriors.