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Abstract

Much of the lending in modern economies is secured by some form of collateral: residential and commercial mortgages and corporate bonds are familiar examples. This paper builds an extension of general equilibrium theory that incorporates durable goods, collateralized securities and the possibility of default to argue that the reliance on collateral to secure loans and the particular collateral requirements chosen by the social planner or by the market have a profound impact on prices, allocations, market structure and the efficiency of market outcomes. These findings provide insights into housing and mortgage markets, including the sub-prime mortgage market.
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1 Introduction

Recent events in financial markets provide a sharp reminder that much of the lending in modern economies is secured by some form of collateral: residential and commercial mortgages are secured by the mortgaged property itself, corporate bonds are secured by the physical assets of the firm, collateralized mortgage obligations and debt obligations and other similar instruments are secured by pools of loans that are in turn secured by physical property. The total of such collateralized lending is enormous: in 2007, the value of U.S. residential mortgages alone was roughly \$10 trillion and the (notional) value of collateralized credit default swaps was estimated to exceed \$50 trillion. The reliance on collateral to secure loans is so familiar that it might be easy to forget that it is a relatively recent innovation: extra-economic penalties such as debtor's prisons, indentured servitude, and even execution were in widespread use in Western societies into the middle of the 19th Century.

Reliance on collateral to secure loans – rather than on extra-economic penalties – avoids the moral and ethical issues of imposing penalties in the event of bad luck, the cost of imposing penalties, and the difficulty of finding the defaulter in order to impose penalties at all. Penalties represent a pure deadweight loss: to the borrower who suffers the penalty and to the society as a whole in administering it. The reliance on collateral, which simply transfers resources from one owner to another, is intended to avoid (some of) this deadweight loss.[1] However, as this paper argues, the reliance on collateral to secure loans can have a profound effect on prices, on allocations,

[1] In practice, seizure of collateral may involve deadweight losses of its own.

on the structure of financial institutions, and especially on the efficiency of market outcomes.

To make these points, we formulate an extension of intertemporal general equilibrium theory that incorporates durable goods (physical or financial assets), collateral, and the possibility of default. To focus the discussion, we restrict attention to a pure exchange framework with two dates but many possible states of nature (representing the uncertainty at time 0 about exogenous shocks at time 1). As is usual in general equilibrium theory, we view individuals as anonymous price-takers; for simplicity, we use a framework with a finite number of agents and divisible loans.[2] Central to the model is that the definition of a security must now include not just its promised deliveries but also the collateral required to back that promise; the same promise backed by a different collateral constitutes a different security and might trade at a different price, because it might give rise to different realized deliveries. We assume that collateral is held and used by the borrower and that forfeiture of collateral is the *only* consequence of default; in particular, there are no penalties for default other than forfeiture of the collateral, and there is no destruction of property in the seizure of collateral. As a result, borrowers will always deliver the minimum of what is promised and the value of the collateral. Lenders, knowing this, need not worry about the identity of the borrowers but only about the future value of the collateral. Our model requires that each security be collateralized by a distinct bundle of assets (usually physical goods); residential mortgages (in the absence of second liens) provide the canonical example of such securities.[3] Although default is suggestive of disequilibrium, our model passes the basic test of consistency: under the hypotheses on agent behavior and foresight that are standard in the general equilibrium literature, equilibrium always exists (Theorem 1). The existence of equilibrium rests on the fact that collateral requirements place an endogenous bound on both long short sales. (The reader will recall that it is the possibility of unbounded short sales that leads to non-existence of equilibrium in the standard model of general equilibrium with incomplete markets. See the discussion following Theorem 1.)

The familiar models of Walrasian equilibrium (WE) and of general equilibrium with

[2]Anonymity and price-taking might appear strange in an environment in which individuals might default. In our context, however, individuals will default when the value of promises exceeds the value of collateral and not otherwise; thus lenders do not care about the identity of borrowers, but only about the collateral they bring. The assumption of price-taking might be made more convincing by building a model that incorporates a continuum of individuals, and the realism of the model might be enhanced by allowing for indivisible loans, but doing so would complicate the model without qualitatively changing the conclusions.

[3][Geanakoplos and Zame, 2010], expands the model to include a broader range of collateralized assets, including pools.

incomplete markets (GEI) tacitly assume that all agents keep all their promises, but ignore the question of *why* agents should keep their promises; implicitly these models assume that there are infinite penalties for breaking promises – so that agents always keep the promises they make and always make only promises that they will be able to keep. Our model of collateral equilibrium (CE) makes explicit the reasons why agents do or do not keep their promises and do or do not make promises that they will not be able to keep – and the reasons why other agents accept these promises, even knowing they may not be kept.

We show (modulo some technical differentiability and interiority assumptions) that whenever CE diverges from GEI, some agent would have borrowed more at the prevailing interest rates if he did not have to put up the collateral to get the loan but still (miraculously) had to maintain the same delivery rates.[4] Credit constraints are the distinguishing characteristic of collateral equilibrium. Somewhat more surprisingly, we show that there is a second distinguishing characteristic of collateral equilibrium: some durable good must trade for a price that is strictly higher than its marginal utility to some agent.[5] When collateral matters, it creates both price and consumption distortions of a particular kind. We identify the deviation in commodity prices as a "collateral value" which leads to commodity prices that are always at least as high as fundamental values and sometimes stricty higher, and the deviation in security prices as a "liquidity value", which leads to security prices that are always at least as high as fundamental values and sometimes strictly higher (Theorem 2).

Collateral and liquidity values have important implications for pricing and production: securities with the same deliveries can sell for different prices (so buyers may not earn the standard "market risk-adjusted" rate of return), and production may be distorted (compared to first best) toward goods that can be used as collateral and away from goods that cannot. They also have important implications for the structure of financial markets: various promises must compete for the underlying collateral, but only those securities that create the maximal liquidity value, equal to the collateral value of the underlying collateral, will be sold; other promises, which might have brought greater welfare gains if they were traded (and miraculously delivered without benefit of collateral) will not be traded because they waste collateral. In extreme cases, financial markets may shut down entirely if agents who want/need to borrow and would be happy to do so at prevailing interest rates are discouraged from

[4]In other words, he would have sold more securities at the going prices if he were freed from the burden of posting collateral.

[5]The agent who did not borrow as much as he would have at the going prices if he did not have to put up collateral (as in GEI) does not do so because the collateral he needs to post trades for a price that exceeds its marginal utility to him.

borrowing because they do not value the collateral enough that they are willing to hold it. In a slightly different vein, whenever CE diverges from WE there must also be divergence from Pareto optimality: CE that are Pareto optimal are necessarily WE (Theorem 3).

These ideas are illustrated in several simple examples. In Example 1 (a mortgage market with no uncertainty) we compute CE as a function of the wealth distribution and collateral requirement and identify parameter regions where CE is Pareto optimal and coincident with WE/GEI and parameter regions where it is not; in the latter regions we identify the distortions that are present. We find that the asset price of the collateral is much more sensitive to the distribution of wealth at time 0 in collateral equilibrium than in Walrasian equilibrium. The price is also very sensitive to exogenously imposed collateral (leverage) requirements. The welfare impact of collateral requirements is ambiguous: lower collateral requirements make it possible for buyers to hold more houses but create more competition for the same houses, thereby driving up the prices.[6] In Examples 2 and 3, we add uncertainty to the basic mortgage market to examine the effects of potential and actual default on outcomes, on welfare and on the market structure. Surprisingly, we find that collateral requirements that lead to default in equilibrium may (*ex ante*) Pareto dominate collateral requirements that do not lead to default; moreover such collateral requirements may be endogenously chosen by the market. This suggests an important implication for the subprime mortgage market: even if it is true that defaults on subprime mortgages led to a crash *ex post*, such mortgages might have been Pareto improving *ex ante*. We cannot characterize the precise conditions under which the market *always* chooses efficient collateral requirements – or more generally, any particular complete or incomplete set of securities – or when there is a welfare-improving role for government, but we do show that government action can be welfare-improving only by taking actions that alter terminal prices (Theorem 4). As long as future prices do not change, no change in lending requirements or production could benefit everyone. Hence any valid welfare-based argument for regulation of down-payment requirements would seem to require that regulators could correctly forecast the price changes that would accompany such regulation.

[6]This seems relevant to a proper understanding of the history of U.S. housing and mortgage markets. Before World War I, mortgage down payment requirements were typically on the order of 50%. The rise of Savings and Loan institutions, later the VHA and FHA – and most recently the sub-prime mortgage market – have all made it easier for (some) consumers to obtain mortgages with much lower down payment requirements. Lower down payment requirements increase competition and drive up housing prices, so some (perhaps very substantial) portion of the boom in housing prices may have over this period should presumably be ascribed to these institutional changes in mortgage markets, rather than to a change in fundamentals. (Contrast [Mankiw and Weil, 1989].)

Following a brief discussion of related literature (below), Section 2 presents the model and Section 3 presents the existence theorem (Theorem 1). Section 4 identifies via Theorem 2 the distortion when collateral equilibrium differs from GEI as arising from a liquidity value and collateral value and shows that efficient collateral equilibrium is Walrasian (Theorem 3). Our simple mortgage market (Example 1) is presented in Section 5 and the variants with uncertainty and default (Examples 2, 3) are presented in Section 6. Section 7 shows that, at least in some circumstances, the market chooses the asset structure – in particular the collateral requirements – efficiently (Theorem 4). Section 6 concludes. The (long) proof of Theorem 1 (existence) is relegated to the Appendix.

Literature

[Hellwig, 1981] provides the first theoretical treatment of collateral and default in a market setting; the focus of that work is on the extent to which the Modigliani–Miller irrelevance theorem survives the possibility of default. [Dubey et al., 1995], [Geanakoplos, 1997] and Geanakoplos and Zame (1997, 2002) (the last of which are forerunners of the present work), provide the first general treatments of a market in which deliveries on financial securities are guaranteed by collateral requirements. Genakoplos (1997) showed how the possibility of default and the need to hold collateral leads to an endogenous choice of securities. The seller of a security is obliged to hold collateral that he might like less than the price he has to pay for it, and this inconvenience hinders many security markets (especially Arrow securities) from becoming active; by explaining which securities will not be traded because of the scarcity of collateral, one explains which are. [Araujo et al., 2002] use a version of our collateral models to show that collateral requirements rule out the possibility of Ponzi schemes in infinite-horizon models, and hence eliminate the need for the transversality requirements that are frequently imposed (Magill and Quinzii, 1994; Hernandez and Santos, 1996; Levine and Zame, 1996). Araujo, Orrillo and Pascoa (2000) and [Araujo et al., 2005] expand the model to allow borrowers to set their own collateral levels, and [Steinert and Torres-Martinez, 2007] expand the model to accommodate security pools and tranching.

[Dubey et al., 2005] is a seminal work in a somewhat different literature, which treats extra-economic penalties for default. (In that particular paper, extra-economic penalties are modeled as direct utility penalties; when penalties are sufficiently severe, that model reduces to the standard model in which enforcement is perfect — and costless, because penalties are never imposed in equilibrium). Default again leads the market to endogenously choose which securities to trade; a seller who defaults might be discouraged from selling because in addition to delivering goods he must deliver penalties. Another central point of that paper, and of [Zame, 1993], which uses a very similar model, is that the possibility of default may promote efficiency (a point that is made here, in a different way, in Example 2). [Kehoe and Levine, 1993] builds a model in which the consequences of default are exclusion from trade in subsequent financial markets, but these penalties constrain borrowing in such a way that there is no equilibrium default. [Sabarwal, 2003] builds a model which combines many of these features: securities are collateralized, but the consequences of default may involve seizure of other goods, exclusion from subsequent financial markets and extra-economic penalties, as well as forfeiture of collateral. [Kau et al., 1994] provide a dynamic model of mortgages as options, but ignore the general equilibrium interrelationship between mortgages and housing prices.

Geanakoplos (2003) argued that as leverage rises and falls, asset prices will rise and fall in a leverage cycle. Fostel and Geanakoplos (2008) introduced the concepts of collateral value and liquidity wedge and showed that they necessarily appeared in a simpler model of collateral equilibrium. They also discussed 'flight to collateral' as an alternative to 'flight to quality'. The notion of liquidity value appears here for the first time.

[Bernanke et al., 1996] and [Holmstrom and Tirole, 1997] are seminal works in a quite different literature that focuses on asymmetric information between borrowers and lenders as the source of borrowing limits. Kiyotaki and Moore (1997) is another seminal and influential paper in the macro literature; it presents a dynamic example of a collateral economy.

A substantial empirical literature examines the effect of bankruptcy and default rules (especially with respect to mortgage markets) on consumption patterns and security prices. [Lin and White, 2001], [Fay et al., 2002], [Lustig and Nieuwerburgh, 2005] and [Girardi et al., 2008] are closest to the present work.

2 Model

As in the canonical model of securities trading, we consider a world with two dates; agents know the present (date 0) but face an uncertain future (date 1). At date 0 agents trade a finite set of commodities and securities. Between dates 0 and 1 the state of nature is revealed. At date 1 securities pay off and commodities are traded again.

2.1 Time & Uncertainty

There are two dates 0 and 1, and S possible states of nature at date 1. We frequently refer to $0, 1, \ldots, S$ as *spots*.

2.2 Commodities, Spot Markets & Prices

There are $L \geq 1$ commodities available for consumption and trade in spot markets at each date and state of nature; the commodity space is $\mathbb{R}^{L(1+S)} = \mathbb{R}^L \times \mathbb{R}^{LS}$. A bundle $x \in \mathbb{R}^{L(1+S)}$ is a claim to consumption at each date and state of the world. For $x \in \mathbb{R}^{L(1+S)}$ and indices s, ℓ, x_s is the bundle specified by x in spot s and $x_{s\ell}$ is the quantity of commodity ℓ specified in spot s. We write $\delta^{s\ell} \in \mathbb{R}^L$ for the commodity bundle consisting of one unit of commodity ℓ in spot s and nothing else. If $x \in \mathbb{R}^L$ then $(x, 0) \in \mathbb{R}^{L(1+S)}$ is the bundle in which x is available at date 0 and nothing is available at date 1. Similarly, if $(x_1, \ldots, x_S) \in \mathbb{R}^{LS}$ then $(0, (x_1, \ldots, x_S)) \in \mathbb{R}^{L(1+S)}$ is the bundle in which x_s is available in state s (for each $s \geq 1$) and nothing is available at date 0. We write $x \geq y$ to mean that $x_{s\ell} \geq y_{s\ell}$ for each s, ℓ; $x > y$ to mean that $x \geq y$ and $x \neq y$; and $x \gg y$ to mean that $x_{s\ell} > y_{s\ell}$ for each s, ℓ.

We depart from the usual intertemporal models by allowing for the possibility that goods are durable. If $x_0 \in \mathbb{R}^L$ is consumed (used) at date 0 we write $F(s, x_0) = F_s(x_0)$ for what remains in state s at date 1. We assume the map $F : S \times \mathbb{R}^L \to \mathbb{R}^L$ is continuous and is linear and positive in consumption. We denote $(F_1(x_0), ..., F_S(x_0)) \in \mathbb{R}^{LS}$ by $F(x_0)$. The commodity 0ℓ is *perishable* if $F(s, \delta^{0\ell}) = 0$ for each $s \geq 1$ and *durable* otherwise. It may be helpful to think of F as being analogous to a production function – except that inputs to production are also consumed.

For each s, there is a spot market for consumption at spot s. Prices at each spot lie in $\mathbb{R}^L_{++}$, so $\mathbb{R}^{L(1+S)}_{++}$ is the space of spot price vectors. For $p \in \mathbb{R}^{L(1+S)}$, p_s is the vector of prices in spot s and $p_{s\ell}$ is the price of commodity ℓ in spot s.

2.3 Consumers

There are I consumers (or types of consumers). Consumer i is described by a consumption set, which we take to be $\mathbb{R}^{L(1+S)}_+$, an endowment $e^i \in \mathbb{R}^{L(1+S)}_+$, and a utility function $u^i : \mathbb{R}^{L(1+S)}_+ \to \mathbb{R}$.

2.4 Collateralized Securities

A *collateralized security* (*security* for short) is a pair $\mathbf{A} = (A, c)$; $A : S \times \mathbb{R}^{L(1+S)}_{++} \to \mathbb{R}_+$ is a continuous function, the *promise* or *face value* (denominated in units of account) and $c \in \mathbb{R}^L_+$ is the *collateral requirement*. In principle, the promise in state s may depend on prices p_s in state s and prices p_0 at date 0 and even on prices $p_{s'}$ in other states. The collateral requirement c is a bundle of date 0 commodities; an agent wishing to sell one share of (A, c) must hold the commodity bundle c. By selling a security, an agent is effectively borrowing the price, while promising the security's face value. Thus we sometimes use the words security and loan interchangeably. The term security emphasizes that we are assuming a perfectly competitive world in which lenders and borrowers meet in large markets, and not a world with a single lender and borrower negotiating with each other.

In our framework, the collateral requirement is the only means of enforcing promises. (Such loans are frequently called *no recourse loans*.) Hence, if agents optimize, the *delivery rate* or *delivery* per share of security (A, c) in state s will not be the face value $A(s, p)$ but rather the minimum of the face value and the value of the collateral in state s:

$$\textsc{Del}((A, c), s, p) = \min\{A(s, p),\ p_s \cdot F(s, c)\}$$

The total delivery on a portfolio $\theta = (\theta_1, \ldots, \theta_J) \in \mathbb{R}^J$ is

$$\textsc{Del}(\theta, s, p) = \sum_j \theta_j \textsc{Del}((A^j, c^j); s, p)$$

We take as given a family of J securities $\mathcal{A} = \{(A^j, c^j)\}$. (The number J of securities might be very large.) Because deliveries never exceed the value of collateral, we assume without loss of generality that $F(s, c^j) \neq 0$ for some s. (Securities that fail this requirement will deliver nothing; in equilibrium the price of such securities will be 0 and trade in such securities will be irrelevant.) Because sales of securities must be collateralized but purchases need not be, it is notationally convenient to distinguish between security purchases and sales; we write $\varphi, \psi \in \mathbb{R}^J_+$ for portfolios of security purchases and sales, respectively.[7] We assume that buying and selling prices for securities are identical; we write $q \in \mathbb{R}^J_+$ for the vector of security prices. An agent who sells the portfolio $\psi \in \mathbb{R}^J_+$ will have to hold (and will enjoy) the collateral bundle $\textsc{Coll}(\psi) = \sum \psi_j c^j$.

Our formulation allows for nominal securities, for real securities, for options and for complicated derivatives. For ease of exposition, our examples focus on real securities.

[7] In principle, agents might go long and short in the same security, although there is no reason why they should do so and equilibrium would not change whether they did so or not.

2.5 The Economy

An *economy (with collateralized securities)* is a tuple $\mathcal{E} = \langle (\{e^i, u^i)\}, \{(A^j, c^j)\}\rangle$, where $\{(e^i, u^i)\}$ is a finite family of consumers and $\{(A^j, c^j)\}$ is a family of collateralized securities. (The set of commodities and the durable goods technology are fixed, so are suppressed in the notation.) Write $\overline{e} = \sum e^i$ for the social endowment. The following assumptions are always in force:

- Assumption 1 $\overline{e} + (0, F(\overline{e}_0)) \gg 0$
- Assumption 2 For each consumer i: $e^i > 0$
- Assumption 3 For each consumer i:
  - (a) u^i is continuous and quasi-concave
  - (b) if $x \geq y \geq 0$ then $u^i(x) \geq u^i(y)$
  - (c) if $x \geq y \geq 0$ and $x_{s\ell} > y_{s\ell}$ for some $s \neq 0$ and some ℓ, then $u^i(x) > u^i(y)$
  - (d) if $x \geq y \geq 0$, $x_{0\ell} > y_{0\ell}$, and commodity 0ℓ is perishable, then $u^i(x) > u^i(y)$

The first assumption says that all goods are represented in the aggregate (keeping in mind that some date 1 goods may only come into being when date 0 goods are used). The second assumption says that individual endowments are non-zero. The third assumption says that utility functions are continuous, quasi-concave, weakly monotone, strictly monotone in date 1 consumption of all goods and in date 0 consumption of perishable goods.[8]

2.6 Budget Sets

Given a set of securities $\mathcal{A}$, commodity prices p and security prices q, a consumer with endowment e must make plans for consumption, for security purchases and sales, and for deliveries against promises. In view of our earlier comments, we assume that deliveries are precisely the minimum of promises and the value of collateral, so we suppress the choice of deliveries. We therefore define the budget set $B(p, q, e, \mathcal{A})$ to be the set of *plans* (x, φ, ψ) that satisfy the budget constraints at date 0 and in each state at date 1 and the collateral constraint at date 0:

[8] We do not require strict monotonicity in durable date 0 goods because we want to allow for the possibility that claims to date 1 consumption are traded at date 0; of course, such claims would typically provide no utility at date 0.

- **at date 0**

$$p_0 \cdot x_0 + q \cdot \varphi \leq p_0 \cdot e_0 + q \cdot \psi$$
$$x_0 \geq \textsc{Coll}(\psi)$$

  In words: expenditures for date 0 consumption and security purchases do not exceed income from endowment and from security sales, and date 0 consumption includes collateral for all security sales.

- **in state** s

$$p_s \cdot x_s + \textsc{Del}(\psi, s, p) \leq p_s \cdot e_s + p_s \cdot F_s(x_0) + \textsc{Del}(\varphi, s, p)$$

  In words: expenditures for state s consumption and for deliveries on promises do not exceed income from endowment, from the return on date 0 durable goods, and from collections on others' promises.

If these conditions are satisfied, we frequently say that the portfolio (φ, ψ) *finances* x at prices p, q.[9]

Note that if security promises in each state depend only on commodity prices in that state and are homogeneous of degree 1 in those commodity prices – in particular, if securities are real (promise delivery of the value of some commodity bundle) – then budget constraints depend only on *relative prices*. In general, however, budget constraints may depend on *price levels* as well as on relative prices.

2.7 Collateral Equilibrium

A *collateral equilibrium* for the economy $\mathcal{E} = \langle (e^i, u^i), \mathcal{A} \rangle$ consists of commodity prices $p \in \mathbb{R}^{L(1+S)}_{++}$, security prices $q \in \mathbb{R}^J_+$ and consumer plans (x^i, φ^i, ψ^i) satisfying the usual conditions:

- **Commodity Markets Clear** [10]

$$\sum x^i = \sum e^i + \sum (0, F(e^i_0))$$

[9] Agents *know* date 0 prices but must *forecast* date 1 prices. Our equilibrium notion implicitly incorporates the requirement that forecasts be correct, so we take the familiar shortcut of suppressing forecasts and treating all prices as known to agents at date 0. See Barrett (2000) for a model in which forecasts might be incorrect.

[10] As in a production economy, the market clearing condition for commodities incorporates the fact that some date 1 commodities come into being from date 0 activities.

- **Security Markets Clear**

$$\sum \varphi^i = \sum \psi^i$$

- **Plans are Budget Feasible**

$$(x^i, \varphi^i, \psi^i) \in B(p, q; e^i, \mathcal{A})$$

- **Consumers Optimize**

$$(x, \varphi, \psi) \in B(p, q, e^i, \mathcal{A}) \Rightarrow u^i(x) \leq u^i(x^i)$$

2.8 WE with Durable Goods

As noted in the Introduction, it is useful to compare/contrast collateral equilibrium (CE) with Walrasian equilibrium (WE) and general equilibrium with incomplete markets (GEI). Here and in the next subsection we record the formalizations of the latter notions in the present durable goods framework. We maintain the fixed structure of commodities and preferences; in particular, date 0 commodities are durable and $F(s, x_0)$ is what remains in state s if the bundle x_0 is consumed at date 0.

A *durable goods economy* is a family $\langle (e^i, u^i) \rangle$ of consumers, specified by endowments and utility functions. We use notation in which a purchase at date 0 conveys the rights to what remains at date 1; hence if commodity prices are $p \in \mathbb{R}^{(1+S)L}_{++}$, the Walrasian budget set for a consumer whose endowment is e is

$$B^W(e, p) = \{x \in \mathbb{R}^{L(1+S)}_{+} : p \cdot x \leq p \cdot e + p \cdot (0, F(x_0))\}$$

A *Walrasian equilibrium* consists of commodity prices p and consumption choices x^i such that

- **Commodity Markets Clear**

$$\sum x^i = \sum e^i + \sum (0, F(e^i_0))$$

- **Plans are Budget Feasible**

$$x^i \in B^W(e^i, p)$$

- **Consumers Optimize**

$$y^i \in B^W(e^i, p) \Rightarrow u^i(y^i) \leq u^i(x^i)$$

2.9 GEI with Durable Goods

In the familiar GEI model, as in our collateral model, goods are traded on spot markets but only securities are traded on intertemporal markets. In the GEI context a *security* is a claim to units of account at each future state s as a function of prices; $D : S \times \mathbb{R}^{L(1+S)} \to \mathbb{R}$. A *GEI economy* is a tuple $\langle (e^i, u^i), \{D^j\} \rangle$ of consumers and securities.

To maintain the parallel with our collateral framework, we keep security purchases and sales separate. Given commodity spot prices $p \in \mathbb{R}^{L(1+S)}_{++}$ and security prices $q \in \mathbb{R}^J$, the budget set $B^{GEI}(p, q, e, \{D^j\})$ for a consumer with endowment e consists of plans (x, φ, ψ) ($x \in \mathbb{R}^{L(1+S)}_{+}$ is a consumption bundle; $\varphi, \psi \in \mathbb{R}^J_+$ are portfolios of security purchases and sales, respectively) that satisfy the budget constraints at date 0 and in each state at date 1:

- **at date 0**

$$p_0 \cdot x_0 + q \cdot \varphi \leq p_0 \cdot e_0 + q \cdot \psi$$

- **in state** s

$$p_s \cdot x_s + \sum_j \psi_j D^j_s(p) \leq p_s \cdot e_s + p_s \cdot F_s(x_0) + \sum_j \varphi_j D^j_s(p)$$

Note that the GEI budget set differs from the collateral budget set in that there is no collateral requirement at date 0 and security deliveries coincide with promises.

A *GEI equilibrium* consists of commodity spot prices $p \in \mathbb{R}^{L(1+S)}_{++}$, security prices $q \in \mathbb{R}^J$, and plans (x^i, φ^i, ψ^i) such that:

- **Commodity Markets Clear**

$$\sum x^i = \sum e^i + \sum (0, F(e^i_0))$$

- **Security Markets Clear**

$$\sum \varphi^i = \sum \psi^i$$

- **Plans are Budget Feasible**

$$(x^i, \varphi^i, \psi^i) \in B(e^i, p, q, \{D^j\})$$

- **Consumers Optimize**

$$(x, \phi, \psi) \in B(e^i, p, q, \{D^j\}) \Rightarrow u^i(x) \leq u^i(x^i)$$

3 Existence of Collateral Equilibrium

Under the maintained assumptions discussed in Section 2, collateral equilibrium always exists; we relegate the proof to the Appendix.

Theorem 1 (Existence) *Under the maintained assumptions, every economy admits a collateral equilibrium.*

Because we allow for real securities, options, derivatives and even more complicated non-linear securities, the proof must deal with a number of issues of varying degrees of subtlety. Because these issues also arise in standard models, where they can lead to the non-existence of equilibrium (see [Hart, 1975] for the seminal example of non-existence of equilibrium with real securities, [Duffie and Shafer, 1985], [Duffie and Shafer, 1986] for generic existence with real securities, and [Ku and Polemarchakis, 1990] for robust examples of non-existence of equilibrium with options), it is useful to understand the similarities and especially the differences in our collateral equilibrium framework.

The discussion is most easily presented in the context of a concrete example. Looking ahead to the framework of Example 1, consider a world with no uncertainty ($S = 1$). There are two goods at each date: food F which is perishable and housing H which is perfectly durable (so that one unit of food at date 0 yields nothing at date 1 while 1 unit of housing at date 0 yields one unit of housing at date 1). There is a single security (A, c) which promises $A = (p_{1H} - p_{1F})^+$, the difference between the date 1 price of housing and the date 1 price of food, if that difference is positive and 0 otherwise, and is collateralized by one unit of date 0 housing $c = \delta^{0H}$. (We make assumptions about consumer endowments below; consumer preferences will not enter the present discussion.)

The first issue concerns possibility of unbounded arbitrage. Suppose the commodity prices are such that $p_{1H} - p_{1F} > 0$ (so that the promise is strictly positive) but that $q = q_{(A,c)} > p_{0H}$. In that case every consumer could short the security an arbitrary amount, use the proceeds to buy the required collateral, and have money left over to buy additional consumption – so there would be an unbounded arbitrage, which would be inconsistent with equilibrium. Of course this particular unbounded arbitrage would not exist if $q < p_{0H}$; the point is only that arbitrage must be ruled out and that whether or not there is an arbitrage in our model depends on both security prices and commodity prices, so that the issue is a bit more subtle than in the standard GEI models. Our proof solves the problem by considering an auxiliary economy in which we impose artificial bounds on portfolio choices (these bounds rule out unbounded arbitrage) and then showing that, at equilibrium, these bounds do not in fact bind.[11]

The second issue concerns a security whose promise is 0. The presence of such a security whose promise is identically 0 would cause no problems: setting its price and volume of trade to 0 could not materially affect equilibrium. However, whether or not the promise A above is 0 depends endogenously on commodity prices, so the list of *potential* prices q must take this into account.[12] The problem of 0 prices also arises in standard GEI models that admit securities whose promises are allowed to be negative (in some states), since the equilibrium prices of such securities could be positive, negative or zero. We find it convenient to solve the problem in our context by solving for equilibrium in auxiliary economies in which security promises are artificially bounded away from 0 and then passing to the limit as the artificial limit is relaxed to go to 0 but other approaches, such as those used in the standard GEI literature, could be used as well.

The third and most serious issue concerns the behavior of budget sets at prices when $p_{1H} = p_{1F}$. To illustrate the problem, suppose $p_{0F} = p_{1F} = 1$, $p_{0H} = 2$, $p_{1H} = 1 + \varepsilon$ and that $q = q_{(A,c)} = \varepsilon$, where $\varepsilon \geq 0$. For $\varepsilon > 0$, a consumer can use (A, c) to shift wealth from date 0 to date 1 or *vice versa*. For example, a consumer with endowment $(e_{0F}, e_{0H}, e_{1F}, e_{1H}) = (1, 0, 0, 0)$ could sell one unit of date 0 food, buy $1/\varepsilon$ shares of the security (A, c), collect the proceeds $(1/\varepsilon)\varepsilon = 1$ and buy 1 unit of date 1 food, obtaining the consumption $(x_{0F}, x_{0H}, x_{1F}, x_{1H}) = (0, 0, 1, 0)$. However when $\varepsilon = 0$, the promise $A = 0$ and the consumer can only shift wealth from date 0 to date 1 by purchasing date 0 housing; at the given prices the largest possible consumption of date 1 food is $x_{1F} = 1/2$ (obtained by purchasing 1/2 units of date 0 housing and then selling the resulting 1/2 unit of date 1 housing). In particular, the consumer's budget set is discontinuous at $\varepsilon = 0$. As the reader will recall, such discontinuities in budget sets lead to non-generic examples of non-existence in economies with real securities (Hart, 1975) and to robust examples of non-existence in economies with options (Ku and Polemarchakis, 1990).

[11] In a model with a continuum of agents, any pre-specified bounds might bind at equilibrium; we would then have to consider the limit of auxiliary economies as the bounds are relaxed to go to infinity, but the argument would go through using arguments that are familiar in the analysis of economies with a continuum of consumers.

[12] If equilibrium prices are such that $p_{1H} - p_{1F} = 0$ then it must also be the case that $q = 0$. To see this note that if $q > 0$ then no one would be willing to buy it but every consumer who held date 0 housing would wish to sell it, so supply could not equal demand. If $q = 0$ then trade in (A, c) might occur – and be indeterminate – but would have no real effects. Note however that if the collateral requirement were different, say $c' = \delta^{0F} + \delta^{0H}$, then the equilibrium price of (A, c') might be positive even if the promise $A = 0$ because no consumer would wish to hold both date 0 food and date 0 housing and hence no consumer could sell (A, c'), and of course no one would be willing to buy it.

However these discontinuities, which present an insuperable obstacle in the more standard models cited, *do not* present an insuperable obstacle in our framework. To see why, notice that in order for a consumer to actually *buy* (rather than just demand) $1/\varepsilon$ shares of (A, c) the consumer must find counterparties who are willing to *sell* an equal number of shares. In the absence of collateral requirements, such counterparties would face no obstacle so long as $\varepsilon > 0$, since selling $1/\varepsilon$ shares of the security requires only the transfer of 1 dollar from future wealth to current wealth. In our framework, however, selling $1/\varepsilon$ shares of the security *also* requires holding $1/\varepsilon$ units of date 0 housing; this will be impossible if ε is small enough that $1/\varepsilon$ exceeds the aggregate supply of housing. Hence the discontinuity in the budget should should not "bind" *at a candidate equilibrium*. As above, this idea is most easily carried through in an auxiliary economy in which we impose an artificial bound on security sales and purchases. In this auxiliary economy, there is no discontinuity in demand so an equilibrium exists; if the artificial bound is sufficiently large (in comparison to the aggregate supply of collateral), it does not bind at equilibrium of the auxiliary economy, so an equilibrium for the auxiliary economy is also an equilibrium for the true economy.[13] Note that a similar argument would *not* work in a standard model in which sales of securities do not need to be collateralized, because the artificial bounds might bind in *every* auxiliary economy and the discontinuity would *reappear* at the candidate equilibrium of the true economy. Indeed this is exactly what happens (non-generically) in economies with real securities and robustly in economies with options.

It may be worth noting that the discontinuity could recur in a way that seems unavoidable if we expand the model to allow for an infinite set of securities. Suppose for example that for each $j = 1, \ldots$ there is a security (A^j, c^j) whose promise is $A^j = j(p_{1H} - p_{1F})^+$ and is collateralized by a single house $c^j = \delta^{0H}$. Fix j and suppose prices are $p_{0F} = p_{1F} = 1$, $p_{0H} = 2$, $p_{1H} = 1 + 1/j$ and $q = q_{(A,c)} = 1/j$. At these prices, any consumer wishing to transfer 1 dollar from current wealth to future wealth (a lender) could do so by purchasing $1/j$ units of the security (A^j, c^j) and finding counterparties (borrowers) willing to transfer 1 dollar from future wealth to current wealth by selling $1/j$ units of the security (A^j, c^j). In contrast to the previous situation, however, taking this position would not pose a problem for the counterparties since in order to take this position the counterparties would need to hold only a *single* unit of date 0 housing. As

[13] Again, the argument could be modified along familiar lines to handle a model with a continuum of consumers. An alternative argument could be constructed along somewhat different lines: If counterparties demanded $1/\varepsilon$ units of date 0 housing and ε is small, this would drive up the price of housing (collateral) beyond the ability/willingness of counterparties to pay for it and again it could be shown that at the candidate equilibrium the discontinuity in the budget set would not "bind". We have chosen our approach only because it is technically less complicated.

$1/j \to 0$, the lender and the borrower would transact only in the security (A^j, c^j), but when $1/j = 0$ neither security transactions nor the corresponding wealth transfers could take place. In this situation, the discontinuity can occur at the candidate equilibrium, so equilibrium might not exist.

4 Distortions

Collateral equilibrium that does not reduce to GEI must involve binding credit constraints. As we will show, if CE does not reduce to GEI then some agent would borrow more (sell more securities) at the going prices (interest rates) if he did not have to put up the collateral to get the loan. He does not do so because the collateral price exceeds its marginal utility to him. When collateral matters, it creates both price and consumption distortions of a particular kind. We identify the deviation in commodity prices as a "collateral value" which leads to commodity prices that are always at least as high as fundamental values and sometimes stricty higher, and the deviation in security prices as a "liquidity value," which leads to security prices that are always at least as high as fundamental values and sometimes strictly higher.

Thoughout this section we fix an economy $\mathcal{E} = \langle \{(e^i, u^i)\}, \{(A^j, c^j)\} \rangle$ and a collateral equilibrium $\langle p, q, (x^i, \varphi^i, \psi^i) \rangle$ for $\mathcal{E}$. To avoid the issues that surround "corner solutions" and to simplify the analysis, we maintain throughout this section the following assumptions for each consumer i:

(a) consumption is non-zero in each spot: $x^i_s > 0$

(b) consumption of date 0 goods *not used* as collateral is non-zero: $x^i_0 > \textsc{Coll}(\psi^i)$

(c) the utility function u^i is continuously differentiable at the equilibrium consumption x^i

(We summarize (a), (b) by saying that equilibrium allocations are *financially interior.*) Note that we do not impose the requirement that consumption of all goods is positive, only that in each state there must be positive consumption of at least one good that is not held as collateral. Hypotheses (a) and (b) would satisfied, for instance, if every agent consumed a positive amount of some perishable good like food in each state.

Given these maintained assumptions, we define various marginal utilities. For each state $s \geq 1$ and commodity k, consumer i's marginal utility for good sk is

$$MU^i_{sk} = \frac{\partial u^i(x^i)}{\partial x_{sk}}$$

By assumption, $x_s \neq 0$ so there is some ℓ for which $x^i_{s\ell} > 0$; define consumer i's *marginal utility of income at state* $s \geq 1$ to be

$$\mu^i_s = \frac{1}{p_{s\ell}} MU^i_{s\ell}$$

(This definition is independent of which ℓ we choose). Durability means that i's utility for $0k$ has two parts: utility from consuming $0k$ at date 0 consumption and utility from the income derived by selling what $0k$ becomes at date 1; hence we can express marginal utility for $0k$ as:

$$MU^i_{0k} = \frac{\partial u^i(x^i)}{\partial x_{0k}} + \sum_{s=1}^{S} \mu^i_s \left[p_s \cdot F_s(\delta^{0k}) \right]$$

For any bundle of goods $y \in \mathbb{R}^L_+$, and any $s \geq 0$, we define

$$MU^i_{sy} = \sum_{k=1}^{S} MU_{sk} y_k$$

By assumption, there is some ℓ for which $x^i_{0\ell} > \textsc{Coll}(\psi^i)_{0\ell}$; define consumer i's *marginal utility of income at date 0* to be

$$\mu^i_0 = \frac{1}{p_{0\ell}} MU^i_{0\ell}$$

(Again, this definition is independent of which ℓ we choose). Finally, define consumer i's *marginal utility for the security* (A, c) in terms of marginal utility generated by actual deliveries at date 1

$$MU^i_{(A,c)} = \sum_{s=1}^{S} \mu^i_s \ \textsc{Del}\Big((A, c), s, p\Big)$$

For each security (A, c) and commodity $0k$ or commodity bundle $y \in \mathbb{R}^L_+$, we follow [Fostel and Geanakoplos, 2008] and define the *fundamental values*, the *collateral value* and the *liquidity value* to consumer i as

$$\begin{aligned}
FV^i_{(A,c)} &= \frac{MU^i_{(A,c)}}{\mu^i_0} \\
FV^i_{0k} &= \frac{MU^i_{0k}}{\mu^i_0} \\
FV^i_{0y} &= \frac{MU^i_{0y}}{\mu^i_0} \\
CV^i_{0k} &= p_{0k} - FV^i_{0k} \\
CV^i_{0y} &= p_0 \cdot y - FV^i_{0y} \\
LV^i_{(A,c)} &= q_{(A,c)} - FV^i_{(A,c)}
\end{aligned}$$

To understand the terminology, note that if we were in the GEI economy in which the security deliveries always coincided with promises and selling the security did not require holding collateral, then the equilibrium price of any security would always coincide with its fundamental value to each consumer while the equilibrium price of each good would always be at least as high as its fundamental value to each consumer and would be equal to its fundamental value to each consumer who holds it. Thus in this GEI economy the fundamental GEI pricing equations would obtain: for each consumer i, commodity sk and security j

$$\frac{MU^i_{sk}}{\mu^i_s} \leq p_{sk} \tag{1}$$

$$\frac{MU^i_{sk}}{\mu^i_s} = p_{sk} \quad \text{if } x^i_{sk} > 0 \tag{2}$$

$$\frac{MU^i_{(A^j,c^j)}}{\mu^i_0} = q_j \tag{3}$$

Hence the liquidity value of a security and the collateral value of a commodity are measures of the price distortion caused (to a particular agent) by the necessity to hold collateral.[14]

Any security sale must be accompanied by the posting of collateral, obtained perhaps through a simultaneous purchase. This simultaneous purchase of a good that serves as collateral for the security sale that helps to finance the purchase is usually called a *leveraged purchase*. We define the fundamental value of a leveraged purchase of bundle of goods c at time 0 via the sale of the security (A, c) as the fundamental value of its residual

$$FV^i_{0c} - FV^i_{(A,c)}$$

The price of the leveraged purchase is the downpayment

$$p_0 \cdot c - q_{(A,c)}$$

With these definitions in hand, we can clarify the relationship between CE and GEI through fundamental values.

Theorem 2 (Fundamental Values) *Under the assumptions maintained in this Section, fundamental values of commodities and securities* never *exceed prices. If some agent i is selling a security j ($\psi^i_j > 0$) then the liquidity value to him is nonnegative and equal to the collateral value to him of the entire bundle that collateralizes the*

[14]Note that if $x^i_{0k} = 0$ then consumer i might find that fundamental pricing holds with $MU^i_{0k}/\mu^i_0 < p_{0k}$ even though he also finds a strictly positive collateral value $CV^i_{0k} > 0$.

security $p_0 \cdot c^j - FV^i_{c^j} = q_j - FV^i_{(A^j,c^j)}$*. Every other security written against the same collateral has equal or smaller liquidity value. Moreover, exactly one of the following must hold:*

- *(i) Fundamental value pricing* holds *for all commodities and securities and the CE is a GEI: Each consumer finds that all date 0 commodities he holds and all securities are priced at their fundamental values, so collateral values and liquidity values are all zero, and* $\langle p, q, x^i, \varphi^i, \psi^i \rangle$ *is a GEI for the incomplete markets economy* $\langle (e^i, u^i), \{D^j\} \rangle$ *(where* D^j *is the security whose deliveries are* $D^j(s,p) = \textsc{Del}((A^j, c^j), s, p)$*);* or
- *(ii) Fundamental value pricing* fails *and the CE is not a GEI: there is a consumer* i*, a security* (A^j, c^j) *and a commodity* $0k$ *such that* $\varphi^i_j = 0$ *(*i *is not buying the security),* $LV^i_{(A^j,c^j)} > 0$ *(*i *finds a strictly positive liquidity value for the security),* $c^j_{0k} > 0$ *(*$0k$ *is part of the collateral requirement for the security) and* $CV^i_{0k} > 0$ *(*i *finds a strictly positive collateral value for* $0k$*).*

Before beginning the proof of Theorem 2 it is convenient to isolate part of the argument as a lemma

Lemma *For each security* (A^j, c^j) *that is traded:*

1. *The price of* (A^j, c^j) *is equal to the fundamental value to every agent* i *who buys it.*

2. *The net price of the leveraged purchase of the bundle of goods* c^j *via the sale of the security* (A^j, c^j) *is equal to the fundamental value of its residual to any agent* i *who buys it:*
$$p_0 \cdot c^j - q_j = FV^i_{c^j} - FV^i_{(A^j,c^j)} \tag{4}$$

3. *The net marginal utility (of the collateral after making the payments on the loan) per dollar of downpayment on* (A^j, c^j) *equals the marginal utility of a dollar spent anywhere else by agent* i*.*
$$\mu^i_0 = \frac{MU^i_{c^j} - MU^i_{(A^j,c^j)}}{p_0 \cdot c^j - q_j}$$

Proof Consider a security (A^j, c^j) that is traded at equilibrium and some agent i who buys it. Agent i can always reduce or increase the amount φ^i_j that he buys by an infinitesimal fraction ε, moving the resulting revenue into or out of consumption that is not used as collateral. Because the agent is optimizing at equilibrium, this marginal move must yield zero marginal utility, which yields (i).

Now consider a security (A^j, c^j) that is traded at equilibrium and some agent i who sells it. Agent i can always reduce or increase *all* his holding of the collateral bundle c^j and the amount ψ^i_j of the security that he sells by a common infinitesimal fraction ε without violating the collateral constraints, moving the resulting revenue into or out of consumption that is not used as collateral. Because the agent is optimizing at equilibrium, this marginal move must yield zero marginal utility. Keeping in mind that μ^i_0 is agent i's marginal utility for income at date 0, it follows that

$$MU^i c^j - MU^i_{(A^j,c^j)} = \mu^i_0 (p \cdot c^j - q_j)$$

Dividing by μ^i_0 yields (ii); dividing by $p_0 \cdot c^j - q_j$ instead yields (iii). ■

Proof of Theorem 2 As we have noted, the budget and market-clearing conditions for CE imply those for GEI. Because utility functions are quasi-concave, in order that the given CE reduce to GEI it is thus necessary and sufficient that the fundamental pricing equations (1), (2), (3) hold for each consumer i, commodity sk and security j. If the given CE does not reduce to GEI then at least one of these equations must fail; we must show that the failure(s) are of the type(s) specified.

Note that the left hand sides of the fundamental pricing equations (1), (2), (3) are just what we have defined as the fundamental values. Because any agent can always consume *less* of some good that she does not use as collateral and use the additional income to buy *more* of any good or of any security, both commodity prices and security prices must weakly exceed fundamental value for every agent.

Now consider a security (A^j, c^j) that is sold at equilibrium and some agent i who sells it. Rearranging equation (4) in the Lemma above yields

$$p_0 \cdot c^j - FV^i_{c^j} = q_j - FV^i_{(A^j,c^j)}$$

As we have already noted, commodity prices are always weakly above fundamental values, so $\sum_{k=1}^{L} p_{0k} c^j_k = p_0 \cdot c^j > FV^i_{c^j} = \sum_{k=1}^{L} FV^i_{0k} c^j_k$ exactly when $p_{0k} > FV^i_{0k}$ for some commodity $0k$ for which $c^j_{0k} > 0$. We conclude that agent i finds a liquidity value for the security (A^j, c^j) he sells if and only if he finds a collateral value for some commodity that is part of the collateral c^j. The price for each good an agent consumes but does not use entirely as collateral in date 0, or consumes in any spot at date 1, must equal its fundamental value to him. Hence if no agent i is selling a security with a liquidity value, then every good is priced at its fundamental value to every agent who holds it.

If there do not exist a security (A^j, c^j) and agent i who sells (A^j, c^j) and finds both a liquidity value and a collateral value, the only remaining distortion possibility is that there is some security $(A^{j'}, c^{j'})$ that is *not* sold at equilibrium and some agent i who finds a liquidity value for $(A^{j'}, c^{j'})$. In that case, agent i could have increased his sales of the security while buying the necessary collateral. Hence there must be a collateral value to him of some good in $c^{j'}$ (which he might not be holding in equilibrium). This completes the proof. ■

The Fundamental Values Theorem shows that at the interest rates that prevail in a collateral equilibrium, agents might want to borrow more money than they actually do – if only they did not have to post collateral. This is indicated precisely by a positive liquidity value for some security, since borrowing is achieved by selling securities (i.e. loans) and the security price defines an interest rate. Agents are constrained from borrowing at the prevailing, attractive interest rates by the inconvenient need to post collateral, and by a positive collateral value which indicates that the collateral price is higher than the marginal utility of the collateral. The theorem has a slightly paradoxical ring to it. One might think that agents who are constrained in their borrowing would be forced to demand fewer durable goods, and that therefore the prices of durable goods might be less than their fundamental values. But the theorem asserts the opposite, namely that the durable goods used as collateral will always sell for more (or at least as much as) their fundamental values. [Kiyotaki and Moore, 1997] show that prices of collateral goods may be *below* fundamental values – but only if *all* date 0 goods are pledged as collateral, a possibility that is ruled out in Theorem 2 by assumption (b), which envisages positive consumption of some non-collateral good like food.

The Lemma shows that the rental price of a durable is always equal to the fundamental value of using it for one period. Suppose the delivery values $D_s^j(p)$ of the promise A^j are equal to the values of the collateral $p_s \cdot F_s(c^j)$ in all states. In this case the leveraged buyer is simply *renting* the collateral for time 0. The fundamental value to the leveraged purchase comes exclusively from the consumption utility of the collateral at time 0. Applying part (iii) of the Lemma shows that the marginal utility per dollar of rental is indeed equal to the marginal utility of money.

4.1 Collateral Value and the Efficient Markets Hypothesis

Theorem 2 tells us that there are two possibilities for a collateral equilibrium. The first is that no agent would choose to sell more of any security even if s/he did not have to put up the collateral (but were still committed to the same delivery rates). In this situation, collateral equilibrium reduces to GEI (with appropriately defined securities payoffs) and fundamental value pricing holds. In this situation the only (but very important) role played by the collateral requirement is that of endogenizing security payoffs. The second is that some agent would choose to sell more of some security if s/he did not have to put up the collateral (but were still committed to the same delivery rates). In that situation, collateral equilibrium does not reduce to GEI and fundamental value pricing fails for at least one agent and one security; moreover, if the same agent is selling that security, then fundamental value pricing fails for at least one durable good as well.

The failure of fundamental value pricing highlights that one must be very careful in applying the general principle that assets with identical payoffs must trade at identical prices. To the contrary, durable assets – either physical assets or financial assets – that yield identical payoffs can trade at different prices *if* one asset is more easily used as collateral. This would seem to be an especially important point in a setting in which some investors are uninformed/unsophisticated. A central implication of the Efficient Markets Hypothesis is that, in equilibrium, prices "level the playing field" for uninformed/unsophisticated investors and so it is not necessary that such investors know or understand everything about an asset because everything relevant will be revealed by its price. However, as Theorem 2 shows, this is not quite true: an uninformed/unsophisticated investor who buys a house, expecting that the price reflects only the consumption value and the future return and forgetting that the price also reflects its collateral value, may be sadly disappointed if he does not leverage his purchase by taking out a big loan against the house or the company. Similarly, a hedge fund that would be eager to buy assets if their purchase could be leveraged may be eager to sell them if they could not be.

4.2 Collateral Value and Overproduction

We have seen that collateral requirements distort consumption decisions, but they may distort production decisions as well. To see this, expand the model by allowing each agent i access to a technology $Y_s^i \subset \mathbb{R}^L$ in each spot s that enables the agent to produce any $y \in Y_s^i$ in spot s. (As usual, we interpret negative components of y as inputs and positive components of output. To be sure that equilibrium exists we can make the usual assumptions on the production technology.) Since intra-period production is by hypothesis instantaneous, every agent i would choose a production vector y_s^i to maximize profits. However, if some goods are better collateral than other goods, profit maximization might lead to technologically inferior production choices.

For instance, suppose that blue houses could be used as collateral while white houses could not be but that blue houses and white houses are otherwise identical (and in particular are perfect substitutes in consumption); suppose further that blue houses require an additional coat of blue paint but otherwise require the same production inputs as white houses. At equilibrium, blue houses will cost more to produce than white houses, the price of blue houses might exceed the price of white houses, and the price difference might exceed the difference in production cost, because the blue houses have an additional collateral value. In that circumstance, only blue houses would be produced – even though that is socially inefficient.

Note that government could ameliorate this inefficiency by changing lending laws so that white houses could serve as collateral. More generally, government might improve welfare by changing lending laws so that more physical goods could be used as collateral, or by creating new goods – government bonds for instance – that could be used as collateral.

4.3 Collateral Value and Credit Rationing

If collateral is the only inducement for delivery, so that security deliveries never exceed collateral payoffs, then the aggregate value of promises traded cannot exceed the aggregate value of collateral. But the *desired* level of promises might be much higher, as can be seen for example in GEI equilibrium of an economy with the same asset payoffs. How, in collateral equilibrium, are agents (collectively) restrained from making more promises? The answer is not immediately obvious, for no single agent is directly constrained from borrowing more. Indeed, as long as agents are consuming positive amounts of food in equilibrium, any one of them could borrow more by buying additional collateral and using it to back another promise. The answer is that each security sale should really be thought of as a purchase of the residual from the attendant collateral. If the value of desired security promises exceeded the value of collateral, there would be excess demand for the collateral. Collateral prices would rise, including collateral values. The premium necessary to pay to hold the collateral eventually would hold desired security sales in check. In short, the scarcity premium or collateral value of the assets serving as collateral limits borrowing.

4.4 Liquidity Value and Endogenous Security Payoffs

Deliveries on promises are altered by collateral in two ways, one obvious and the other less obvious but even more important. Without any incentive to deliver beyond the collateral, security payoffs will be shaped to some extent by the collateral, since they are the minimum of promises and collateral values. For example, if the collateral has no value in some state s, then there will be no deliveries in state s. But it would be completely wrong to presume that total security deliveries are equal or proportional to total collateral payoffs. For one thing, security payoff types may look very different from collateral payoffs. For another, consider a "financial asset" that provides no utility at time 0 to its owner. There would be no point in using that asset as collateral for a loan that promises the whole collateral in every state; the owner could just as easily sell the asset. Similarly, there would be no point to a loan that promised the proportion $\lambda < 1$ of the collateral in every state: the owner could sell $\lambda < 1$ of the collateral instead. Thus deliveries on securities backed by financial assets will look very different from the payoffs of those financial assets.

Once we have redefined each promise by its delivery rate, the question still remains: which promise will be traded? As Geanakoplos (1997) put it, not every promise type is rationed the same amount: many potential security types are rationed to zero.

The reason so many kinds of marketed promises are not traded is that many potential loans must compete for the same collateral, and according to the Fundamental Value theorem, all the loans with smaller liquidity value than the corresponding collateral value will not be actively traded in equilibrium at all – even though they are available and priced by the market. Such loan types 'waste' collateral.

4.5 Liquidity Value and Inefficient Security Choices

The market "chooses" the actively traded securities guided by the available collateral, and not by which security could create the greatest gains to trade per dollar expended. There is no reason that the security that maximizes gains to trade per dollar would have the biggest liquidity value. The security with the largest liquidity value per unit of the collateral, not the largest liquidity value per dollar of the security, will be traded. For example, an Arrow-like security (that promises delivery of the entire collateralizing asset in exactly one state) might provide large gains to trade per dollar of the security yet have smaller liquidity value than some other security that promises payoffs in many states. The liquidity value of a security must alway be less than its market price, and if there are many states in which the Arrow security promises zero, then the Arrow security price might be low and it might well have a smaller liquidity value than some other security. In that circumstance the other security might completely choke off trade in the Arrow security, despite providing smaller gains to trade. Example 3 in Section 6 illustrates just this point (among others).

4.6 Efficient Collateral Equilibria are Walrasian

When markets are incomplete, GEI allocations are generically inefficient – Pareto suboptimal – but in those circumstances in which GEI allocations happen to be Pareto optimal they are in fact Walrasian [Elul, 1999]. Since CE coincide with GEI when there are no distortions, it should not come as a surprise that when CE allocations happen to be Pareto optimal they are also Walasian.

Theorem 3 *Assume the given CE allocation is Pareto optimal and that, in addition to the maintained assumptions, assume that there is at least one consumer h who consumes a strictly positive amount of* every *good: $x^h_{s\ell} > 0$ for every s, ℓ. Then the CE allocation is Walrasian. Indeed, if we define prices π by $\pi_{s\ell} = MU^h_{s\ell}/\mu^h_0$ then $\langle \pi, x^i \rangle$ is a WE for the Walrasian economy $\langle (e^i, u^i) \rangle$.*

Proof There is no loss in assuming that all contracts (A^j, c^j) are traded in equilibrium. (Otherwise, simply delete non-traded contracts.) If agent i is buying the contract (A^j, c^j) then $q_j = FV^i_j$ (otherwise i should have bought more or less of this contract). Whether or not consumer h had been a buyer of this contract, we must have $q_j = FV^h_j$, for otherwise h could "buy" a little of (A^j, c^j) or "sell" a little to one of the buyers i of (A^j, c^j) (keeping in mind that there must be buyers, since every contract is traded), making or receiving payment of value q_j in date 0 goods that i is consuming at date 0, and delivering (in goods i is consuming in equilibrium) in each state s a tiny bit more in value than $\text{Del}((A_j, c_j), s, p)$. (This is feasible because h is consuming strictly positive amounts of all goods, and so can make the deliveries by reducing his consumption.) This would make *both* h and i better off, which would contradict Pareto efficiency. As in the proof of Theorem 2 it follows that for all goods k, $p_{0k} = FV^h_{0k} \equiv \pi_{0k}$. And of course from the fact that h is optimizing in the collateral equilibrium and chose positive consumption of each good, it must be that p_s is proportional to π_s for all $s \geq 1$.

To see that $\langle \pi, x \rangle$ must be a Walrasian equilibrium, choose $a^j \in \mathbb{R}^{L(1+S)}$ so that $q_j = -p_0 \cdot a^j_0$ and $\text{Del}((A^j, c^j), s, p) = p_s \cdot a^j_s$ for all $s \geq 1$. Because $q_j = FV^h_j$, it follows that $\pi \cdot a^j = 0$, and hence that for each agent i, $x^i \in B^W(e^i, \pi)$. Since (x^i) is a Pareto efficient allocation, $\langle \pi, x \rangle$ must be a Walrasian equilibrium. If any agent $i \neq h$ could improve his utility in his Walrasian budget set, he could improve it with a very small change while spending strictly less (since his utility is quasi-concave and monotonic). Since $x^h >> 0$, and differentiable, and since $\pi_{s\ell} = MU^h_{s\ell}/\mu^h_0$, agent h could take the opposite of the trade and also be strictly better off.■

5 A Simple Mortgage Market

In this section we offer a simple example that illustrates the working of our model and the distortions quantified in Section 4.

Example 1 Consider a world with no uncertainty ($S = 1$). There are two goods at each date: food F which is perishable and housing H which is perfectly durable. There are two consumers (or two types of consumers, in equal numbers); endowments and utilities are:

$$\begin{aligned} e^1 &= (18 - w, 1; 9, 0) \qquad & u^1 &= x_{0F} + x_{0H} + x_{1F} + x_{1H} \\ e^2 &= (w, 0; 9, 0) \qquad & u^2 &= \log x_{0F} + 4x_{0H} + x_{1F} + 4x_{1H} \end{aligned}$$

Consumer 1 finds food and housing to be perfect substitutes and has constant marginal utility of consumption; Consumer 2 likes housing more than Consumer 1, finds date 0 housing and date 1 housing to be perfect substitutes, but has decreasing marginal utility for date 0 food. We take $w \in (0, 18)$ as a parameter representing different initial distributions of wealth. In a moment we shall add another parameter α representing exogenously imposed borrowing constraints. The example illustrates that in collateral equilibrium the price of the durable collateral good housing is very sensitive to the distribution of wealth and to borrowing constraints, ranging from far below the Walrasian price to far above the Walrasian price. By contrast, in Walrasian equilibrium, the price of housing is nearly impervious to the distribution of wealth in period 0.

As a benchmark, we begin by recording the unique Walrasian equilibrium $\langle \tilde{p}, \tilde{x} \rangle$, leaving the simple calculations to the reader. If we normalize so that $\tilde{p}_{0F} = 1$ then equilibrium prices, consumptions and utilities are:

$$\tilde{p}_{0F} = 1 \ , \ \tilde{p}_{1F} = 1 \ , \ \tilde{p}_{0H} = 8 \ , \ \tilde{p}_{1H} = 4$$

$$\begin{aligned} \tilde{x}^1 &= (17, 0; 18 - w, 0) \qquad & \tilde{u}^1 &= 35 - w \\ \tilde{x}^2 &= (1, 1; w, 1) \qquad & \tilde{u}^2 &= 8 + w \end{aligned}$$

Consumer 2 likes housing much more than Consumer 1 and is rich in date 1, so, whatever her date 0 endowment, she buys all the date 0 housing and consumes one unit of food at date 0 – borrowing from her date 1 endowment if necessary, and of course repaying if she does so. Note that the distribution of food at date 1 and individual utilities all depend on w but that that consumption of food and housing at date 0 and the price of housing do not depend on w. Equilibrium social utility is always 43, which is the level it must be at any Pareto efficient allocation in which both agents consume food in date 1. (Both agents have constant marginal utility of 1 for date 1 food, so utility is transferable in the range where both consume date 1 food.) When $w < 9$, Consumer 2 borrows (so Consumer 1 lends) to finance date 0 consumption; when $w > 9$, Consumer 2 lends (so Consumer 1 borrows) in order to finance date 1 consumption

In the GEI world, in which securities always deliver precisely what they promise and security sales do not need to be collateralized, the Walrasian outcome will again obtain when there are at least as many independent securities as states of nature – here, at least one security whose payoff is never 0.

However, in the world of collateralized securities, Walrasian outcomes need not obtain. When w is small, Consumer 2 is poor at date 0 and so would like to borrow – but the amount she can borrow is constrained by the fact that she will never be required to repay more than the *future* value of the collateral. When w is large, Consumer 2 is rich at date 0 and so would like to lend – but the amount she can lend is constrained by the fact that the borrower would necessarily need to hold collateral.

Using housing to collateralize its own purchase is leveraging; regulating this leveraging can be accomplished by setting collateral requirements. To see the macroeconomic effects of regulating leverage, we introduce another exogenous parameter $\alpha \in [0, 4]$ that specifies the size of the security promise that can be made using a house as collateral. We assume that only one security $(A_\alpha, c) = (\alpha p_{1F}, \delta^{0H})$ is available for trade; (A_α, c) promises the value of α units of food in date 1 and is collateralized by 1 unit of date 0 housing.[15] As we shall see the nature of collateral equilibrium depends on the parameters w, α; Figure 1 depicts the various equilibrium regions and the price of housing as functions of these parameters. Note that even this simplest of settings is quite rich.

[15]In our formulation, the security promise and collateral requirement are specified exogenously and the security price is determined endogenously. A more familiar formulation would specify the security price and the down payment requirement exogenously and have the interest rate (hence the security promise) be determined endogenously. Of course, the two formulations are equivalent: the down payment requirement d, interest rate r, house price p_{0H}, security price q_α and promise α are related by the obvious equations: $d = (p_{0H} - q_\alpha)/p_{0H}$, $r = (\alpha - q_\alpha)/q_\alpha$.

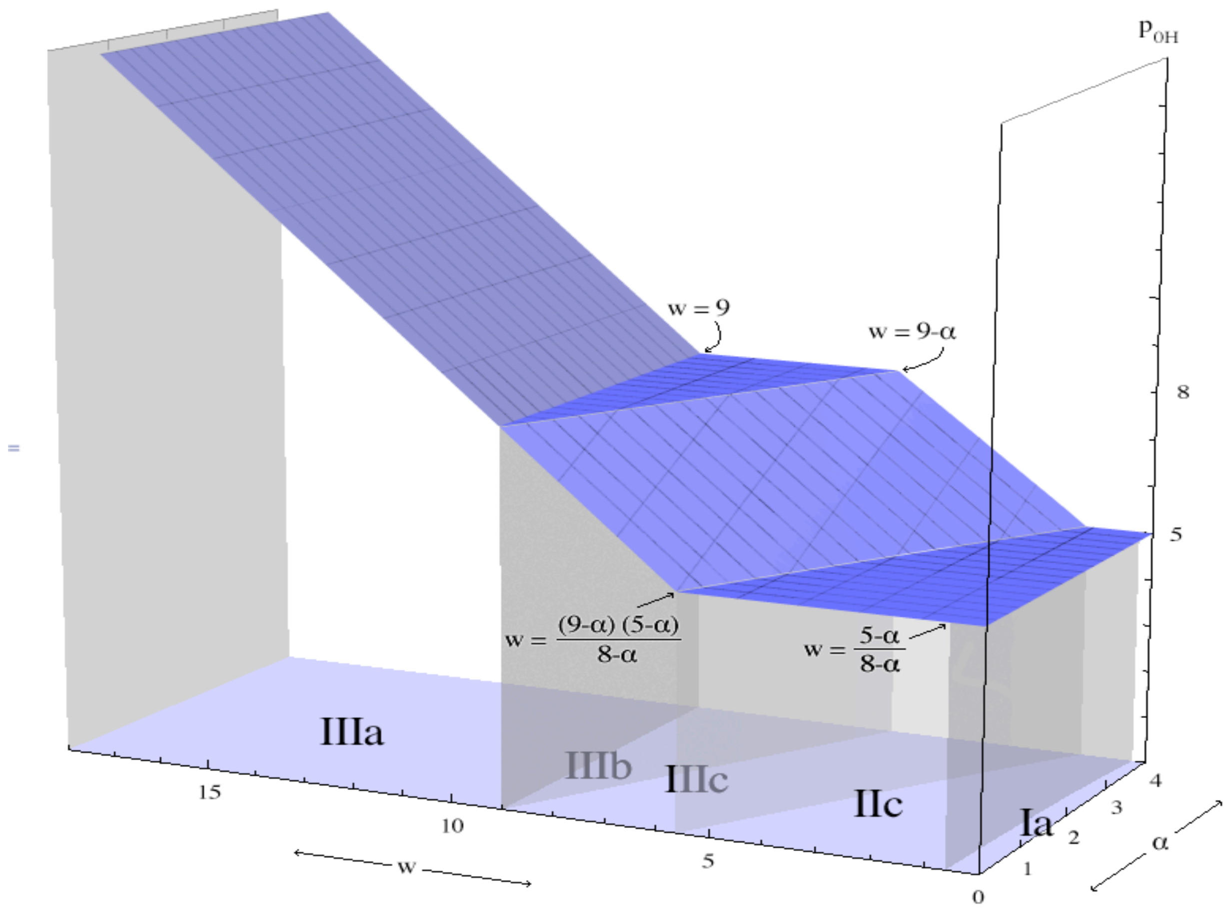

Figure 1: Equilibrium Regions and Date 0 Housing Prices

Before beginning the calculations (which are perhaps surprisingly delicate), we make a useful observation: Increasing α enables Consumer 2 to back more borrowing with the same collateral and hence to buy more housing with borrowed money.[16] Since Consumer 2 loves housing and is rich in the last period, enabling her to borrow makes her better off, all else equal. But all else need not be equal: when all the Consumer 2 types borrow, competition will then raise the price of housing. We trace out the effects of these opposite forces on her welfare by computing equilibrium for each parameter pair (w, α).

Because we compute equilibria via first order conditions, especially those of Consumer 2, it is convenient to classify equilibria according to the quantity of housing and the amount of borrowing capacity exercised by Consumer 2; by definition the borrowing capacity ψ^2 cannot exceed housing held, so this leads to 9 potential types of equilibria, as in Table 1 – but because the collateral requirement entails that $\psi^2 \leq x^2_{0H}$, there are no equilibria of types Ib, Ic so that only 7 types of equilibria are actually possible.

Table 1: Types of Equilibrium

	$\psi^2 = 0$	$0 < \psi^2 < x^2_{0H}$	$0 < \psi^2 = x^2_{0H}$
$x^2_{0H} = 0$	**Ia**	**Ib**	**Ic**
$x^2_{0H} \in (0,1)$	**IIa**	**IIb**	**IIc**
$x^2_{0H} = 1$	**IIIa**	**IIIb**	**IIIc**

For the given functional forms, we shall see that there are no equilibria of type IIb (although there would be equilibria of type IIb for some other functional forms and parameter values). For all the other types, we solve simultaneously for the equilibrium variables and the region in the parameter space in which an equilibrium of that type obtains. We find that these regions are disjoint and partition the parameter space, and that there is a unique equilibrium for each parameter pair (w, α). For many of the variables, the equilibrium values do not depend on the parameters w, α, and we find these first; then we sketch the calculations for the remaining equilibrium variables in types IIc, IIIc, and IIIa, leaving the details and calculations for other types to the reader. We present quite a lot of detail because the calculations are surprisingly complicated and illustrate well the notions discussed in Section 4.

To solve for equilibrium, we begin by normalizing (as we are free to do) so that $p_{0F} = 1$. Because (A_α, c) is a real security we are also free to normalize so that

[16] As we shall show, $p_{1H} = 4$ in every equilibrium, so if $\alpha > 4$ an agent who sells (A_α, c) will default, delivery will be 4 rather than α and the resulting equilibrium will coincide with the equilibrium that would prevail with $\alpha = 4$.

$p_{1F} = 1$. In collateral equilibrium no agent can begin with less wealth in any state $s \geq 1$ than his inital endowment. At date 1 the prices and allocations that prevail are those in the standard exchange economy that results after endowments are adjusted to reflect asset deliveries. It follows that Consumers 1 and 2 each consume food in date 1 ($x^1_{1F} > 0, x^2_{1F} > 0$), that Consumer 2 acquires all the housing at date 1, ($x^1_{1H} = 0, x^2_{1H} = 1$), that the price of housing in period 1 is 4, ($p_{1H} = 4$) and that the marginal utilities of money to both agents at date 1 are one ($\mu^1_1 = \mu^2_1 = 1$). Because $\alpha \in [0, 4]$, the date 1 value of collateral (weakly) exceeds the promise A_α, so $\textsc{Del}(A_\alpha, p) = \alpha$; hence $MU^1_{(A_\alpha,c)} = MU^2_{(A_\alpha,c)} = \alpha$. The marginal utility to Consumer 2 of owning the house in date 0 is obviously 8, since she can live in it at both dates. The marginal utility to Consumer 1 of owning the house at date 0 is 5, since he can live in it at date 0 and sell it for 4 units of food in period 1.

We assert that in every equilibrium, no matter who or whether the security is sold, the security price $q_\alpha \geq \alpha$. To see this, suppose $q_\alpha < \alpha$. Because Consumer 1's marginal utility for food is 1 in both dates, optimality of his equilibrium consumption means that it must not be possible for him to shift from food consumption to holding the security, so necessarily $x^1_{0F} = 0$. But then $x^2_{0F} = 18$ so it is possible for Consumer 2 to make this shift; since Consumer 2's marginal utility for date 0 food is $1/x^2_{0F} = 1/18$ and her marginal utility for date 1 food is 1, this is a contradiction. So we conclude that $q_\alpha \geq \alpha$, as asserted.

Next we assert that $p_{0H} \geq 5$. If $p_{0H} < 5$ then Consumer 1 would strictly prefer to buy date 0 housing rather than date 0 food so optimality implies that $x^1_{0F} = 0$. Since Consumer 1 initially owns the entire housing stock and a strictly positive amount of date 0 food, he must be spending some of his date 0 income on purchasing the security at price $q_\alpha \geq \alpha$, from which (like food) he gets at most one utile per dollar spent (since $\mu^1_1 = 1$). From this contradiction we conclude that $p_{0H} \geq 5$, as asserted.

Finally, we assert that in equilibrium Consumer 1 could never be a net borrower (sell more of the security than he buys), even in cases where he is very poor in state 0 and Consumer 2 is very rich. If he did, then Consumer 2 would have to be a net lender, which by the fundamental pricing lemma implies that

$$\frac{1/x^2_{0F}}{1} = \frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{(A_\alpha,c)}}{q_\alpha} = \frac{\alpha\mu^2_1}{q_\alpha} = \frac{\alpha}{q_\alpha}$$

or $x^2_{0F} = q_\alpha/\alpha \geq 1$. But then from Consumer 2's optimization involving a positive amount of food,

$$\frac{1/x^2_{0F}}{1} = \frac{MU^2_{0F}}{p_{0F}} \geq \frac{MU^2_{0H}}{p_{0H}} = \frac{8}{p_{0H}}$$

which implies that $p_{0H} \geq 8x^2_{0F}$. If Consumer 1 is borrowing, then (since he began with all the housing stock) $x^1_{0F} > 0$ and $\mu^1_0 = 1$. Moreover, in order to borrow he must continue to hold housing. He will only desire that if the fundamental value of the residual from the leveraged purchase of housing is at least as high as its price,

$$MU^1_{0H} - \alpha\mu^1_1 \geq \mu^1_0(p_{0H} - q_\alpha)$$
$$5 - \alpha \geq p_{0H} - q_\alpha \geq 8x^2_{0F} - \alpha x^2_{0F} = (8-\alpha)x^2_{0F} \geq 8 - \alpha$$

which is a contradiction. Thus Consumer 1 cannot be a net borrower in equilibrium, and so we can take $\psi^1 = \varphi^2 = 0$.

Summarizing: for all $w \in (0, 18)$, all $\alpha \in [0, 4]$, and in every equilibrium, we have

$$p_{0F} = 1, p_{1F} = 1, p_{1H} = 4, p_{0H} \geq 5, q_\alpha \geq \alpha, \psi^1 = 0, \varphi^2 = 0, \mu^1_0 = \mu^1_1 = \mu^2_1 = 1 \quad (5)$$

Furthermore, if there is trade in the securities market, then Consumer 1 must be the lender and $q_\alpha = \alpha$.

Lastly we observe that since housing gives higher utility to consumer 2 than to Consumer 1, and since $\mu^1_0 = \mu^1_1 = \mu^2_1 = 1$, the only way they could both hold housing at date 0 is if $x^2_{0F} < 1$ so that $\mu^2_0 > 1$. But in that case, Consumer 2 would borrow as much as he could using his housing as collateral. This rules out equilibria of type IIb and also rules out equilibria of type IIa except in the trivial case $\alpha = 0$.

With these preliminary observations out of the way, we shall proceed through cases in which Consumer 2 gets progressively richer and the price of housing gets progressively higher. We begin by analyzing equilibrium of the first interesting type, IIc, in which Consumer 1 and Consumer 2 both hold housing and Consumer 2 borrows all he can on his housing. Since Consumer 1 is lending, we showed already that $q_\alpha = \alpha$. Since Consumer 1 could always buy food, obtaining one utile per dollar expended, and since he does not have any collateral reason to hold housing, we must have $p_{0H} \leq 5$. Combined with the above demonstration that $p_{0H} \geq 5$, we deduce $p_{0H} = 5$.

To solve for the remaining equilibrium variables we use Consumer 2's date 0 first order conditions – but the correct first order conditions may not be obvious. Because Consumer 2 holds food and housing at date 0, it might appear by analogy with standard first order conditions that

$$\frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{0H}}{p_{0H}} \quad (6)$$
$$\frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{(A_\alpha,c)}}{q_\alpha} \quad (7)$$

Consumer 2 enjoys 4 utils from living in the house at each date, so $MU^2_{0H} = 8$. In view of our earlier calculations, it follows from (6) that $MU^2_{0F} = 8/5$ and from (7) that $MU^2_{0F} = 1$, which is nonsense.

The error in this analysis is that (6) and (7) are *not* the correct first order conditions for Consumer 2. They neglect the collateral value for housing and the liquidity value for securities, respectively. Consumer 2 can borrow against date 1 income by selling the security, but selling the security requires holding collateral. By assumption, at equilibrium $x^2_{0H} = \psi^2$, so Consumer 2 is exercising all of her borrowing power; hence she cannot hold less housing without simultaneously divesting herself of some of the security and cannot sell more of the security without simultaneously acquiring more housing.

The correct first order conditions for Consumer 2 derive from the equality between the fundamental value to Consumer 2 of the residual of the leveraged purchase and its price. In other words, buying an additional infinitesimal amount ε of housing costs $p_{0H}\varepsilon$, but of this cost $q_\alpha\varepsilon = \alpha\varepsilon$ can be borrowed by selling α units of the security, using the additional housing as collateral, so the net payment is only $(p_{0H} - q_\alpha)\varepsilon = (p_{0H} - \alpha)\varepsilon$. However, doing this will require repaying the loan in date 1, so the additional utility obtained will not be $MU^2_{0H}\varepsilon = 8\varepsilon$ but rather $(MU^2_{0H} - \alpha)\varepsilon = (8 - \alpha)\varepsilon$. On the other hand, selling an additional ε units of food generates income of $p_{0F}\varepsilon$ at a utility cost of $MU^2_{0F}\varepsilon$. Hence the correct first order condition for Consumer 2 is not (6), but rather

$$\frac{1}{x^2_{0F}} = \frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{0H} - \alpha}{p_{0H} - \alpha} = \frac{8 - \alpha}{5 - \alpha} \tag{8}$$

Consumer 2's date 0 budget constraint is

$$(5 - \alpha)x^2_{0H} + x^2_{0F} = w \tag{9}$$

Solving yields

$$x^2_{0F} = \frac{5 - \alpha}{8 - \alpha}\ ,\ x^2_{0H} = \frac{w - \frac{5-\alpha}{8-\alpha}}{5 - \alpha}$$

From this we can solve for all the equilibrium consumptions and utilities and security trades

$$
\begin{aligned}
p_{0H} &= 5 \\
q_\alpha &= \alpha \\
x^1 &= \left(18 - \frac{5-\alpha}{8-\alpha}, 1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}; 9 + \alpha\left[\frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right] + 4\left[1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right], 0\right) \\
u^1 &= 32 - w \\
x^2 &= \left(\frac{5-\alpha}{8-\alpha}, \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}; 9 - \alpha\left[\frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right] - 4\left[1 - \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}\right], 1\right) \\
u^2 &= 8 + \log(5-\alpha) - \log(8-\alpha) + \left(\frac{8-\alpha}{5-\alpha}\right) w \\
\varphi^1 &= \psi^2 = x^2_{0H} = \frac{w - \frac{5-\alpha}{8-\alpha}}{5-\alpha}
\end{aligned}
$$

Finally, the region in which equilibria are of type IIc is defined by the requirement that $x^2_{0H} \in (0,1)$, so

$$
\text{Region IIc } = \left\{(w,\alpha) : \frac{5-\alpha}{8-\alpha} < w < \frac{(5-\alpha)(9-\alpha)}{8-\alpha}\right\}
$$

In equilibria of type IIIc, $x^2_{0H} = 1$ and $\psi^2/x^2_{0H} = 1$ so Consumer 1 no longer holds housing in date 0, and we cannot guess in advance what the price of housing will be in period 0, but must solve for it along with the other variables. Reasoning as above, we see that Consumer 2's date 0 first-order condition and budget constraint are

$$
\begin{aligned}
\frac{8-\alpha}{p_{0H}-\alpha} &= \frac{1}{x^2_{0F}} \\
p_{0H} - \alpha + x^2_{0F} &= w
\end{aligned}
$$

Solving yields:

$$
\begin{aligned}
p_{0H} &= \alpha + \left(\frac{8-\alpha}{9-\alpha}\right) w \\
q_\alpha &= \alpha \\
x^1 &= \left(18 - \frac{w}{9-\alpha}, 0; 9+\alpha, 0\right) \\
u^1 &= 27 + \alpha - \frac{w}{9-\alpha} \\
x^2 &= \left(\frac{w}{9-\alpha}, 1; 9-\alpha, 1\right) \\
u^2 &= \log\left(\frac{w}{9-\alpha}\right) + 17 - \alpha \\
\varphi^1 &= \psi^2 = x^2_{0H} = 1
\end{aligned}
$$

The region in which equilibria are of type IIIc is determined by the requirements that it be optimal for Consumer 2 to borrow the maximum amount possible, whence $x_{0F} \leq 1$, and that Consumer 1 not wish to buy housing, whence $p_{0H} \geq 5$. Putting these together yields:

$$
\text{Region IIIc } = \left\{(w,\alpha) : \left(\frac{5-\alpha}{8-\alpha}\right)(9-\alpha) \leq w \leq (9-\alpha)\right\}
$$

In region IIIa, the security is not traded but Consumer 2 holds all the housing and some food at date 0, so Consumer 2 must be rich enough at date 0 to buy all the housing without borrowing, and must be indifferent to trading date 0 food for date 0 housing or for the security. Her first order conditions and budget constraint at date 0 are:

$$
\frac{MU^2_{(A,c)}}{q_\alpha} = \frac{MU^2_{0F}}{p_{0F}} \tag{10}
$$

$$
\frac{MU^2_{0F}}{p_{0F}} = \frac{MU^2_{0H}}{p_{0H}} \tag{11}
$$

$$
1 \cdot x^2_{0F} + p_{0H} \cdot 1 = w \tag{12}
$$

Solving yields

$$
p_{0H} = \frac{8w}{9} \ , \ x^2_{0F} = \frac{w}{9} \ , \ q_\alpha = \frac{\alpha w}{9}
$$

At these prices, Consumer 1 would like to *sell* the security, but is deterred from doing so by the requirement to hold (expensive) collateral. Equilibrium consumptions and utilities are:

$$
\begin{aligned}
x^1 &= (26 - w, 0; 9, 0) \\
u^1 &= 35 - w \\
x^2 &= (\frac{w}{9}, 1; 9, 1) \\
u^2 &= \log\left(\frac{w}{9}\right) + 17 \\
\varphi^1 &= \psi^2 = 0
\end{aligned}
$$

Finally, region IIIa is defined by the requirement that Consumer 2 be rich enough to buy all the housing at date 0:

$$\text{Region IIIa} = \{w : w \geq 9\}$$

We summarize the description of equilibrium in the remaining regions Ia, IIa, and IIb below.

- **Ia** $\quad 0 < w \leq (5 - \alpha)/(8 - \alpha)$

$$
\begin{array}{ccc}
p_{0H} & = & 5 \\
q_\alpha & = & \alpha \\
x^1 & = & (18 - w, 1; 13, 0) \\
u^1 & = & 32 - w \\
x^2 & = & (w, 0; 5, 1) \\
u^2 & = & \log w + 9 \\
\varphi^1 = \psi^2 & = & 0
\end{array}
$$

- **IIa** $\quad \alpha = 0$, $5/8 < w < 45/8$

$$
\begin{aligned}
p_{0H} &= 5 \\
q_\alpha &= \alpha \\
x^1 &= \left(18 - \frac{5}{8}, 1 - (\frac{w}{5} - \frac{1}{8}); 13 - 4(\frac{w}{5} - \frac{1}{8}), 0\right) \\
u^1 &= 32 - w \\
x^2 &= \left(\frac{5}{8}, \frac{w}{5} - \frac{1}{8}; 5 + (\frac{w}{5} - \frac{1}{8}), 1\right) \\
u^2 &= \log(\frac{5}{8}) + 8 + \frac{8w}{5} \\
\varphi^1 &= \psi^2 = 0
\end{aligned}
$$

- **IIIb** $\quad 9-\alpha < w < 9$

$$\begin{aligned} p_{0H} &= 8 \\ q_\alpha &= \alpha \\ x^1 &= (17, 0; 18-w, 0) \\ u^1 &= 35 - w \\ x^2 &= (1, 1; w, 1) \\ u^2 &= 8 + w \\ \varphi^1 = \psi^2 &= \frac{9-w}{\alpha} \end{aligned}$$

Example 1 reinforces a number of the points made in Section 4.

- In Walrasian equilibrium the price of housing $p_{0H} = 8$ no matter what the distribution of income in period 0. If Consumer 2 is poor in period 0, she borrows against her big income in period 1. By contrast, in collateral equilibrium, the price of housing rises from $p_{0H} = 5$ to $p_{0H} = 16$ as w rises from 0 to 18. The price of housing also rises when α rises, that is, when more borrowing is allowed. In this example, asset prices are much more volatile in collateral equilibrium than in Walrasian equilibrium.

- CE allocations may be inefficient – and in particular, differ from Walrasian allocations – even when financial markets are "complete." Because Example 1 represents a transferable utility economy, an allocation is Pareto efficient if and only if the sum of individual utilities is 43; these allocations are precisely those for which Consumer 2 holds all the housing in both dates and exactly one unit of date 0 food: $x^2_{0H} = x^2_{1H} = 1$ and $x^2_{0F} = 1$. Hence, CE is Pareto efficient exactly when $w \in [9-\alpha, 9]$; that is, in Region IIIb and in portions of the boundaries of Regions IIIa and IIIc. And, as asserted in Theorem 3, wherever CE is efficient, it is equivalent to WE.

- For $w \notin [5, 9]$ collateral equilibrium is inefficient no matter what collateral requirement is set; indeed, collateral equilibrium will be inefficient no matter what collateralized securities are available for trade. (See [Geanakoplos and Zame, 2010], Theorem 3).

- As asserted in Theorem 2, in every region where CE $\neq$ GEI, there are distortions and some consumer experiences a collateral value and liquidity value. In Regions IIc and IIIc this is Consumer 2. To see the distortions more concretely, fix

$w = 7/2$. For $\alpha \in (0,2)$ parameter values are in region IIc and for $\alpha \in [2,4]$ parameter values are in region IIIc but in both cases the collateral requirement distorts Consumer 2's consumption choice, leading her to hold "too much" housing, given the price. To see this, compare marginal utilities per dollar for date 0 food and date 0 housing. In Region IIc we have

$$\frac{MU^2_{0F}}{p_{0F}} = \frac{8-\alpha}{5-\alpha} > \frac{8}{5} = \frac{MU^2_{0H}}{p_{0H}}$$

while in Region IIIc we have

$$\frac{MU^2_{0F}}{p_{0F}} = \frac{2(9-\alpha)}{7} > \frac{16}{7(\alpha + \frac{8-\alpha}{9-\alpha})} = \frac{MU^2_{0H}}{p_{0H}}$$

This distortion can be seen in prices as well: to say that Consumer 2's marginal utility per dollar for date 0 food exceeds her marginal utility per dollar for date 0 housing is to say that the price of date 0 housing is too high. Consumer 2 is willing to pay the higher price of date 0 housing because holding housing enables her to borrow; that is, she derives a collateral value from housing as well as a consumption value. Similarly, Consumer 2 finds the marginal utility per dollar for date 0 food to be higher than the marginal utility of making the payments on the security; the price of the security is "too high" as well – she experiences a liquidity value.

In the portion of Region IIIa where $w > 9$, it is Consumer 1 who experiences collateral values and a liquidity value, although Consumer 1 neither holds housing nor sells the security. In this region, at the prevailing interest rate (asset price), Consumer 1 would be delighted to borrow (sell the asset) but is discouraged from doing so because he would have to hold collateral, which he does not wish to do. In this region the effect of the collateral distortion is to shut down the borrowing/lending market entirely.

- The effects of collateral requirements on welfare are subtle. Again, fix $w = 7/2$. Increases in α (equivalently, decreases in the down payment requirement) make it possible for consumers of type 2 to access more date 1 wealth. For $\alpha \in [0,2)$, this makes it possible for consumers of type 2 to afford more housing; the net result is Pareto improving. For $\alpha \in [2,4]$, however, consumers of type 2 already own all the available houses, so increasing α only leads to more competition among them, which serves only to drive up the price of date 0 housing (from $p_{0H} = 5$ when $\alpha = 2$ to $p_{0H} = 34/5$ when $\alpha = 4$). This price increase makes Consumers of type 1 better off but makes Consumers of type 2 *worse off*.

- In this Example, we insisted that only one security is offered, a mortgage collateralized by a single house and promising the value of α units of food. However, nothing would change if we allowed for various mortgages, each collateralized by a single house (this is just a normalization) but with different promises. The reason is that only the security with the largest liquidity value will be traded; this is the (unique) mortgage with the largest promise.

6 Default, Crashes and Welfare

Default and crashes are suggestive of inefficiency. As the Examples presented in this Section show, this need not be true: both default and crashes may be welfare enhancing.[17] Indeed, as Example 2 shows, levels of collateral that are socially optimal may lead to default with positive probability.

More generally, there is a link between collateral requirements and future prices. Lower collateral requirements lead buyers to take on more debt; the difficulties of servicing this debt can lead to reduced demand and lower prices – and even to crashes – in the future. Importantly, such crashes occur precisely *because* lower collateral requirements encourage borrowers to take on more debt than they can service. And yet, despite these crashes, lower collateral requirements may be welfare enhancing.

A final point made by these Examples is that although the set of securities *available for trade* is given exogenously as part of the data of the model, the set of securities that are *actually traded* is determined endogenously at equilibrium. Thus, we may view the financial structure of the economy as chosen by the competitive market. As Example 3 shows, if many collateral levels are available, the market may choose levels of collateral that lead to default with positive probability, and this choice may be efficient; morever, even if all possible securities are available for trade, the market may choose an incomplete set to actually be traded at equilibrium.

Example 2 (Default and Crashes) We construct a variant on Example 1. Rather than present a full-blown analysis as in Example 1, we fix endowments and take only the security promise as a parameter, making it easier to focus on the points of interest.

There are two states of nature and two goods: Food, which is perishable, and Housing, which is durable. There are two (types of) consumers, with endowments

[17] That default may be welfare enhancing is a point that has been made, in different contexts, by [Zame, 1993], [Sabarwal, 2003] and [Dubey et al., 2005].

and utility functions:

$$
\begin{aligned}
e^1 &= (29/2, 1; 9, 0; 9, 0) \\
u^1 &= x_{0F} + x_{0H} + (1/2)(x_{1F} + x_{1H}) + (1/2)(x_{2F} + 3x_{2H}) \\
e^2 &= (7/2, 0; 9, 0; 5/2, 0) \\
u^2 &= \log x_{0F} + 4x_{0H} + (1/2)(x_{1F} + 4x_{1H}) + (1/2)(x_{2F} + 4x_{2H})
\end{aligned}
$$

Note the only differences from Example 1 are that Consumer 1 likes housing better in state 2 and Consumer 2 is poor in state 2 – the 'bad' state.

A single security – a mortgage – $\mathbf{A}_\alpha = (\alpha p_{1F}, \alpha p_{2F}; \delta^{0H})$, promising the value of α units of food and collateralized by 1 unit of housing, is available for trade; we take $\alpha \in [0, 4]$ as a parameter.[18] (Equivalently, we could consider securities that promise to deliver the value of one unit of food and are collateralized by $1/\alpha$ units of housing.) We distinguish four regions; in each there is a unique equilibrium. In Region I, α is sufficiently small that Consumer 2 cannot borrow enough to buy all the housing at date 0, but buys the remaining housing in date 1. In Region II, α is large enough that Consumer 2 can buy all the housing at date 0 but small enough that she will be able to honor her promises in both states at date 1 and retain all the housing at date 1. In Region III, Consumer 2 will honor her promises but will not be able to retain all the housing. In Region IV, Consumer 2 will default. Finally, at the boundary of Regions II and III, prices and equilibrium consumptions are indeterminate. The calculations in Regions I, II are almost identical to those in Example 1; the calculations for Regions III, IV follow the same method with the appropriate changes to incorporate default. In all regions we can take the price of food to be 1 in every state $s = 0, 1, 2$.

- **Region I:** $\alpha \in [\mathbf{0}, \mathbf{2})$
  Consumers 1 and 2 both hold date 0 housing; Consumer 2 borrows as much as she can in date 0 and honors her promises in both states at date 1. In both states at date 1, Consumer 2 buys the remaining housing out of her remaining balance of food. Hence the price of housing in both states is 4, and the marginal utility of a dollar to each consumer is 1 in both states. It follows that $q_\alpha = \alpha$, and that $p_{0H} = 5$. The key equations are the marginal condition for Consumer 2 on the residual from the leveraged purchase of the house and the budget equation that says that all income not spent on food must be spent on housing:

$$
\frac{1/x_{0F}^2}{p_{0F}} = \frac{8-\alpha}{5-\alpha} \Rightarrow x_{0F}^2 = \frac{5-\alpha}{8-\alpha}
$$

$$
x_{0H}^2 = \frac{7/2 - \frac{5-\alpha}{8-\alpha} + \alpha x_{0H}^2}{5}
$$

[18] As before, the case $\alpha > 4$ reduces to the case $\alpha = 4$.

Solving yields

$$x^2_{0H} = 1 - \frac{7}{2(5-\alpha)} + \frac{1}{8-\alpha}$$

Hence equilibrium prices and consumptions are:

$$\begin{aligned}
p &= (1, 5; 1, 4; 1, 4) \\
x^1 &= \left(18 - \frac{5-\alpha}{8-\alpha}, 1 - x^2_{0H}\,; 13 + (\alpha-4)x^2_{0H}, 0\,; 13 + (\alpha-4)x^2_{0H}, 0\right) \\
x^2 &= \left(\frac{5-\alpha}{8-\alpha}, x^2_{0H}\,; 5 - (\alpha-4)x^2_{0H}, 1\,; -\frac{3}{2} - (\alpha-4)x^2_{0H}, 1\right)
\end{aligned}$$

(Lest date 1 food consumptions seem strange, remember that $(\alpha - 4) < 0$.) As α increases (the collateral requirement becomes less stringent) in this range, there is no effect on date 0 housing prices $p_{0H} = 5$ but Consumer 2 is able to borrow more which also gives her an incentive to shift date 0 consumption from food to housing, so her date 0 food consumption decreases and her date 0 housing consumption increases: $x^2_{0F} = 5/8, x^2_{0H} = 23/40$ when $\alpha = 0$; $x^2_{0F} = 1/2, x^2_{0H} = 1$ when $\alpha = 2$.

- **Region II:** $\alpha \in [\mathbf{2}, \mathbf{5/2})$
  Consumer 2 holds all the housing at both dates; Consumer 2 borrows as much as she can in date 0 and honors her promises in both states at date 1. Again the two key equations are:

$$\begin{aligned}
\frac{1/x^2_{0F}}{p_{0F}} &= \frac{8-\alpha}{p_{0H}-\alpha} \Rightarrow x^2_{0F} = \frac{p_{0H}-\alpha}{8-\alpha} \\
p_{0H} &= 7/2 - \frac{p_{0H}-\alpha}{8-\alpha} + \alpha
\end{aligned}$$

  Equilibrium prices and consumptions are:

$$\begin{aligned}
p &= \left(1, \alpha + \left(\frac{8-\alpha}{9-\alpha}\right)\left(\frac{7}{2}\right); 1, 4; 1, 4\right) \\
q_\alpha &= \alpha \\
x^1 &= \left(18 - \frac{7}{2(9-\alpha)}, 0; 9+\alpha, 0; 9+\alpha, 0\right) \\
x^2 &= \left(\frac{7}{2(9-\alpha)}, 1; 9-\alpha, 1; \frac{5}{2} - \alpha, 1\right)
\end{aligned}$$

  As α increases in this range, date 0 housing prices rise but Consumer 2 is able to borrow more; she is already holding all the housing at date 0 but can now consume more food as well: $x^2_{0F} = 1/2, x^2_{0H} = 1, p_{0H} = 5$ when $\alpha = 2$, $x^2_{0F} = 7/13, x^2_{0H} = 1, p_{0H} = 71/13$ in the limit as $\alpha \to 2.5$.

- **Boundary between Regions II, III: $\alpha = 5/2$**
Consumer 2 holds all the housing at both dates; Consumer 2 borrows as much as she can in date 0; Consumer 2 honors her promises in both states at date 1. In the bad state, Consumer 2 holds all the housing and no food so p_{2H} is indeterminate. This makes the crucial equations a bit more complicated, also because $\mu_2^2 = 4/p_{2H}$

$$\frac{1/x_{0F}^2}{p_{0F}} = \frac{4 + .5(4-\alpha) + .5(p_{2H}-\alpha)\frac{4}{p_{2H}}}{p_{0H}-\alpha} \Rightarrow x_{0F}^2 = \frac{p_{0H}-\alpha}{8-(.5+\frac{2}{p_{2H}})\alpha}$$
$$p_{0H} = 7/2 - \frac{p_{0H}-\alpha}{8-(.5+\frac{2}{p_{2H}})\alpha} + \alpha$$

This gives

$$p_{0H} = \alpha + \frac{7}{2}\frac{8-(.5+\frac{2}{p_{2H}})\alpha}{9-(.5+\frac{2}{p_{2H}})\alpha} = \frac{5}{2} + \frac{7}{2}\frac{27/4-\frac{5}{p_{2H}}}{31/4-\frac{5}{p_{2H}}}$$
$$x_{0F}^2 = \frac{7}{2}\frac{1}{9-(.5+\frac{2}{p_{2H}})\alpha} = \frac{7}{2}\frac{1}{31/4-\frac{5}{p_{2H}}}$$

Notice that as p_{2H} falls (in its indeterminate range), x_{0F}^2 rises and p_{0H} falls. Equilibrium prices and consumptions are:

$$p = \left(1, \frac{5}{2} + \left(\frac{7}{2}\right)\left(\frac{27p_{2H}-20}{31p_{2H}-20}\right); 1, 4; 1, p_{2H}\right)$$
$$p_{2H} \in [3,4]$$
$$q_\alpha = 5/2$$
$$x^1 = \left(18 - \frac{14p_{2H}}{31p_{2H}-20}, 0; \frac{23}{2}, 0; \frac{23}{2}, 0\right)$$
$$x^2 = \left(\frac{14p_{2H}}{31p_{2H}-20}, 1; \frac{13}{2}, 1; 0, 1\right)$$

When $\alpha = 5/2$ date 0 food consumption is indeterminate at equilibrium but all other consumptions are determinate. For the equilibrium with $p_{2H} = 4$ (which is the limit of equilibria as α converges to 5/2 from below), $x_{0F}^1 = 317/13, x_{0F}^2 = 7/13, p_{0H} = 71/13$; for the equilibrium with $p_{2H} = 3$ (which is the limit of equilibria as α converges to 5/2 from above), $x_{0F}^1 = 1272/73, x_{0F}^2 = 42/73, p_{0H} = 396/73$. Hence Consumer 1 is better off "just before" the crash and Consumer 2 is better off "just after" the crash. Total utility is higher "just after" the crash,

- **Region III:** $\alpha \in (\mathbf{\frac{5}{2}}, \mathbf{3}]$
  Consumer 2 holds all the housing at date 0; Consumer 2 borrows as much as she can in date 0; Consumer 2 honors her promises in the good state at date 1. In the bad state the price of housing falls to $p_{2H} = 3$; Consumer 2 (who has assets of endowment plus housing $= (5/2) + 3$) sells the house, repays her debt, and then buys all the housing she can afford at the price $p_{2H} = 3$; Consumer 1 holds the remaining housing. The crucial equations are now a bit more complicated because $p_{2H} = 3$ and $\mu_2^2 = 4/3$:
  $$\frac{1/x_{0F}^2}{p_{0F}} = \frac{4 + .5(4-\alpha) + .5(3-\alpha)\frac{4}{3}}{p_{0H} - \alpha} \implies x_{0F}^2 = \frac{p_{0H} - \alpha}{8 - \frac{7}{6}\alpha}$$
  $$p_{0H} = 7/2 - \frac{p_{0H} - \alpha}{8 - \frac{7}{6}\alpha} + \alpha$$
  Equilibrium prices and consumptions are:
  $$p = \left(1, \alpha + \left(\frac{7}{2}\right)\left(\frac{48 - 7\alpha}{54 - 7\alpha}\right); 1, 4, 1, 3\right)$$
  $$q_\alpha = \alpha$$
  $$x^1 = \left(18 - \frac{21}{54 - 7\alpha}, 0; 9 + \alpha, 0; \frac{23}{2}, \frac{2\alpha - 5}{6}\right)$$
  $$x^2 = \left(\frac{21}{54 - 7\alpha}, 1; 9 - \alpha, 1; 0, \frac{11 - 2\alpha}{6}\right)$$
  As α increases in this range, date 0 housing prices rise again but Consumer 2 is again able to borrow more and her consumption of date 0 food rises: $p_{0H} = 396/73, x_{0F}^2 = 42/73$ when $\alpha = \frac{5}{2}$, $p_{0H} = 129/22, x_{0F}^2 = 7/11$ when $\alpha = 3$.

- **Region IV** $\alpha \in (\mathbf{3}, \mathbf{4}]$
  Consumer 2 holds all the housing at date 0; Consumer 2 borrows as much as she can in date 0; Consumer 2 honors her promises in the good state at date 1. In the bad state the price of housing falls to $p_{2H} = 3$; Consumer 2, who has assets of endowment plus housing $= (5/2) + 3$, delivers the house (which is worth 3) instead of her promise (which is $\alpha > 3$), defaults on her debt, and then buys all the housing she can afford at the price $p_{2H} = 3$; Consumer 1 holds the remaining housing. Now the price of debt is $q_\alpha = (.5\alpha + .5(3))$ and the crucial equilibrium equations are
  $$\frac{1/x_{0F}^2}{p_{0F}} = \frac{4 + .5(4-\alpha) + .5(3-3)\frac{4}{3}}{p_{0H} - q_\alpha} \Rightarrow x_{0F}^2 = \frac{p_{0H} - 1.5 - \alpha/2}{6 - \alpha/2}$$
  $$p_{0H} = 7/2 - \frac{p_{0H} - 1.5 - \alpha/2}{6 - \alpha/2} + .5\alpha + 1.5 = 5 - \frac{p_{0H} - 1.5 - \alpha/2}{6 - \alpha/2} + \alpha/2$$

These give

$$
\begin{aligned}
(7-\alpha/2)(p_{0H}-\alpha/2) &= 5(6-\alpha/2)+1.5 = 63/2-5\alpha/2\\
p_{0H} &= \frac{63/2-5\alpha/2}{7-\alpha/2}+\alpha/2 = \frac{63-5\alpha}{14-\alpha}+\alpha/2\\
x^2_{0F} &= \frac{\frac{63-5\alpha}{14-\alpha}-1.5}{6-\alpha/2} = \frac{\frac{42-3.5\alpha}{14-\alpha}}{6-\alpha/2} = \frac{84-7\alpha}{(14-\alpha)(12-\alpha)} = \frac{7}{14-\alpha}
\end{aligned}
$$

Equilibrium prices and consumptions are:

$$
\begin{aligned}
p &= \left(1, \frac{\alpha}{2}+\left(\frac{63-5\alpha}{14-\alpha}\right); 1, 4; 1, 3\right)\\
q_\alpha &= \frac{\alpha+3}{2}\\
x^1 &= \left(\frac{29}{2}-\left(\frac{7}{14-\alpha}\right), 0; 9+\alpha, 0; \frac{23}{2}, \frac{1}{6}\right)\\
x^2 &= \left(\frac{7}{14-\alpha}, 1; 9-\alpha, 1; 0, \frac{5}{6}\right)
\end{aligned}
$$

After default, the date 0 housing price and Consumer 2's date 0 food consumption rise: $p_{0H} = 129/22, x^2_{0F} = 7/11$ when $\alpha = 3$ and $p_{0H} = 63/10, x^2_{0F} = 7/10$ when $\alpha = 4$.

We want to make two very important points about this example:

- As α rises past $\alpha = 5/2$ the price of housing in the bad state falls precipitously from 4 to 3: there is a *crash*. The crash occurs despite the fact that all agents are perfectly rational, have perfect foresight and hold the same beliefs: the low collateral requirement (equivalently low down payment requirement) provides incentive for consumers of type 2 to take on more debt than they can service. Strikingly, the crash occurs when $\alpha < 3$ – *before* consumers of type 2 default on their promises. Perfect foresight entails that the crash is rationally anticipated, so it leads to a sudden drop in the price of housing at date 0, from $71/13 = 5.46$ to $396/73 = 5.42$.

- As in Example 1 this is a transferable utility economy (Consumers 1 and 2 have constant and equal marginal utilities for food in the good state 1), so we may identify social welfare with the sum of individual utilities. The effects of collateral requirement on welfare are complicated; direct computation shows that there are a number of different regimes:

– $\mathbf{0} \leq \alpha < \mathbf{2}$: welfare of both types of consumers is increasing; social welfare is increasing

– $\mathbf{2} < \alpha \leq \mathbf{5/2}$: welfare of consumers of type 1 is increasing, welfare of consumers of type 2 is decreasing; social welfare is decreasing

– $\alpha = \mathbf{5/2}$: welfare of consumers of type 1 jumps down, welfare of consumers of type 2 jumps up; social welfare jumps up

– $\mathbf{5/2} < \alpha \leq \mathbf{3}$: welfare of consumers of type 1 is increasing, welfare of consumers of type 2 is decreasing; social welfare is decreasing

– $\mathbf{3} < \alpha \leq \mathbf{4}$: welfare of consumers of type 1 is increasing, welfare of consumers of type 2 is decreasing; social welfare is increasing

In particular, social welfare is higher after the crash than immediately before, and social welfare is higher after default than immediately before, so *collateral levels that lead to a perfectly foreseen crash or to perfectly foreseen default can be welfare enhancing.* It is thus hasty to presume that default and crashes are, all things considered, destructive to welfare.

In our framework, the set of securities available for trade is given exogenously, but the set actually traded is determined endogenously at equilibrium. Because the former set might be very large (conceptually, all conceivable securities) we can view the security market structure itself as determined by the action of the competitive market. As we see below, the result can be default at equilibrium (even when securities that do not lead to default are available for trade).

Example 3 (Which Securities are Traded?) We maintain the entire structure of Example 2, except that some arbitrary set $\{(A^j, c^j)\}$ of securities is available for trade. To be consistent with our framework, we assume the set of available securities is finite, but, at least conceptually, we might imagine that *all possible* securities are offered. Because only housing is durable, we assume that only housing is used as collateral; there is no loss in normalizing so that $c^j = \delta^{0H}$ for each j.

In this setting, only Consumer 2 will sell securities (borrow); as Theorem 1 showed, Consumer 2 will sell only that security which offers her the largest liquidity value. (If more than one security offers Consumer 2 the largest liquidity value, Consumer 2 might sell any or all of them.) To see this, suppose that at equilibrium Consumer 2 sells (A^j, c^j) but that $LV^2_{(A^k,c^k)} > LV^2_{(A^j,c^j)}$. Because both securities require the same collateral Consumer 2 could sell ε fewer shares of (A^j, c^j) and ε more shares of (A^k, c^k) without violating her collateral constraint. The definition of liquidity value means that this change would strictly improve Consumer 2's utility, which would contradict the requirement that Consumer 2 optimizes at equilibrium.

Because liquidity values depend on the equilibrium prices and consumptions, it is not in general possible to order *a priori* the liquidity values of given securities and hence to know which securities will be traded and which will not be. Instead, we analyze two particularly interesting scenarios.

Suppose first that mortgages with various promises – but no other securities – are available. As above, write $\mathbf{A}_\alpha = (\alpha p_{1F}, \alpha p_{2F}; \delta^{0H})$. We claim that only the mortgage with the greatest promise (not exceeding 4 – the maximum value of housing in date 1, hence the maximum delivery that will be made on any security) will be traded at equilibrium. To show this it suffices to show that if $\alpha < \beta \leq 4$ then $LV_\alpha = LV^2_{\mathbf{A}_\alpha} < LV^2_{\mathbf{A}_\beta} = LV_\beta$. To this end, we estimate marginal utilities of income μ^2_s in the various spots, then fundamental values, security prices, then liquidity values.

- In state 1, prices are $p_{1F} = 1, p_{1H} = 4$ so $\mu^2_1 = (1/2)1$. In state 2, prices are $p_{2F} = 1, p_{2H} \geq 3$ so $\mu^2_2 \leq (1/2)4/3$. We claim that $\mu^2_0 > 7/6$. To see this, note that μ^2_0 is at least as high as the maximum of marginal utility per dollar for food and marginal utility per dollar for housing. The former strictly exceeds $7/6$ unless $x^2_{0F} \geq 6/7$ and the latter weakly exceeds $8/5$ if $x^2_{0H} < 1$, in which case $p_{0H} = 5$. Hence to establish that $\mu^2_0 > 7/6$ it remains only to consider the case in which $x^2_{0F} \geq 6/7$ and $x^2_{0H} = 1$. In this case, we have

$$p_{0H} = p_{0H}x^2_{0H} \leq \frac{7}{2} + 4 - x^2_{0F} \leq \frac{15}{2} - \frac{6}{7} = \frac{93}{14}$$

  so that the marginal utility per dollar for housing is at least $8/(93/14) = 112/93 > 7/6$, as asserted.

- Because only Consumer 1 buys securities, security prices coincide with expected actual deliveries. Write α_s, β_s for the deliveries $\mathbf{A}_\alpha, \mathbf{A}_\beta$ in state s so prices are $q_\alpha = (1/2)(\alpha_1 + \alpha_2)$ and $q_\beta = (1/2)(\beta_1 + \beta_2)$.

- We have shown above that $\mu^2_0 > 7/6$, and that $\mu^2_1 = 1/2$ and $\mu^2_2 \leq 2/3$. Using these estimates and the definitions, we obtain

$$\begin{aligned}
\mu^2_0(LV_\beta - LV_\alpha) &= [\mu^2_0 q_\beta - (\mu^2_1\beta_1 + \mu^2_2\beta_2)] - [\mu^2_0 q_\alpha - (\mu^2_1\alpha_1 + \mu^2_2\alpha_2)] \\
&= [\mu^2_0(\beta_1 + \beta_2)/2 - (\mu^2_1\beta_1 + \mu^2_2\beta_2)] - [\mu^2_0(\alpha_1 + \alpha_2)/2 - (\mu^2_1\alpha_1 + \mu^2_2\alpha_2)] \\
&= [(\mu^2_0/2) - \mu^2_1][\beta_1 - \alpha_1] + [(\mu^2_0/2) - \mu^2_2][\beta_2 - \alpha_2] \\
&> (1/12)(\beta_1 - \alpha_1) - (1/12)(\beta_2 - \alpha_2)
\end{aligned}$$

Because the delivery on any security will be the minimum of its promise and the value of collateral it follows that $\beta_1 - \alpha_1 = \beta - \alpha$ and $\beta_2 - \alpha_2 \leq \beta - \alpha$. Hence $\mu_0^2(LV_\beta - LV_\alpha) > 0$, whence $LV_\beta - LV_\alpha > 0$, as asserted.

In particular, if $\mathbf{A}_4$ – the mortgage with the largest promise – is offered, then only this mortgage will be traded, *even though this leads to default in equilibrium.*

Now suppose that *all* possible securities are offered. We assert that in equilibrium only those securities (with collateral δ^{0H}) whose deliveries are 4 in state 1 and 3 in state 2 will actually be traded, and that equilibrium commodity prices and consumptions coincide with the equilibrium when only the security $\mathbf{A}_4$ above is traded. To see this fix an equilibrium. First suppose the security $\mathbf{B}$ is traded and that $\textsc{Del}(\mathbf{B}, 1, p) < 4$. Let $\mathbf{B}'$ be any security with the same collateral and state 2 promise (hence delivery) as $\mathbf{B}$, but which promises (hence delivers) 4 units of account in state 1. Arguing exactly as above, we see that $\mathbf{B}'$ offers Consumer 2 (the only seller of $\mathbf{B}$) a strictly greater liquidity value than does $\mathbf{B}$. Hence Consumer 2 would strictly prefer to sell $\mathbf{B}'$ rather than $\mathbf{B}$, which would be a contradiction. We conclude that if $\mathbf{B}$ is traded then $\textsc{Del}(\mathbf{B}, 1, p) = 4$. Now suppose two securities $\mathbf{B}, \mathbf{B}'$ are traded, that $\textsc{Del}(\mathbf{B}, 1, p) = \textsc{Del}(\mathbf{B}', 1, p) = 4$ but that $\beta_2 = \textsc{Del}(\mathbf{B}, 2, p) < \textsc{Del}(\mathbf{B}', 2, p) = \beta_2'$. Arguing as before, we see that Consumer 2's date 0 first-order conditions require:

$$\frac{8 - \frac{1}{2}(4) - \mu_2^2(\beta_2)}{p_{0H} - \frac{1}{2}(4 + \beta_2)} = \frac{1}{x_{0F}} = \frac{8 - \frac{1}{2}(4) - \mu_2^2(\beta_2')}{p_{0H} - \frac{1}{2}(4 + \beta_2')}$$

However, this would entail $\beta_2 = \beta_2'$ which would be a contradiction. We conclude that all securities traded at equilibrium deliver 4 in state 1 and some common $\beta \leq 4$ in state 2.

Finally, consider the magnitude of β. If $\beta > 3$ we could argue exactly as in Example 2 to show that default would occur in state 2, which would entail that $p_{2H} = 3$, and hence that actual delivery would be only 3 – a contradiction. If $\beta < 2.5$ we could argue as before to show that the equilibrium price $p_{2H} = 4$, and hence that any security whose delivery is 4 in state 1 and $\beta' \in (\beta, 3)$ in state 2 would offer Consumer 2 a greater liquidity value which would be a contradiction. If $2.5 \leq \beta < 3$ the argument is a little more delicate, in part because the price p_{2H} might be indeterminate in the interval $[3, 4]$, but the conclusion would be the same: any security whose delivery is 4 in state 1 and $\beta' \in (\beta, 3)$ in state 2 would offer Consumer 2 a greater liquidity value, which again would be a contradiction.

To see the last assertion, note first that if $2.5 \leq \beta < 3$ then Consumer 2's behavior is the same as in Example 2: at date 0 she borrows as much as she can to buy all the housing; in the good state she delivers the full promise of 4 from her endowment, and holds all the housing; in the bad state she delivers the promise of β from her endowment of 5/2 and the value p_{2H} of the housing she owns, and uses the remainder of her wealth to buy back as much housing as she can. If $\beta > 2.5$ she cannot buy back *all* the housing so Consumer 1 buys some of it, whence $p_{2H} = 3$; if $\beta = 2.5$ the price of housing is indeterminate: $p_{2H} \in [3, 4]$. As in Example 2, equilibrium is defined by Consumer 2's first order conditions and budget constraints; this yields the equations:

$$\begin{aligned}\frac{1}{x^2_{0F}} &= \frac{4 + (1/2)[(4-4) + (p_{2H} - \beta)(4/p_{2H})]}{p_{0H} - q} \\ p_{0F}x^2_{0F} + p_{0H}x^2_{0H} &= (7/2) + qx^2_{0H}\end{aligned}$$

where q is the security price. Since the security delivers 4 in the good state and β in the bad state its price is $q = (1/2)(4 + \beta)$. We have normalized $p_{0F} = 1$; consumer 2 buys all the housing at date 0 so $x^2_{0H} = 1$. Plugging all these in gives

$$\begin{aligned}\frac{1}{x^2_{0F}} &= \frac{6 - (2\beta/p_{2H})}{p_{0H} - (1/2)(4+\beta)} \\ x^2_{0F} + p_{0H} &= (7/2) + (1/2)(4+\beta)\end{aligned}$$

Solving the second equation for p_{0H}, plugging into the first equation and then solving yields

$$\begin{aligned}x^2_{0F} &= \frac{(7/2)}{7 - (2\beta/p_{2H})} \\ \frac{1}{x^2_{0F}} &= (2/7)[7 - (2\beta/p_{2H})]\end{aligned}$$

Since $\beta \in [2.5, 3)$ and $p_{2H} \in [3, 4]$ it follows $1/x^2_{0F} > 10/7$. Because $\mu^2_0 \geq 1/x^2_{0F}$ it follows that $\mu^2_0 > 10/7$ as well. Now we can argue as before to see that any security that delivers 4 in state 1 and $\beta' \in (\beta, 3)$ in state 2 would yield Consumer 2 a higher liquidity value; since this is a contradiction, we conclude that $\beta = 3$, as asserted.

We conclude that only those securities (with collateral δ^{0H}) whose deliveries are 4 in state 1 and 3 in state 2 will actually be traded. It follows immediately that equilibrium commodity prices and consumptions coincide with the equilibrium when only the security $\mathbf{A}_4$ (see Example 2) is traded.

Note that even though all possible securities are available, the set of securities that are traded at equilibrium is *endogenously incomplete*. Note in particular that Arrow securities are not traded, even though they are available, because they make extremely inefficient use of the available collateral.

7 Is the Security Market Efficient?

In the scenario analyzed in Example 3, the market chooses socially efficient promises – even though those promises may lead to default. Characterizing situations when the market does or does not choose socially efficient sets of securities – or at least Pareto undominated sets of securities – seems an important and difficult question, to which we do not know the answer. (Indeed, because multiple equilibria are possible, it is not entirely clear precisely how to formulate the question.) However, we can give an unambiguous answer about Pareto domination in at least one important case: if date 1 prices do not depend on the choices of securities.

Theorem 4 (Constrained Efficiency) *Every set of collateral equilibrium plans is Pareto undominated among all sets of plans that:*

(a) are socially feasible

(b) given date 0 decisions, respect each consumer's budget set at every state s at date 1 at the given *equilibrium prices*

(c) call for deliveries on securities that are the minimum of the promise and the value of collateral

Proof Let $\langle p, q, (x^i, \varphi^i, \psi^i)\rangle$ be an equilibrium, and suppose that $(\hat{x}^i, \hat{\varphi}^i, \hat{\psi}^i)$ is a family of plans meeting the given conditions that Pareto dominates the equilibrium set of plans. By assumption, all the alternative plans are feasible, meet the budget constraints at each state at date 1, and call for deliveries that are the minumum of promises and the value of collateral, Optimality of the equilibrium plans at prices p, q means, therefore, that *all* the alternative plans $(\hat{x}^i, \hat{\varphi}^i, \hat{\psi}^i)$ fail the budget constraints at date 0. Because the alternative set of plans is socially feasible, summing over consumers yields a contradiction. ■

A particular implication of constrained efficiency is that prohibiting trade in certain securities – for example, those that are leveraged above some threshold – cannot lead to a Pareto improvement if it does not lead to a change in date 1 prices. Put differently: among security structures that lead to the same date 1 prices, the market chooses efficiently. In an environment or model in which only one good is available for consumption at date 1 and the price of that good is taken to be 1, it is tautological that all security structures lead to the same date 1 prices, so in that situation collateral equilibrium will *always* be constrained efficient and the market will *always* choose the security structure efficiently; compare [Kilenthong, 2006].[19]

8 Conclusion

Collateral requirements are almost omnipresent in modern economies, but the effects of these collateral requirements have received little attention except in circumstances where there is actual default. This paper argues that collateral requirements have important effects on every aspect of the economy – even when there is no default. Collateral requirements inhibit lending, limit borrowing, and distort consumption decisions.

When all borrowing must be collateralized, the supply of collateral becomes an important financial constraint. If collateral is in short supply the necessity of using collateral to back promises creates incentives to create collateral and to stretch existing collateral. The state can (effectively) create collateral by issuing bonds that can be used as collateral and by promulgating law and regulation that make it easier to seize goods used as collateral.[20] The market's attempts to stretch collateral have driven much of the financial engineering that has rapidly accelerated over the last three-and-a-half decades (beginning with the introduction of mortgage-backed securities in the early 1970's) and that has been designed specifically to stretch collateral by making it possible for the same collateral to be used several times: allowing agents to collateralize their promises with other agents' promises (pyramiding) and allowing the same collateral to back many different promises (tranching). These two innovations are at the bottom of the securitization and derivatives boom on Wall Street, and have greatly expanded the scope of financial markets. We address many of these issues in a companion paper [Geanakoplos and Zame, 2010] that expands the model presented here to allow for pyramiding, pooling and tranching. That work characterizes those efficient allocations that can be supported in equilibrium when financial innovation is possible but borrowing must be collateralized; a central finding is that robust inefficiency is an inescapable possibility.

The model offered here abstracts away from transaction costs, informational asym-

[19] Note that constrained efficiency continues to hold even if we add the possibility of instantaneous production, as discussed in Subsection 4.2. Thus although the market may lead investors to produce technologically inferior goods – because of their collateral value – this production will be in the social interest, given the set of possible goods that can be used as collateral, and provided that period 1 prices are not affected.

[20] Similarly, state regulations concerning seizure can have an enormous influence on bankruptcies; see [Lin and White, 2001] and [Fay et al., 2002] for instance.

metries and many other frictions that play an important role in real markets. It also restricts attention to a two-date world, and so does not address issues such as default at intermediate dates. All these are important questions for later work.

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Appendix: Proof of Theorem 1

Proof In constructing an equilibrium for $\mathcal{E} = ((e^i, u^i), \mathcal{A})$, we must confront the possibility that security promises, hence deliveries, may be 0 at some commodity spot prices.[21] (An option to buy gold at \$400/ounce will yield 0 if the the spot price of gold is below \$400/ounce.) Because of this, the argument is a bit delicate. We construct, for each $\rho > 0$, an auxiliary economy $\mathcal{E}^\rho$ in which security promises are bounded below by ρ; in these auxiliary economies, equilibrium security prices will be different from 0. We then construct an equilibrium for $\mathcal{E}$ by taking limits as $\rho \to 0$.

For each $s = 0, 1, \ldots, S$, choose and fix an arbitary *price level* $\beta_s > 0$. (Because promises are functions of prices, choosing price levels is not the same thing as choosing price normalizations, and we *do not* assert that equilibrium is independent of the price levels — only that for every set of price levels there exists an equilibrium.) Write $\mathbf{1}_0 = (1, \ldots, 1) \in R^L_+$ and define:

$$\begin{aligned}
\Delta_s &= \{(p_{s\ell}) \in \mathbb{R}^L_{++} : \sum p_{s\ell} = \beta_s\} \\
\Delta &= \Delta_0 \times \ldots \times \Delta_S \\
\mathbb{Q} &= \{q \in \mathbb{R}^J_+ : 0 \le q^j \le 2\beta_0 \mathbf{1}_0 \cdot c^j\}
\end{aligned}$$

For each $\rho > 0$, define an security $(A^{\rho j}, c^j)$ whose promise is $A^{\rho j} = A^j + \rho$. Let $\mathcal{A}^\rho = \{(A^{\rho 1}, c^1) \ldots, (A^{\rho J}, c^J)\}$. Define the auxiliary economy $\mathcal{E}^\rho = \langle (e^i, u^i), \mathcal{A}^\rho \rangle$, so $\mathcal{E}^\rho$ differs from $\mathcal{E}$ only in that security promises have been increased by ρ in every state and for all spot prices. We construct equilibria (for the auxiliary economies and then for our original economy) with commodity prices in Δ and security prices in $\mathbb{Q}$.

We first construct truncated budget sets and demand and excess demand correspondences in this auxiliary economy. By assumption, collateral requirements for each security are non-zero. Choose a constant μ so large that $\mu c^j \not\le \bar{e}_0$ for each j. (Thus, to sell μ units of the security $\mathbf{A}^{\rho j}$ would require more collateral than is actually available to the entire economy.) For each $(p, q) \in \Delta \times \mathbb{Q}$ and each consumer i, define the truncated budget set and the individual truncated demand correspondence

$$\begin{aligned}
B^i_0(p, q) &= \{\pi \in B^i(p, q, e^i \tilde{\mathcal{A}}^\rho) : 0 \le \varphi^i_j \le \mu I \ , \ 0 \le \psi^i_j \le \mu I \text{ for each } j \\
d^i(p, q) &= \{\pi = (x, \varphi, \psi) \in B^i_0(p, q) : \pi \text{ is utility optimal in } B^i_0(p, q)\}
\end{aligned}$$

(Note that truncated demand exists at every price (p, q), because we bound security purchases and sales. Absent such a bound, demands would certainly be undefined at some prices. For instance, if $q^j = 2\beta_0 \mathbf{1}_0 \cdot c^j$, agents could sell $A^{\rho j}$ for enough to

[21] If collateral requirements are not zero and promises are not 0 then deliveries cannot be 0 either.

finance the purchase of its collateral requirement c^j, so there would be an unlimited arbitrage. Bounding security sales bounds the arbitrage.) Write

$$D(p,q) = \sum d^i(p,q)$$

for the aggregate demand correspondence.

For each plan π, we define security excess demand and commodity excess demands $z_s(\pi)$ in each spot:

$$z_a(\pi) = \varphi - \psi \ ; \ z_s(\pi) = x_s - \bar{e}_s$$

Write $z(\pi) = (z_0(\pi), \ldots, z_S(\pi); z_a(\pi)) \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J$, and define the aggregate excess demand correspondence

$$Z : \Delta \times \mathbb{Q} \to \mathbb{R}^{L(1+S)} \times \mathbb{R}^J \ ; \ Z(p,q) = z(D(p,q))$$

It is easily checked that $Z(p,q)$ is non-empty, compact, and convex for each p, q and that the correspondence Z is upper hemi-continuous. Because consumptions security sales are bounded, Z is also bounded below. Because utility functions are monotone, a familiar argument [Debreu, 1959] shows that Z satisfies the usual boundary condition:

$$||Z(p,q)|| \to \infty \text{ as } (p,q) \to \text{bdy}\Delta \times \mathbb{Q}$$

(It doesn't matter which norm we use.)

Now fix $\varepsilon > 0$, and set

$$\Delta^\varepsilon = \{p \in \Delta : p_{s\ell} \geq \varepsilon \text{ for each } s, \ell\}$$

Because Z is an upper hemi-continuous correspondence, it is bounded on $\Delta^\varepsilon \times \mathbb{Q}$; set

$$\mathbb{Z}^\varepsilon = \{z \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J : ||z|| \leq \sup_{(p,q) \in \Delta^\varepsilon \times Q} ||Z(p,q)||\}$$

Define the correspondence

$$\begin{aligned} &F^\varepsilon : \Delta^\varepsilon \times \mathbb{Q} \times \mathbb{Z}^\varepsilon \to \Delta^\varepsilon \times Q \times \mathbb{Z}^\varepsilon \\ &F^\varepsilon(p,q,z) = \text{argmax}\,\{(p^*,q^*) \cdot z : (p^*,q^*) \in \Delta^\varepsilon \times Q\} \times Z(p,q) \end{aligned}$$

For prices $(p,q) \in \Delta \times \mathbb{Q}$ and a vector of excess demands $z \in \mathbb{R}^{L(1+S)} \times \mathbb{R}^J$, $(p,q) \cdot z$ is the value of excess demands. We caution the reader that, in this setting, Walras' law need not hold for arbitrary prices: the value of excess demand need not be 0. We shall see, however, that the value of excess demand *is* 0 at the prices we identify as candidate equilibrium prices

Our construction guarantees that F^ε is an upper-hemicontinuous correspondence, with non-empty, compact convex values. Kakutani's theorem guarantees that F^ε has a fixed point. We assert that for some $\varepsilon_0 > 0$ sufficiently small, the correspondences $F^\varepsilon, 0 < \varepsilon < \varepsilon_0$ have a *common fixed point*. To see this, write $G^\varepsilon \subset \Delta^\varepsilon \times Q \times \mathbb{Z}^\varepsilon$ for the set of all fixed points of F^ε; G^ε is a non-empty compact set. We show that for some $\varepsilon_0 > 0$ sufficiently small, the sets G^ε are nested and decrease as ε decreases; that is, $G^{\varepsilon_1} \subset G^{\varepsilon_2}$ whenever $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$.

To see this, note first that security deliveries are bounded, because deliveries never exceed the value of collateral. Hence individual expenditures at budget feasible plans (and in particular at plans in the truncated demand set) are bounded, independent of prices (because income from endowments is bounded, security prices and sales are bounded, and security purchases and deliveries are bounded). Choose an upper bound $M > 0$ on individual expenditures at budget feasible plans. Because commodity demands are non-negative, individual excess demands are bounded below; choose a lower bound $-R < 0$ on individual exess demands.

Because excess demand is the sum of individual demands less the sum of endowments, it follows that if $z \in Z(p,q)$ then

$$(p,q)\cdot z \le MI \quad \text{and} \quad z_{s\ell} \ge -RI \text{ for each commodity } s\ell$$

A familiar argument (based on strict monotonicity of preferences) shows that if commodity prices tend to the boundary of Δ then aggregate commodity excess demand blows up. If the price of some security tends to 0 but the value of its collateral does not, then deliveries on that security do not tend to 0, whence demand for that security and consequent aggregate commodity excess demand again blow up. Hence we can find $\varepsilon_0 > 0$ such that if $(p,q) \in \Delta \times \mathbb{Q}$, $z \in Z(p,q)$, and $p_{s_0\ell_0} < \varepsilon_0$ for some spot s_0 and commodity ℓ_0 then there is some spot s_1 and commodity ℓ_1 such that

$$z_{s_1\ell_1} > \frac{1}{\beta_{s_1} - (L-1)\varepsilon}\left[MI + \varepsilon R(L-1) + (\max_s \beta_s)RI\right] \tag{13}$$

We assert that if $0 < \varepsilon < \varepsilon_0$ then $G^\varepsilon \subset \Delta^{\varepsilon_0} \times \mathbb{Q} \times \mathbb{Z}^{\varepsilon_0}$. To see this, suppose that $(p,q,z) \in G^e p$ and $p \notin \Delta^{\varepsilon_0}$. Define $\tilde{p} \in \Delta$ by

$$\tilde{p}_{s\ell} = \begin{cases} \varepsilon & \text{if } s = s_0, \ell \ne \ell_0 \\ \beta_s - (L-1)\varepsilon & \text{if } s = s_0, \ell = \ell_0 \\ \beta_s/L & \text{otherwise} \end{cases}$$

Direct calculation using equation (13) shows that $(\tilde{p}, 0)\cdot z > MI$, which is a contradiction. We conclude that $p \in \Delta^{\varepsilon_0}$ and hence that $(p,q,z) \in G_0^\varepsilon$ as desired.

The definition of F^ε implies that if $0 < \varepsilon_1 < \varepsilon_2$ and $G^{\varepsilon_1} \subset \Delta^{\varepsilon_2} \times \mathbb{Q} \times \mathbb{Z}^{\varepsilon_2}$ then $G^{\varepsilon_1} \subset G^{\varepsilon_2}$. Hence, for $0 < \varepsilon < \varepsilon_0$ the sets G^ε are nested and decrease as ε decreases.

A nested family of non-empty compact sets has a non-empty intersection so we may define the non-empty set G:

$$G = \bigcap_{\varepsilon < \varepsilon_0} G^\varepsilon$$

Let $(p, q, z) \in G$; we assert that $z = 0$ and that p, q constitute equilibrium prices for the economy $\mathcal{E}^\rho$.

We first show that excess security demand $z_a^j = 0$ for each j. If $z_a^j > 0$, the requirement that (p, q) maximize the value of excess demand would imply that q_j is as big as possible: $q_j = 2\beta_0 \mathbf{1}_0 \cdot c^j$. But then agents could sell $A^{\rho j}$ for enough to finance the purchase of the collateral requirement, whence the excess demand for $A^{\rho j}$ would be negative, a contradiction. We conclude that security excess demand must be non-positive. If the excess demand for security j were strictly negative, the requirement that (p, q) maximize the value of excess demand would imply that q_j is as small as possible: $q_j = 0$. But if the price of $A^{\rho j}$ were 0 then every agent would wish to buy it because its delivery would be $\min\{\rho, p_s \cdot F_s(c^j)\} > 0$. Hence the excess demand for A_j^ρ must be positive, a contradiction.[22] We conclude that $z_a = 0$.

We claim that Walras' law holds: $(p, q) \cdot z = 0$. To see this, choose individual demands $\pi^i \in d^i(p, q)$ for which the corresponding aggregate excess demand is z: $Z(\sum \pi^i) = z$. For each agent i, the plan π^i lies in the budget set at prices (p, q), so the date 0 expenditure required to carry out the plan π^i is no greater than the value of date 0 endowment. Because utility is strictly monotone in date 0 perishable commodities and in all commodities in state s, optimization implies that all individuals spend all their income at date 0, so we conclude that the date 0 expenditure required to carry out the plan π^i is precisely equal to the value of date 0 endowment; i.e., the value of date 0 excess demand is 0 for each individual. Summing over all individuals shows that the value of date 0 aggregate excess demand is 0: $p_0 \cdot z_0 + q \cdot z_a = 0$. Now consider any state $s \geq 1$ at date 1. We can argue exactly as above to conclude that the value of each individual's excess demand is equal to the net of deliveries on purchases and sales of securities. Thus, the value of aggregate excess demand in state s is the net of deliveries on aggregate purchases and sales of securities. However, $z_a = 0$ so aggregate purchases and sales of securities are equal, and so the value of aggregate excess demand in state s is 0. Summing over all spots we conclude that $(p, q) \cdot z = 0$, as asserted.

We show next that $z = 0$. If not, Walras' law entails that excess demand for some

[22] Note that we could not obtain this conclusion in the original economy, because, at the prices (p, q) the security A^j might promise 0 in every state.

commodity is positive; say $z_{s_0\ell_0} > 0$. Define commodity prices $\tilde{p}$ by:

$$\tilde{p}_{s\ell} = \begin{cases} p_{s\ell} & \text{if} \quad s \neq s_0 \\ \varepsilon & \text{if} \quad s = s_0, \ell \neq \ell_0 \\ 1 - (L-1)\varepsilon & \text{if} \quad s = s_0, \ell = \ell_0 \end{cases}$$

Because $(p, q) \cdot z = 0$ and $z_{s_0\ell_0} > 0$, $(\tilde{p}, q) \cdot z$ will be strictly positive if ε is small enough. However, this would contradict our assumption that $(p, q, z) \in G$ and hence is a fixed point of F^ε for *every* sufficiently small ε. We conclude that $z = 0$. Hence $\langle p, q, (\pi^i)\rangle$ is an equilibrium for the economy $\mathcal{E}^\rho$.

It remains to construct an equilibrium for the original economy $\mathcal{E}$. To this end, let $p(\rho), q(\rho), (\pi^i(\rho))$ be equilibrium prices and plans for $\mathcal{E}^\rho$ and let $\rho \to 0$. By construction, prices and plans lie in bounded sets, so we may choose a sequence $(\rho_n) \to 0$ for which the corresponding prices and plans converge; let the limits be $p, q, (\pi^i)$. Commodity prices p do not lie on the boundary of Δ (for otherwise the excess demands at prices $p(\rho_n), q(\rho_n)$ would be unbounded, rather than 0). It follows that $\pi^i(\rho)$ is utility optimal in consumer i's budget set at prices (p, q). Because the collection of plans (π^i) is the limit of collections of socially feasible plans, it follows that they are socially feasible and hence that the artificial bounds on security purchases and sales do not bind at the prices p, q. Hence $\langle p, q, (\pi^i)\rangle$ is an equilibrium for $\mathcal{E}$. ■