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Asset prices, debt constraints and inefficiency ☆

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Abstract

We consider (possibly non-stationary) economies with endogenous solvency constraints under uncertainty over an infinite horizon, as in Alvarez and Jermann (2000) [5]. A sort of Cass Criterion (Cass, 1972 [10]) completely characterizes constrained inefficiency under the hypothesis of uniform gains from risk-sharing (which is always satisfied in stationary economies when the autarchy is constrained inefficient). Uniform gains from risk-sharing also guarantee a finite value of the intertemporal aggregate endowment at a constrained optimum. Hence, no equilibrium exhibits a null interest rate in the long run. Finally, constrained inefficiency occurs if and only if there exists a feasible redistribution producing a welfare improvement at all contingencies.

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1. Introduction

Debt limits prevent the economy from attaining a social optimum because individuals are unable to exploit all trading opportunities and disparities in subjective evaluations of risk persist at equilibrium. The interesting issue is whether competitive equilibria are constrained optimal, that is, whether there may be benefits from redistributions under the condition that debt constraints cannot be removed. This notion of optimality is particularly relevant when debt limits are endogenous, since, most likely, policy intervention fails in sidestepping the incentive constraints from which debt limits arise. The purpose of this work is to verify whether, at a competitive equilibrium, the mere observation of prices completely reveals constrained inefficiency.

In a recent literature, debt constraints are self-enforcing under limited commitment, *i.e.*, they are generated by reservation utilities (the value of debt repudiation for individuals). Alvarez and Jermann [5], following Kehoe and Levine [17] and Kocherlakota [19], postulate that debt repudiation induces a permanent exclusion from financial markets, though individuals maintain labor incomes. Their notion of not-too-tight debt constraints corresponds to participation constraints at reservation utilities ensuring that individuals would not benefit, at every contingency, from reverting permanently to autarchy. Within this framework, a natural notion of *constrained inefficiency* is given by a feasible welfare improvement subject to participation constraints at autarchic reservation utilities. A hypothetical planner, thus, is restricted by constraints analogous to those inducing market incompleteness.

Inspired by Cass [10], we adopt the view that a failure of optimality is to be revealed by observable prices alone, not relying on any direct knowledge of preferences or of subjective evaluations of risk. We show that, provided that gains from risk-sharing are uniformly positive at the autarchy, constrained inefficiency occurs if and only if *low implied interest rates* prevail at equilibrium, that is, when interest rates (net of growth) are recurrently and sufficiently negative conditionally on some non-negligible event. The intuition is straightforward at steady states: in every period of trade, a planner might reduce current consumption of an unconstrained individual for an equal compensation in the following period of trade; when the rate of interest is strictly negative, as the marginal rate of substitution coincides with gross interest rate, this reallocation yields an increase in the welfare of this individual; and, as this balanced redistribution can be continued indefinitely over the infinite horizon of the economy, it proves a failure of constrained optimality. Under uncertainty and when interest rates fluctuate over time, a Modified Cass Criterion serves to precisely identify the domain of low implied interest rates. It corresponds to an extension of Aiyagari and Peled's [2] Dominant Root Characterization to non-stationary allocations. In its more transparent formulation, the Modified Cass Criterion requires the existence of a sequence of bounded positive transfers of commodities, $\{v_t\}$, satisfying, for some ρ in $(0, 1)$,

$$\rho \mathbb{E}_t m_{t,t+1} v_{t+1} \geqslant v_t,$$

where $\{m_{t,t+1}\}$ is the sequence of stochastic discount factors commonly used for asset pricing in macroeconomics. The value of the parameter ρ might be interpreted as an upper bound on the average safe (gross) interest rate prevailing in the long period conditionally on some non-negligible event, whereas the identified transfers yield a sequential welfare improvement when redistributed across unconstrained individuals. Thus, when constrained efficiency fails, prices retain all the information about relevant welfare improving feasible redistributions. Instead, in order to produce a welfare improvement at a constrained optimum, the reallocation of risk need necessarily depend on unobservable marginal evaluations of individuals.

Sufficiency of the Modified Cass Criterion for constrained inefficiency obtains without additional restrictions on fundamentals. Necessity, instead, is established under the hypothesis of *uniform gains from risk-sharing* at the autarchy. This requires the existence of a feasible welfare improving redistribution of risk, subject to participation, notwithstanding a partial uniform destruction of aggregate resources. The logic underlying this strengthening of constrained inefficiency is to avoid borderline cases: the Modified Cass Criterion uncovers directions for first-order feasible welfare improvements; (unconstrained) individuals evaluate intertemporal transfers at market interest rates, whereas feasibility requires a null shadow interest rate; when a null interest rate prevails over the long run, first-order welfare evaluation remains ambiguous. It is worth remarking that our method allows us to show that, in stationary economies, any constrained inefficient autarchic allocation exhibits uniform gains from risk-sharing. Thus, necessity of the Modified Cass Criterion emerges without any additional restriction on fundamentals in stationary economies.

In stationary economies, Alvarez and Jermann [5] also provide a characterization of constrained optimality. The method consists in exploiting a sort of equivalence with non-sequential equilibria of Kehoe and Levine [17]. They show that *high implied interest rates* (*i.e.*, a finite present value of intertemporal aggregate endowment) are sufficient and, to some extent, necessary for constrained optimality. In particular, necessity is obtained by means of Kehoe and Levine's [17] Second Welfare Theorem. This delivers a non-sequential quasi-equilibrium for some supporting abstract Arrow–Debreu prices. Whenever supporting Arrow–Debreu prices are strictly positive, a quasi-equilibrium is, as a matter of fact, an equilibrium and high implied interest rates obtain. Kehoe and Levine [17, Lemma 3] prove that this holds in some circumstances (sufficiently productive assets in positive net supply), whereas Alvarez and Jermann [5, Proposition 4.10] identify an alternative condition (some risk-sharing is possible).

We find that, in general, high implied interest rates are not necessary for constrained efficiency. In fact, in a non-stationary economy, we provide an example of a non-autarchic constrained efficient allocation exhibiting a null interest rate and, hence, violating high implied interest rates. However, we also show that, when the autarchy admits uniform gains from risk-sharing, high implied interest rates are necessary for constrained efficiency. Therefore, under the domain of uniform gains from risk-sharing, constrained efficiency is fully characterized by high implied interest rates and constrained inefficiency by low implied interest rates. Consistently, no equilibrium exhibits a null interest rate over the long run. We again remark that, in stationary economies, any constrained inefficient autarchic allocation involves uniform gains from risk-sharing.

The Modified Cass Criterion maintains an autonomous role in identifying welfare improving policies and in revealing a stronger characterization of constrained inefficiency under uniform gains from risk-sharing. Indeed, constrained inefficiency occurs if and only if there exists a feasible redistribution of risk that (weakly) increases utilities with respect to planned consumptions at all date-events. In other terms, the sort of inefficiency uncovered by the Modified Cass Criterion does not depend on the reservation utilities that individuals can guarantee themselves under default (private values of debt repudiation). When the Modified Cass Criterion is satisfied, the planner might sequentially, or *ex post*, distribute utility gains independently of initial endowments. This might be interpreted as a sort of failure of social transversality.

The essay is organized as follows. In Section 2, we present the basic assumptions on fundamentals. In Section 3, we introduce the notions of constrained inefficiency, equilibrium and price support, and we define uniform gains from risk-sharing. In Section 4, we provide the characterization of constrained inefficiency in terms of equilibrium prices (Modified Cass Criterion) and we prove the equivalence between constrained inefficiency and *ex-post* inefficiency. In Section 5, we compare our analysis with the one in Alvarez and Jermann [5] and, in particular, we establish the necessity of high implied interest rates for a constrained optimum. In Section 6, we provide some concluding remarks and discuss possible extensions. In Appendix A, we show that, in a stationary economy, any constrained inefficient autarchic allocation involves uniform gains from risk-sharing. In Appendix B, in a non-stationary economy, we provide an example of a constrained efficient allocation satisfying Alvarez and Jermann's [5] hypothesis of gains from risk-sharing and violating high implied interest rates. All proofs are collected in Appendix C.

2. Fundamentals

2.1. Time and uncertainty

Time and uncertainty are represented by an event-tree $\mathcal{S}$, a countably infinite set, endowed with ordering $\succcurlyeq$. For a date-event σ in $\mathcal{S}$, $t(\sigma)$ in $\mathcal{T}=\{0,1,2,\ldots,t,\ldots\}$ denotes its date and

$$\sigma_+=\bigl\{\tau\in\mathcal{S}(\sigma)\colon\ t(\tau)=t(\sigma)+1\bigr\}$$

is the non-empty finite set of all immediate direct successors, where

$$\mathcal{S}(\sigma)=\{\tau\in\mathcal{S}\colon\ \tau\succcurlyeq\sigma\}.$$

The initial date-event is ϕ in $\mathcal{S}$, with $t(\phi)=0$, that is, $\sigma\succcurlyeq\phi$ for every σ in $\mathcal{S}$. This construction is canonical (Debreu [13, Chapter 7]).

2.2. Vector space notation and terminology

We basically adhere to Aliprantis and Border [3, Chapters 5–8] for notation and terminology concerning vector spaces. Consider the vector space of all real maps on $\mathcal{S}$, $\mathbb{R}^{\mathcal{S}}$, endowed with the canonical (product) ordering. An element v of $\mathbb{R}^{\mathcal{S}}$ is positive, strictly positive and uniformly strictly positive if, respectively, for every σ in $\mathcal{S}$, $v_\sigma\geqslant 0$, $v_\sigma>0$ and $v_\sigma\geqslant\epsilon>0$. For a positive element v of $\mathbb{R}^{\mathcal{S}}$, we simply write $v\geqslant 0$ and, when v in $\mathbb{R}^{\mathcal{S}}$ is non-null, $v>0$. For an element v of $\mathbb{R}^{\mathcal{S}}$, v^+ in $\mathbb{R}^{\mathcal{S}}$ and v^- in $\mathbb{R}^{\mathcal{S}}$ are, respectively, its positive part and its negative part, so that $v=v^+-v^-$ in $\mathbb{R}^{\mathcal{S}}$ and $|v|=v^++v^-$ in $\mathbb{R}^{\mathcal{S}}$. Also, for an arbitrary collection $\{v^j\}_{j\in\mathcal{J}}$ of elements of $\mathbb{R}^{\mathcal{S}}$, its supremum and its infimum in $\mathbb{R}^{\mathcal{S}}$, if they exist, are denoted, respectively, by

$$\bigvee_{j\in\mathcal{J}}v^j\quad\text{and}\quad\bigwedge_{j\in\mathcal{J}}v^j.$$

Finally, the positive cone of any vector subspace F of $\mathbb{R}^{\mathcal{S}}$ is $\{v\in F\colon\ v\geqslant 0\}$.

2.3. Commodity space

There exists a single commodity that is traded and consumed at every date-event. The (*reduced*) *commodity space* is L, the (Riesz) vector space of all *bounded* real maps on $\mathcal{S}$. The vector space L is endowed with the supremum norm given by

$$\|v\|=\inf\bigl\{\lambda>0\colon\ |v|\leqslant\lambda u\bigr\},$$

where here u denotes the unit of L. Notice that, as far as the aggregate endowment is bounded, the restriction to a reduced commodity space only serves to simplify our presentation. Furthermore, growth could be straightforwardly encompassed in our analysis by strengthening the hypotheses on preferences.

2.4. Preferences

There is a finite set $\mathcal{J}$ of individuals. For every individual i in $\mathcal{J}$, the *consumption space* X^i is the positive cone of L. A consumption plan x^i in X^i is *interior* if it is uniformly strictly positive. Though more general preferences can be encompassed in our analysis, it simplifies to assume time additively separable utilities. Preferences of individual i in $\mathcal{J}$ on X^i are represented by

$$U^i(x^i) = \sum_{\sigma \in \mathcal{S}} \pi^i_\sigma u^i(x^i_\sigma),$$

where π^i is a strictly positive element of $\mathbb{R}^{\mathcal{S}}$, such that $\sum_{\sigma \in \mathcal{S}} \pi^i_\sigma$ is finite, and $u^i : \mathbb{R}_+ \to \mathbb{R}$ is a bounded, smooth, smoothly strictly increasing and smoothly strictly concave per-period utility function. For every date-event σ in $\mathcal{S}$, let

$$U^i_\sigma(x^i) = \frac{1}{\pi^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} \pi^i_\tau u^i(x^i_\tau).$$

This is the continuation utility beginning from date-event σ in $\mathcal{S}$.

2.5. Uniform impatience

We assume that there exists a sufficiently small $1 > \eta > 0$ satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\pi^i_\sigma \geqslant \eta \sum_{\tau \in \mathcal{S}(\sigma)} \pi^i_\tau.$$

This hypothesis imposes uniform impatience across individuals at interior consumption plans (see, for instance, Levine and Zame [21, Assumption 5] or Santos and Woodford [25, Assumption 2]).

2.6. Allocations

An *allocation* x is an element of $X = \prod_{i \in \mathcal{J}} X^i$. An allocation x in X is interior if, for every individual i in $\mathcal{J}$, the consumption plan x^i in X^i is interior. The hypothesis of interiority is stronger than necessary and is maintained only to simplify the presentation.

2.7. Subjective prices

For an individual i in $\mathcal{J}$, at an interior consumption plan x^i in X^i, the *subjective price* p^i is an element of P^i, the set of all strictly positive elements p^i of $\mathbb{R}^{\mathcal{S}}$ such that $\sum_{\sigma \in \mathcal{S}} p^i_\sigma$ is finite, satisfying, at every consumption plan z^i in X^i,

$$U^i(z^i) - U^i(x^i) \leqslant \sum_{\sigma \in \mathcal{S}} p^i_\sigma (z^i_\sigma - x^i_\sigma).$$

Subjective prices exist under the maintained hypotheses on preferences at interior consumption plans. Indeed, for every individual i in $\mathcal{J}$,

$$(p^i_\sigma)_{\sigma \in \mathcal{S}} = (\pi^i_\sigma \partial u^i(x^i_\sigma))_{\sigma \in \mathcal{S}}.$$

3. Efficiency, equilibrium and prices

3.1. Constrained inefficiency

Constrained inefficiency occurs when there exists a welfare improving feasible redistribution subject to sequential participation constraints, that is, guaranteeing the autarchic utility to individuals beginning from every date-event. This notion of constrained inefficiency was introduced by Kehoe and Levine [17] and adopted by Alvarez and Jermann [5].

Given an allocation e in X, we define the set $X^*(e)$ of all allocations x in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U_\sigma^i(x^i) \geqslant U_\sigma^i(e^i).$$

This is the set of allocations guaranteeing given reservation utilities to individuals across date-events. Also, we consider the set $X(e)$ containing all allocations x in $X^*(e)$ satisfying

$$\sum_{i\in\mathcal{J}} x^i = \sum_{i\in\mathcal{J}} e^i.$$

This is the set of all allocations satisfying participation and exhausting aggregate resources. An allocation x in $X(e)$ is *constrained inefficient* if it is Pareto dominated by an allocation z in $X(e)$.

3.2. Uniform gains from risk-sharing

The autarchic allocation is constrained inefficient when a feasible risk-sharing yields a welfare improvement without violating participation constraints. In part of the analysis, we need to postulate that such utility gains are uniformly positive. Formally, an allocation e in X admits *uniform gains from risk-sharing* if there exists an alternative allocation y in $X^*(e)$ satisfying, for some $1 > \epsilon > 0$,

$$\sum_{i\in\mathcal{J}} y^i \leqslant (1-\epsilon)\sum_{i\in\mathcal{J}} e^i.$$

Thus, a feasible welfare improvement obtains, subject to participation, even though a constant share of the aggregate endowment is destroyed.

In a tradition of general equilibrium (Debreu [12]), a measure of inefficiency is given by the *coefficient of resource utilization*, that is, by the largest share of the aggregate endowment whose destruction is consistent with a feasible welfare improving redistribution. Uniform gains from risk-sharing basically require that Debreu's measure of inefficiency be positive at the autarchy. The precise role of this restriction will be clarified in the following analysis. A positive Debreu's measure, in a sense, serves to capture a robust form of inefficiency.

3.3. Price support

Trade occurs sequentially. Individuals participate into *complete* financial markets subject to debt constraints. More precisely, private liabilities cannot exceed some predefined quantitative limits. To the purpose of simplification, we here only retain equilibrium restrictions in terms of price support, that is, of optimality of consumption plans subject to budget constraints for *some* initial endowment of resources and *some* debt limits. Consistently, financial investments are to be interpreted as *net* variations with respect to equilibrium positions and debt limits correspond to *unexhausted* debt opportunities at equilibrium.

A price p is an element of P, the set of all strictly positive elements of $\mathbb{R}^{\mathcal{S}}$. It represents present-value prices of contingent consumption. For every individual i in $\mathcal{J}$, debt constraints f^i are an element of F^i, the set of all positive elements of $\mathbb{R}^{\mathcal{S}}$, and a financial plan v^i is an element of V^i, the set of all elements of $\mathbb{R}^{S}$ vanishing at the initial date-event (that is, $v^i_\phi = 0$, as initial claims, or liabilities, are inherited from the unrepresented past and cannot be varied). The budget set $B^i_p(x^i, f^i)$ contains all consumptions plans z^i in X^i satisfying, for some financial plan v^i in V^i, at every date-event σ in $\mathcal{S}$, budget constraint,

$$\sum_{\tau\in\sigma_+} p_\tau v^i_\tau + p_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leqslant p_\sigma v^i_\sigma,$$

and debt constraints,

$$-\left(v^i_\tau + f^i_\tau\right)_{\tau\in\sigma_+} \leqslant 0.$$

Hence, the budget set identifies all consumption plans that are affordable, subject to budget restrictions, by varying financial positions without violating debt limits.

Across individuals, debt constraints are f in F. Debt constraints f in F are *non-trivial* if, at every date-event σ in $\mathcal{S}$, $\sum_{i\in\mathcal{J}} f^i_\sigma > 0$. An allocation x in X is (*non-trivially*) *supported* by price p in P at (non-trivial) debt constraints f in F if, for every individual i in $\mathcal{J}$, the consumption plan x^i in X^i is U^i-optimal in the budget constrain $B_p(x^i, f^i)$. Non-trivial support requires that, at every date-event, some individual is allowed to borrow (*i.e.*, to reduce savings), so ruling out fundamentally autarchic equilibria.

Price support admits a first-order characterization based on elementary variational arguments, as in Alvarez and Jermann [5]. First, for every individual, the subjective evaluation of transfers at succeeding date-events cannot exceed their market evaluation (*). Second, whenever an individual is allowed to borrow against income at some succeeding date-event, subjective and market evaluations need coincide (**).

Lemma 1 *(First-order conditions). An interior allocation x in X is supported by price p in P at debt constraints f in F only if, at every date-event σ in $\mathcal{S}$,*

$$\bigvee_{i\in\mathcal{J}}\left(\frac{p^i_\tau}{p^i_\sigma}\right)_{\tau\in\sigma_+} \leqslant \left(\frac{p_\tau}{p_\sigma}\right)_{\tau\in\sigma_+} \tag{*}$$

and

$$\sum_{\tau\in\sigma_+}\left(\frac{p^i_\tau}{p^i_\sigma}\right) f^i_\tau = \sum_{\tau\in\sigma_+}\left(\frac{p_\tau}{p_\sigma}\right) f^i_\tau, \tag{**}$$

where, for every individual i in $\mathcal{J}$, p^i in P^i is the subjective price at interior consumption plan x^i in X^i. Furthermore, an interior allocation x in X is supported by price p in P at bounded *debt constraints f in F if, at every date-event σ in $\mathcal{S}$, conditions* (*)–(**) *are satisfied.*

In Alvarez and Jermann [5], debt constraints reflect solvency requirements. Namely, every individual is allowed to borrow up to the point at which the continuation utility is equal to her reservation utility (at the corresponding date-event). More formally, given an autarchic allocation e in X, an allocation x in $X(e)$ is (*Alvarez–Jermann*) *supported* by price p in P if it is supported by price p in P at debt constraints f in F satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U^i_\sigma\left(x^i\right) - U^i_\sigma\left(e^i\right) > 0 \quad \text{only if } f^i_\sigma > 0.$$

The underlying logic of this notion is that, whenever subjective welfare exceeds autarchic utility at some date-event, debt constraints allow for borrowing at that date-event, that is, for a (locally) unrestricted participation into financial markets.

4. Cass Criterion

We here provide a characterization of constrained inefficiency in terms of supporting prices. In particular, we exploit a Modified Cass Criterion, exactly as in economies of overlapping generations. This is a variation of the original criterion proposed by Cass [10] for capital theory and extended by Chattopadhyay and Gottardi [11] to stochastic overlapping generations economies. It was initially introduced by Demange and Laroque [14] and by Bloise and Calciano [7] for stochastic overlapping generations economies (they also discuss alternative equivalent formulations and compare with the previous literature).

Formally, a price p in P fulfills the *Modified Cass Criterion* if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every date-event σ in $\mathcal{S}$,

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geqslant p_\sigma v_\sigma.$$

Similarly, it fulfills the *Weak Modified Cass Criterion* if there exists a non-null positive element v of L satisfying, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau \geqslant p_\sigma v_\sigma.$$

The Modified Cass Criterion represents an extension of Aiyagari and Peled's [2] Dominant Root Characterization for stationary allocations to non-stationary allocations. At a non-stochastic steady state, it simply requires a negative (net) rate of interest.

Prices fulfilling the Modified Cass Criterion might be regarded as involving *low implied interest rates*. Prices exhibit *high implied interest rates*, according to the terminology borne out by Alvarez and Jermann [5], when they are summable, that is, when

$$\sum_{\sigma \in \mathcal{S}} p_\sigma \text{ is finite.}$$

Clearly, high interest rates are inconsistent with the (Weak) Modified Cass Criterion. Finally, prices involve *neither high nor low implied interest rates* when they neither satisfy the Modified Cass Criterion (though they do satisfy the Weak Modified Cass Criterion) nor are summable. The latter circumstance reveals a null interest rate in the long period and corresponds to a *golden rule* in the terminology for overlapping generations economies. High interest rates, in turn, guarantee a finite pricing of all intertemporal consumption profiles, so preserving the duality between commodity and price spaces. As our characterization of inefficiency exploits low implied interest rates, prices are consistent with an efficient allocation of resources even when not involving high interest rates and, hence, when inducing an infinite value of the aggregate endowment.

We begin with proving sufficiency of the Modified Cass Criterion.

Lemma 2 *(Sufficiency). Given an autarchic allocation e in X, an interior allocation x in $X(e)$, with non-trivially supporting price p in P, is constrained inefficient if price p in P satisfies the Modified Cass Criterion.*

The logic underlying welfare improvement is extremely simple. The Modified Cass Criterion identifies transfers. At every date-event, a positive amount of resources might be taken from an unconstrained individual for a compensation at the immediately following date-events. As such an individual is unconstrained (at the following date-events), the subjective (first-order) evaluation of this variation in consumptions coincides with the market evaluation. Hence, by the Modified Cass Criterion, transfers yield a welfare improvement. As this adjustment policy might be continued indefinitely over the infinite horizon, it produces a feasible welfare improvement. Indeed, a uniform $1 > \rho > 0$ over date-events guarantees that first-order welfare benefits translate into utility gains. In particular, at every date-event, this redistribution (weakly) increases utilities with respect to planned consumptions and, hence, with respect to autarchy.

Necessity of the Modified Cass Criterion is stated in the following lemma.

Lemma 3 *(Necessity). Given an autarchic allocation e in X, an interior allocation x in $X(e)$, with supporting price p in P, is constrained inefficient only if price p in P satisfies the Weak Modified Cass Criterion. Furthermore, given an autarchic allocation e in X, admitting uniform gains from risk-sharing, an interior allocation x in $X(e)$, with supporting price p in P, is constrained inefficient only if price p in P satisfies the Modified Cass Criterion.*

Necessity obtains by means of an extremely transparent argument. For every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau\in\sigma_+} p_\tau^i v_\tau^i + p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right) = p_\sigma^i v_\sigma^i.$$

Here, v^i in L represents the subjectively evaluated (first-order) benefit, in terms of current consumption, from the redistribution. For an individual i in $\mathcal{J}$, v^i in L need not be positive at all date-events, though it is positive at the initial date-event (indeed, at some non-initial date-event, subjective welfare might fall below utility at planned consumptions). However, notice that, when an individual is constrained in transferring resources at a date-event, consistent debt constraints ensure that the individual will positively benefit, with respect to planned consumptions, from the redistribution at that date-event. Hence, exploiting the fact that subjective evaluation cannot exceed market evaluation, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau\in\sigma_+} p_\tau v_\tau^i + p_\sigma\left(z_\sigma^i - x_\sigma^i\right) \geqslant p_\sigma v_\sigma^i.$$

Only market prices appear in this inequality. Aggregating across individuals,

$$\sum_{\tau\in\sigma_+} p_\tau\left(\sum_{i\in\mathcal{J}} v_\tau^i\right) + p_\sigma\sum_{i\in\mathcal{J}}\left(z_\sigma^i - x_\sigma^i\right) \geqslant p_\sigma\left(\sum_{i\in\mathcal{J}} v_\sigma^i\right).$$

Feasibility proves the claim, as a constant share of aggregate resources remains undistributed under uniform gains from risk-sharing.

The above lemmas establish a complete characterization of constrained inefficiency under the hypothesis of uniform gains from risk-sharing at the autarchy. In particular, constrained inefficiency corresponds to low implied interest rates. As uniform gains from risk-sharing play an important role, we show that this hypothesis is necessarily satisfied in *stationary* economies (that is, in economies where uncertainty is represented as a Markov chain over a finite state space with strictly positive transition probabilities) when the autarchy is constrained inefficient (see Appendix A).

Proposition 1 *(Modified Cass Criterion). Given an autarchic allocation e in X, admitting uniform gains from risk-sharing, an interior allocation x in $X(e)$, with non-trivially supporting price p in P, is constrained inefficient if and only if price p in P satisfies the Modified Cass Criterion.*

It is worth observing that, as a matter of fact, the Modified Cass Criterion allows for a stronger characterization of constrained inefficiency. Indeed, constrained inefficiency occurs if and only if there exists a feasible redistribution that (weakly) increases utilities with respect to planned consumptions at all date-events. A hypothetical planner might sequentially, or *ex post*, redistribute utility gains independently of initial endowments. Formally, an allocation x in X is *ex-post* inefficient if it is Pareto dominated by an alternative allocation z in $X(x)$, where $X(x)$ denotes the set of feasible allocations when allocation x in X is regarded as the *status quo*. Intuitively, *ex-post* inefficiency might be interpreted as defining a failure of social transversality.

Proposition 2 *(Violation of transversality). Given an autarchic allocation e in X, admitting uniform gains from risk-sharing, an interior allocation x in $X(e)$, with non-trivially supporting price p in P, is constrained inefficient if and only if it is* ex-post *inefficient.*

A relevant implication of this coincidence between constrained and *ex-post* inefficiency is that minimal information is needed for the implementation of welfare improving policies. Indeed, equilibrium prices reveal some beneficial feasible redistributions without requiring any precise knowledge of preferences, consumptions and endowments (and, in particular, of utilities at the autarchy): it is only necessary to separate constrained from unconstrained individuals. Furthermore, these redistributions are immune to renegotiations: individuals might not profit from *ex-post* recontracting the terms of participation, when financial markets remain open.

5. High implied interest rates

It is natural to compare our characterization of constrained inefficiency with that provided by Alvarez and Jermann [5]. Indeed, they show, on the one side, that every equilibrium allocation involving high implied interest rates is constrained efficient [5, Corollary 4.7] and, on the other side, that every non-autarchic constrained efficient allocation involves high implied interest rates [5, Proposition 4.10]. Therefore, according to Alvarez and Jermann [5], *high implied interest rates* fully characterize non-autarchic constrained efficiency. Our analysis, instead, demonstrates that a violation of high implied interest rates is, in general, consistent with constrained efficiency. We provide an example of a non-autarchic constrained efficient equilibrium exhibiting a constant null interest rate (Appendix B). This example requires non-stationary initial endowments, which might be of interest for applications to the sustainability of sovereign debt, when some countries face a decline, or a deindustrialization, and some other countries an expansion, or an industrialization; or to the diversification of risk induced by temporary shocks, vanishing in the long run with positive probability. The analysis in this section is devoted to clarify the necessity of high implied interest rates at constrained efficient allocations.

A well-established tradition in general equilibrium shows that efficiency is equivalent to the existence of supporting positive linear functionals. This is also the approach in Kehoe and Levine [17], exploited by Alvarez and Jermann [5]. Define Φ as the set of all non-vanishing (continuous) positive linear functionals φ on L. Given an autarchic allocation e in X, an allocation x

in $X(e)$ is *supported by a positive linear functional* φ in Φ if, for every allocation z in $X^*(e)$ that Pareto dominates allocation x in $X(e)$,

$$\sum_{i\in\mathcal{J}}\varphi\left(z^i-x^i\right)\geqslant 0,$$

where $X^*(e)$ contains all allocations z in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U^i_\sigma\left(z^i\right)\geqslant U^i_\sigma\left(e^i\right).$$

Support by linear functionals corresponds to a quasi-equilibrium (with transfers) in Kehoe and Levine's [17] non-sequential representation of trades. Any positive linear functional φ in Φ admits a unique decomposition as $\varphi=\varphi^f+\varphi^b$, where φ^f is a positive order-continuous linear functional on L and φ^b is a purely-additive linear functional on L. The first part admits a sequential representation, that is, for every v in L,

$$\varphi^f(v)=\sum_{\sigma\in\mathcal{S}}\varphi^f_\sigma v_\sigma,$$

where $(\varphi^f_\sigma)_{\sigma\in\mathcal{S}}$ is an element of $\mathbb{R}^{\mathcal{S}}$ such that $\sum_{\sigma\in\mathcal{S}}|\varphi^f_\sigma|$ is finite. The second part is a sort of bubble component vanishing on all elements v of L such that $\{\sigma\in\mathcal{S}\colon |v_\sigma|>0\}$ is a finite subset of $\mathcal{S}$. A linear functional φ in Φ is strictly positive if, for every non-null positive element v on L, $\varphi(v)>0$. A Kehoe and Levine's [17] equilibrium (with transfers) obtains if and only if the supporting linear functional is strictly positive (and, typically, the bubble component vanishes).

We here show that support by positive linear functionals is a vacuous property when gains from risk-sharing are not uniform at the autarchy. Indeed, any feasible allocation is supported, independently of constrained efficiency. Clearly, at a constrained inefficient allocation, supporting linear functionals are not strictly positive and involve some non-negligible bubble component. Hence, they do not correspond to prices.

Lemma 4 *(Failure of the Second Welfare Theorem). Given an autarchic allocation e in X, violating uniform gains from risk-sharing, any allocation x in $X(e)$ is supported by a positive linear functional φ in Φ.*

When the autarchy admits uniform gains from risk-sharing, only constrained efficient allocations are supported by linear functionals. Furthermore, the supporting linear functional admits a sequential representation and, hence, corresponds to prices.

Lemma 5 *(Validity of the Second Welfare Theorem). Given an autarchic allocation e in X, admitting uniform gains from risk-sharing, an allocation x in $X(e)$ is supported by a positive linear functional φ in Φ if and only if it is constrained efficient. Furthermore, at an interior allocation x in $X(e)$, any of such supporting linear functionals is strictly positive and order-continuous.*

The established Second Welfare Theorem allows for a stricter characterization of constrained efficiency under uniform gains from risk-sharing at the autarchy. On the one side, by the Modified Cass Criterion, if an allocation is supported by prices involving high implied interest rates, it cannot be constrained inefficient. On the other side, by the Second Welfare Theorem, any constrained efficient allocation is supported by prices exhibiting high implied interest rates. Hence, high implied interest rates obtain if and only if equilibrium is constrained efficient.

Proposition 3 *(High implied interest rates). Given an autarchic allocation e in X, admitting uniform gains from risk-sharing, an interior allocation x in $X(e)$, with non-trivially supporting price p in P, is constrained efficient if and only if price p in P exhibits high implied interest rates.*

6. Conclusion and extensions

Our contribution is three-fold. First, we show that a failure of constrained optimality in the allocation of resources is completely revealed by observable prices, without requiring any precise knowledge of fundamentals (consumption plans, endowments, preferences). Inefficiency corresponds to low implied interest rates as captured by the Modified Cass Criterion. Furthermore, this criterion exploits properties of the stochastic discount factor, commonly used in macroeconomic theory, and is suitable for empirical studies (along the lines, for instance, of Abel, Mankiw, Summers and Zeckhauser [1] and Barbie, Hagedorn and Kaul [6]). Second, we prove that constrained inefficiency coincides with a feasible recursive welfare improvement independently of private values of debt repudiation, that is, with a feasible redistribution yielding (weakly) higher utilities at all contingencies along the infinite horizon of the economy. This might be interpreted as a failure of social transversality. Third, we clarify the necessity of high implied interest rates (*i.e.*, a positive interest rate over the long period) at a constrained efficient equilibrium.

A natural generalization of limited enforcement of debt contracts can be encompassed in our analysis by postulating that default guarantees individuals some given state-contingent reservation utilities. Different specifications of such reservation utilities induce different private values of debt repudiation. Indeed, the punishment for default might depend on legislation, conventions, bargaining power and other institutional aspects (see Hellwig and Lorenzoni [15], Bulow and Rogoff [9], Phelan [24], Kiyotaki and Moore [18], Lustig [22,23] and Krueger and Uhlig [20] as examples of different formulations of outside values for borrowers). In all these instances, a natural notion of constrained inefficiency is given by a feasible welfare improvement subject to participation constraints at reservation utilities sustaining endogenous debt limits. A hypothetical planner, in other terms, is restricted by constraints analogous to those inducing restrictions to trades in financial markets. Admittedly, this form of constrained optimum is unsatisfactory to some extents, as it is *conditional* on reservation utilities that might be *endogenously* determined at equilibrium (as they would also be in Kehoe and Levine [17] and Alvarez and Jermann [5] with multiple commodities). Thus, constrained efficient equilibria might still be Pareto ranked because of pecuniary externalities. This additional source of inefficiency, however, is not in general revealed by the mere observation of prices and, in order to be ascertained, requires a precise knowledge of fundamentals.

Our method based on the Modified Cass Criterion allows for a characterization of (robust forms of) constrained inefficiency in all these economies with limited enforcement of debt contracts (Bloise and Reichlin [8]). Also, it permits to establish a coincidence of constrained inefficiency (conditional on reservation utilities which might be endogenously determined at equilibrium) with *ex-post* inefficiency (independent of reservation utilities). This coincidence is of some interest, as *ex-post* inefficiency might regarded as a prudential criterion. Under limited commitment, reservation utilities vary from autarchy (the most severe punishment, when private endowment cannot be confiscated) to welfare evaluated at planned consumptions (the most lenient punishment). They might not be completely or unambiguously identified by legislation or by some public signals, as individuals might have different opportunities of renegotiating debt contracts or obtaining loans from other financial intermediaries after default. *Ex-post* inefficiency does not rely on a precise knowledge of reservation utilities of individuals: a welfare improvement occurs under the most lenient punishment for debt repudiation and, hence, under the most severe participation constraints for the planner.

Appendix A. Stationarity

We here restrict attention to stationary economies, where uncertainty is represented by a simple Markov process on a finite state space S. To simplify notation, we assume that there exists a map $c: \mathcal{S} \to S$ such that, for every date-event σ in $\mathcal{S}$,

$$c(\sigma_+) = S.$$

In other terms, at every date-event σ in $\mathcal{S}$, $c(\sigma)$ is the current state in S and any other state in S might be reached (with strictly positive probability) in the following period. Fundamentals are *measurable* with respect to the finite state space S. In particular, it follows that the stationary autarchic allocation e in X is non-trivially supported by a price p in P fulfilling, for some strictly positive matrix Q in $\mathbb{R}^{S \times S}$, at every date-event σ in $\mathcal{S}$,

$$(p_\tau)_{\tau \in \sigma_+} = (Q_{(c(\sigma), c(\tau))})_{\tau \in \sigma_+} p_\sigma.$$

Here, $Q_{(s,\bar{s})}$ is the price in current state s in S for the delivery of one unit of account in state $\bar{s}$ in S in the following period.

Proposition 4 *(Stationary economies). Under stationarity, an interior autarchic allocation e in X admits uniform gains from risk-sharing if and only if it is constrained inefficient.*

We need to show that a constrained inefficient autarchic allocation exhibits uniform gains from risk-sharing. The proof is developed in several steps. We preliminarily demonstrate that, at a constrained optimum, utility gains are uniformly positive.

Claim 1. *There exists $\epsilon > 0$ satisfying, for every constrained efficient allocation z in $X(e)$, at every date-event σ in $\mathcal{S}$,*

$$\bigvee_{i \in \mathcal{J}} U_\sigma^i(z^i) - U_\sigma^i(e^i) \geqslant \epsilon.$$

Proof. Given welfare weights θ in Θ, for every date-event σ in $\mathcal{S}$, consider the value function

$$g_\sigma(\theta) = \max_{z \in X(e)} \sum_{i \in \mathcal{J}} \theta^i \left(U_\sigma^i(z^i) - U_\sigma^i(e^i)\right),$$

where

$$\Theta = \left\{\theta \in \mathbb{R}_+^{\mathcal{J}}: \sum_{i \in \mathcal{J}} \theta^i = 1\right\}.$$

By stationarity, as a matter of fact, the value function is measurable with respect to states s in S. As states are recurrent and the value function is continuous in θ in Θ, it is straightforward to verify that there exists $\epsilon > 0$ such that, at every date-event σ in $\mathcal{S}$,

$$\min_{\theta \in \Theta} g_\sigma(\theta) \geqslant \epsilon.$$

Furthermore, by the Principle of Optimality, every constrained efficient allocation z in $X(e)$ satisfies, at every date-event σ in $\mathcal{S}$,

$$g_\sigma(\theta) \geqslant \sum_{i \in \mathcal{J}} \theta^i \big(U^i_\sigma\big(z^i\big) - U^i_\sigma\big(e^i\big)\big)$$

and, for some welfare weights θ in Θ,

$$g_\sigma(\theta) = \sum_{i \in \mathcal{J}} \theta^i \big(U^i_\sigma\big(z^i\big) - U^i_\sigma\big(e^i\big)\big).$$

Supposing that the claim is false, then there exists a constrained efficient allocation z in $X(e)$ such that, at some date-event σ in $\mathcal{S}$, for every individual i in $\mathcal{J}$,

$$U^i_\sigma\big(z^i\big) - U^i_\sigma\big(e^i\big) < \epsilon.$$

Thus, for some welfare weights θ in Θ,

$$\epsilon \leqslant g_\sigma(\theta) = \sum_{i \in \mathcal{J}} \theta^i \big(U^i_\sigma\big(z^i\big) - U^i_\sigma\big(e^i\big)\big) < \epsilon,$$

which is a contradiction. □

Claim 2. *There exist a uniformly strictly positive element v of L and a non-null positive element w of L satisfying, for some sufficiently small $\lambda > 0$, at every date-event σ in $\mathcal{S}$,*

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau \geqslant v_\sigma + w_\sigma \tag{A.1}$$

and

$$\frac{1}{p_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau w_\tau \geqslant \lambda. \tag{A.2}$$

Proof. Preliminarily observe that, for consumptions varying in an interior compact interval of $\mathbb{R}_+$, there exists a sufficiently small $\mu > 0$ satisfying, for every individual i in $\mathcal{J}$,

$$u^i\big(c^*\big) - u^i(c) \leqslant \partial u^i(c)\big(c^* - c\big) - \mu \partial u^i(c)\big|c^* - c\big|^2.$$

This shows a sort of quadratic concavity of utility. Also, by Claim 1, exploiting convexity of preferences, at no loss of generality, it can be assumed that there exists an interior allocation z in $X(e)$ such that, at every date-event σ in $\mathcal{S}$,

$$\bigvee_{i \in \mathcal{J}} U^i_\sigma\big(z^i\big) - U^i_\sigma\big(e^i\big) \geqslant \epsilon.$$

Thus, utility gains are uniformly positive in the aggregate.

For every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$, define

$$v^i_\sigma = \frac{1}{p^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p^i_\tau\big(z^i_\tau - e^i_\tau\big) - \mu \frac{1}{p^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p^i_\tau\big|z^i_\tau - e^i_\tau\big|^2.$$

Notice that, for every individual i in $\mathcal{J}$, v^i is a positive element of L. Indeed, this follows from feasibility, the hypothesis of uniform impatience and quadratic concavity. Define $v = \sum_{i \in \mathcal{J}} v^i$, an element itself of L, and observe that, by uniform positivity of utility gains and quadratic concavity, $v = \sum_{i \in \mathcal{J}} v^i$ is a uniformly strictly positive element of L. Finally, define the non-null positive element w of L by setting, at every date-event σ in $\mathcal{S}$,

$$w_\sigma = \mu \sum_{i \in \mathcal{J}} \left| z_\sigma^i - e_\sigma^i \right|^2.$$

To prove restriction (A.2), assume that the left-hand side is finite, for otherwise the claim would be trivially satisfied. Notice that, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left| z_\tau^i - e_\tau^i \right| \geqslant v_\sigma^i.$$

Also, by Hölder's Inequality (see Aliprantis and Border [3, Proposition 13.2] and Shannon and Zame [26] for an analogous argument),

$$\left(\frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left| z_\tau^i - e_\tau^i \right|^2 \right) \left(\frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \right) \geqslant \left(\frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left| z_\tau^i - e_\tau^i \right| \right)^2.$$

By uniform impatience and bounded derivatives, for some sufficiently small $\Delta_* > 0$ and some sufficiently large $\Delta^* > 0$,

$$\frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left| z_\tau^i - e_\tau^i \right|^2 \geqslant \eta \left(\frac{\Delta_*}{\Delta^*} \right) \left(\frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left| z_\tau^i - e_\tau^i \right| \right)^2 \geqslant \eta \left(\frac{\Delta_*}{\Delta^*} \right) \left(v_\sigma^i \right)^2,$$

where $1 > \eta > 0$ is given by the assumption of uniform impatience. Using the fact that subjective prices are dominated by market prices,

$$\frac{1}{p_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau \left| z_\tau^i - e_\tau^i \right|^2 \geqslant \eta \left(\frac{\Delta_*}{\Delta^*} \right) \left(v_\sigma^i \right)^2.$$

Aggregating across individuals, by uniformly strict positivity of v in L, this proves the validity of restriction (A.2) for some sufficiently small $\lambda > 0$.

For every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma^i} \sum_{\tau \in \sigma_+} p_\tau^i v_\tau^i + \left(z_\sigma^i - e_\sigma^i \right) \geqslant v_\sigma^i + \mu \left| z_\sigma^i - e_\sigma^i \right|^2.$$

By the first-order characterization of supporting price p in P, as v^i is a positive element of L,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^i + \left(z_\sigma^i - e_\sigma^i \right) \geqslant v_\sigma^i + \mu \left| z_\sigma^i - e_\sigma^i \right|^2.$$

Aggregating across individuals, by feasibility, this proves the validity of restriction (A.1). □

We now exploit the characterization provided by Claim 2 in order to prove that the Perron–Frobenius eigenvalue of the strictly positive matrix Q in $\mathbb{R}^{S \times S}$ is larger than the unity. It is well established that this eigenvalue coincides with the spectral radius of Q in $\mathbb{R}^{S \times S}$,

$$r(Q) = \lim_{n \in \mathbb{N}} \sqrt[n]{\left\| Q^n \right\|_\infty},$$

where $\mathbb{R}^S$ is endowed with the supremum norm. Equally well known is that

$$\lim_{n\in\mathbb{N}}\left(\frac{1}{r(Q)}\right)^n \|Q^n\|_\infty \text{ is finite.}$$

See, for instance, Horn and Johnson [16, Theorem 8.2.8].

Claim 3. *The spectral radius, $r(Q)$, of the strictly positive matrix Q in $\mathbb{R}^{S\times S}$ is larger than the unity.*

Proof. To simplify notation, we introduce the positive linear operator $T : L \to L$ that is defined, at every date-event σ in $\mathcal{S}$, by

$$T(v)_\sigma = \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau.$$

This operator, being positive, is norm-continuous. Also, it can be directly verified that it is order-continuous. Restrictions in Claim 2 might be equivalently expressed as

$$T(v) \geqslant v + w \tag{A.3}$$

and

$$\sum_{m=0}^{\infty} T^m(w) \geqslant \lambda u, \tag{A.4}$$

where u is the unitary element of L. Notice that the norms of the considered positive linear operators, for every n in $\mathbb{N}$, fulfill $\|T^n\| = \|T^n(u)\|$ and $\|Q^n\|_\infty = \|Q^n(\vec{u})\|_\infty$, where u is the unit of L and $\vec{u}$ is the unit of $\mathbb{R}^S$ (see, for instance, Aliprantis and Burkinshaw [4, Exercise 2 at p. 270]). Also, by direct computation, it can be verified that, at every n in $\mathbb{N}$, $\|T^n\| = \|Q^n\|_\infty$.

Suppose that the left-hand side of (A.4) converges, that is, as all terms in the infinite sum are positive elements of L,

$$\sum_{m=0}^{\infty} T^m(w) = \bigvee_{n=0}^{\infty}\sum_{m=0}^{n} T^m(w) \text{ is an element of } L.$$

By positivity of the linear operator $T : L \to L$, for every n in $\mathbb{N}$,

$$\sum_{m=0}^{\infty} T^{n+m}(w) \geqslant \lambda T^n(u).$$

However, by inequality (A.3),

$$\|v\| T^n(u) \geqslant T^n(v) \geqslant v + \sum_{m=0}^{n-1} T^m(w) \geqslant \sum_{m=0}^{n-1} T^m(w).$$

The two inequalities produce a contradiction, as the remainders of the series are larger than or equal to a constant share of its partial sums,

$$\sum_{m=0}^{\infty} T^{n+m}(w) \geqslant \left(\frac{\lambda}{\|v\|}\right)\sum_{m=0}^{n-1} T^m(w).$$

Therefore, noticing that

$$\|T^n\| \geqslant \|T^n(u)\| \geqslant \left\|T^n\left(\frac{v}{\|v\|}\right)\right\| \geqslant \left\|\sum_{m=0}^{n-1} T^m\left(\frac{w}{\|v\|}\right)\right\|,$$

the sequence $\{\|T^n\|\}_{n\in\mathbb{N}}$ diverges and so does the sequence $\{\|Q^n\|_\infty\}_{n\in\mathbb{N}}$. Therefore, by Perron–Frobenius Theorem, $r(Q) > 1$. □

By Perron–Frobenius Theorem (*e.g.*, Horn and Johnson [16, Theorem 8.2.2]), the strictly positive matrix Q in $\mathbb{R}^{S\times S}$ admits a strictly positive eigenvector $\vec{v}$ in $\mathbb{R}^S$ satisfying

$$Q(\vec{v}) = r(Q)\vec{v}.$$

This, reproducing the proof of Lemma 2, suffices to show that uniform gains from risk-sharing exist.

Claim 4. *There exist uniform gains from risk-sharing.*

Proof. It suffices to observe that there exists a uniformly strictly positive element v of L satisfying, for some sufficiently large $1 > \rho > 0$ and some sufficiently small $1 > \epsilon > 0$, at every date-event σ in $\mathcal{S}$,

$$(1-\epsilon)\rho \sum_{\tau\in\sigma_+} p_\tau v_\tau \geqslant p_\sigma v_\sigma.$$

This is, indeed, obtained by setting, at every date-event σ in $\mathcal{S}$, $v_\sigma = \vec{v}_{c(\sigma)}$, jointly with

$$\frac{1}{r(Q)} = (1-\epsilon)\rho.$$

Thus, by the proof of Lemma 2, this suffices to yield the result. □

Appendix B. Example

In this appendix, we provide an example of a constrained efficient allocation, according to Alvarez and Jermann [5], violating the hypothesis of high implied interest rates. In particular, a null interest rate sustains a stationary allocation as non-autarchic equilibrium at not-too-tight debt constraints. Initial endowments are non-stationary and are constructed so as to approach the equilibrium stationary allocation in the long period. Non-stationarity of the economy is necessary for a non-autarchic constrained optimum not to involve high interest rates, as shown in Appendix A.

Before presenting the example, we shall produce necessary conditions for constrained inefficiency. To simplify, we shall assume that there is no uncertainty, that is, $\mathcal{S}$ can be identified with $\mathcal{T}$; also, that there is a common discount factor, $1 > \delta > 0$, and that the common per-period utility function $u:\mathbb{R}_+ \to \mathbb{R}$ is smooth on $\mathbb{R}_+$ (that is, to be precise, it can be extended as a twice continuously differentiable function on some open set containing $\mathbb{R}_+$); finally, that $u'(1) < \delta u'(0)$. Recall that, given an initial allocation e in X, an interior allocation x in $X(e)$ is supported by price p in P if, for every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p_{t+1}}{p_t} \geqslant \frac{p^i_{t+1}}{p^i_t} \tag{*}$$

and

$$\frac{p_{t+1}}{p_t} = \frac{p^i_{t+1}}{p^i_t} \quad \text{if } U^i_{t+1}(x^i) - U^i_{t+1}(e^i) > 0, \tag{**}$$

where p^i in P^i is the subjective price at interior consumption plan x^i in X^i.

Claim 5. *Given an autarchic allocation e in X, an interior allocation x in $X(e)$, with supporting price p in P, is Pareto dominated by an alternative allocation z in $X(e)$ only if there exists a strictly positive element v of L satisfying, for some sufficiently small $\epsilon > 0$, at every t in $\mathcal{T}$,*

$$\frac{p_{t+1}}{p_t} v_{t+1} \geqslant v_t + \epsilon \sum_{i \in \mathcal{J}} (z^i_t - x^i_t)^2$$

and

$$\sum_{s \in \mathcal{T}} \delta^s \sum_{i \in \mathcal{J}} |z^i_{t+s} - x^i_{t+s}| \geqslant \epsilon v_t.$$

Proof. Preliminarily observe that, for consumptions varying in a compact interval of $\mathbb{R}_+$, there exists a sufficiently small $\epsilon > 0$ satisfying

$$u(c') - u(c) \leqslant u'(c)(c' - c) - \epsilon u'(c)(c' - c)^2.$$

This shows a sort of quadratic concavity of intertemporal utility.

For every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$, define

$$v^i_t = \frac{1}{p^i_t} \sum_{s \in \mathcal{T}} p^i_{t+s}(z^i_{t+s} - x^i_{t+s}) - \epsilon \frac{1}{p^i_t} \sum_{s \in \mathcal{T}} p^i_{t+s}(z^i_{t+s} - x^i_{t+s})^2.$$

Notice that, for every individual i in $\mathcal{J}$, v^i is an element of L. Indeed, this follows from feasibility and uniform impatience. Define $v = \sum_{i \in \mathcal{J}} v^i$, an element itself of L, and observe that, by Pareto dominance and quadratic concavity, $v_0 = \sum_{i \in \mathcal{J}} v^i_0 > 0$. In addition, at every t in $\mathcal{T}$,

$$\frac{1}{\epsilon} \sum_{i \in \mathcal{J}} \sum_{s \in \mathcal{T}} \delta^s |z^i_{t+s} - x^i_{t+s}| \geqslant \sum_{i \in \mathcal{J}} \frac{1}{p^i_t} \sum_{s \in \mathcal{T}} p^i_{t+s} |z^i_{t+s} - x^i_{t+s}| \geqslant v_t,$$

where the first inequality, as $\epsilon > 0$ can be assumed to be arbitrarily small, follows from bounded derivatives of per-period utility $u : \mathbb{R}_+ \to \mathbb{R}$ over a compact interval of $\mathbb{R}_+$.

For every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p^i_{t+1}}{p^i_t} v^i_{t+1} + (z^i_t - x^i_t) \geqslant v^i_t + \epsilon (z^i_t - x^i_t)^2.$$

As debt constraints are not-too-tight,

$$\frac{p_{t+1}}{p_t} > \frac{p^i_{t+1}}{p^i_t} \quad \text{only if } U^i_{t+1}(x^i) - U^i_{t+1}(e^i) = 0.$$

Hence, as $U^i_{t+1}(z^i) - U^i_{t+1}(x^i) \geqslant 0$, $v^i_{t+1} \geqslant 0$. We consistently conclude that, for every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p_{t+1}}{p_t} v^i_{t+1} + (z^i_t - x^i_t) \geqslant v^i_t + \epsilon (z^i_t - x^i_t)^2.$$

Aggregating across individuals, by feasibility, this proves our claim. □

For the example, it suffices to consider only two individuals, $\mathcal{J} = \{e, o\}$, associated with even, e, and odd, o, periods of trade. Let $x_e > 0$ and $x_o > 0$ satisfy $x_e + x_o = 1$ and

$$u'(x_e) = \delta u'(x_o). \tag{B.1}$$

Allocation x in X is given by

$$x^e = (x_e, x_o, x_e, x_o, \ldots),$$
$$x^o = (x_o, x_e, x_o, x_e, \ldots).$$

At allocation x in X, the supporting price p in P is

$$(p_t)_{t\in\mathcal{T}} = (1, 1, 1, \ldots, 1, \ldots),$$

whereas the subjective price p^i in P^i of individual i in $\mathcal{J}$ is given by

$$\left(p^i_t\right)_{t\in\mathcal{T}} = \left(\delta^t u'\left(x^i_t\right)\right)_{t\in\mathcal{T}}.$$

We need to construct initial endowments e in X which are consistent with price support at not-too-tight debt constraints.

Claim 6. *There exists an autarchic allocation e in X, satisfying*

$$\sum_{i\in\mathcal{J}} U^i_t\left(x^i\right) - \sum_{j\in\mathcal{J}} U^i_t\left(e^i\right) > 0 \quad \text{at every } t \in \mathcal{T},$$

such that allocation x lies in $X(e)$ and is supported by price p in P.

Proof. Consider the (local) difference equation

$$h(\xi_t, \xi_{t+1}) = u(x_e) + \delta u(x_o) - u(x_e + \xi_t) - \delta u(x_o - \xi_{t+1}) = 0. \tag{B.2}$$

It is easy to verify that this difference equation admits a strictly positive solution $(\xi_t)_{t\in\mathcal{T}}$ in L satisfying $\lim_{t\in\mathcal{T}} \xi_t = 0$. (Indeed, observe that $\xi > 0$ implies $h(\xi, \xi) > 0$ and $h(\xi, 0) < 0$, so that $h(\xi, \xi') = 0$ for some $\xi > \xi' > 0$ by the Intermediate Value Theorem.) Endowments e in X are given by

$$e^e = (x_e + \xi_0, x_o - \xi_1, x_e + \xi_2, x_o - \xi_3, \ldots),$$
$$e^o = (x_o - \xi_0, x_e + \xi_1, x_o - \xi_2, x_e + \xi_3, \ldots).$$

In addition, because of restriction (B.2), at every t in $\{0, 2, 4, \ldots\}$,

$$U^e_t\left(x^e\right) = U^e_t\left(e^e\right)$$

and

$$U^o_t\left(x^o\right) \geqslant u(x_o) + \delta U^o_{t+1}\left(x^o\right) > u(x_o - \xi_t) + \delta U^o_{t+1}\left(e^o\right) \geqslant U^o_t\left(e^o\right);$$

at every t in $\{1, 3, 5, \ldots\}$,

$$U^o_t\left(x^o\right) = U^o_t\left(e^o\right)$$

and

$$U^e_t\left(x^e\right) \geqslant u(x_o) + \delta U^e_{t+1}\left(x^e\right) > u(x_o - \xi_t) + \delta U^e_{t+1}\left(e^e\right) \geqslant U^e_t\left(e^e\right).$$

Because of restriction (B.1), this suffices to prove the claim. □

We now conclude that allocation x in X is a constrained optimum at initial allocation e in X.

Claim 7. *Given the constructed initial allocation e in X, allocation x in $X(e)$ is constrained efficient.*

Proof. Supposing not, we can apply the characterization of Claim 5. Exploiting the stationarity of supporting price p in P, this characterization imposes the existence of a strictly positive element v of L satisfying, for some sufficiently small $\epsilon > 0$, at every t in $\mathcal{T}$,

$$v_{t+1} \geqslant v_t + \epsilon \sum_{i \in \mathcal{J}} \left(z_t^i - x_t^i\right)^2 \tag{B.3}$$

and

$$\sum_{s \in \mathcal{T}} \delta^s \sum_{i \in \mathcal{J}} \left|z_{t+s}^i - x_{t+s}^i\right| \geqslant \epsilon v_t. \tag{B.4}$$

Clearly, the sequence $(v_t)_{t \in \mathcal{T}}$ in L converges, so that condition (B.3) yields

$$\lim_{t \in \mathcal{T}} v_{t+1} \geqslant v_0 + \epsilon \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{J}} \left(z_t^i - x_t^i\right)^2.$$

Therefore,

$$\lim_{t \in \mathcal{T}} \sum_{i \in \mathcal{J}} \left|z_t^i - x_t^i\right| = 0.$$

This is inconsistent with condition (B.4) as the sequence $(v_t)_{t \in \mathcal{T}}$ in L is (weakly) increasing. □

Summing up, we have provided an example of a constrained optimum, according to Alvarez and Jermann [5], which is not autarchic and does not involve high implied interest rates, as supporting prices exhibit a null interest rate. It is to be remarked that, *strictu sensu*, this is not a counterexample to Proposition 4.10 of Alvarez and Jermann [5], as they also assume stationary endowments, though, in the proof, stationarity of endowments seems not being exploited.

Appendix C. Proofs

Proof of Lemma 1. Necessity of this first-order characterization is established by Alvarez and Jermann [5]. To prove sufficiency, for an individual i in $\mathcal{J}$, consider any consumption plan z^i in the budget set $B_p^i(x^i, f^i)$. It follows that, for some financial plan v^i in V^i, at every date-event σ in $\mathcal{S}$,

$$-p_\sigma^i \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma}\right) f_\tau^i + p_\sigma^i \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma}\right) \left(v_\tau^i + f_\tau^i\right) + p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \leqslant p_\sigma^i v_\sigma^i,$$

where p^i in P^i is the subjective price at consumption plan x^i in X^i. Using condition (*), along with debt constraints, this yields

$$-p_\sigma^i \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma}\right) f_\tau^i + \sum_{\tau \in \sigma_+} p_\tau^i \left(v_\tau^i + f_\tau^i\right) + p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \leqslant p_\sigma^i v_\sigma^i.$$

Using condition (**), this finally becomes

$$-\sum_{\tau \in \sigma_+} p_\tau^i f_\tau^i + \sum_{\tau \in \sigma_+} p_\tau^i \left(v_\tau^i + f_\tau^i\right) + p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \leqslant p_\sigma^i \left(v_\sigma^i + f_\sigma^i\right) - p_\sigma^i f_\sigma^i.$$

Adding up, one obtains

$$-\sum_{\sigma\in\mathcal{S}_t}\sum_{\tau\in\sigma_+} p^i_\tau f^i_\tau + \sum_{\sigma\in\mathcal{S}^t} p^i_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leqslant 0,$$

where, for every t in $\mathcal{T}$, $\mathcal{S}_t = \{\sigma \in \mathcal{S}\colon t(\sigma) = t\}$ and $\mathcal{S}^t = \{\sigma \in \mathcal{S}\colon t(\sigma) \leqslant t\}$. Observing that debt constraints f in F are bounded and subjective price p^i in P^i defines an order-continuous linear functional on L,

$$\sum_{\sigma\in\mathcal{S}} p^i_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leqslant 0.$$

This, by concavity of utility, suffices to prove the claim. □

Proof of Lemma 2. At no loss of generality, as x in X is an interior allocation, it can be assumed that

$$\bigwedge_{i\in\mathcal{J}} x^i \geqslant v.$$

Also, by the Modified Cass Criterion, there exists a non-null positive element v of L satisfying, for some sufficiently large $1 > \rho > 0$ and for some sufficiently small $1 > \epsilon > 0$, at every date-event σ in $\mathcal{S}$,

$$(1-\epsilon)\rho\sum_{\tau\in\sigma_+} p_\tau v_\tau \geqslant p_\sigma v_\sigma.$$

For future reference, we shall prove a more general claim: there exists an alternative allocation z in $X^*(x)$ fulfilling

$$\epsilon v + \sum_{i\in\mathcal{J}} z^i \leqslant \sum_{i\in\mathcal{J}} e^i, \tag{C.1}$$

where $X^*(x)$ denotes the set of all allocations z in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U^i_\sigma\left(z^i\right) \geqslant U^i_\sigma\left(x^i\right) \geqslant U^i_\sigma\left(e^i\right).$$

Consider any partition $(\mathcal{P}^i)_{i\in\mathcal{J}}$ of the set of date-events $\{\sigma \in \mathcal{S}\colon v_\sigma > 0\}$ such that, for every non-initial date-event σ in $\mathcal{S}$, σ belongs to $\mathcal{P}^i$ only if $f^i_\sigma > 0$. This construction is consistent as price support is non-trivial. Also, for every individual i in $\mathcal{J}$, let $\mathcal{N}^i = \{\sigma \in \mathcal{S}\colon \sigma_+ \cap \mathcal{P}^i \neq \emptyset\}$. Finally, for every date-event σ in $\mathcal{S}$, define $\mathcal{P}^i(\sigma) = \mathcal{P}^i \cap \mathcal{S}(\sigma)$ and $\mathcal{N}^i(\sigma) = \mathcal{N}^i \cap \mathcal{S}(\sigma)$. Notice that, by the Modified Cass Criterion, if a date-event σ belongs to $\mathcal{P}^i$ for some individual i in $\mathcal{J}$, then it belongs to $\mathcal{N}^j$ for some other individual j in $\mathcal{J}$.

For every individual i in $\mathcal{J}$, define

$$z^i = x^i + (1-\epsilon)\sum_{\sigma\in\mathcal{P}^i} v_\sigma - \sum_{\sigma\in\mathcal{N}^i}\left(\frac{\sum_{\tau\in\sigma_+\cap\mathcal{P}^i} p_\tau v_\tau}{\sum_{\tau\in\sigma_+} p_\tau v_\tau}\right) v_\sigma.$$

Here, for notational convenience, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \bigoplus_{\sigma\in\mathcal{S}} \mathbb{R}_\sigma$. For every individual i in $\mathcal{J}$, the underlying redistribution increases consumption at date-events in $\mathcal{P}^i$ and decreases consumption at date-events in $\mathcal{N}^i$. Clearly, by construction, z in X is an allocation satisfying the strong form of feasibility (C.1). Also, notice that, by construction, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\begin{aligned}\sum_{\nu\in\mathcal{N}^i(\sigma)}\left(\frac{\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p_\tau v_\tau}{\sum_{\tau\in\nu_+}p_\tau v_\tau}\right)p_\nu^i v_\nu &\leqslant \sum_{\nu\in\mathcal{N}^i(\sigma)}p_\nu^i\left(\frac{p_\nu v_\nu}{\sum_{\tau\in\nu_+}p_\tau v_\tau}\right)\frac{1}{p_\nu}\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p_\tau v_\tau\\ &\leqslant \sum_{\nu\in\mathcal{N}^i(\sigma)}\left(\frac{p_\nu v_\nu}{\sum_{\tau\in\nu_+}p_\tau v_\tau}\right)\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p_\tau^i v_\tau\\ &\leqslant (1-\epsilon)\rho\sum_{\nu\in\mathcal{N}^i(\sigma)}\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p_\tau^i v_\tau\\ &\leqslant (1-\epsilon)\rho\sum_{\tau\in\mathcal{P}^i(\sigma)}p_\tau^i v_\tau.\end{aligned}$$

The first inequality is a simple manipulation; the second inequality uses the fact that subjective and market evaluations coincide; the third inequality is a consequence of the Modified Cass Criterion; the last inequality uses the construction of subsets $\mathcal{P}^i$ and $\mathcal{N}^i$ of $\mathcal{S}$. Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\begin{aligned}\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left(z_\tau^i-x_\tau^i\right) &\geqslant (1-\epsilon)(1-\rho)\sum_{\tau\in\mathcal{P}^i(\sigma)}p_\tau^i v_\tau\\ &\geqslant (1-\rho)\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left(z_\tau^i-x_\tau^i\right)^+.\end{aligned}\tag{C.2}$$

Manipulating inequality (C.2), we obtain

$$\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left(z_\tau^i-x_\tau^i\right)\geqslant\left(\frac{1-\rho}{\rho}\right)\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left(z_\tau^i-x_\tau^i\right)^-\geqslant(1-\rho)\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left(z_\tau^i-x_\tau^i\right)^-.$$

Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left(z_\tau^i-x_\tau^i\right)\geqslant\left(\frac{1-\rho}{2}\right)\sum_{\tau\in\mathcal{S}(\sigma)}p_\tau^i\left|z_\tau^i-x_\tau^i\right|.\tag{C.3}$$

Condition (C.3) guarantees a first-order positive welfare effect beginning from every date-event σ in $\mathcal{S}$. To obtain a welfare improvement, we show that higher order effects are uniformly bounded. As allocation x in X is interior, for a sufficiently small $\eta>0$, any allocation y in $B_\eta(x)$ is also interior, where

$$B_\eta(x)=\left\{y\in X\colon \sum_{i\in\mathcal{J}}\left\|y^i-x^i\right\|\leqslant\eta\right\}.$$

Notice that per-period utility $u^i:\mathbb{R}_+\to\mathbb{R}$ exhibits a bounded second-order term over any compact interior interval of $\mathbb{R}_+$. Thus, it can be assumed that there exists a sufficiently large $\mu>0$ satisfying, given any allocation y in $B_\eta(x)$, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$u^i\left(y_\sigma^i\right)-u^i\left(x_\sigma^i\right)\geqslant\partial u^i\left(x_\sigma^i\right)\left(y_\sigma^i-x_\sigma^i\right)-\left(\frac{\mu}{2}\right)\left|y_\sigma^i-x_\sigma^i\right|\partial u^i\left(x_\sigma^i\right)\left|y_\sigma^i-x_\sigma^i\right|.$$

Also, possibly contracting v in L, at no loss of generality,

$$\bigvee_{i\in\mathcal{J}}\left\|z^i-x^i\right\|\leqslant\|v\|\leqslant\eta\wedge\left(\frac{1-\rho}{\mu}\right).$$

Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$u^i\left(z^i_\sigma\right) - u^i\left(x^i_\sigma\right) \geqslant \partial u^i\left(x^i_\sigma\right)\left(z^i_\sigma - x^i_\sigma\right) - \left(\frac{1-\rho}{2}\right)\partial u^i\left(x^i_\sigma\right)\left|z^i_\sigma - x^i_\sigma\right|.$$

This, because of condition (C.3), shows that, at every date-event σ in $\mathcal{S}$, $U^i_\sigma(z^i) \geqslant U^i_\sigma(x^i)$, which suffices to prove the claim. □

Proof of Lemma 3. At no loss of generality, it can be assumed that allocation x in $X(e)$ is strongly Pareto dominated by an alternative allocation z in $X^*(e)$ satisfying

$$\sum_{i\in\mathcal{J}} z^i \leqslant \sum_{i\in\mathcal{J}} e^i.$$

For every individual i in $\mathcal{J}$, define, at every date-event σ in $\mathcal{S}$,

$$v^i_\sigma = \frac{1}{p^i_\sigma}\sum_{\tau\in\mathcal{S}(\sigma)} p^i_\tau\left(z^i_\tau - x^i_\tau\right).$$

By Pareto dominance, at the initial date-event ϕ in $\mathcal{S}$, $\sum_{i\in\mathcal{J}} v^i_\phi > 0$. In addition, as allocation x in X is interior, by uniform impatience, for every individual i in $\mathcal{J}$, v^i is an element of L. In addition, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\frac{1}{p^i_\sigma}\sum_{\tau\in\sigma_+} p^i_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma\right) = v^i_\sigma. \tag{C.4}$$

Observe that, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$v^i_\sigma < 0 \quad \text{implies} \quad U^i_\sigma\left(z^i\right) - U^i_\sigma\left(x^i\right) < 0.$$

Therefore, as allocation z lies in $X(e)$,

$$v^i_\sigma < 0 \quad \text{implies} \quad U^i_\sigma\left(x^i\right) - U^i_\sigma\left(e^i\right) > 0.$$

Using the consistency of debt constraints, this yields

$$v^i_\sigma < 0 \quad \text{implies} \quad f^i_\sigma > 0.$$

Hence, by first-order conditions (*)–(**), condition (C.4) delivers, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma\right) \geqslant v^i_\sigma.$$

Summing up across individuals,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau + \sum_{i\in\mathcal{J}}\left(z^i_\sigma - x^i_\sigma\right) \geqslant v_\sigma, \tag{C.5}$$

where $v = \sum_{i\in\mathcal{J}} v^i$ in L. By feasibility, condition (C.5) delivers, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v^+_\tau \geqslant v^+_\sigma,$$

where v^+ in L is the non-null positive part of v in L. Hence, the Weak Modified Cass Criterion is satisfied.

When gains from risk-sharing are uniform, possibly replacing allocation z in $X^*(e)$ with allocation $z + \lambda(y - z)$ in $X^*(e)$ for some sufficiently small $1 > \lambda > 0$, where allocation y in $X^*(e)$ is given by the hypothesis of uniform gains from risk-sharing, it can be assumed that, for some sufficiently small $\epsilon > 0$, at every date-event σ in $\mathcal{S}$,

$$\sum_{i \in \mathcal{J}} \left(z_\sigma^i - x_\sigma^i\right) \leqslant -\epsilon.$$

Also, as v is a bounded element in L, for some sufficiently large $1 > \rho > 0$, at every σ in $\mathcal{S}$,

$$\epsilon \geqslant \left(\frac{1-\rho}{\rho}\right) v_\sigma.$$

Define recursively a non-null positive element w of L by means of $w_\phi = v_\phi^+ > 0$ and, at every σ in $\mathcal{S}$, $(w_\tau)_{\tau \in \sigma_+} = (v_\tau^+)_{\tau \in \sigma_+}$, if $w_\sigma > 0$, and $(w_\tau)_{\tau \in \sigma_+} = 0$, if $w_\sigma = 0$. By construction, at every σ in $\{\sigma \in \mathcal{S}\colon\ w_\sigma = 0\}$,

$$\rho \sum_{\tau \in \sigma_+} p_\tau w_\tau \geqslant p_\sigma w_\sigma;$$

at every σ in $\{\sigma \in \mathcal{S}\colon\ w_\sigma > 0\}$, exploiting condition (C.5),

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau w_\tau \geqslant \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau \geqslant v_\sigma + \epsilon \geqslant \left(\frac{1}{\rho}\right) w_\sigma,$$

so that

$$\rho \sum_{\tau \in \sigma_+} p_\tau w_\tau \geqslant p_\sigma w_\sigma.$$

This proves the claim. □

Proof of Proposition 1. It is an immediate implication of Lemmas 2–3. □

Proof of Proposition 2. If allocation x in $X(e)$ is *ex-post* inefficient, then it is also constrained inefficient. Assuming constrained inefficiency, Lemma 3 implies the Modified Cass Criterion and the proof of Lemma 2 shows *ex-post* inefficiency. □

Proof of Lemma 4. Take any allocation x in $X(e)$ and let $X^*(e, x)$ be the set of all allocations z in $X^*(e)$ that weakly Pareto dominates allocation x in $X(e)$. The set

$$V = \left\{ \left(\sum_{i \in \mathcal{J}} x^i - \sum_{i \in \mathcal{J}} z^i \right) \in L\colon\ z \in X^*(e, x) \right\}$$

is a non-empty convex set that cannot intersect the interior of the positive cone of L. The empty intersection follows from the violation of the hypothesis of uniform gains from risk-sharing at the autarchic allocation e in X. By a canonical application of the Separating Hyperplane Theorem (as in Kehoe and Levine [17, Proposition 5]), there exists a positive linear functional φ in Φ supporting allocation x in $X(e)$. □

Proof of Lemma 5. Consider a constrained optimum x in $X(e)$. By the Separating Hyperplane Theorem (Kehoe and Levine [17, Proposition 5]), there exists a positive linear functional φ in Φ

supporting allocation x in $X(e)$. Consider any constrained inefficient allocation x in $X(e)$ and, in order to obtain a contradiction, suppose that it is supported by a positive linear functional φ in Φ. At no loss of generality, allocation x in $X(e)$ is strongly Pareto dominated by an alternative allocation z in $X(e)$. Also, by continuity and convexity of preferences, for every sufficiently small $1 > \lambda > 0$, allocation x in $X(e)$ is Pareto dominated by allocation $z + \lambda(y - z)$ in $X^*(e)$, where allocation y in $X^*(e)$ is given by the hypothesis of uniform gains from risk-sharing. Hence,

$$0 \leqslant (1-\lambda)\sum_{i\in\mathcal{J}} \varphi\big(z^i - x^i\big) + \lambda \sum_{i\in\mathcal{J}} \varphi\big(y^i - x^i\big) \leqslant -\epsilon\lambda\varphi\bigg(\sum_{i\in\mathcal{J}} e^i\bigg) < 0,$$

where the existence of a sufficiently small $1 > \epsilon > 0$ is given by uniform gains from risk-sharing at the autarchy.

We now prove strict positivity of the supporting linear functional φ in Φ at a constrained efficient allocation x in $X(e)$. Suppose that allocation z in $X^*(e)$ Pareto dominates allocation x in $X(e)$ and

$$\sum_{i\in\mathcal{J}} \varphi\big(z^i - x^i\big) = 0.$$

By strict monotonicity of preferences, strict Pareto dominance involves no loss of generality. For some sufficiently small $1 > \lambda > 0$, consider the allocation

$$(1-\lambda)z + \lambda y \in X^*(e),$$

where allocation y in $X^*(e)$ is given by the hypothesis of uniform gains from risk-sharing. By continuity, this alternative allocation Pareto dominates allocation x in $X(e)$. Therefore, by separation,

$$(1-\lambda)\varphi\bigg(\sum_{i\in\mathcal{J}} z^i - \sum_{i\in\mathcal{J}} x^i\bigg) \geqslant -\lambda\varphi\bigg(\sum_{i\in\mathcal{J}} y^i - \sum_{i\in\mathcal{J}} x^i\bigg) \geqslant \epsilon\lambda\varphi\bigg(\sum_{i\in\mathcal{J}} e^i\bigg).$$

This yields

$$0 \geqslant \varphi\bigg(\sum_{i\in\mathcal{J}} z^i - \sum_{i\in\mathcal{J}} x^i\bigg) \geqslant \epsilon\bigg(\frac{\lambda}{1-\lambda}\bigg)\varphi\bigg(\sum_{i\in\mathcal{J}} e^i\bigg) > 0,$$

which is a contradiction. Therefore, for every allocation z in $X^*(e)$ that Pareto dominates allocation x in $X(e)$,

$$\sum_{i\in\mathcal{J}} \varphi\big(z^i - x^i\big) > 0.$$

Because of strict monotonicity of preferences, for every $v > 0$ in L, $\varphi(v) > 0$, thus proving strict positivity.

To show the remaining claim, we argue as follows. Given any period t in $\mathcal{T}$, v^t in L (respectively, v_t in L) is given by $v^t_\tau = v_\tau$, for every τ in $\mathcal{S}$ with $t(\tau) \geqslant t$ (respectively, with $t(\tau) = t$), and $v^t_\tau = 0$, otherwise. Also, let u be the unit of L and, at no loss of generality, suppose that $\sum_{i\in\mathcal{J}} e^i \geqslant u$. Given any sufficiently small $1 > \lambda > 0$, by concavity of utilities, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}_t$,

$$U^i_\sigma\big(x^i + u_t - \lambda\big(x^i - y^i\big)^t\big) - U^i_\sigma\big(x^i\big) \geqslant \Delta_* - \lambda\frac{1}{\pi^i_\sigma}\sum_{\tau\in\mathcal{S}(\sigma)} \pi^i_\tau \Delta^* \big\|x^i - y^i\big\|,$$

where $\Delta^* > \Delta_* > 0$ are, respectively, the upper and the lower bound on marginal utilities (restricted on the feasible set). Using uniform impatience, this yields

$$U_\sigma^i\left(x^i + u_t - \lambda\left(x^i - y^i\right)^t\right) - U_\sigma^i\left(x^i\right) \geqslant \Delta_* - \lambda\left(\frac{\Delta^*}{\eta}\right)\left\|x^i - y^i\right\|.$$

Thus, provided that $1 > \lambda > 0$ is sufficiently small,

$$\Delta_* - \lambda\left(\frac{\Delta^*}{\eta}\right)\bigvee_{i\in\mathcal{J}}\left\|x^i - y^i\right\| > 0.$$

Consider the alternative allocation z in X defined, for every individual i in $\mathcal{J}$,

$$z^i = x^i + u_t - \lambda\left(x^i - y^i\right)^t.$$

Allocation z in X Pareto dominates allocation x in $X(e)$ and satisfies participation, thus belonging to $X^*(e)$. By separation,

$$\begin{aligned}\varphi\left(Nu_t - \lambda\epsilon u^t\right) &\geqslant \varphi\left(Nu_t - \lambda\left(\sum_{i\in\mathcal{J}} x^i - \sum_{i\in\mathcal{J}} y^i\right)^t\right)\\ &\geqslant \varphi\left(\sum_{i\in\mathcal{J}} z^i - \sum_{i\in\mathcal{J}} x^i\right) > 0,\end{aligned}$$

where N in $\mathbb{N}$ denotes the cardinality of $\mathcal{J}$ and $1 > \epsilon > 0$ is given by the hypothesis of uniform gains from risk-sharing. This delivers, at every t in $\mathcal{T}$,

$$\varphi(u_t) \geqslant \left(\frac{\lambda\epsilon}{N}\right)\varphi\left(u^t\right),$$

which suffices to prove the claim, as in Kehoe and Levine [17, Lemma 3]. □

Proof of Proposition 3. Assuming high implied interest rates, the Modified Cass Criterion is violated and, hence, constrained efficiency obtains. Assuming constrained efficiency, by Lemma 5, allocation x in $X(e)$ is supported by an order-continuous strictly positive linear functional φ in Φ. It is straightforward to prove that, at every date-event σ in $\mathcal{S}$,

$$\bigvee_{i\in\mathcal{J}}\left(\frac{p_\tau^i}{p_\sigma^i}\right)_{\tau\in\sigma_+} \leqslant \left(\frac{\varphi_\tau}{\varphi_\sigma}\right)_{\tau\in\sigma_+},$$

where p^i in P^i is the subjective price of individual i in $\mathcal{J}$ at interior consumption plan x^i in X^i. Also, as price p in P is non-trivially supporting, at every date-event σ in $\mathcal{S}$,

$$\bigvee_{i\in\mathcal{J}}\left(\frac{p_\tau^i}{p_\sigma^i}\right)_{\tau\in\sigma_+} = \left(\frac{p_\tau}{p_\sigma}\right)_{\tau\in\sigma_+}.$$

This suffices to prove the claim. □

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