OPTIMAL PRICE SETTING WITH OBSERVATION AND MENU COSTS*

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We study the price-setting problem of a firm in the presence of both observation and menu costs. The firm optimally decides when to "review" costly information on the adequacy of its price. Upon each review, the firm chooses whether to adjust its price, one or more times, before the next price review. Each price adjustment entails paying a menu cost. The firm's choices map into several statistics: the frequency of price reviews, the frequency of price adjustments, the size distribution of price changes, and the hazard rate of price adjustments. The simultaneous presence of observation and menu costs produces complementarities that change the predictions of simpler models featuring one cost only. For instance, infrequent observations may reflect a high menu cost rather than high observation costs: in spite of these complementarities, we show that the ratio of the two costs is identified by several statistics on price observations and adjustments. *JEL* Codes: E31, E50.

I. INTRODUCTION

A large literature modeling infrequent adjustment has focused on either of two costs: one is a standard fixed cost of adjusting the state, the other is a fixed cost of observing the state. The effects of these adjustment costs have been thoroughly analyzed in a variety of contexts. An example of the fixed adjustment cost is the canonical *sS* problem. The analysis of the

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implications of costly observation of the relevant state is more recent, yet examples abound.[1] This article solves a price setting problem in the presence of both costs. This delivers three main novel results. First, we develop an analytical characterization of the optimal information-gathering and adjustment policies, as well as of their implications for several observable statistics on the times of information gathering and adjustments, and on the size distribution of adjustments. Second, the complementary nature of both costly activities (i.e., reviews and adjustments) yields new implications that are in neither of the models featuring only one cost. Third, the sharp analytical characterization of the implications of the model with both costs provides a mapping that can be used to measure the relative size of these costs.

We develop this model with several applications in mind, ranging from the households' portfolio decision, to the firm's investment choice, or the price-setting problem of a monopolist. For concreteness, because of its inherent simplicity, and because of its importance in macroeconomics, we focus on the price-setting problem.[2] In this problem the firm optimally decides when to collect costly information on the adequacy of its price, an activity we refer to as a price "review." In several papers Reis provides a broader interpretation of this activity and discusses its relationship with the rational inattention literature; see, for example, section 2.1 of Reis (2006a). In our setup, upon a costly review at time t the firm decides the time of the next review $t + T$, as well as when and how many times to adjust its price, subject to a menu cost, within the interval $[t, t + T)$. We follow the literature and refer to multiple adjustments between observations as a *price plan*. We analyze when price plans are optimal. Furthermore, for the case where price plans are not optimal, we study the mapping between the two adjustment frictions (observation and menu cost) and the frequency of reviews, the frequency of price adjustments, the size distribution of the adjustments, and the hazard rate of price adjustments.

1. For examples with a fixed observation cost see Caballero (1989), Duffie and Sun (1990), Reis (2006a, 2006b), Abel, Eberly, and Panageas (2007, 2009), and more generally the related "rational inattention" literature as in Moscarini (2004), Sims (2003).

2. See Duffie (2010) for applications of rational inattention to asset pricing. Alvarez, Guiso, and Lippi (2011) analyze a version of the consumption-savings and portfolio choice model studied by Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007). That paper uses a novel data set to measure the frequency of observation and adjustment of portfolios for Italian investors.

Our model embeds the two "polar" cases of menu cost (e.g., Barro 1972; Dixit 1991) and observation cost only (e.g., Caballero 1989; Reis 2006b). These models produce different policy rules: the menu cost yields "state-dependent" rules, as in, for example, Golosov and Lucas (2007), while the observation cost yields "time-dependent" rules, as in Mankiw and Reis (2002, 2006, 2007). Because these rules have different implications for the response of the economy to aggregate shocks, identifying the nature of the frictions underlying sticky prices is important. However, there is no agreed-on method to discriminate between these competing hypothesis. Some researchers have conducted on field analyses aimed at uncovering the nature of the frictions underlying price setting, for example, Levy et al. (1997) and Zbaracki et al. (2004). Our analysis provides a theoretical framework to quantify these frictions: with suitable data, our theory can be used to estimate the magnitude of the menu and the observation cost. An application of these ideas can be found in the work in progress by Cavallo (2010) and Cavallo and Rigobon (2010), who study a huge novel data set of prices using, among other things, the theory developed in this article. Finally, the distinction between the two activities allows us to consider the economics behind "price plans" or "sticky plans," an assumption that has been shown to have implications for monetary policy, for example, by Mankiw and Reis (2002) and Burstein (2006).

I.A. A Preview of the Price-Setting Problem

The firm minimizes the expected discounted value of a per-period loss function plus the expected discounted sum of the fixed costs incurred. The firm's instantaneous loss function is $B(p-p^*)^2$, where p is the current (log) price, p^* the "target" (log) price that maximizes current profits, and the parameter B depends on the curvature of the profit function. The target p^* follows a random walk with drift, arising from innovations in the firm's marginal costs: the drift, μ, is the inflation rate and the innovations are idiosyncratic shocks with variance σ^2. We refer to the difference between p and p^* as the price gap, denoted by $\tilde{p} \equiv p - p^*$. The firm faces two fixed costs. The first is a standard menu cost, ψ, that applies to any change in price. The second is an observation cost, θ, that the firm bears to discover p^* and hence $\tilde{p}$. The firm chooses stopping times at which to observe the value of p^* and stopping times at which to adjust its price p.

The combination of a Brownian motion for the target price, a quadratic objective, and no general equilibrium feedbacks are important to obtain sharp analytical results. While these assumptions are common in the literature, and they are used in the polar cases where analytical results are obtained, they are not without cost: for instance, they imply that there is no invariant distribution of relative prices.[3]

Next, we describe the main analytical results concerning the policy rules and the implications for the observable statistics. Our analytical results are based on approximations that are valid for small values of the discount rate ρ and of the fixed cost $\frac{\psi}{B}$, as done in the analytical literature covering the polar cases.

I.B. Summary of Results

In Section III we show that if the inflation rate μ is small, due to the fixed cost of changing prices, it is optimal to have at most one price adjustment between reviews, which occurs immediately after the review. Thus *price plans* are not optimal for low inflation. Moreover, in Section IV we use the symmetry between the effects of inflation and deflation to show that the frequency of price reviews and adjustments, as well as its hazard rate, are insensitive to inflation around $\mu = 0$. This prediction is consistent with the evidence in Gagnon (2009) and Alvarez, Gonzalez-Rozada, Neumeyer, and Beraja (2011). We also show that inflation has only a second-order effect on the unconditional expectation of the net profits, as well as on the variance of price changes and on the distribution for the *absolute* value of price changes. Instead, it has a first-order effect on the expected value of price changes.

Based on these findings, in Section V we focus on the analytical characterization of the decision rules for the case of zero inflation. The optimal policy with observation and menu costs combines both time-dependent and state-dependent features: upon observing $\tilde{p}$ the firm optimally chooses not to adjust the price if $\tilde{p}$ falls in the inaction region $(-\bar{p}, \bar{p})$. In this case it chooses the optimal time for the next observation, given by the function $\mathrm{T}(\tilde{p}) = \tau - (\frac{\tilde{p}}{\sigma})^2$. When $\tilde{p}$ falls outside the inaction region the firm sets the price gap to 0 and the next observation occurs in τ periods.

3. Danziger (1999) general equilibrium monetary economy shares several features with ours: the fixed menu cost is proportional to current profits, and in equilibrium the firm's current payoff is a quadratic function of the ratio between target and current price, where the target price is the ratio between the money supply and the idiosyncratic productivity, both random walks with drift in logs.

Hence the firm's policy is described by two parameters: $\bar{p}, \tau$. We characterize the mapping between these policy parameters and the structural parameters, σ, B, ψ, θ, including elasticities. The decision rule has in common with the menu cost model that there is an inaction range with a minimum size of the price change, namely $\bar{p}$, and in common with the observation cost model, that information gathering is infrequent, with a maximum time elapsed between reviews of τ. The combination of the two frictions implies a novel feature of the decision rule: namely, that the optimal review times, $\mathrm{T}(\tilde{p})$, are a function of the state, with an inverted U-shape in the inaction region. The reason for this is clear: as $|\tilde{p}|$ gets close to $\bar{p}$, it is more likely to subsequently fall outside the inaction region, and hence it is optimal for the next observation to occur sooner than if $\tilde{p} = 0$.

The economic implications of these results differ sharply from the workings of models with one cost only. The key novelty is that through $\bar{p}$ and τ, the times of review and the times of adjustment depend on *both* the observation costs and the menu cost. This has several novel implications, for instance, the fact that reviews are more frequent than adjustments does not imply, as one might guess from a model with only one cost, that adjusting prices is costlier than observing, that is, that menu costs are larger than observation costs. We show that more frequent reviews may be consistent with observations costs much bigger than the menu costs.

In Section VI we use the firm's optimal policy to analytically derive the implications for the average frequency of price reviews, n_r, the average frequency of price adjustments, n_a, the size distribution of price changes, $w(\Delta p)$, and the hazard rate of price adjustments, $\mathbf{h}(t)$. Each of these four statistics depends only on two parameters. The first one is $\alpha \equiv \frac{\psi}{\theta}$, the ratio of the menu to the observation cost. The second parameter measures, fixing α, the cost/benefit ratio of observing and/or adjusting. This parameter is $\frac{\theta\sigma^2}{B}$ for $w(\Delta p)$, and it is $\frac{\theta}{(B\sigma^2)}$ for n_a, n_r and $\mathbf{h}(t)$. The economics behind this result is simple: the two activities—price reviews and price adjustments—have different costs but are complementary, since firms only gather information to know whether to adjust and if so, by how much. Because of this complementarity, α determines how many observations per adjustment are made. We comment on four novel implications of the model with two costs, that is, $0 < \alpha < \infty$.

First, the distribution of price changes $w(\Delta p)$ is normal in the model with observation cost ($\alpha = 0$), and it is degenerate bimodal in the menu cost model ($\alpha \to \infty$), with all its mass concentrated on the thresholds $\pm\bar{p}$. In the model with 2 costs ($0 < \alpha < \infty$) the distribution of price changes resembles a normal whose mass is chopped over the inaction interval, and with long decreasing tails outside the inaction boundaries. Such a bimodal distribution is interesting because it allows the model to reproduce several facts about the distribution of price changes: a large mass of small adjustments, two modes, and a large standard deviation, see Midrigan (2007); Cavallo (2010); Cavallo and Rigobon (2010).

Second, although in each of the models with only one cost the hazard rate is monotone increasing, in our model it is not monotone. In particular, the instantaneous hazard rate $\mathbf{h}(t)$ is continuous, strictly increasing, and it has an asymptote in the menu cost model, whereas it is degenerate in the model with observation cost only. For $0 < \alpha < \infty$, the hazard rate shares some properties with the observation cost model, like an initial value of 0 for $t \in [0, \tau)$, and a spike at $t = \tau$. But unlike that model, it has a finite continuos nonzero hazard rate for higher values of t. Loosely speaking, the shape of the hazard rate function has some periodicity, in that it looks like a series of nonmonotone functions around durations that are multiple of τ. The reason for this nonmonotonicity comes from the fact that reviews happen at unequal length of time—given by the function $\mathrm{T}(\cdot)$—and adjustment occurs depending on whether the price gap is larger than the threshold $\bar{p}$ at the time of observation. We find the nonmonotonicity appealing because most empirical studies fail to find evidence for increasing hazard rates, which is the implication stemming from the polar cases with only one cost.[4] Also, periodical spikes in the hazard rate have been estimated in several datasets by researchers, for example, Nakamura and Steinsson (2008), who interpret it as evidence of time dependence in firms' pricing decisions or alternatively seasonality in costs or demand. In Section VI.C. we use our model to discuss the aggregation biases that this nonmonotonicity can originate, even after controlling for the average size and for the average duration of price changes.

4. Klenow and Kryvtsov (2008) estimate a flat hazard rate on U.S. CPI data. A downward sloping hazard is estimated by Nakamura and Steinsson (2008) on a similar U.S. data set using a different methodology, and by Alvarez, Burriel, and Hernando (2005) for the Euro area. In contrast, Cavallo (2010) estimates hump-shaped hazard functions for four Latin American countries.

Third, another unique prediction of the model with two costs ($0 < \alpha < \infty$) is that the frequency of price reviews is higher than the frequency of price adjustment, that is that the ratio $\frac{n_r}{n_a} > 1$. This happens because the reviews where the price gap is in the inaction region do not produce a price adjustment. We find this prediction interesting because it seems to be typical in the data, see Fabiani et al. (2007) for statistics on price reviews and adjustments for firms across several European countries, and Alvarez, Guiso, and Lippi (2011) for statistics on portfolio reviews and adjustments for Italian households.

Fourth, our theory identifies several observable statistics that can be used to measure the relative size of the menu cost versus the observation cost: $\alpha \equiv \frac{\psi}{\theta}$. For instance, the ratio between the frequency of price reviews and price adjustment $\frac{n_r}{n_a}$ is a function only of α, and so are the following moments from the distribution of price changes: *std* $\frac{|\Delta p|}{\mathbb{E}|\Delta p|}$, *mode* $\frac{|\Delta p|}{\mathbb{E}|\Delta p|}$. We characterize these mappings analytically. Matched with suitable data, for example, after accounting for structural heterogeneity across firms size and industries, these formulas can be used to estimate α. Thus, the model provides a theory that can be used to quantify the observation and menu costs.

I.C. Related Literature

A good summary of the literature on price setting with imperfect information is in Section 7.1 of Mankiw and Reis (2010). A strand of this literature studies price-setting decisions in models where both the size of the price change and its timing are endogenous in the presence of costly observation. In Bonomo and Carvalho (2004) and Woodford (2009) the firm optimally chooses the times of observation and adjustment, under the assumption that the information and the menu cost are lumped together. As in our setup, these models predict infrequent information review and price adjustment. But since the observation and menu cost are lumped, every price review triggers a price adjustment: this differs from our model and, as a consequence, yields different predictions concerning pricing behavior as observed in the micro data. In Bonomo and Carvalho observations/adjustments are equally spaced in time, the distribution of price changes is unimodal, and the hazard rate of price changes is constant at 0 with a spike at the time of the observation. In Woodford's model it is assumed that the firm cannot keep track of the time elapsed since the last observation/adjustment but that it receives noisy signals on the price gap. This implies that price adjustments are triggered by the signals (by construction they are independent of the time elapsed since the last price change). The distribution of price changes has either one or two modes, depending on the size of the observation/adjustment cost. For small levels of the cost, which are needed to reproduce a large mass of small price adjustments, the distribution of price changes has a single mode. By contrast, our model with separate observation and menu costs produces a distribution of price changes that is bimodal and features a large mass of small adjustments and large standard deviation of price changes. Finally, Gorodnichenko (2008) presents numerical solutions of a model where the firm faces a fixed cost to acquire information and a fixed cost to change the price. Differently from our article, he focuses on aggregate uncertainty and also assumes that after one period the firm learns the true value of the state for free. We remark that although these papers are related to our work, their aim is different. Their main question is to analyze the macro-economic effect of monetary shocks. The aim here is more modest: to derive a mapping between the model parameters and several statistics, to gauge the magnitude of menu costs and observation costs from the micro data.

II. The Price-Setting Problem

We analyze the quadratic tracking problem of a firm facing an instantaneous loss function given by $B\ \left(p\left(t\right)-p^{*}\left(t\right)\right)^{2}$ where $p(t)$ is a decision for the firm and $p^{*}(t)$ is the log of the random target, that is, the optimal value that she would set with full knowledge of the state of the problem and without any adjustment friction. The target changes stochastically, and we assume that the firm must pay a fixed cost θ to observe the state $p^{*}(t)$, and that she minimizes expected discounted losses. We refer to the argument of the loss function as the price gap: $\tilde{p}(t) \equiv p(t) - p^{*}(t)$. The constant B measures the cost elasticity to price deviations from the target. Moreover, it is assumed that the firm faces a physical cost ψ associated with resetting the price (a "menu cost").[5] The simplification of using a quadratic approximation to the profit

5. The Online Appendix discusses one case that is useful to interpret the units of B, θ, ψ, σ. The instantaneous loss can be derived as the second-order approximation of the profit function relative to the static optimal profits, where $p(t)$ is the log-price of a monopolistic firm and $p^{*}(t)$ its corresponding static optimal level. With a constant demand elasticity $\eta > 1$ and constant returns to scale,

function has been used in the seminal work on price-setting problem with menu cost by Caplin and Leahy (1997) and others as discussed by Stokey (2008) for menu cost models, and also by Caballero (1989) and Moscarini (2004) for costly observation models.

The log of the target price $p^*(t)$ follows a random walk with drift μ, with normal innovations with variance σ^2 per unit of time, so starting from a price gap $\tilde{p}(t_0)$ at time t_0, the uncontrolled evolution of the price gap is

$$\tilde{p}(t_0 + t) = \tilde{p}(t_0) \; - \mu \, t \; - s \, \sigma \, \sqrt{t}, \tag{1}$$

where s is a standard normal.[6] Hence $\mathbb{E}_{t_0}[\tilde{p}(t_0 + t)] = \tilde{p}(t_0) - \mu t$, $Var_{t_0}[\tilde{p}(t_0 + t)] = \sigma^2 t$, and $\mathbb{E}_{t_0}(\tilde{p}(t_0 + t))^2 = (\tilde{p}\,(t_0) - \mu t)^2 + \sigma^2 t$. We assume the firm controls the price gap by setting the price $p(t_0)$ in nominal terms, so that the drift μ can be interpreted as the inflation rate.

The problem faced by the firm with observation ($\theta > 0$) and menu cost ($\psi > 0$) can be stated as follows. Upon paying the cost θ, and finding the value of the price gap $\tilde{p}(t)$, the firm decides the time until the next observation: T. Between observations, the information of the firm is summarized by $\tilde{p}$. During this time the firm faces several choices, including not changing prices at all, changing prices multiple times, changing prices only once at the time of the observation, or delaying the first price change with respect to the observation. Next we develop the notation required to discuss these choices. We denote the number of times that the firm adjusts its price between observations by $J \in \mathbb{N}$. If $J = 0$ there is no adjustment and a period of length $T > 0$ elapses until the next observation. If $J \geq 1$ then one or more price adjustments take place between observations, each occurring t_i periods after the observation (with $i = 1, \ldots, J$). Upon each adjustment the firm pays the cost ψ and chooses a price level such that the expected value of the price gap on adjustment is $\hat{p}_i$.[7]

$B = \frac{1}{2}\,\eta\,(\eta - 1)$. In this case θ and ψ are measured as a proportion of static optimal profits per unit of time.

6. In the interpretation derived before, $p^*(t)$ is equal to the log of a constant markup over nominal marginal cost, where shocks to the nominal marginal cost are due to shocks to firm-specific productivity. Bils and Klenow (2004) estimate very persistent specific productivity process using U.S. data, they cannot reject the hypothesis of log-productivity follows a random walk with drift.

7. Sections B 2.1, B 2.2, and B 2.3 in the Online Appendix present detailed analyses of some special cases of this model featuring one cost only: the menu cost

Without loss of generality, we define the value function $V(\tilde{p})$ of the firm at the time of an observation of the price gap: $\tilde{p}$. Let $V_J(\tilde{p})$ denote a conditional value function, namely, the best value that the firm can achieve by making J price adjustments between observations. The unconditional value function for the firm problem is:

$$V(\tilde{p}) = \min_{J \geq 0} V_J(\tilde{p}) \quad \text{so that } J^*(\tilde{p}) = \arg\min_{J \geq 0} V_J(\tilde{p}), \tag{2}$$

which states that the firm will choose the optimal number of price adjustments and where we denote the minimizing choice as $J^*(\tilde{p})$.

We next describe the Bellman equations for the conditional value functions $V_J(\tilde{p})$. For $J = 0$, that is, conditional on no price adjustment between observations, we have

$$V_0(\tilde{p}) = \theta + \min_T \int_0^T e^{-\rho t} B\left[(\tilde{p} - \mu t)^2 + \sigma^2 t\right] dt + e^{-\rho T} \times \int_{-\infty}^{\infty} V\left(\tilde{p} - \mu T - s\sigma\sqrt{T}\right) dN(s)\,, \tag{3}$$

in this case the only choice for the firm is the time elapsed until the next review: T. The first integral on the right side gives the cost arising from the expected second moment of the price gap, while the second integral term denotes the expected continuation value. This continuation value uses that the price gap T periods ahead is normally distributed with expected value $\tilde{p} - \mu T$ and variance $\sigma^2 T$.

If $J \geq 1$, the function $V_J(\tilde{p})$ gives the optimal value *conditional* on making $J = 1, 2, 3, \ldots$ price adjustments. In this case the firm chooses the time until the next review, T, the times t_i of each price adjustment and the (expected) value of the price gap on each adjustment $\hat{p}_i$ (where $i = 1, 2, \ldots, J$). The Bellman equation is

$$V_J(\tilde{p}) = \theta + \min_{T, \{\hat{p}_i, t_i\}_{i=1}^J} \int_0^{t_1} e^{-\rho t} B\ (\tilde{p} - \mu t)^2\, dt + \int_0^T e^{-\rho t} B\sigma^2 t\, dt + \sum_{i=1}^{J-1} e^{-\rho t_i}\left[\psi + \int_0^{t_{i+1}-t_i} e^{-\rho t}\, B\,(\hat{p}_i - \mu t)^2\, dt\right] + e^{-\rho t_J}\left[\psi + \int_0^{T-t_J} e^{-\rho t} B\,(\hat{p}_J - \mu t)^2\, dt\right]$$

model ($\psi = 0$), the costly observation model ($\theta = 0$), and the Sheshinski and Weiss (1977) model ($\sigma = 0$).

$$+ e^{-\rho T} \int_{-\infty}^{\infty} V\left(\hat{p}_J - \mu(T - t_J) - s\sigma\sqrt{T}\right) dN(s)$$

$$\text{for} \quad J = 1, 2, 3, \ldots \tag{4}$$

where $0 \leq t_1 < \ldots < t_i < t_{i+1} < \ldots < t_J < T$. Note that if $t_1 > 0$ then the price gap $\tilde{p}$ is not adjusted immediately, that is, the first price adjustment takes place some time after the observation. In this case the first integral on the right side of the equality gives the losses due to the continuation of $\tilde{p}$ into the period, which is the only reason $\tilde{p}$ is an argument of $V_J(\cdot)$. Instead, if the firm chooses $t_1 = 0$, the price gap is adjusted immediately on review (and the value of this integral is 0). The last term in the first line measures the costs due to the expected variance term $\sigma^2\ t$, just like in Equation (3).

The second and third lines contain the losses corresponding to the J price adjustments, all of which are done using the same information: $\tilde{p}$. Each adjustment occurs t_i periods after the observation, involves a (discounted) menu cost ψ, and a reset of the (expected) price gap at the new value $\hat{p}_i$ (where $i = 1, \ldots, J$). After the adjustment, the expected value of the price gap $\hat{p}_i$ will depreciate with the inflation rate μ during the time interval until the next adjustment: $t_{i+1} - t_i$. The integral in the fourth line gives the expected continuation value. This continuation value uses that the price gap T periods ahead is normally distributed with expected value $\hat{p}_J - \mu\,(T - t_J)$ and variance $\sigma^2 T$.

We briefly comment on the nature of the trade-offs faced by the firm. If the price gap has a trend ($\mu \neq 0$), then by making several price adjustments between reviews (i.e., a large value of J) the firm is tracking the expected value of the price target more closely. Given J, the expected discounted value of the terms $B(\hat{p}_i - \mu t)^2$, measuring the average value recorded by the price gap between reviews, can be made smaller by an appropriate choice of $\hat{p}_i$ and t_i. On the other hand, a large number of price adjustments comes at the expense of paying ψ, the fixed menu cost, J times. The other firm decision concerns the time elapsed until the next observation T. The shorter this time, the more often the observation cost θ is paid. On the other hand, the shorter this time, the smaller the variance of the deviation from the price target that accumulates until the next observation is gathered, which corresponds to smaller values of the integral of the discounted value of the terms $B\sigma^2 t$. Essentially, the choice of J can be used to control the expected value of the price gap between observations, while the choice of T can be used to control the mean variance of the price gap. A numerical illustration of the value functions V_J is provided in the next section. Our analysis of this problem is divided in three sections. First, we study when multiple adjustments between observations are optimal. Second, we study the sensitivity of the problem to inflation around zero inflation. Third, we conduct an in-depth analysis of the case of zero inflation.

III. On the Optimality of Price Plans

We first argue that multiple price adjustments between price observations are a form of indexation to inflation. Let J^* defined in Equation (4) be $J^* \geq 3$, and fix i in $1 < i < J^*$. Consider the first-order condition for $\hat{p}_i, \hat{p}_{i-1}$, and t_i. The restriction on i means that there is at least one price adjustment before and one after time t_i. Simple algebra (letting $\rho \downarrow 0$ for simplicity), gives:

$$\hat{p}_{i-1} = \mu\,\frac{t_i - t_{i-1}}{2}\ ,\ \hat{p}_i = \mu\,\frac{t_{i+1} - t_i}{2}\ ,\ \text{and}\ \ t_i = \frac{t_{i+1} + t_{i-1}}{2}\ .$$

This shows the sense in which the optimal policy is an instance of indexation: adjustments are equally spaced in time, and prices increase with accumulated inflation, so that the expected price gap front loads $\frac{1}{2}$ of the inflation expected over the period until the next price adjustment.

Next, we turn to the discussion of when price plans are optimal. There are two extreme cases nested in our parametrization that imply infinitely many adjustments between observations. First, let the adjustment cost $\psi \downarrow 0$. In this case the firm will be adjusting infinitely often between observations ($J^* \to \infty, t_{i+1} - t_i \downarrow 0$, and $T > 0$), so that the price gap is expected to be 0 between observations. The value function converges to the one for the problem with observation cost only and no drift, which is akin to Reis (2006b). The second case is one where $\frac{\sigma}{\mu} \downarrow 0$. As $\sigma \downarrow 0$ the benefit of observing is tiny relative to its cost $\theta > 0$, and hence observations will be very infrequent. Yet with μ bounded away from 0, the benefits of adjustment are important, so there will be multiple adjustments between observations. This case is akin to the classic Sheshinski and Weiss (1977) model.

Next we characterize when multiple adjustments between reviews are optimal.

PROPOSITION 1. Let $\theta > 0$, $\psi > 0$, and $\sigma > 0$. There exists a threshold $\bar{\mu} > 0$ such that for all inflation rates $|\mu| < \bar{\mu}$: (i) $J^*(\tilde{p}) \leq 1$ for all $\tilde{p} \in \mathbb{R}$. (ii) if $\tilde{p}$ is such that $J^*(\tilde{p}) = 1$ then $t_1 = 0$. (iii) as $\rho \downarrow 0$ the threshold satisfies $\frac{\bar{\mu}}{\sigma^2} = m\left(\frac{\theta\sigma^2}{B}, \frac{\psi}{\theta}\right)$, where the function $m\left(\frac{\theta\sigma^2}{B}, \cdot\right)$ is strictly increasing in the neighborhood of $\frac{\psi}{\theta} = 0$.

Part (i) shows that it is optimal to adjust prices at most once between successive reviews, and part (ii) shows that price adjustments occur immediately after the price review for inflation rates below the threshold $\bar{\mu}$. Parts (i) and (ii) say that price plans are not optimal for inflation rates below the threshold $\bar{\mu}$. Part (iii) shows that the threshold is strictly increasing in the size of the menu cost ψ. In short, the proposition establishes a condition under which price plans are optimal. This is important because Mankiw and Reis (2002) and Burstein (2006) find that nominal aggregate shocks have larger and more persistent real effects when the firm adjustment rule is a price plan.

We computed $\bar{\mu}$ for three values of the ratio of the menu to the observation cost: $\alpha \equiv \frac{\psi}{\theta}$. This ratio will turn out to be useful in characterizing the optimal policy rules in later sections. We assume a markup over costs of about 15%, so that $B \cong 20$, and a volatility $\sigma = 0.15$, which are similar to the values chosen by Golosov and Lucas (2007).[8] We fix the observation cost at $\theta = 0.03$, so that the model-implied cost of a review amounts approximately to 0.4% of revenues. This magnitude is comparable to estimates by Zbaracki et al. (2004) for the managerial cost of changing prices associated to gathering information. For these parameters, the threshold value for μ is: $\bar{\mu} = 0.06$ for $\psi = 0.005$, $\bar{\mu} = 0.10$ for $\psi = 0.015$, and $\bar{\mu} = 0.13$ for $\psi = 0.030$. [9] Although Proposition 1 only establishes that $\bar{\mu}$ is increasing in ψ in a neighborhood of 0, in

8. Golosov and Lucas (2007) set the standard deviation of idiosyncratic productivity shocks to be 0.11 per year; Burstein and Hellwig (2006) set the standard deviation between 0.06 and 0.25 at the monthly frequency, depending on the specification of the model; Eichenbaum, Jaimovich, and Rebelo (2008) estimate the median standard deviation in cost to be 0.11 per week on micro data from a large U.S. retailer.

9. These three values of ψ correspond to $\alpha = 0.17$, $\alpha = 0.50$, and $\alpha = 1.0$, respectively. The case of $\psi = 0.015$ corresponds to an average of about 1.6 price adjustments per year for relatively low values of the drift μ, which is consistent with estimates by Nakamura and Steinsson (2008).

these examples—as well as in all the ones we have computed—it is increasing also away from 0.

Next we describe how the patterns for the frequency of price adjustments and reviews change for values of inflation above the threshold $\bar{\mu}$. Table I reports statistics for the frequencies of price adjustments and reviews as a function of $\mu > \bar{\mu}$. These statistics are obtained by solving the model numerically at the parameter values already discussed for the case of $\psi = 0.015$.[10] We want to emphasize two results. First, the level of the drift μ has to be substantially larger than the threshold $\bar{\mu} = 0.10$ before it is optimal to plan more than one adjustment between two consecutive reviews, that is, $J^*(\tilde{p}) \geq 2$ for some $\tilde{p}$. In fact, for intermediate values of the drift μ it is optimal to delay some of the price adjustments, that is, $t_1 > 0$ for some $\tilde{p}$, but it is not optimal to adjust prices more than once between consecutive reviews, that is, $J^*(\tilde{p}) \leq 1$ for all $\tilde{p}$. For instance, at the parametrization of Table I, we find that the size of the drift has to be larger than 50% before it is optimal to plan more than one adjustment between two consecutive reviews, $J^*(\tilde{p}) \geq 2$ for some $\tilde{p}$. In fact, for values of the drift above $\bar{\mu}$ but smaller than 50%, the fraction of price adjustments occurring at a different time from the observation date increases in the size of the drift, but it is never optimal to adjust prices more than once between consecutive observation dates. Second, the ratio of the frequency of price reviews to the

TABLE I

STATISTICS ON THE TIME OF ADJUSTMENTS AS A FUNCTION OF INFLATION (μ)

	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.5$	$\mu = 0.6$
n_a	1.65	1.88	2.21	2.60	2.96	4.07
$n_{a\|t_1>0}$	0.00	0.33	0.73	1.16	1.58	1.21
$n_{a\|J\geq 2}$	0.00	0.00	0.00	0.00	0.00	2.98
$\frac{n_r}{n_a}$	1.52	1.43	1.28	1.15	1.06	0.66

Note. Parameters values are $B = 20$, $\sigma = 0.15$, $\theta = 0.03$, and $\psi = \alpha\ \theta$; n_a denotes the average number of adjustments per year; $n_{a|t_1>0}$ denotes the average number of delayed adjustments per year; $n_{a|J\geq 2}$ denotes the average number of price adjustments conditional on at least two adjustments between consecutive reviews; n_r denotes the average number of reviews per year.

10. The Online Appendix reports the statistics from a parametrization that uses a smaller value of σ. Bonomo, Carvalho, and Garcia (2010) provide yet another solution to the problem presented in Equations (2)–(4) under a different parametrization.

frequency price adjustments, $\frac{n_r}{n_a}$, is decreasing in the size of the drift $|\mu|$, but $\frac{n_r}{n_a}$ is larger than 1 as long as the drift is small enough that multiple price adjustments between consecutive observation dates are not optimal. As a consequence, the size of the drift needed to make price adjustments more frequent than price reviews can be substantially larger than $\bar{\mu}$.

Finally, we discuss how the choice of price plans depends on the level of the price gap upon observation. To this aim, we plot in Figure I the value functions $V_J(\tilde{p})$ for $J = 0, 1, 2$, evaluated at the optimal policy, for the same parametrization of Table I in the case of $\mu = 0.6$. We find the discussion of the optimal policy at this high parametrization of μ useful to understand how the drift affects the decision problem. Figure I shows that for relatively high or relatively low values of the price gap $\tilde{p}$, it is optimal to plan two price adjustments before observing the state again, that is, $J^* = 2$. In this case, the first adjustment occurs immediately on observation to reduce the size of the price gap,

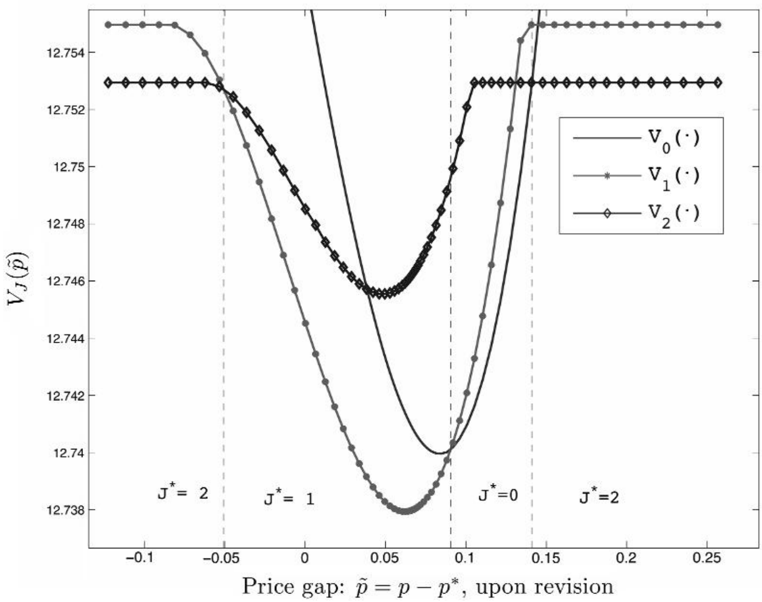

FIGURE I

Value Function $V_J(\tilde{p})$ in the Case of High Inflation: $\mu = 0.6$

Parameters values are $B = 20$, $\sigma = 0.15$, $\theta = 0.03$, and $\psi = 0.015$.

while the second adjustment occurs before the next observation date to reduce losses from the expected decrease in the price gap due to the positive drift. For intermediate values of $\tilde{p}$, two different cases arise. If the price gap is small enough, the firm plans to have one delayed price adjustment, that is, $J^* = 1$ and $t_1 > 0$, because the positive drift reduces the expected price gap, eventually increasing expected losses to the point that an adjustment is optimal before the next planned observation date. In the other case, instead, the price gap is large enough that it is optimal not to adjust the price before the next observation, that is, $J^* = 0$. In this case, despite inflation, the expected price gap does not decrease enough to justify a price adjustment before the next observation date.

IV. The Case of Low Inflation

In this section we analyze the effect of inflation on the frequency of price adjustments and the size distribution of price changes under the assumption that the inflation rate μ is positive but small. We restrict attention to $\sigma > 0$, $\psi > 0$, and $|\mu| < \bar{\mu}$, so that following Proposition 1 there is at most one price adjustment between observations, which occurs immediately after the observation.

In the case of $|\mu| < \bar{\mu}$, the general problem set up in Section II is given by Equations (2) and (3) and by the conditional value function for $J = 1$ which becomes:

$$V_1 = \theta + \psi + \min_{T,\hat{p}_1} \int_0^T e^{-\rho t} B\left[(\hat{p}_1 - \mu t)^2 + \sigma^2 t\right] dt + e^{-\rho T} \times \int_{-\infty}^{\infty} V\left(\hat{p}_1 - \mu T - s\sigma\sqrt{T}\right) dN(s), \tag{5}$$

where $\tilde{p}$ is *not* an argument of V_1 since $t_1 = 0$. The decision rules for this case are as follows. Since $\psi > 0$, there are thresholds $\underline{p} < \bar{p}$ defining the range of inaction where $V_1 > V_0(\tilde{p})$ for $\tilde{p} \in (\underline{p}, \bar{p})$, and satisfying $V_0(\underline{p}) = V_0(\bar{p}) = V_1$. Thus, prices are not adjusted if, immediately after an observation, $\tilde{p} \in (\underline{p}, \bar{p})$. Otherwise, if upon an observation the price gap is outside the range of inaction $(\underline{p}, \bar{p})$, the cost ψ will be paid and prices will be adjusted to set the price gap to $\hat{p}_1$. Because in this case there is at most one adjustment, we refer to the optimal return point simply as $\hat{p}$. Finally, we use $\mathrm{T}(\tilde{p})$ to denote the optimal time until the next review decided on observing a price gap $\tilde{p}$.

We use the optimal decision rule summarized by $\underline{p}$, $\bar{p}$, $\hat{p}$, and $\mathrm{T}(\tilde{p})$ to define a first-order Markov process for the price gap, $\tilde{p}$, whose transition function is as follows. If $\tilde{p} \in (\underline{p}, \bar{p})$, there is no price adjustment, and the value of the price gap $\mathrm{T}(\tilde{p})$ periods ahead is normally distributed with variance $\sigma^2 \mathrm{T}(\tilde{p})$ and expected value $\tilde{p} - \mu \mathrm{T}(\tilde{p})$. Otherwise, when $\tilde{p} \notin (\underline{p}, \bar{p})$, there is a price change of size $\Delta p = \hat{p} - \tilde{p}$, the next observation is in $\mathrm{T}(\tilde{p})$ periods from now, and the price gap is normally distributed with variance $\sigma^2 \mathrm{T}(\tilde{p})$ and expected value $\hat{p} - \mu \mathrm{T}(\tilde{p})$. This process for the price gap can be used to define several statistics of interest: the expected time between observations of the price gap, and its reciprocal, that is, the average frequency of price reviews which we denote by n_r; the expected time between price adjustments; and the associated average frequency of adjustment denoted by n_a. Likewise, we can define the hazard rate of price changes, as a function of the time elapsed since the last price adjustment, denoted by $h(t)$. Finally, we can define the distribution of (nonzero) price changes Δp, with density $w(\Delta p)$, and the density of the absolute value of price changes $v(|\Delta p|)$.

For the next proposition we explicitly write μ as an argument of the value function $V(\tilde{p}, \mu)$, and of the statistics such as $n_a(\mu)$, $\mathbb{E}[\Delta p, \mu]$. We also define $\mathbb{E}[V](\mu)$ to be the expected value of the value function under the invariant distribution of the price gaps $g(\cdot)$ as $\mathbb{E}[V](\mu) \equiv \int_{-\infty}^{\infty} V(\tilde{p}, \mu) g(\tilde{p}, \mu) d\tilde{p}$. We have the following.

PROPOSITION 2. Assume that $\sigma > 0$ and that all the functions below are differentiable:

$$(a)\ \frac{\partial}{\partial \mu} n_a(\mu)\big|_{\mu=0} = \frac{\partial}{\partial \mu} n_r(\mu)\big|_{\mu=0} = 0,$$

$$\text{and } \frac{\partial}{\partial \mu} h(t, \mu)\big|_{\mu=0} = 0 \text{ for all } t \geq 0,$$

$$(b)\ \frac{\partial}{\partial \mu} \mathbb{E}[\Delta p, \mu]\big|_{\mu=0} = \frac{1}{n_a(0)} > 0 \text{ and } \frac{\partial^2}{\partial \mu^2} \mathbb{E}[\Delta p, \mu]\big|_{\mu=0} = 0,$$

$$(c)\ \frac{\partial}{\partial \mu} \mathbb{E}\left[(\Delta p - \mathbb{E}[\Delta p])^{2k}, \mu\right]\Big|_{\mu=0} = 0, \text{ for } k = 1, 2, \ldots,$$

$$(d)\ \frac{\partial}{\partial \mu} v(|\Delta p|, \mu)\big|_{\mu=0} = 0 \text{ for all } |\Delta p| > \bar{p}(0), \text{ and}$$

$$(e)\ \frac{\partial}{\partial \mu} \mathbb{E}[V](\mu)\big|_{\mu=0} = 0.$$

Part (a) shows that the average number of adjustments per unit of time, $n_a(\mu)$, is insensitive to inflation at $\mu = 0$. The average frequency of price reviews, $n_r(\mu)$, and the whole hazard rate function of price adjustment, $h(t,\mu)$, are also insensitive to inflation at $\mu = 0$. Part (b) states that the expected value of price changes increases linearly with μ with slope $\frac{1}{n_a(0)}$, at least for small values of $\mu = 0$. This follows from (a) and from the identity: $\mu = n_a(\mu)\,\mathbb{E}[\Delta p, \mu]$, that is, that the product of the average price change times the number of adjustments equals the inflation rate.

The result that the "intensive" margin of price adjustment is insensitive to inflation at $\mu = 0$ applies to the special case of models with menu cost only ($\theta = 0$), as illustrated in the numerical results reported in Figure 3 of Golosov and Lucas (2007), when $\sigma > 0$. The proof of each of these results, as well as of the other parts of this proposition, is based on the symmetry of the problem, stated precisely in Lemma 1 in the Appendix. Given the symmetry of the loss function and the distribution of shocks, is it easy to see that $n_a(\mu)$=$n_a(-\mu)$. Thus, if n_a is differentiable at 0, then it must be flat. Indeed for the case of $\theta = \sigma = 0$, which corresponds to the model in Sheshinski and Weiss (1977), the "insensitivity result" does not hold, because although the function n_a is symmetric, it has a kink at $\mu = 0$. We conjecture, but have not proven, that as long as $\sigma >$ 0, all these functions are differentiable at $\mu = 0$. The economics are clear: the effect of small inflation is swamped by idiosyncratic shocks when $\sigma > 0$. This is consistent with (unreported) numerical results showing that for a higher σ, the function $n_a(\mu)$ remains flat for a bigger interval of inflation rates μ.

Proposition 2 studies the sensitivity of several statistics to inflation in the vicinity of $\mu = 0$. Figure II complements this proposition by computing several of these statistics for values of μ between 0 and 5% annual inflation for the parameters described at the end of Section III. For instance, the insensitivity of n_a with respect to inflation is illustrated in the top left panel of Figure II, where it is clearly seen that $n_a(\mu)$ is constant for a range up to 5% annual inflation and three values of $\alpha = \frac{\psi}{\theta}$.

The theoretical result about the insensitivity of n_a—and the associated linearity of $\mathbb{E}[\Delta p]$—is supported by the evidence in Gagnon (2009) who, among others, finds that when inflation is low (say below 10–15%), the frequency of price changes is almost unrelated to inflation, and that the average magnitude of price changes has a tight linear relationship with inflation.

To understand (c) and (d) it is useful to realize that for $\mu = 0$ the distribution of price changes is symmetric around 0, a consequence of the symmetry of the loss function and of

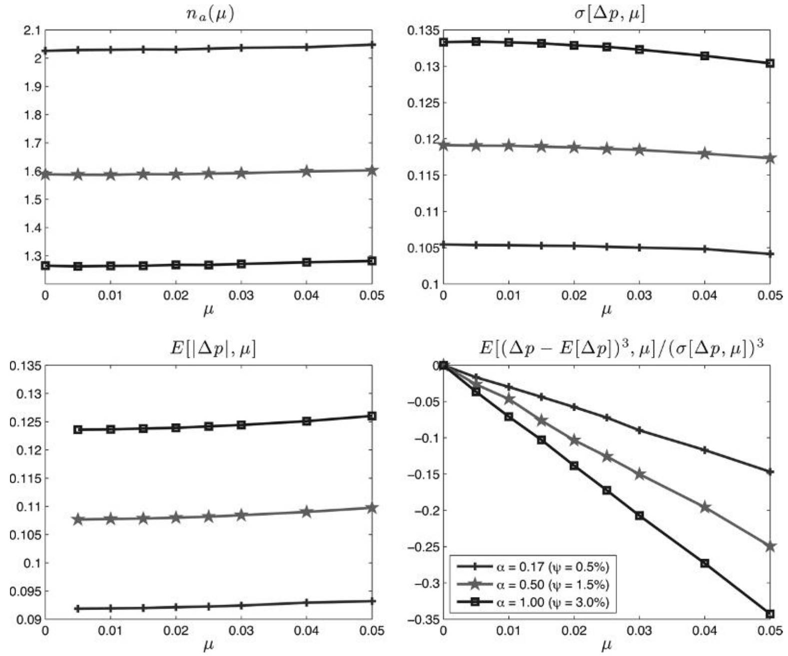

FIGURE II

Statistics from the Distribution and Frequency of Price Changes

Parameters values are $B = 20$, $\sigma = 0.15$, $\theta = 0.03$, and $\psi = \alpha\,\theta$.

the distribution of the shocks. Part (c) shows that all the even centered moments are approximately the same for 0 and low inflation. This is illustrated in the top right panel of Figure II where we plot the standard deviation of price changes for annual inflation rates below 5% and three values of α. Yet inflation has a first-order effect on other aspects of the distribution of price changes, which is illustrated in the bottom right panel where we plot its skewness, which starts at 0 when there is no inflation and decreases with μ. Part (d) shows that the whole distribution of the *absolute value* of price changes is approximately the same for low and zero inflation, which is illustrated in the numerical example in the bottom left panel of Figure II for the expected value of this distribution. Finally, part (e) shows that while inflation has a first-order effect on some features of the decision rules, such as $(\underline{p}, \bar{p}, \hat{p})$, it has only a second-order effect on the expected value function. Equivalently, inflation causes a second-order increase in the unconditional expectation of losses for the firm.

These results show that the expected losses of the firm as well as the frequency and several moments of the size distribution of price changes are insensitive to inflation at $\mu = 0$. Thus, the analysis of the problem in a low-inflation environment is well approximated by studying the case of zero inflation, to which we turn next.

V. Optimal Decision Rules for the Case of Zero Inflation

Proposition 1 and the assumption of zero inflation imply that it is optimal to have at most one adjustment, $J^*(\tilde{p}) \in \{0, 1\}$, and that conditional on adjustment there is no delay, $t_1 = 0$. We write the special case of Equations (2)–(3) and Equation (5) for $\mu = 0$ as:

$$V_0(\tilde{p}) = \theta + \min_T B \int_0^T e^{-\rho t}\,[\tilde{p}^2 + \sigma^2 t]\,dt + e^{-\rho T} \times \int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{T}\right) dN(s), \tag{6}$$

$$V_1 = \psi + \theta + \min_{T,\hat{p}} B \int_0^T e^{-\rho t}\,[\hat{p}^2 + \sigma^2 t]\,dt + e^{-\rho T} \times \int_{-\infty}^{\infty} V\left(\hat{p} - s\sigma\sqrt{T}\right) dN(s), \tag{7}$$

$$V(\tilde{p}) = \min\{V_0(\tilde{p})\,,\ V_1\}, \tag{8}$$

for $\tilde{p} \in (-\infty, \infty)$, where T in Equations (6) and (7) is the optimal decision rule for the time between observations, a policy function that will be described by the function $\mathrm{T}(\tilde{p})$. It is immediate that the value function V is symmetric around $\tilde{p} = 0$ and increasing around it, hence if a price adjustment takes place the optimal reset price is $\hat{p} = 0$. Hence the optimal time between reviews conditional on adjustment is $\tau \equiv \mathrm{T}(0)$, the optimal choice in Equation (7). We show that $\mathrm{T}(\tilde{p})$ has a maximum at $\tilde{p} = 0$, attaining the value τ, and that it is symmetric around 0 with an inverted U-shape. The symmetry established in Lemma 1 for $\mu = 0$ implies that $\underline{p} = -\bar{p}$. We summarize these results in the next proposition.

Proposition 3. Let $\mu = 0$. The value function V is symmetric around $\tilde{p} = 0$, and V is strictly increasing in $\tilde{p}$ for $0 < \tilde{p} < \bar{p}$. The optimal policy conditional on adjustment is $\hat{p} = 0$. The derivative of $V_0(\tilde{p})$ for $0 \le \tilde{p}$ satisfies $0 \le V_0'(\tilde{p})$ with strict inequality if $\mathrm{T}(\tilde{p}) > 0$. Thus $V'(0) = V_0'(0) = 0$, $V''(0) > 0$, and $V'(\tilde{p}) = 0$, for $\tilde{p} > \bar{p}$ and hence V is not differentiable at $\tilde{p} = \bar{p}$.

The proposition also shows that at the boundary of the range of inaction, $\bar{p}$, the value function has a kink, that is, there is *no smooth pasting*. This differs from the model with menu cost which only features the smooth pasting property, typical of continuous-time fixed-cost models, see for example Dixit (1993) and Stokey (2008).

The next proposition characterizes the function $\mathrm{T}(\cdot)$. The proof is based on a second-order expansion of the first-order condition with respect to T, and expansions of $\mathrm{T}(\cdot)$ and $V(\cdot)$ around $\tilde{p} = 0$, as well as their symmetry.

PROPOSITION 4. As $\rho \downarrow 0$ the optimal rule for the time to the next revision $\mathrm{T}(\tilde{p})$ is:

$$\mathrm{T}(\tilde{p}) = \tau - \left(\frac{\tilde{p}}{\sigma}\right)^2 + o(|\tilde{p}^3|) \text{ for } \tilde{p} \in (-\bar{p},\ \bar{p}),$$
$$\text{and} \quad \mathrm{T}(\tilde{p}) = \tau \text{ otherwise.} \tag{9}$$

Figure III displays $\mathrm{T}(\cdot)$ for the numerical example discussed at the end of Section III. We found the approximation for $\mathrm{T}(\cdot)$ in Proposition 4 to be precise for a large range of economically interesting parameters, as the comparison in Figure III shows (see the Online Appendix for more documentation). A few comments are in order. First, the shape of the optimal decision rule depends only on σ, and not on the other parameters: B, θ, and ψ. Second, if the agent finds herself after a review with a price gap $\tilde{p} = 0$, she will set $\mathrm{T}(0) = \tau$, since the optimal adjustment would have implied a post adjustment price gap of 0. Third, the function $\mathrm{T}(\tilde{p})$ is decreasing in (the absolute value of) $\tilde{p}$. If on a review the agent finds the price gap close to the boundary of the range of inaction, she plans for a relatively early review, since the target is likely to cross the threshold $\bar{p}$. Fourth, the price gap is normalized by σ, the standard deviation of the changes in (the log of) the target price. This is also natural, because the interest of the decision maker is on the likelihood that the price target will deviate and hit the barriers, so that for a lower σ she is prepared to wait more for the same price gap $\tilde{p}$.

Next, we compute an analytical approximation to the value function and optimal policies. The approximation relies on the fact that $V(\cdot)$ is symmetric around $\tilde{p} = 0$, that is, $V(\tilde{p}) = V(-\tilde{p})$, and hence all the derivatives of odd order are 0. The approximation uses a quadratic expansion of $V_0(\tilde{p})$ around $\tilde{p} = 0$, because the

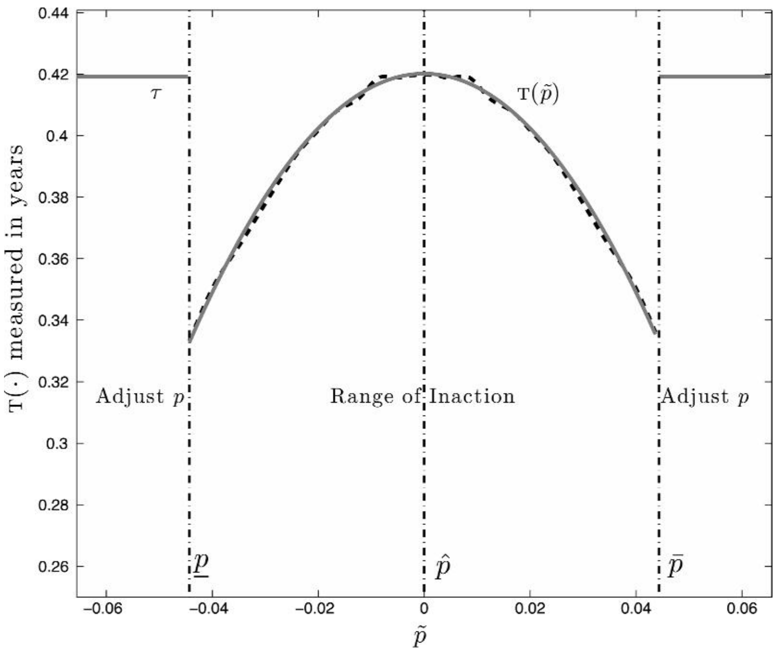

FIGURE III

Optimal Decision Rule $\tau(\tilde{p})$

Parameter values: $B = 20$, $\sigma = 0.15$, $\theta = 0.03$, $\psi = 0.015$. Vertical dashed-dotted lines: thresholds and optimal return point. Solid line: function in Equation (9); dashed line: numerical solution to optimal policy.

conditional value function V_1 is constant:

$$V_0(\tilde{p}) = V_0(0) + \frac{1}{2}V_0''(0)\,(\tilde{p})^2 + o(|\tilde{p}|^3) \approx V_0(0) + \frac{1}{2}V_0''(0)\,(\tilde{p})^2,$$

since $V_0'(0) = V_0'''(0) = 0$ and $V_0''(0) > 0$. We refer to the left-hand side of this expression as the quadratic approximation even though, since $V_0'''(0) = 0$, the remainder is of order smaller than $|\tilde{p}^3|$. The other source of approximation, to simplify the analytical expressions, is that we let ρ converge to 0.[11] The quadratic approximation for the value function is globally accurate if the range of inaction, $[-\bar{p},\, \bar{p}]$, is small. Since $\bar{p}$ converges to 0 as the menu cost ψ goes to 0, the approximation will be accurate for small values of ψ relatively to θ. We discuss the accuracy of these

11. This second approximation has negligible effects on the accuracy of the solution given the small discount rates that are appropriate for this problem.

approximations shortly. Proposition 5 uses these approximations to characterize the values of $\bar{p}$ and τ. To this end, it is convenient to define the variable $\phi \equiv \frac{\bar{p}}{(\sigma\sqrt{\tau})}$, which measures the minimum size of the innovation of a standard normal required to get out of the inaction region, $[-\bar{p}, \bar{p}]$, after resetting the price to $\hat{p} = 0$. We refer to the variable ϕ as the determinant of the "normalized range of inaction," $[-\phi, \phi]$.

PROPOSITION 5. Define $\alpha \equiv \frac{\psi}{\theta}$ and $\phi \equiv \frac{\bar{p}}{(\sigma\sqrt{\tau})}$, and assume that $\theta > 0$ and $\psi > 0$ and $\alpha < \left(\frac{1}{2} - 2(1 - N(1))\right)^{-1} \approx 5.5$. As $\rho \downarrow 0$, there exists a unique solution for $\bar{p}$ and τ, in that solution ϕ is a function of 2 arguments: the normalized costs $\left(\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B}\right)$. For small values of $\sigma^2 \frac{\psi}{B}$, the solution $\phi(\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B})$ is approximated by $\varphi(\alpha)$ which solves

$$(10) \qquad 1 = \varphi(\alpha)^2 \left(\frac{2}{\alpha} + 4\left[1 - N\left(\varphi(\alpha)\right)\right]\right) \text{ , with elasticity}$$

$$(11) \qquad \frac{\partial \log \varphi(\alpha)}{\partial \log \alpha} = \frac{1}{2} \text{ at } \alpha = 0 \text{ and } \frac{\partial \log \varphi(\alpha)}{\partial \log \alpha} < \frac{1}{2} \text{ for } \alpha > 0\ ,$$

such that $\varphi(\alpha) - \phi(\sigma^2 \frac{\theta}{B}, \sigma^2 \frac{\psi}{B}) = o\left(\left[\sigma^2 \frac{\psi}{B}\right]\right)$. The optimal values for the time until the next revision after an adjustment, τ, and the width of the range of inaction, $\bar{p}$, are given by

$$(12) \qquad \tau = \sqrt{\frac{\theta}{\sigma^2\, B}}\ \frac{\sqrt{\alpha}}{\varphi(\alpha)} \quad > \quad \tau\ |_{\psi=0} \ = \ \sqrt{\frac{\theta}{\sigma^2\, B}\, 2}$$

$$(13) \qquad \bar{p} = \left[\sigma^2 \frac{\psi}{B}\right]^{\frac{1}{4}} \ \sqrt{\varphi(\alpha)} \ < \ \bar{p}\ |_{\theta=0} \ = \ \left[\sigma^2 \frac{\psi}{B}\, 6\right]^{\frac{1}{4}} \ .$$

While the proof of the proposition involves some algebra, the logic follows three simple steps. First, we develop a system of two equations in two unknowns, the equations are the first-order condition for τ and the value matching condition at $\bar{p}$, that is, $V_0(\bar{p}) = V_1$. These equations are simplified by using the quadratic approximation for V_0 and by letting $\rho \downarrow 0$. Second, a bit of analysis of these equations shows that under the conditions stated in the proposition, the solution is unique and well defined (i.e., it implies $\mathrm{T}(\bar{p}) > 0$). Third, we obtain an approximation for ϕ, namely, φ.

Proposition 5 shows that the expressions in Equations (12) and (13) are the generalizations of the corresponding formulas for the case in which there is only an observation or a menu cost, respectively: for a given ratio of the cost α, they have the same functional form. The equations show that the length of time until the next revision, τ, is higher in the model with both costs than in the model with observation cost only, and the width of the inaction band, $\bar{p}$, is smaller than in the menu cost model.[12] Notice that τ is higher because the introduction of the menu cost increases the cost of one price adjustment (from θ to $\theta+\psi$) but not the benefit. As a consequence firms optimally economize on the number of times they pay the cost. The reason $\bar{p}$ is smaller than in the menu cost case is more subtle. In the pure menu cost model observations are free, that is, the firm can monitor when the state crosses the threshold at no cost. But with an observation cost this is not true, and when the firm discovers to be "sufficiently close" to the barrier it prefers to adjust rather than having to pay again for observing when exactly the barrier is crossed. In other words, both barriers shift inwards. More specifically, using Equations (11) into (12) and (13) we obtain the following elasticities with respect to the two costs:

$$0 \leq \frac{\partial \log \tau}{\partial \log \psi} = \frac{1}{2} - \frac{\partial \log \varphi(\alpha)}{\partial \log \alpha} \leq \frac{1}{2}$$

$$\text{and} \quad 0 < \frac{\partial \log \tau}{\partial \log \theta} = \frac{\partial \log \varphi(\alpha)}{\partial \log \alpha} \leq \frac{1}{2}, \tag{14}$$

$$0 \leq \frac{\partial \log \bar{p}}{\partial \log \psi} = \frac{1}{2}\left(\frac{1}{2} + \frac{\partial \log \varphi(\alpha)}{\partial \log \alpha}\right) \leq \frac{1}{2}$$

$$\text{and} \quad 0 > \frac{\partial \log \bar{p}}{\partial \log \theta} = -\frac{1}{2}\frac{\partial \log \varphi(\alpha)}{\partial \log \alpha} \geq -\frac{1}{2}. \tag{15}$$

Equations (14) and (15) show that the time to the next review after a price adjustment, τ, is increasing in θ; similarly, the width of the inaction band is increasing in ψ. In fact, as the direct cost associated to a review (adjustment) increases, the frequency with which the firm reviews the state (adjust the price) decreases. This is a result inherited from models with observation cost or menu cost only. However, our model with both costs has additional implications for the interaction between the optimal values of τ and $\bar{p}$ and the costs θ and ψ. Equation (14) shows that τ is *increasing* in ψ, because at the time of deciding the next

12. Also note that, from Equations (11) and (12) it follows that τ is weakly increasing in the ratio of the cost α, with elasticity less than $\frac{1}{2}$.

observation after a price adjustment, an agent facing a higher adjustment cost maximizes the chances of adjusting the price on observation by delaying the next observation. Equation (15) shows that the width of the inaction band is *decreasing* in θ, since at the time of an observation an agent facing a higher cost minimizes the chances of paying further observation cost by narrowing the range of inaction. Moreover, the elasticity of the width of the inaction region, $\bar{p}$, with respect to ψ is smaller than in the menu cost model. In particular, the larger the ratio of θ to ψ, the smaller the elasticity of $\bar{p}$, with respect to ψ. Similarly, the elasticity of the time to the next review after a price adjustment, τ, with respect to θ is smaller than in the model with observation cost only, and it is decreasing in the ratio of ψ to θ. We note that the width of the inaction band has elasticity $\frac{1}{4}$ with respect to $\frac{\sigma^2}{B}$, as in the menu cost model, and that the time to the next review after a price adjustment has elasticity equal to $-\frac{1}{2}$ with respect to $B\sigma^2$, as in the model with observation cost only.

To summarize, we conclude this section with a discussion of the nature of our approximations derived in Propositions 4 and 5 and their accuracy. As asserted in the propositions, as well as in the comments that follow them, our analytical approximations are valid for small values of the discount rate ρ and of the fixed cost $\frac{\psi}{B}$, and for values of $\frac{\theta}{B}$ and σ that are bounded away from 0. These approximations are common in the literature.[13]

Now we discuss the accuracy of our analytical approximate solution by comparing it with a numerical solution of the original problem obtained on a very tight grid. To understand the type of error produced by the value function approximation, recall that we are using that $\rho \downarrow 0$ and V_0 is quadratic in $\tilde{p}$. The effect of the discount rate is almost negligible for any reasonable value of ρ, a result found in many similar problems. We focus on the other assumption, which is more specific to our problem. Since the function V_0 is symmetric and has a minimum at $\tilde{p} = 0$, a quadratic approximation is accurate around $\tilde{p} = 0$. Also, recall that Proposition 3 shows that the function is increasing for all $\tilde{p} < \bar{p}$. Thus, because $\bar{p}$ tends to 0 as ψ goes to 0, the relevant range of V_0, given by $[-\bar{p}, \bar{p}]$, is small, and hence the approximation is accurate if ψ is small. We find the approximation for $\bar{p}$ to be

13. For instance, the small ρ and small $\frac{\psi}{B}$ are already used in the analytical results for the case of menu cost only in Dixit (1991). Analogously, Reis (2006b) consider a "perturbation," which is equivalent to our assumptions.

accurate for values of $\alpha = \frac{\psi}{\theta} < 2$.[14] On the other hand, when ψ is large relative to θ, the quality of the approximation deteriorates. In particular, as θ goes to 0, the problem converges to the menu cost model studied by Dixit (1991). The value function V_0 in this model is convex close to $\tilde{p} = 0$, and concave around $\bar{p}$, to satisfy smooth pasting. It is easy to see that this implies that $V_0''''(0) < 0$. Thus, as θ becomes small relative to ψ, the value function V_0 becomes closer to the one of the menu cost, and hence our quadratic approximation becomes worse, especially for values of $\tilde{p}$ away from 0. In particular, since the quadratic approximation has $V_0''''(0) = 0$, it produces higher values of $V_0(\tilde{p})$ for $\tilde{p}$ away from 0, and consequently the value of $\bar{p}$ that we obtain tends to be smaller than the true one when $\frac{\psi}{\theta}$ is large.

VI. Statistics for the Case of Zero Inflation

In this section we maintain the assumption of zero inflation and characterize the implications of the optimal policy rule for the following statistics of interest: the frequency of price revisions, the frequency of price adjustment, the distribution of price adjustment, and the hazard rate for price changes. The decision rules described by the threshold $\bar{p}$ and function $\mathrm{T}(\cdot)$ imply a stationary Markov process for the price gap on review (and before adjustment). Assume the price gap immediately after an observation at time t_0 is $\tilde{p} \in \mathbb{R}$. Define the price adjustment rule $\Delta(\tilde{p})$ as 0 in the inaction region, and $-\tilde{p}$ otherwise. The next observation will take place in $T' = \mathrm{T}(\tilde{p})$ periods. Let s be a standard normal random variable and let $\tilde{p}'$ be the price gap upon the next review (and before adjustment) at time $t_0 + \mathrm{T}(\tilde{p})$. Then:

$$
(16) \quad p' \, , \, T' = \begin{cases} \tilde{p} - s\,\sigma\,\sqrt{\tau - \left(\frac{\tilde{p}}{\sigma}\right)^2} \, , \; \tau - \left(\frac{\tilde{p}}{\sigma}\right)^2 & \text{if } \tilde{p} \in (-\bar{p}, \bar{p}). \\ -s\,\sigma\,\sqrt{\tau} \, , \quad \tau & \text{if } \tilde{p} \notin (-\bar{p}, \bar{p}). \end{cases}
$$

We denote the density of the invariant distribution for the price gap on review (and before adjustment) as $g(\cdot)$. We define the expected time between reviews, denoted by $\mathcal{T}_r$, as the expected value of T under $g(\cdot)$. Thus the average number of reviews per unit of time is $n_r = \frac{1}{\mathcal{T}_r}$. Tracking the process starting right after a price change, one can compute the probability distribution of the times

14. See the Online Appendix for more documentation.

t elapsed between consecutive price changes. This distribution implies the expected time between price changes, denoted by $\mathcal{T}_a$, and thus the expected number of adjustments per unit of time $n_a = \frac{1}{\mathcal{T}_a}$ and more generally the hazard rate of price changes $h(t)$. Finally, this process implies a distribution of price changes, whose density we denote by $w(\Delta p)$. As mentioned, price changes occur when the price gap falls in the action region, in which case $\Delta p = -\tilde{p}$, and thus the density $w(\cdot)$ can be easily computed using the density $g(\cdot)$ and the price adjustment rule.

We define the normalized price gap as $y \equiv \frac{\tilde{p}}{\sigma\sqrt{\tau}}$. Using the approximation for $\mathrm{T}(\,\cdot\,)$ given in Equation (9) and the evolution of the price gap in Equation (16), the law of motion for the normalized price gap $\{y\}$ is

$$y'\ ,\ T' = \begin{cases} y - s\sqrt{1-y^2}\,, & \tau(1-y^2) \quad \text{if } y \in (-\phi, \phi) \\ \\ -s\,\sigma\,\sqrt{\tau} \quad , & \tau \quad \text{if } y \notin (-\phi, \phi). \end{cases} \tag{17}$$

where we use the notation $\phi \equiv \frac{\bar{p}}{(\sigma\sqrt{\tau})}$ for the barriers introduced in Proposition 5, which showed that in the approximate solution the normalized barrier depends only on the ratio of the costs, $\phi = \varphi(\alpha)$, and that τ depends on $\alpha = \frac{\psi}{\theta}$ as well as on $\frac{\theta}{(\sigma^2 B)}$. Equation (17) reveals an interesting property of the model: only the parameter ϕ, and the time interval τ appear in the law of motion of the normalized price gap. We now use this result and the law of motion for the price gaps derived above to state a useful result.

PROPOSITION 6. Fix $\alpha = \frac{\psi}{\theta}$. Use the decision rules in Proposition 5 to define the parameters $\mathbf{P} \equiv \left(\frac{\theta\sigma^2}{B}\right)^{\frac{1}{4}} \left(\frac{\sqrt{\alpha}}{\varphi(\alpha)}\right)^{\frac{1}{2}}$ and $\mathbf{T} \equiv \left(\frac{\theta}{B\sigma^2}\right)^{\frac{1}{2}} \left(\frac{\sqrt{\alpha}}{\varphi(\alpha)}\right)$. The density function of the price gaps upon review $g(\tilde{p})$, the density function of the price changes $w(\Delta p)$, and the hazard rate of price changes $h(t)$, are functions only of two parameters and are homogeneous of degree -1 with respect to their argument and one parameter, as follows:

$$g\,(\tilde{p}\ ;\ \mathbf{P}, \alpha) = \frac{1}{\mathbf{P}}\, g\left(\frac{\tilde{p}}{\mathbf{P}}\ ;\ 1,\ \alpha\right) \text{ for all } \tilde{p} \in \mathbb{R}\,,$$

$$w\,(\Delta p\ ;\ \mathbf{P}, \alpha) = \frac{1}{\mathbf{P}}\, w\left(\frac{\Delta p}{\mathbf{P}}\ ;\ 1, \alpha\right) \text{ for all } \Delta p \in \mathbb{R}\,,$$

$$h\,(t\ ;\ \mathbf{T}, \alpha) = \frac{1}{\mathbf{T}}\, h\left(\frac{t}{\mathbf{T}}\ ;\ 1, \alpha\right) \text{ for all } t \geq 0\,.$$

As shown in the proof, the new parameters are related to the optimal policy rules determined in Proposition 5, in particular $\mathbf{P} = \sigma\sqrt{\tau}$ and $\mathbf{T} = \tau$. The economics of this proposition is that the "shape" of the statistics that we are studying depends only on α, even though the "level" of those statistics also depends on a scaling factor.

VI.A. Average Frequency of Price Changes and Reviews

Let us consider the expected time between price changes $\mathcal{T}_a$ and the expected time between reviews $\mathcal{T}_r$. The proposition implies that these are functions only of α and $\mathbf{T}$, and that they are homogeneous of degree 1 with respect to $\mathbf{T}$, as follows:

$$\mathcal{T}_a(\alpha, \mathbf{T}) = \mathbf{T}\, \mathcal{T}_a(\alpha, 1)\ ,\ \ \mathcal{T}_r(\alpha, \mathbf{T}) = \mathbf{T}\, \mathcal{T}_r(\alpha, 1)\ .$$

We now comment on the two implications concerning the elasticity of the average review and adjustment frequencies with respect to its two arguments: $\frac{\theta}{(B\ \sigma^2)}$ and α. First, using the definition of $\mathbf{T}$ it is immediate that the elasticity with respect to $\frac{\theta}{(B\ \sigma^2)}$ is $\frac{1}{2}$: an increase of θ and ψ in the same percentage, which keeps α constant, decreases both frequencies by half of that percentage. This $\frac{1}{2}$ elasticity is present in the models that feature either information cost only or menu cost only. A novel element of our model relative to the polar cases with only one cost is the complementarity between observation and adjustment, which are fully captured in the statistics by the separate argument α. For instance, for a fixed θ, it can be shown that the number of reviews is decreasing in α, that is, more expensive menu costs will induce the firm to review less often.[15]

The second implication is that since the frequency of review $n_r = \frac{1}{\mathcal{T}_r(\alpha, \mathbf{T})}$ and the frequency of adjustment, $n_a = \frac{1}{\mathcal{T}_a(\alpha, \mathbf{T})}$, are proportional to the factor $\frac{1}{\mathbf{T}}$, then their ratio depends *only* on α:

$$\frac{n_r}{n_a} = \mathcal{F}(\alpha)\ \in [1, +\infty]\ . \tag{18}$$

This function is plotted in the left panel of Figure IV. The plot shows that $\frac{n_r}{n_a} = 1$ for $\alpha = 0$. This is immediate because in the absence of menu cost every observation produces an adjustment.

15. The propositions 9, 10 and 11 in Alvarez, Lippi, and Paciello (2010) provide analytical expressions to compute the values of n_a and n_r and characterize analytically the partial derivatives of these functions with respect to θ, ψ, α.

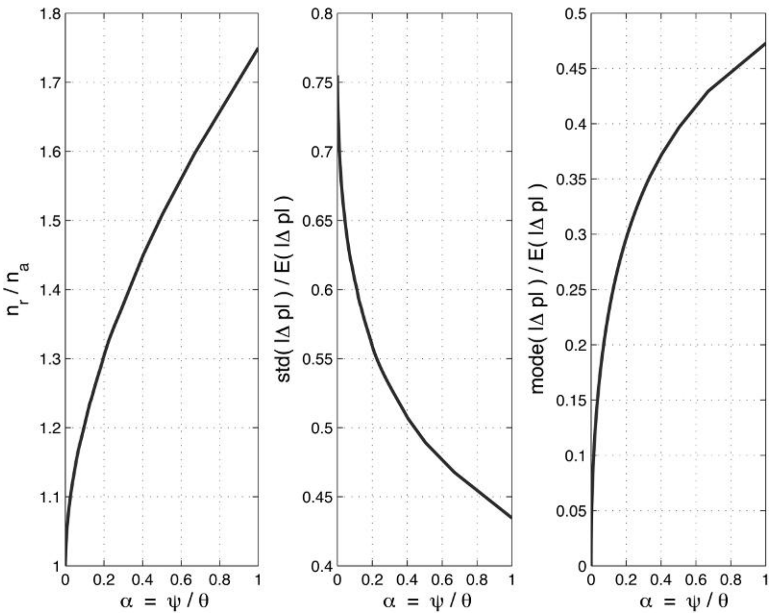

FIGURE IV

Mapping between α and Three Observable Statistics

As α increases the number of reviews per adjustment increases, that is, more reviews per adjustment occur. The figure shows that the function $\left(\frac{n_r}{n_a} - 1\right)$ is well approximated by the square root of α, so that its elasticity is approximately $\frac{1}{2}$. Matched with suitable observations on $\frac{n_r}{n_a}$, this function can be used to estimate the magnitude of α. We return to this point shortly.

As an application of the complementarity between reviews and adjustments, consider Figure V, which plots the sector average frequencies of price reviews and changes across different sectors in different countries.[16] From the fact that reviews are more frequent than adjustments, one could naively infer (using the models with only one cost) that adjusting prices is costlier than observing and the menu cost is the most relevant margin in the firm's problem. Yet as the left panel of Figure IV shows, the fact that reviews are more frequent than adjustments is consistent with observations cost that are much bigger than menu costs,

16. The sources of the data are the surveys described by Fabiani et al. (2007) and Greenslade and Parker (2008). Sectors are classified according to two-digit NACE. See our Data Appendix for more documentation.

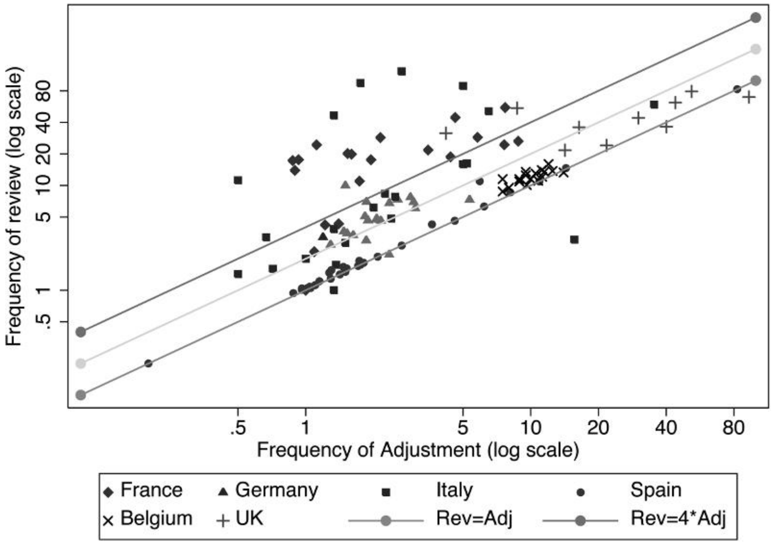

FIGURE V

Average Industry Frequency of Price Changes vs. Adjustments

Each data point is the mean number of price changes and price reviews per year in a country/industry pair. See the Data Appendix for more documentation.

that is, α much smaller than 1. This can help reconcile the view of researchers that observation costs are much bigger than pure menu costs, see, for example, Reis (2006a) and Zbaracki et al. (2004), with the statistics of Figure V.

The characterization of n_r, n_a and $\frac{n_r}{n_a}$ is useful to interpret the data of Figure V. The intercept of a line in the $(\log n_a, \log n_r)$ plane is related to α, with a positive elasticity. For a given intercept, that is, fixing α, the position of an observation along the 45° line is given by the value of the scalar $\mathbf{T}$, that is, with an elasticity of $\frac{1}{2}$ with respect to the $\frac{\theta}{(B\ \sigma^2)}$ parameter.

VI.B. Distribution of Price Changes

We now turn to the implications of results in Proposition 6 for the statistics computed from the distribution of price changes. For instance, the average size of price changes is homogeneous of degree 1 in $\mathbf{P}$:

$$\mathbb{E}[\ |\Delta p|\ ] = \mathbf{P}\ \mathcal{E}(\alpha)\ , \tag{19}$$

where $\mathcal{E}(\alpha) = 2 \int_{\varphi(\alpha)}^{\infty} |x|\; w(x; 1, \alpha)\, dx$. Notice that, holding constant the ratio of the two costs, α, the expression in Equation (19) has the same comparative statics with respect to a change in both cost (ψ, θ), and to a change in (B, σ^2) than the ones from the models with observation cost only or with menu cost only.

Moreover, the homogeneity of the density of price changes $w\,(\Delta p\ ;\ \mathbf{P}, \alpha)$ implies that, fixing α, the moments computed from this density for different values of the parameter $\mathbf{P}$ will only differ by a constant scaling factor. In other words, fixing α and $\mathbf{P}$, any ratio of moments of the same order computed for Δp, such as the coefficient of variation or the ratio of the mode to the mean, will depend only on α, for instance:

$$\text{(20)} \qquad \frac{\text{mode } |\Delta p|}{\mathbb{E}[\ |\Delta p|\ ]} = \mathcal{M}(\alpha) \;\text{ and }\; \frac{\text{std}[|\Delta p|]}{\mathbb{E}[\ |\Delta p|\ ]} = \mathcal{S}(\alpha)\ ,$$

where $\mathcal{M}(\alpha)$ and $\mathcal{S}(\alpha)$ are monotonic functions that only depend on α.[17] Thus, as illustrated in Figure VI, the shape of the distribution depends on the ratio of the two costs. In particular, the shape of the distribution of normalized price changes ranges from a standard normal when $\alpha = 0$ to a bimodal distribution in the case of $\alpha \to \infty$.

This result provides an additional identification scheme to measure α, using the distribution of price changes.[18] More generally, the implications of Proposition 6 are useful because they produce a mapping between some observable statics and α that could be used in the data to estimate the relative cost of review to adjustment. Figure IV plots the ratio of the average frequency of review to adjustment, $\frac{n_r}{n_a}$ (left panel), the coefficient of variation of the absolute value of price changes, $\frac{\text{std}[|\Delta p|]}{\mathbb{E}[\ |\Delta p|\ ]}$ (middle panel), and the ratio of the mode to the mean for the absolute value of price changes (right panel). As shown in Equations (18) and (20), these observable statistics depend only on α. Hence they provide an overidentifying restriction for our model.

Finally notice that the shape of the distribution of price gaps on observation, $g(\tilde{p}\ ;\ \mathbf{P},\alpha)$, only depends on α. This distribution is

17. These functions are given by $\mathcal{M}(\alpha) = \frac{\varphi(\alpha)}{\mathcal{E}(\alpha)}$, and $\mathcal{S}(\alpha) = \frac{\left(2\int_{\varphi(\alpha)}^{\infty} x^2\; w(x;1,\alpha)\; dx - (\mathcal{E}(\alpha))^2\right)^{\frac{1}{2}}}{\mathcal{E}(\alpha)}$. Numerical results show that $\mathcal{M}(\alpha) \in [0, 1]$ is increasing in α, and $\mathcal{S}(\alpha) \in \left[0, \sqrt{\frac{\pi}{2} - 1}\right]$ is decreasing in α.

18. See Cavallo (2010) and Cavallo and Rigobon (2010) for an application of our theory to statistics on the distribution of price changes.

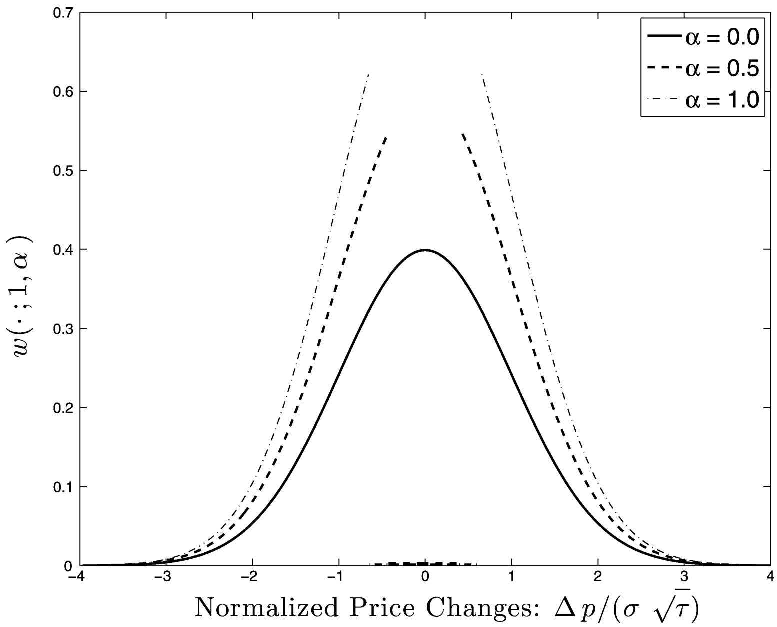

FIGURE VI

Distribution of Nonzero Normalized Price Changes

potentially important for the aggregate implications of our model as it determines the mass of firms near the adjusting thresholds.[19] This shape ranges from a normal distribution in the case of observation cost only, that is, $\alpha = 0$, to a triangular distribution having the mode at $\tilde{p} = 0$ in the case of a menu cost model, that is, $\alpha \to \infty$.

VI.C. The Hazard Rate of Price Changes

In our model with both costs, the hazard rate function is *not* monotone in the time elapsed since the last price change and, according to Proposition 6, its shape is determined by the relative size of the two costs, $\alpha = \frac{\psi}{\theta}$. In fact, the hazard function ranges between two extreme cases: in the model with observation cost only, that is, $\alpha = 0$, the hazard rate is 0 for all durations with the exception of a single positive spike to infinity after τ units of time since the last price adjustment; in contrast, the

19. See Caplin and Spulber (1987), Caplin and Leahy (1991), Golosov and Lucas (2007), and Midrigan (2007) for more details in the case of menu cost models.

hazard rate is increasing but quickly converging to a constant and strictly positive rate in the menu cost model, that is, for $\alpha \to \infty$. We are able to characterize analytically the hazard function for durations shorter than $\tau + \min\{\tau, 2\underline{\tau}\}$, as described in the following proposition.

PROPOSITION 7. Let t be the time elapsed since the last price adjustment. The hazard rate of price adjustments is 0 for $t \in [0, \ \tau)$, it jumps to infinity at $t = \tau$, it returns to 0 in the segment $t \in (\tau, \tau + \underline{\tau})$, it jumps to a positive value at $\tau + \underline{\tau}$. In the segment $[\tau + \underline{\tau}, \tau + \min\{\tau, 2\underline{\tau}\})$ it is strictly positive and, if $\underline{\tau} > \left(\frac{1}{2}\right)\tau$, it is strictly increasing and tends to infinity at the end of this segment and returns to 0 right after.

Proposition 7 does not characterize the hazard rate when durations are longer than $\tau + \min\{\tau, 2\underline{\tau}\}$. Although an expression can be developed for larger durations, it becomes increasingly complex because a price change can happen after several combinations of previous reviews. Indeed, the larger the value of t, the larger the number of combinations of different duration of previous reviews that can happen. The effect of this feature is that the hazard rate for larger values of elapsed time t will tend to be smaller but without the "holes" between the different waves of price adjustments.

Figure VII plots the normalized hazard function, $h(\frac{t}{\tau}; 1, \alpha)$, versus normalized time units, $\frac{t}{\tau}$, for two different values of α: a high value $\alpha = 5$ in the left panel, a small value $\alpha = 0.5$ in the right panel. The hazard function is obtained through simulations of the model's decision rule, Equation (16). For each value of α, the other parameter determining the hazard, τ, is chosen to match the median frequency of regular price changes and the average absolute size of price changes found by Klenow and Kryvtsov (2008) on U.S. data.[20] This exercise highlights that products with identical average frequency and size of price changes may still be characterized by very different shapes of the hazard function.

The hazard rate is characterized by "waves" of price adjustments, which are more evident for smaller values of α such as the one reported in the right panel of Figure VII. After τ units of time all the adjustments occur simultaneously, so the hazard rate has a spike in both cases. The subsequent waves are less

20. We use a median frequency of 1.6 per year, a mean absolute price change of 0.11.

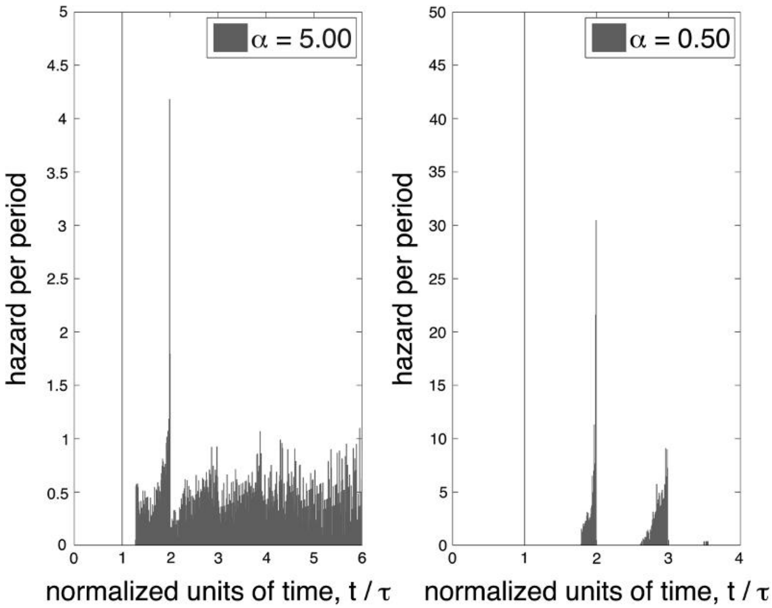

FIGURE VII

Normalized Hazard Rate of Price Changes for Different Values of $\alpha = \frac{\psi}{\theta}$

The hazard rate was obtained by simulating the model in normalized daily units. Dividing $h\left(\frac{t}{\tau}; 1, \alpha\right)$ by 365 gives the probability of a price change in an interval of length τ. See Proposition 6.

concentrated around a single value, and hence the hazard rates have smaller spikes, which occur approximately every additional τ units of time. The larger α, the smaller the spikes of the different waves of price adjustments, and the smaller the distance between the different waves of price adjustment. In fact, at $\alpha = 5$ the different waves of price adjustments are harder to distinguish, so the hazard function is flatter. Intuitively, as $\alpha \to \infty$ increases, the hazard rate in our model with both costs resembles more and more hazard of the model with menu cost only.

An important implication of the example in Figure VII is that accounting for cross-products heterogeneity in the ratio of menu to observation cost is important to correctly estimate the shape of the hazard function. In fact, the example shows that two firms with identical average frequency and average size of price changes may give rise to very different shapes of the hazard if characterized by very different values of $\alpha = \frac{\psi}{\theta}$. This prediction is in stark contrast to standard models with menu cost or observation cost only where controlling for the heterogeneity in the frequency of price changes is enough to account for heterogeneity in the hazard function across different products.[21] This result provides theoretical support to Nakamura and Steinsson (2008) who find substantial heterogeneity in the shape of the hazard function across major groups of products, even after controlling for product-specific heterogeneity in the level of the hazard.[22]

VII. Extensions and Future Research

We discuss several extensions of the baseline model and some areas for future research. First, the information arrival in our model is discrete: no new information arrives between reviews. We briefly describe one extension where firms receive free signals on the realization of the price gap, in the spirit of Woodford (2009), Gorodnichenko (2008), and Bonomo, Carvalho, and Garcia (2010). This model is intuitively appealing because if there is a large change in the firm's price gap, the firm is likely to learn about it, even if it does not pay the observation cost θ. In our extension the signal received by the firm equals the value of the price gap plus an i.i.d. normal noise.[23] As in the baseline model, if the firm "reviews the price," paying the observation cost θ, the price gap is revealed perfectly. In this extension the state of the problem is given by two variables: the best forecast of the price gap and its variance, which evolves deterministically given the Gaussian structure. Two general points emerge from this extension. The first point is that free signals are a substitute for costly reviews. Everything else the same, the presence of free information on the price gap reduces the need to incur in (costly) reviews, relative to the number of adjustments. This is natural because the only

21. See Campbell and Eden (2005) for a description of how the frequency of price changes helps accounting for heterogeneity in models where only the adjustment cost is present. See the Online Appendix for an analytical derivation of the hazard function in the models with menu and observation cost only.

22. Nakamura and Steinsson (2008) argue that failing to account for this heterogeneity might partially explain the differences in results relative to the shape of the hazard with respect to Klenow and Kryvtsov (2008).

23. In the Online Appendix we state the problem formally, solve for the Riccati equation of the variance of the signals explicitly, characterize analytically four limiting cases, and set up a numerical procedure to solve the model.

purpose of a review is to obtain information on the price gap. The second point is that even with $\mu = 0$, there are regions of the state space where it is optimal to have multiple adjustments between observations. This outcome is not possible in the model without signals with $\mu = 0$, since the forecast of the price gap is constant between observations. These features imply that if the signals are very informative, then costly reviews occur less frequently than price adjustments, potentially leading to $\frac{n_r}{n_a} < 1$. This feature appears inconsistent with the evidence for the European countries collected in Fabiani et al. (2007).

Another extension to be explored in future research is to apply the baseline setup of this article with both costs to the problem of a multiproduct firm. The key assumption, introduced by Lach and Tsiddon (1996) and Midrigan (2007, 2009), is that once the menu cost is paid the firm can adjust the price of *all* its products. Similar to our model, this setup can generate arbitrarily small price changes even with a strictly positive menu cost. In Alvarez and Lippi (2010) we provide an analytical solution for a firm selling n goods but facing no observation cost. In this case we show that the distribution of price changes is bimodal only for $n \leq 2$, it is single peaked for $n \geq 4$, and converges to the normal as $n \to \infty$. Thus, provided that $n \geq 3$, the multigood model without observation cost and the one-good model with both costs have different implications. We also show that for all n the hazard rate is increasing, it has an asymptote, and it ranges from the menu cost case (when $n = 1$) to the deterministic one in Taylor's model (as $n \to \infty$). The stark differences between the shape of the hazard rate of price changes in the menu-cost framework (inclusive of the multi-product extension) and the two-cost model can be used empirically to discriminate between the competing hypothesis generating the small price changes.

Two further extensions concern quantitative applications of the model with two costs. The first one is to estimate the relative size of the menu and the observation cost. In Section VI we showed that the shape of the distribution of price adjustment is informative of α. A rigorous analysis of this point must deal with the issue of aggregating sectors that are heterogeneous in the parameters $(\sigma^2, \frac{\theta}{B}, \frac{\psi}{B})$ in the steady state of an economy. The aggregation issue is important: witness of this claim is the large variation in the frequencies of price adjustment and reviews across sectors, displayed in Figure V.

Finally, the model with two costs can be used to set up a general equilibrium economy along the lines of Mankiw and Reis (2002) and Golosov and Lucas (2007), and numerically study the impulse response of the economy to aggregate monetary shocks.[24] We think that this is interesting because state-dependent rules, generated by model with only menu costs as in Golosov and Lucas (2007), feature strong selection effects that tend to predict a smaller output effect than is predicted by models with observation cost only, as in Mankiw and Reis (2002). Since our model combines elements of both, we think that an educated parametrization might provide new information on this important topic.

APPENDIX: PROOFS

A Useful Lemma: Symmetry

The next lemma highlights a symmetry property of the value function, associated decision rules, and statistics of the problem with respect to the inflation rate μ. To state it, we add μ explicitly as an argument of the value function, policies, and statistics.

LEMMA 1. Let $\mu \in \mathbb{R}$. (a) For each $J \geq 0$ and $\tilde{p} \in \mathbb{R}$: $V_J(\tilde{p}, \mu) = V_J(-\tilde{p}, -\mu)$, $T(\tilde{p}, \mu) = T(-\tilde{p}, -\mu)$ and for $1 \leq i \leq J : \hat{p}_i(\mu) = -\hat{p}_i(-\mu)$, and $t_i(\tilde{p}, \mu) = t_i(-\tilde{p}, -\mu)$. ($b$) For all $\tilde{p} \in \mathbb{R} : V(\tilde{p}, \mu) = V(-\tilde{p}, -\mu)$. ($c$) For all $\tilde{p} \in \mathbb{R} : J^*(\tilde{p}, \mu) = J^*(-\tilde{p}, -\mu)$. ($d$) For all $\Delta p \in \mathbb{R} : w(\Delta p, \mu) = w(-\Delta p, -\mu)$, $h(t, \mu) = h(t, -\mu)$ for all $t \geq 0$, $n_a(\mu) = n_a(-\mu)$, and $n_r(\mu) = n_r(-\mu)$.

Proof. Parts (a)–(b) follows from a guess and verify strategy: assuming the symmetry of (b) on the value function on the right-hand side of Equations (3) and (4), it follows directly using the symmetry of the quadratic loss function with respect to $\tilde{p} = 0$ and the symmetry of the density of the standard normal $n(\cdot)$. Part (c) follows directly from parts (a)–(b). Part (d) follows from the symmetry of the polices described in parts (a)–(c). Part (e) follows from differentiating the expressions in part (d). ■

24. While the presence of both cost makes the problem for the firm more complicated than the one in Golosov and Lucas (2007), it still has one *idiosyncratic state*, namely, the price gap. Hence, one can use exactly the same numerical technique used by Golosov and Lucas's (2007) to solve for an impulse response to an *aggregate* monetary shock.

Proof of Proposition 1

Proof. First consider the case where $\mu = 0$. In this case, if $\psi > 0$ and $\sigma > 0$, it follows from Lemma 1 that the optimal policy is to reset the price gap to 0, $\hat{p} = 0$. Because the expected value of the price gap remains at $\hat{p} = 0$ between reviews, there are no gains from price adjustment without the information produced by a review, but there are strictly positive losses of size $\psi > 0$. Likewise, there are no gains from delaying the adjustment after a review. The argument against the optimality of a delayed adjustment is that during the first part of the period, of length $t_1 > 0$, the firm is having a loss of the order of the price gap. In particular the first-order condition for t_1 in Equation (4) is $Be^{-\rho t_1}\left[(\tilde{p} - \mu t_1)^2 - (\hat{p}_1 - \mu t_1)^2\right]$, which for $\mu = 0$ and the optimal value $\hat{p}_1 = 0$ gives $Be^{-\rho t_1}\tilde{p}^2 > 0$. Since adjustments occur only outside the range of inaction the price gap in that case is "large," that is, $\tilde{p}^2 > \bar{p}^2$ and thus there is a strictly positive loss in delaying the adjustment. To summarize, for $\mu = 0$, having multiple adjustment between observations or delayed adjustments would strictly increase the losses of the decision maker. Now we consider the case of $|\mu| > 0$. We note that the period return function evaluated between reviews and the law of motion of the price gap are continuous with respect to μ, and hence the value function, by a careful application of the theorem of maximum, is continuous with respect to μ. Using the continuity with respect to μ we will establish the two required results. First, because the benefits for multiple adjustment between observations are continuous on μ, and hence for μ close to 0 they are close to 0. On the other hand, an adjustment between reviews increases the cost in a discrete amount, $\psi > 0$. Thus, there exists a $\bar{\mu}_1 > 0$ for which if $|\mu| < \bar{\mu}_1$ it is not optimal to adjust between reviews. The second result of the proposition is that for small but positive inflation, the optimal price adjustment occurs immediately on review, $t_1 = 0$. Again by continuity on μ, notice that adjustment on the price will happen for values of the price gap bounded away from 0, that is, the first-order condition for t_1 will continue to hold as an inequality outside the inaction region. Thus, there exists a $|\bar{\mu}_2| > 0$ for which adjustments will not be delayed. Taking $\bar{\mu} = \min\{\bar{\mu}_1, \bar{\mu}_2\}$ we obtain the desired result. That $\frac{\bar{\mu}}{\sigma^2}$ is a function of $(\frac{\theta\sigma^2}{B}, \frac{\psi}{\theta})$ follows from the homogeneity of the value function and policies with respect to (B, θ, ψ) and $(\theta, \psi, \mu, \sigma^2, \rho)$. Clearly for $\theta > 0, \sigma > 0$ we have $\lim m(\cdot, \psi/\theta) = 0$ as $\psi \downarrow 0$ since adjustments are becoming free.

Since we have established that $m > 0$ for $\psi > 0$ then it must be locally increasing. ■

Proof of Proposition 2

To obtain (a) we use the symmetry property of $n_a(\cdot)$, $n_r(\cdot)$ and $h(\cdot)$ described in Lemma 1 as well as differentiability of these functions at $\mu = 0$. To obtain (b) we differentiate (twice) with respect to μ the identity $n_a(\mu)\mathbb{E}[\Delta p, \mu] = \mu$, and use that $\mathbb{E}[\Delta p, 0] = \frac{\partial}{\partial \mu} n_a(\mu) = 0$, and $n_a(0) > 0$. For the rest of the results we use the following.

LEMMA 2. Let $f(p,q), h(p,\mu)$ be two functions continuous with respect to the first argument, differentiable at 0 with respect to the second, and symmetric around $(0,0)$. Let $h(\cdot,\mu)$ be a density function. Let $q(\mu)$ be a differentiable function with $q(0) = 0$ and $|q'(0)| < \infty$. Define $H(\mu) \equiv \int_{-\infty}^{\infty} f(p, q(\mu)) h(\cdot, \mu)\, dp$. Then $\frac{\partial}{\partial \mu} H(0) = 0$.

To prove the lemma, differentiate the integral with respect μ obtaining:

$$\frac{\partial}{\partial \mu} H(\mu) = \int_{-\infty}^{\infty} \frac{\partial}{\partial q} f(p, q(\mu))\, q'(\mu)\, h(p,\mu)\, dp + \int_{-\infty}^{\infty} f(p, q(\mu))\, \frac{\partial}{\partial \mu} h(\cdot, \mu)\, dp.$$

Using the symmetry of f and g, note we have for all p:

$$\frac{\partial}{\partial q} f(p,0) = -\frac{\partial}{\partial q} f(-p,0)\,,\ \frac{\partial}{\partial \mu} g(p,0) = -\frac{\partial}{\partial \mu} g(-p,0)$$
$$f(p,0) = f(-p,0) \ \text{ and } \ g(p,0) = g(-p,0)\,,$$

breaking the integral into positive and negative values of p and using these results we prove the lemma.

To obtain (c) we use the lemma for $f(\Delta p, q) = (\Delta p - q)^{2k}$ for $k = 1, 2, \ldots,$ $h(\Delta p, \mu) = w(\Delta p, \mu)$ and $q(\mu) = \mathbb{E}[\Delta p, \mu]$. To obtain (d) we use the lemma for $f(\Delta p, q) = |\Delta p|^k$ for $k = 1, 2, \ldots,$ $h(\Delta p, \mu) = w(\Delta p, \mu)$ and $q(\mu) = \mathbb{E}[\Delta p, \mu]$. Because the derivative of all the moments of the absolute value of price changes is 0, so must be the derivative of the density. To obtain (e) we use the lemma for $f(\tilde{p}, \mu) = V(\tilde{p}, \mu)$ and $h(\tilde{p}, \mu) = g(\tilde{p}, \mu)$.

Proof of Proposition 3

Proof. Under the conjecture that $\hat{p} = 0$ and that $V(\cdot)$ is symmetric around 0, and by the symmetry of the normal density, we can rewrite the Bellman equations (6) and (7) using only the positive range for $\tilde{p} \in [0\ ,\ \infty)$ as:[25]

$$V_0(\tilde{p}) = \theta + \min_{\tau} B \int_0^{\tau} e^{-\rho t}\, [\tilde{p}^2 + \sigma^2 t]\, dt \quad + \\ e^{-\rho\tau} \int_{-\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V\left(\tilde{p} + s\sigma\sqrt{\tau}\right) dN(s) \\ + e^{-\rho\tau} \int_{\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V\left(-\tilde{p} + s\sigma\sqrt{\tau}\right) dN(s) \tag{21}$$

$$V_1 = \psi + \theta + \min_{\tau} B \int_0^{\tau} e^{-\rho t}\, [\sigma^2 t]\, dt + e^{-\rho\tau} 2 \\ \times \int_0^{\infty} V\left(s\sigma\sqrt{\tau}\right) dN(s). \tag{22}$$

We use the corollary of the contraction mapping theorem. First, notice that if the V in the right side of Equation (7) is symmetric around $\tilde{p} = 0$, with a minimum at $\tilde{p} = 0$, then it is optimal to set $\hat{p} = 0$. Second, notice that if the function V in the right side of Equation (6) is symmetric with a minimum at $\tilde{p} = 0$, then the value function in the left side is also symmetric, and hence V in Equation (8) is symmetric. Third, using the symmetry, we show that if $V(\tilde{p})$ is weakly increasing, then the right side of Equation (8) is weakly increasing. It suffices to show that $V_0(\tilde{p})$ given by the right side of (21) is increasing in $\tilde{p}$ for a fixed arbitrary value of τ. We do this in two steps. The first step is to notice that the expression containing $\tilde{p}^2$ in (21) is obviously increasing in $\tilde{p}$. For the second step, without loss of generality, we assume that V is differentiable almost everywhere and compute the derivative with respect to $\tilde{p}$ of the remaining two terms involving the expectations of $V(\,\cdot\,)$ in (21). This derivative is:

$$\frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}} \left[V(0)\, n\left(\frac{-\tilde{p}}{\sigma\sqrt{\tau}}\right) - V(0)\, n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}}\right) \right]$$

25. Equation (21) uses that $\int_{-\infty}^{\infty} V(p - s)dN(s) = \int_{-p}^{\infty} V(p + s)dN(s) + \int_{p}^{\infty} V(-p + s)dN(s)$.

$$+ e^{-\rho\tau}\left[\int_{\frac{-\tilde{p}}{(\sigma\sqrt{\tau})}}^{\infty} V'(\tilde{p} + s(\sigma\sqrt{\tau}))dN(s) - \int_{\frac{\tilde{p}}{(\sigma\sqrt{\tau})}}^{\infty} V'(-\tilde{p} + s(\sigma\sqrt{\tau}))dN(s)\right]$$
$$= e^{-\rho\tau}\left[\int_0^{\infty} V'(z)\frac{1}{\sigma\sqrt{\tau}}\left(dN\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}}\right) - dN\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}}\right)\right)\right]$$
$$= e^{-\rho\tau}\left[\int_0^{\infty} V'(z)\frac{1}{\sigma\sqrt{\tau}\sqrt{2\,\pi}}\left(e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}}\right)^2}\right)dz\right] \geq 0,$$

where the term involving $V(0)$ is 0 due symmetry of $dN(s) = n(s)ds$, where $n(\cdot)$ is the density of a standard normal, and where the inequality follows since $e^{-\frac{1}{2}\left(\frac{x-\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} - e^{-\frac{1}{2}\left(\frac{x+\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} > 0$ for $x > 0$ and $\tilde{p} > 0$. Thus the derivative of $V_0(\tilde{p})$ for $0 \leq \tilde{p}$ is

$$0 \leq V_0'(\tilde{p}) = 2\,B\,\tilde{p}\,\frac{1 - e^{-\rho\mathrm{T}(\tilde{p})}}{\rho} + e^{-\rho\mathrm{T}(\tilde{p})}$$
$$\times \int_0^{\infty} V'(z)\,\frac{e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\mathrm{T}(\tilde{p})}}\right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\mathrm{T}(\tilde{p})}}\right)^2}}{\sigma\,\sqrt{\mathrm{T}(\tilde{p})\,2\,\pi}}\,dz$$

Notice that the inequality is strict if $\tilde{p} > 0$ and $V'(x) > 0$ in a segment of strictly positive length. If $\tilde{p} = 0$, then the slope is 0.

Finally, differentiating the value function twice, and evaluating at $\tilde{p} = 0$ we get

$$V''(0) = 2\,B\,\frac{1 - e^{-\rho\tau}}{\rho} + 2\,\frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}}\int_0^{\bar{p}} V'(z)\,z\,\frac{e^{-\frac{1}{2}\frac{z^2}{\sigma^2\tau}}}{\sigma\sqrt{\tau\,2\,\pi}}\,dz > 0\,.$$

■

Proof of Proposition 4

Proof. The expression is based on a second-order expansion of $\mathrm{T}(\cdot)$ around $\tilde{p} = 0$. The first-order condition for τ can be written as:

$$F(\mathrm{T};\tilde{p}) \equiv e^{-\rho\tau}\left(B\,(\tilde{p}^2 + \sigma^2\tau) - \rho\int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s)\right.$$
$$\left. + \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\,\frac{-s\;\sigma}{2\,\sqrt{\tau}}\,dN(s)\right).$$

At a minimum $F(\mathrm{T}(\tilde{p})\,;\,\tilde{p}) = 0$ and $F_\tau(\mathrm{T}(\tilde{p})\,;\,\tilde{p}) \geq 0$. We have $\left.\frac{\partial\mathrm{T}(\tilde{p})}{\partial\tilde{p}}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}}}{F_\tau} = 0$. That $\frac{\partial\mathrm{T}}{\partial\tilde{p}} = 0$ follows from the symmetry of $\mathrm{T}(\cdot)$

around $\tilde{p}$, which is verified directly by checking that $F_{\tilde{p}} = 0$ (see below). Totally differentiating $F_\tau \mathrm{T}' + F_{\tilde{p}}$ we obtain:

$$0 = F_{\tau\tau}(\mathrm{T}')^2 + F_{\tau\tilde{p}}\mathrm{T}' + F_\tau \mathrm{T}'' + F_{\tilde{p}\tau}\mathrm{T}' + F_{\tilde{p}\tilde{p}} \ ,$$

using that $\mathrm{T}' = 0$ we get the second derivative:

$$\left.\frac{\partial^2 \mathrm{T}(\tilde{p})}{(\partial\tilde{p})^2}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}\tilde{p}}}{F_\tau} = 0.$$

To compute this second derivative we first compute:

$$\begin{aligned} F_\tau(\mathrm{T};\tilde{p}) = -\rho F(\tau;\tilde{p}) + e^{-\rho\tau}\Bigg(B\sigma^2 - \rho\int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\frac{-s\sigma}{2\sqrt{\tau}}dN(s) \\ - \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\tau^{\frac{-3}{2}}}{4}\ dN(s) \\ + \int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s)\Bigg) \end{aligned}$$

Taking $\rho \downarrow 0$, using that at the optimum $F = 0$, that in the approximation $\bar{V}'(\tilde{p}) = V''(0)\,\tilde{p}$ and that $\bar{V}''(\tilde{p}) = V''(0)$ we obtain:

$$\begin{aligned} F_\tau(\mathrm{T};0) &= B\ \sigma^2 - \int_{-\infty}^{\infty} V'\left(-s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\ \tau^{-\frac{3}{2}}}{4}\ dN(s) \\ &\quad + \int_{-\infty}^{\infty} V''\left(-s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s) \\ &= B\ \sigma^2 - \int_{-\infty}^{\infty} V''(0)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s) \\ &\quad + \int_{-\infty}^{\infty} V''(0)\frac{s^2\sigma^2}{4\ \tau}\ dN(s) \ \ = B\ \sigma^2\ . \end{aligned} \tag{23}$$

We also have:

$$\begin{aligned} F_{\tilde{p}}(\mathrm{T};\tilde{p}) &= e^{-\rho\tau}\Bigg(2B\tilde{p} - \rho\int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s) \\ &\quad + \int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\Bigg) \\ F_{\tilde{p}\tilde{p}}(\tau;\tilde{p}) &= e^{-\rho\tau}\Bigg(2B - \rho\int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s) \\ &\quad + \int_{-\infty}^{\infty} V'''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\Bigg)\,. \end{aligned}$$

Evaluating $F_{\tilde{p}\tilde{p}}$ at $\tilde{p} = 0$ for $\rho \downarrow 0$ and the approximation with $V'''(0) = 0$ gives:

$$F_{\tilde{p}\tilde{p}}(\mathrm{T}; 0) = 2B \; . \tag{24}$$

Expanding $\mathrm{T}(\cdot)$ around $\tilde{p} = 0$, using that its first derivative is 0 and the second derivative is the negative of the ratio of the expressions in Equations (23) and (24) we obtain:

$$\mathrm{T}(\tilde{p}) = \mathrm{T}(0) + \mathrm{T}'(0)(\tilde{p}) + \frac{1}{2}\mathrm{T}''(0)(\tilde{p})^2 = \tau - \frac{1}{2}\frac{F_{\tilde{p}\tilde{p}}}{F_\tau}(\tilde{p})^2 = \tau - \left(\frac{\tilde{p}}{\sigma}\right)^2 .$$

which appears in the proposition. ■

Proof of Proposition 5

Proof. We begin by establishing two lemmas that are useful to characterize the solution for $\bar{p}$ and τ. The proofs for this lemmas are given at the end of this section.

LEMMA 3. Let $\phi \equiv \frac{\bar{p}}{\sigma\sqrt{\tau}}$, then $V''(0)$, $\bar{p}$, and τ solve the recursive system:

$$\sigma^2\frac{\psi}{B} = f(\phi), \sigma^2\tau = h(\phi), \text{ and } V''(0) = 2\,\frac{\psi}{\bar{p}^2} ,$$

where $f(\cdot)$ and $h(\cdot)$ are the following known functions of ϕ and of two parameters $(\sigma^2\frac{\theta}{B}, \sigma^2\frac{\psi}{B})$:

$$\sigma^2\frac{\psi}{B} = f(\phi) \equiv \frac{\phi^2\,[h(\phi)]^2}{1 - 2\sqrt{h(\phi)}\int_0^\phi s^2\,dN(s)} \tag{25}$$

$$\tau = \frac{h(\phi)}{\sigma^2} \equiv \sqrt{2\,\frac{\theta + 2\,\psi(1 - N(\phi))}{\sigma^2 B}} \; . \tag{26}$$

Equations (25) and (26) can be thought of as the optimality conditions for $\bar{p}$ and τ. An immediate corollary of Lemma 3 is that the optimal values of ϕ and $\sigma^2\tau$ are only functions of two parameters $\sigma^2\frac{\theta}{B}$ and $\sigma^2\frac{\psi}{B}$. Notice also that the expression for τ in Equation (26) is the same as the square root formula for the problem with observation cost only (see Section 4.1 of Alvarez, Lippi, and Paciello 2010), except that the cost θ is replaced by the "expected" cost $\theta + \psi\,2\,(1 - N(\phi))$.

Lemma 3 gives a recursive system of equations whose solution is the optimal value of $(\tau, \bar{p})$. The next lemma gives a sufficient condition for the existence and uniqueness of the system, and provides some comparative statics. In fact, it turns out that the approximations used in this section can only be used globally—that is, for all $\tilde{p}$ – if $\phi \in (0, 1)$, as is clear from Proposition 4. Thus, the next lemma restricts attention to parameter settings so that there is a unique solution in this range.

LEMMA 4. Let $\phi \equiv \frac{\bar{p}}{(\sigma\sqrt{\tau})}$. Assume that $\frac{\psi}{\theta} \leq \left(\frac{1}{2} - 2(1 - N(1))\right)^{-1} \approx 5.5$. Then there exists a unique value $\phi \in (0, 1)$ that solves $\sigma^2 \frac{\psi}{B} = f(\phi)$ defined in Equation (25). Also let τ be the solution of $\tau = \frac{h(\phi)}{\sigma^2}$ defined in Equation (26). Then,

1. ϕ is decreasing in θ, and τ is increasing in θ,
2. ϕ is decreasing in $\frac{\sigma^2}{B}$, and $\sigma^2\tau$ is increasing in $\frac{\sigma^2}{B}$ with an elasticity $\geq \frac{1}{2}$,
3. $\frac{\partial\phi}{\partial\frac{\sigma^2}{B}} = 0$ evaluated at $\frac{\sigma^2}{B} = 0$,
4. $\frac{\partial\phi}{\partial\psi} > 0$ if $\sigma^2\frac{\psi}{B}$ is small relative to θ.

The assumption is that the observation cost must be sufficiently large relative to the menu cost ($\frac{\psi}{\theta} < 5.5$) for the approximation to be globally valid—that is, for $\phi < 1$—for arbitrary values of $\frac{\sigma^2}{B}$. The reason for this assumption is that the problem formulation presumes that after adjusting the price the firm waits for $\tau > 0$ periods before the next review because observation has a non-negligible cost relative to the adjustments. For instance, when $\theta = 0$ the problem formulation is incorrect as the model becomes the menu cost model where τ=0 and price reviews happen continuously.

We now use these lemmas to establish the result in Proposition 5. Write the solution as $\lambda(\psi\frac{\sigma^2}{B}, \alpha) \equiv \phi(\psi\frac{\sigma^2}{B}, \psi\frac{\frac{\sigma^2}{B}}{\alpha})$. Then fixing α we can write $\lambda(\psi\frac{\sigma^2}{B}, \alpha) = \lambda(0, \alpha) + \lambda_1(0, \alpha)\psi\frac{\sigma^2}{B} + o(\psi\frac{\sigma^2}{B})$ where $\lambda(0, \alpha) = \varphi(\alpha)$ and where by part 3 of Lemma 4: $\lambda_1(0, \alpha) = \frac{\partial\phi}{\partial\frac{\sigma^2}{B}} = 0$.

Some algebra, using the implicit definition of $\varphi(\alpha)$ by Equation (10), gives

$$\frac{\partial \log\varphi}{\partial \log\alpha} = \frac{1 - \frac{4\alpha(1-N(\varphi))}{2+4\alpha(1-N(\varphi))}}{2 - \frac{4\alpha(1-N(\varphi))}{2+4\alpha(1-N(\varphi))}\frac{n(\varphi)\varphi}{(1-N(\varphi))}}. \tag{27}$$

Since, $\varphi \to 0$ as $\alpha \to 0$, then $\frac{\partial \log\varphi}{\partial \log\alpha} \to \frac{1}{2}$. For values of $\alpha > 0$, we have that

$$\frac{\partial \log \varphi}{\partial \log \alpha} < \frac{1}{2} \iff \frac{n(\varphi)\varphi}{(1 - N(\varphi))} < 2, \tag{28}$$

which is a property of the normal distribution for values of $\varphi < 1$. Finally, inequality (12) follows from the definition of φ in Equation (10) and because $2\frac{\varphi}{\alpha} = [1 - \varphi^2 4(1 - N(\varphi))] < 1$. Inequality (13) follows because $\varphi < 1$ and Equation (10).

Proof of Lemma 3

First we notice that using the quadratic approximation into the definition of $\bar{p}$ given by $V_0(\bar{p}) = V_1$ implies

$$\psi = \frac{1}{2} V''(0) \, (\bar{p})^2 \ . \tag{29}$$

Second we derive Equation (26) as the first-order condition for τ. To this end, use the Bellman equation (7) for a fixed $\tau > 0$ evaluated at the optimal $\hat{p} = 0$, the symmetry of $V(\tilde{p})$, and the approximation

$$V(\tilde{p}) = \min\{V_1 \ , \ V(0) + \frac{1}{2} V''(0) \, (\tilde{p})^2\}$$

to write:

$$\begin{aligned} V(0) = V_1 - \psi &= \theta + B\sigma^2 \int_0^\tau e^{-\rho t} t \, dt + e^{-\rho\tau} \int_{-\infty}^{\infty} V\left(s\sigma\sqrt{\tau}\right) dN(s) \\ &= \theta + B\sigma^2 \int_0^\tau e^{-\rho t} t \, dt + e^{-\rho\tau} \ V(0) + \psi \ e^{-\rho\tau} \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right] \\ &\quad + e^{-\rho\tau} \ V''(0) \ \sigma^2 \tau \ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} \ s^2 \, dN(s). \end{aligned}$$

Thus

$$\rho V(0) = \frac{\theta + B\sigma^2 \int_0^\tau e^{-\rho t} t \, dt + \psi \ e^{-\rho\tau} \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right] + e^{-\rho\tau} \ V''(0) \ \sigma^2 \tau \ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} \ s^2 \, dN(s)}{\frac{(1 - e^{-\rho\tau})}{\rho}},$$

letting $\rho \downarrow 0$ gives

$$\lim_{\rho \downarrow 0} \ \rho V(0) = \frac{\theta + \psi \ 2 \left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right]}{\tau} + B\sigma^2 \frac{\tau}{2} + V''(0) \ \sigma^2 \ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} \ s^2 \, dN(s).$$

Maximizing the right side of this expression gives

$$0 = -\frac{\theta + \psi\, 2\left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right]}{\tau^2} + \frac{B\sigma^2}{2} + \left(\psi\, 2\, n\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\,\left(\frac{\bar{p}}{\sigma}\right)(\tau)^{-3/2}\right)\frac{1}{\tau}$$
$$- V''(0)\,\sigma^2\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)^2 n\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\left(\frac{\bar{p}}{\sigma}\right)(\tau)^{-\frac{3}{2}},$$

where we use $n(\,\cdot\,)$ for the density of the standard normal. Using that $V''(0) = \frac{2\psi}{\bar{p}^2}$, this expression simplifies to

$$0 = -\frac{\theta + \psi\, 2\left[1 - N\left(\frac{\bar{p}}{\sigma\sqrt{\tau}}\right)\right]}{\tau^2} + \frac{B\sigma^2}{2}.$$

Rearranging and using the definition of ϕ gives $\sigma^2\tau = h(\phi)$ of Equation (26).

Third, we obtain an expression for $V''(0)$. Differentiating the value function twice, and evaluating it at $\tilde{p} = 0$ we get

$$V''(0) = 2\,B\,\frac{1 - e^{-\rho\tau}}{\rho} + 2\,\frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}}\int_0^{\bar{p}} V'(z)\,z\,\frac{e^{-\frac{1}{2}\frac{z^2}{\sigma^2\tau}}}{\sigma\sqrt{\tau\,2\,\pi}}\,dz.$$

With a change in variable $s = \frac{z}{(\sigma\sqrt{\tau})}$ we have:

$$V''(0) = 2\,B\,\frac{1 - e^{-\rho\tau}}{\rho} + 2\,e^{-\rho\tau}\int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} V'\left(\sigma\sqrt{\tau}\,s\right)\,s\,\frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\,\pi}}\,ds\,.$$

Using the third-order approximation $V(\tilde{p}) = V(0) + \frac{1}{2}V''(0)\,(\tilde{p})^2$ around $\tilde{p} = 0$ we obtain:

$$V''(0) = 2\,B\,\frac{1 - e^{-\rho\tau}}{\rho} + e^{-\rho\tau}\,V''(0)\,2\,\sigma\sqrt{\tau}\int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2\,\frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\,\pi}}\,ds,$$

or collecting terms:

$$V''(0) = \frac{2\,B\,\frac{1-e^{-\rho\tau}}{\rho}}{1 - e^{-\rho\tau}\,2\,\sigma\sqrt{\tau}\,\int_0^{\frac{\bar{p}}{\sigma\sqrt{\tau}}} s^2\,\frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\,\pi}}\,ds},$$

and letting $\rho \downarrow 0$, using the definition of ϕ and N for the CDF of a standard normal:

$$V''(0) = \frac{2\,B\,\tau}{1 - \;2\,\sigma\sqrt{\tau}\,\int_0^{\phi} s^2\,dN(s)}\,. \tag{30}$$

Using Equation (29) to replace $V''(0)$ into Equation (30), using the definition of ϕ, and using $\sigma^2\tau = h(\phi)$ to replace τ and $\sqrt{\tau}$ we obtain Equation (25).

Proof of Lemma 4

Begin defining

$$\hat{f}(\phi) = \frac{\hat{h}(\phi)\,\phi^2}{1 - 2\sqrt{h(\phi)}\int_0^{\phi} s^2 dN(s)} \quad \text{where } \hat{h}(\phi) \equiv \frac{B}{\sigma^2}\,[h(\phi)]^2$$
$$= 2\,(\theta + 2\psi\,(1 - N(\phi)))\ , \text{ so that}$$
$$\hat{f}(\phi) = \frac{2\,\phi^2\ (\theta + 2\psi\,(1 - N(\phi)))}{1 - 2\left[2\sigma^2\frac{\theta}{B} + 4\sigma^2\frac{\psi}{B}\,(1 - N(\phi))\right]^{\frac{1}{4}}\int_0^{\phi} s^2 dN(s)}\,,$$

and noting that $\psi = \hat{f}(\phi)$ is the same as the solution of Equations (25) and (26).

First we turn to the existence and uniqueness of the solution. We show that it follows from an application of the intermediate function theorem, together with monotonicity. We show that if $\frac{\theta}{\psi} > \frac{1}{2} - 2(1 - N(1)) \approx 0.1827$ then: there is a value $0 < \phi' \leq 1$ so that: (i) the function $\hat{f}$ is continuous and increasing in $\phi \in [0, \phi')$, (ii) $\hat{f}(0) = 0$, (iii) $\hat{f}(\phi') > \psi$, (iv) $\hat{f}(\phi) < 0$ for $\phi \in (\phi', 1]$.

The value of ϕ' is given by the minimum of 1 or the solution to

$$1 = 2\left[2\sigma^2\frac{\theta}{B} + 4\sigma^2\frac{\psi}{B}\,(1 - N(\phi'))\right]^{1/4}\int_0^{\phi'} s^2 dN(s)\,, \tag{31}$$

so that if $\phi' < 1$, the function $\hat{f}$ has a discontinuity going from being positive and tending to $+\infty$ to being negative and tending to $-\infty$.

The rest of the proof fills in the details: Step (1): Show that $\hat{h}(\phi)^2 \cdot (\phi)^2$ is increasing in ϕ if $\frac{\theta}{\psi} > 0.1667$ for $\phi < 1$. Step (2): Show that $\sqrt{h(\phi)} \cdot \int_0^{\phi} s^2 dN(s)$ is increasing in ϕ if $\phi < 1$. Step (3): Using (1) and (2) the function $\hat{f}$ is increasing in ϕ for values of ϕ that are smaller than 1, provided that its denominator is positive.

Step (1) follows from totally differentiating $h(\phi)^2 \cdot (\phi)^2$ with respect to ϕ. Collecting terms we obtain that the derivative is proportional to $\theta + 2 \cdot \psi(1 - N(\phi) - \phi \cdot N'(\phi))$. Since the function $1 - N(\phi) - \phi \cdot N'(\phi)$ is positive for small values of ϕ and negative for large values, we evaluate it at its upper bound for the relevant region, obtaining: $\theta + 2\psi(1 - N(1) - N'(1)) > 0$ or $\theta > \psi 2[N(1) + N'(1) - 1] \approx \psi\ 0.1667$. But notice that this condition is implied by the assumption: $\theta > \psi[\frac{1}{2} - 2(1 - N(1))] \approx \psi\ 0.1827$.

Step (2) follows from totally differentiating $\sqrt{h(\phi)} \cdot \int_0^{\phi} s^2 dN(s)$ with respect to ϕ. Collecting terms we obtain that the derivative is proportional to $\phi^2 - \int_0^{\phi} s^2 dN(s)\ \frac{\psi}{2}/(\theta + 2\psi\ (1 - N(\phi)))$. This expression is greater than $\phi^2 - \int_0^{\phi} s^2 dN(s)\ /(4(1 - N(\phi)))$, which is obtained by setting θ to 0. This integral is positive for the values of ϕ in $(0, 1)$.

Now we turn to the comparative statics results. That ϕ is decreasing in θ follows since $\hat{f}$ is increasing in θ. That $\sigma^2\tau$ is increasing follows from the previous result and inspection of h. That ϕ is decreasing in $\frac{\sigma^2}{B}$ follows because $\hat{f}$ is increasing in $\frac{\sigma^2}{B}$. That $\sigma^2\tau$ is increasing follows from the previous result and inspection of h. That $\frac{\partial\phi}{\partial\frac{\sigma^2}{B}} = 0$ at $\frac{\sigma^2}{B} = 0$ follows from differentiating $\hat{f}$ with respect to $\frac{\sigma^2}{B}$ and verifying that that derivative is 0 when evaluated at at $\frac{\sigma^2}{B} = 0$. That ϕ is strictly increasing in ψ when $\frac{\sigma^2}{B}$ is small relative to θ it follows from differentiating $\frac{\hat{f}}{\psi}$ with respect to ψ. That derivative is strictly negative and continuous on the parameters, when evaluated at $\theta > 0$ and $\frac{\sigma^2}{B} = 0$. ■

Proof of Proposition 6

Proof. First the result for the density $g(\cdot)$. Note from Proposition 5 that the decision rules can be written as $\sigma\sqrt{\tau} = \mathbf{P}$ and $\tau = \mathbf{T}$. Equation (17) shows that the density for the normalized process y, which we denote by $g(y, 1, \alpha)$, depends exclusively on the parameter α (through ϕ). Recall that $y = \frac{\tilde{p}}{\sigma\sqrt{\tau}}$. The change in variable from the normalized process y to the price gap $\tilde{p}$ shows that $g(\tilde{p}, \mathbf{P}, \alpha) = \left(\frac{1}{\mathbf{P}}\right)\ g(y, 1, \alpha)$. This proves that the density $g(\tilde{p}, \mathbf{P}, \alpha)$ is homogeneous of degree -1 in $\tilde{p}$ and $\mathbf{P}$. A straightforward extension of this logic shows that the same is true for the density of price changes $w(\Delta p)$.

Now we turn to the hazard rate: The time until next adjustment is the first observation time t where $y(t) > \phi$ or $y(t) < -\phi$. The expression $T'(y) = \tau\,(1 - y^2)$ immediately shows that $T'(y)$ is homogeneous of degree 1 in $\tau = \mathbf{T}$. Let $S(t)$ be the survival function and recall hazard rate definition $h(t) = -\left(\frac{1}{S(t)}\right)\frac{\partial}{\partial t}S(t)$. Consider the following change of variable: $\hat{t} = \mathbf{T}\,t$. This gives that $\frac{\partial}{\partial t}S(t, 1) = \frac{\partial}{\partial t}S(\hat{t}, \mathbf{T})\,\mathbf{T}$ which shows that the function $\frac{\partial}{\partial t}S(\hat{t}, \mathbf{T})$ rate is homogeneous of degree -1 in $(\hat{t}, \mathbf{T})$. It is then immediate to verify, using the hazard rate definition, that $h(\hat{t}, \mathbf{T})$ is also homogeneous of degree -1. ■

Proof of Proposition 7

Let t denote the time elapsed since the last price adjustment, and let $S(t)$ be the survival probability, that is, the fraction of spells of unchanged prices that are of length t or longer. The instantaneous hazard rate is defined as $\mathbf{h}(t) = \frac{-S'(t)}{S(t)}$. First, notice that until τ units of time no firm will review its price, and hence no adjustments will take place, so that $S(t) = 1$ and the hazard rate is 0 for $t \in [0, \tau)$. At τ all the firms review their prices, and a fraction of them adjusts. This fraction is $2(1 - N(\phi))$, that is, the probability that after the review the target is outside the range of inaction. Thus, there is a jump down in the survival function to $S(\tau) = 2\,N(\phi) - 1$, and thus the instantaneous hazard rate is infinite at this point. For the remaining firms the time of the next review depends on the current price gap $\tilde{p}$. The earliest next review among these firms occurs $\underline{\tau}$ periods after the first review, these are the firms that have a price gap inside the range of inaction but arbitrarily close to its boundary, that is, very close to $\bar{p}$ or $-\bar{p}$. We describe the number of firms that change prices in their second review, between times $\tau + \tilde{t}$ and $\tau + \tilde{t} + \Delta$, as approximately $\frac{\partial S(\tau+\tilde{t})}{\partial t} \times \Delta$, satisfying:

$$\frac{\partial S(\tau + \tilde{t})}{\partial t} = \left[1 - N\left(\frac{\bar{p} - p(\tilde{t})}{\sigma\sqrt{\tilde{t}}}\right) + N\left(\frac{-\bar{p} - p(\tilde{t})}{\sigma\sqrt{\tilde{t}}}\right)\right] 2\frac{\partial p(\tilde{t})}{\partial t}\ \frac{n\left(\frac{p(\tilde{t})}{\sigma\sqrt{\tau}}\right)}{\sigma\sqrt{\tau}},$$

for $\underline{\tau} < \tilde{t} < \tau$ where $p(\tilde{t}) \equiv \mathrm{T}^{-1}(\tilde{t})$ denotes the inverse of $\mathrm{T}(\cdot)$, so that $\mathrm{T}^{-1}(\tilde{t}) : [\underline{\tau}, \tau] \to [0, \bar{p}]$. The first term in brackets is the fraction of those firms that had price gap $\tilde{p} > 0$ at time τ and that after the second review are outside the range of inaction, and hence adjust their price (this expression is multiplied by 2 to include the firms with $\tilde{p} < 0$ at time τ). The remaining term counts the number of firms that have a price gap $\tilde{p} = p(\tilde{t})$ so that they will adjust their price at $\tau + \tilde{t}$. This, in turn, is made of two terms. The second ratio is the density of innovations from time 0 to time τ necessary to end up in the required value of the price gap $p(\tilde{t})$. The derivative, $\frac{\partial p(\tilde{t})}{\partial t}$, comes from a change of variables formula, to convert the density of prices into a density expressed with respect to times. If $\tau + \underline{\tau} > 2\tau$, the expression for $\frac{\partial S'(\tilde{t}+\tau)}{\partial dt}$ is valid for all $t \in [\tau + \underline{\tau}, 2\tau]$. In this case, since the symmetry of $\mathrm{T}(\cdot)$ implies that $\frac{\partial \mathrm{T}(0)}{\partial p} = 0$, then $\frac{\partial p(\tau)}{\partial t} = \infty$, and thus the hazard rate tends to infinity at the end of this interval, and reverts to 0 afterward. If this condition is not satisfied, the expression for the derivative of S for values higher than $\tau + 2\underline{\tau}$ is more complex because a price change can occur at exactly the same time after two or three reviews.

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SUPPLEMENTARY MATERIAL

An Online Appendix for this article can be found at QJE online (qje.oxfordjournals.org).

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