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**The Demand of Liquid Assets
with Uncertain Lumpy Expenditures**

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Abstract

We consider an inventory model for a liquid asset where the per-period net expenditures have two components: one that is frequent and small and another that is infrequent and large. We give a theoretical characterization of the optimal management of liquid asset as well as of the implied observable statistics. We use our characterization to interpret some aspects of households' currency management in Austria, as well as the management of demand deposits by a large sample of Italian investors.

JEL Classification Numbers: E5

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1 Introduction

This paper studies some implications of uncertain lumpy purchases for the management of liquid assets in the context of inventory theoretical models. By lumpy purchases we mean large-sized expenditures that must be paid with a liquid asset. The paper accomplishes three objectives. First, it shows that some of the theoretical predictions of this problem are in stark contrast compared to those of canonical inventory models. In particular, a novel feature introduced by the lumpy purchases is the *possibility* that liquidity gets withdrawn and spent immediately. This feature changes the relationship between the size of liquidity withdrawals and the average liquidity holdings compared to canonical models. Equivalently, this affects the relationship between the average cash holdings and its “scale variable”, e.g. the average expenditures in a period. Second, our analysis of the optimal policy breaks some new ground on the mathematical analysis of inventory models, as the solution of the model with jumps turns out to be non-trivial. For instance, we show that the standard boundary conditions used to characterize the optimal policy are necessary but not sufficient, for an optimum. Third, the paper brings new evidence to bear on the model predictions concerning households’ liquidity management. We use two novel datasets of Austrian and Italian households to summarize the main patterns in the data concerning the management of currency and of broader liquid assets. We show that our model can explain some empirical regularities that traditional models cannot account for. Although this paper focuses on households, mostly because of data availability, our model has clear implications for the management of liquid assets by firms, and reserve management by banks, which we discuss in the concluding section.

The standard inventory model solves the problem of an agent trading off the holding cost of an inventory with the cost of adjusting the inventory. In simple models the inventories are assumed to be needed to finance an exogenous consumption flow (for households), or net sales (for firms). These models are typically set up in continuous time and the uncontrolled dynamics of inventories is described by a process with continuous paths. The classic examples

of this setup when applied to households are [Baumol \(1952\)](#), [Tobin \(1956\)](#), where the process describes the household's consumption that needs to be paid with currency. When applied to firms, as in [Miller and Orr \(1966\)](#), the process is the firm's net cash revenue. In this case one cost of the inventory is the low return of the liquid asset. This paper studies the implications for cash-management of having lumpy uncertain purchases: thus we explore the consequences of departing from the assumption that net cash consumption has continuous paths, allowing the unregulated inventories to follow a jump process.

The ideas in this paper can be sketched in the context of a simple model. Suppose that in each period the agent must finance a consumption in cash of size c per unit of time and also that with a probability κ per unit of time the agent must make a cash payment of size z . In this case the expected consumption to be financed with cash is $e = c + \kappa z$ per unit of time. One strategy for the agent is to withdraw enough money so that, at least if this happens soon after the time of the withdrawal, all payments can be financed with the cash at hand. This strategy has the advantage of saving on the adjustment cost, but it has the disadvantage of incurring a holding cost on the inventory of cash. The inventory cost increases with the purchase size z , since the agent has to withdraw more money. An alternative strategy is to withdraw money when the purchase z happens. This strategy saves in holding cost, since the agent spends the money right away, but it involves paying the adjustment cost more frequently. This strategy is preferred when the probability κ is small. Thus, as the size of the large purchases z increases, and/or as they become infrequent (κ small), the optimal policy is to withdraw every time a large purchase occurs. In this simple extreme case the expected value of large purchases κz has no effect on the average cash holdings (M). The size of the withdrawal (W), when triggered by a large purchase, increases by the amount of this purchase, and an amount z of cash is spent immediately. These results, in some sense, turn the logic of the classical inventory model up-side down: cash is not spent slowly and adjustment is not triggered by the crossing of some sS bands. Instead, it is the arrival of a large purchase that triggers an adjustment and a simultaneous withdrawal and large use of

cash. While κz has no effect on the average cash holdings, its magnitude affects the average size of withdrawals, and hence W/M is increasing in κz . The high value of W/M also implies that the number of withdrawals per unit of time n is small relative to the benchmark of the Baumol-Tobin model for an agent financing the same consumption e . The economics is simple: the withdrawals that are triggered by large purchases account for a large share of cash expenditures e and contribute nothing to average money holdings. Additionally, if every random large expenditures triggers a withdrawal, agents on average hold cash at the time of withdrawals, a behavior that can be described as “precautionary” and which is clearly apparent in the data.

There is a large literature on inventory models applied to liquid assets. Most of the literature assumes that the cumulated net cash consumption has continuous path, hence not allowing for lumpy purchases or sales. Examples are [Baumol \(1952\)](#), [Tobin \(1956\)](#), [Miller and Orr \(1966\)](#) among many others.¹ One exception is the work by [Bar-Ilan, Perry, and Stadjje \(2004\)](#). The set up of that paper includes lumpy purchases and sales using a more general specification than the one in this paper. They compute the value of selected policies, but do not characterize the nature of the optimal decision rules. In this paper, instead, we give a characterization of the optimal policy which, from the technical point of view, is not a trivial matter since it has to address two issues: (i) the form of the inaction set, i.e. whether it is a single interval or the union of disjoint ones; and (ii) whether the necessary boundary conditions are also sufficient, which turns out not to be the case for this problem. Additionally, the focus of our paper is different, we concentrate on the implications for the cash management statistics and on the differences with standard models.

We present two empirical applications of our model. The first one uses two surveys on currency management, from Italian and Austrian households. Both surveys contain information on the patterns of cash management: the average consumption paid with cash per

¹Other examples are [Eppen and Fama \(1969\)](#), [Constantinides \(1978\)](#), [Constantinides and Richard \(1978\)](#), [Frenkel and Jovanovic \(1980\)](#), [Harrison, Sellke, and Taylor \(1983\)](#), [Harrison and Taskar \(1983\)](#), [Bar-Ilan \(1990\)](#), [Duffie and Sun \(1990\)](#), [Abel, Eberly, and Panageas \(2007\)](#), [Baccarin \(2009\)](#), [Bensoussan, Chutani, and Sethi \(2009\)](#), and [Alvarez and Lippi \(2009\)](#).

period e , the average cash holdings M , average withdrawal size W , average number of withdrawals per unit of time n , and average cash holdings at the time of withdrawal \underline{M} . Besides that, the Austrian data set contains information on the patterns of purchase size, recorded in a consumption diary held by the same individuals to whom the survey was administered (see the description in [Mooslechner, Stix, and Wagner \(2006\)](#)). The diary data show that, for a non-negligible fraction of individuals, the assumption that large purchases are paid in currency is realistic. We use the diary and survey information to investigate some of the predictions of our model by comparing individuals that differ in the importance of the lumpy component of their expenditures paid with currency. In [Section 4.1](#) we present evidence on several statistics, such as the frequency and size of withdrawal relative to the average holding of currency holdings, that is supportive of the mechanism highlighted in the model. The second application, in [Section 4.2](#), focuses on a broader liquid assets, close to $M2$, using data from a sample of Italian customers of a large commercial bank described in [Alvarez, Guiso, and Lippi \(2011\)](#). We argue that accounting for the lumpy nature of purchases, as in the case of e.g. durable purchases, seems important to understand the management of liquid assets.

The paper is organized as follows. [Section 2](#) outlines the main idea of the paper using a simple deterministic model. A stochastic version of the model is discussed in [Section 3](#), where various specifications are explored. [Section 4](#) illustrates two empirical applications of the model. [Section 5](#) has conclusions and directions for future research.

2 A deterministic model with lumpy purchases

This section develops a simple version of the Baumol-Tobin (BT) model where the consumption paid in cash has two deterministic components, one continuous at the rate c per unit of time and the other discontinuous, with jumps of size z , exactly every $1/\kappa$ periods of time. Thus, total consumption per unit of time is $e \equiv c + z\kappa$, the sum of the cumulative consumption at the rate c plus the κ jumps in consumption, each of them of size z . These jumps on

the cash consumption in the model are meant to be a simple representation of the fact that households' purchases varies in size. The objective function, as in BT, is to minimize the cost V given by $V = RM + bn$ where M is the average cash balances, R is the opportunity cost of the cash balances, n is the number of withdrawals per unit of time, and b is the fixed cost paid for each withdrawal.

It turns out that the optimal policy is of one of three types, depending on parameters. When κ is small, the agent withdraws every $1/(i \kappa)$ units of time, where $i \geq 1$. In this case, there are i withdrawals between jumps in cumulative consumption, and $n > \kappa$. One of the withdrawals will happen just before the jump z , and hence financing the discontinuous part of consumption is done at no cost. If κ is large, the agent makes a withdrawal every j/κ units of time, where $j \geq 1$. In this case there are j jumps in cumulated consumption between successive withdrawals, or $n < \kappa$. Thus the agent will only "save" on the opportunity cost of the associated consumption z once every j jumps between withdrawals. For intermediate values of κ , withdrawals happen exactly every $1/\kappa$ periods of time, or $n = \kappa$.

We define two thresholds $\underline{\kappa}$ and $\bar{\kappa}(z)$, which determine the patterns of cash management:

$$\underline{\kappa} \equiv \sqrt{\frac{R c}{2 b}} \leq \bar{\kappa}(z) \equiv \frac{R z + \sqrt{(R z)^2 + 8 b R c}}{4 b}.$$

Note that $\underline{\kappa} = \bar{\kappa}(0)$ and that $\bar{\kappa}$ is strictly increasing in z . To simplify the description we will assume that a certain combination of parameters takes on integer values. Define u as follows:

$$u \equiv \max \left\{ \sqrt{\frac{R c}{2 b \kappa^2}}, \sqrt{\frac{\kappa^2 2 b}{R (c + \kappa z)}} \right\}.$$

For the description of the optimal policy we let W be the average withdrawal size, so W/M is the ratio of average withdrawal to average stock of money. We have:

PROPOSITION 1. Assume that if $u > 1$, then u is an integer. Then the optimal decision

rules and the value of the objective function V are given by:

$$\begin{aligned}
\text{If } \kappa < \underline{\kappa} & : n = \sqrt{\frac{R c}{2 b}} > \kappa, \frac{W/M}{2} = \frac{c + \kappa z}{c}, \text{ and } V = \sqrt{2 R b c}, \\
\text{If } \underline{\kappa} \leq \kappa \leq \bar{\kappa}(z) & : n = \kappa, \frac{W/M}{2} = \frac{c + \kappa z}{c} \text{ and } V = \frac{R c}{2} + b \kappa, \\
\text{If } \kappa > \bar{\kappa}(z) & : n = \sqrt{\frac{R (c + z \kappa)}{2 b}} < \kappa, \frac{W/M}{2} = \frac{c + \kappa z}{c + z(\kappa - n)} \text{ and} \\
& V = \sqrt{2 R b (c + \kappa z)} - \frac{R z}{2}.
\end{aligned}$$

Using the accounting identity $W n = c + \kappa z$ and the expressions in the proposition one can find the values of W and M separately. The interpretation of this proposition is as follows: when κ is small relative to what determines the frequency of withdrawals in BT, then the jumps can be made coincide with one of the many withdrawals. In this case, the agent will withdraw the extra amount z and spend it immediately. Thus the cash expenditure associated with the jump does not incur into any opportunity cost since that cash is held only for an instant. Also, the agent does not incur any extra fixed cost b , since it has to withdraw to finance the continuous expenditure anyway. As a consequence, the decision for the agent on the number of withdrawals n is exactly as in BT, but it is as if the consumption to be financed is c , instead of $c + \kappa z$. This can be seen from the expression for n and V . The expression for W/M is larger than 2, since at the time of a jump in consumption the agent withdraws an extra amount z , which is spent immediately and does not contribute to the average money holdings M . On the other hand, consider the case where κ is large, so that the agent will like to withdraw several times between jumps. In this case, only the first of the jump, the one that occurs immediately after a withdrawal will have no opportunity cost associated with it. Otherwise, the decisions are as if the agent has to finance $c + \kappa z$ in the BT model. This can be seen in the expression for n , which is the same as in BT, and in the one for V , which is identical, except that it subtracts the “savings” in the opportunity cost

for one jump per period. The ratio of W/M depends on how large these jumps are, i.e. on z . The following extreme case may help to understand the model for large κ . Assume that κ is very large, but z is very small, so there are very frequent jumps of small size, keeping the product $z\kappa = \gamma$ positive and finite. As the jumps become very small, the model is identical to BT, with total consumption $c + \kappa z$. This can be seen in the case where $\kappa > \bar{\kappa}(z)$, and taken the limit to z to zero. We now develop the comparison with BT in detail.

Let us denote by $n_{BT} = \sqrt{R(c + z\kappa)/(2b)}$ the optimal decision if one were to measure the total cash consumption $c + z\kappa$ and assume that it is continuous as in BT. Also we recall that in BT the ratio $W/M = 2$ since cash consumption is constant per unit of time ($z = 0$). We then compute two ratios, n/n_{BT} and $(W/M)/2$, as functions of κ . These are useful to compare the prediction with BT. We have

$$\begin{aligned} \text{If } \kappa \leq \underline{\kappa} &\implies \frac{n}{n_{BT}} = \sqrt{\frac{c}{c + \kappa z}}, \quad \frac{W/M}{2} = \frac{c + \kappa z}{c}, \\ \text{If } \underline{\kappa} < \kappa < \bar{\kappa}(z) &\implies \frac{n}{n_{BT}} = \sqrt{\frac{\kappa^2 2b}{R(c + \kappa z)}} \leq 1, \quad \frac{W/M}{2} = \frac{c + \kappa z}{c}, \\ \text{If } \kappa \geq \bar{\kappa}(z) &\implies \frac{n}{n_{BT}} = 1, \quad 1 \leq \frac{W/M}{2} = \frac{c + \kappa z}{c + \kappa z - z\sqrt{\frac{R(c + \kappa z)}{2b}}} < \frac{c + \kappa z}{c}. \end{aligned}$$

Notice that in terms of the statistics W/M and n/n_{BT} the implications of the model when $\kappa < \underline{\kappa}$ depend only on the value of $z\kappa$, and not separately on κ and z , a feature that will be shared by the model in [Section 3.2.1](#). There are two extreme cases that are useful to highlight. Keeping the product $z\kappa = \gamma > 0$, strictly positive and finite, we have:

$$\begin{aligned} \lim z = 0, \lim \kappa = \infty &\implies \frac{n}{n_{BT}} = 1, \quad \frac{W/M}{2} = 1, \\ \lim z = \infty, \lim \kappa = 0 &\implies \frac{n}{n_{BT}} = \sqrt{\frac{c}{c + \gamma}}, \quad \frac{W/M}{2} = \frac{c + \gamma}{c}. \end{aligned}$$

The first line describes the case of an economy in which all consumption is continuous (i.e. no jumps ever occur). In this case the model coincides with BT. The second line describes

the limiting case of an economy in which the lumpy expenditures is concentrated in a single jump (the probability of a jump per unit of time is thus zero). In this case the number of withdrawal is smaller, and the W/M ratio higher, than in BT.

Finally, note that the money demand M as function of R/b , the interest rate relative to the fixed cost, is decreasing (keeping the other parameters c, κ, z fixed). The shape of the money demand as a function of R/b depends on the value of the interest rates. For this purpose we divide the values of the interest rates R/b into three segments, using two cut-off points defined as follows: $\bar{R}(\kappa)$ solves $\kappa = \underline{\kappa}(R)$ and $\underline{R}(\kappa, z)$ solves $\kappa = \bar{\kappa}(R, z)$, where $\underline{\kappa}$ and $\bar{\kappa}$ are the thresholds defined above, which we now write as functions of the interest rate. Since $\underline{\kappa}(R) < \bar{\kappa}(R)$, and since both are increasing in R , starting at 0 and going to ∞ , the thresholds $\underline{R}(\kappa) < \bar{R}(\kappa)$ are well defined and are both decreasing in κ , and \underline{R} is decreasing in z . Using these thresholds, [Proposition 1](#), and the accounting identity $Wn = c + \kappa z$ we have:

$$\frac{\partial \log M}{\partial \log R} = \begin{cases} -\frac{1}{2} - \frac{1}{2} \frac{zn}{c+z(\kappa-n)} & \text{if } 0 < R < \underline{R}(\kappa, z) & \text{where } n < \kappa \\ 0 & \text{if } \underline{R}(\kappa, z) < R < \bar{R}(\kappa) & \text{where } n = \kappa \\ -\frac{1}{2} & \text{if } \bar{R}(\kappa) < R & \text{where } n > \kappa \end{cases} \quad (1)$$

Thus the elasticity of the money demand is non-monotone on R . For small values of R is more elastic than $1/2$, for intermediate values it is inelastic, and for high values of R the elasticity is $1/2$.

3 A stochastic model

We consider a model where consumption has three components: one is deterministic at a constant rate c per unit of time, - as in our previous model. The second component represent large purchases: we assume that the jump process occurs with probability κ per unit of time, and that when it happens cumulated consumption increases by an amount given by the parameter $z > 0$. With this parameterization, expected consumption per period, say

per year, equals $e = c + \kappa z$. The third component introduces random variation in the net cash consumption, with variance σ^2 per unit of time. This is to capture, as in the seminal model of cash management of firms by [Miller and Orr \(1966\)](#), income that is received in cash, a feature that is mostly associated with firms' cash flows. Moreover this gives rise to cash deposits as well as withdrawals. If we denote cumulative consumption paid in cash by $C(t)$ we assume that $dC(t) = cdt + zdN + \sigma dB$, where $N(t)$ is the poisson counter, and $B(t)$ is an standard Brownian motion. If we interpret dC as the consumption during a period of length dt , we note that, when $\sigma > 0$, it can be negative.

We also assume that with a Poisson arrival rate p per unit of time, the agent has an opportunity to adjust her cash balances without paying the cost b . We introduce the free adjustment opportunities as a way to model the possibility of cash replenishment due to the ATM network, an feature that we have explored in [Alvarez and Lippi \(2009\)](#). The reason to include it here is that it shares some implications with the model with large purchases. Specifically, as shown below, that cash management data alone (such as frequency and size of withdrawals, average cash holdings, etc.) cannot identify separately p from κ . On the other hand, having data on the size distribution of cash purchases can help identify these parameters.

The introduction of the large purchases is useful to explore the implications of an alternative reason for “precautionary” type of behavior. In this model, there are three types of withdrawals, those that occur when m reaches zero, those that occur at the time of a jump in consumption if $m < z$, and those that occur if the agent has a free withdrawal opportunity. The idea is that at times when cumulated consumption jumps (i.e. a large purchase occurs), if the money balances at hand m are not large enough to pay for the sudden increase in cash consumption, i.e. if $m < z$, then the agent will withdraw cash, even if cash has not reached zero. Otherwise, the nature of the optimal policy is the same, after withdrawal agents set their cash balances to the optimal replenishment level m^* .

The standard inventory model has unregulated inventory following a process with con-

tinuous paths.² Yet there are several exceptions. Milbourne’s (1983) model is set up in discrete time and makes no special assumptions about the process for net cash-holdings. If we let C be the cumulated unregulated process, and we let N be the counter of a Poisson process, we have that $dC(t) = z dN$ from state dependent $F(x, z)$ in Song and Zipkin (1993) (where the state x is a finite Markov chain) and in Archibald and Silver (1978). A paper with a closely related, but more general, set up is Bar-Ilan, Perry, and Stadje (2004), which assumes $dC(t) = \mu dt + \sigma dB + z^u dN^u - z^d dN^d$ and where B is a standard Brownian motion, $z^i \geq 0$ are the up and down jumps, and N^i are the counters of two Poisson processes with possible different constant intensities, and where the jump sizes have general distributions which include the exponential distribution. Their paper also has a more general adjustment cost, including fixed and variable cost, that differs for deposits and withdrawals, for which the authors present an analytical solution for the value of following a type of sS policy.³

We show below that solving the Bellman equation is more involved than in the standard case where the unregulated inventory (cash in this case) follows a process with continuous path. This requires to solve a delay-differential equation, as opposed to an ordinary differential equation. While in Section 3.1 we present an algorithm to solve for the parameters that fully characterize the Bellman equation, we do not have a simple closed form solution for the thresholds that describe the optimal policy (m^*, m^{**}) , as we did for the case with no jumps in Alvarez and Lippi (2009). Of course if the jumps were small, i.e. if z was small, the statistics of interest would not be affected. In particular, we show that in the limit as $z \rightarrow 0$ while keeping κz constant, the model reduces to the one with continuous consumption. Thus, in Section 3.2.1 we will concentrate on the case of large but infrequent jumps, i.e. large

² If we let C be the cumulated unregulated process, we have that $C(t) = c t$ for constant $c > 0$ as in Baumol (1952), Tobin (1956), Jovanovic (1982), Alvarez and Lippi (2009); or $C(t) = \sigma B_t$ for constant $\sigma > 0$ and B_t a standard BM in Miller and Orr (1966), Eppen and Fama (1969); or $C(t) = ct + \sigma B_t$ for constants $c, \sigma > 0$: Constantinides and Richard (1978), Constantinides (1978), Frenkel and Jovanovic (1980), Harrison, Sellke, and Taylor (1983), Harrison and Taskar (1983), Bar-Ilan (1990), and $dC(t) = c(x)dt + \sigma(x)dB$ in Baccarin (2009).

³ Apparently unaware of this work, Sato and Suzuki (2011) have recently obtained a similar set of equations for value of following an sS policy. Davis, Guo, and Wu (2010) show the smoothness of the value function in a very general set-up.

z and small κ , which echoes the deterministic case of $\kappa < \underline{\kappa}$. We will describe the nature of the optimal policy for this case, as well as the implications for several cash management statistics.

3.1 Bellman Equation

We consider a trigger policy described by two thresholds: m^* , the value of cash after and adjustment, and m^{**} , the value of cash that triggers a deposit. Non-negativity of cash triggers a withdrawal at $m = 0$. After a deposit or a withdrawal, agents return to the value m^* , so that the size of a deposit is $m^{**} - m^*$ whereas, due to the consumption jumps and the free adjustment opportunities, withdrawals have a random size. The Bellman equation in the interior of the range of inaction, given by $0 < m < m^{**}$, becomes:

$$\begin{aligned} rV(m) &= Rm + p \left[\min_{\hat{m}} V(\hat{m}) - V(m) \right] + \frac{\sigma^2}{2} V''(m) \\ &\quad + \kappa \min \left[b + \min_{\hat{m}} V(\hat{m}) - V(m), V(m - z) - V(m) \right] \\ &\quad + V'(m) (-c - \pi m) \end{aligned}$$

where π is the inflation rate. The term $\min [b + \min_{\hat{m}} V(\hat{m}) - V(m), V(m - z) - V(m)]$ takes into account that after the jump in consumption the agent can decide to withdraw cash, or otherwise her cash balances becomes $m - z$. We let

$$m^* \equiv \arg \min_{\hat{m}} V(\hat{m}), \text{ and } V^* \equiv V(m^*) . \quad (2)$$

If the value function is differentiable, we have that

$$V'(m^*) = 0 . \quad (3)$$

Non-negativity of cash implies that

$$V(m) = V^* + b \text{ for } m \leq 0. \quad (4)$$

For the range $0 \leq m \leq z$ we look for a solution of the form of an Ordinary Differential Equation (ODE):

$$(r + p + \kappa) V(m) = Rm + (p + \kappa) V^* + \kappa b + V'(m)(-c - \pi m) + \frac{\sigma^2}{2} V''(m) \quad (5)$$

since in this range every jump triggers a withdrawal. This feature is as in [Bar-Ilan, Perry, and Stadje \(2004\)](#), who refer to it as adjustment triggered by downcrosses. Instead for the range $z \leq m \leq m^{**}$, we have a Delay-Differential Equation (DDE):

$$(r + p + \kappa) V(m) = Rm + pV^* + \kappa V(m - z) + V'(m)(-c - \pi m) + \frac{\sigma^2}{2} V''(m), \quad (6)$$

since in this range after a jump cash balances are positive. If cash reaches the value of m^{**} , then it triggers a deposit of size $m^{**} - m^*$ after paying the fixed cost b . Thus we have:

$$V(m^{**}) = V(m^*) + b \quad \text{and} \quad V(m^{**}) = V(m) \text{ for all } m \geq m^{**}. \quad (7)$$

If $V(\cdot)$ is differentiable at $m = m^{**}$, then we get that

$$V'(m^{**}) = 0, \quad (8)$$

a condition typically referred as to “smooth pasting”. We notice that, in general, it will not be differentiable at this point if $\sigma = 0$.

We can further characterize the Bellman equation for V for given policy described by thresholds (m^*, m^{**}) by splitting the range of inaction in intervals of length z . The idea is that, at a given point m , the value function depends on the local evolution around m and

on the value that it will take after a jump, i.e. at $m - z$. But since cash is non-negative, for $m \in [0, z]$ any jump will lead to a withdrawal, and hence, given V^* , the value function only depends on its local evolution, i.e. it is a second order (first order if $\sigma = 0$) linear ODE described by [equation \(5\)](#). Then, *given the solution of the value function in the lower segment*, one can construct the segments corresponding to higher values of m recursively, which themselves solve a system of ODE's described by [equation \(6\)](#). In the case of $\pi = 0$ the ODEs have constant coefficients. The value matching [equation \(2\)](#), [equation \(4\)](#), [equation \(7\)](#) provide three boundary conditions. The continuity of the level, first derivative (and second if $\sigma > 0$) across each segment, provide additional boundary conditions.

PROPOSITION 2. Assume $\pi = 0$ and $\sigma \geq 0$. Given two thresholds $0 < m^* < m^{**}$ the value of following such a policy, i.e. the solution of [equation \(4\)](#), [\(5\)](#), [\(6\)](#) and [\(7\)](#), can be described by J functions V_j :

$$V(m ; m^*, m^{**}) = V_j(m) \text{ for } m \in [zj, \min \{z(j+1), m^{**}\}] , \quad (9)$$

where

$$V_j(m) = A_j + D_j(m - zj) + \sum_{k=1,2} \sum_{i=0}^j B_{j,i}^k e^{\lambda_k(m-zj)} (m - zj)^i \quad (10)$$

where λ_k is a solution of $r + p + \kappa = -c\lambda + \frac{\sigma^2}{2}\lambda^2$ for $k = 1, 2$ and where the constants $A_j, D_j, B_{j,i}^k$ for $j = 0, 1, 2, \dots, J-1$, $i = 1, \dots, j$, and $k = 1, 2$ solve a block recursive system of linear equations described in the proof.

Note that for $\sigma = 0$ there is only one root λ and hence one set of coefficients $\{B_{j,i}\}$, together with $\{A_j, D_j\}$. Using [Proposition 2](#) we can write the optimality of the return point

equation (3) and the smooth pasting condition equation (8) as

$$0 = V'(m^*; m^*, m^{**}) = D_{j^*} + \sum_{k=1,2} \sum_{i=0}^{j^*} B_{j^*,i}^k e^{\lambda_k(m^* - zj^*)} [\lambda_k + i(m^* - zj^*)i - 1], \quad (11)$$

$$0 = V'(m^{**}; m^*, m^{**}) = D_{J-1} + \sum_{k=1,2} \sum_{i=0}^{J-1} B_{J-1,i}^k e^{\lambda_k(m^{**} - z(J-1))} [\lambda_k + i(m^{**} - z(J-1))i - 1], \quad (12)$$

where j^* is the smallest integer such that $m^* < (j^* + 1)z$, and where equation (12) applies only if $\sigma > 0$. Since Proposition 2 shows that, for the case of $\sigma > 0$ the constants $\{A_j, D_j, B_{j,i}^k\}$ are a function of (m^*, m^{**}) , we can regard equation (11) and equation (12) as a system of two non-linear equations determining (m^*, m^{**}) . Instead if $\sigma = 0$ the constants $\{A_j, D_j, B_{j,i}^k\}$ are a function of m^* , so we can regard equation (11) as a one non-linear equation determining m^* . Intuitively, when $\sigma = 0$ cash balances can only go down and cash deposits never occur, so that the threshold m^{**} drops out of the problem. The value function $V(m; m^*, m^{**})$ is then simplified to $V(m; m^*)$, i.e. it is indexed by only 1 parameter (the optimal return point m^*).

Notice that for given *arbitrary* values of the thresholds (m^*, m^{**}) a stochastic process for m is completely determined. Given this process, one can use straightforward computer simulations to derive the statistics of interest on the cash management, such as the frequency of withdrawals n , the average money holdings M , the withdrawal size W , the money holdings at the time of a withdrawal \underline{M} , which will obviously depend of the chosen thresholds. The *optimal* values of (m^*, m^{**}) have to satisfy two non-linear equations, namely equation (11) and equation (12). The ultimate objective of course is to compare the statistics implied by the optimal policy with the same statistics from actual data, such as the ones discussed in Section 4. The one caveat, discussed below in Section 3.2.2, is that the conditions implied by equation (11) and equation (12) are necessary but not sufficient, so that care must be taken in ensuring that the values that are chosen are the ones corresponding to a global minimum for the value function, and not just a local extreme. This can be done, for instance, by simple inspection of the known value function evaluated at a fixed value of m , e.g. m^* , as a function

of the (m^*, m^{**}) thresholds, as we do in [Figure 1](#).

Next we present a proposition showing that the limit of small and frequent jumps is the case with continuous consumption.

PROPOSITION 3. Consider the solution of the value function as $z \rightarrow 0$ and $0 < \gamma \equiv \lim_{z \rightarrow 0} z \times \kappa < \infty$. This solution coincides with the one without jumps, i.e. $\kappa = z = 0$ but with continuous consumption at the rate $c + \gamma$.

The logic of the proof of this proposition is straightforward, so we only sketch the argument here. First, notice that path for the cumulative consumption accounted for jumps $zN(t)$ goes to γt with probability one. Second, notice that the contribution of these jumps to the value function, given by $\kappa(V(m - z) - V(m))$ when $m > z$ can be written as $\kappa(V(m) - V'(m)z + o(z))$. Assuming that we can permute the limit of the derivative with the derivative of the limit, we obtain that in the limit the contribution of this term is $-\gamma V'(m)$, a term analogous to the contribution from c . The contribution of the segment $m > z$ is negligible as z goes to zero. This result can be useful to make contact with the data. The issue is not whether consumption transactions occur as discrete events or not, which of course they do. The previous result states that small frequent purchases can be approximated by the continuous model. The issue is whether the continuous consumption model is a good approximation given the observed size of purchases. Thus, if the purchases using cash are small and frequent, the model with a continuous path may be a good idealization. On the other hand, intuitively, a model with infrequent and large purchases, will be the most different case, a set up to which we will turn in [Section 3.2.1](#).

We briefly discuss the empirical counterparts and interpretations of the model with $\sigma > 0$ vs $\sigma = 0$. Notice that for $\sigma = 0$ the model predicts no deposits and only withdrawals: the inaction region is given by $[0, m^*]$, and cash inside this region only moves down, either at a constant rate per unit of time, or with jumps of size z and frequency κ . Consistent with this behavior notice that [Table 1](#) shows that Italian households whose head is not self-employed make very few deposits relative to withdrawals: the average ratio of the number of deposits

to the number of withdrawals is less than 10%, with a large fraction of households with exactly zero deposits. Thus for the non-self employed households a model with $\sigma = 0$, which generates the extreme case of no deposits, seems a reasonable approximation. In contrast, consider the statistics for the self-employed households, which appear in square brackets in the table. Their deposits are more frequent than for the rest of the households: the average ratio of number of deposits to the number of withdrawals of about 66%. The model with $\sigma > 0$ accounts for this fact. Moreover we think that it does so for the right reason: these households look like firms in the sense that their net cash consumption can be negative (i.e. they receive income in cash). We believe that this is the first empirical direct evidence on the prediction, implied by [Miller and Orr \(1966\)](#) models, that deposits are larger but less frequent than withdrawals.

We finish this section with a brief comment on the optimality of the class of trigger policies considered here. First, in the case where $\sigma = 0$, it is easy to show that the ergodic distribution of m lies in $[0, m^*]$, whose interior contains the inaction region. Second, in the case with no jumps ($\kappa = 0$), $\sigma > 0$ and $p = 0$, it has been shown by e.g. [Constantinides and Richard \(1978\)](#) that trigger policies of this type are optimal. The extension to the case of $p > 0$ should be relatively straightforward. The third, more subtle, case is the combination of jumps (so that $\kappa > 0, z > 0$) and a brownian motion (so that $\sigma > 0$) for cumulated net cash consumption. The potential complication comes about when there are discrete changes in the unregulated state, i.e. discrete changes in m in our case. This case has been studied in discrete time with finite but arbitrary horizon by [Neave \(1970\)](#). He showed that the decision rule will, in general, have an inaction region close to the optimal return point, that outside the inaction region there is a set of intervals where either adjustment or inaction is optimal, and that for large values there is an open ended interval for which adjustment is optimal. [Bar-Ilan \(1990\)](#) has also produced a counterexample in the case of two periods two jumps, one up and one down. More recently [Chen and Simchi-Levi \(2009\)](#) have a slightly fuller characterization of this case and an analysis of a more general case. Hence, the issue in

our continuous time model with jumps, which make the model mathematically very close to a discrete time model, is whether there could be several inaction and adjustment regions. Thus, while the form of the optimal policy for a model where the state follows the sum of a diffusion and a more general jump component, such as in the specification of [Bar-Ilan, Perry, and Stadje \(2004\)](#) has not been characterized, our set-up is special enough so that the decision rules, in the ergodic set for m , are of the “sS” form considered above. The features that make our problem special are that the state the jumps are all downwards and of the same size (i.e. $z > 0$) and that the state is non-negative. In our case, if the state reaches m^{**} then it is controlled to be set at m^* . Importantly, since the jumps in net cash consumption are *all* downwards, the state can only reach m^{**} at time $t = \tau$ if it was below, but very close, at times arbitrary close to τ . On the other side, the boundary at $m = 0$ follows from non-negativity of cash and from the fact that the period return function attains its minimum at $m = 0$. Thus, the value of m^{**} is defined as the smallest strictly positive value of the state for which adjustment is optimal.⁴

3.2 Solving for M, W, n in the case of no Brownian shocks: $\sigma = 0$

In this section we concentrate on the special case of the model where the net cumulated cash consumption is the sum of a deterministic constant consumption per unit of time and random jumps, i.e. we set the brownian component to have zero variance or $\sigma = 0$. We concentrate on this case for two reasons. The first is that our data sets for Italy and Austria focuses on households whose head is not self employed for whom, as discussed above, the case of $\sigma = 0$ is more appropriate. The second reason for using $\sigma = 0$ is its simplicity. In particular, when $\pi = \sigma = 0$, the linear equations for the coefficients in [Proposition 2](#) simplify

⁴We can write a discrete time version of our model in the notation of [Neave \(1970\)](#) and [Chen and Simchi-Levi \(2009\)](#) as follows. To simplify we write the version with $p = 0$. Let Δ be the length of the time period. The period return function is $l(m) = +\infty$ if $m < 0$ and otherwise $l(m) = \Delta R m$. The i.i.d. process for unregulated cash, $\xi = -\Delta c + \sqrt{\Delta}\sigma s - z dN$, where s is a symmetric binomial with zero mean and standard deviation one, where $dN = 1$ with probability $\kappa\Delta$ and zero otherwise, and where dN and s are independent. The discount factor is $\gamma = 1/(1+r\Delta)$. The cost function has $K = Q = b$, and no proportional cost, $k = q = 0$. In term of their notation we have, as we let $\Delta \downarrow 0$, the decision rules satisfy: $U = T = m^*$, $0 = t = t^+$, and $u^- = m^{**}$.

considerably, and since the range of inaction becomes $[0, m^*]$, there is one non-linear equation in one unknown (see [Appendix B.1](#) for the relevant equations for this case). Remember that, as discussed above, when $\sigma = 0$ the value function $V(m; m^*, m^{**})$ simplifies to $V(m; m^*)$, i.e. it is indexed by only 1 parameter (the optimal return point m^*).

First, and for completeness, we consider the case where the large purchases are frequent. As in the deterministic case, if for a given size of the purchases z , the frequency κ is high enough, it is optimal to increase the size of the withdrawal and finance -in expected value- several purchases with each withdrawal.

PROPOSITION 4. Let $\pi = \sigma = 0$, $z > 0$, $c \geq 0$, $p \geq 0$ and $b/R > 0$. There exist $\bar{\kappa}(z)$, which is increasing in z , and $\bar{r} > 0$ such that for any $\kappa > \bar{\kappa}(z)$ and $r < \bar{r}$, the optimal threshold satisfies $m^*(p, \kappa, z, c) > z$.

The logic of this proposition is the same as in the deterministic case, so the proof is omitted.⁵ Next, we further specialize the problem to the more tractable case in which the primitive parameters are such that the size of the jump z is larger than m^* , so that in this case every jump in expenditures will trigger a withdrawal.

3.2.1 The case of infrequent large purchases: $z > m^*$

We continue with the analysis of the case of no inflation and no Brownian component, i.e. $\pi = \sigma = 0$. Furthermore we solve the model and the cash holding statistics M, W, n, \underline{M} for a configuration of parameters that corresponds to the case of small κ , (i.e. smaller than $\underline{\kappa}$) in the deterministic model of [Section 2](#) and large value of z . We found this case instructive for two reasons: first, based on the result in [Proposition 3](#), the case where z is large and κ small presents some interesting differences compared to the problem with consumption expenditures are continuous. Second we think that the parameters for which this case applies,

⁵ We believe that the assumption that r is small is not required for the results, but it simplifies the constructive aspect of the proof. [Appendix B](#) provides the algorithms to compute value function and several cash management statistics of interest for the case of $\sigma = 0$ and $\pi = 0$ for any configuration of the lumpy purchase parameters: z, k . The logic is the same one used to solve for the value function in [Proposition 2](#).

which concern size and frequency of the large purchases, seem to be empirically appropriate for modeling the currency management behavior for households in Austria, as argued below.

Abusing notation, we let $V(m; m', p, \kappa, z, c)$ denote the value function of the model analyzed in [Section 3.2](#) when current cash is m and the return point for cash is m' for the parameters (p, κ, z, c) . We also let $m^*(p, \kappa, z, c)$ be the value of the optimal return point for these parameters, and let $M(p, \kappa, z, c)$, $W(p, \kappa, z, c)$, $\underline{M}(p, \kappa, z, c)$ and $n(p, \kappa, z, c)$ be the corresponding cash-management statistics, described in [Section 3.2](#). For future reference we let

$$V^*(m'; p, \kappa, z, c) \equiv V(m'; m', p, \kappa, z, c) \tag{13}$$

the value of following a policy with return threshold m' when cash is at this value. Recall that at the optimal threshold value $V^*(m^*(p, \kappa, z, c); p, \kappa, z, c)$ is the smallest value of the value function.

The next proposition studies the effect of the presence of large purchases (i.e. whether or not $C(t)$ jumps) in a model with free withdrawal opportunities. We note that if $z = 0$ or $\kappa = 0$ the model with no jumps corresponds to a version of Baumol-Tobin where there are p free withdrawal opportunities per unit of time. We have characterized the solution of that model and estimated it for a cross section of Italian households in [Alvarez and Lippi \(2009\)](#). The free withdrawal opportunities of that model imply that, relative to the prediction in Baumol-Tobin, agents makes more withdrawals (say $n/(c/2M) \equiv n/n_{BT} > 1$) and they are smaller in size (say, $W/M < 2$). Also differently to Baumol-Tobin, that model implies that in average agents withdraw when they have strictly positive real balances, i.e. $\underline{M} > 0$. We use the notation $m^*(p', 0, 0, c)$ to denote the optimal return threshold for the model with no jumps, with a rate of p' free adjustment opportunities per unit of time, and with consumption at a constant rate c .

PROPOSITION 5. Assume that $\pi = \sigma = 0$, $c > 0$, $p \geq 0$, $b/R > 0$, $z > 0$ and $r > 0$. There

exists $\underline{\kappa} > 0$ such that for any $\kappa < \underline{\kappa}$ and $z > m^*(p + \kappa, 0, 0, c)$ we have:

$$\begin{aligned}
m^*(p, \kappa, z, c) &= m^*(p + \kappa, 0, 0, c) , \\
V^*(m^*(p, \kappa, z, c); p, \kappa, z, c) &= V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) + \frac{\kappa b}{r} , \\
M(p, \kappa, z, c) &= M(p + \kappa, 0, 0, c) , \\
W(p, \kappa, z, c) &= W(p + \kappa, 0, 0, c) + \frac{\kappa z}{n} , \\
\underline{M}(p, \kappa, z, c) &= \underline{M}(p + \kappa, 0, 0, c) , \\
n(p, \kappa, z, c) &= n(p + \kappa, 0, 0, c) .
\end{aligned}$$

Moreover, the conclusion holds for the same value of $\underline{\kappa}$ for all $z' > z$.

Part of the proof of the proposition is straightforward. In particular, if it is optimal to set $m^*(p, \kappa, z, c) < z$, then the value of the threshold equals the one in a model with no jumps, i.e. with $z = 0$, but with $p + \kappa$ free adjustment opportunities, i.e. $m^*(p + \kappa, 0, 0, c)$. In other words, it is always a local minimum to set the return threshold equal to $m^*(p + \kappa, 0, 0, c)$. In the case of no jumps in consumption at each free withdrawal opportunity cash balances are set to m^* right after the adjustment. The consequences of a free adjustment opportunity bear many similarities with those that follow a jump in consumption. Both cases occur independently of the cash balances, and in both cases cash balances go to m^* after the adjustment. The difference is that upon a free withdrawal opportunity the agent does it because it saves the cost b , while upon a consumption jump the agent does it because of the binding non-negativity of consumption. For the exact equivalence we require that the rate at which the free adjustment opportunities arrives is $p + \kappa$, so that the rate at which this type of adjustment occurs is equally likely. The two value functions differ only by a constant term measuring the present value of the cost saved by the free withdrawal opportunities. It follows that the average cash holdings, average cash at withdrawals and average number of adjustments are equal to the ones obtained in a model with $z = 0$, $p + \kappa$ free adjustment opportunities and consumption equal to c . The average withdrawal size differs, because the

jumps creates an extra withdrawal of size z every κ/n withdrawals. Finally, to show that the threshold value $m^*(p + \kappa, 0, 0, c) < z$ is optimal provided that κ is small, we use an argument that is analogous to the one of the deterministic model of [Section 2](#).

To understand the hypothesis that $z \geq m^*$ used in [Proposition 5](#) it is useful to give a characterization of $m_0^* \equiv m^*(p + \kappa, 0, 0, c)$. In [Alvarez and Lippi \(2009\)](#) we show that m_0^* is the unique positive solution to:

$$\frac{b}{cR} = \left(\frac{m_0^*}{c}\right)^2 \left[1 + \sum_{i=1}^{\infty} \left(\frac{m_0^*}{c}(r + p + \kappa)\right)^i \frac{1}{(2+i)!}\right] \quad (14)$$

which we denote by $m_0^* = \varphi(b/(Rc), p + r)$. Clearly m_0^* is a strictly increasing function of b/R , which goes from 0 to ∞ as b/R varies in the same range, and it is decreasing in p . The limit as $r + p + \kappa \downarrow 0$ is the familiar Baumol-Tobin expression $m_0^*/c = \sqrt{2b/(cR)}$. Finally, m_0 is increasing in c with an elasticity between 1/2 and 1. Thus, for a fixed z , one of the hypothesis of [Proposition 5](#) holds, for a small enough fixed cost relative to opportunity cost, i.e. small b/R . Also, since m_0^* is decreasing in $p + \kappa + r$, thus $\sqrt{2bc/R} \geq c \varphi(\frac{b}{cR}, r + p + \kappa)$. Hence a sufficient conditions for $z \geq m^*(p + \kappa, 0, 0, c)$ is that $z \geq \sqrt{2bc/R} \geq c$.

A direct implication of [Proposition 5](#) is that data on M, W, \underline{M}, n and e can *not* identify κ, p and z separately. These data can only identify κz and $\kappa + p$.⁶ We now describe a condition that the ratios W/M and \underline{M}/M must satisfy to be consistent with the behaviour described in the hypothesis of [Proposition 5](#). Additionally, we describe how to identify $\kappa z/e$ and $\kappa + p$, as long as the previous condition is met. To do so, we first describe all the implications of the observable statistics M, W, \underline{M}, n and e for the model's parameters under the hypothesis of [Proposition 5](#).

PROPOSITION 6. Assume that the assumptions of [Proposition 5](#) hold. Then model implies the following relationship between the five observable statistics $(M, W, \underline{M}, n, c + \kappa z)$, the four

⁶ To see this, note that for any pair of $\kappa + p$ and $z\kappa$ that are consistent with M, W, \underline{M}, n and e , there are several other pairs $\kappa' + p'$ and $z'\kappa'$, with $\kappa + p = \kappa' + p'$ and $z\kappa = z'\kappa'$, with $z' > z$ and $\kappa' < \kappa$ that produce the same observations.

structural parameters $(c, \kappa z, p + \kappa, b/R)$, and the threshold m^* :

$$n = \frac{\kappa + p}{1 - \exp(-(\kappa + p)\frac{m^*}{c})}, \quad (15)$$

$$W + \underline{M} = m^* + \frac{\kappa}{n}z, \quad (16)$$

$$n \frac{\underline{M}}{M} = \kappa + p, \quad (17)$$

$$c + \kappa z = nW, \quad (18)$$

$$\frac{m^*}{c} = \varphi\left(\frac{b/c}{R}, r + p + \kappa\right). \quad (19)$$

The proof of this proposition is straightforward. Equation (15) follows by noting that the time between withdrawals is distributed as a truncated exponential with parameter $\kappa + p$, truncated at time $\bar{t} \equiv m^*/c$, the time it will take to deplete money holdings with continuous consumption. The fundamental theorem of renewal theory then implies that the frequency is the reciprocal of the expected time between adjustments. Equation (16) follows by taking expected values of the cash flows at time of a withdrawal. It states that on average, after a withdrawal, an agent has balances m^* , which is the sum of the average cash at the time of withdrawal (\underline{M}) and the withdrawal size (W) net of the fraction κ/n of the withdrawals where a consumption jump of size z is financed. Equation (17) follows by computing the average cash holdings at the time of an adjustment. A fraction $1 - (p + \kappa)/n$ of the withdrawals the agent has reached zero cash holdings at the time of a withdrawal. A fraction $(p + \kappa)/n$ of the withdrawals the agents has strictly positive cash holdings, and since the occurrence of these adjustment are independent of the level of cash holdings, in these cases the agents has the average cash holdings. Equation (18) is simply the budget constraint. Up to here the implications follow from the form of the optimal decision rules.⁷ Finally, equation (19), already presented and discussed in equation (14) ensures that the value of the threshold m^*

⁷One can also add an expression that computes the value of average value of M , using n, m^* and parameters, namely $M/c = (n m^*/c - 1)/(\kappa + p)$. Yet, this equation is implied by equations (15)-(18).

is optimal. Using [Proposition 5](#) we have replaced here p by $p + \kappa$.

We use the expressions in [Proposition 6](#) to solve for the fraction of expenditures that corresponds to jumps, i.e. $\kappa z/(c + \kappa z)$, the value of m^* , which gives a lower bound to z in order for the proposition to apply, and the value of $p + k$. We have:

$$\frac{\kappa z}{\kappa z + c} = 1 + \frac{1}{W/M} \left(\frac{\underline{M}/M}{\frac{\log(1-\underline{M}/M)}{\underline{M}/M} + 1} \right), \quad (20)$$

$$\kappa \leq p + \kappa = n \frac{\underline{M}}{M} \leq n, \quad (21)$$

$$z \geq m^* = M \left(\frac{\log(1 - \underline{M}/M)}{1 + \frac{\log(1-\underline{M}/M)}{\underline{M}/M}} \right). \quad (22)$$

The next proposition summarizes the implications for W/M and \underline{M}/M of γ , $b/(cR)$ and $p + \kappa$ in the case where $m^* \leq z$. To simplify the expressions we take the limit as $r \downarrow 0$, in which case, given γ all the other other parameters combine in a single index to explain the effects on W/M and \underline{M}/M .

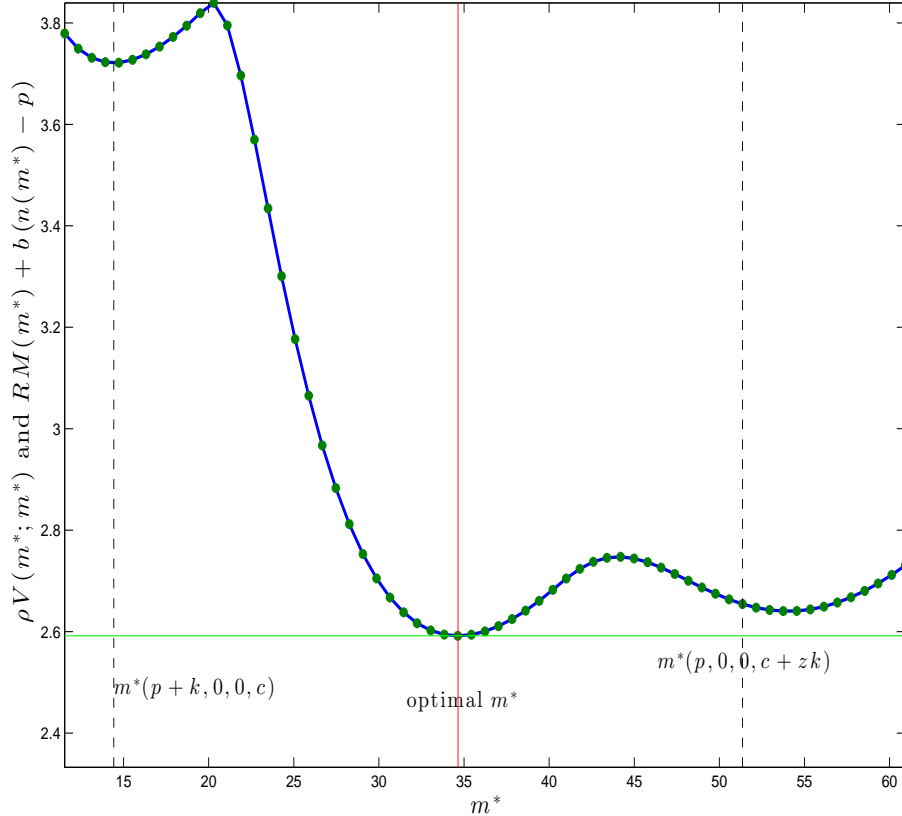
PROPOSITION 7. Let $r \downarrow 0$ and assume that z , $b/(cR)$, and $p + \kappa$ are such that $m^* \leq z$. Define $\gamma \equiv z\kappa/(z\kappa + c)$. Then

$$\frac{W}{M} = \frac{1}{1 - \gamma} \omega \left(\frac{b/c}{R} (p + \kappa)^2 \right) \quad \text{and} \quad \frac{\underline{M}}{M} = \mu \left(\frac{b/c}{R} (p + \kappa)^2 \right) \quad (23)$$

where $\omega : \mathbb{R}_+ \rightarrow [0, 2]$ is strictly decreasing and $\mu : \mathbb{R}_+ \rightarrow [0, 1]$ is strictly increasing. The functions ω and μ depend on no other parameters.

The proof uses the equivalence results of [Proposition 5](#). Since every jump triggers a withdrawal when $z > m^*$ then the model is equivalent to the model with continuous consumption and no jumps, except that p (the arrival rate of free adjustments opportunity) is now $p + k$, and the consumption scale variable is simply c , not $c + kz$. Given this equivalence, the functions ω and μ are those described in Section 4.3 (namely in proposition 6 and Figure 1) in [Alvarez and Lippi \(2009\)](#).

Figure 1: Value function under policy threshold rule m^* evaluated at $m = m^*$.



The parameters for this problem are $z = 20, k = 60, c = 365, p = 15$.

3.2.2 On the first order conditions for m^*

We conclude with an illustration on the necessary, but not sufficient, nature of the boundary conditions for the optimal return point in our problem. Remember that when $\sigma = 0$ the value function $V(m; m^*, m^{**}) = V(m; m^*)$ is indexed by only 1 parameter, the optimal return point m^* . Figure 1 helps to understand the different cases covered in Proposition 4 and Proposition 5 depending on the value of κ . The figure plots the value of following a policy characterized by a threshold m^* , evaluated at $m = m^*$, for different values of this threshold. The best policy is given by the value of m^* that minimizes $V(m^*, m^*)$. Interestingly this function is *not* single peaked. This example shows that simply adding the boundary condition $V'(m^*; m^*) = 0$ does *not* insure the optimality of the given policy. Note the difference with

models without the jump component, such as Constantinides (1978), where a verification theorem states that any function that solves the relevant ODE and boundary conditions is a solution of the problem. The parameter values considered for this figure correspond to an “intermediate” value for κ and z for which the optimal threshold has $m^* \approx 35 > z = 20$. Note that while setting $m^* = m^*(p + \kappa, 0, 0, c) = 15 < z = 20$, so that every jump would trigger a withdrawal, is a local minimum of $V(m^*; m^*)$ but it is not the global minimum. In other words, the values for this example do not satisfy the hypothesis that $\kappa < \underline{\kappa}$ of Proposition 5. Notice also that setting $m^* = 51$, which is the optimal threshold for the case of no jumps, but a larger continuous consumption equal to $c + \kappa z$ is also not the optimal for the case with jumps, but it is also close to a local minimum.

4 Two Empirical Applications

In this section we describe two applications of the ideas developed above. The first one applies the model to currency management using survey and diary data from a sample of Austrian households. In this case the relevant liquid asset is currency, and the relevant notion of expenditures are those paid with cash. The second investigation applies the model to the management of liquid assets using a panel of administrative records for the customers of a large Italian bank. In this case we take liquid asset to be a concept similar to M2, and the expenditures to include both durable and non-durable purchases.⁸

Table 1: Currency management statistics in Italy and Austria

	ATM Card	Italy (2002)	Austria (2005)
Expenditure share paid w. currency	w/o	0.65 ^a	0.96 ^A
	w.	0.52 ^a	0.73 ^A
Currency: M/e (e per day)	w/o	17 ^b	15 ^B
	w.	13 ^b	15 ^B
M per Household	w/o	410 ^c	332 ^C
	w.	330 ^c	206 ^C
Currency at withdrawals ^d : \underline{M}/M	w/o	0.46	0.22
	w.	0.41	0.26
Withdrawal ^e : W/M	w/o	2.0	2.4
	w.	1.3	1.6
Withdrawal / Deposit ^{h,i} : W/D	w.	0.66 [0.66]	n.a.
	w/o	1.10 [0.99]	n.a.
# of withdrawals: n (per year) ^f	w/o	23	21
	w.	58	68
Normalized: $\frac{n}{n_{BT}} = \frac{n}{e/(2M)}$ (e per year) ^f	w/o	1.7	1.4
	w.	3.9	5.4
# of deposits / # withdrawals ^h : n_D/n	w.	0.09 [0.66]	n.a.
Fraction of households with $W/M > 2$	w/o	0.25	0.29
	w.	0.13	0.19
Fraction of households with $\frac{n}{n_{BT}} \equiv \frac{n}{e/(2M)} < 1$	w/o	0.50	0.57
	w.	0.19	0.31
# of observations	w/o	2,275 ^g	153 ^G
	w.	3,729 ^g	895 ^G

Entries are sample means. The unit of observation is the household for Italy; for Austria, the subject of the survey are men and women 14 years and older, not households. Only households with a checking account (both Austria and Italy) and whose head is not self-employed (Italy) are included, with the exception of data in square brackets $[\cdot]$, which are computed only for households whose head is self-employed (approximately 17% of all households with a bank account).

Notes for Italian data: Source: Bank of Italy - *Survey of Household Income and Wealth*.

-^aRatio of expenditures paid with currency to total expenditures (durables, non-durables and services). -^bAverage currency held by the household during the year divided by daily expenditures paid with currency. -^cIn 2004 euros. -^dAverage currency at the time of withdrawal as a ratio to average currency. -^eAverage withdrawal during the year as a ratio to average currency. -^fThe entries with $n = 0$ are coded as missing values. -^gNumber of respondents for whom the currency and the currency-consumption data are available in each survey. Data on withdrawals are supplied by a smaller number of respondents. -^hSample average over 1993-2000. -ⁱComputed for households reporting $D > 0$.

Notes for Austrian data. Source: Austrian National Bank - *OeNB*.

-^ANumerator and denominator of the ratio are based on transactions collected in a diary kept for 7 days. The diary excludes automatic payments and likely misses large transactions (a broader measurement would produce smaller values for the ratio). -^BAverage currency carried by the individual (sum of currency with them and currency available at home; items 18 and 18a in questionnaire), divided by daily expenditures paid with currency. Respondents keep a large fraction of currency balances at home; The average of the ratio of currency at home to total currency held is about 60%. -^CIn 2005 Euros; -^GNumber of respondents with a bank deposit account and non-zero values for M , W , e , n . This accounts for about 87% of the sample.

4.1 Currency Management of Austrian and Italian Households

In this section we present evidence on the cash management of Italian and Austrian households, and then concentrate on the latter for which we also have data on the nature of their consumption paid with cash. We present evidence that there are some large expenditures paid with cash, and that the households for which these expenditures are more important exhibit some of the behavior predicted by the theory in terms of their frequency and size of withdrawals.

Table 1 displays some statistics from two households surveys, one from Italy and one from Austria. We present all statistics splitting the sample between households with an ATM card and those without, as a rough way to control for the consequences of the “free withdrawals” opportunities, namely a large number of withdrawals relative to Baumol-Tobin, as explained above and in [Alvarez and Lippi \(2009\)](#). We display the mean across households (individuals for Austria) of several statistics: share of consumption that is paid using currency for both countries, the average amount of currency held M , the average amount of currency held at the time of a withdrawal \underline{M} , the average size of a withdrawal W , the average size of deposits D , the number of deposits per year n_D and number of withdrawals per year n . Several of the statistics are computed as ratios, which helps in interpreting them in terms of the model. For instance, we use M/e , the average money to daily cash consumption, W/M the average withdrawal size to average currency held, the ratio of the average size of deposits to withdrawals W/D , the ratio of n to n_{BT} , where the latter is the frequency implied by the assumption that withdrawals occur when cash is zero and that $W = 2M$, so that $n_{BT} = e/(2M)$. The statistics displayed in **Table 1** show that the Austrian and Italian households cash management is similar in several dimensions. The Italian survey data have

⁸While in this paper we focus on the implication of these large purchases for cash management, a related interesting literature, both empirical and theoretical, studies the choice of means of payments, especially in relation to the purchases size, as in e.g. [Whitesell \(1989\)](#), [Bounie, Francois, and Houy \(2007\)](#) and [Mooslechner, Stix, and Wagner \(2006\)](#). The problem of the choice of means of payment differ across economies. In particular we conjecture that for developing economies and less developed countries, where alternative to cash are less prevalent, more people will be paying for large purchases using cash, and hence the issues discussed in this paper are more relevant. Preliminary work using panel data from rural Thailand in [Alvarez, Pawasutpaisit, and Townsend \(2011\)](#) supports this hypothesis.

been used in several paper studying cash management, such as [Attanasio, Guiso, and Jappelli \(2002\)](#), [Lippi and Secchi \(2009\)](#), [Alvarez and Lippi \(2009\)](#). The Austrian dataset is smaller in size but includes some additional information concerning the size distribution of purchases that we are going to use below.

Using the outcomes of the case in which $\sigma = 0$, analyzed in [Proposition 5](#) and [Proposition 6](#), we compute some statistics to measure the degree of large infrequent cash purchases for individuals in Austria. The data comes from two related sources: a diary of daily expenditures of the Austrian households and a retrospective survey of the same households, described in [Mooslechner, Stix, and Wagner \(2006\)](#). The diary asked individuals to record all purchases made in the following week. The survey contains several questions on cash management, as well as on method and pattern of purchases. We split the sample between agents with and without ATM cards because, at least using only the cash management statistics M, W, \underline{M}, e , the model does not identify separately p and κ . Yet the value of p should be related (positively) to the density and availability of ATMs.⁹ Thus, the split between those with and without ATM cards serves as a way to “control” for the value of p .

Table 2: Currency as the usual payment method for purchases of different size

	All (1048 Obs.)	w/ATM card (895 Obs.)	w/o ATM card (153 Obs.)
% Individuals that use currency ^a			
Purchases between [0, 10] euros	95.6	94.9	100.0
Purchases between [11, 30] euros	83.3	80.6	99.4
Purchases between [30, 50] euros	70.4	65.6	98.7
Purchases between [51, 100] euros	58.1	51.5	97.7
Purchases between [100, 200] euros	47.6	39.2	97.7
Purchases between [201, 400] euros	43.2	24.2	96.1
Purchases between [401, ∞] euros	45.9	27.4	96.1

Using survey of Austrian individuals described in [Mooslechner, Stix, and Wagner \(2006\)](#).

- ^a Percentage of individuals that answer that currency is the usual method of payments for purchases for each different size. The alternatives are currency or other method. Based on 1048 responses for each purchase size.

⁹For empirical support of this hypothesis in a model with $\kappa = z = 0$ see [Alvarez and Lippi \(2009\)](#).

Table 2 shows that the small size purchases are made using currency by almost all individuals, but that less than half of the individuals use currency as the usual means of payments for the large purchases (400 euros or more). These statistics are presented separately for those with ATM cards and for those without, which shows a clear difference. Almost all individuals without an ATM card use cash as the usual payments regardless of the size of the purchases, whereas the use of cash falls sharply with the size of the purchase for individuals with an ATM card.

Table 3: Cash Management and Large Purchases (> 400 euros) in Austria

	All (1048 Obs.)		w/ATM card (895 Obs.)		w/o ATM card (153 Obs.)	
Individuals who usually make large purchases in cash ^a						
% persons that use cash for large purchases	46%		37%		96%	
	mean	median	mean	median	mean	median
Withdrawal to Money: W/M	2.0	1.1	1.9	1.0	2.1	1.3
# withdrawals relative to BT: ^b n/n_{BT}	3.5	1.2	4.4	1.5	1.5	0.7
Normalized cash at withdrawals: ^c $n\bar{M}/M$	13.4	4.5	17.5	6.3	4.0	2.6
Normalized size of cash expenditures: ^d z/m^*	5.9	1.7	5.7	1.8	6.9	1.2
Individuals who usually do NOT make large purchases in cash ^a						
% persons that do not use cash for large purchases	54%		63%		4%	
	mean	median	mean	median	mean	median
Withdrawal to Money: W/M	1.6	1.0	1.5	1.0	10.4	2.1
# withdrawals relative to BT: ^b n/n_{BT}	5.9	1.9	6.0	1.9	0.9	0.7
Normalized cash at withdrawals: ^c $n\bar{M}/M$	20.6	7.8	20.8	8.0	3.8	2.9

Using survey of Austrian individuals described in Mooslechner, Stix, and Wagner (2006).

- ^a Based on a question about how individual usually paid for items that cost more than 400 euros. Two options are available, either currency or other payment methods. Total number of respondents is 1048.

- ^b # of withdrawals n relative to Baumol-Tobin benchmark, $n_{BT} = e/(2M)$ Based on a diary of all transactions during a week. This the week is right after the month corresponding to the question on large transactions above.

- ^c The variable $n\bar{M}/M$ is the product of the number of withdrawals n and the ratio of the average cash at the time of withdrawal, \bar{M} to the average cash holdings.

- ^d The statistics is computed for individuals with $z > 0$, measured with the diary data (see footnote a), m^* computed using equation (22) and the survey data on \bar{M}/M .

Table 3 displays some cash management statistics for individuals who use cash for large purchases (top panel) and for those that do not (bottom panel). These data are useful to compare our model with the canonical one with continuous consumption. These statistics are also presented separately for those with ATM cards. We display the ratio of the average size of withdrawals to the average cash holdings, W/M , as well as the ratio of the number

of withdrawals relative to the one implied by Baumol-Tobin, n/n_{BT} .¹⁰ We interpret the model as having implications for the comparison between individuals for whom currency is the usual means of payment for large purchases versus those for whom it is not. For those that use currency for large purchases we expect W/M to be larger, and n/n_{BT} to be smaller. Comparing the top and the bottom panels of [Table 3](#) we find some support for this prediction for all the individuals and for those with ATM cards.¹¹ With respect to $n\underline{M}/M$ recall that, from [equation \(21\)](#), this statistic equals $p + \kappa$. So under the reasonable assumption that individuals with ATM cards have a higher value of p , consistent with the evidence in [Alvarez and Lippi \(2009\)](#), this statistic should be larger in this group.

The last row of the first panel of [Table 3](#) reports a statistic that shows how the threshold of 400 euros, chosen independently by the survey designers as a threshold for large purchases, is a reasonable approximation for the value of z which satisfies $z > m^*$. The row reports the mean and median value of z/m^* , where the variable z is measured using the diary data, computed for those individuals for whom $z > 0$ in the diary data. The value of m^* is computed using [equation \(22\)](#) and the observations on \underline{M}/M . It appears that, for all the cases considered the value is greater than one, confirming the empirical appropriateness of the assumption $z > m^*$.

In [Table 4](#) we use the diary expenditures data (recorded during a week). We first note that, given how infrequent are the large purchases in cash per month, there is not enough information in a week of expenditures to measure heterogeneity across individuals by using the 400 euros threshold. Therefore we measure the lumpiness of expenditures using the ratio of the average cash purchase to the median cash purchase both for those with and without ATM cards. This indicator is given by the ratio between the average and the median size

¹⁰ The accounting identity $Wn = e$ implies that the product of these statistics should be 2. We present both statistics because the identity does not hold exactly for each unit of observation. We interpret this discrepancy as measurement error. Consistent with this interpretation we find that the patterns of violation of the identity are symmetric and centered around zero (see [Alvarez and Lippi \(2009\)](#)).

¹¹ The prediction is not verified when comparing across individuals without ATM cards. But notice that there are only 6 individuals w/o ATM who did not use cash as the payment method, so this statistic is likely noisy.

Table 4: Statistics on the lumpiness of cash purchases in Austria (based on diary data)

	with ATM card		without ATM card	
	mean	median	mean	median
Purchase size: average / median: ^a e_a/e_m	1.6	1.3	1.6	1.3
Average purchase size / income: ^b e_a/y	2.2	1.4	4.4	2.3
Number of purchases per week	10	9	10	9

Based on a diary of all transactions during a week (right after the month corresponding to the question on large transactions described in the survey).

- ^a The ratio uses the average and median transaction size paid with currency for each individual during the week of the diary.

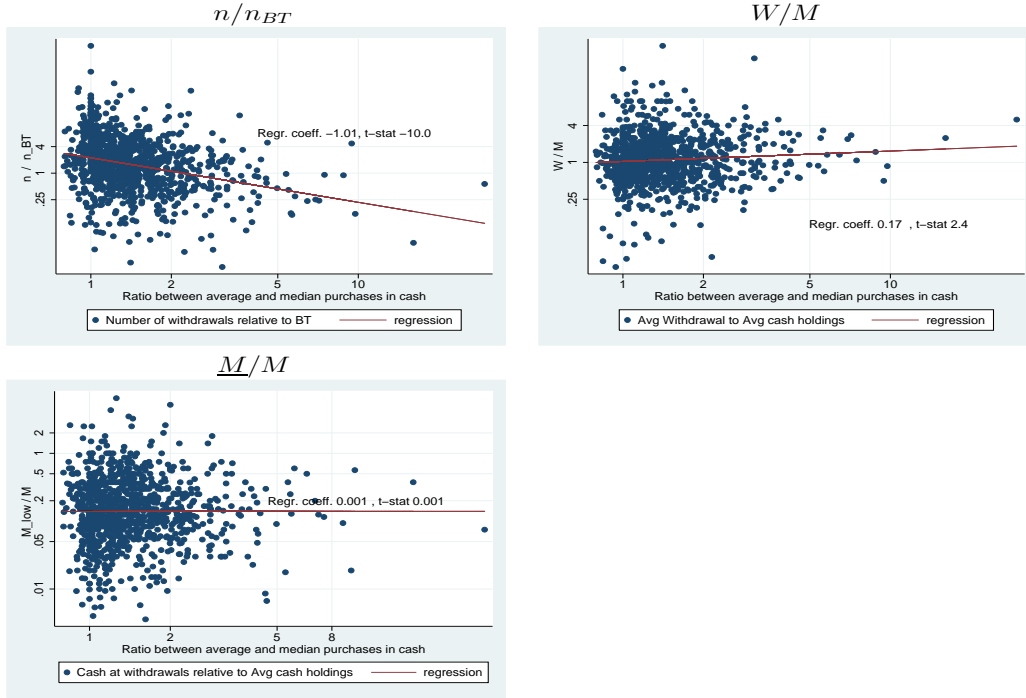
- ^b Value of the average purchase made with currency (e_a) divided by the monthly income of the individual (y).

of the household purchase, denoted as e_a/e_m . Notice that in our model this statistic is $e_a/e_m = (c + \kappa z)/c$ so that it is increasing in the share of lumpy purchases.¹² Additionally, we report a statistic for the number of purchases made with cash during a week. The table shows that the measure of lumpy purchases are essentially constant across ATM ownership.

Next we use our proxy for the lumpiness of purchases, e_a/e_m , and correlate it to patterns of cash-management statistics. We first present cash management statistics pooling all Austrian individuals. We find some patterns that are broadly consistent with what the model predicts as a consequence of variation of z across agents. Recall that, using the discrete time interpretation of the model outlined above, the skewness in the size distribution of purchases is increasing in z , so that we interpret e_a/e_m as a proxy for z . We find that e_a/e_m is negatively correlated with n/n_{BT} and positively with W/M (see [Figure 2](#)). We did not find any correlation between e_a/e_m and \underline{M}/M , which is consistent with the hypothesis that the ratio is determined by the variation of $p + \kappa$ rather than that of z , and that it is dominated by the

¹² To see this it is helpful to consider a discrete-time version of the model, with a period of length 1 broken into small subperiods of length dt , so that $1/dt$ is the number of subperiods. In each subperiod there is exactly one “small” purchase of size $c dt$. Moreover, one large purchase of size z occurs with probability κdt and no large purchase occurs with probability $1 - \kappa dt$. Large purchases are independently distributed across subperiods. In this discrete-time model there are exactly $1/dt$ purchases of size $c dt$ in a period of length 1, and a number between 0 and $1/dt$ purchases of size z , with a binomial distribution. (There are $1/dt$ purchases of size $c dt$ and a probability $(\kappa dt)^j (1 - \kappa dt)^{1/dt-j} j! / (1/dt - j)! / (1/dt!)$ of having exactly j “large purchases” of size z for $j = 0, \dots, 1/dt$). The average number of large purchases over a period of length 1 is κ , so that for sufficiently small time periods (i.e. $1/dt > \kappa$) the median purchase size is $e_m \equiv c dt$ while the average purchase size per period is $e_a = (c + \kappa z) dt$, so that $e_a/e_m = (c + \kappa z)/c$ gives a measure of the skewness of the size distribution of expenditures.

Figure 2: Austria: the inventory model statistics vs. e_a/e_m



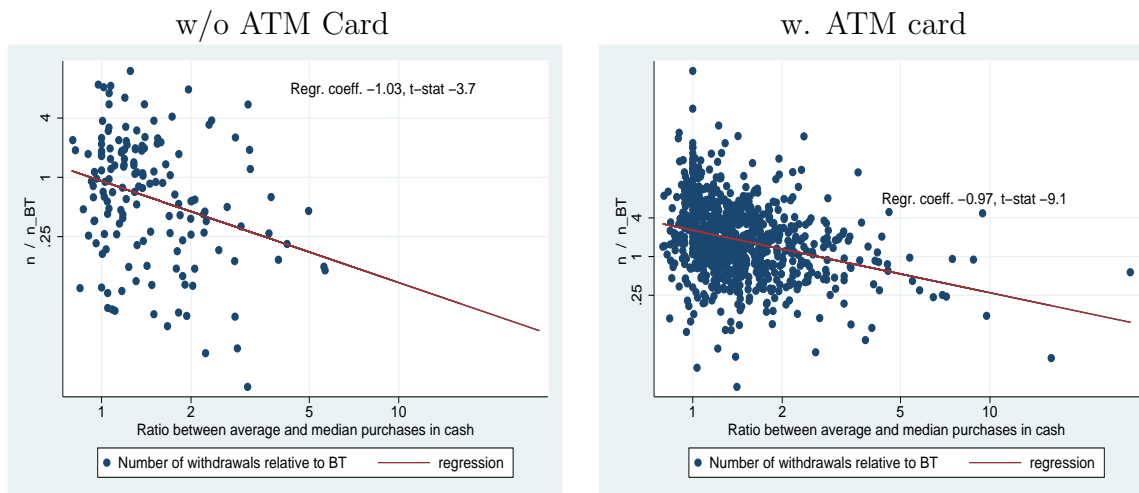
variation on p .

We now turn to a comparison of the cash management statistics across ATM ownership groups. **Figure 3** plots the normalized number of withdrawals, n/n_{BT} against the skewness measure e_a/e_m . The negative correlation displayed is consistent with what the model predicts as a consequence of variation of z across agents.

4.2 Management of Liquid Assets by Italian Investors

This section applies the lumpy-purchases hypothesis to the modeling of the household management of a broad liquid asset, close to $M2$, using data from a sample of Italian households. The information source is a panel of Italian households (investors), whose transactions were recorded in the administrative data of a large commercial bank. We document that following liquidations of high return asset into liquid asset, a disproportionate amount (relative to the frequency of adjustments) is spent early on, a pattern that is predicted by our model where

Figure 3: Austria: normalized withdrawal frequency, n/n_{BT} , vs. lumpiness, e_a/e_m



Note: Log scale. The vertical axis reports the normalized number of withdrawals n/n_{BT} .

lumpy expenditures triggers withdrawals.

In Alvarez, Guiso, and Lippi (2011) we analyze a class of models where households must use a liquid asset to pay for all their expenditures and they face information and/or transaction cost to transfer money from high-yield illiquid assets to the low-yield liquid asset. In that paper we consider specifications with either non-durable consumption or durable consumption. In the specification where all the expenditures are in non-durable goods – a version of Duffie and Sun (1990) or Abel, Eberly, and Panageas (2007) – then the expenditures occur at a constant rate between the adjustments of liquid assets. This implies an average holding of liquid asset similar to the one in Tobin (1956)-Baumol (1952). Indeed in Alvarez, Guiso, and Lippi (2011) we found that the cross section distribution of the ratio of M2 times the frequency of financial trades relative to the rate of consumption of non-durables is totally at odds with this prediction. On the other hand, in the specification where all the expenditures are in durable goods – a variation of Grossman and Laroque (1990) – then the expenditures are lumpy and occur infrequently, implying that they can be paid without holding any liquid asset between adjustments. This implies that the average holdings of liquid assets tend to zero as the model time period shrinks. A more realistic model will have expenditures both

in non-durable goods – so that they are continuous – as well as on durable goods -which with either transaction cost of indivisibilities, becomes lumpy and infrequent. The model in this paper analyzes such a set up. Hence while the model generalizes the one considered in [Alvarez, Guiso, and Lippi \(2011\)](#) by having both type of expenditures, it simplifies the set up in two dimensions: the process for expenditures is exogenous and the adjustment cost does not include observation costs.

The analysis uses a panel data of administrative records from Unicredit, one of the largest Italian commercial banks. The administrative data contain information on the stocks and the net flows of 26 assets categories that investors have at Unicredit. These data are available at a monthly frequency for 35 months beginning in December 2006.¹³ Since the administrative record registers both the stock of each asset category at the end of the period as well as the net trading flow into that category, we can directly identify trading decisions, which would not be possible if only assets valuation at the end of period were available. One of the 26 assets is the checking account. In what follows we distinguish assets into two categories: liquid assets, which we identify with the checking account, and investments the sum of the remaining 25 assets classes. We also experimented, with no change on the results, with a broader definition of liquid assets, and hence a narrower definition of financial assets.

The data we are interested in concern the household flows of financial investment liquidations and purchases, and the changes in the liquid asset holdings (i.e. checking account). The key hypothesis to be explored is whether expenditures occur at a constant rate between liquidations, i.e. whether the liquid assets are depleted at a roughly constant rate, as implicit

¹³ See Appendix F in [Alvarez, Guiso, and Lippi \(2011\)](#) for a detailed description of the data. There are two samples. The first is a sample of about 40,000 investors that were randomly drawn from the population of investors at Unicredit and that served as a reference sample from extracting the investors to be interviewed in the 2007 survey. We refer to this as the large sample. We do not have direct access to the administrative records for the large sample; calculations and estimates on this sample were kindly done at Unicredit. The second which we call the survey sample has the same administrative information for the investors that actually participated in the 2007 survey. We do have access to the survey-sample data which can additionally be matched with the information from the 2007 survey. A description of the merged data is in [Guiso, Sapienza and Zingales \(2010\)](#). Since some households left Unicredit after the interview the administrative data are available for 1,541 households instead on the 1,686 in the 2007 survey. Notice that both the large sample and the survey sample are balanced panel data.

in inventory models with a continuous consumption process. For this we use the temporal patterns of asset sales and liquid asset changes in our panel data to show that the spending rate of liquid assets that comes from asset sales is at least twice as fast as the one consistent with a model with steady expenditure financed with cash and the observed frequency of asset liquidations.

Table 5: Liquidity and portfolio transactions by Italian investors

Dependent variable: Change in the checking account in the month of the transaction		
Regressors	Coefficient	Standard Error
Flow of investment sales:		
β_0 : current	0.703***	0.0057
β_1 : lag 1	-0.23***	0.0062
β_2 : lag 2	-0.16***	0.0065
β_3 : lag 3	0.002	0.006
β_4 : lag 4	-0.03	0.0065
Flow of investment purchases:		
γ_0 : current	-0.65***	0.0065
γ_1 : lag 1	0.020***	0.007
γ_2 : lag 2	-0.076***	0.007
γ_3 : lag 3	0.056***	0.007
γ_4 : lag 4	-0.011**	0.006
Investor total assets:		
δ :	0.092***	0.0025
N. observations	31,622	
R^2	0.47	

OLS regressions of the net flow into the checking account on the net flow of investments sales and purchases. Estimates include investors fixed effects. Three or two asterisks denote that the coefficients are significant at the 1% or less and 5% confidence level, respectively. Source: *Unicredit survey sample*, monthly administrative records (35 months) of 26 accounts for each of 1,541 investors.

Next we illustrate the empirical exercise. We run a regression between C_{jt} -the net euro-flow of the checking account of investor j in month t , and the net investments flows F_{jt} distinguishing between the net flow of investment sales F_{jt}^S and investment purchases F_{jt}^P ,

also in euro amounts during the same month, as well as with lags. Empirically four lags are sufficient to characterize the dynamics. We notice that by construction F_{jt}^S and F_{jt}^P are either zero or positive. So for instance $F_{jt}^S = 100$ (or $F_{jt}^P = 100$) means that over that month there is a net investment sale (or purchase) of 100 euros. Thus F_{jt}^S and F_{jt}^P are never positive (at the same time) for investor j and are both zero if there are no trades with a net cash flow. The regression we run is

$$C_{jt} = \sum_{k=0}^4 \beta_k F_{jt-k}^S + \sum_{k=0}^4 \gamma_k F_{jt-k}^P + \delta W_{jt} + h_j + u_{jt}$$

where W_{jt} is investor j total financial assets, h_j is an investor j fixed effect and u_{jt} an error term. We use the estimated coefficients, shown in [Table 5](#), to characterize the pattern of liquidity management by a household who sells an asset. The implied impulse response is readily computed using the point estimates of the β_k coefficients. Following an investment sale, about 30 cents per dollar are spent in the same month. In the two months following the sale, approximately 60 cents per dollar are spent (the sum of $(1-0.703)+0.23+0.16$).¹⁴

Table 6: Summary statistics for the average annual number of asset sales trades

	All Asset Sales N_{Sj}		Asset sales ≥ 500		Asset sales ≥ 1000	
	Median	Mean (sd)	Median	Mean (sd)	Median	Mean (sd)
Total sample	1.03	1.40 (1.29)	1.03	1.17 (1.11)	0.70	1.06 (1.03)
Stockholders (total)	1.71	1.81 (1.28)	1.37	1.53 (1.13)	1.02	1.40 (1.07)
Stockholders (direct)	1.71	1.97 (1.30)	1.37	1.69 (1.19)	1.37	1.55 (1.12)

Source: *Unicredit* survey sample, monthly administrative records (35 months) of 26 accounts for each of 1,541 investors.

To assess whether these patterns are consistent with a steady depletion of the liquid asset, such as the one implied by the models with continuous consumption described above, we need

¹⁴For comparison if we use a broader definition of liquid asset that includes time deposits (hence excludes them from financial investments) we obtain that the results are essentially the same. In particular the pattern of coefficients of the lags of investments sales on change in liquid asset account, i.e. of the coefficients β_k for $k = 0, 1, \dots, 4$ are 0.70, -0.22 , -0.16 , 0.002 and -0.03 respectively.

to use the information on the frequency of asset sales computed in our dataset, reported in [Table 6](#). The table shows that the annual frequency of asset sales for the median household is around one sale per year. Hence, if assets sales were used mostly to finance a steady flow of consumption expenditures, one would expect that the liquidity obtained from the asset sale should be spent out at a rate of roughly $1/12$ per month.¹⁵ This would imply that in the first month one should see an increase in the checking account of about 0.92 cents per euro of investments liquidation, and a negative effect of about 0.08 cents in the subsequent months. Instead, the estimated pattern indicates a much larger liquidity reduction in the first month (0.7 vs 0.9) than is implied by the steady consumption hypothesis. Likewise, the rate at which liquidity further decrease in the following two months is faster than the one predicted by the steady consumption hypothesis. Considering the mean frequency of asset sales, as opposed to the median, makes the picture a bit less striking, since the mean frequency of asset sales is higher (about 1.4 trades per year), though the observed dynamics of the checking account remain inconsistent with the steady consumption hypothesis (as e.g. the observed value of 0.7 is smaller than $1-1.4/12 = 0.88$).

5 Concluding remarks

We presented an inventory model for the demand of liquid assets that allows for the possibility that, in addition to a continuous (deterministic or random) component, the law of motion for the liquid assets might record a jump when left controlled. These jumps may be caused by lumpy expenditures, such as the purchase of durable goods by households. We showed that a key difference compared to canonical inventory models is that, since the liquidity used to finance these jumps has infinite velocity, then the lumpy expenditure component does not enter as the “scale variable” for the average demand of liquidity and only affects some

¹⁵An even $1/12$ is an upper bound since our statistics are based on records of the balance of the investor’s accounts at the end of every month. Thus, for example, if the sale of assets happens during the middle of the month the fraction of the sale of asset that should be consumed during the remaining period of the first month should be $1/24$. This would imply an increase during the first month in the checking account of about 0.96.

cash managements statistics, such as size and frequency of withdrawals. We showed that accounting this phenomenon is useful to interpret cash management patterns in Austria and in Italy.

Other applications of the ideas in this paper concerns models of liquidity management by firms and by banks. First, consider the case of firms which have large expenditures that must be paid with a liquid asset. Our model predicts that these expenditures will not affect the average firm holdings of liquid assets, for exactly the same reasons discussed in the model for households: these expenditures have an infinite velocity. This prediction is potentially testable by examining a panel data of liquid asset holding. We find some confirmation of our hypothesis in the existing literature. [Bates, Kahle, and Stulz \(2009\)](#) run several specifications of panel regressions to explain the ratio of liquid asset to total asset for of U.S. manufacturing firms from 1980 to 2006. After controlling for other determinants of liquid asset holding, they find a negative coefficient on the ratio of acquisitions to assets, which can be interpreted as a measure of large and infrequent disbursement of liquid assets. The second application is the management of reserves of banks in accounts at the Federal Reserve System. It has been documented that the distribution of payments made through Fedwire is highly skewed.¹⁶ Thus our framework, adapted to take into account some specific features of this market, would be suitable to study and interpret this data. We leave the investigation of these topics for future work.

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¹⁶In 2008 the median value Fedwire payments was about \$24,000 and its average about \$5.8 millions, see [Board of Governors of the Federal Reserve System \(2012\)](#).

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A Proofs

A.1 Proof of Proposition 1.

Proof. We consider two possible patterns for cash management, depending of the relative frequencies of withdrawals and jumps in cumulative consumption. In the first, the agent makes a withdrawal every j/κ units of time, where $j \geq 1$. In this case there are j jumps in cumulated consumption between withdrawals. In the second, the agent withdraws every $1/(i \kappa)$ units of time, where $i \geq 1$. In this case, there are i withdrawals between jumps in cumulative consumption.

Case I: withdrawals every j/κ units of time (j jumps between withdrawals)

We consider policies where agents make a withdrawal every j/κ units of time, and thus there are $j \geq 1$ jumps in cumulated consumption between two withdrawals. Thus the number of withdrawals per unit of time, the size of the withdrawal, and the average cash balances per unit of time:

$$\begin{aligned} n(j) &= \frac{\kappa}{j}, \\ W(j) &= c \frac{j}{\kappa} + z j = \frac{c}{n} + \frac{\kappa z}{n}, \\ M(j) &= \frac{1}{2} c \frac{j}{\kappa} + \frac{1}{2} z (j - 1) = \frac{(c + z\kappa)}{2 n} - \frac{1}{2} z. \end{aligned}$$

for and integer $j \geq 1$. the number of withdrawals per unit of time n is the reciprocal of the time between withdrawals. The expression for withdrawal size W accounts for flow of c during the time period of length j/κ . In this case the cash balances between two withdrawals decrease continuously with consumption c and discontinuously at times that are multiples of $1/\kappa$ by the amount z . The first part of the expression for average cash balances M contains the contribution to required average cash balances from the continuous consumption c , an expression identical to the one in Baumol-Tobin. The second part contains the contribution due to the “jumps”, or discontinuous consumption. The last part of the expression contains only $(j - 1)$ because the last consumption jump is financed by the corresponding withdrawal.

The objective function is then

$$C_I \equiv \min_{j \geq 1} R M(j) + b n(j) \equiv \min_{\kappa/n \in \mathbb{I}_+} R \frac{(c + z\kappa)}{2 n} + b n - \frac{R z}{2}$$

Notice that, except for the constraint on n the relevant problem to decide n is as in Baumol-Tobin, but with total consumption equal to $c + \kappa z$. Thus the optimal decision rule and objective function, ignoring integer constraint on j , but imposing that $j \geq 1$, or equivalently

$n \leq \kappa$, can be written as

$$\text{If } 2 b \kappa^2 \geq R (c + z\kappa) \implies n_I = \sqrt{\frac{R (c + z\kappa)}{2 b}} \text{ and } C_I = \sqrt{2 R b (c + \kappa z)} - \frac{R z}{2}$$

$$\text{If } 2 b \kappa^2 < R (c + z\kappa) \implies n_I = \kappa \text{ and } C_I = \frac{R c}{2} + b\kappa$$

The ratio of average withdrawal to average money stock is given by:

$$\frac{W}{M} = 2 \frac{c + \kappa z}{c + z(\kappa - n)} \geq 2 .$$

$$\text{If } 2 b \kappa^2 < R (c + z\kappa) \implies 1 < \frac{W/M}{2} = \frac{c + \kappa z}{c + \kappa z - z \sqrt{\frac{R (c + \kappa z)}{2 b}}} < \frac{c + \kappa z}{c}$$

$$\text{If } 2 b \kappa^2 \geq R (c + z\kappa) \implies \frac{W/M}{2} = \frac{c + \kappa z}{c}$$

Case II: withdrawals every $1/(i\kappa)$ units of time (i withdrawals between jumps)

Consider policies where agents make i withdrawals in a period of length $1/\kappa$, where i is an integer. Thus, the time between withdrawals is $1/(i\kappa)$. In this case the number of withdrawals per unit of time is $i\kappa$. The size of withdrawals varies, since 1 every i withdrawals includes the amount for a consumption jump. Finally, the average average money holdings are identical to those of the Baumol Tobin model where consumption is given by the continuous component only. Hence we have:

$$\begin{aligned} n(i) &= i\kappa , \\ W(i) &= \frac{z}{i} + \frac{c}{i\kappa} = \frac{\kappa z}{n} + \frac{c}{n} , \\ M(i) &= \frac{c}{2i\kappa} = \frac{c}{2n} . \end{aligned}$$

The agent solves:

$$C_{II} = \min_{i \geq 1} R M(i) + b n(i) = \min_{n/\kappa \in \mathbb{I}_+} R \left[\frac{c}{2n} \right] + b n .$$

Notice that, except for the constraint on n the relevant problem to decide n is as in Baumol-Tobin, but with total consumption equal to c . Thus the optimal decision rule and objective function, ignoring the integer constraints on n , but imposing that $n \geq \kappa$ can be written as

$$\text{If } 2 b \kappa^2 \leq R c \implies n_{II} = \sqrt{\frac{R c}{2 b}} \text{ and } C_{II} = \sqrt{2 R b c}$$

$$\text{If } 2 b \kappa^2 > R c \implies n_{II} = \kappa \text{ and } C_{II} = \frac{R c}{2} + b\kappa$$

The ratio of average withdrawal to average money stock is given by:

$$\frac{W}{M} = 2 \frac{c + \kappa z}{c} \geq 2 .$$

Optimal Policy: combining Case I and Case II

Now we can obtain the decision rule, combining the case where there are multiple jumps between withdrawals (case I) and there are multiple withdrawals between jumps (case II). We first find an expression for the thresholds $(\underline{\kappa}, \bar{\kappa}(z))$ for which the constraint on $i \geq 1$ and $j \geq 1$ binds, for case II and I respectively.

$$\underline{\kappa} = \sqrt{\frac{R c}{2 b}} \leq \bar{\kappa}(z) = \frac{R z + \sqrt{(R z)^2 + 8 b R c}}{4 b} .$$

Note that $\underline{\kappa} = \bar{\kappa}(0)$ and that $\bar{\kappa}$ is strictly increasing in z . We can then write:

$$\text{If } \kappa \leq \underline{\kappa} \implies n = n_{II} = \sqrt{\frac{R c}{2 b}} \text{ and } C = C_{II} = \sqrt{2 R b c}$$

$$\text{If } \underline{\kappa} < \kappa < \bar{\kappa}(z) \implies n = \kappa \text{ and } C = C_I = C_{II} = \frac{R c}{2} + b \kappa$$

$$\text{If } \kappa \geq \bar{\kappa}(z) \implies n = n_I = \sqrt{\frac{R (c + z \kappa)}{2 b}} \text{ and } C = C_I = \sqrt{2 R b (c + \kappa z)} - \frac{R z}{2}$$

For simplicity, in the characterization of the optimal policies, we disregarded the constraint that either j or i are integers. A necessary and sufficient conditions to disregard that constraint is the following. Define u as follows:

$$u \equiv \max \left\{ \sqrt{\frac{R c}{2 b \kappa^2}}, \sqrt{\frac{\kappa^2 2 b}{R (c + \kappa z)}} \right\} .$$

The condition is that if $u > 1$, then u is an integer. In this case the constraint that n is an integer is not binding.

A.2 Proof of Proposition 2.

Proof. *Step 1: Deriving a system of ODE's*

Taking as given the values of m^* and m^{**} . Using these values, we split the range of inaction for V , given by $[0, m^{**}]$, into J intervals. The first $J - 1$ intervals are of width z and are given by $[jz, (j + 1)z]$ for $j = 0, 1, \dots, J - 1$. The last interval is given by $[Jz, \min \{m^{**}, (J + 1)z\}]$. We also define j^* as the smallest integer such that $z(j^* + 1) \geq m^*$, so that $z j^* m^* < z(j^* + 1)$. We index the solution of each of the ODE's by j . We start with $V_0 : [0z] \rightarrow \mathbb{R}$ which solves the linear second order (first order if $\sigma = 0$) ODE:

$$(r + p + \kappa) V_0(m) = Rm + (p + \kappa) V^* + \kappa b + V_0'(m) (-c - \pi m) + \frac{\sigma^2}{2} V_0''(m) \quad (\text{A-1})$$

for $0 \leq m \leq z$. For $1 \leq j \leq J$, taking as given the function $V_{j-1}(\cdot)$, we have the following

linear second order (first order if $\sigma = 0$) ODE for $V_j : [jz, \min\{(j+1)z, m^{**}\}] \rightarrow \mathbb{R}$:

$$(r + p + \kappa)V_j(m) = Rm + pV^* + \kappa V_{j-1}(m - z) + V_j'(m)(-c - \pi m) + \frac{\sigma^2}{2}V_j''(m) \quad (\text{A-2})$$

for $j = 1, \dots, J - 1$. We also have the following value matching at $m = 0$, $m = m^{**}$ and $m = m^*$:

$$V_0(0) = V^* + b, \quad (\text{A-3})$$

$$V_{J-1}(m^{**}) = V^* + b, \quad (\text{A-4})$$

$$V_{j^*}(m^*) = V^*. \quad (\text{A-5})$$

Since V is twice differentiable for $\sigma > 0$, (once if $\sigma = 0$) we require that

$$V_j(zj) = V_{j-1}(zj) \text{ and if } \sigma > 0, V_j'(zj) = V_{j-1}'(zj),$$

for $j = 1, 2, \dots, J - 1$. Notice that for $\sigma > 0$, if the V_j solve their corresponding ODE, then these pair of equalities implies that the second derivatives of V_j and V_{j-1} agree at each point for $j \geq 1$. This can be shown, recursively, starting from $j = 1$. Instead if $\sigma = 0$, if $V_j(zj) = V_{j-1}(zj)$, then the first derivative agrees on these points.

Step 2: Solving the system of ODEs

Up to here, given (m^*, m^{**}) we have a system of second order (first order if $\sigma = 0$) linear differential equations, with exactly as many boundary conditions to find a unique solution, which we denote by $V(\cdot; m^*, m^{**})$ on the range $[0, m^{**}]$. The function $V(\cdot; m^*, m^{**})$ is the value of following a policy indexed by (m^*, m^{**}) . In particular, for any given V^* , there is a two parameter family (one parameter if $\sigma = 0$) that solves V_0 in $[0, z]$. Thus, fixing V^* and these two parameters (one if $\sigma = 0$) we can use V_0 to solve for V_1 in the range, $[z, 2z]$. These second order (first order if $\sigma = 0$) ODE uses the two (one if $\sigma = 0$) boundary conditions: $V_0(z) = V_1(z)$ and (if $\sigma > 0$) $V_0'(z) = V_1'(z)$. We continue recursively, solving for V_j on $[jz, (j+1)z]$ for $j = 1, \dots, J - 1$, using the previously found solution for V_{j-1} on $[(j-1)z, jz]$, each time using the two (one if $\sigma = 0$) boundary conditions $V_{j-1}(jz) = V_j(jz)$ and (if $\sigma > 0$) $V_{j-1}'(jz) = V_j'(jz)$. At the end of this procedure we have functions V_0, V_1, \dots, V_{J-1} depending on V^* and two parameters (one if $\sigma > 0$). We can solve for these three numbers (two if $\sigma = 0$) imposing the value matching conditions [equations \(A-3\)-\(A-4\)-\(A-5\)](#).

We note that in the case in which $\pi = 0$, the homogeneous linear second order (first order if $\sigma = 0$) ODE have constant coefficients. Hence the solution of the homogenous is given by the linear combination of two exponential functions. The solution of the non-homogenous solution is given by sum of the product of each of the solution of the homogeneous and other function. This can be computed recursively, starting from $j = 0$. This gives the following solution in [equation \(10\)](#).

Step 3: Deriving the linear equations for the coefficients of the solution of [equation \(10\)](#)

First we will take as given the two values of B_{00}^k for $k = 1, 2$ and develop a system of equation for $\{A_j, D_j\}$ and the remaining $\{B_{j,i}^k\}$.

The variable V^* can be eliminated of the system using the value matching at $m = 0$,

equation (A-3) and the form of V_0 , namely

$$V^* = A_0 + \sum_{k=1,2} B_{0,0}^k - b. \quad (\text{A-6})$$

We solve for the coefficients of V_0 on the constant and multiplying m on both sides of the ODE We have

$$D_0 = \frac{R}{(r+p+\kappa)}, \quad A_0 = \frac{(p+\kappa)V^* + \kappa b - c D_0}{r+p+\kappa} \quad (\text{A-7})$$

Replacing the conjectured form of V_j in equation (10) on both sides of the ODE equation (A.2) for $j = 1, 2, \dots, J-1$:

$$\begin{aligned} & (r+p+\kappa) \left(A_j + D_j(m-zj) + \sum_{k=1,2} \sum_{i=0}^j B_{j,i}^k e^{\lambda_k(m-zj)} (m-zj)^i \right) \\ &= Rm + pV^* - c \left(D_j + \sum_{k=1,2} \sum_{i=0}^j B_{j,i}^k e^{\lambda_k(m-zj)} \left(\lambda_k (m-zj)^i + i (m-zj)^{i-1} \right) \right) \\ &+ \kappa \left(A_{j-1} + D_{j-1}(m-zj) + \sum_{k=1,2} \sum_{i=0}^{j-1} B_{j-1,i}^k e^{\lambda_k(m-zj)} (m-zj)^i \right) \\ &+ \frac{\sigma^2}{2} \left(\sum_{k=1,2} \sum_{i=0}^j B_{j,i}^k e^{\lambda_k(m-zj)} \left(\lambda_k^2 (m-zj)^i + 2\lambda_k i (m-zj)^{i-1} + i(i-1) (m-zj)^{i-2} \right) \right) \end{aligned} \quad (\text{A-8})$$

For $1 \leq j \leq J-1$: matching the constant and coefficients on m on both sides of the ODE for V_j equation (A-8) we have:

$$D_j = \frac{R}{(r+p+\kappa)} + \frac{\kappa}{(r+p+\kappa)} D_{j-1}, \quad A_j = \frac{pV^* - c D_j - \kappa z j D_{j-1} + \kappa A_{j-1}}{r+p+\kappa} + D_j z j. \quad (\text{A-9})$$

Thus, using equation (A-6), equation (A-7) and equation (A-9) we can solve for all $\{D_j, A_j\}_{j=1, \dots, J-1}$ as a functions of B_{00}^k for $k = 1, 2$.

Now we match the coefficients of the terms involving $e^{\lambda_k(m-zj)}(m-zj)^i$ in both sides of equation (A-8), the ODE for V_j . Fixing an ODE $j = 1, \dots, J-1$, we have coefficients for $i = 0, 1, \dots, j$. Matching the coefficient for $e^{\lambda_k(m-zj)}(m-zj)^j$ gives no additional restrictions, given the expression for λ_k . The coefficient for $e^{\lambda_k(m-zj)}(m-zj)^{j-1}$ gives the following difference equation for $B_{j,j}^k$:

$$B_{j,j}^k j (c - \sigma^2 \lambda_k) = \kappa B_{j-1,j-1}^k \text{ for } j = 1, \dots, J-1, k = 1, 2. \quad (\text{A-10})$$

Thus using equation (A-6) we can solve for $\{B_{j,j}^k\}_{j=1, \dots, J-1}$ given B_{00}^k for $k = 1, 2$.

Likewise matching the coefficients for $e^{\lambda_k(m-zj)}(m-zj)^i$ for $i = 0, 1, \dots, j-2$, canceling

some terms due to the expression for λ_k , gives

$$(c - \sigma^2 \lambda_k)(i+1)B_{j,i+1}^k = \kappa B_{j-1,i}^k + \frac{\sigma^2}{2} B_{j,i+2}^k (i+1)(i+2) \text{ for } j = 2, \dots, J-1, i = 0, \dots, j-2, k = 1, 2. \quad (\text{A-11})$$

Imposing that the level and the first derivative of the functions V_{j-1} and V_j agree when evaluated at zj we obtain:

$$A_j + D_j + \sum_{k=1,2} B_{j,0}^k = A_{j-1} + D_{j-1}z + \sum_{k=1,2} \sum_{i=0}^{j-1} B_{j-1,i}^k e^{\lambda_k z} z^i \quad (\text{A-12})$$

$$D_j + \sum_{k=1,2} B_{j,0}^k \lambda_k = D_{j-1} + \sum_{k=1,2} \sum_{i=0}^{j-1} B_{j-1,i}^k e^{\lambda_k z} [\lambda_k z^i + i z^{i-1}] , \quad (\text{A-13})$$

for $j = 1, 2, \dots, J-1$.

We will now solve for $\{B_{j,i}^k\}$ for $j = 1, \dots, J-1$ and $i = 0, \dots, j-1$, for each $k = 1, 2$. First, we can use [equation \(A-12\)](#) and [equation \(A-13\)](#) for $j = 1$ to solve for $B_{1,0}^k$. Using these values and $\{B_{2,2}^k\}$ we can use [equation \(A-11\)](#) and [equation \(A-12\)](#)-[equation \(A-13\)](#) for $j = 2$ to solve for $\{B_{2,1}^k, B_{2,0}^k\}$. In general, on one hand, fixing $k = 1, 2$ and $j = 2, \dots, J-1$ if $\{B_{j-1,i}^k\}_{i=0, \dots, j-2}$ are known, [equation \(A-11\)](#) can be solve for $\{B_{j,i}^k\}_{i=0, \dots, j-1}$ using $B_{j,j}^k$ as a known boundary condition. On the other hand, we can use [equation \(A-12\)](#) and [equation \(A-13\)](#) for $j = 2, \dots, J-1$ to obtain two extra linear equations for $\{B_{j,0}^k\}_{i=0, \dots, j, k=1,2}$, given the values of $\{B_{j-1,i}^k\}_{i=0, \dots, j-1}$.

At this point we have solved for $\{D_j, A_j, B_{j,i}^k\}$ as functions of $B_{0,0}^k$. We can now solve for $B_{0,0}^k$ using two more linear equations: the value at V_{j^*} at m^* and of V_{J-1} at m^{**} has to satisfy:

$$V^* = A_{j^*} + D_{j^*}(m^* - zj^*) + \sum_{k=1,2} \sum_{i=0}^{j^*} B_{j^*,i}^k e^{\lambda_k(m^* - zj^*)} (m^* - zj^*)^i , \quad (\text{A-14})$$

$$V^* + b = A_{J-1} + D_{J-1}(m^{**} - z(J-1)) + \sum_{k=1,2} \sum_{i=0}^{J-1} B_{J-1,i}^k e^{\lambda_k(m^{**} - z(J-1))} (m^{**} - z(J-1))^i \quad (\text{A-15})$$

We note that while V^* appears in these equation, it can be replaced by using [equation \(A-6\)](#) in terms of A_0, B_{00}^k .

A.3 Proof of [Proposition 3](#).

Proof. Consider the Bellman equation for the case of continuous consumption at the rate $\gamma + c$ and no jumps, so $z = 0$. If $0 < m < m^{**}$ it reads:

$$(r + p)V(m) = Rm + pV^* - V'(m)(c + \gamma).$$

Instead he Bellman equation for the case of jumps, when $z < m < m^{**}$ is

$$(r + p)V(m; z) = Rm + pV^*(z) - V'(m; z)c + \kappa (V(m; z) - V(m - z; z)).$$

Where the z as a second argument is included to emphasize that it is the problem with jumps. We want to show that as $z \rightarrow 0$ both Bellman equations coincide, which is the same than showing that:

$$V'(m)\gamma = \lim_{z \rightarrow 0} \kappa (V(m; z) - V(m - z; z)).$$

This instead holds by writing $V(m - z; z) = V(m; z) - V'(m; z)z + o(z)$. Taking the limit as $z \rightarrow 0$ and using that $\kappa z = \gamma$.

A.4 Proof of Proposition 5.

Proof. Consider the function $V(m; m', p, \kappa, z, c)$, the value of following a policy with an upper threshold m' at the current value m . This function has been characterized fully in Proposition 8 in Appendix B for the appropriate setting of θ . Recall that the solution of the agent's problem $V(\cdot; m^*, p, \kappa, z, c)$ is minimized at m^* . Thus, we can find the optimal threshold m^* by minimizing $V^*(m'; p, \kappa, z, c) \equiv V(m'; m', p, \kappa, z, c)$.

The condition that $m^*(p + \kappa, 0, 0, c) < z$ ensures that

$$V^*(m', p, \kappa, z, c) \geq V^*(m^*(p + \kappa, 0, 0, c); p, \kappa, z, c) + \frac{\kappa b}{r}$$

for all $m' \leq z$ every jump triggers a withdrawal, and hence, apart from the cost b , the jumps behaves exactly as free withdrawal opportunities. Thus in this range the value function satisfies

$$V^*(m'; p, \kappa, z, c) = V^*(m', p + \kappa, 0, 0, c) + \kappa b/r.$$

The value function $V(\cdot; \cdot, p, 0, 0, c)$ is the value function for the problem studied in Alvarez and Lippi (2009), with no jumps. Either using the results in Alvarez and Lippi (2009), or solving the relevant ODE and boundary condition in Proposition 8 for the case where $m' < z$ we have the following explicit solution

$$V^*(m', p + \kappa, 0, 0, c) = \frac{r + p + \kappa}{r} \left[\frac{m' \frac{R}{r+p+\kappa} + b}{1 - e^{-(r+p+\kappa)m'/c}} + b \right]$$

This function is single peaked, attains its minimum at $m^*(p + \kappa, 0, 0, c)$ and is strictly increasing in m' for values $m' > m^*(p + \kappa, 0, 0, c)$.

The argument for values $m' > z$ uses that for any $\epsilon > 0$, we can find $\underline{\kappa}$ such that for $\kappa < \underline{\kappa}$:

$$\left| V^*(m^*(p, \kappa, z, c); p, \kappa, z, c) - V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) - \frac{b\kappa}{r} \right| < \epsilon$$

The result will follow by choosing ϵ to be

$$\epsilon(z) = V^*(z; p, \kappa, z, c) - V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) - \kappa b/r .$$

That $\epsilon > 0$ follows from the assumption that $m^*(p + \kappa, 0, 0, c) < z$. Since $V^*(z; p, \kappa, z, c)$ is increasing in z , then so is $\epsilon(z)$.

Now we show the required continuity. Consider a policy where, regardless of whether $m < z$ or not, if a jump takes place the agent makes a withdrawal. The expected discounted cost of this policy equals $V^*(m'; p + \kappa, 0, 0, c) + \kappa b/r$. The first term is the expected discounted cost of financing a constant cash consumption of c and having free withdrawal opportunities at the rate $p + \kappa$. The second term is the expected discounted cost of all the withdrawals that occur every time a jump occur. Since withdrawals occur even if $m \geq z$ we have for all m' :

$$V^*(m'; p, \kappa, z, c) \leq V^*(m'; p + \kappa, 0, 0, c) + \kappa b/r .$$

From the optimality of $m^*(p + \kappa, 0, 0, c)$ and $m^*(p, 0, 0, c)$ we have that for all m' :

$$\begin{aligned} V^*(m'; p + \kappa, 0, 0, c) &\geq V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) \\ V^*(m'; p, 0, 0, c) &\geq V^*(m^*(p, 0, 0, c); p, 0, 0, c) . \end{aligned}$$

Since the cost is increasing in each component of the cash expenditures:

$$V^*(m'; p, \kappa, z, c) \geq V^*(m'; p, 0, 0, c) \geq V^*(m^*(p, 0, 0, c); p, 0, 0, c) .$$

Collecting these inequalities we have that for any m' :

$$\begin{aligned} V^*(m^*(p, 0, 0, c); p, 0, 0, c) &\leq V^*(m'; p, 0, 0, c) \leq V^*(m'; p, \kappa, z, c) \\ &\leq V^*(m^*(p + \kappa, 0, 0); p + \kappa, 0, 0, c) + \frac{\kappa b}{r} \end{aligned}$$

Finally since we show in [Alvarez and Lippi \(2009\)](#) that $V^*(m^*(\cdot, 0, 0, c); \cdot, 0, 0, z)$ is continuous and $\kappa b/r$ is continuous on κ , we have that for any $c, b/R$, and p , there exist a $\underline{\kappa}$ such that for all $\kappa < \underline{\kappa}$,

$$\left| V^*(m^*(p + \kappa, 0, 0); p + \kappa, 0, 0, c) + \frac{\kappa b}{r} - V^*(m^*(p, 0, 0, c); p, 0, 0, c) \right| < \epsilon(z) .$$

We note that the absolute value in the previous expression is independent of z . Hence for all $\kappa < \underline{\kappa}$:

$$\left| V^*(m^*(p + \kappa, 0, 0); p + \kappa, 0, 0, c) + \frac{\kappa b}{r} - V^*(m^*(p, \kappa, c, z); p, \kappa, z, c) \right| < \epsilon(z) .$$

Finally, since $\epsilon(z)$ is increasing in z , but the upper and lower bounds on $V^*(m^*(p, \kappa, c, z); p, \kappa, z, c)$ are not, we have that the conclusion of the proposition holds also for $z' > z$.

Online Appendices - Not for publication

The demand for liquid assets with uncertain lumpy expenditures

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B The model with $\pi = \sigma = 0$

B.1 Solution of the Bellman equation for the $\pi = \sigma = 0$ case.

In this case the linear system described in the proof of [Proposition 2](#) further simplifies. In particular we have that there is only one root λ so we omit k from all the expression.

Since $\sigma = 0$, the range of inaction is $[0, m^*]$. Let j^* the smallest integer such that $(j^* + 1)z \geq m^*$. The value function in each segment $V_j : [zj, z(j + 1)] \rightarrow \mathbb{R}$ for $j = 0, \dots, j^*$:

$$V_j(m) = A_j + D_j(m - zj) + \exp(\lambda(m - zj)) \sum_{i=0}^j B_{j,i} (m - zj)^i$$

where the constants V^*, λ, A_j, D_j and $B_{j,i}$ satisfy the following set of linear equations:

$$\begin{aligned} \lambda &= \frac{r + p + \kappa}{-c}, \quad D_0 = \frac{R}{(r + p + \kappa)}, \\ (r + p + \kappa)A_0 &= (p + \kappa)V^* + \kappa b - \frac{c}{r + p + \kappa}, \quad B_{0,0} = b + V^* - A_0. \end{aligned}$$

and for $j = 0, 1, \dots, j^* - 1$:

$$\begin{aligned} D_{j+1} &= \frac{-1}{\lambda} \left[\frac{R}{c} + \frac{\kappa}{c} D_j \right], \quad A_{j+1} = \frac{1}{\lambda} \left(D_{j+1} - \frac{pV^*}{c} - \frac{\kappa}{c} [A_j - D_j z(j + 1)] \right) + D_{j+1} z(j + 1), \\ B_{j+1,0} &= A_j + D_j z + e^{\lambda z} \sum_{i=0}^j B_{j,i} z^i - A_{j+1}, \quad B_{j+1,i+1} = \frac{1}{i+1} \frac{\kappa}{c} B_{j,i} \text{ for } i = 0, 1, 2, \dots, j \text{ and} \\ V^* &= A_{j^*} + D_{j^*} (m^* - zj^*) + e^{\lambda(m^* - zj^*)} \sum_{i=0}^{j^*} B_{j^*,i} (m^* - zj^*)^i \end{aligned}$$

Finally, the optimality of the threshold m^* implies that:

$$0 = D_{j^*} + e^{\lambda(m^* - zj^*)} \left[\sum_{i=0}^{j^*} B_{j^*,i} \left(\lambda (m^* - zj^*)^i + i (m^* - zj^*)^{i-1} \right) \right] \quad (\text{A-16})$$

B.2 Computing M, W, n for any z, k when $\pi = \sigma = 0$

This section computes the value function and several cash management statistics of interest for the case of $\sigma = 0$ and $\pi = 0$ for any configuration of the lumpy purchase parameters: z, k . Besides the value function $V(m)$ we also define the functions $M(m), w(m), \underline{m}(m)$ and $n(m)$ as the expected discounted (at rate ρ) integral of the respective quantities (cash balances, withdrawals size, cash-at-withdrawal and deposit indicator), conditional on the current value of m , where cash holding follow the law of motion that corresponds to the optimal decision rule of the model with $\sigma = \pi = 0$ and optimal return m^* . If there is no free withdrawal

opportunity, cash balances evolve as

$$dm(t) = -c dt - z dN(t) \text{ for } m(t) \in (0, m^*] \quad (\text{A-17})$$

where $N(t)$ is the counter of a Poisson process with arrival at the rate κ . If there is a free adjustment, which occurs with Poisson arrival rate p per unit of time. We let \underline{m}_t the cash at the time of a withdrawal, and w_t the amount of the withdrawal. In an instant an adjustment (withdrawal) can happen in either of the following three cases: i) $m(t)$ reaches zero in which case, $\underline{m}_t = 0$ and $w_t = m^*$, ii) a jump in cash consumption occurs when $m(t) < z$, then $\underline{m}_t = m(t)$ and $w_t = z + m^* - m(t)$, iii) a free withdrawal opportunity takes place, then $\underline{m}_t = m(t)$ and $w_t = m^* - m(t)$. We use the functions $M(m)$, $w(m)$, $\underline{m}(m)$ and $n(m)$ to compute the expected value under the invariant distribution of the money holdings M , the average withdrawal size W , the average cash holdings at the time of withdrawal \underline{M}/M , and the average number of withdrawals per unit of time n . We define the unconditional expected values (M, W, \underline{M}, n) by multiplying the expected discounted value by the discount rate ρ . This adjustment converts these quantities in a flow. For these cash holding statistics we take the discount rate to zero, to obtain the corresponding expected value under the invariant distribution of the process for $\{m(t)\}$, as explained below.¹⁷ In particular let:

$$\begin{aligned} M(m) &= \mathbb{E} \left[\rho \int_0^\infty e^{-\rho t} m(t) dt \mid m_0 = m \right] \\ w(m) &= \mathbb{E} \left[\rho \sum_{j=0}^\infty e^{-\rho \tau_j} (m(\tau_j^+) - m(\tau_j^-) + z I_{\tau_j}) \mid m_0 = m \right] \\ \underline{m}(m) &= \mathbb{E} \left[\rho \sum_{j=0}^\infty e^{-\rho \tau_j} m(\tau_j^-) \mid m_0 = m \right] \\ n(m) &= \mathbb{E} \left[\rho \sum_{j=0}^\infty e^{-\rho \tau_j} \mid m_0 = m \right] \\ V(m) &= \frac{1}{\rho} [R M(m) + b (n(m) - p)] , \end{aligned}$$

where τ_j are the times at which a withdrawal happens (which may coincide with a free withdrawal opportunity or with a jump in consumption or not) and where I_t is an indicator that a cash consumption jump has occurred at time τ_j when $m(\tau_j^-) \leq z$. The expectations are taken with respect to the process for $\{m(t)\}$ generated by [equation \(A-17\)](#). Notice that the value function V is the sum of the expected discounted cost of holding cash plus the expected discounted cost of the adjustments. The factor $1/\rho$ corrects the flow nature of the definitions for M and n . Since the adjustments n include those that are free, the last terms subtracts the expected discounted value of them. In the case of the value function we let

¹⁷ Alternatively, the limits of M, W, \underline{M} and n can be computed by solving for the invariant distribution of m , say h , and the expected number of withdrawals, n and using them to define the remaining statistics (M, \underline{M} and W). We do so in the [Online Appendix C](#), but the derivation and calculations of n and h are more involved and specialized.

$\rho = r$. For the cash holding statistics we are interested in

$$M = \lim_{\rho \rightarrow 0} M(m), \quad w = \lim_{\rho \rightarrow 0} w(m), \quad \underline{m} = \lim_{\rho \rightarrow 0} \underline{m}(m), \quad n = \lim_{\rho \rightarrow 0} n(m) .$$

As implicit in the notation, as $\rho \downarrow 0$ the functions do *not* depend on m . Note that w is the expected value of the total amount of withdrawals during a period of length 1, and hence the average withdrawal size is $W = \frac{w}{n}$. Likewise, \underline{m} is the expected value of the total amount of cash at the time of withdrawal in a period of length 1, and hence the average cash at the time of a withdrawal is $\underline{M} = \frac{\underline{m}}{n}$.

These functions satisfy the following system of ODE equations, which, in order to simplify the solution, we only write for the case of $\pi = 0$. The logic for them is the same as the one for the value function in the general case discussed in [Section 3.1](#). We let j^* be the smallest integer for which $m \leq (j^* + 1)z$. Thus, all these functions will be defined in segments of the form $[zj, z(j + 1)]$. For $m \in [0, z]$ we have:

$$(\rho + \kappa + p) F_0(m) = \rho \nu_0 m + \rho \alpha_0 - F_0'(m) c + (\kappa + p) F^*$$

for suitable choices of the constants ν_0 and α_0 (see the [Appendix B.4](#) for details). We follow the notational convention that the function evaluated right after hitting the barrier m^* , i.e. at $m(t^+) = m^*$, is denoted with a $*$, say for instance $M(m^*) = M^*$. For $m \in [zj, z(j + 1)]$ for $j = 1, 2, \dots, j^*$

$$(\rho + \kappa + p) F_j(m) = \rho \nu m + \rho \alpha - F_j'(m) c + \kappa F_{j-1}(m - z) + p F^* ,$$

for some suitable choices of α and ν (see [Appendix B.4](#) for details). Continuity at $m = zj$ for $j = 1, \dots, j^*$

$$F_j(zj) = F_{j-1}(zj)$$

for $j = 1, 2, \dots, j^*$. The conditions at $m = 0$ are

$$F_0(0) = \rho \alpha^* + F^* ,$$

for a suitable choice of α^* (see the [Appendix B.4](#) for details). Now we can write the solution for F as a function of $m^*, \nu, \nu_0, \alpha, \alpha_0$ and α^* .

PROPOSITION 8. Assume that $c > 0$, $\rho > 0$, and $\rho + p + \kappa > 0$. The ODE-DDE for F has the following solution. Let m^* and $\theta \equiv (\nu, \nu_0, \alpha, \alpha_0, \alpha^*)$ be given. Define j^* as the smallest integer so that $(j^* + 1)z \geq m^*$. Then $F_j(\cdot; m^*, \theta) : [zj, z(j + 1)] \rightarrow \mathbb{R}$ has the form:

$$F_j(m; m^*, \theta) = G_j + S_j(m - zj) + \sum_{i=0}^j H_{j,i} e^{\lambda(m - zj)} (m - zj)^i$$

where given the constant $\lambda = \frac{\rho + \kappa + p}{-c}$, and the values for G_j, S_j, H_{ij} for $j = 1, \dots, j^*$ and $i = 1, \dots, j$ solve a block recursive system of linear equations described in the proof.

We use this general set-up to develop a non-linear equation to find the value of the optimal return point m^* . This equation reflects that m^* is chosen optimally, and hence it must satisfy

that $V'(m^*) = V'_{j^*}(m^*) = 0$. This can be written as:

$$0 = S_{j^*} + \sum_{i=0}^{j^*} H_{j^*i} e^{\lambda(m^* - zj^*)} i (m^* - zj^*)^{i-1} + \lambda \sum_{i=0}^{j^*} H_{j^*i} e^{\lambda(m^* - zj^*)} (m^* - zj^*)^i$$

B.3 Proof of Proposition 8

First we describe the system of linear equations that the coefficients for G_j , S_j , H_{ij} for $j = 1, \dots, j^*$ and $i = 1, \dots, j$ and F^* solve:

$$\begin{aligned} S_0 &= \frac{\rho}{\rho + \kappa + p} \nu_0, \\ G_0 &= \frac{\rho}{\rho + \kappa + p} \alpha_0 - \frac{c}{\rho + \kappa + p} S_0 + \frac{(\kappa + p)}{\rho + \kappa + p} F^*, \\ H_{00} &= \rho \alpha^* + F^* - G_0, \end{aligned}$$

for $j = 0, 1, 2, \dots, j^* - 1$:

$$\begin{aligned} S_{j+1} &= \frac{\rho}{\rho + \kappa + p} \nu + \frac{\kappa}{\rho + \kappa + p} S_j, \\ G_{j+1} &= \frac{\rho \alpha + p F^*}{\rho + \kappa + p} + \frac{\kappa}{\rho + \kappa + p} [G_j - S_j z (j + 1)] - \frac{c}{\rho + \kappa + p} S_{j+1} + S_{j+1} z (j + 1), \\ H_{j+1,0} &= G_j + S_j z + \sum_{i=0}^j H_{j,i} e^{\lambda z} z^i - G_{j+1} \text{ where} \end{aligned}$$

$$H_{j+1,i} = \frac{1}{i} \frac{\kappa}{c} H_{j,i-1} \text{ for } i = 1, 2, \dots, j + 1, \text{ and}$$

$$F^* = G_{j^*} + S_{j^*} (m^* - zj^*) + \sum_{i=0}^{j^*} H_{j^*i} e^{\lambda(m^* - zj^*)} (m^* - zj^*)^i$$

Second, we derive this equations. We do this in two steps.

Step I. Solution for $j = 0$.

We have

$$F_0(m) = G_0 + S_0 m + H_{00} \exp(\lambda m)$$

and

$$F'_0(m) = S_0 + \lambda H_{00} \exp(\lambda m)$$

or

$$(\rho + \kappa + p) [G_0 + S_0 m + H_{00} e^{\lambda m}] = \rho \nu_0 m + \rho \alpha_0 - c [S_0 + \lambda H_{00} e^{\lambda m}] + (\kappa + p) F^*$$

Now we solve for λ , S_0 , G_0 and H_{00} .

For λ we have:

$$(\rho + \kappa + p) \exp(\lambda m) = -c \lambda H_{00} \exp(\lambda m)$$

or

$$\lambda = \frac{\rho + \kappa + p}{-c},$$

For S_0 :

$$(\rho + \kappa + p) S_0 = \rho \nu_0$$

or

$$S_0 = \frac{\rho}{\rho + \kappa + p} \nu_0.$$

For G_0

$$(\rho + \kappa + p) G_0 = \rho \alpha_0 - c S_0 + (\kappa + p) F^*$$

or

$$G_0 = \frac{\rho}{\rho + \kappa + p} \alpha_0 - \frac{c}{\rho + \kappa + p} S_0 + \frac{(\kappa + p)}{\rho + \kappa + p} F^*$$

Using the boundary condition $F_0(0) = \rho \alpha^* + F^*$,

$$F_0(0) = G_0 + H_{00} = \rho \alpha^* + F^*$$

or

$$H_{00} = \rho \alpha^* + F^* - G_0.$$

Step II. Solution for $1 \leq j \leq j^*$.

We have

$$F'_j(m) = S_j + e^{\lambda(m-zj)} \sum_{i=0}^j H_{j,i} \left[\lambda(m-zj)^i + i(m-zj)^{i-1} \right]$$

so that

$$\begin{aligned} & (\rho + \kappa + p) \left[G_{j+1} + S_{j+1}(m - z(j+1)) + \sum_{i=0}^{j+1} H_{j+1,i} \exp(\lambda(m - z(j+1))) (m - z(j+1))^i \right] \\ = & \rho \nu m + \rho \alpha + p F^* \\ & + \kappa \left[G_j + S_j(m - z(j+1)) + \sum_{i=0}^j H_{j,i} \exp(\lambda(m - z(j+1))) (m - z(j+1))^i \right] \\ & - c \left[S_{j+1} + \exp(\lambda(m - z(j+1))) \sum_{i=0}^{j+1} H_{j+1,i} \left[\lambda(m - z(j+1))^i + i(m - z(j+1))^{i-1} \right] \right] \end{aligned}$$

Matching the coefficients for $\exp(\lambda(m - z(j + 1)))$ requires:

$$\begin{aligned}
& (\rho + \kappa + p) \sum_{i=0}^{j+1} H_{j+1,i} (m - z(j + 1))^i \\
&= \kappa \sum_{i=0}^j H_{j,i} (m - z(j + 1))^i \\
&\quad - c \sum_{i=0}^{j+1} H_{j+1,i} \left[\lambda (m - z(j + 1))^i + i (m - z(j + 1))^{i-1} \right]
\end{aligned}$$

and using the expression for λ

$$\begin{aligned}
0 &= \kappa \sum_{i=0}^j H_{j,i} (m - z(j + 1))^i \\
&\quad - c \sum_{i=0}^{j+1} H_{j+1,i} i (m - z(j + 1))^{i-1}
\end{aligned}$$

and matching the coefficients for $(m - z(j + 1))^{i-1}$:

$$0 = \kappa H_{j, i-1} - c i H_{j+1, i}$$

so that

$$H_{j+1, i} = \frac{1}{i} \frac{\kappa}{c} H_{j, i-1}$$

for $i = 1, 2, \dots, j + 1$.

Matching the coefficients of m :

$$(\rho + \kappa + p) S_{j+1} = \rho \nu + \kappa S_j$$

or

$$S_{j+1} = \frac{\rho}{\rho + \kappa + p} \nu + \frac{\kappa}{\rho + \kappa + p} S_j$$

Matching the coefficient for the constants:

$$\begin{aligned}
& (\rho + \kappa + p) [G_{j+1} - S_{j+1} z(j + 1)] \\
&= \rho \alpha + p F^* + \kappa [G_j - S_j z(j + 1)] - c S_{j+1}
\end{aligned}$$

or

$$G_{j+1} = \frac{\rho \alpha + p F^*}{(\rho + \kappa + p)} + \frac{\kappa}{\rho + \kappa + p} [G_j - S_j z(j + 1)] - \frac{c}{(\rho + \kappa + p)} S_{j+1} + S_{j+1} z(j + 1)$$

Finally using the continuity at zj :

$$F_{j+1}(z(j+1)) = G_{j+1} + S_{j+1} + \sum_{i=0}^{j+1} H_{j+1,i}(0)^i = G_{j+1} + H_{j+1,0}$$

$$F_j(z(j+1)) = G_j + S_j z + \sum_{i=0}^j H_{j,i} \exp(\lambda z) (z)^i$$

so that

$$G_{j+1} + H_{j+1,0} = G_j + S_j z + \sum_{i=0}^j H_{j,i} \exp(\lambda z) (z)^i$$

or

$$H_{j+1,0} = G_j + S_j z + \sum_{i=0}^j H_{j,i} \exp(\lambda z) (z)^i - G_{j+1} .$$

Finally we require that

$$F^* = F_{j^*}(m^*) \text{ or}$$

$$F^* = G_{j^*} + S_{j^*}(m^* - zj^*) + \sum_{i=0}^{j^*} H_{j^*,i} \exp(\lambda(m^* - zj^*)) (m^* - zj^*)^i$$

B.4 Bellman equations for $V, M, \underline{M}, W, n$ and its coefficients

In the range $[0, z]$ we have:

$$\begin{aligned} (\rho + \kappa + p) M_0(m) &= \rho m - M'_0(m) c + (\kappa + p) M^* \\ (\rho + \kappa + p) w_0(m) &= \rho(\kappa + p)(m^* - m) + \rho\kappa z - w'_0(m) c + (\kappa + p) w^* \\ (\rho + \kappa + p) \underline{m}_0(m) &= \rho(\kappa + p) m - \underline{m}'_0(m) c + (\kappa + p) \underline{m}^* \\ (\rho + \kappa + p) n_0(m) &= \rho(\kappa + p) - n'_0(m) c + (\kappa + p) n^* \\ (\rho + \kappa + p) V_0(m) &= Rm + \kappa b - V'_0(m) c + (\kappa + p) V^* \end{aligned}$$

We follow the notational convention that the function evaluated right after hitting the barrier m^* , i.e. at $m(t^+) = m^*$, is denoted with a $*$, say for instance $M(m^*) = M^*$. For $m \in [zj, z(j+1)]$ for $j = 1, 2, \dots, j^*$

$$\begin{aligned} (\rho + \kappa + p) M_j(m) &= \rho m - M'_j(m) c + \kappa M_{j-1}(m - z) + pM^* \\ (\rho + \kappa + p) w_j(m) &= \rho p(m^* - m) - w'_j(m) c + \kappa w_{j-1}(m - z) + p w^* \\ (\rho + \kappa + p) \underline{m}_j(m) &= \rho pm - \underline{m}'_j(m) c + \kappa \underline{m}_{j-1}(m - z) + p \underline{m}^* \\ (\rho + \kappa + p) n_j(m) &= \rho p - n'_j(m) c + \kappa n_{j-1}(m - z) + pn^* \\ (\rho + \kappa + p) V_j(m) &= Rm - V'_j(m) c + \kappa V_{j-1}(m - z) + pV^* \end{aligned}$$

Continuity of these function across the segments gives:

$$\begin{aligned}
M_j(z_j) &= M_{j-1}(z_j), \\
w_j(z_j) &= w_{j-1}(z_j), \\
\underline{m}_j(z_j) &= \underline{M}_{j-1}(z_j), \\
n_j(z_j) &= n_{j-1}(z_j), \\
V_j(z_j) &= V_{j-1}(z_j),
\end{aligned}$$

The boundary conditions at $m = 0$: are

$$\begin{aligned}
M_0(0) &= M^*, \\
w_0(0) &= \rho m^* + w^*, \\
\underline{m}_0(0) &= \underline{m}^*, \\
n_0(0) &= \rho + n^*, \\
V_0(0) &= V^* + b,
\end{aligned}$$

We display the expressions to map the equations for the general formulation F to each of the 4 variables: M , w , \underline{m} , and n . They are

$$\begin{aligned}
M_0 &: \nu_0 = 1, \alpha_0 = 0, \\
w_0 &: \nu_0 = -(\kappa + p), \alpha_0 = (\kappa + p)m^* + \kappa z, \\
\underline{m}_0 &: \nu_0 = \kappa + p, \alpha_0 = 0, \\
n_0 &: \nu_0 = 0, \alpha_0 = (\kappa + p), \\
V_0 &: \nu_0 = R/\rho, \alpha_0 = \kappa b/\rho,
\end{aligned}$$

$$\begin{aligned}
M_j &: \nu = 1, \alpha = 0, \\
w_j &: \nu = -p, \alpha = pm^* \\
\underline{m}_j &: \nu = p, \alpha = 0, \\
n_j &: \nu = 0, \alpha = p, \\
V_j &: \nu = R/\rho, \alpha = 0,
\end{aligned}$$

and

$$\begin{aligned}
M_0 &: \alpha^* = 0, \\
w_0 &: \alpha^* = m^* \\
\underline{m}_0 &: \alpha^* = 0, \\
n_0 &: \alpha^* = 1, \\
V_0 &: \alpha^* = b/\rho.
\end{aligned}$$

C Invariant distribution of cash holdings h

We derive analytically the stationary distribution of cash holdings. The pdf of the invariant distribution h solves the ODE

$$h(m)(p + \kappa) = h'(m)(c + \pi m)$$

for $m^* \geq m \geq m^* - z$ and the DDE

$$h(m)(p + \kappa) = h'(m)(c + \pi m) + \kappa h(m + z)$$

for $0 \leq m \leq m^* - z$.

Proof of the ODE-DDE for h .

Take a discrete time version of the law motion with time period of length Δ . For $m^* \geq m \geq m^* - z - (c + m^*\pi)\Delta$ we have:

$$h(m, t + \Delta) = (1 - (p + \kappa)\Delta) h(m + (c + m\pi)\Delta, t)$$

and for $0 \leq m < m^* - z - (c + m^*\pi)\Delta$ we have:

$$h(m, t + \Delta) = (1 - (p + \kappa)\Delta) h(m + (c + m\pi)\Delta, t) + \kappa\Delta h(m + z + (c + m\pi)\Delta, t)$$

This gives:

$$h(m)(p + \kappa) = h'(m)(c + \pi m)$$

In steady state,

$$h(m) = (1 - (p + \kappa)\Delta) h(m + (c + m\pi)\Delta)$$

and

$$h(m) = (1 - (p + \kappa)\Delta) h(m + (c + m\pi)\Delta) + \kappa\Delta h(m + z + (c + m\pi)\Delta)$$

or

$$h(m) = (1 - (p + \kappa)\Delta) [h(m) + h'(m)\Delta(c + \pi m) + o(\Delta)]$$

and

$$h(m) = (1 - (p + \kappa)\Delta) [h(m) + h'(m)\Delta(c + \pi m) + o(\Delta)] + \kappa\Delta h(m + z + (c + m\pi)\Delta)$$

or

$$h(m)((p + \kappa)) = (1 - (p + \kappa)\Delta) \left[h'(m)(c + \pi m) + \frac{o(\Delta)}{\Delta} \right]$$

and

$$h(m)(p + \kappa) = (1 - (p + \kappa)\Delta) h'(m)(c + \pi m) + \frac{o(\Delta)}{\Delta} + \kappa h(m + z + (c + m\pi)\Delta)$$

and taking $\Delta \rightarrow 0$:

$$h(m)(p + \kappa) = h'(m)(c + \pi m)$$

for $m^* \leq m \leq m - z$ and

$$h(m)(p + \kappa) = h'(m)(c + \pi m) + \kappa h(m + z)$$

for $0 \leq m \leq m^* - z$.

Characterization of h

As in the case of the value function, we can further characterize h by splitting its support $[0, m^*]$ into J intervals, where J is the smallest integer for which $Jz \geq m^*$. The first $J - 1$ intervals have width z , and are given by $[m^* - (j + 1)z, m^* - zj]$, for $j = 0, 1, \dots, J - 2$, so that

$$h_j : [m^* - (j + 1)z, m^* - zj] \rightarrow R_+$$

for $j = 0, \dots, J - 2$. The last one may be smaller, and is given by $[0, m^* - z(J - 1)]$, so that

$$h_{J-1} : [0, m^* - z(J - 1)] \rightarrow R_+$$

For the first interval we have an ODE

$$h_0(m)(p + \kappa) = h'_0(m)(c + \pi m)$$

for $m \in [m^* - z, m^*]$. Notice that, except for a multiplicative constant of integration, h_0 can be solved for in this interval. For the next intervals we take as given h_{j-1} and solve for h_j solving the following ode:

$$h_j(m)(p + \kappa) = h'_j(m)(c + \pi m) + \kappa h_{j-1}(m + z)$$

for $m \in [\max\{m^* - z(j + 1), 0\}, m^* - zj]$.

We impose that the function h is continuous everywhere, so that

$$h_j(m^* - zj) = h_{j-1}(m^* - zj)$$

for $j = 1, 2, \dots, J - 1$. Notice that this implies that the derivatives of h_j and h_{j-1} agree at these points for $j \geq 2$.

Hence, by splitting the domain in this way we turn the DDE into the solution of several ODE's.

Finally, since h is a density we have:

$$1 = \int_0^{m^*} h(m) dm = \sum_{j=0}^{J-1} \int_{m^* - z(j+1)}^{m^* - zj} h_j(m) dm + \int_0^{m^* - z(J-1)} h_{J-1}(m) dm.$$

Solution of h for the case of $\pi = 0$.

The solution for h is of the following form:

$$h(m) \left(\frac{p + \kappa}{c} \right) = h'(m)$$

for $m^* \geq m \geq m^* - z$ and otherwise

$$h(m) = \frac{c}{p + \kappa} h'(m) + \frac{\kappa}{p + \kappa} h(m + z)$$

We have that

$$h_0(m) = K_{00} \exp(\mu (m + z - m^*))$$

since

$$h'_0(m) = \mu K_{00} \exp(\mu (m + z - m^*))$$

Thus the ODE is satisfied setting

$$\mu = \frac{p + \kappa}{c}.$$

for any value of K_{00} .

For $j = 1, 2, \dots, J - 1$ we have

$$h_j(m) = \frac{c}{p + \kappa} h'_j(m) + \frac{\kappa}{p + \kappa} h_{j-1}(m + z)$$

for $m \in [\max\{m^* - z(j + 1), 0\}, m^* - zj]$.

We guess the solution of the form:

$$h_j(m) = \exp(\mu [m + z(j + 1) - m^*]) \sum_{i=0}^j K_{j,i} (m + z(j + 1) - m^*)^i$$

and thus

$$h'_j(m) = \exp(\mu [m + z(j + 1) - m^*]) \sum_{i=0}^j K_{j,i} \left[\mu (m + z(j + 1) - m^*)^i + i (m + z(j + 1) - m^*)^{i-1} \right]$$

Replacing our guess in the ode:

$$\begin{aligned} & \exp(\mu [m + z(j + 2) - m^*]) \sum_{i=0}^{j+1} K_{j+1,i} (m - z(j + 2) - m^*)^i \\ = & \frac{c}{p + \kappa} \exp(\mu [m + z(j + 2) - m^*]) \sum_{i=0}^{j+1} K_{j+1,i} \left[\mu (m + z(j + 2) - m^*)^i + i (m + z(j + 1) - m^*)^{i-1} \right] \\ & + \frac{\kappa}{p + \kappa} \exp(\mu [m + z(j + 2) - m^*]) \sum_{i=0}^j K_{j,i} (m - z(j + 2) - m^*)^i \end{aligned}$$

Simplifying:

$$\begin{aligned}
& \sum_{i=0}^{j+1} K_{j+1,i} (m - z(j+2) - m^*)^i \\
= & \frac{c}{p + \kappa} \sum_{i=0}^{j+1} K_{j+1,i} \left[\mu (m + z(j+2) - m^*)^i + i (m + z(j+2) - m^*)^{i-1} \right] \\
& + \frac{\kappa}{p + \kappa} \sum_{i=0}^j K_{j,i} (m - z(j+2) - m^*)^i
\end{aligned}$$

or using $\mu = (p + \kappa) / c$:

$$\begin{aligned}
0 = & \frac{c}{p + \kappa} \sum_{i=0}^{j+1} K_{j+1,i} i (m + z(j+2) - m^*)^{i-1} \\
& + \frac{\kappa}{p + \kappa} \sum_{i=0}^j K_{j,i} (m - z(j+2) - m^*)^i
\end{aligned}$$

Matching the coefficients of term with $(m - z(j+1) - m^*)^{i-1}$ for $i = 1, 2, \dots, j+1$

$$\frac{c}{p + \kappa} K_{j+1,i} i = -\frac{\kappa}{p + \kappa} K_{j,i-1}$$

or

$$K_{j+1,i} = -\frac{1}{i} \frac{\kappa}{c} K_{j,i-1}$$

for $i = 1, 2, \dots, j+1$.

For $K_{j+1,0}$ we use that

$$h_{j+1}(m^* - z(j+1)) = h_j(m^* - z(j+1))$$

or

$$\begin{aligned}
h_j(m^* - z(j+1)) &= \exp(\mu \cdot 0) \sum_{i=0}^j K_{j,i} 0^i = K_{j,0} \\
h_{j+1}(m^* - z(j+1)) &= \exp(\mu z) \sum_{i=0}^{j+1} K_{j+1,i} (z)^i
\end{aligned}$$

$$K_{j,0} = \exp(\mu z) \sum_{i=0}^{j+1} K_{j+1,i} (z)^i$$

or

$$K_{j+1,0} = \frac{K_{j,0}}{\exp(\mu z)} - \sum_{i=1}^{j+1} K_{j+1,i} (z)^i$$

Finally K_{00} is obtained by requiring that

$$1 = \int_0^{m^*} h(m) dm = \sum_{j=0}^{J-1} \int_{m^*-z(j+1)}^{m^*-zj} h_j(m) dm + \int_0^{m^*-z(J-1)} h_{J-1}(m) dm.$$

We use that for $0 \leq j < J-1$,

$$\begin{aligned} & \int_{m^*-z(j+1)}^{m^*-zj} h_j(m) dm \\ &= \int_{m^*-z(j+1)}^{m^*-zj} \sum_{i=0}^j \left(\exp(\mu [m + z(j+1) - m^*]) K_{j,i} (m + z(j+1) - m^*)^i \right) dm \\ &= \sum_{i=0}^j K_{j,i} \int_{m^*-z(j+1)}^{m^*-zj} \left(\exp(\mu [m + z(j+1) - m^*]) (m + z(j+1) - m^*)^i \right) dm \\ &= \sum_{i=0}^j K_{j,i} \int_0^z \exp(\mu \hat{m}) (\hat{m})^i d\hat{m} \end{aligned}$$

and

$$\begin{aligned} & \int_0^{m^*-z(J-1)} h_{J-1}(m) dm \\ &= \int_0^{m^*-z(J-1)} \left(\sum_{i=0}^{J-1} \exp(\mu [m + z(J-2) - m^*]) K_{J-1,i} (m + z(J-2) - m^*)^i \right) dm \\ &= \sum_{i=0}^{J-1} K_{J-1,i} \int_0^{m^*-z(J-1)} \exp(\mu [m + z(J-2) - m^*]) (m + z(J-2) - m^*)^i dm \\ &= \sum_{i=0}^{J-1} K_{J-1,i} \int_0^{m^*-z(J-1)} \exp(\mu [m + z(J-2) - m^*]) (m + z(J-2) - m^*)^i dm \end{aligned}$$

we have

$$\begin{aligned} 1 &= \sum_{j=0}^{J-2} \sum_{i=0}^j K_{j,i} \int_0^z \exp(\mu \hat{m}) (\hat{m})^i d\hat{m} \\ &+ \sum_{i=0}^{J-1} K_{J-1,i} \int_0^{m^*-z(J-1)} \exp(\mu [m + z(J-2) - m^*]) (m + z(J-2) - m^*)^i dm \end{aligned}$$

Letting

$$\begin{aligned} L(i) &= \int_0^z \exp(\mu m) m^i dm = \frac{\exp(\mu m) m^i}{\mu} \Big|_0^z - \frac{i}{\mu} \int_0^z \exp(\mu m) m^{i-1} dm \\ &= \frac{\exp(\mu z) z^i}{\mu} - \frac{i}{\mu} L(i-1) \end{aligned}$$

so that

$$L(i) = \frac{\exp(\mu z) z^i}{\mu} - \frac{i}{\mu} L(i-1)$$

for $i = 1, 2, \dots$, with

$$L(0) = \int_0^z \exp(\mu m) dm = \frac{\exp(\mu z)}{\mu}.$$

Likewise letting

$$\begin{aligned} G(i) &= \int_0^{m^*-z(J-1)} \exp(\mu [m + z(J-2) - m^*]) (m + z(J-2) - m^*)^i dm \\ &= \frac{\exp(\mu [m + z(J-2) - m^*]) (m + z(J-2) - m^*)^i}{\mu} \Big|_0^{m^*-z(J-1)} \\ &\quad - \frac{i}{\mu} \int_0^z \exp(\mu [m + z(J-2) - m^*]) (m + z(J-2) - m^*)^{i-1} dm \\ &= \frac{\exp(\mu z) (z)^i}{\mu} - \frac{\exp(\mu [z(J-2) - m^*]) (z(J-2) - m^*)^i}{\mu} \\ &\quad - \frac{i}{\mu} G(i-1) \end{aligned}$$

with

$$\begin{aligned} G(0) &= \int_0^{m^*-z(J-1)} \exp(\mu [m + z(J-2) - m^*]) dm \\ &= \frac{\exp(\mu z)}{\mu} - \frac{\exp(\mu [z(J-2) - m^*])}{\mu} \end{aligned}$$

Now we can write

$$1 = \sum_{j=0}^{J-2} \sum_{i=0}^j K_{j,i} L(i) + \sum_{i=0}^{J-1} K_{J-1,i} G(i)$$

Expected number of withdrawals n

The expected number of withdrawals n is obtained by computing the reciprocal of the expected time between withdrawals. To do we first compute the expected time until the next withdrawal, as a function of the current level of cash m . We denote such a function as $t(m)$. Below we show that this function solves the following DDE:

$$t(m)(p + \kappa) = 1 + t'(m)(c + \pi m) + \kappa t(m - z)$$

for $m^* \geq m > z$ and the ODE

$$t(m)(p + \kappa) = 1 + t'(m)(c + \pi m)$$

for $0 \leq m < z$, with $t(0) = 0$. Since after any withdrawal real balances go to m^* , the expected time between successive withdrawals is $t(m^*)$ and by the fundamental theorem of

renewal theory the average number of withdrawals is:

$$n = \frac{1}{t(m^*)}$$

Proof. Consider a discrete time version with periods of length Δ of the system. In this case the time until adustment solves

$$t(m) = (1 - (p + \kappa) \Delta) [\Delta + t(m - \Delta c - \pi m \Delta)] + \kappa \Delta t(m - \Delta c - \pi m \Delta - z)$$

for $m > z$ and

$$t(m) = (1 - (p + \kappa) \Delta) [\Delta + t(m - \Delta c - \pi m \Delta)]$$

for $m < z$, with boundary condition

$$t(0) = 0.$$

This law of motion gives:

$$t(m) (1 - [1 - (p + \kappa) \Delta]) = (1 - (p + \kappa) \Delta) [\Delta + t'(m) (c + \pi m) \Delta + o(\Delta)] + \kappa \Delta t(m - \Delta c - \pi m \Delta - z)$$

or

$$t(m) (p + \kappa) = (1 - (p + \kappa) \Delta) \left[1 + t'(m) (c + \pi m) + \frac{o(\Delta)}{\Delta} \right] + \kappa t(m - \Delta c - \pi m \Delta - z)$$

and taking $\Delta \rightarrow 0$:

$$t(m) (p + \kappa) = 1 + t'(m) (c + \pi m) + \kappa t(m - z)$$

for $m^* \geq m > z$ and

$$t(m) (p + \kappa) = 1 + t'(m) (c + \pi m)$$

for $0 \leq m < z$.

Characterization of the solution for t

As in the case of the value function, we solve for $t(\cdot)$ by dividing the domain in J intervals, where again J is the smallest integer for which $Jz \geq m^*$. The first $J - 1$ intervals are of length z , denoted them by $[zj, z(j + 1)]$ for $j = 0, 1, \dots, J - 2$. The last interval is $[z(J - 1), m^*]$. We then find $t_0 : [0, z] \rightarrow R_{++}$ that solves the ODE:

$$t_0(m) = \frac{1}{p + \kappa} + t'_0(m) \left(\frac{c + \pi m}{p + \kappa} \right)$$

for $m \in [0, z]$, and given t_{j-1} we solve for $t_j : [zj, z(j + 1)] \rightarrow R_{++}$

$$t_j(m) = \frac{1}{p + \kappa} + t'_j(m) \left(\frac{c + \pi m}{p + \kappa} \right) + \frac{\kappa}{p + \kappa} t_{j-1}(m - z)$$

for $m \in [zj, z(j+1)]$ for $j = 0, 1, \dots, J-2$. For $j = J-1$, the function $t_{J-1} : [z(J-1), m^*] \rightarrow R_+$ solves the same ode than for the other $j \geq 1$. Finally the continuity of t requires that

$$t_{j+1}(z(j+1)) = t_j(z(j+1))$$

for $j = 0, 1, \dots, J-2$. Hence, by splitting the domain in this way we turn the solution of a DDE into the solution of several ODE's.

Solving $t(m)$ for $\pi = 0$.

In this case we have:

$$t_0(m) = \frac{1}{p+\kappa} + t'_0(m) \left(\frac{c}{p+\kappa} \right)$$

for $m \in [0, z]$, and given t_{j-1} we solve for t_j

$$t_j(m) = \frac{1}{p+\kappa} + t'_j(m) \left(\frac{c}{p+\kappa} \right) + \frac{\kappa}{p+\kappa} t_{j-1}(m-z)$$

for $m \in [zj, z(j+1)]$ for $j = 0, 1, \dots, J-2$, and for $j = J-2$, then $m \in [z(J-1), m^*]$.

I. Solution for $j = 0$. We guess a solution for t_0 of the form:

$$t_0(m) = C_0 + T_{0,0} \exp(\mu m)$$

so that

$$C_0 + T_{0,0} \exp(\mu m) = \frac{1}{p+\kappa} + T_{0,0} \mu \exp(\mu m) \left(\frac{c}{p+\kappa} \right)$$

and hence:

$$\begin{aligned} \mu &= \frac{p+\kappa}{c}, \\ C_0 &= \frac{1}{p+\kappa}. \end{aligned}$$

We impose that $t(0) = 0$ obtaining

$$\frac{1}{p+\kappa} + T_{00} \exp\left(\frac{p+\kappa}{c} 0\right) = 0$$

or

$$T_{00} = -\frac{1}{p+\kappa}.$$

II. Solution for $j \geq 1$. For $m \in [zj, z(j+1)]$ and $1 \leq j \leq J-2$ or $m \in [m(J-1), m^*]$ we guess

$$t_j(m) = C_j + \exp(\mu(m-zj)) \sum_{i=0}^j T_{j,i} (m-zj)^i$$

and hence:

$$t'_j(m) = \exp(\mu(m - zj)) \sum_{i=0}^j T_{j,i} \left[\mu (m - zj)^i + i (m - zj)^{i-1} \right]$$

Substituting in the ode we have:

$$\begin{aligned} & C_{j+1} + \sum_{i=0}^{j+1} T_{j+1,i} \exp(\mu(m - z(j+1))) (m - z(j+1))^i \\ &= \frac{1}{p + \kappa} + \left(\frac{c}{p + \kappa} \right) \exp(\mu(m - zj)) \sum_{i=0}^{j+1} T_{j+1,i} \left[\mu (m - z(j+1))^i + i (m - z(j+1))^{i-1} \right] \\ & \quad + \frac{\kappa}{p + \kappa} \left[C_j + \exp(\mu(m - z(j+1))) \sum_{i=0}^j T_{j,i} (m - z(j+1))^i \right] \end{aligned}$$

Matching coefficients we have the following conditions. For the constant:

$$C_{j+1} = \frac{1}{p + \kappa} + \frac{\kappa}{p + \kappa} C_j$$

For $\exp(\mu(m - z(j+1)))$

$$\begin{aligned} & \sum_{i=0}^{j+1} T_{j+1,i} (m - z(j+1))^i \\ &= \left(\frac{c}{p + \kappa} \right) \sum_{i=0}^{j+1} T_{j+1,i} \left[\mu (m - z(j+1))^i + i (m - z(j+1))^{i-1} \right] \\ & \quad + \frac{\kappa}{p + \kappa} \sum_{i=0}^j T_{j,i} (m - z(j+1))^i \end{aligned}$$

and using $\mu = \frac{p+\kappa}{c}$,

$$\begin{aligned} 0 &= \left(\frac{c}{p + \kappa} \right) \sum_{i=0}^{j+1} T_{j+1,i} i (m - z(j+1))^{i-1} \\ & \quad + \frac{\kappa}{p + \kappa} \sum_{i=0}^j T_{j,i} (m - z(j+1))^i \end{aligned}$$

Matching the coefficients of $(m - z(j+1))^i$:

$$\left(\frac{c}{p + \kappa} \right) T_{j+1,i+1} (i+1) = -\frac{\kappa}{p + \kappa} T_{j,i}$$

or

$$T_{j+1,i+1} = \frac{1}{(i+1)} \left(\frac{\kappa}{c}\right) T_{j,i}$$

for $i = 0, 1, \dots, j$.

Finally we use the continuity of t at the edges of the intervals

$$t_{j+1}(z(j+1)) = t_j(z(j+1))$$

for all $j = 0, 1, \dots, J-2$. This gives

$$\begin{aligned} t_j(z(j+1)) &= C_j + \exp(\mu z) \sum_{i=0}^j T_{j,i}(z)^i \\ t_{j+1}(z(j+1)) &= C_{j+1} + \exp(\mu 0) \sum_{i=0}^{j+1} T_{j+1,i}(0)^i = C_{j+1} + T_{j+1,0} \end{aligned}$$

thus

$$C_j + \exp(\mu z) \sum_{i=0}^j T_{j,i}(z)^i = C_{j+1} + T_{j+1,0}$$

or

$$T_{j+1,0} = C_j + \exp(\mu z) \sum_{i=0}^j T_{j,i}(z)^i - C_{j+1}$$

Finally evaluating $t_{J-1}(\cdot)$ at m^* gives the desired quantity.

Average Withdrawals W

We characterize the average withdrawal size W . To do so, notice that we can divide the withdrawals in three types: i) those that happens when $m = 0$, ii) those that happens because a jump in consumption, i.e. the arrival of a consumption jump when $m \leq z$ and iii) those that happens because the arrival of a free opportunity to withdraw. In average there are n withdrawals per unit of time, out of which p are of type iii), $\kappa \int_0^z h(m) dm$ of type ii) and hence $n - p - \kappa \int_0^z h(m) dm$ are of type i. The size of the withdrawals is different in each case, so the average withdrawal is given by:

$$\begin{aligned} W &= \left[\frac{n - p - \kappa \int_0^z h(m) dm}{n} \right] m^* + \frac{p}{n} \int_0^{m^*} (m^* - m) h(m) dm \\ &\quad + \left[\frac{\kappa \int_0^z h(m) dm}{n} \right] \frac{\int_0^z (m^* + z - m) h(m) dm}{\int_0^z h(m) dm} \end{aligned}$$

or

$$\begin{aligned} W &= \left[\frac{n - p - \kappa \int_0^z h(m) dm}{n} \right] m^* + \frac{p}{n} \int_0^{m^*} (m^* - m) h(m) dm \\ &\quad + \left[\frac{\kappa}{n} \right] \int_0^z (m^* + z - m) h(m) dm \end{aligned}$$

C.1 Supplemental Material

Table 7: Cash management statistics across payment technology - Italy

<i>Stats:</i>	1993	1995	1998	2000	2002	2004
<i>Share of cash expenditure</i>						
HH w. Demand deposit - No card	70	68	66	68	67	64
HH w. Debit / ATM card - No credit	65	63	60	57	54	50
HH w. Credit card	57	57	55	52	50	44
<i>W / M</i>						
HH w. Demand deposit - No card	2.3	1.7	1.9	2.1	2.0	1.9
HH w. Debit / ATM card - No credit	1.6	1.2	1.4	1.4	1.3	1.5
HH w. Credit card	1.3	1.0	1.2	1.3	1.2	1.3
<i>n / n_{BT} = 2 M n / e</i>						
HH w. Demand deposit - No card	1.2	1.4	2.6	2.0	1.7	2.0
HH w. Debit / ATM card - No credit	2.1	2.5	3.7	3.4	3.6	4.0
HH w. Credit card	3.3	3.1	4.1	4.5	4.2	4.4
<i>M / M</i>						
HH w. Demand deposit - No card	.41	.31	.47	.47	.45	
HH w. Debit / ATM card - No credit	.41	.30	.37	.43	.40	
HH w. Credit card	.47	.32	.41	.48	.44	

Source SHIW. Entries are sample means. The unit of observation is the household whose head is not self-employed.