# IO in I-O: Competition and Volatility in Input-Output Networks


Basile Grassi*

Nuffield College, University of Oxford


JOB MARKET PAPER


November 20, 2016 – Version 1.20

[Latest Version]



**Abstract**

There is a growing literature suggesting that firm level productivity shocks can help understand macroeconomic level outcomes. However, existing models are very restrictive regarding the nature of competition within sector and its implication for the propagation of shocks across the input-output (I-O) network. The goal of this paper is to offer a more comprehensive understanding of how firm level shocks can shape aggregate dynamics. To this end, I build a tractable multi-sector heterogeneous firm general equilibrium model featuring oligopolistic competition and an I-O network. It is shown that a positive shock to a large firm increases both the average productivity and the Herfindahl Index in its sector. By reducing the sector price, the change in average productivity propagates only to downstream sectors. Conversely, the change in the Herfindahl Index, by increasing price and reducing demand for intermediate inputs, propagates both to downstream and upstream sectors. The sensitivity of aggregate volatility to firms' shocks is determined by the sector's (i) Herfindahl Index, which measures the volatility of the sector, (ii) position in the input-output network, which measures the direct and indirect importance of this sector for the household, and (iii) relative market power in the supply chain, which relates to the changes in demand to upstream sectors.



**Keywords:** Oligopoly, Imperfect Competition, Input-Output Network, Firm Heterogeneity, Random Growth, Granularity, Volatility



*I would like to acknowledge helpful comments from Axelle Arquié, James Best, Florin Bilbiie, Paul Beaudry, Ariel Burstein, Vasco M. Carvalho, Nuno Coimbra, Martin Ellison, Jean Flemming, François Fontaine, Xavier Gabaix, Clément Imbert, Jean Imbs, Julien Labonne, Isabelle Mejean, Ezra Oberfield, Dan Quigley, David Ronayne and Andre Veiga.

> *"[I] think that the issue of growing market power deserves increased attention from economists and especially from macroeconomists."*
>
> *Larry Summers, Washington Post, March 2016*

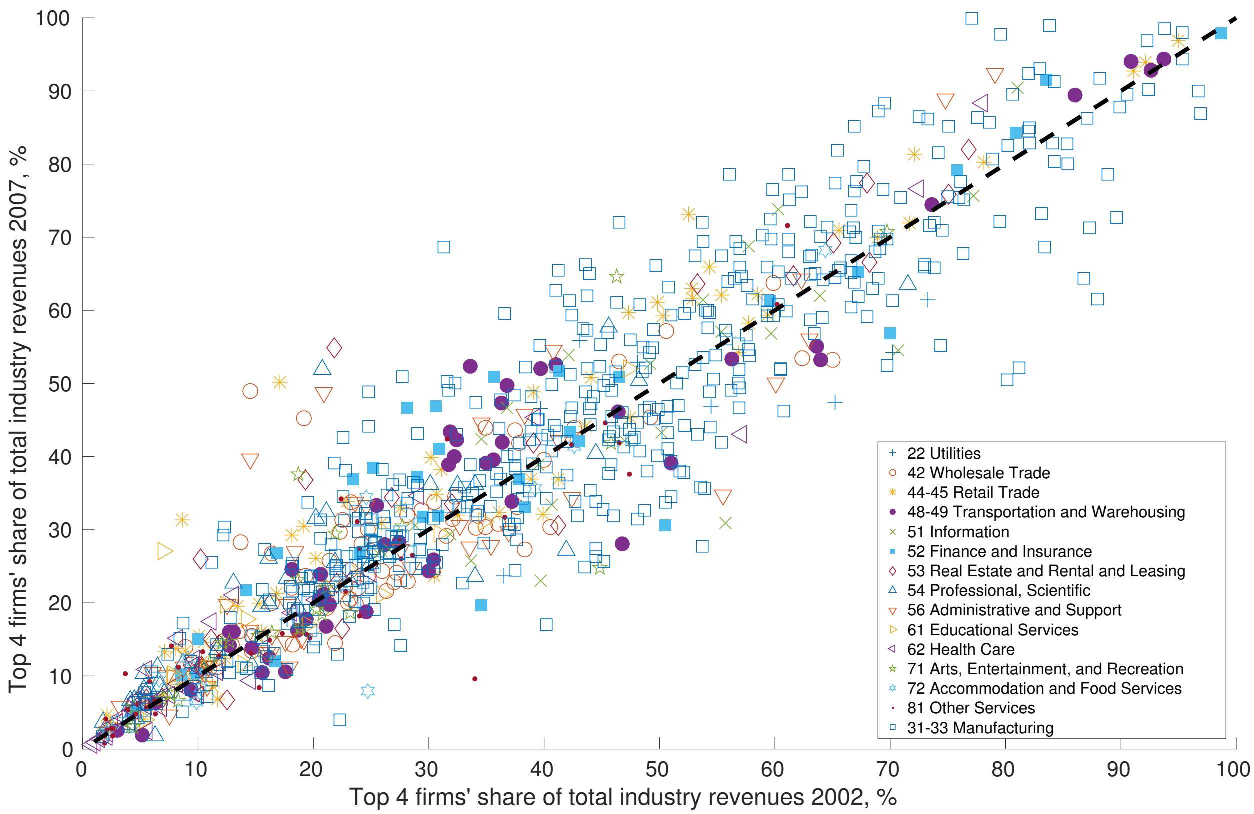
## 1. Introduction

A growing literature suggests that firm-level productivity shocks can explain movement in prices and output at the sector and macroeconomic level[1]. This literature relies on the idea that a handful of large firms represent a large share of a sector, and thus shocks hitting these large firms cannot be balanced out by those affecting smaller firms. However, these models are very restrictive regarding the nature of competition within a sector: firms are large enough to have a systemic importance but these firms do not internalized this when they make their decisions. This paper explores the alternative oligopolistic market structure where firms do take into account the effect of their decisions on sector-level price and quantity in order to study the propagation of firm-level shocks to other sectors through the Input-Output (I-O) network. The properties of the propagation that arise under oligopolistic competition – relative to the monopolistic case – are shown to increase the response of aggregate volatility to firm-level shocks.

Figures 1 and 2 motivate this paper: sectors are concentrated and linked through a "small world" I-O network. Figure 1 shows the top four firms' share

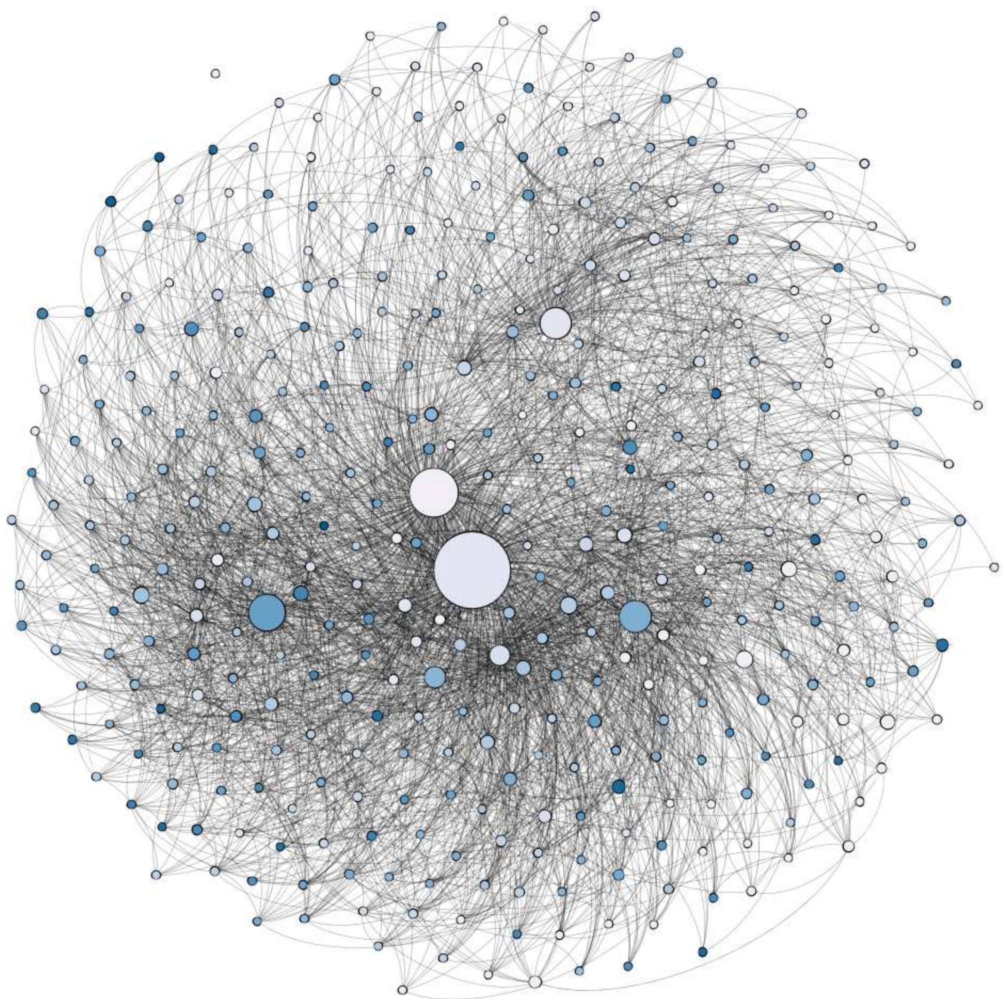
[1]An important paper is the seminal work by Gabaix (2011) where he shows that when the firm-size distribution is fat-tailed, firm-level shocks do not wash out at the aggregate. Building on this seminal work, Carvalho and Grassi (2015) show that firm dynamic models contain a theory of business cycle as soon as the continuum of firms' assumption is relaxed. Acemoglu et al. (2012), Carvalho (2010, 2014) and Baqaee (2016) build on the multi-sector business cycle framework of Long and Plosser (1983) to show how shocks on sectors linked through an I-O network can translate into aggregate fluctuations. Earlier contributions include Jovanovic (1987), Durlauf (1993) and Bak et al. (1993).

Figure 1: Sector Concentration




Note: Top four firms' share of total revenues in 2002 and in 2007 for 6-digit NAICS industry. The mean value is 35.37% in 2002 and 37.21% in 2007. 970 industries. Source: Census Bureau.

Figure 2: Sectors' Concentrations and the I-O Network in 2007



Note: larger nodes of the network represent sectors supplying inputs to many other sectors. A darker color represents higher top four firms' share of total revenues in 2007 (sectors without available data are left white). There are 389 sectors. Source: Bureau of Economic Analysis, detailed I-O table for 2007 and Census Bureau. The figure is drawn with the software package Gephi.

of industry revenue in 2002 against the same measure in 2007 for around 970 industries. Industry revenue accounted for by the top four firms varies from almost zero to close to 100% with a median value close to 33% in 2007. The first thing to note is that large firms represent an important share of revenue of the median sector. Secondly, as concentration is a widely used measure of a sector's competition intensity, this figure also suggests that different sectors have different competition levels. Finally, the fact the sectors are not all on the 45° line shows that concentration is not constant across time. While confirming the "granular" nature of these sectors, this figure emphasizes the importance of the oligopolistic nature of competition within sector and its evolution across time. Besides, these sectors are not independent from each other: production in one sector relies on a complex and interlocking supply-chain. Figure 2 displays the I-O network among 389 sectors for the US in 2007. This is a "small world" network: a few nodes are connected to many other nodes. In such production networks, as shown by Acemoglu et al. (2012), Carvalho (2010, 2014) and Baqaee (2016), sector-level shocks translate into aggregate volatility. In this paper, I study how firm-level shocks affect sector-level productivity and competition and how changes in the level of productivity and competition propagate in the I-O network and thus shape the aggregate dynamics.

To this end, I build a tractable multi-sector heterogeneous firm general equilibrium model featuring oligopolistic competition and an I-O network. Within each sector, a finite number of heterogeneous firms are subject to oligopolistic competition and set variable markups à la Atkeson and Burstein (2008). Up to an approximation, two sector-level sufficient statistics, the sectors' average productivity and the productivity Herfindahl Index – a concentration measure, entirely characterize the equilibrium of this economy. When oligopolistic competition is taken into account, in a sector with a sales Herfindahl of 0.18 (above this level merger law in the U.S. starts to apply), I show that the effect of firm-level shocks on aggregate volatility increases by 80% relative to the monopolistic competition case.

The mechanism is as follows. Firm-level shocks affect both the sector's average productivity and concentration. To see this, take a sector with a finite number of heterogeneous firms and assume that an already large firm is subject to a positive productivity shock. Following this shock, the sector's average productivity becomes larger since the productivity of one firm has become larger. Since this firm was already large before the shock hit, the sector becomes even more concentrated. This firm-level shock has two opposite effects on price and output at the sector level. First, because of the increase in average productivity, the sector good is cheaper and output increases. Second, because of the increases in concentration, competition in the sector decreases: this large firm is larger and can use its size to extract even more profit. It follows that the sector price increases and output decreases. These changes in prices and output propagate to the other sectors through the I-O network. The increase in productivity, resulting in a decrease in price, reduces the marginal cost of downstream sectors. Indeed the downstream sectors use this good as an input to produce. The decrease in competition, resulting in an increase in price, propagates downstream as it increases the marginal cost of downstream sectors. But it also propagates to upstream sectors as it reduces the share of sector's income used to pay for intermediate inputs and thus the demand for upstream sectors' goods. The propagation of this shock downstream ultimately affects the price of goods purchased by the household and thus the real wage. The stronger is the effect, the more the sector's good is directly and indirectly (through other sectors) consumed by the household. The propagation of shocks upstream ultimately affects the profit rebated to the households as it reduces demand for upstream goods. The stronger is the effect, the higher is the sector's market power relative to its supply-chain market power. The above example described the effect of one shock on an already large firm but, in this paper, each firm's productivity is subject to persistent idiosyncratic shocks which make these two sufficient statistics follow $AR(1)$-type processes, as in Carvalho and Grassi (2015). Each sector's price and quantity are thus stochastic which translate into aggregate volatility thanks to the "small world" nature of the I-O network.

I show that the effect of firm volatility in a given sector on aggregate volatility is a function of three characteristics. First, the concentration which determines how important large firms are in that sector and thus how much shocks to these firms create volatility at the sector level and hence at the aggregate level. Second, the sector centrality which measures that sector's direct and indirect importance in the household's consumption bundle. This characteristic relates to the transmission of firm-level shocks to downstream sectors. Third, the sector's relative market power over its supply chain which measures how much profit is captured by that sector relative to how much profit its whole supply chain generates. This characteristic relates to the propagation of firm level shocks to upstream sectors.

Thanks to the high tractability of the model and the fact that the equilibrium is characterized by two sector-level sufficient statistics, I calibrate the model by relying on the choice of a few deep parameters. This calibration allows me to decompose the contribution of firm-level volatility in a given sector on aggregate volatility. For a sector with a sales Herfindahl index of 0.18, which is the level above which merger law applies in the U.S, the effect of firm-level shocks on wage volatility increases by 80% when oligopolistic competition is taken into account relative to a version of the model with monopolistic competition. This number ranges from 19% to 65% for the ten sectors where firm-level volatility affects aggregate volatility the most.

**Related Literature** This paper contributes to the emerging literature on the micro-origin of aggregate fluctuations. This literature is based on two mains ideas: the "granular hypothesis" and the I-O network. For the former, seminal work by Gabaix (2011) shows that whenever the firm-size distribution is fat-tailed, idiosyncratic shocks do not average out quickly enough and therefore translate into sizable aggregate fluctuations. Carvalho and Grassi (2015) ground the "granular hypothesis" in a well-specified firm dynamic setup. In the latter, Acemoglu et al. (2012) and Carvalho (2010) show that when the distribution of sectors' centrality in the I-O network is fat tailed then sector level perturbations also generate sizable aggregate fluctuations. Relative to these papers, I present the first framework that includes both components explicitly. The "granular hypothesis" leads to sector-level fluctuations whereas the I-O network structure translates sector-level fluctuations into aggregate fluctuations. An important drawback of this literature is that firms are supposed to be large enough to influence the aggregate but also small enough to not be strategic. In Carvalho and Grassi (2015) framework such assumptions were made because firms interacted in a perfectly competitive labor market. Here, I present the first model of strategic pricing where aggregate fluctuations arise from purely idiosyncratic shocks.

This paper also contributes to the literature on the propagation of shocks in I-O networks. This literature has studied the transmission of well-identified shocks in the I-O network: Acemoglu et al. (2015) study the transmission of well identified supply and demand shocks, Carvalho et al. (2016) and Boehm et al. (2016) study the firm level impact of supply chain disruptions occurring in the aftermath of the Great East Japan Earthquake in 2011, while Barrot and Sauvagnat (2016) look at the effect of natural disasters. Baqaee (2016) studies theoretically the effect of shocks on entry cost. My paper contributes to this literature by studying the propagation of (endogenous) changes in the sectors' levels of competition, which act as supply shocks to downstream sectors and demand shocks to upstream sectors.

Finally, this paper also contributes to the literature on imperfect competition among heterogeneous firms. Krugman (1979), Ottaviano et al. (2002), Melitz and Ottaviano (2008), Bilbiie et al. (2012) and Zhelobodko et al. (2012) study demand-side pricing complementary whereas I look at supply-side pricing complementaries as in Atkeson and Burstein (2008) but in an I-O context. Furthermore I show that such a model is highly tractable and that firm heterogeneity can be summarized at the sector level by just two sufficient statistics.

**Outline** The paper is organized as follows. In section 2, I describe and solve the household's and firm's problem. In section 3, I first aggregate firm behavior at the sector level and show that firm heterogeneity can be summarized by two sufficient statistics. I then solve for the stochastic dynamics of these two statistics. In section 4, I describe the equilibrium and show that the model can be entirely solved at the sector level. In section 5, I first solve analytically for the equilibrium in the no-capital case and show how concentration and centrality determine the response of the economy to shocks. Section 6 concludes.

## 2. Model

In this section, I describe the structure of the economy and I solve for the household and firm's problem. A representative household consumes, supplies labor and invests in productive capital. Sectors are linked by a production network, firms compete within a sector and set their price (or quantities) strategically. Firms are subject to idiosyncratic shocks that generate uncertainty on sector's productivity. These sectoral dynamics generate aggregate uncertainty.

## 2.1. Household

The representative household consumes, invests and supplies labor. The household problem is

$$\text{Max}\left\{\mathbb{E}_0\sum_{t=0}^{\infty}\rho^t\left(\frac{C_t^{1-\eta}}{1-\eta}-\theta\frac{L_t^{\chi}}{\chi}\right)\right\}$$

subject to the budget constraint $P_t^C C_t + P_t^I I_t \leq w_t L_t + r_t K_t + Pro_t$. Where $P_t^C$ is the price index of the composite consumption good, $P_t^I$ is the price index of the composite investment good, $w_t$ is the wage rate, $r_t$ is the rental rate of capital, $K_t$ is the capital stock, $C_t$ is the composite consumption good, $I_t$ is the composite investment good, $L_t$ is the labor supplied and $Pro_t$ is the total profit made by the firms. Capital accumulation is subject to adjustment cost $K_{t+1} = (1-\delta)K_t + I_t - \phi\left(\frac{I_t}{K_t}\right)K_t$, where $\delta$ is the depreciation rate and $\phi(.)$ is the adjustment cost of capital. The composite consumption good and composite investment good are Cobb-Douglas aggregators of each sector's goods: $C_t = \prod_{t=0}^{N} C_{k,t}^{\beta_k}$ and $I_t = \prod_{t=0}^{N} I_{k,t}^{\nu_k}$ where $C_{k,t}$ (resp. $I_{k,t}$) is the amount of good $k$ consumed (resp invested) by the household at time $t$. $\beta_k$ and $\nu_k$ are the Cobb-Douglas shares of each goods in the composite consumption and investment goods respectively. Sector $k$'s good is a CES composite of $N_k$ varieties produced by each firm within the sector $k$:

$$C_{k,t} = \left(N_k^{-\zeta_k}\sum_{t=0}^{N_k} C_t(k,i)^{\frac{\varepsilon_k-1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k-1}} \quad \text{and} \quad I_{k,t} = \left(N_k^{-\zeta_k}\sum_{t=0}^{N_k} I_t(k,i)^{\frac{\varepsilon_k-1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k-1}}$$

$C_t(k,i)$ (resp. $I_t(k,i)$) is the amount of sector $k$'s variety $i$ consumed (resp invested) by the household at time $t$, $\varepsilon_k$ is the elasticity of substitution between two varieties in sector $k$, $\zeta_k$ controls the love-for-variety effect in sector $k$. The proposition below describes the optimal intra and inter-temporal allocation of the representative household.

**Proposition 2..1** (Household's Optimal Allocation): *The household's optimal allocation is described below.*

1. *The labor supply and the intertemporal choice:*

$$\frac{w_t}{P_t^C} = \theta \frac{L_t^{\chi-1}}{C_t^{-\eta}} \qquad q_t = \frac{1-\phi'(\frac{I_t}{K_t})}{P_t^I}$$

$$q_t = \mathbb{E}_t \left[ \rho \left( \frac{C_{t+1}}{C_t} \right)^{-\eta} \frac{P_t^C}{P_{t+1}^C} \left[ r_{t+1} + q_{t+1} \left( 1 - \delta - \phi \left( \frac{I_{t+1}}{K_{t+1}} \right) + \frac{I_{t+1}}{K_{t+1}} \phi' \left( \frac{I_{t+1}}{K_{t+1}} \right) \right) \right] \right]$$

*where $q_t$ is the replacement cost of capital.*

2. *The intra-temporal allocation among sectors is:*

$$\frac{P_{k,t}C_{k,t}}{P_t^C C_t} = \beta_k \quad \text{and} \quad \frac{P_{k,t}I_{k,t}}{P_t^I I_t} = \nu_k$$

*where the consumption and investment price indices are:*

$$P_t^C = \prod_{k=1}^{N} P_{k,t}^{\beta_k} \quad \text{and} \quad P_t^I = \prod_{k=1}^{N} P_{k,t}^{\nu_k}$$

3. *The intra-temporal allocation among varieties in sector $k$ is characterized by*

$$C_t(k,i) = N_k^{-\varepsilon_k \zeta_k} \left( \frac{P_t(k,i)}{P_{k,t}} \right)^{-\varepsilon_k} C_{k,t} \quad \text{and} \quad I_t(k,i) = N_k^{-\varepsilon_k \zeta_k} \left( \frac{P_t(k,i)}{P_{k,t}} \right)^{-\varepsilon_k} I_{k,t}$$

*with the sector $k$'s price index*

$$P_{k,t} = \left( N_k^{-\varepsilon_k \zeta_k} \sum_{i=1}^{N_k} P_t(k,i)^{1-\varepsilon_k} \right)^{\frac{1}{1-\varepsilon_k}}$$

## 2.2. Firms

Firm $i$ in sector $k$ produces a variety $i$ of sector $k$'s good. There are $N_k$ firms in the sector $k$ that compete either monopolistically, à la Cournot or à la Bertrand. The firm $i$ in sector $k$ maximizes its profit:

$$\pi_t(k,i) = p_t(k,i)y_t(k,i) - \sum_{l=1}^{N}\sum_{j=1}^{N_l} p_t(l,j)x_t(k,i,l,j) - w_t L_t(k,i) - r_t K_t(k,i)$$

where $p_t(k,i)$ is the price charged by firm $i$ in sector $k$, $y_t(k,i)$ is the quantity produced, $x_t(k,i,l,j)$ is the quantity of variety $j$ of good $l$ used by firm $i$ in sector $k$, $L_t(k,i)$ is the labor input and $K_t(k,i)$ is the amount of capital rented from household. The firms in sector $k$ have access to the following constant return to scale technology:

$$y_t(k,i) = A_k\left(\Big(Z_t(k,i)L_t(k,i)\Big)^{1-\alpha_k} K_t(k,i)^{\alpha_k}\right)^{\gamma_k}\left(\prod_{l=1}^{N} x_t(k,i,l)^{\omega_{k,l}}\right)$$

$\gamma_k$ is the share of primary inputs (labor and capital) in the production, $A_k$ is a normalization constant[2], $\alpha_k$ is the capital share in that primary input, $Z_t(k,i)$ is the labor-augmented productivity specific to the firm $i$ in sector $k$, $\omega_{k,l}$ is the input share of sector $l$'s goods needed in sector $k$'s production. The $(N \times N)$ matrix $\Omega = \{\omega_{k,l}\}_{k,l}$ represents the input-output network. Because of constant return to scale the $k$th rows of $\Omega$ sum to $\gamma_k$: $\sum_{l=1}^{N}\omega_{k,l} = \gamma_k$. Furthermore, $x_t(k,i,l)$ is a composite of sector $l$'s varieties:

$$x_t(k,i,l) = \left(N_l^{-\zeta_l}\sum_{j=1}^{N_l} x_t(k,i,l,j)^{\frac{\varepsilon_l - 1}{\varepsilon_l}}\right)^{\frac{\varepsilon_l}{\varepsilon_l - 1}}$$

[2]This normalization constant makes the mathematics simpler it is equal to $A_k = \gamma_k^{-\gamma_k}\prod_{l=1}^{N}\omega_{k,l}^{-\omega_{k,l}}$.

$x_t(k,i,l,j)$ is the quantity of the variety $j$ of sector $l$'s good that is used for the production of variety $i$ of sector $k$'s good. Note that the elasticity of substitution among varieties in a sector is the same for firms and the representative household. Even if this assumption seems extreme it is used to keep the mathematics simple and the model tractable.

In the remaining of this section, I describe the firm's problem solution including its pricing behavior and I derive an useful approximation. Finally I describe the process that productivity $Z_t(k,i)$ follows.

### 2.2.1. Firm's Problem

The firm $i$ in sector $k$ chooses its inputs to minimize the cost of producing $y_t(k,i)$ units of its variety. Proposition 2..2 describes the solution of the cost minimization problem of a firm.

**Proposition 2..2** (Firm's Cost Minimization): *To produce $y_t(k,i)$ units of its variety, firm $i$ in sector $k$ uses at time $t$ the following inputs:*

$$L_t(k,i) = \gamma_k(1-\alpha_k)\left(\frac{w_t}{\lambda_t(k,i)}\right)^{-1} y_t(k,i) \quad \textit{and} \quad K_t(k,i) = \gamma_k\alpha_k\left(\frac{r_t}{\lambda_t(k,i)}\right)^{-1} y_t(k,i)$$
$$x_t(k,i,l,j) = N_l^{-\zeta_l\varepsilon_l}\left(\frac{p_t(l,j)}{P_{l,t}}\right)^{-\varepsilon_l} x_t(k,i,l) \quad \textit{where} \quad x_t(k,i,l) = \omega_{k,l}\left(\frac{P_{l,t}}{\lambda_t(k,i)}\right)^{-1} y_t(k,i)$$

*The marginal cost $\lambda_t(k,i)$ faced by that firm is:*

$$\lambda_t(k,i) = \left(Z_t(k,i)^{\gamma_k(\alpha_k-1)}\left(\frac{w_t}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r_t}{\alpha_k}\right)^{\gamma_k\alpha_k}\right)^{\gamma_k}\prod_{l=1}^{N} P_{l,t}^{\omega_{k,l}}$$

*where $P_{l,t} = \left(N_l^{-\zeta_l\varepsilon_l}\sum_{j=1}^{N_l} p_t(l,j)^{1-\varepsilon_l}\right)^{\frac{1}{1-\varepsilon_l}}$ and where $p_t(l,j)$ is the price charged by firm $j$ in sector $l$.*

Note that the labor-capital ratio is constant across firms in a sector $k$ and equal to $\frac{1-\alpha_k}{\alpha_k}\frac{r_t}{w_t}$. The labor-augmented productivity $Z_t(k,i)$ is heterogeneous across firms in the sector $k$. It follows that the marginal cost $\lambda_t(k,i)$ is also heterogeneous across firm in sector $k$.

Firm $i$ in sector $k$ faces the demand $y_t(k,i) = N_k^{-\zeta_k \varepsilon_k} \left(\frac{p_t(k,i)}{P_{k,t}}\right)^{-\varepsilon_k} Y_{k,t}$ where $Y_{k,t}$ is the total demand faced by sector $k$: $Y_{k,t} = C_{k,t}+I_{k,t}+\sum_{l=1}^{N}\sum_{j=1}^{N_l} x_t(l,j,k) = \left(N_k^{-\zeta_k}\sum_{i=1}^{N_k} y_t(k,i)^{\frac{\varepsilon_k-1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k-1}}$. Using the household allocation for $C_{k,t}$ and $I_{k,t}$ and the demand for composite intermediate good by firms, it follows:

$$P_{k,t}Y_{k,t} = \beta_k P_t^C C_t + \nu_k P_t^I I_t + \sum_{l=1}^{N}\sum_{j=1}^{N_l} \omega_{l,k}\lambda_t(l,j)y_t(l,j) \tag{1}$$

Note that in an economy with no investment and no input-output linkages ($\nu_k = \omega_{l,k} = 0$), the sector $k$'s revenue would write $P_{k,t}Y_{k,t} = \beta_k P^C C$ which is a sector specific parameter times aggregate output. Instead $P_{k,t}Y_{k,t}$ is a weighted sum of aggregate quantities where the weights are sector specific parameters. The firm's problem is $\mathrm{Max}\,\{p_t(k,i)y_t(k,i) - \lambda_t(k,i)y_t(k,i)\}$ subject to the demand

$$y_t(k,i) = N_k^{-\zeta_k\varepsilon_k} p_t(k,i)^{-\varepsilon_k} P_{k,t}^{\varepsilon_k-1} P_{k,t}Y_k \quad \Leftrightarrow \quad p_t(k,i) = N_k^{-\zeta_k} y_t(k,i)^{\frac{1}{\varepsilon_k}-1} P_{k,t}Y_{k,t}$$

where $p_t(k,i)$ is the price set by firm $i$ in sector $k$, $y_t(k,i)$ is the quantity produced by firm $i$ in sector $k$, $\lambda_t(k,i)$ is the marginal cost of firm $i$ in sector $k$, $N_k$ is the number of firms in sector $k$, $P_{k,t}$ is the price index in sector $k$, $Y_{k,t}$ is the quantity of good produced by sector $k$ and where:

$$Y_{k,t} = \left(N_k^{-\zeta_k}\sum_{i}^{N_k} y_t(k,i)^{\frac{\varepsilon_k-1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k-1}} \quad \text{and} \quad P_{k,t} = \left(N_k^{-\zeta_k\varepsilon_k}\sum_{i}^{N_k} p_t(k,i)^{1-\varepsilon_k}\right)^{\frac{1}{1-\varepsilon_k}}$$

In the next proposition, firms are assumed to take $P_{k,t}Y_{k,t}$ as given i.e they don't internalize their price/quantity choice on $P_{k,t}Y_{k,t}$. The reason is that $P_{k,t}Y_{k,t}$ is a weighted sum of aggregate quantities where the weights are parameters as discussed above. Firms are assumed to take into account the effect of their choices on sector level quantity but not on aggregate quantities (in the Cournot and Bertrand competition case). I follow here Atkeson and Burstein (2008). In their paper, there is a continuum number of sectors and thus in equation (1) the $\sum_{l=1}^{N}$ is replaced by an integral. Atkeson and Burstein (2008)'s assumption is replaced here by the assumption on the limited firms' ability to internalize their effect on aggregate quantities. The following proposition characterizes the pricing of firm $i$ in sector $k$. For ease of notation, I am dropping the time subscript.

**Proposition 2..3** (Firm's Pricing): *When $\varepsilon_k > 1$, the firm $i$ in sector $k$ sets a price $p(k,i)$, has a sale share $s(k,i)$ and a subjective demand elasticity $\varepsilon(k,i)$ that satisfy the following system:*

$$p(k,i) = \frac{\varepsilon(k,i)}{\varepsilon(k,i)-1}\lambda(k,i)$$

$$s(k,i) = \frac{p(k,i)y(k,i)}{P_k Y_k} = N_k^{-\zeta_k \varepsilon_k}\left(\frac{p(k,i)}{P_k}\right)^{1-\varepsilon_k}$$

$$\varepsilon(k,i) = \begin{cases} \varepsilon_k & \textit{Under Monopolistic Competition} \\ \varepsilon_k(1-s(k,i)) + s(k,i) = \varepsilon_k - (\varepsilon_k - 1)s(k,i) & \textit{Under Bertrand Competition} \\ \left(\frac{1-s(k,i)}{\varepsilon_k} + s(k,i)\right)^{-1} = \left(\frac{1}{\varepsilon_k} + (1-\frac{1}{\varepsilon_k})s(k,i)\right)^{-1} & \textit{Under Cournot Competition} \end{cases}$$

Let us define the firm level markup $\mu(k,i) = \frac{\varepsilon(k,i)}{\varepsilon(k,i)-1}$. Note that $\frac{d\mu(k,i)}{d\varepsilon(k,i)} < 0$, since $\varepsilon(k,i)$ is decreasing in $s(k,i)$ for the Bertrand and the Cournot case when $\varepsilon_k > 1$. It implies that the firm level markup is increasing in its size $s(k,i)$ measured by the sale share in its sector. Larger firms charge a higher markup. Note also that the subjective demand elasticity is a weighted average of the elasticity of substitution across varieties $\varepsilon_k$ and the elasticity of substitution across sector which is here equal to one.

Unfortunately, this system of equations does not admit an analytical solution. Therfore, it is not possible to aggregate the firms' behavior at the sector level. However, in the proposition below, I show that one can approximate the solution of this system of equation by the sales share under monopolistic competition which will allow to solve the model at the sector level.

**Proposition 2..4** (Firm's Pricing Approximation): *At the third order, when $\widehat{s}(k,i) \to 0$, the sales share of firm $i$ in sector $k$ is:*

- *Under Bertrand Competition,*

$$s(k,i) = \widehat{s}(k,i) - \left(1 - \frac{1}{\varepsilon_k}\right)\widehat{s}(k,i)^2 + \left(1 - \frac{1}{1-\varepsilon_k}\right)\left(1 - \frac{1}{\varepsilon_k}\right)^2 \widehat{s}(k,i)^3 + o\Big(\widehat{s}(k,i)^4\Big)$$

- *Under Cournot competition,*

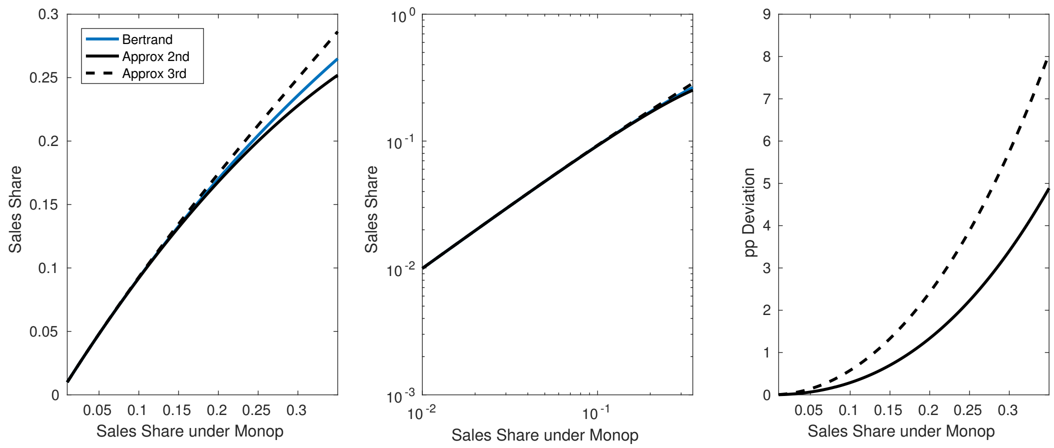
$$s(k,i) = \widehat{s}(k,i) - (\varepsilon_k - 1)\,\widehat{s}(k,i)^2 + \left(3 + \frac{1}{1-\varepsilon_k}\right)(\varepsilon_k - 1)^2\,\widehat{s}(k,i)^3 + o\Big(\widehat{s}(k,i)^4\Big)$$

*where $\widehat{s}(k,i) = \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{1-\varepsilon_k} N_k^{-\zeta_k \varepsilon_k} \left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k}$ is the sales share of firm $i$ in sector $k$ under Monopolistic competition.*

Proposition 3 shows that when the sales share are not too large, the firm's pricing problem can be approximated easily by the sales share under monopolistic competition. In Figure 3, I plot the second and third order approximations along a numerical solution of the firm pricing problem of proposition 2..3. On this Figure one can see that, for the calibrated value of $\varepsilon_k = 5$, the approximation holds for sales share up to 35%. Moreover the third order does not add much precision to this approximation. In the remaining of the paper, I am assuming that firms behave as the second order approximation as described in assumption 1.

**Assumption 1** (Firm's Pricing): *Agents are assumed to make a second order*

Figure 3: Firm's Pricing Approximation




Note: For $\varepsilon_k = 5$. Left panel shows the Bertrand sales share using a numerical solver (blue), the second (black) and the third (dashed black) order approximation as a function of the Monopolistic sales share. The middle panel plot is on a log-log scale, the right panel shows percentage deviation of both approximations with respect to the numerical solution. For the Cournot case, see appendix.

*approximation of Firm $i$'s sales share in sector $k$ around the monopolistic case,*

$$s(k,i) = \begin{cases} \widehat{s}(k,i) & \textit{under Monopolistic Competition} \\ \widehat{s}(k,i) - \left(1 - \frac{1}{\varepsilon_k}\right)\widehat{s}(k,i)^2 + o\left(\widehat{s}(k,i)^3\right) & \textit{under Bertrand Competition} \\ \widehat{s}(k,i) - \left(\varepsilon_k - 1\right)\widehat{s}(k,i)^2 + o\left(\widehat{s}(k,i)^3\right) & \textit{under Cournot Competition} \end{cases}$$

*where* $\widehat{s}(k,i) = \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{1-\varepsilon_k} N_k^{-\zeta_k \varepsilon_k} \left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k}$ *is the sales share of firm $i$ in sector $k$ under monopolistic competition.*

### 2.2.2. Firm's Productivity Dynamics

The (labor-augmented) productivity of firm $i$ in sector $k$ is $Z_t(k,i)$ and is heterogeneous among firms in sector $k$. I assume that this firm level productivity follows a sector specific Markov chain over the discrete state space $\Phi = \{1, \varphi_k, \varphi_k^2, \cdots, \varphi_k^n, \cdots, \varphi_k^{M_k}\} = \{\varphi_k^n\}_{n\in\{0,1,\ldots,M_k\}}$ for $\varphi_k > 1$ which is evenly distributed in logs[3]. This Markov chain is described by the transition probabilities $\mathcal{P}^{(k)}_{n,n'}$, where $\mathcal{P}^{(k)}_{n,n'} = \mathbb{P}\left(\varphi_k^{n_{t+1,k,i}} = \varphi_k^{n'} | \varphi_k^{n_{t,k,i}} = \varphi_k^n\right)$ is the probability that a firm $i$ in sector $k$ jumps from productivity level $\varphi_k^n$ to $\varphi_k^{n'}$ between time $t$ and time $t+1$. I assume a specific Markovian chain as described in the assumption below.

**Assumption 2** (Random Growth): *Firm level productivity in sector $k$ follows the Markov chain with transition probability:*

$$\mathcal{P}^{(k)} = \begin{pmatrix} a_k + b_k & c_k & 0 & \cdots & \cdots & 0 & 0 \\ a_k & b_k & c_k & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_k & b_k & c_k \\ 0 & 0 & 0 & \cdots & 0 & a_k & b_k + c_k \end{pmatrix}$$

*where* $a_k + b_k + c_k = 1$.

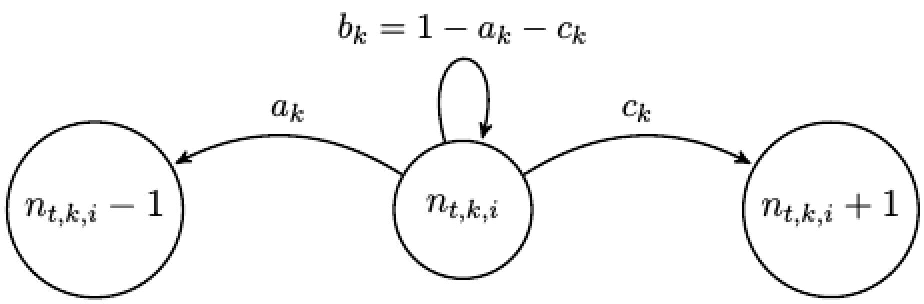
This Markovian process is a discretization of a random growth process and is taken from Córdoba (2008). Figure 4 represents these transition probabilities. The properties 2..1 below show that such a process implies random growth for productivity, i.e that the growth rate of productivity is independent of its level.

**Properties 2..1** (Gibrat's Law): *The growth rate of the firm's productivity which follows the Markov chain described in assumption 2 is independent of the level of productivity (for $M_k > n_{t,k,i} > 0$):*

$$\mathbb{E}\left[\frac{\varphi_k^{n_{t+1,k,i}}}{\varphi_k^{n_{t,k,i}}}\right] = a_k\varphi_k^{-1} + b_k + c_k\varphi_k \quad \textit{and} \quad \mathbb{V}ar\left[\frac{\varphi_k^{n_{t+1,k,i}}}{\varphi_k^{n_{t,k,i}}}\right] = a_k\varphi_k^{-2} + b_k + c_k\varphi_k^2 - \rho_k^2$$

[3]This means that $\frac{\varphi_k^{n+1}}{\varphi_k^n} = \varphi_k$.

Figure 4: Productivity Process




Note: A representation of the transition probabilities in assumption 2 of a firm $i$ in sector $k$ for $M_k > n_{t,k,i} > 0$.

## 3. Sectors

In this section, I describe how the firm behavior is aggregated at the sector level. In the first part of this section, I show that, given aggregate prices and quantities, the sector equilibrium can be entirely described by two (endogenous) variables per sector: the sectors' markups and the sectors' productivities. Both are weighted average of firm-level markups and productivities respectively. Given these two variables, one can solve for the price, the size and the profit of each sector. Furthermore, under assumption 1 and given aggregate prices and quantities, I show that the sector equilibrium is characterized by two sufficient statistics per sector: the cross-sectional average firm's productivity and the cross-sectional firm's productivity concentration. Note that in this part, I am abstracting from the time subscript for clarity.

In the second part of this section, I derive the law of motion of the sectors' productivity distributions under the random growth (assumption 2). For a given sector, the productivity distribution is a stochastic object that hovers around its stationary value. Therefore, the two sufficient statistics (the cross-sectional average firm's productivity and the cross-sectional firm's productivity concentration) are also stochastic. I also characterize the law of motion of these statistics. Finally, I articulate these results and conclude.

## 3.1. Sector Level's Aggregation

In this part, I first define the sector's markup and show how it relates to moments of the sector's firm size distribution. I then solve for each sector's price and size. I show that they are related to different centrality measures of the Input-Output network. I then explain how these centralities are related to the double marginalization. Finally, under a second order approximation (assumption 1), I can solve analytically for each sector's size and price, given aggregate prices and quantities. I use these results to derive comparative statistics.

I first define the sector level markup as the sector's price divided by the sector's marginal cost. To do so, let us first look at the the total cost of sector $k$, $TC_k$, which is the sum of the total cost of firms in sector $k$: $TC_k = \sum_{i=1}^{N_k} \lambda(k,i)y(k,i) = \left(\sum_{i=1}^{N_k} N_k^{-\zeta_k\varepsilon_k}\lambda(k,i)p(k,i)^{-\varepsilon_k}P_k^{\varepsilon_k}\right)Y_k$ after substituting for the total demand faced by each sector $k$'s firm. By definition the sector $k$'s marginal cost is $\lambda_k = \frac{dTC_k}{dY_k} = \sum_{i=1}^{N_k} N_k^{-\zeta_k\varepsilon_k}\lambda(k,i)p(k,i)^{-\varepsilon_k}P_k^{\varepsilon_k}$. It follows that the sector $k$'s markup is $\mu_k = \frac{P_k}{\lambda_k} = \left(\sum_{i=1}^{N_k} N_k^{-\zeta_k\varepsilon_k}\lambda(k,i)p(k,i)^{-\varepsilon_k}P_k^{\varepsilon_k-1}\right)^{-1}$. I use the firm's pricing rule - the firm level price is equal to the firm's marginal cost times the firm's markup $\lambda(k,i) = \mu(k,i)^{-1}p(k,i)$ - and the expression of the sales share $s(k,i) = N_k^{-\zeta_k\varepsilon_k}\left(\frac{p(k,i)}{P_k}\right)^{1-\varepsilon_k}$ to find the following expression of the sector $k$'s markup:

$$\mu_k = \frac{P_k}{\lambda_k} = \left(\sum_{i=1}^{N_k} \mu(k,i)^{-1}s(k,i)\right)^{-1}$$

The sector's markup is a sales share weighted harmonic average of firm level markups[4]. This expression is valid for any firm level pricing rule that implies a markup over marginal cost $p(k,i) = \mu(k,i)\lambda(k,i)$, under monopolistic, Bertrand or Cournot competition. In the proposition 3..1, I show that the sector level markup is a function of moments of the sector's firm size distribution. Especially, I focus on the impact of the Herfindahl index on this sector's markup.

**Proposition 3..1** (Sector Level Markup): *The sector $k$'s markup is equal to*

$$\mu_k = \begin{cases} \frac{\varepsilon_k}{\varepsilon_k - 1} & \text{Under Monopolistic competition} \\ \left(1 - \frac{1}{\varepsilon_k}\sum_{n=0}^{\infty}\left(1-\frac{1}{\varepsilon_k}\right)^n HK_k^{n+1}(n+1)\right)^{-1} & \text{Under Bertrand competition} \\ \left(\frac{\varepsilon_k - 1}{\varepsilon_k} - \frac{1}{\varepsilon_k}(\varepsilon_k - 1)HHI_k\right)^{-1} & \text{Under Cournot competition} \end{cases}$$

*where* $HHI_k = \left(\sum_{i=1}^{N_k} s(k,i)^2\right)$ *is the sector $k$'s Herfindahl index, and* $HK_k(n+1) = \left(\sum_{i=1}^{N_k} s(k,i)^n\right)^{1/n}$ *is the Hannah and Kay (1977) concentration index.*

*In addition, the sector level markup is an increasing function of the sector's Herfindahl index.*

$$\frac{\partial \mu_k}{\partial HHI_k} = \begin{cases} 0 & \text{Under Monopolistic competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k^2}\mu_k^2 > 0 & \text{Under Bertrand competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\mu_k^2 > 0 & \text{Under Cournot competition} \end{cases}$$

*NB:* $HK_k(2)^2 = HHI_k$ *the Herfindahl index is the square of the second Hannah and Kay (1977) concentration index.*

The above proposition first shows that under Monopolistic competition the sector level markup is constant and equal to the firm level markups. This is

[4]Note that the sector $k$'s marginal cost is a quantity share weighted average of firm level marginal cost $\lambda_k = \sum_{i=1}^{N_k} \lambda(k,i)\frac{y(k,i)}{Y_k}$.

obvious since the sector's markup is an average of firms' markups and as under monopolistic competition all the firms in a given sector charge the same markup. Secondly, as soon as pricing becomes strategic, the sales share distribution in this sector plays a crucial role. Under Cournot competition for example, the second moment of this sector's sales share distribution, i.e the Herfindahl index, entirely determines the sector's markup. The intuition is as follows, when the sector's concentration is high, i.e the Herfindahl index is high, large firms have a higher market share and thus they can use this higher market power to charge a higher markup which in turn aggregate to a higher sector's markup.

The second part of this proposition derives some comparative statics of the markup with respect to the Herfindahl Index while keeping everything else constant. Under Bertrand and Cournot competition, a higher sector's Herfindahl index always implies a higher sector's markup. The effect is stronger for low competitive, high markup sectors. Finally the sensitivity of the sector's markup to the sector's Herfindahl index is stronger under Cournot that under Bertrand competition.

After studying the sector level markup, I describe how the sectors' prices depend on other sectors' markups and productivities. Crucially, these interdependences are driven by the input-output network.

**Proposition 3..2** (Sector's Price): *The sector's prices satisfy the following system of equations:*

$$\Big\{\log P_k\Big\}_k = (I-\Omega)^{-1}\left\{\log\left(\mu_k\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} Z_k^{\gamma_k(\alpha_k-1)}\right)\right\}_k$$

*where $\mu_k$ is the sector $k$ markup and $Z_k$ is a measure of sector $k$'s productivity defined by $Z_k^{\gamma_k(\alpha_k-1)} = \sum_{i=1}^{N_k} Z(k,i)^{\gamma_k(\alpha_k-1)}\frac{y(k,i)}{Y_k}$.*

To build intuition, let us first focus on the case where there are no input-output trade: $\Omega = 0$. In that case the sector $k$'s price is equal to the sector $k$'s markup times the sector $k$'s marginal cost of (primary) inputs $\widetilde{\lambda}_k := \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k \alpha_k} Z_k^{\gamma_k(\alpha_k-1)}$. In the presence of input-output trade, the sector $k$'s price becomes[5]:

$$\log P_k = \log \mu_k + \log \widetilde{\lambda}_k + \sum_{l=1}^{N} \omega_{k,l} \log \mu_l \widetilde{\lambda}_l + \dots$$

where I write only the impact of direct suppliers of sector $k$. The sector $k$'s price is still equal to the sector $k$'s markup times the sector $k$'s marginal cost. However, the sector $k$'s marginal cost is now equal the sector $k$'s marginal cost of primary inputs times the cost of primary inputs upstream weighted by the input shares in sector $k$'s production function $\omega_{k,l}$ (which is the intensity of sector $l$'s good used in sector $k$'s production). The latter is a function of markups in upstream sectors since sector $k$ pays the price charged by its upstream sectors which have some market power. This intuition generalizes to supplier of the suppliers of sector $k$ and so on through the following terms.

This expression captures the double marginalization between sectors. Indeed, note that under perfect competition ($\forall l, \mu_l = 1$), the sector $k$'s price is equal to:

$$\log P_k^{Perfect} = \log \widetilde{\lambda}_k + \sum_{l=1}^{N} \omega_{k,l} \log \widetilde{\lambda}_l + \dots$$

Sector $k$'s price are then equal to the sector $k$'s *technological* marginal cost which is the average of all the upstream marginal cost of primary inputs weighted by the intensity of (direct and indirect) input-output linkages. Under imperfect competition, the sector $k$'s charges a price higher than its *technological* marginal cost. The difference between the price charged and the *technological* marginal cost is $\log \mu_k + \sum_{l=1}^{N} \omega_{k,l} \log \mu_l + \dots$ which depends on upstream sectors' market powers. This capture the idea of double marginal-

[5]Note that $(I - \Omega)^{-1} = I + \Omega + \Omega^2 + \Omega^3 + \dots$.

ization.

After studying a sector's markup and price, I now turn to this sector size. I show that the sector size is determined by the sectors' markups, aggregate quantities and the input-output network.

**Proposition 3..3** (Sector's Size = Sector's Supplier Centrality): *The sector $k$'s size measure by its sales is*

$$P_kY_k = \widetilde{\beta}_k P^C C + \widetilde{\nu}_k P^I I$$

*where* $\widetilde{\beta}' = \beta' \left(I - \mu^{-1}\Omega\right)^{-1}$ *and* $\widetilde{\nu}' = \nu' \left(I - \mu^{-1}\Omega\right)^{-1}$ *are respectively the final consumption supplier centrality and the investment supplier centrality.* $\Psi^{(s)} = \left(I - \mu^{-1}\Omega\right)^{-1}$ *is the supply-side influence matrix and* $\mu^{-1} = diag(\{\mu_k^{-1}\}_k)$ *is the diagonal matrix where* $\mu_k$ *is the sector* $k$*'s markup.*

What determines sector's size? Each sector's good is either consumed or invested by the household, or is used as input by other sectors. To see that let us write the first few terms in the expression of $\widetilde{\beta}_k$ when there is no investment:

$$P_kY_k = \widetilde{\beta}'_k P^C C = \beta'_k P^C C + \sum_{l=1}^{N} \frac{\beta_l P^C C \omega_{l,k}}{\mu_l} + \ldots$$

The first term $\beta'_k$ captures the contribution to sector $k$'s sales of the household's (direct) consumption. The second term, $\sum_{l=1}^{N} \frac{\beta_l P^C C \omega_{l,k}}{\mu_l}$, captures the contribution to sector $k$'s sales of (direct) downstream sectors. The latter is determined by $\omega_{l,k}$ the share of sector $k$'s good used in the production of sector $l$ good and $\beta_l$ the household spending share on good $l$. The term $\beta_l P^C C \omega_{l,k}$ captures the demand of good $k$ that comes from the household through the sector $l$. And this intuition goes through for the customer of the customer of sector $k$ through the next term and so on and so forth.

An important thing to note is that this expression is affected by the markups of sectors downstream of sector $k$. Indeed, a sector $l$ only use a share $\frac{1}{\mu_l}$ of its revenue to pay for inputs. Thus the amount of sector $l$'s revenue that goes to sector $k$ is $\frac{\beta_l P^C C \omega_{l,k}}{\mu_l}$. When a sector charges a high markup, it keeps more profit from its sales and less is left to pay for inputs among which sector $k$ goods. This problem is somehow similar to the demand side of the double marginalization problem described above: double marginalization means that whenever a upstream sector charges a higher markup, then downstream sectors charge a higher price. Here, whenever *downstream* sector charges a higher markups, the demand of the *upstream* sectors goes down. Market power accumulates over the supply chain: upstream as demand shifter and downstream as supply shifter.

In a perfect competitive framework, Acemoglu et al. (2012) show that the sector size (in term of sales share) is determined by the Bonacich's centrality measure of that sector in the input-output network: $\beta'(I - \Omega)^{-1}$. The Bonacich's centrality is a network statistic that captures the impact of indirect connections among sectors. If sector $A$ needs inputs from sector $B$ that itself needs inputs from sector $C$, the Bonacich's centrality measure of sector $A$ takes into account the first degree connections (sector $B$ demand from sector $A$) and the second degree connections (sector $C$ demand from sector $B$ and the resulting demand from sector $A$). In a perfect competitive framework such statistic is entirely determined by the input-output structure $\Omega$. Baqaee (2016) has shown in an imperfect competition framework, similar my framework, that the size of a sector is determined by a Katz centrality measure that is related to the Bonacich's centrality but modified by sectors' markups. In Baqaee (2016), the markups are endogenous because of the entry and exit margin and thus the relevant centrality becomes also endogenous. In my paper, the relevant centrality measure is also endogenous because of the endogenous sectors' markups. However, the latter endogeneity of markups is due to the firms' heterogeneity rather than the extensive margin. Note also, that because sectors' goods can also be invested, the sector size is determined by both the final consumption supplier centrality $\widetilde{\beta}$ and the investment supplier centrality $\widetilde{\nu}$.

Finally, for completeness I solve for the sector's level profit and I show how this profit is rebated among firms.

**Proposition 3..4** (Sector's and Firm's Profit): *The sector $k$'s profit is* $pro_k = \frac{\mu_k - 1}{\mu_k}\left(\widetilde{\beta}_k P^C C + \widetilde{\nu}_k P^I I\right)$ *while the profit of firm $i$ in sector $k$ is* $pro(k,i) = \frac{\mu(k,i)-1}{\mu(k,i)}\frac{\mu_k}{\mu_k - 1}s(k,i)pro_k$ *where* $s(k,i) = \frac{p(k,i)y(k,i)}{P_k Y_k}$ *is the sales share of firm $i$ in sector $k$.*

The total profit of sector $k$ is then a function of its total sales. As show in proposition 3..3, this is determined by the supplier centralities $\widetilde{\beta}_k$ and $\widetilde{\nu}_k$. Through these centralities, the input-output structure and the market power of downstream sectors determines sectors' profits. Furthermore, the sector $k$'s market power enhances its profit. Indeed, $1 - \frac{1}{\mu_k}$ can be seen as the inverse of the effective sector $k$'s elasticity of demand[6], it also the share of sector $k$'s revenue that goes to profit. Finally, this profit is distributed among firms in in a sector depending on the sales share of that firm and its ratio between the sector wide effective demand elasticity, $\frac{\mu_k}{\mu_k - 1}$, and the firm (subjective) effective demand elasticity, $\frac{\mu(k,i)}{\mu(k,i)-1} = \varepsilon(k,i)$.

The above propositions emphasize the key role played by the sectors' markups and productivities in the determination of sectors' prices, sizes and profits. However, each sector's markup and productivity are endogenous variables, they are weighted average of their firm level counterpart. In the following of this part of this section, I now show that under a second order approximation (assumption 1) the sectors' markups and the productivities are entirely pinned down by two sector level statistics: the cross-sectional average productivity and the cross-sectional productivity Herfindahl Index.

[6]For the monopolistic case, where firms charge constant markup, $\frac{\mu_k - 1}{\mu_k} = \frac{1}{\varepsilon_k}$.

**Proposition 3..5** (Sector under Assumption 1): *Under assumption 1 the sectors' prices are equal to:*

$$\{\log P_k\}_k =$$
$$(I-\Omega)^{-1}\left\{\log\left(\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} N_k^{\zeta_k\frac{\varepsilon_k}{\varepsilon_k-1}}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\Delta_k\right)\right)^{\frac{1}{\varepsilon_k-1}}\right)\right\}_k$$

*the sector $k$'s markup is equal to:*

$$\mu_k = \frac{\varepsilon_k}{\varepsilon_k - f_k\left(\Delta_k\right)}$$

*while the sector $k$'s productivity is equal to:*

$$Z_k^{\gamma_k(\alpha_k-1)} = N_k^{\zeta_k\frac{\varepsilon_k}{\varepsilon_k-1}}\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\Delta_k\right)\right)^{\frac{1}{\varepsilon_k-1}}\left(\frac{\varepsilon_k - f_k\left(\Delta_k\right)}{\varepsilon_k-1}\right)$$

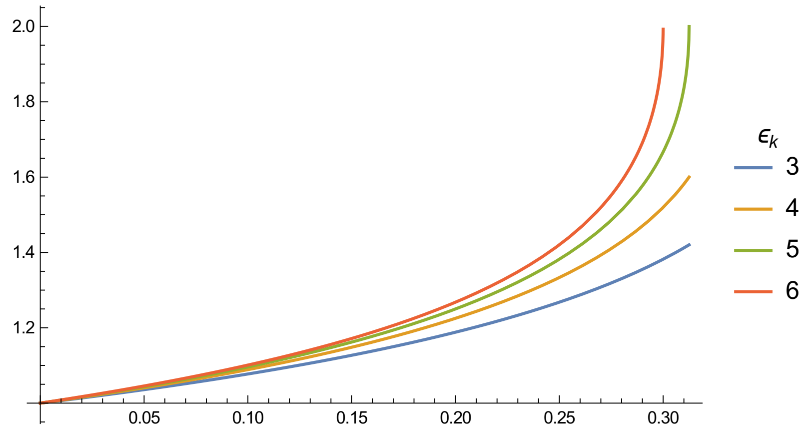
*where*

$$f_k(x) = \begin{cases} 1 & \textit{Under Monopolistic} \\ \frac{1-\sqrt{1-4\left(1-\frac{1}{\varepsilon_k}\right)x}}{2\left(1-\frac{1}{\varepsilon_k}\right)x} \quad \textit{for } x\in\left[0,\frac{1}{4\left(1-\frac{1}{\varepsilon_k}\right)}\right] & \textit{Under Bertrand} \\ \frac{1-\sqrt{1-4(\varepsilon_k-1)x}}{2(\varepsilon_k-1)x} \quad \textit{for } x\in[0,\frac{1}{4(\varepsilon_k-1)}] & \textit{under Cournot} \end{cases}$$

*and where $\Delta_k = \left(\frac{\overline{Z_k^{(2)}}}{\overline{Z_k^{(1)}}}\right)^2$ is a concentration measure: the productivity Herfindahl index while $\overline{Z_k^{(n)}} = \left(\sum_i^{N_k} Z(k,i)^{n(\varepsilon_k-1)\gamma_k(1-\alpha_k)}\right)^{\frac{1}{n}}$ is the $n$th moment of the sector $k$'s productivity distribution.*

First note that $\Delta_k = \left(\frac{\overline{Z_k^{(2)}}}{\overline{Z_k^{(1)}}}\right)^2 = \sum_i^{N_k}\left(\frac{Z(k,i)^{2(\varepsilon_k-1)\gamma_k(1-\alpha_k)}}{\overline{Z_k^{(1)}}}\right)^2$ where $\overline{Z_k^{(1)}}$ is equal to $\overline{Z_k^{(1)}} = \sum_i^{N_k} Z(k,i)^{(\varepsilon_k-1)\gamma_k(1-\alpha_k)}$. Thus $\Delta_k$ is the sum of the sector $k$'s firm productivity share squared, in other words this is the sector $k$'s productivity Herfindahl index. Higher $\Delta_k$ implies a higher dispersion of productivity i.e.

Figure 5: Deviation from Monopolistic Competition




Note: Bertrand Case, $f_k : x \mapsto \frac{1-\sqrt{1-4(1-1/\varepsilon_k)x}}{2(1-1/\varepsilon_k)x}$ for different value of $\varepsilon_k$. For the Cournot case, see appendix.

a higher concentration. As one can see in proposition 3..5 this concentration measure determines the distortion introduced by the strategic pricing through the term $f_k(\Delta_k)$.

It easy to show that when $\Delta_k \to 0$, we have $f_k(\Delta_k) \to 1$ for the Bertrand or the Cournot case. When the productivity heterogeneity across firms in a sector is going to zero the Bertrand (resp. Cournot) case nests the monopolistic case: sector's price, markup and productivity converge to their monopolistic counterpart. This is very intuitive, since firms are strategic and use their relative market power in order to influence the sector price and extract more profit. If firms are all identical, firms cannot do so and we are back to the Monopolistic Dixit and Stiglitz (1977) case.

In the figure 5, I plot the function $x \mapsto f_k(x)$ for different values of $\varepsilon_k$. One can see that this function takes values above one and is increasing. In other words, a higher heterogeneity within a sector creates a higher deviation from the monopolistic case.

The term $\left(f_k\left(\Delta_k\right)\right)^{\frac{-1}{\varepsilon_k-1}}$ captures the effect of the firm strategic pricing on the sector level price. This term *does not* only capture the change of sector level markup but it also captures the change in the productivity implied by the strategic pricing. A higher concentration implies that $(i)$ the sector is less competitive, i.e charges a higher markup and $(ii)$ is more productive. As is shown in corollary 3..1. If the concentration is higher in a sector, large firms can extract more profit by using their market power to influence the sector's price. However, in a more concentrated sector, large firms are even larger compared to the monopolistic case. These large firms use more primary input in a more efficient way. The former of these two opposite effects dominates as shown in the corollary below.

**Corollary 3..1** (Concentration, Competition and Productivity): *Under assumption 1, a sector with higher concentration is less competitive but more productive. Formally, keeping the average productivity $\overline{Z_k^{(1)}}$ constant, we have*

$$\left.\frac{d\mu_k}{d\Delta_k}\right|_{\overline{Z_k^{(1)}}} > 0 \quad \textit{and} \quad \left.\frac{dZ_k}{d\Delta_k}\right|_{\overline{Z_k^{(1)}}} > 0 \quad \textit{and} \quad \left.\frac{d\left(\mu_k Z_k^{\gamma_k(\alpha_k-1)}\right)}{d\Delta_k}\right|_{\overline{Z_k^{(1)}}} < 0$$

Finally note that the deviation from monopolistic competition affects other sectors through the input-output matrix $\Omega$. In corollary 3..2, I describe how an increase in concentration of a sector affects other sectors' centrality.

**Corollary 3..2** (Concentration and Supplier Centrality): *Under assumption 1, the sector $l \neq k$'s centrality is non-increasing in the sector $k$'s concentration.*

*Formally, keeping the average productivity $\overline{Z_k^{(1)}}$ constant, we have*

$$\left.\frac{d\log\widetilde{\beta}_l}{d\log\Delta_k}\right|_{\overline{Z_k^{(1)}}}=-\left(\Psi^{(s)}_{k,l}-\mathbb{I}_{k,l}\right)\frac{f_k(\Delta_k)}{\varepsilon_k-f_k(\Delta_k)}e_k$$

*and*

$$\left.\frac{d\log\widetilde{\nu}_l}{d\log\Delta_k}\right|_{\overline{Z_k^{(1)}}}=-\left(\Psi^{(s)}_{k,l}-\mathbb{I}_{k,l}\right)\frac{f_k(\Delta_k)}{\varepsilon_k-f_k(\Delta_k)}e_k$$

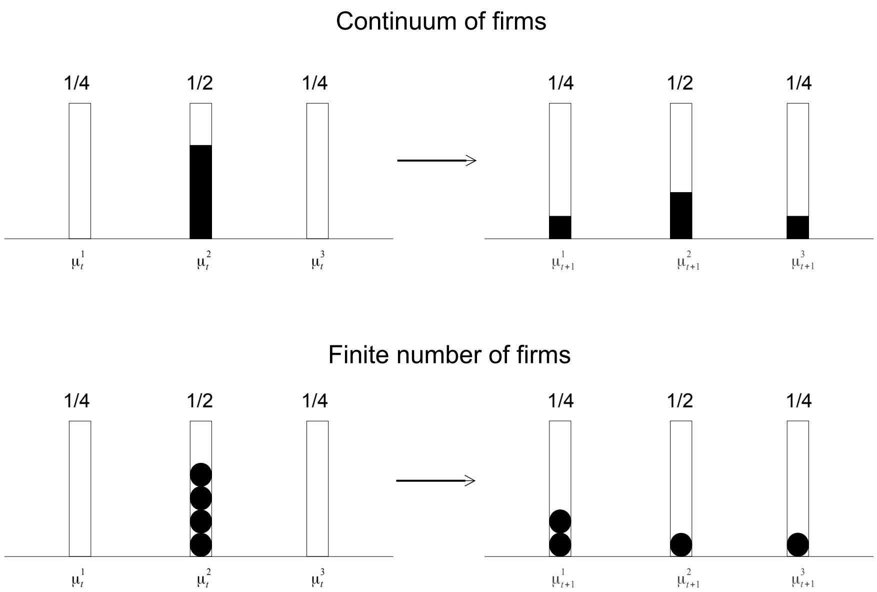
*where $e_k=\frac{d\log f_k(\Delta_k)}{d\log\Delta_k}$ is the elasticity of $f_k$ to the sector $k$'s concentration index $\Delta_k$. $\Psi^{(s)}=(I-\mu^{-1}\Omega)$ is the supplier influence matrix and $\mathbb{I}_{k,l}=1$ if $k=l$ and zero otherwise.*

Note that $\frac{f_k(\Delta_k)}{\varepsilon_k-f_k(\Delta_k)}e_k$ is the elasticity of the sector $k$'s markup to sector $k$'s concentration $\frac{d\log\mu_k}{d\log\Delta_k}$. For $l\neq k$, the expression becomes $\left.\frac{d\log\widetilde{\beta}_l}{d\log f_k(\Delta_k)}\right|_{\overline{Z_k^{(1)}}}=-\frac{d\log\mu_k}{d\log\Delta_k}\Psi^{(s)}_{k,l}\leq 0$. In words, the centrality of other sectors decreases when concentration in a sector $k$ increases while its cross-sectional average productivity stays the same. The intuition is as follows: when the concentration in sector $k$ increases, the sector $k$'s markup increases (see corollary 3..1); sector $k$ demands less intermediate inputs to achieve the same sales; thus if the sector $l$ is upstream of sector $k$ (measured by $\Psi^{(s)}_{k,l}$), sector $l$ suffers a decrease in its sales share, i.e it supplier centrality decreases. The effect is stronger $(i)$ if sector $k$ is an important (direct or indirect) consumer of sector $l$'s good ($\Psi^{(s)}_{k,l}$ is high) or $(ii)$ sector $k$'s markup is very sensitive to the concentration ($\frac{d\log\mu_k}{d\log\Delta_k}$ is high).

## 3.2. Sector Dynamics

In this subsection, I describe the evolution of a generic sector's productivity distribution. Let us define the vector $g_t^{(k)}=\{g_{t,n}^{(k)}\}_{0\leq n\leq M}$ where $g_{t,n}^{(k)}$ is the number of firms at productivity level $\varphi^n$ at time $t$ in sector $k$. The vec-

Figure 6: An illustrative example of the productivity distribution dynamics




Note: Top panel, with a continuum of firms we have a deterministic transition. Bottom Panel, with a finite number of firms we have a stochastic transition.

tor $g_t^{(k)}$ is thus the firm's productivity distribution in sector $k$. Recall that in sector $k$ there is an integer number of firms $N_k$. Following Carvalho and Grassi (2015), this assumption implies that the productivity distribution is a stochastic object. To understand the intuition, let us study a simple example.

Assume there are only three levels of productivity and four firms. At time period $t$ these firms are distributed according to the bottom-left panel of Figure 6, i.e. all four firms produce with the intermediate level of productivity. Further assume that these firms have an equal probability of $1/4$ of going up or down in the productivity ladder and that the probability of staying at the same intermediate level is $1/2$. That is, the transition probabilities are given by $(1/4, 1/2, 1/4)'$. First note that, if instead of four firms we had assumed a continuum of firms, the law of large numbers would hold such that at $t+1$ there would be exactly $1/4$ of the (mass of) firms at the highest level of productivity, $1/2$ would remain at the intermediate level and $1/4$ would transit to the lowest level of productivity (top panel of Figure 6). This is not the case here, since the number of firms is finite. For instance, a distribution of firms such as the one presented in the bottom-right panel of Figure 6 is possible with a positive probability. Of course, many other arrangements would also be possible outcomes. Thus, in this example, the number of firms in each productivity bin at $t+1$ follows a multinomial distribution with a number of trials of $4$ and an event probability vector $(1/4, 1/2, 1/4)'$.

In this simple example, all firms are assumed to have the same productivity level at time $t$. It is easy however to extend this example to any initial arrangement of firms over productivity bins. Indeed, for any initial number of firms *at a given* productivity level, the distribution of *these* firms across productivity levels next period follows a multinomial. Therefore, the *total* number of firms in each productivity level next period, is simply a sum of multinomials, i.e. the result of transitions from *all* initial productivity bins. The following proposition generalizes this example.

**Proposition 3..6** (Sector $k$'s Productivity Distribution Dynamics): *Under assumption 2, the Sector $k$'s Productivity Distribution satisfies the following law of motion*

$$g_{t+1}^{(k)} = (\mathcal{P}^{(k)})' g_t^{(k)} + \epsilon_t^{(k)}$$

*where $\epsilon_t^{(k)} = \left\{\epsilon_{t,n}^{(k)}\right\}_{0 \le n \le M}$ is a mean zero random vector with the following* *variance-covariance structure:*

$$\mathbb{V}ar_t\left[\epsilon^{(k)}_{t,n}\right] = \begin{cases} a_k(1-a_k)g^{(k)}_{t,n+1} + b_k(1-b_k)g^{(k)}_{t,n} + c_k(1-c_k)g^{(k)}_{t,n-1} & \textit{for } n > 0 \\ (1-c_k)c_k g^{(k)}_{t,0} + (1-a_k)a_k g^{(k)}_{t,1} & \textit{for } n = 0 \\ (1-a_k)a_k g^{(k)}_{t,M} + (1-c_k)c_k g^{(k)}_{t,M-1} & \textit{for } n = M \end{cases}$$

$$\mathbb{C}ov_t\left[\epsilon^{(k)}_{t,n};\epsilon^{(k)}_{t,n'}\right] = \begin{cases} 0 & \textit{for } |n-n'| > 2 \\ -b_k c_k g^{(k)}_{t,n} - a_k b_k g^{(k)}_{t,n+1} & \textit{for } n' = n+1 \\ -a_k c_k g^{(k)}_{t,n+1} & \textit{for } n' = n+2 \end{cases}$$

$$\mathbb{C}ov_t\left[\epsilon^{(k)}_{t,0};\epsilon^{(k)}_{t,n'}\right] = \begin{cases} 0 & \textit{for } n' > 2 \\ -(1-c_k)c_k g^{(k)}_{t,0} - a_k b_k g^{(k)}_{t,1} & \textit{for } n' = 1 \\ -a_k c_k g^{(k)}_{t,1} & \textit{for } n' = 2 \end{cases}$$

$$\mathbb{C}ov_t\left[\epsilon^{(k)}_{t,M};\epsilon^{(k)}_{t,n'}\right] = \begin{cases} 0 & \textit{for } n' < M-2 \\ -b_k c_k g^{(k)}_{t,M-1} - a_k(1-a_k)g^{(k)}_{t,M} & \textit{for } n' = M-1 \\ -a_k c_k g^{(k)}_{t,M-1} & \textit{for } n' = M-2 \end{cases}$$

**Stationary Distribution** The dynamics of the firms' productivity distribution within a sector implies that the distribution hovers around a stationary distribution. This stationary distribution is given by the stationary distribution of the Markovian process that firms' productivity follows in that sector. Here I solve for this object. Note that, since in the model this distribution is the state variable, the stationary distribution is the steady state of this economy.

**Proposition 3..7** (Sector $k$'s Productivity Stationary Distribution): *Under* *assumption 2 and if $a_k < c_k$ then the stationary distribution of firm level productivity in sector $k$ is Pareto and equal to $g_n^{(k)} = N_k \frac{1-\varphi_k^{-\delta_k}}{1-\left(\varphi_k^{M_k+1}\right)^{-\delta_k}} (\varphi_k^n)^{-\delta_k}$ where $\delta_k = \frac{\log \frac{a_k}{c_k}}{\log \varphi_k}$ is the tail index.*

The above proposition shows that the stationary distribution is Pareto with tail $\delta_k$. It is a well established fact that random growth process generates Pareto distribution when there is some perturbation for low productivity level (see Gabaix, 1999 or Gabaix, 2009 for a review). In the context of assumption 2, this result is due to Córdoba (2008).

**Dynamics of Moments of the Productivity Distribution** In the previous proposition, I described the dynamics of the sector's productivity distribution, the state variable of this economy. Propositions 3..3 and 3..5 show that to solve for the sectors' prices and quantities, only two moments of the sectors' productivity distributions are needed, namely the cross-sectional average productivity $\overline{Z_{t,k}^{(1)}}$ and the cross-sectional productivity Herfindahl index, $\Delta_{t,k}$. The proposition 3..8 below describes the dynamics of these two sectors level sufficient statistics.

**Proposition 3..8** (Dynamics of $\overline{Z_{t,k}^{(1)}}$ and $\Delta_{t,k}$): *Under random growth (assumption 2), the moments $\overline{Z_{t,k}^{(1)}}$ and $\Delta_{t,k}$ of the sector $k$'s productivity distribution satisfy the following dynamics:*

$$\frac{\overline{Z_{t+1,k}^{(1)}}}{\overline{Z_{t,k}^{(1)}}} = \rho_k^{(1)} + \frac{o_{t,k}^{M,(1)}}{\overline{Z_{t,k}^{(1)}}} + \sqrt{\varrho_k^{(1)} \Delta_{t,k} + o_{t,k}^{\sigma,(1)}} \quad \varepsilon_{t+1,k}^{(1)}$$

$$\left(\frac{\overline{Z_{t+1,k}^{(1)}}}{\overline{Z_{t,k}^{(1)}}}\right)^2 \frac{\Delta_{t+1,k}}{\Delta_{t,k}} = \rho_k^{(2)} + \frac{o_{t,k}^{M,(2)}}{\Delta_{t,k}} + \sqrt{\varrho_k^{(2)} \kappa_{t,k} + o_{t,k}^{\sigma,(2)}} \quad \varepsilon_{t+1,k}^{(2)}$$

*where $\varepsilon_{t+1,k}^{(1)}$ and $\varepsilon_{t+1,k}^{(2)}$ are random variables following a $\mathcal{N}(0,1)$ with a covari-*

*ance*

$$\mathbb{C}ov_t\left[\varepsilon^{(1)}_{t+1,k};\varepsilon^{(2)}_{t+1,k}\right]=\frac{\overline{\varrho_k}Skew_{t,k}+o^{C,(2)}_{t,k}}{\left(\varrho^{(1)}_k\Delta_{t,k}+o^{\sigma,(1)}_{t,k}\right)^{1/2}\left(\varrho^{(2)}_k\kappa_k+o^{\sigma,(2)}_{t,k}\right)^{1/2}}$$

*Where,* $\kappa_{t,k}=\frac{\left(\overline{Z^{(4)}_{t,k}}\right)^4}{\left(\overline{Z^{(2)}_{t,k}}\right)^4}$ *and* $Skew_{t,k}=\frac{\left(\overline{Z^{(3)}_{t,k}}\right)^3}{\left(\overline{Z^{(2)}_{t,k}}\right)^2\left(\overline{Z^{(1)}_{t,k}}\right)}$ *while* $o^{M,(1)}_{t,k},o^{M,(2)}_{t,k},o^{\sigma,(1)}_{t,k},o^{\sigma,(2)}_{t,k}$ *and* $o^{C,(2)}_{t,k}$ *are predeterminated at time* $t+1$*. The constants* $\rho^{(n)}_k=a_k\varphi_k^{-n(\varepsilon_k-1)\gamma_k(1-\alpha_k)}+b_k+c_k\varphi_k^{n(\varepsilon_k-1)\gamma_k(1-\alpha_k)}$ *and* $\varrho^{(n)}_k=a_k\varphi_k^{-2n(\varepsilon_k-1)\gamma_k(1-\alpha_k)}+b_k+c_k\varphi_k^{2n(\varepsilon_k-1)\gamma_k(1-\alpha_k)}-(\rho^{(n)}_k)^2$ *are respectively the mean and variance of the growth rate of firm* $i$ *in sector* $k$ *productivity measure* $Z(k,i)^{n(\varepsilon_k-1)\gamma_k(1-\alpha_k)}$.

Proposition 3..8 is similar to the theorem 2 in Carvalho and Grassi (2015). It shows that the dynamics of these moments of the sector $k$'s productivity distribution are persistent, and that the persistent parameters $\rho^{(1)}_k$ and $\rho^{(2)}_k$ depend on the firm level productivity process through $a_k$, $b_k$ and $c_k$. The intuition is that since the firm level productivity is itself persistent, this persistence is aggregated at the sector level. The higher is the firm level persistence, higher is the sector level persistence as shown in Carvalho and Grassi (2015).

Moreover, the (conditional) variance of the growth rate of the sector $k$'s cross-sectional average productivity, $\overline{Z^{(1)}_{k,t+1}}$, is time varying and is determined by the concentration $\Delta_{t,k}$. Here as in Gabaix (2011) and Carvalho and Grassi (2015), any volatility of the sector level productivity is due to idiosyncratic shocks at the firm level. When a sector is concentrated, shocks to large firms do not wash out at the aggregate level. When the concentration is high at the sector level, i.e when large firms are more important, shocks to these large firms generate larger movements in the average. Thus a higher concentration implies larger large firms and thus more volatility of the cross-sectional mean due to idiosyncratic shocks.

Proposition 3..8 also shows that the growth rate (normalized) of the cross-sectional productivity Herfindahl index, $\Delta_{t,k}$, is normally distributed with a standard error governed by $\kappa_{k,t} = \left(\overline{Z_{t,k}^{(4)}}\right)^4 / \left(\overline{Z_{t,k}^{(2)}}\right)^4$. This statistic is the (empirical) Kurtosis of the sector $k$'s productivity distribution. Intuitively, the Kurtosis measures how much of the variance is due to extreme events. Therefore a higher Kurtosis implies a higher volatility of the variance, since a higher Kurtosis implies more extremely large firms.

## 3.3. Taking Stock

In this section, the firm-level behavior are aggregated at the sector-level. Proposition 3..3 and 3..2 show how the sectors' markups and productivities affect the sales share and the price of other sectors. These results also emphasize the role played by the double marginalization and how it affects sector prices and sales share.

Under a second order approximation (assumption 1) of the firm level behavior, proposition 3..5 shows that both the sector level markup and productivity are function of only two moments of the sector's cross-sectional productivity distribution, namely $\overline{Z_{t,k}^{(1)}}$, the cross-sectional average, and $\Delta_{t,k}$, the cross-sectional productivity Herfindahl index. The former measures how much a representative firm is productive, while the latter measures how strongly the sector is concentrated. While the average productivity does not affect the sector level markup, the concentration does. Given these two statistics (and aggregate prices and quantities), the sector prices and quantities are entirely analytically solved for.

In order to characterize a given sector dynamics, I need to characterize how the sector's productivity distribution evolves. Under random growth (assumption 2), the law of motion of a sector's productivity distribution turns out to be very tractable. Proposition 3..6 shows that this distribution is a stochastic object and describes its law of motion. The sector's productivity distribution hovers around its stationary distribution which is solved closed form in proposition 3..7. Since the sector's productivity distribution is stochastic, it implies that its moments are also stochastic. Since the two sufficient statistics $\overline{Z_{t,k}^{(1)}}$ and $\Delta_{t,k}$ are moments of the productivity distribution, sector's price and quantity are themselves random variables. To completely describe the dynamics of a sector, proposition 3..8 describes the evolution of these two sufficient statistics: $\overline{Z_{t,k}^{(1)}}$, the cross-sectional average, and $\Delta_{t,k}$, the cross-sectional Herfindahl index.

## 4. Equilibrium

In this section, I describe the factors' markets clearing conditions. Finally I show that under some assumptions, the within-sector firm level heterogeneity can be entirely summarized by the two sufficient statistics $\overline{Z_{t,k}^{(1)}}$ and $\Delta_{t,k}$. It follows that the equilibrium of this economy can be defined at the sector level.

First let us write the factors' markets clearing conditions.

**Proposition 4..1** (Factor's Market Clearing Conditions): *The labor market clearing condition is* $L = w^{-1}\sum_{k=1}^{N}(1-\alpha_k)\gamma_k\mu_k^{-1}\left(\widetilde{\beta}_k P^C C + \widetilde{\nu}_k P^I I\right)$ *and the capital market clearing condition is* $K = r^{-1}\sum_{k=1}^{N}\gamma_k\alpha_k\mu_k^{-1}\left(\widetilde{\beta}_k P^C C + \widetilde{\nu}_k P^I I\right)$

In proposition 4..1, note that the sectors' supplier centralities $\widetilde{\beta}_k$ and $\widetilde{\nu}_k$ determine the factor demand. It is because these centralities are a measure of the sector size as shown in proposition 3..3. The primary input shares $\gamma_k$ and the share of each factors $\alpha_k$ also determine the factor demand. Finally the sector's markup $\mu_k$ determines the share of revenue that is used to pay inputs. By Walras law, only one of the market clearing condition is enough to solve for the equilibrium. Note that the right hand side of the factor market clearing condition depends only on sectors' markups either directly or through the centrality measures $\widetilde{\beta}_k$ and $\widetilde{\nu_k}$. Under assumption 1, the sector level markup is only a function of the sector productivity Herfindahl index $\Delta_{t,k}$. Once again, one need to know this statistic for each sector to compute the factor market equilibrium condition.

It is now possible to define the equilibrium of this economy at the sector level. Let us normalize the price of the composite consumption good $P_t^C = 1$.

**Definition 4..1** (Equilibrium at the Sector Level):

*A* Monopolistic *(resp. Bertrand, Cournot) equilibrium at the sector level, under assumptions 1 and 2 is a set of sequences of aggregate prices* $(w_t, r_t, q_t, P_t^I)$*, sector level prices* $(P_{t,k})$*, aggregate quantities* $(C_t, I_t, K_t, L_t)$*, sector level quantities* $(C_{t,k}, I_{t,k}, Y_{t,k}, \mu_{t,k})$ *and sector's productivity distribution* $(g_t^{(k)})$ *such that:*

1. *The representative consumer behavior satisfies by the capital accumulation, the labor supply, the intertemporal choice, the intra-temporal allocation among sectors (proposition 2..1: 1 and 2)*
2. *Sectors' prices and markups satisfy proposition 3..5*
3. *Sectors' Sales shares satisfy proposition 3..3*
4. *Markets for labor and capital clear (proposition 4..1)*
5. *Each sector's productivity distribution evolves according to proposition 3..6*

There are 4 equations for the intertemporal allocation of households (Law of motion of capital, and proposition 2..1: 1 and 2), $2 \times N$ equations for the intratemporal allocation of consumption goods and investment goods at the sector level, 2 price indices' equations for the final consumption and the investment goods (proposition 2..1), $N$ equations for the sectors' markups (proposition 3..5), $N$ for the sectors' sizes (proposition 3..3), $N$ equations for the sectors' prices (proposition 3..5), and $N$ equations on the evolution of the sectors' productivity distributions (proposition 3..6) and one equation for the household budget constraint. Finally there are 2 factor market clearing conditions (proposition 4..1), one of which is redundant by Walras law. The total number of equations is then $6 \times N + 8$. The number of variables is as follows: aggregate prices $(w_t, r_t, q_t, P_t^I)$, sector level prices $(P_{t,k})_k$, aggregate quantities $(K_t, I_t, L_t, C_t)$ and sector-level quantities $(C_{t,k}, I_{t,k}, Y_{t,k}, \mu_{t,k})_k$ and the sectors' productivity distributions $g_t^{(k)}$, hence $6 \times N + 8$ variables.

## 5. A Special Case

In this section, I study the special case where the only primary input is labor $\forall k, \alpha_k = 0$ and where $\forall k, \zeta_k = 0$. I also simplify the household problem by setting $\theta$ to one. With these assumptions, the intertemporal choice of the household is no longer relevant for the equilibrium. The model becomes a repeated static economy. In this case, I can solve analytically for the aggregate consumption and the wage. I then use this framework to build intuition and derive clear comparative statics. In this section, I first show the solution for the aggregate consumption and wage. I then derive comparative statics and define a statistic that summarizes the impact of firm level shocks on the economy. Finally, I study some examples of network structure to illustrate their impact on the propagation of firm level shocks across sectors and on the aggregate.

The tractability of this special case allows to calibrate the model using only a few parameters. I use this calibration to quantify the elasticity of the aggregate profit, wage and output with respect to concentration. I first assume that the elasticity of substitution across varieties in a sector is equal to 5 for all sectors. Using the detailed I-O table for 2007 of Bureau of Economic Analysis described in the data appendix, I recover the matrix $\Omega$ and the household spending share $\beta_k$ for each sector. Using the concentration data of the Census Bureau which give for each manufacturing sector the Herfindahl-Hirschman-Index (HHI) for the 50 largest firms, I compute the implied value of the productivity concentration $\Delta_k$ by inverting the formula $HHI_k = \frac{f_k(\Delta_k)-1}{1-1/\varepsilon_k}$ for each sector. Unfortunately, the HHI is only available for manufacturing sectors, I then assume for the other sector Dixit-Stiglitz competition. Finally, I choose a value of the labor supply elasticity $\chi$ of 2 and a value of $\eta$, the relative risk aversion, of 1.5 in lines with usual assumption in the business cycle literature.

## 5.1. Role of Concentration

To solve for the aggregate, proposition 5..1 solves for the aggregate profit and show how labor income and profit are distributed.

**Proposition 5..1** (Aggregate Profit): *When $\forall k, \alpha_k = 0$ and $\forall k, \zeta_k = 0$, the profit and labor income shares are:*

$$\frac{Pro_t}{P_t^C C_t} = \widetilde{\beta}_t' \left(\frac{\mu_t - 1}{\mu_t}\right) \quad \textit{and} \quad \frac{w_t L_t}{P_t^C C_t} = 1 - \widetilde{\beta}_t' \left(\frac{\mu_t - 1}{\mu_t}\right)$$

*where $\widetilde{\beta}_t = (I - \mu_t^{-1}\Omega)^{-1}$, $\mu_t^{-1} = diag(\{\mu_{t,k}\}_k)$ and $\frac{\mu_t - 1}{\mu_t}$ is the vector $\{\frac{\mu_{t,k}-1}{\mu_{t,k}}\}_k$.*

Proposition 5..1 shows that the profit income share $\frac{Pro}{P^C C} = \widetilde{\beta}' \left(\frac{\mu-1}{\mu}\right)$ is only a function of the sectors' markups. Under assumption 1, these markups are entirely determined by the sectors' concentrations since $\mu_{t,k} = \frac{\varepsilon_k}{\varepsilon_k - f_k(\Delta_{t,k})}$ (see proposition 3..5). This is very intuitive, since the markup determines how much of the sector's income is distributed across profit and inputs.

First, note that a change in the cross-sectional average productivity in sector $k$, $\overline{Z_{t,k}^{(1)}}$, does not affect the profit income share. Indeed such a shock affects all the firms in sector $k$ in the same way, and thus does not increase the market power of one versus the others. Thus the sectors' markups and profits are not affected. However, a change in sector $k$'s concentration - measure by the productivity Herfindahl index, $\Delta_{t,k}$ - does affect the relative market power of firms in sector $k$. This change in relative market power affects the sector $k$'s markup, which in turn affects sales and profits of others sectors through the input-output network. Formally, I show that

$$\frac{\partial\left(\frac{Pro_t}{P_t^C C_t}\right)}{\partial \Delta_{t,k}} = \mu_{t,k}\widetilde{\beta}_{t,k}\left(1-\sum_{l=1}^{N}\Psi_{k,l}^{(s)}\frac{\mu_{t,l}-1}{\mu_{t,l}}\right)\frac{f_k'(\Delta_k)}{\varepsilon_k}$$
$$\frac{\partial\left(\log\frac{Pro_t}{P_t^C C_t}\right)}{\partial \log \Delta_{t,k}} = \mu_{t,k}\frac{pro_k}{Pro}\left(1-\sum_{l=1}^{N}\Psi_{k,l}^{(s)}\frac{\mu_{t,l}-1}{\mu_{t,l}}\right)e_k$$

where $\Psi^{(s)} = (I-\mu_t^{-1}\Omega)^{-1}$ is the supplier influence matrix. To understand the intuition, let us focus on the case where there is no input-output trade, $\Omega = 0$ then $\Psi^{(s)} = I$ and $\widetilde{\beta}_{t,k} = \beta_k$. In that case $\frac{d\left(\frac{Pro_t}{P_t^C C_t}\right)}{d\Delta_{t,k}} = \frac{\beta_k f_k'(\Delta_{k,t})}{\varepsilon_k}$. The change of profit is governed by the importance of that sector for household consumption and the elasticity of substitution in that sector, which measures how much this sector's markup is sensitive to concentration[7]. The larger the sector is (as measured by the household spending share $\beta_k$), and the more the markup is sensitive to concentration, the higher the change in profit share is.

For the case where $\Omega \neq 0$, the same intuition applies. The importance of a sector is now measured by the supplier centrality measure $\widetilde{\beta}_{t,k}$ (which is equal to the sales share of that sector by proposition 3..3) , while the sensitivity of the sector $k$'s markup to the concentration is still measured by $\frac{f_k'(\Delta_k)}{\varepsilon_k}$. Here, a change in markup in sector $k$ also affects its payment for intermediate inputs to upstream sectors. The intensity of that change is governed

[7]Note that $\frac{d\mu_k^{-1}}{d\Delta_k} = \frac{-1}{\varepsilon_k} f_k'(\Delta_k)$.

by the share of intermediate inputs (direct and indirectly) used in the production of sector $k$'s good. This is measured by the term $(k, l)$ of the supplier influence matrix: $\Psi^{(s)}_{k,l}$. If the concentration of a sector using intensively intermediate inputs (directly and indirectly) increases then its demand of upstream sectors' goods is reduced, which in turn decreases the profits of upstream sectors. The total effect on aggregate profit share of this increase of concentration is thus reduced. I define the markup centrality that summarized, for a given sector, the market power of its upstream sectors.

**Definition 5..1** (Markup Centrality): *The markup centrality is defined as:*

$$\widetilde{\mu}_{k,t} = \frac{\widetilde{\varepsilon}_k}{\widetilde{\varepsilon}_k - 1}$$

*where* $\widetilde{\varepsilon}^{-1}_k$ *is the* $k$*th element of the vector* $\widetilde{\varepsilon}^{-1}$ *defined as*

$$\widetilde{\varepsilon}^{-1} = (I - \mu^{-1}\Omega)^{-1} \left\{ \frac{f_k(\Delta_k)}{\varepsilon_k} \right\}_k = \sum_{l=1}^{N} \Psi^{(s)}_{k,l} \frac{\mu_{t,l} - 1}{\mu_{t,l}}$$

With this definition of the markup centrality, the effect of concentration on profit share can be rewritten as:

$$\frac{\partial \left( \frac{Pro_t}{P^C_t C_t} \right)}{\partial \Delta_{t,k}} = \widetilde{\beta}_{t,k} \frac{\mu_{t,k}}{\widetilde{\mu}_{k,t}} \frac{f'_k(\Delta_k)}{\varepsilon_k} \quad \text{or} \quad \frac{\partial \left( \log \frac{Pro_t}{P^C_t C_t} \right)}{\partial \log \Delta_{t,k}} = \frac{pro_k}{Pro} \frac{\mu_{t,k}}{\widetilde{\mu}_{k,t}} e_k \tag{2}$$

where the term $\frac{\mu_{t,k}}{\widetilde{\mu}_{k,t}}$ measures the markup of a sector relative to the markup centrality i.e. the markup of its upstream sectors. This term summarizes the implied change of profit in sector $k$'s upstream sector following the increase in sector $k$'s concentration. If the concentration in a sector were a policy instrument, and if the government wanted to reduce the income share of profit, then that government should reduce concentration in the large sectors (as measured by its profit share) that have a high markup relative to

Table 1: 10 Highest Value of $\frac{\partial\left(\log \frac{Pro}{PC_C}\right)}{\partial \log \Delta_k}$

| Rank | Description | (1) $\frac{\partial\left(\log \frac{Pro}{PC_C}\right)}{\partial \log \Delta_k}$ | (2) $\frac{pro_k}{Pro}$ | (3) $\frac{\mu_k}{\bar{\mu}_k}$ | (4) $e_k = \frac{\partial \log f_k(\Delta_k)}{\partial \log \Delta_k}$ |
|---|---|---|---|---|---|
| 1 | Petroleum refineries | 0.1825 | 3.2022 | 82.0782 | 6.9436 |
| 2 | Pharmaceutical preparation | 0.09641 | 2.379 | 91.0579 | 4.4504 |
| 3 | Automobile | 0.091837 | 0.82031 | 79.1659 | 14.1417 |
| 4 | Distilleries | 0.043212 | 0.19743 | 93.3785 | 23.4397 |
| 5 | Dog and cat food | 0.042521 | 0.21589 | 80.4335 | 24.4871 |
| 6 | Animal (except poultry) slaughtering, [...] | 0.037498 | 0.93887 | 73.5278 | 5.4318 |
| 7 | Breakfast cereal | 0.034594 | 0.13358 | 85.3834 | 30.3305 |
| 8 | Computer terminals and other computer [...] | 0.033749 | 0.20891 | 89.6474 | 18.0203 |
| 9 | Soap and cleaning compound | 0.033614 | 0.35909 | 89.2162 | 10.4924 |
| 10 | Soft drink and ice | 0.030435 | 0.52436 | 79.4411 | 7.3063 |

Note: $\varepsilon_k = 5$. Columns (1),(2),(3) and (4) are percentage points. Source: Bureau of Economic Analysis (detailed I-O table for 2007) and Census Bureau (Herfindahl-Hirschman index for the 50 largest firms). Only Manufacturing 31-33 industries. See Data Appendix for more details.

their markup centrality. Table 1 displays the 10 sectors where the elasticity of the aggregate profit share with respect to concentration is the highest. A decrease in concentration of 1% in the Petroleum refineries sector leads to a decrease in aggregate profit share of 0.0018%. This sector is large (as its profit share is close to 3.2%), captures 82% of the markup along its supply chain and is very concetrated. If the governement wanted to decrease the aggregate profit share and had to choose one sector where to decrease concentration, this governement should thus focus on the Petroleum refineries sector.

After solving for the distribution of income across profit and labor, the equilibrium is now solved for using the second order approximation (assumption 1) in Proposition 5..2. This proposition shows that the wage, $w_t$ and the aggregate consumption $C_t$ are function of $2 \times N$ statistics: the sectors' cross-sectional average productivities $\overline{Z_k^{(1)}}$ and Herfindahl Indices, $\Delta_{t,k}$.

In this proposition I droppe the time subscript.

**Proposition 5..2** (A Special Case: Wage and Consumption): *Under assumption 1 and when $\theta = 1$ and $\forall(k,k'), \alpha_k = 0, \zeta_k = 0$, the equilibrium wage is*

$$\log w = -\overline{\beta}' \left\{ \log \left( \frac{\varepsilon_k}{\varepsilon_k - 1} \left( \overline{Z_k^{(1)}} \right)^{\frac{-1}{\varepsilon_k - 1}} \left( f_k\left(\Delta_k\right) \right)^{\frac{1}{\varepsilon_k - 1}} \right) \right\}_k$$

*and aggregate consumption is*

$$\begin{aligned} \log C = & \frac{-\chi}{\chi + \eta - 1} \overline{\beta}' \left\{ \log \left( \frac{\varepsilon_k}{\varepsilon_k - 1} \left( \overline{Z_k^{(1)}} \right)^{\frac{-1}{\varepsilon_k - 1}} \left( f_k\left(\Delta_k\right) \right)^{\frac{1}{\varepsilon_k - 1}} \right) \right\}_k \dots \\ & \dots - \frac{\chi - 1}{\chi + \eta - 1} \log \left( 1 - \widetilde{\beta}' \left\{ \frac{f_k\left(\Delta_k\right)}{\varepsilon_k} \right\}_k \right) \end{aligned}$$

*where $\overline{\beta}' = \beta'(I - \Omega)^{-1}$ and $\widetilde{\beta}' = \beta'(I - \mu^{-1}\Omega)^{-1}$ with $\mu^{-1} = diag\{\mu_k^{-1}\}_k = diag\left\{1 - \frac{f_k(\Delta_k)}{\varepsilon_k}\right\}_k$ and where, the function $f_k : x \mapsto f_k(x)$ is*

$$f_k(x) = \begin{cases} 1 & \textit{Under Monopolistic} \\ \frac{1 - \sqrt{1 - 4\left(1 - \frac{1}{\varepsilon_k}\right)x}}{2\left(1 - \frac{1}{\varepsilon_k}\right)x} \quad \textit{for } x \in \left[0, \frac{1}{4\left(1 - \frac{1}{\varepsilon_k}\right)}\right] & \textit{Under Bertrand} \\ \frac{1 - \sqrt{1 - 4(\varepsilon_k - 1)x}}{2(\varepsilon_k - 1)x} \quad \textit{for } x \in [0, \frac{1}{4(\varepsilon_k - 1)}] & \textit{under Cournot} \end{cases}$$

*where $\Delta_k = \left(\frac{\overline{Z_k^{(2)}}}{\overline{Z_k^{(1)}}}\right)^2$ is a productivity concentration measure (the productivity Herfindahl index) while $\overline{Z_k^{(n)}} = \left(\sum_i^{N_k} Z(k,i)^{n(\varepsilon_k - 1)\gamma}\right)^{\frac{1}{n}}$ is the $n$th moment of the sector $k$'s productivity distribution.*

The proposition 5..2 entirely characterizes the equilibrium given the sectors' cross-sectional average productivities $\overline{Z_k^{(1)}}$ and Herfindahl Indices $\Delta_k$. The equilibrium wage and consumption are determined by two centrality measures of the input-output network namely $\widetilde{\beta}' = \beta'(I - \mu^{-1}\Omega)^{-1}$ and $\overline{\beta}' = \beta'(I - \Omega)^{-1}$. The former is a measure of the equilibrium sector's sales share since $\widetilde{\beta}_k = \frac{P_k Y_k}{P^C C}$ (proposition 3..3). The latter determines the elasticity of the wage and consumption to the cross-sectional average productivity (while keeping constant the concentration). Indeed, when concentration is kept constant, it is easy to show that:

$$\left.\frac{\partial \log C}{\partial \log \overline{Z_k^{(1)}}}\right|_{\Delta_k} = \frac{\chi}{\chi + \eta - 1}\frac{\overline{\beta}_k}{\varepsilon_k - 1} > 0 \quad \text{and} \quad \left.\frac{\partial \log w}{\partial \log \overline{Z_k^{(1)}}}\right|_{\Delta_k} = \frac{\overline{\beta}_k}{\varepsilon_k - 1} > 0$$

Note however that the elasticity of output to a sector's cross-sectional average productivity is also function of the labor supply elasticity. The higher is the labor supply elasticity, the higher is the increase in labor supply following a cross-sectional average productivity increase (as long as $\eta > 1$).

This model is different from Acemoglu et al. (2012) because the output elasticity to sectors' cross-sectional average productivities is *not* equal to sales share of the sector. If one is interested in quantifying the impact of a sector wide shock that increases the productivity of *all* the firms in that sector, one should look at the centrality measure of this sector $\overline{\beta}_k$ rather than at its sales share. Indeed, this centrality is purely technological as it is only a function of the input-output matrix $\Omega$ and the household's preferences $\beta$. As shown in Baqaee (2016), this is due to the imperfect competition and the accumulation of markups along the supply chain.

However, unlike Baqaee (2016) the firm heterogeneity and the deviation from monopolistic competition introduce a role for the sector's concentration $\Delta_{t,k}$. It is easy to show that the elasticity of the wage with respect to sector $k$'s concentration (while keeping the cross-sectional average productivity constant) is equal to:

$$\left.\frac{\partial \log w}{\partial \log \Delta_{t,k}}\right|_{\overline{Z_k^{(1)}}} = \frac{-\overline{\beta}_k}{\varepsilon_k - 1} e_k < 0$$

When the concentration in sector $k$ increases then the real wage decreases. The higher is the effect, the higher is the sector $k$'s centrality $\overline{\beta}_k$. The intuition is as follows. As shown in corollary 3..1, when the sector $k$'s concentration $\Delta_{t,k}$ increases, the sector's markup also increases. The increases the price of the sector $k$'s good and make it more expensive for the household. However, this sector $k$'s price increase also pushes the marginal cost of sectors downstream to sector $k$. These downstream sectors also increase their price which makes their goods more expensive too, due to double marginalization. Therefore what determines the elasticity of the real wage to sector $k$'s centrality is the importance of sector's $k$ good directly and indirectly (through others sectors) in the consumption of households which is measured by $\overline{\beta}_k$. Table 2 show the 10 sectors where the value of the elasticity of the wage with respect to concentration is the smallest. An increase in concentration of 1% in the Petroleum refineries sector reduce the wage by almost -0.08%, a similar reduction in wage can be attained by a reduction the sector average productivity by 7% (=0.08/1.14).

Concentration has a more ambiguous effect on aggregate consumption because an increase in sector $k$'s concentration reduces the real wage but also increases the profit share:

$$\frac{\partial \log C}{\partial \log \Delta_k} = \frac{-\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}e_k + \frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\frac{\mu_k}{\widetilde{\mu}_k}e_k$$

Following an increase in the concentration of sector $k$, two effects arise $(i)$ a negative effect on the wage (first term in the right hand side) and $(ii)$ a positive income effect (second term in the right hand side). The former goes as follows, as concentration increase in sector $k$, the real wage decreases: the price of the composite consumption good is affected directly and indirectly (through other sectors). Leisure becomes relatively cheaper and thus household substitute toward leisure. The strength of this substitution effect is stronger for high labor supply elasticity $\chi$ and high consumer centrality

Table 2: 10 Lowest Value of $\frac{\partial \log w}{\partial \log \Delta_k}$

| Rank | Description | (1) $\frac{\partial \log w}{\partial \log Z_k^{(1)}}$ | (2) $\frac{\partial \log w}{\partial \log \Delta_k}$ | (3) $e_k = \frac{\partial \log f_k(\Delta_k)}{\partial \log \Delta_k}$ |
|---|---|---|---|---|
| 1 | Petroleum refineries | 1.1486 | -0.079758 | 6.9436 |
| 2 | Automobile | 0.25446 | -0.035985 | 14.1417 |
| 3 | Pharmaceutical preparation | 0.80699 | -0.035915 | 4.4504 |
| 4 | Animal (except poultry) slaughtering, [...] | 0.33918 | -0.018424 | 5.4318 |
| 5 | Dog and cat food | 0.062438 | -0.015289 | 24.4871 |
| 6 | Distilleries | 0.059772 | -0.01401 | 23.4397 |
| 7 | Soap and cleaning compound | 0.12041 | -0.012634 | 10.4924 |
| 8 | Soft drink and ice | 0.17149 | -0.01253 | 7.3063 |
| 9 | Computer terminals and other computer [...] | 0.068267 | -0.012302 | 18.0203 |
| 10 | Soybean and other oilseed processing | 0.047027 | -0.011734 | 24.9513 |

Note: $\varepsilon_k = 5$. Columns (1),(2) and (3) are percentage points. Source: Bureau of Economic Analysis (detailed I-O table for 2007) and Census Bureau (Herfindahl-Hirschman index for the 50 largest firms). Only Manufacturing 31-33 industries. See Data Appendix for more details.

$\overline{\beta}_k$. The positive income effect, $(ii)$, is due to the fact that the increase in concentration pushes aggregate profit up which is ultimately rebates to the household who thus increases its consumption. This effect is stronger the higher is the change of profit share due to the increase in concentration of sector $k$ (see equation 2), the higher is total profit relative to labor income and the higher is the labor supply elasticity $\chi$. Table 3 shows the ten sectors with the lowest elasticity of consumption with respect to concentration. An increase in concentration of 1% in the Petroleum refineries sector reduces output by 0.036%, whereas an increase of average producitivity of 1% in this sector leads to an increase in consumption of 0.92%.

With the above results that characterize the aggregate consumption and wage, it is easy to solve for the sectors' outputs. Indeed, propositions 3..5 and 3..3 together with the proposition 5..2 I can solve for the equilibrium output of a given sector.

Table 3: 10 Lowest Value of $\frac{\partial \log C}{\partial \log \Delta_k}$

| Rank | Description | (1) $\frac{\partial \log C}{\partial \log \Delta_k}$ | (2) $\frac{\chi}{\chi+\eta-1} \frac{\partial \log w}{\partial \log \Delta_k}$ | (3) $\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right) \frac{\partial \log \frac{Pro}{PC\,C}}{\partial \log \Delta_k}$ | (4) $\frac{\partial \log C}{\partial \log Z_k}$ |
|---|---|---|---|---|---|
| 1 | Petroleum refineries | -0.035854 | -0.063806 | 0.027952 | 0.91892 |
| 2 | Automobile | -0.014722 | -0.028788 | 0.014066 | 0.20357 |
| 3 | Pharmaceutical preparation | -0.013965 | -0.028732 | 0.014767 | 0.64559 |
| 4 | Animal (except poultry) slaughtering, [...] | -0.0089958 | -0.014739 | 0.0057433 | 0.27135 |
| 5 | Soybean and other oilseed processing | -0.006204 | -0.0093872 | 0.0031832 | 0.037622 |
| 6 | Dog and cat food | -0.0057188 | -0.012231 | 0.0065127 | 0.049951 |
| 7 | Soft drink and ice | -0.0053622 | -0.010024 | 0.0046615 | 0.13719 |
| 8 | Fluid milk and butter | -0.0050734 | -0.0091334 | 0.00406 | 0.081068 |
| 9 | Poultry processing | -0.0050546 | -0.0085552 | 0.0035006 | 0.10557 |
| 10 | Soap and cleaning compound | -0.0049585 | -0.010107 | 0.0051485 | 0.096327 |

Note: $\varepsilon_k = 5, \chi = 2$ and $\eta = 1.5$. Columns (1),(2),(3) and (4) are percentage points. Source: Bureau of Economic Analysis (detailed I-O table for 2007) and Census Bureau (Herfindahl-Hirschman index for the 50 largest firms). Only Manufacturing 31-33 industries. See Data Appendix for more details.

**Corollary 5..1** (A Special Case: Sectors' Output): *Under assumption 1 and when $\theta = 1$ and $\forall (k,k'), \alpha_k = 0, \zeta_k = 0$, the sector $k$'s output is equal to*

$$\log Y_k = \log \widetilde{\beta}_k - \sum_{l=1} \Psi^{(d)}_{k,l} \log \left( \frac{\varepsilon_l}{\varepsilon_l - 1} \left( \overline{Z^{(1)}_l} \right)^{\frac{-1}{\varepsilon_l - 1}} \left( f_l \left( \Delta_l \right) \right)^{\frac{1}{\varepsilon_l - 1}} \right) \dots$$
$$\dots + \frac{1-\eta}{\chi + \eta - 1} \log w - \frac{\chi - 1}{\chi + \eta - 1} \log \left( 1 - \frac{Pro}{P^C C} \right)$$

*where $\widetilde{\beta} = \beta'(I - \mu^{-1}\Omega)^{-1}$ i.e $\widetilde{\beta}_k = \sum_{l=1}^N \beta_l \Psi^{(s)}_{l,k}$ with $\Psi^{(s)} = (I - \mu^{-1}\Omega)^{-1}$, the supllier influence matrix and $\Psi^{(d)} = (I - \Omega)^{-1}$ the demand-side influence matrix.*

Each terms are easily interpretable. The first term is the share of aggregate demand that goes to sector $k$ directly and indirectly, it capture the importance of sector $k$ as a supplier to the final consumer. The second term captures the cost of inputs used directly and indirectly, it capture the role played by the sector $k$ as a customer of other sectors' goods. The last two terms capture the aggregate demand. The elasticity of sector $k$'s output with respect to sector $l$'s average productivity is equal to:

$$\frac{\partial \log Y_k}{\partial \log \overline{Z^{(1)}_l}} = \frac{\Psi^{(d)}_{k,l}}{\varepsilon_l - 1} + \frac{1-\eta}{\chi + \eta - 1} \frac{\overline{\beta_l}}{\varepsilon_l - 1}$$

The first term captures the change in price of intermediate inputs $l$ used by sector $k$. Indeed, $\Psi^{(d)}_{k,l}$ is the amout of goods $l$ used by sector $k$ directly and indirectly. Whenever productivity in sector $l$ increases, the price of of sector $l$'s good fall which in turn increases production in sector $k$. The wage increases following an increase in sector $l$'s productivity and this has two distinct effects: the cost of labor increases which reduces sector $k$'s output while households are richer and consume more. This effect is capture by the second term. The change in productivity in a sector has thus an effect on sector downstream and a general equilibrium effect. To evaluate the impast of a change in concentration in a given sector on other sectors, I compute the elasticity of sector $k$'s output, $Y_k$ with respect to sector $l$'s Herfindahl, $\Delta_l$:

$$\frac{\partial \log Y_k}{\partial \log \Delta_l} = -\left(\Psi^{(s)}_{l.k} - \mathbb{I}_{l,k}\right)\frac{f_l(\Delta_l)}{\varepsilon_l - f_l(\Delta_l)}e_l - \frac{\Psi^{(d)}_{k,l}}{\varepsilon_l - 1}e_l \ldots$$
$$\ldots - \frac{1-\eta}{\chi+\eta-1}\frac{\overline{\beta_l}}{\varepsilon_l - 1}e_l + \frac{\chi-1}{\chi+\eta-1}\frac{Pro}{wL}\frac{pro_k}{Pro}\frac{\mu_l}{\widetilde{\mu}_l}e_l$$

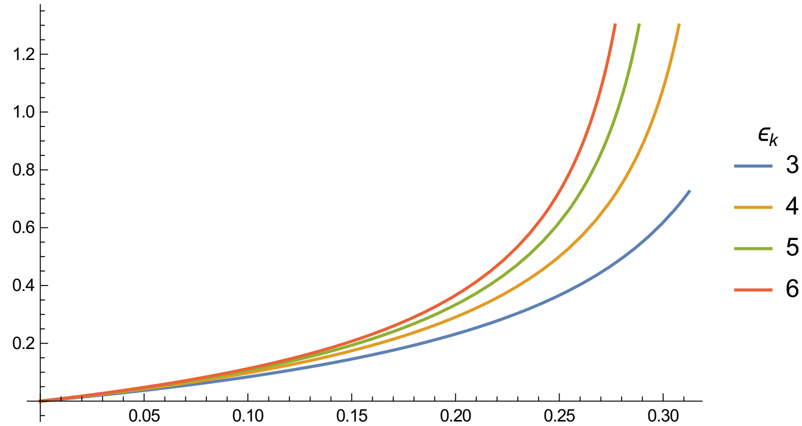
Once again each term can be easily interpreted. The first term is due to the fact that following a change in sector $l$'s Herfindahl, the share of aggregate spending going to sector $k$ through sector $l$ is reduced: sector $l$ captures more profit. Indeed, $\frac{d \log \mu_l^{-1}}{d \log \Delta_l} = -\frac{f_l(\Delta_l)}{\varepsilon_l - f_l(\Delta_l)}e_l$ is the change in the share of sector $l$'s income that is used to pay for intermediate inputs while $\Psi^{(s)}_{l.k}$ represents the share of this payement that goes to sector $k$. The second term is due to the fact that the change in concentration $\Delta_l$ affects sector $l$'s productivity: this affects sector $k$ through the consumption of sector $l$ goods by sector $k$, $\Psi^{(d)}_{k,l}$. Finally the last two terms are the general equilibrium effect on both wage and profit income.

Note that the elasticity of the strategic pricing distortion with respect to concentration, $e_k$, is key to evaluate the effect of an increases in concentration on any variables. Figure 7 displays this elasticity for the Bertrand cases as a function of $\Delta_k$ for different values of $\varepsilon_k$. For $\varepsilon_k = 5$ and for a concentration $\Delta_k$ equal to $0.1376$, which corresponds to a sales Herfindahl index of $0.18$[8], this elasticity is about $0.17$[9].

[8]Merger laws in the U.S. apply for sales Herfindahl index above $0.18$.

[9]In the Cournot case to have a sales Herfindahl Index of $0.18$, $\Delta_k$ has to be equal to $0.061$ for $\varepsilon_k = 5$. And the elasticity $e_k$ is about $2.58$.

Figure 7: Elasticity of the Distortion w.r.t. the Concentration




Note: Bertrand Case, elasticity $\frac{d \log f_k(x)}{d \log x}$ of $f_k : x \mapsto \frac{1-\sqrt{1-4(1-1/\varepsilon_k)x}}{2(1-1/\varepsilon_k)x}$ for different value of $\varepsilon_k$. Cournot Case, see appendix.

## 5.2. Sensitivity to Firm's Volatility

In this part, I study the effect of firm level volatility on aggregate volatility. To do so I define two centralities: the firm's volatility wage (resp. consumption) centrality is defined as the derivative of the variance of the growth rate of the wage (resp. consumption) with respect to the variance of the growth rate of firm level productivity $\varrho_k^{(1)} := \mathbb{V}ar_t \left[ Z_{t+1}^{(\varepsilon_k-1)\gamma}(k,i)/Z_t^{(\varepsilon_k-1)\gamma}(k,i) \right]$.

**Definition 5..2** (Firm's Volatility Centralities): *The Firm's volatility wage (resp. consumption) centrality is:*

$$\breve{\beta}_{t,k}^{w} := \frac{\partial \mathbb{V}ar_t \left[ \log \frac{w_{t+1}}{w_t} \right]}{\partial \varrho_k^{(1)}} \quad \textit{and} \quad \breve{\beta}_{t,k}^{C} := \frac{\partial \mathbb{V}ar_t \left[ \log \frac{C_{t+1}}{C_t} \right]}{\partial \varrho_k^{(1)}}$$

*where* $\varrho_k^{(1)} := \mathbb{V}ar_t \left[ Z_{t+1}^{(\varepsilon_k-1)\gamma}(k,i)/Z_t^{(\varepsilon_k-1)\gamma}(k,i) \right]$.

These centralities measure the effect of firm's productivity volatility on the volatility of the wage and the aggregate consumption.

**Proposition 5..3** (Firm's Volatility Centrality): *Under assumptions 1 and 2, when* $\forall k, \alpha_k = 0, \zeta_k = 0$*, the firm's volatility wage and consumption centralities* $\breve{\beta}^w_t$ *and* $\breve{\beta}^C_t$ *are*

$$\breve{\beta}^w_{t,k} = \left(\frac{\overline{\beta}_k}{\varepsilon_k - 1}\right)^2 \Delta_{t,k}\left(4e^2_{t,k} + 4e_{t,k} + 1\right)$$

*and*

$$\breve{\beta}^C_{t,k} = \left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k - 1}\right)^2 \Delta_{t,k}(4e^2_{t,k} + 4e_{t,k} + 1)\dots$$
$$\dots + 4\left(\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro_t}{w_tL_t}\right)\left(\frac{pro_{t,k}}{Pro_t}\right)\frac{\mu_{t,k}}{\widetilde{\mu}_{t,k}}\right)^2 \Delta_{t,k}e^2_{t,k}$$
$$\dots - 4\frac{\chi(\chi-1)}{(\chi+\eta-1)^2}\frac{\overline{\beta}_k}{\varepsilon_k - 1}\left(\frac{Pro_t}{w_tL_t}\right)\left(\frac{pro_{t,k}}{Pro_t}\right)\frac{\mu_{t,k}}{\widetilde{\mu}_{t,k}}\Delta_{t,k}\left(2e_{t,k}+1\right)e_{t,k}$$

*where* $\overline{\beta}' = \beta'(I-\Omega)^{-1}$, $\widetilde{\mu}_{t,k} = \frac{\widetilde{\varepsilon}_{t,k}}{\widetilde{\varepsilon}_{t,k}-1}$ *and where* $\left\{\widetilde{\varepsilon}^{-1}_{t,k}\right\}_k = (I-\mu_t^{-1}\Omega)^{-1}\left\{\frac{f_k(\Delta_{t,k})}{\varepsilon_k}\right\}_k$. $\Delta_{t,k}$ *is the sector* $k$*'s productivity Herfindahl index and* $e_{t,k}$ *is the elasticity of* $f_k$ *with respect to* $\Delta_{t,k}$ *at time* $t$.

Unlike Baqaee (2016), the sector's volatility is due to firm level shocks only. Under random growth (assumption 2), each sectors' fluctuations are driven by fluctuations in moments of the sector's productivity distribution and especially the cross-sectional average productivity $\overline{Z^{(1)}_{t,k}}$ and its Herfindahl index $\Delta_{t,k}$. These two key sufficient statistics evolve according to proposition 3..8. Taking into account the fluctuations in these two statistics, the effect of firm's volatility on aggregate volatility is given by the proposition 5..3.

This effect depends crucially on the elasticity of the wage to sector $k$'s cross-sectional average productivity shocks $\overline{\beta}_k$ which measures the importance of that sector in the composite consumption good price index. Furthermore, the influence of sector $k$'s firm level volatility is increasing in the sector $k$'s concentration $\Delta_{t,k}$ through three channels. The first one is captured by the term $\left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2\Delta_{t,k}$ and is due to the fact that the cross-sectional average of productivity $\overline{Z_{t,k}^{(1)}}$ is higher when the concentration is higher: larger firms are even larger and shocks to these large firms matters more. The second channel is due to the fact, that $\Delta_{t,k}$ is itself volatile and that its volatility is increasing in its level (this is the term $\left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2 \Delta_{t,k} 4e_{t,k}^2$). Finally, the third channel, is due to the fact that these two statistics are correlated and this correlation is also increasing in sector $k$'s concentration: firm level shocks affect both the cross-sectional average and the dispersion of sector productivity at the same time (this is the term $\left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2 \Delta_{t,k} 4e_{t,k}$).

In a version of the model with monopolitic competition only, it is easy to show that $\breve{\beta}_{t,k}^w = \left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2 \Delta_{t,k}$, it follows that the term $4e_{t,k}^2 + 4e_{t,k}$ capture the extra effect due to the oligopolistic competition. For a sector with a sales Herfindahl of 0.18, which the level above which merge law applies in the U.S, and for an elasticity of subtitution across varieties of 5 the term $4e_{t,k}^2 + 4e_{t,k}$ is equal to 0.79. In other words, oligopolistic competition increases the effect of firm volatility on aggregate volatility by 80% relative to a model with monopolistic competition. Figure 8 displays the value of $4e_k^2 + 4e_k$ as a function of $\Delta_k$ for different value of $\varepsilon_k$. Table 4 show the sector for which $\breve{\beta}_{t,k}^w$ is the highest. Columns (1) and (2) give the value of $\breve{\beta}_{t,k}^w$ relative to the case where $\Omega = 0$ and monopolistic comeptition is assumed in all sectors. Column (3) give the value $1 + 4e_{t,k}^2 + 4e_{t,k}$ in these sectors. It follow that by taking into account I-O trade the effect of firm-level volatility on wage volatility is increased by 89% with monopolistic competition (column 1) and by 146% with oligopolistic competition (column 2).

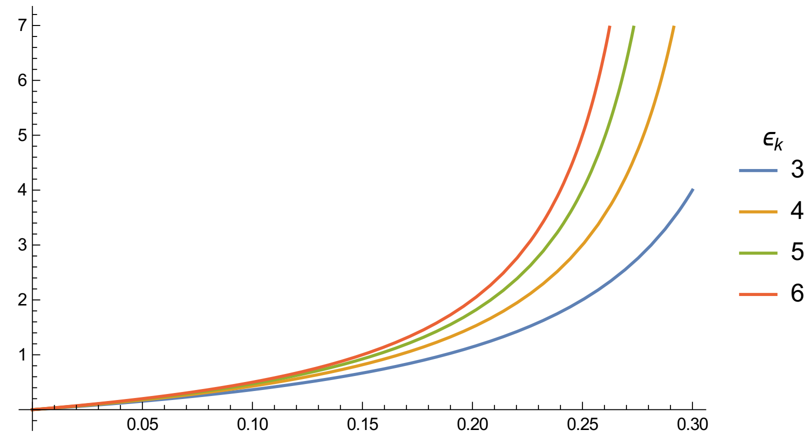
The same intuition applies for the effect of firm's volatility on aggregate consumption volatility. However, in addition, fluctuations of $\Delta_{t,k}$ shift the profit income share, which affects aggregate consumption through an income effect (the term $4\left(\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro_t}{w_t L_t}\right)\left(\frac{pro_{t,k}}{Pro_t}\right)\frac{\mu_{t,k}}{\widetilde{\mu}_{t,k}}\right)^2 \Delta_{t,k} e_{t,k}^2$). The last term reflects the correlation between cross-sectional average productivity and the concentration.

Table 4: 10 Highest Value of $\frac{\partial \mathbb{V}ar_t\left[\log \frac{w_{t+1}}{w_t}\right]}{\partial \sigma_k}$

| Rank | Description | (1) I-O + Monop | (2) I-O + Oligop | (3) (2)/(1) |
|---|---|---|---|---|
| 1 | Petroleum refineries | 1.8946 | 2.4574 | 1.297 |
| 2 | Pharmaceutical preparation | 1.2266 | 1.4546 | 1.1859 |
| 3 | Automobile | 1.0711 | 1.7626 | 1.6457 |
| 4 | Animal (except poultry) slaughtering, [...] | 2.3338 | 2.8685 | 1.2291 |
| 5 | Soft drink and ice | 1.0876 | 1.4287 | 1.3136 |
| 6 | Toilet preparation | 1.1496 | 1.4624 | 1.2721 |
| 7 | Soap and cleaning compound | 1.6268 | 2.3812 | 1.4637 |
| 8 | Audio and video equipment | 1.4817 | 1.7208 | 1.1614 |
| 9 | Poultry processing | 1.5209 | 2.0538 | 1.3504 |
| 10 | Biological product (except diagnostic) | 95.4241 | 126.9477 | 1.3304 |

Note: $\varepsilon_k = 5$. $\frac{\partial \mathbb{V}ar_t\left[\log \frac{w_{t+1}}{w_t}\right]}{\partial \sigma_k}$: (1) and (2) relative to the no I-O and monopolitic competition case; (3) relative to the I-O and monopolitic competition case. Source: Bureau of Economic Analysis (detailed I-O table for 2007) and Census Bureau (Herfindahl-Hirschman index for the 50 largest firms). Only Manufacturing 31-33 industries. See Data Appendix for more details.

Figure 8: Effect of Oligopolistic Competition w.r.t. the Concentration: $4e_k^2 + 4e_k$



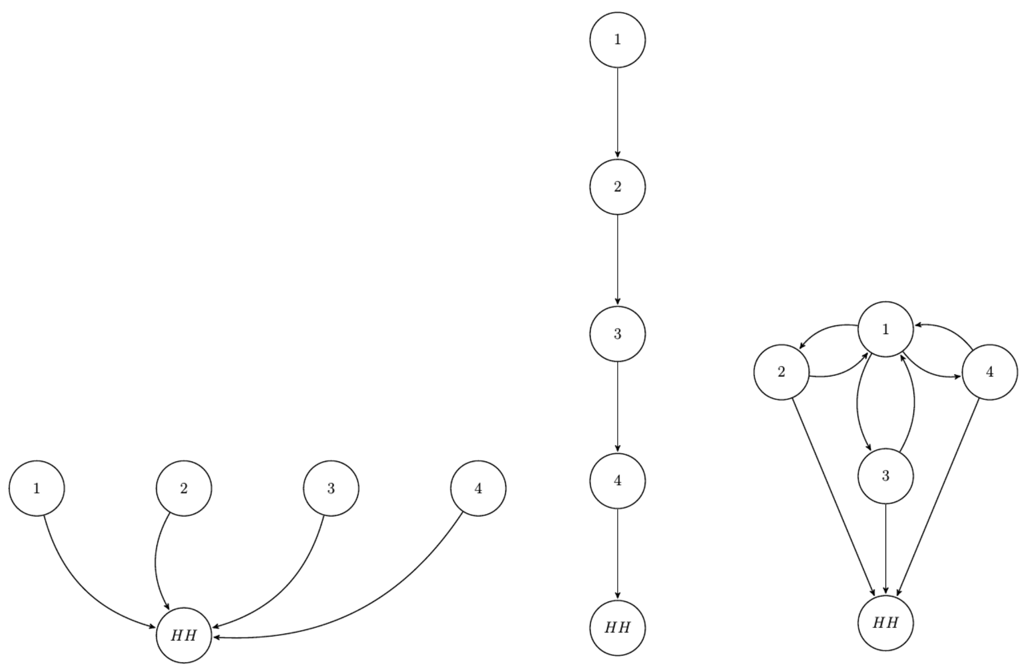

Note: $4e_{t,k}^2 + 4e_{t,k}$ for the Bertrand Case where $e_k = \frac{d \log f_k(x)}{d \log x}$ with $f_k : x \mapsto \frac{1-\sqrt{1-4(1-1/\varepsilon_k)x}}{2(1-1/\varepsilon_k)x}$ for different value of $\varepsilon_k$. For a sales Herfindahl of 0.18 and for $\varepsilon_k = 5$ then $\Delta_k = 0.1376$ and $4e_k^2 + 4e_k = 0.79$. Cournot Case, see appendix.

## 5.3. Shock in the Horizontal, Vertical and Star Economies

I describe the effect of a positive shock on a large firm for three Input-Output Networks: the horizontal, the vertical and the star economies. These three economies are represented in Figure 9. The horizontal economy (left panel) is characterized by no input-output trade and all sectors are supplying the household equally. The vertical economy (middle panel) has a source, here sector $1$, and a sink, here the household. The star economy (right panel) has a central sector, here sector $1$, whereas the other sectors are supplying equally the household. The centrality $\bar{\beta}_1$ of the sector $1$ is smaller in the vertical economy than in the horizontal economy which is smaller than in the star economy.

In each economy, a positive shock on a large firm (top $20\%$) in sector $1$ puts this firm at the top $1\%$ of the productivity distribution. In Figure 10,

Figure 9: Three Production Networks with Four Sectors




Note: From left to right: a horizontal economy with no input trade, a vertical economy with a source and a sink, and a star economy with a central node. Source: Carvalho (2014) and Bigio and LaO (2016).

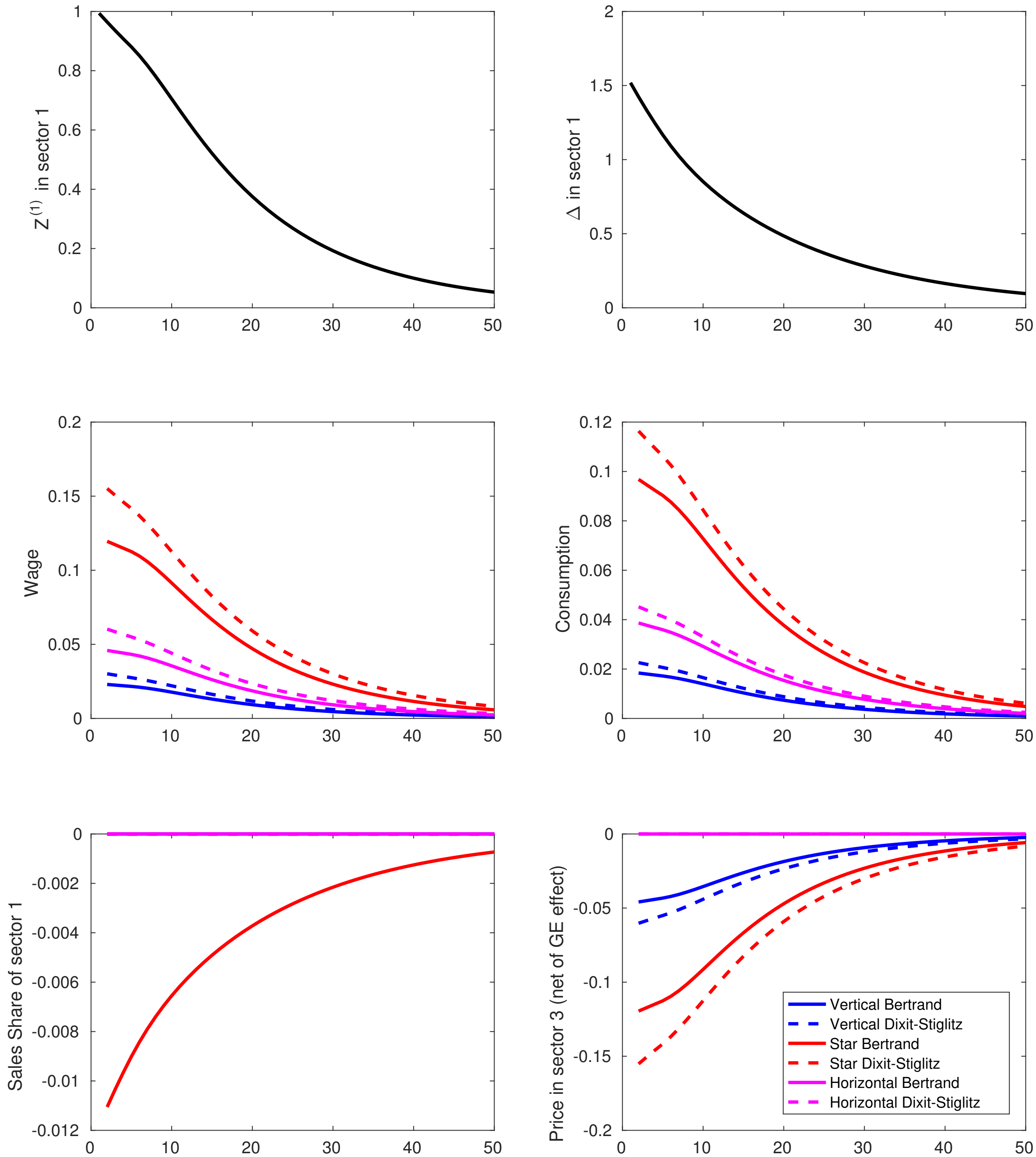
I plot the response of different variables to this shock for each economy: the cross sectional average productivity $\overline{Z^{(1)}_{t,1}}$ in sector $1$ (top left panel), the concentration $\Delta_{t,1}$ in sector $1$ (top right panel), the wage and consumption (middle left and right panels resp.), the sales share $P_{t,1}Y_{t,1}/P^C_t C_t$ in sector $1$ (bottom left panel) and the price $P_{t,3}$ in sector $3$ net of the effect of the wage (bottom right panel). The dashed lines are the responses under Dixit-Stiglitz competition while the full lines are the responses under Bertrand competition. In Figure 11, I plot the responses of the same variables where the cross-sectional average productivity is kept constant while the evolution of the concentration is identical as the one in Figure 10. This Figure allows to look only at the effect of the change in concentration. These economies are calibrated such that the initial level of concentration in sector $1$ across these economies is identical and such that the response of $\overline{Z^{(1)}_{t,1}}$ and $\Delta_{t,1}$ are also identical.

The first thing to note is that such a shock has a positive effect on both the cross-sectional average productivity, $\overline{Z^{(1)}_{t,1}}$, and the concentration, $\Delta_{t,1}$. It is because an already large firm becomes even more productive which increases the average productivity *and* the concentration. As the shocked firm goes back to it initial productivity level, these two statistics are converging back to their long-term average.

The aggregate response of the wage and consumption to this shock is higher in the star economy than in the horizontal economy which is also higher than in the vertical economy. Note that there are two effects at play here: $(i)$ the increase in average productivity has a positive effect on the wage and output and $(ii)$ the increase in concentration results in an increase in the sector $1$'s markup which decreases the wage and the output. Here the positive effect of the increase of the average productivity dominates the negative effect of the increase in concentration. In the middle panels of Figure 11, we can see the negative response of the wage and output due to the increase in concentration. The stronger are these effects, the higher is the centrality $\bar{\beta}_1$. Indeed, the centrality measures the importance as a supplier of the sector $1$ to the household. Thus the responses is stronger in the star economy than in the horizontal and even stronger than in the vertical economy.

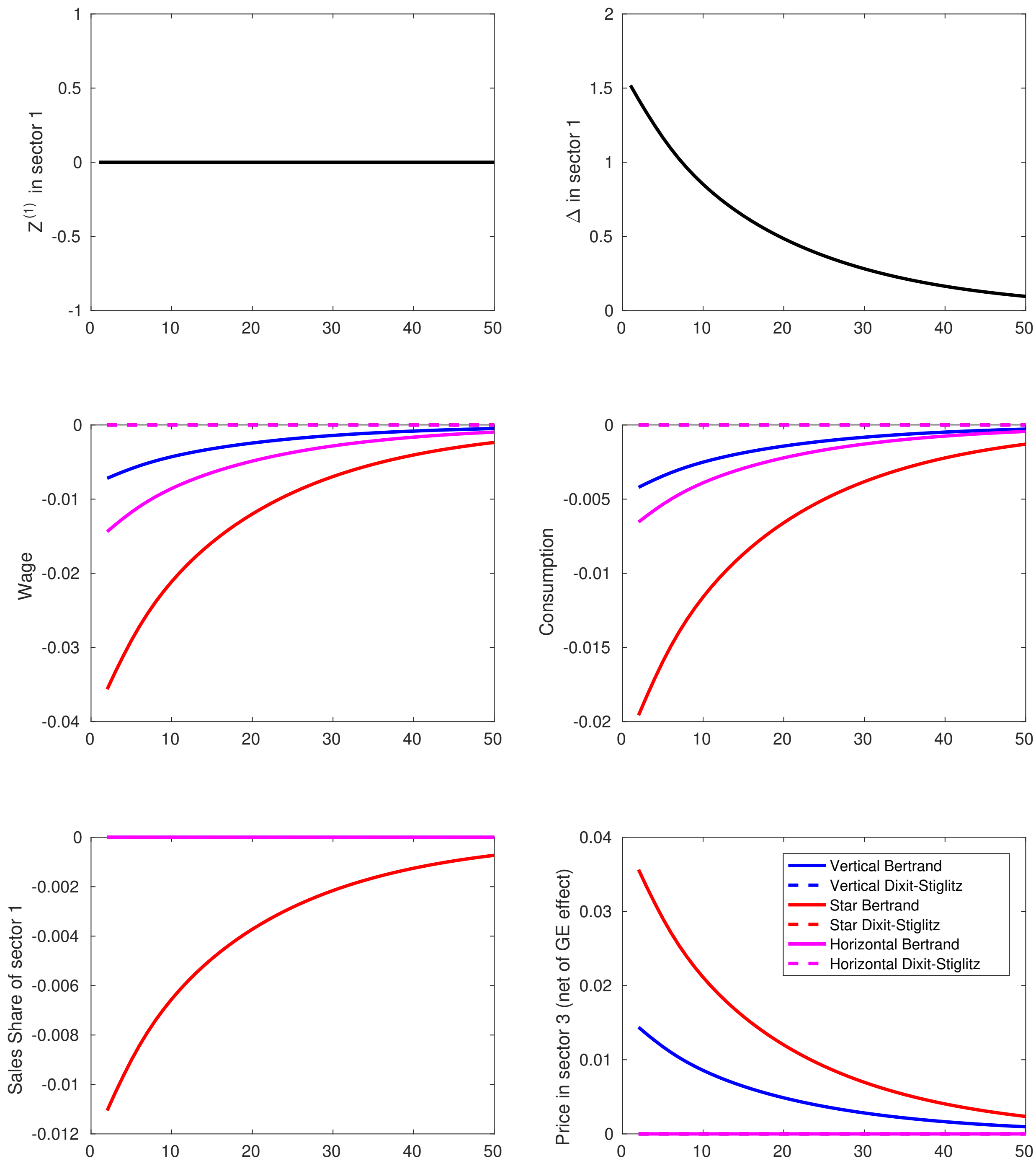
To study the transmission of this shock to the rest of the economy, let us focus on the bottom panel of Figures 10 and 11. The sales share of sector $1$ (bottom left panel) is affected by this shock only in the star economy. According to proposition 3..3, the sales share is affected by markups of downstream sectors. Sector $1$ is a downstream sector of itself in the star economy. Since the other sectors buy and sell their goods to sector $1$. Furthermore, under Dixit-Stiglitz competition, sector $1$'s markup is constant and thus does not affect any sales share. According to proposition 3..2, sector prices are affected by upstream sectors. In the vertical and in the star economies, sector $1$ is a (direct or indirect) supplier of sector $3$ while it is not in the horizontal economy. It follows that price in sector $3$ is only affected in the vertical and the star economies. There are again two opposite effects. On one hand, sector $1$ becomes more productive and thus sell its good at a lower price to its downstream sectors since sector $1$ is part of the marginal cost of sector $3$, which in turn charges a lower prices. One the other hand, concentration is higher in sector $1$ and thus it charges a higher markup and thus a higher price. Then its suppliers faces a higher marginal cost and thus charges a higher price. This is the result of double marginalization. The bottom right panel of Figure 11 shows the latter effect. However, since the wage enters the marginal cost of each sector, prices are increasing accordingly. In both Figures, I have reported the part of the price which is not due to the increase in the wage.

## Figure 10: Shock on a Large Firm: Horizontal, Vertical, Star




Note: Response to a positive shock on a top $20\%$ firms in sector $1$ for the Horizontal, Vertical and Star economies. Top left: mean productivity; Top right: concentration; Middle Left: wage; Middle right: Consumption; Bottom left: Sales share in sector $1$; Bottom right: Price in sector 3 minus the effect of the wage. Calibration: $\varepsilon_k = 5, \alpha_k = 0$ for all sectors.

Figure 11: Shock on Concentration: Horizontal, Vertical, Star




Note: Response to a concentration shock in sector 1 for the Horizontal, Vertical and Star economies. Top left: mean productivity; Top right: concentration; Middle Left: wage; Middle right: Consumption; Bottom left: Sales share in sector 1; Bottom right: Price in sector 3 minus the effect of the wage. Calibration: $\varepsilon_k = 5, \alpha_k = 0$ for all sectors.

# 6. Conclusion

In this paper, I study how firm-level shocks affect sector-level productivity and competition and how changes in the level of productivity and competition propagate in the input-output network. Changes in the level of competition act as supply shocks to downstream sectors and demand shocks to upstream sectors. The position of a sector in the input output network determines the elasticity of wages and output to changes in both average productivity and concentration. The relative market power of a sector in its supply chain affects the elasticity of profit income share and aggregate consumption to changes in the level of competition. Finally, I show that firms in highly central, highly concentrated sectors and sectors that capture most of the profit along the supply chain are the most important for aggregate volatility.

The fact that in the framework described in this paper, changes in the level of competition shifts the distribution of income between primary inputs and profit hints at the potential impact of these changes on inequality. As soon as households are heterogeneous in their holdings of firm stocks, these changes in competition will create distributional effects. I leave this question open for future work.

Throughout the paper, I completely abstract from the effect of competition on growth. Indeed, rent seeking behavior has been shown to be a driving force for R&D investment and endogenous growth as in the seminal work of Aghion and Howitt (1992). Imbs and Grassi (2015) study the interaction of growth and volatility arising from firm-level shocks in a model of ideas flows à la Lucas (2009). The introduction of imperfect competition and rent-seeking behavior deserves further research.

# References


Acemoglu, Daron, Ufuk Akcigit, and William Kerr, *Networks and the Macroeconomy: An Empirical Exploration*, University of Chicago Press, June

—, Vasco M. Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, "The Network Origins of Aggregate Fluctuations," *Econometrica*, 09 2012, *80* (5), 1977–2016.

Aghion, Philippe and Peter Howitt, "A Model of Growth through Creative Destruction," *Econometrica*, March 1992, *60* (2), 323–51.

Atkeson, Andrew and Ariel Burstein, "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, December 2008, *98* (5), 1998–2031.

B, Jr Long John and Charles I Plosser, "Real Business Cycles," *Journal of Political Economy*, February 1983, *91* (1), 39–69.

Bak, Per, Kan Chen, Jose Scheinkman, and Michael Woodford, "Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics," *Ricerche Economiche*, March 1993, *47* (1), 3–30.

Baqaee, David Rezza, "Cascading Failures in Production Networks," mimeo 2016.

Barrot, Jean-Noel and Julien Sauvagnat, "Input Specificity and the Propagation of Idiosyncratic Shocks in Production Networks," *The Quarterly Journal of Economics*, 2016, *131* (3), 1543–1592.

Bigio, Saki and Jennifer LaO, "Financial Frictions in Production Networks," Working Paper 22212, National Bureau of Economic Research April 2016.

Bilbiie, Florin O., Fabio Ghironi, and Marc J. Melitz, "Endogenous Entry, Product Variety, and Business Cycles," *Journal of Political Economy*, 2012, *120* (2), 304 – 345.

Boehm, Christoph E., Aaron Flaaen, and Nitya Pandalai-Nayar, "Input Linkages and the Transmission of Shocks: Firm-Level Evidence from the 2011 Tohoku Earthquake," 2016.

Carvalho, Vasco M, "Aggregate Fluctuations and the Network Structure of Intersectoral Trade," mimeo 2010.

Carvalho, Vasco M., "From Micro to Macro via Production Networks," *Journal of Economic Perspectives*, Fall 2014, *28* (4), 23–48.

_ and Basile Grassi, "Large Firm Dynamics and the Business Cycle," *CEPR*, May 2015, (DP10587).

_ , Makoto Nirei, Yukiko Saito, and Alireza Tahbaz-Salehi, "Supply Chain Disruptions: Evidence from the Great East Japan Earthquake," 2016.

Córdoba, Juan Carlos, "A Generalized Gibrat's Law," *International Economic Review*, November 2008, *49* (4), 1463–1468.

Dixit, Avinash K and Joseph E Stiglitz, "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, June 1977, *67* (3), 297–308.

Durlauf, Steven N., "Nonergodic Economic Growth," *The Review of Economic Studies*, 1993, *60* (2), 349–366.

Gabaix, Xavier, "Zipf'S Law For Cities: An Explanation," *The Quarterly Journal of Economics*, August 1999, *114* (3), 739–767.

_ , "Power Laws in Economics and Finance," *Annual Review of Economics*, 05 2009, *1* (1), 255–294.

_ , "The Granular Origins of Aggregate Fluctuations," *Econometrica*, 05 2011, *79* (3), 733–772.

Hannah, L. and J.A. Kay, *Concentration in modern industry: theory, measurement, and the U.K. experience*, Macmillan, 1977.

Imbs, Jean and Basile Grassi, “Why Risky Sectors Grow Fast,” Technical Report, Paris School of Economics 2015.

Jovanovic, Boyan, “Micro Shocks and Aggregate Risk,” *The Quarterly Journal of Economics*, May 1987, *102* (2), 395–409.

Krugman, Paul R., “Increasing returns, monopolistic competition, and international trade,” *Journal of International Economics*, November 1979, *9* (4), 469–479.

Lucas, Robert E., “Ideas and Growth,” *Economica*, 02 2009, *76* (301), 1–19.

Melitz, Marc J. and Gianmarco I. P. Ottaviano, “Market Size, Trade, and Productivity,” *Review of Economic Studies*, 2008, *75* (1), 295–316.

Ottaviano, Gianmarco, Takatoshi Tabuchi, and Jacques-Franois Thisse, “Agglomeration and Trade Revisited,” *International Economic Review*, May 2002, *43* (2), 409–436.

Severini, Thomas A., *Elements of Distribution Theory*, Cambridge University Press, 2005.

Zhelobodko, Evgeny, Sergey Kokovin, Mathieu Parenti, and JacquesFranois Thisse, “Monopolistic Competition: Beyond the Constant Elasticity of Substitution,” *Econometrica*, November 2012, *80* (6), 2765–2784.

# A Data Appendix

In this paper, I use two types of data at the sector level. The first one is the I-O data of the Bureau of Economic Analysis. The second one is the concentration data of the Census Bureau.

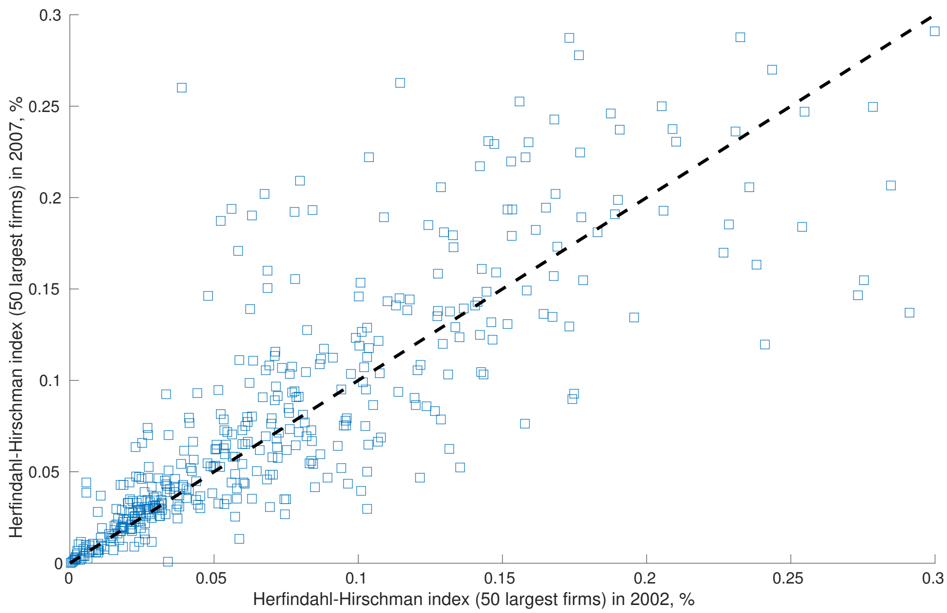
The Bureau of Economic Analysis provide Input-Output information at different level of aggregation. I use here the detailed I-O table from 2007 which gives information on 389 sectors. They do not provide direct requirement Industry-by-Industry table but instead total Industry-by-Industry requirement table. I then use the formula $\Omega = (TOT - I)TOT^{-1}$ to find the direct requirement of an industry input to produce one dollar of its output. To find the value of household consumption share, I use the USE table of the Bureau of Economic Analysis, which gives for each commodity how much the household buy of this commodity. I then recover the share of income spend by the household on each industry by premultiplying these commodity spending share by the MAKE table. The MAKE table gives for each industry how much of each commodity is needed to produce one dollar of output.

The Census Bureau provides concentration measure for different level of aggregation for all sectors except for Agriculture, Forestry, Fishing and Hunting (11); Mining, Quarrying, and Oil and Gas Extraction (21); Construction (23); Management of Companies and Enterprises (55); Public Administration (92). The measure of concentration are the top 4,8,20 and 50 firms' share of total industry revenues in 2002, 2007 and 2012. For manufacturing (31-33), the census bureau also gives the Herfindahl-Hirschman Index among the 50 largest firms. I use these measures in Figures 1,12 and 13.

Using the correspondance table given by the Bureau of Economic Analysis between the I-O sectors classification and the NAICS 2007 classification, I matched these two data source to plot Figure 2 and to calibrate the model in section 5..

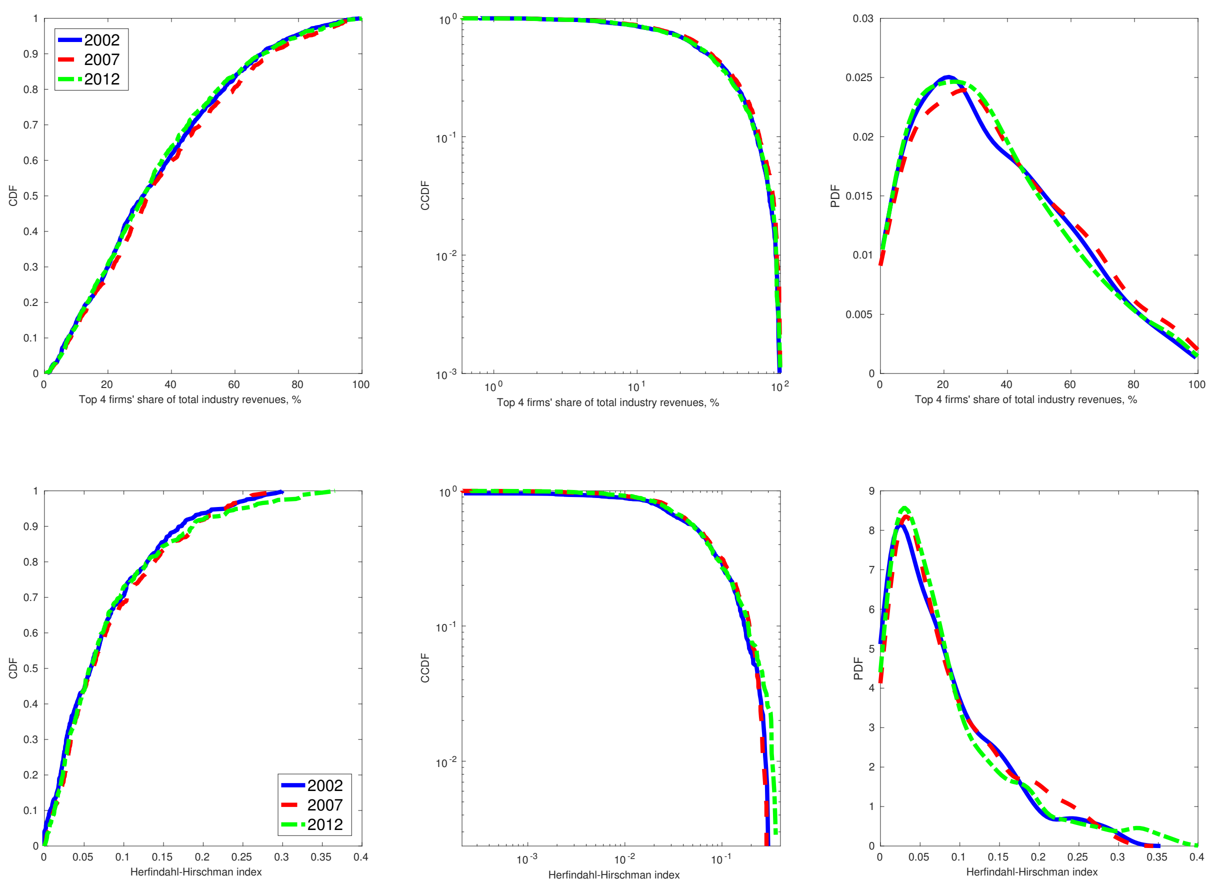
# B Figures Appendix

Figure 12: Sectors' Concentrations - Manufacturing



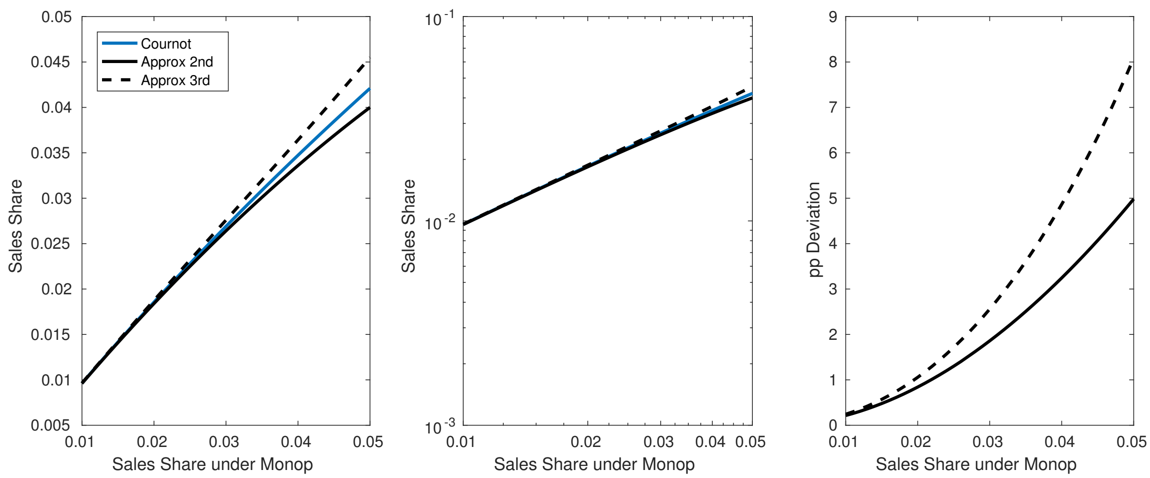

Note: Herfindahl-Hirschman index for the 50 largest companies in 2002 and in 2007 for 6 digits NAICS manufacturing industry (31-33). 448 industries. Source: Census Bureau.

Figure 13: Sectors' Concentration Distribution



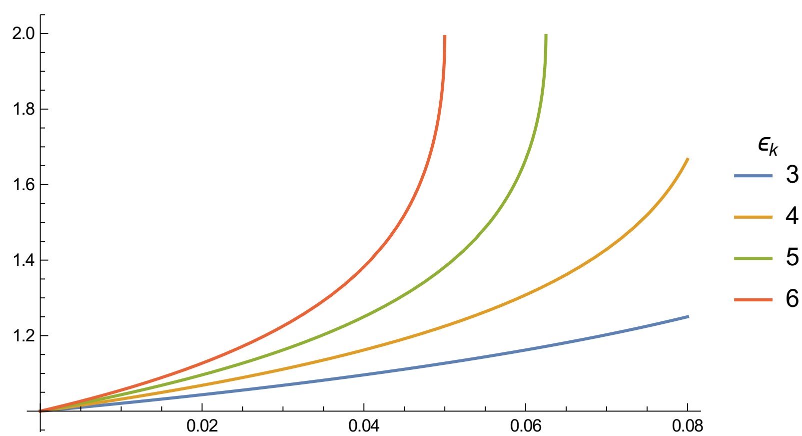

Note: Empirical cumulative distribution function (left), counter cumulative distribution function (center), and Kernel smoothing function estimate of the probability distribution function (right) of top four firms' share of total revenues for 6 digits NAICS industry (top panel) and of Herfindahl-Hirschman index for the 50 largest companies for 6 digits NAICS manufacturing industries (31-33). Source: Census Bureau.

Figure 14: Firm's Pricing Approximation: Cournot case



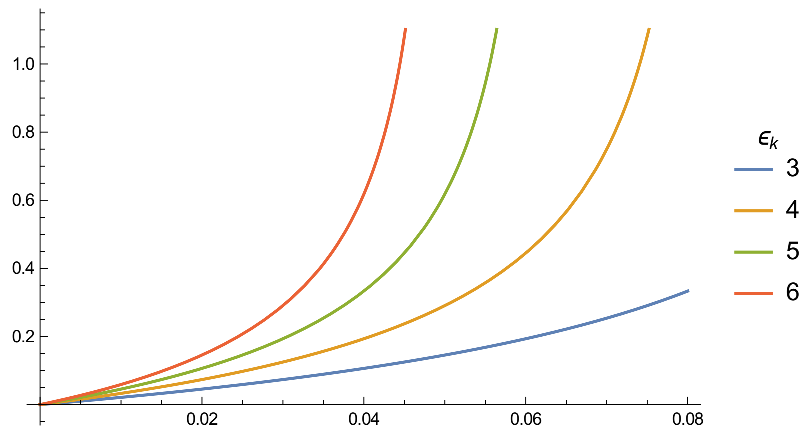

Note: For $\varepsilon_k = 5$. Left panel shows the Cournot sales share using a numerical solver (blue), the second (black) and the third (dashed black) order approximation as a function of the Monopolistic sales share. The middle panel plot is on a log-log scale, the right panel show percentage deviation of both approximations with respect to the numerical solution. For the Bertrand case see main text.

Figure 15: Deviation from Monopolistic Competition: Cournot




Note: Cournot Case, $f_k : x \mapsto \frac{1-\sqrt{1-4(\varepsilon_k-1)x}}{2(\varepsilon_k-1)x}$ for different value of $\varepsilon_k$. For the Bertrand case, see main text.

Figure 16: Elasticity of the Distortion w.r.t. the Concentration




Note: Cournot Case, elasticity $\frac{d \log f_k(x)}{d \log x}$ of $f_k : x \mapsto \frac{1-\sqrt{1-4(\varepsilon_k-1)x}}{2(\varepsilon_k-1)x}$ for different value of $\varepsilon_k$. Bertrant Case, see main text.

# C Proofs Appendix

## C1. Firms

**Proof of Proposition 2..4.** Firm's Pricing Approximation

Note that this proof is written for any elasticity of subtitution across intermediate inputs $\sigma$. The case study in the main text is for $\sigma = 1$, the whole proof goes through. Let us defined the following system of equation for a given parameter $\chi$:

$$p(k,i) = \frac{\varepsilon(k,i)}{\varepsilon(k,i)-1}\lambda(k,i)$$
$$s(k,i) = \frac{p(k,i)y(k,i)}{p_k Y_k} = N_k^{-\zeta_k \varepsilon_k}\left(\frac{p(k,i)}{p_k}\right)^{1-\varepsilon_k}$$
$$\varepsilon(k,i) = \begin{cases} \varepsilon_k & \text{Under Monopolistic Competition} \\ \varepsilon_k - \chi(\varepsilon_k - \sigma)s(k,i) & \text{Under Bertrand Competition} \\ \left(\frac{1}{\varepsilon_k} + \chi(\frac{1}{\sigma} - \frac{1}{\varepsilon_k})s(k,i)\right)^{-1} & \text{Under Cournot Competition} \end{cases}$$

Let us rewrite the system of equation describing the pricing of the firm $i$ in sector $k$ by subsituting the expression of $\varepsilon(i,k)$ and $p(k,i)$:

$$s(k,i) = \begin{cases} \left(1 - \frac{1}{\varepsilon_k - \chi(\varepsilon_k - \sigma)s(k,i)}\right)^{\varepsilon_k - 1} N_k^{-\zeta_k \varepsilon_k}\left(\frac{\lambda(k,i)}{p_k}\right)^{1-\varepsilon_k} & \text{Under Bertrand} \\ \left(1 - \frac{1}{\varepsilon_k} - \chi(\frac{1}{\sigma} - \frac{1}{\varepsilon_k})s(k,i)\right)^{\varepsilon_k - 1} N_k^{-\zeta_k \varepsilon_k}\left(\frac{\lambda(k,i)}{p_k}\right)^{1-\varepsilon_k} & \text{Under Cournot} \end{cases}$$

Let us described the system of equation with the unknown $X(\omega,\chi) = s(k,i)^\xi$ with $\omega = N_k^{-\zeta_k \varepsilon_k}\left(\frac{\lambda(k,i)}{p_k}\right)^{1-\varepsilon_k}$ by the function $\mathcal{H}(X,\omega,\chi)$ such that

$$\mathcal{F}(\omega,\chi) = \mathcal{H}(X(\omega,\chi),\omega,\chi) = 0 \tag{3}$$

with

$$\mathcal{H}(X,\omega,\chi) = \begin{cases} X - \left(1 - \frac{1}{\varepsilon_k - \chi(\varepsilon_k - \sigma)X^{1/\xi}}\right)^{\xi(\varepsilon_k - 1)} \omega^\xi & \text{Under Bertrand} \\ X - \left(1 - \frac{1}{\varepsilon_k} - \chi(\frac{1}{\sigma} - \frac{1}{\varepsilon_k})X^{1/\xi}\right)^{\xi(\varepsilon_k - 1)} \omega^\xi & \text{Under Cournot} \end{cases}$$

Note that $X(\omega,0) = \widehat{s}(k,i)^\xi$ is the solution under monopolistic competition. The solution of this system $X(\omega,\chi)$ satisfies at the second order:

$$X(\omega,\chi) = X(\omega,0) + \chi X'(\omega,0) + \chi^2 X''(\omega,0) + o(\chi^2)$$

where $X'(\omega,\chi) := \frac{\partial X}{\partial \chi}(\omega,\chi)$ and $X''(\omega,\chi) := \frac{\partial X'}{\partial \chi}(\omega,\chi)$.

For $\chi = 1$ it yields an approximation of the Cournot and Bertrand solution:

$$X(\omega,1) \approx X(\omega,0) + X'(\omega,0) + X''(\omega,0)$$

Let us compute these derivatives by differentiating equation 3:

$$\mathcal{F}'_\chi(\omega,\chi) = 0 = X'(\omega,\chi)\mathcal{H}'_X(X(\omega,\chi),\omega,\chi) + \mathcal{H}'_\chi(X(\omega,\chi),\omega,\chi)$$
$$\mathcal{F}''_\chi(\omega,\chi) = 0 = X''(\omega,\chi)\mathcal{H}'_X(X(\omega,\chi),\omega,\chi) + (X'(\omega,\chi))^2\mathcal{H}''_{XX}(X(\omega,\chi),\omega,\chi) + 2X'(\omega,\chi)\mathcal{H}''_{\chi X}(X(\omega,\chi),\omega,\chi)$$

From which it follows:

$$X'(\omega,\chi) = -\frac{\mathcal{H}'_\chi(X(\omega,\chi),\omega,\chi)}{\mathcal{H}'_X(X(\omega,\chi),\omega,\chi)}$$

$$X''(\omega,\chi) = -\frac{(X'(\omega,\chi))^2\mathcal{H}''_{XX}(X(\omega,\chi),\omega,\chi) + 2X'(\omega,\chi)\mathcal{H}''_{\chi X}(X(\omega,\chi),\omega,\chi)}{\mathcal{H}'_X(X(\omega,\chi),\omega,\chi)}$$

and evaluating this at $(\omega, 0)$:

$$X'(\omega,0) = -\frac{\mathcal{H}'_\chi(X(\omega,0),\omega,0)}{\mathcal{H}'_X(X(\omega,0),\omega,0)}$$

$$X''(\omega,0) = -\frac{(X'(\omega,0))^2\mathcal{H}''_{XX}(X(\omega,0),\omega,0) + 2X'(\omega,0)\mathcal{H}''_{\chi X}(X(\omega,0),\omega,0)}{\mathcal{H}'_X(X(\omega,0),\omega,0)}$$

We are left to compute the derivative of $\mathcal{H}(X,\omega,\chi)$ and substitute, which yields:

$$X'(\omega,0) = \begin{cases} -\xi(1-\frac{\sigma}{\varepsilon_k})X(\omega,0)^{1/\xi+1} & \text{Under Bertrand} \\ -\xi(\frac{\varepsilon_k}{\sigma}-1)X(\omega,0)^{1/\xi+1} & \text{Under Cournot} \end{cases}$$

$$X''(\omega,0) = \begin{cases} \xi(1-\frac{\sigma}{\varepsilon_k})^2(\xi-\frac{1}{\varepsilon_k-1})X(\omega,0)^{2/\xi+1} & \text{Under Bertrand} \\ \xi(\frac{\varepsilon_k}{\sigma}-1)^2(2+\xi-\frac{1}{\varepsilon_k-1})X(\omega,0)^{2/\xi+1} & \text{Under Cournot} \end{cases}$$

which yields:

$$X(\omega,1) \approx \begin{cases} X(\omega,0)\left(1-\xi(1-\frac{\sigma}{\varepsilon_k})X(\omega,0)^{1/\xi}+\xi(1-\frac{\sigma}{\varepsilon_k})^2(\xi-\frac{1}{\varepsilon_k-1})X(\omega,0)^{2/\xi}\right) & \text{Under Bertrand} \\ X(\omega,0)\left(1-\xi(\frac{\varepsilon_k}{\sigma}-1)X(\omega,0)^{1/\xi}+\xi(\frac{\varepsilon_k}{\sigma}-1)^2(2+\xi-\frac{1}{\varepsilon_k-1})X(\omega,0)^{2/\xi}\right) & \text{Under Cournot} \end{cases}$$

By substituing $X(\omega,1) = s(k,i)^\xi$ and $X(\omega,0) = \widehat{s}(k,i)^\xi$, we get the result. □

## C2. Sectors Aggregation

### C2.1. Sector Markup, Price, Size and Profit

**Proof of Proposition 3..3.** Sector's Size = Sector's Supplier Centrality

In this section, we are going to solve for some measure of sector size, namely

$P_k^\sigma Y_k$. From equation (1), we have

$$P_k Y_k = \beta_k P^C C + \nu_k P^I I + \sum_{l=1}^{N} \omega_{l,k} Y_l \left( \sum_{j=1}^{N_l} \lambda(l,j) \frac{y(l,j)}{Y_l} \right)$$

Remember that the sector level marginal cost $\lambda_l = \sum_{j=1}^{N_l} \lambda(l,j) \frac{y(l,j)}{Y_l}$ and of the sector level markup $\mu_l = \left( \sum_{j=1}^{N_l} \mu(l,j)^{-1} \frac{P(l,j)y(l,j)}{P_l Y_l} \right)^{-1}$. Using that $\lambda(l,j) = \mu(l,j)^{-1} p(l,j)$, it has been shown that $P_l = \mu_l \lambda_l$. Let us substitute this equality in the previous equation.

$$P_k Y_k = \beta_k P^C C + \nu_k P^I I + \sum_{l=1}^{N} \omega_{l,k} \mu_l^{-1} P_l Y_l$$

Let us define $s_k = P_k Y_k$, the vectors $s = \{s_k\}_k$, $\beta = \{\beta_k\}_k$, $\nu = \{\nu_k\}_k$, and the diagonal matrix $\mu^{-1} = diag(\{\mu_k^{-1}\}_k)$. The previous equation becomes in matrix form

$$s' = \beta' P^C C + \nu' P^I I + s' \mu^{-1} \Omega$$

Solving this equation in $s$ yields

$$s' = \beta' \Psi^{(s)} P^C C + \nu' \Psi^{(s)} P^I I$$

with $\Psi^{(s)} = \left( I - \mu^{-1} \Omega \right)^{-1}$. □

**Proof of Proposition 3..2.** Sector's Price We have that

$$\lambda(k,i) = h(k,i)^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$$

with $h(k,i) = \left( \frac{w}{1-\alpha_k} \right)^{(1-\alpha_k)} \left( \frac{r}{\alpha_k} \right)^{\alpha_k} Z(k,i)^{(\alpha_k - 1)}$ by summing over firms in sector $k$ time their output share $\frac{y(k,i)}{Y_k}$ we get

$$\lambda_k = \sum_{i=1}^{N_k} \lambda(k,i) \frac{y(k,i)}{Y_k} = \left( \sum_{i=1}^{N_k} h(k,i)^{\gamma_k} \frac{y(k,i)}{Y_k} \right) \prod_{l=1}^{N} P_l^{\omega_{k,l}}$$

Note that

$$\sum_{i=1}^{N_k} h(k,i)^{\gamma_k} \frac{y(k,i)}{Y_k} = \left( \frac{w}{1-\alpha_k} \right)^{\gamma_k (1-\alpha_k)} \left( \frac{r}{\alpha_k} \right)^{\gamma_k \alpha_k} \sum_{i=1}^{N_k} Z(k,i)^{\gamma_k (\alpha_k - 1)} \frac{y(k,i)}{Y_k}$$

$$= \left( \frac{w}{1-\alpha_k} \right)^{\gamma_k (1-\alpha_k)} \left( \frac{r}{\alpha_k} \right)^{\gamma_k \alpha_k} Z_k^{\gamma_k (\alpha_k - 1)}$$

It follows that

$$\lambda_k = \sum_{i=1}^{N_k} \lambda(k,i)\frac{y(k,i)}{Y_k} = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} Z_k^{\gamma_k(\alpha_k-1)} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$$

Using the fact that $\lambda_k = \mu_k^{-1}P_k$, we have

$$P_k = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} \mu_k Z_k^{\gamma_k(\alpha_k-1)} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$$

taking logs writing these equations in matrix form yields:

$$\log P = \left\{\log\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} \mu_k Z_k^{\gamma_k(\alpha_k-1)}\right\}_k + \Omega \log P$$

solving this matrix equation yields the result.

$$\log P = (I-\Omega)^{-1}\left\{\log\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} \mu_k Z_k^{\gamma_k(\alpha_k-1)}\right\}_k$$

□

**Proof of Proposition** 3..4. Sector's and Firm's Profit

By definition of the sector $k$ profit, we have

$$pro_k = \sum_{i=1}^{N_k} pro(k,i) = \sum_{i=1}^{N_k} p(k,i)y(k,i) - \sum_{i=1}^{N_k} \lambda(k,i)y(k,i)$$

using the definition of $\lambda_k$ and its relationship with $\mu_k$, we have

$$pro_k = P_kY_k - \lambda_kY_k = (1-\mu_k^{-1})P_kY_k$$

using proposition 3..3:

$$pro_k = (1-\mu_k^{-1})P_kY_k = \frac{\mu_k - 1}{\mu_k}\left(\widetilde{\beta}_k(P^C)^\sigma C + \widetilde{\nu}_k(P^I)^\sigma I\right)$$

Firm $i$ profit in sector $k$ is

$$\begin{aligned} pro(k,i) &= \frac{\mu(k,i)-1}{\mu(k,i)} \frac{p(k,i)y(k,i)}{P_kY_k} P_kY_k \\ &= \frac{\mu(k,i)-1}{\mu(k,i)} \frac{\mu_k}{\mu_k - 1} \frac{p(k,i)y(k,i)}{P_kY_k} pro_k \\ &= \frac{\mu(k,i)-1}{\mu(k,i)} \frac{\mu_k}{\mu_k - 1} s(k,i) pro_k \end{aligned}$$

where $s(k,i)=\frac{p(k,i)y(k,i)}{P_kY_k}$ is the sales share of firm $i$ in sector $k$. $\square$

### C2.2. Under a Second Order Approximation 1

### Proof of proposition 9

**Lemma 1** (Sector's Markup under Assumption 1):
*Under assumption 1, the sector $k$'s markup satisfies*

$$\mu_k^{-1}=\begin{cases}\frac{\varepsilon_k-1}{\varepsilon_k} & \text{Under Monopolistic}\\ \frac{\varepsilon_k-1}{\varepsilon_k}-\frac{1}{\varepsilon_k}\left(1-\frac{1}{\varepsilon_k}\right)N_k^{-2\zeta_k}H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}}\left(\overline{Z_k^{(2)}}\right)^2 & \text{Under Bertrand}\\ \frac{\varepsilon_k-1}{\varepsilon_k}-\frac{1}{\varepsilon_k}\Big(\varepsilon_k-1\Big)N_k^{-2\zeta_k}H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}}\left(\overline{Z_k^{(2)}}\right)^2 & \text{Under Cournot}\end{cases}$$

*where* $H_k=N_k^{-\zeta_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k}P_k^{\varepsilon_k}\left(\frac{w}{1-\alpha_k}\right)^{-\varepsilon_k\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{-\varepsilon_k\gamma_k\alpha_k}\left(\prod_{l=1}^N P_l^{-\varepsilon_k\omega_{k,l}}\right)$ *and* $\overline{Z_k^{(n)}}=\left(\sum_{i=1}^{N_k}Z(k,i)^{n(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)^{\frac{1}{n}}$ *is a moment of the sector $k$'s firm productivity distribution.*

**Proof of Lemma 1.** Sector's Markup under Assumption 1

*Bertrand Competition*
Under Bertrand Competition, we have

$$\mu_k^{-1}=1-\frac{1}{\varepsilon_k}\sum_{n=0}^{\infty}\left(1-\frac{1}{\varepsilon_k}\right)^n HK_k^{n+1}(n+1)$$
$$=1-\frac{1}{\varepsilon_k}HK_k^1(1)-\frac{1}{\varepsilon_k}\left(1-\frac{1}{\varepsilon_k}\right)HK_k^2(2)-\frac{\left(1-\frac{1}{\varepsilon_k}\right)^2}{\varepsilon_k}\sum_{n=2}^{\infty}\left(1-\frac{1}{\varepsilon_k}\right)^{n-2}HK_k^{n+1}(n+1)$$

The Hannah-Kay centrality measure is such that

$$HK_k^{n+1}(n+1)=\sum_{i=1}^{N_k}s(k,i)^n$$
$$HK_k^1(1)=\sum_{i=1}^{N_k}s(k,i)=1$$
$$HK_k^2(2)=\sum_{i=1}^{N_k}s(k,i)^2=HHI_k$$

under assumption 1 (i.e without term of order higher than $s(k,i)^3$), the (inverse) of the markup is

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k} - \frac{1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k}\right)\sum_{i=1}^{N_k} s(k,i)^2$$

From assumption 1, we have when $\frac{\varepsilon_k}{\sigma} \to 1$, $s(k,i)^2 = \hat{s}(k,i)^2$ where $\hat{s}(k,i)$ is the sales share of firm $i$ in sector $k$ under Monopolistic competition: $\hat{s}(k,i) = N_k^{-\zeta_k \varepsilon_k} P_k^{\varepsilon_k - 1}\left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{1-\varepsilon_k} \lambda(k,i)^{1-\varepsilon_k}$. Substituing the expression for the marginal cost yields

$$\hat{s}(k,i) = N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k - 1}{\varepsilon_k}} Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k - 1)}$$

Using the above equations give us

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k} - \frac{1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k}\right) N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k - 1}{\varepsilon_k}} \left(\overline{Z_k^{(2)}}\right)^2$$

*Cournot Competition*

From proposition 3..1, we have under the Cournot case

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k} - \frac{1}{\varepsilon_k}(\varepsilon_k - 1)\sum_{i=1}^{N_k} s(k,i)^2$$

Under assumption 1, we have

$$s(k,i)^2 = \hat{s}(k,i)^2$$

where $\hat{s}(k,i)$ is the sales share of firm $i$ in sector $k$ under Monopolistic competition: $\hat{s}(k,i) = N_k^{-\zeta_k \varepsilon_k} P_k^{\varepsilon_k - 1}\left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{1-\varepsilon_k} \lambda(k,i)^{1-\varepsilon_k}$. Substituing the expression for the marginal cost yields

$$\hat{s}(k,i) = N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k - 1}{\varepsilon_k}} Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k - 1)}$$

Using the above equations give us

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k} - \frac{1}{\varepsilon_k}(\varepsilon_k - 1) N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k - 1}{\varepsilon_k}} \sum_{i=1}^{N_k} Z(k,i)^{2(1-\varepsilon_k)\gamma_k(\alpha_k - 1)}$$

□

**Lemma 2** (Sector's Productivity under Assumption 1):

*Under assumption 1, the sector level productivity* $Z_k = \left(\sum_{i=1}^{N_k} Z(k,i)^{\gamma_k(\alpha_k - 1)} \frac{y(k,i)}{Y_k}\right)^{\frac{1}{\gamma_k(\alpha_k - 1)}}$

*satisfies*

$$Z_k^{\gamma_k(\alpha_k-1)} = \begin{cases} H_k\left(\overline{Z_k^{(1)}}\right) & \text{Under Monopolistic} \\ H_k\left(\overline{Z_k^{(1)}}\right) - \frac{\varepsilon_k}{\varepsilon_k-1}\left(1-\frac{1}{\varepsilon_k}\right)H_k^{-1/\varepsilon_k}N_k^{-\zeta_k}H_k^2\left(\overline{Z_k^{(2)}}\right)^2 & \text{Under Bertrand} \\ H_k\left(\overline{Z_k^{(1)}}\right) - \frac{\varepsilon_k}{\varepsilon_k-1}\left(\varepsilon_k-1\right)H_k^{-1/\varepsilon_k}N_k^{-\zeta_k}H_k^2\left(\overline{Z_k^{(2)}}\right)^2 & \text{Under Cournot} \end{cases}$$

*where* $H_k = N_k^{-\zeta_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k}P_k^{\varepsilon_k}\left(\frac{w}{1-\alpha_k}\right)^{-\varepsilon_k\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{-\varepsilon_k\gamma_k\alpha_k}\left(\prod_{l=1}^N P_l^{-\varepsilon_k\omega_{k,l}}\right)$ *and* $\overline{Z_k^{(n)}} = \left(\sum_{i=1}^{N_k} Z(k,i)^{n(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)^{\frac{1}{n}}$ *is a moment of the sector* $k$*'s firm productivity distribution.*

**Proof of Lemma 2**. Sector's Productivity under Assumption 1

*Monopolistic Competition:*

$$Z_k^{\gamma_k(\alpha_k-1)} = \sum_{i=1}^{N_k} Z(k,i)^{\gamma_k(\alpha_k-1)}\frac{y(k,i)}{Y_k}$$

Using the demand face by firm $i$ in sector $k$ and $p(k,i) = \frac{\varepsilon_k}{\varepsilon_k-1}\lambda(k,i)$ yields

$$\frac{y(k,i)}{Y_k} = N_k^{-\varphi_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k}\lambda(k,i)^{-\varepsilon_k}P_k^{\varepsilon_k}$$

The firm's $i$ in sector $k$ marginal cost is equal to $\lambda(k,i) = Z(k,i)^{\gamma_k(\alpha_k-1)}\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\prod_{l=1}^N P_l^{\omega_{k,l}}$. Substituing this last equation in the expression of the output share yields

$$\frac{y(k,i)}{Y_k} = N_k^{-\varphi_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k}P_k^{\varepsilon_k}\left(\frac{w}{1-\alpha_k}\right)^{-\varepsilon_k\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{-\varepsilon_k\gamma_k\alpha_k}\left(\prod_{l=1}^N P_l^{-\varepsilon_k\omega_{k,l}}\right)Z(k,i)^{-\varepsilon_k\gamma_k(\alpha_k-1)}$$

From which it follows that

$$Z_k^{\gamma_k(\alpha_k-1)} = N_k^{-\varphi_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k}P_k^{\varepsilon_k}\left(\frac{w}{1-\alpha_k}\right)^{-\varepsilon_k\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{-\varepsilon_k\gamma_k\alpha_k}\left(\prod_{l=1}^N P_l^{-\varepsilon_k\omega_{k,l}}\right)\sum_{i=1}^{N_k} Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}$$

*Bertrand Competition:*

Since $p(k,i)^{-1} = \mu(k,i)^{-1}\lambda(k,i)^{-1}$, we have $\frac{y(k,i)}{Y_k} = P_k\lambda(k,i)^{-1}\mu(k,i)^{-1}s(k,i)$. Under Bertrand competition

$$\mu_k^{-1} = 1-\varepsilon(k,i)^{-1} = 1-\frac{1}{\varepsilon_k}\left(1-\left(1-\frac{1}{\varepsilon_k}\right)s(k,i)\right)^{-1} = 1-\frac{1}{\varepsilon_k}\sum_{n=0}^{\infty}\left(1-\frac{1}{\varepsilon_k}\right)^n s(k,i)^n$$

thus

$$\frac{y(k,i)}{Y_k} = P_k\lambda(k,i)^{-1}\left(s(k,i)-\frac{1}{\varepsilon_k}\sum_{n=0}^{\infty}\left(1-\frac{1}{\varepsilon_k}\right)^n s(k,i)^{n+1}\right)$$

Under assumption 1, we have (i.e without term of higher order than $\hat{s}(k,i)^3$), we have

$$\frac{y(k,i)}{Y_k} = P_k\lambda(k,i)^{-1}\left(\hat{s}(k,i)\frac{\varepsilon_k-1}{\varepsilon_k} - \left(1-\frac{1}{\varepsilon_k}\right)\hat{s}(k,i)^2\right)$$

From there the proof goes as in the Cournot case by substituting $(\varepsilon_k - 1)$ by $\left(1-\frac{1}{\varepsilon_k}\right)$.

*Cournot Competition:*

We have that $\frac{y(k,i)}{Y_k} = \frac{P_k}{p(k,i)}s(k,i)$ and

$$\begin{aligned} p(k,i)^{-1} &= \mu(k,i)^{-1}\lambda(k,i)^{-1} = \left(1-\varepsilon(k,i)^{-1}\right)^{-1}\lambda(k,i)^{-1} \\ &= \left(\frac{\varepsilon_k-1}{\varepsilon_k} - \frac{1}{\varepsilon_k}(\varepsilon_k-1)s(k,i)\right)\lambda(k,i)^{-1} \end{aligned}$$

thus

$$\frac{y(k,i)}{Y_k} = P_k\lambda(k,i)^{-1}\left(\frac{\varepsilon_k-1}{\varepsilon_k}s(k,i) - \frac{1}{\varepsilon_k}(\varepsilon_k-1)s(k,i)^2\right)$$

Under assumption 1, we have $s(k,i) = \hat{s}(k,i) - \left(\frac{\varepsilon_k}{\sigma}-1\right)\hat{s}(k,i)^2$ and $s(k,i)^2 = \hat{s}(k,i)^2$ where $\hat{s}(k,i)$ is the sales share of firm $i$ in sector $k$ under Monopolistic competition: $\hat{s}(k,i) = N_k^{-\zeta_k\varepsilon_k}P_k^{\varepsilon_k-1}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{1-\varepsilon_k}\lambda(k,i)^{1-\varepsilon_k}$. Substituing these in the output share yields

$$\begin{aligned} \frac{y(k,i)}{Y_k} &= P_k\lambda(k,i)^{-1}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\left(\hat{s}(k,i) - (\varepsilon_k-1)\,\hat{s}(k,i)^2\right) - \frac{1}{\varepsilon_k}(\varepsilon_k-1)\hat{s}(k,i)^2\right) \\ &= P_k\lambda(k,i)^{-1}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\hat{s}(k,i) - (\varepsilon_k-1)\,\hat{s}(k,i)^2\right) \end{aligned}$$

and thus

$$\begin{aligned} \frac{y(k,i)}{Y_k} &= P_k\lambda(k,i)^{-1}\hat{s}(k,i)\left(\frac{\varepsilon_k-1}{\varepsilon_k} - (\varepsilon_k-1)\,\hat{s}(k,i)\right) \\ &= N_k^{-\zeta_k\varepsilon_k}P_k^{\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{1-\varepsilon_k}\lambda(k,i)^{-\varepsilon_k}\left(\frac{\varepsilon_k-1}{\varepsilon_k} - (\varepsilon_k-1)\,N_k^{-\zeta_k\varepsilon_k}P_k^{\varepsilon_k-1}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{1-\varepsilon_k}\lambda(k,i)^{1-\varepsilon_k}\right) \\ &= N_k^{-\zeta_k\varepsilon_k}P_k^{\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k}\lambda(k,i)^{-\varepsilon_k}\left(1 - \frac{\varepsilon_k}{\varepsilon_k-1}(\varepsilon_k-1)\,N_k^{-\zeta_k\varepsilon_k}P_k^{\varepsilon_k-1}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{1-\varepsilon_k}\lambda(k,i)^{1-\varepsilon_k}\right) \end{aligned}$$

Since the firm $i$ in sector $k$ marginal cost is

$$\lambda(k,i) = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\left(\prod_{l=1}^{N}P_l^{\omega_{k,l}}\right)Z(k,i)^{\gamma_k(\alpha_k-1)}$$

thus,

$$\frac{y(k,i)}{Y_k} = H_k Z(k,i)^{-\varepsilon_k\gamma_k(\alpha_k-1)}\left(1-\frac{\varepsilon_k}{\varepsilon_k-1}(\varepsilon_k-1)\,N_k^{-\zeta_k\varepsilon_k}N_k^{\zeta_k(\varepsilon_k-1)}H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}}Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)$$

$$= H_k Z(k,i)^{-\varepsilon_k\gamma_k(\alpha_k-1)}\left(1-\frac{\varepsilon_k}{\varepsilon_k-1}(\varepsilon_k-1)\,N_k^{-\zeta_k}H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}}Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)$$

$$Z(k,i)^{\gamma_k(\alpha_k-1)}\frac{y(k,i)}{Y_k} = H_k Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\left(1-\frac{\varepsilon_k}{\varepsilon_k-1}(\varepsilon_k-1)\,N_k^{-\zeta_k}H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}}Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)$$

Summing over all the firms in sector $k$, yields

$$\sum_{i=1}^{N_k} Z(k,i)^{\gamma_k(\alpha_k-1)}\frac{y(k,i)}{Y_k} = \sum_{i=1}^{N_k}\left(H_k Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}-\frac{\varepsilon_k}{\varepsilon_k-1}(\varepsilon_k-1)\,N_k^{-\zeta_k}H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}+1}Z(k,i)^{2(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)$$

$$= \left(H_k\sum_{i=1}^{N_k} Z(k,i)^{(1-\varepsilon_k)\gamma_k(\alpha_k-1)}-\frac{\varepsilon_k}{\varepsilon_k-1}(\varepsilon_k-1)\,N_k^{-\zeta_k}H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}+1}\sum_{i=1}^{N_k} Z(k,i)^{2(1-\varepsilon_k)\gamma_k(\alpha_k-1)}\right)$$

□

**Proof of Proposition 3..5.** Sector Assumption 1

From proposition 3..2, we have

$$(I-\Omega)\log P = \left\{\log\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\mu_k Z_k^{\gamma_k(\alpha_k-1)}\right\}_k$$

by taking the row $k$ of the above vector equation yields

$$\log P_k - \sum_{k=1}^{N}\omega_{k,l}\log P_l = \log\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\mu_k Z_k^{\gamma_k(\alpha_k-1)}$$

$$\log P_k + \sum_{k=1}^{N}\log P_l^{-\omega_{k,l}} = \log\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\mu_k Z_k^{\gamma_k(\alpha_k-1)}$$

$$\log\left(P_k\prod_{k=1}^{N}P_l^{-\omega_{k,l}}\right) = \log\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\mu_k Z_k^{\gamma_k(\alpha_k-1)}$$

$$P_k\left(\prod_{k=1}^{N}P_l^{\omega_{k,l}}\right)^{-1} = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\mu_k Z_k^{\gamma_k(\alpha_k-1)}$$

*Cournot Competition*

From lemma 1,

$$\mu_k^{-1} = \frac{\varepsilon_k-1}{\varepsilon_k}-\frac{\varepsilon_k-1}{\varepsilon_k}N_k^{-2\zeta_k}H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}}\left(\overline{Z_k^{(2)}}\right)^2$$

and from lemma 2,

$$Z_k^{\gamma_k(\alpha_k-1)} = H_k\left(\overline{Z_k^{(1)}}\right) - \varepsilon_k H_k^{-1/\varepsilon_k} N_k^{-\zeta_k} H_k^2 \left(\overline{Z_k^{(2)}}\right)^2$$

where $H_k = N_k^{-\zeta_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k} P_k^{\varepsilon_k}\left(\frac{w}{1-\alpha_k}\right)^{-\varepsilon_k\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{-\varepsilon_k\gamma_k\alpha_k}\left(\prod_{l=1}^N P_l^{-\varepsilon_k\omega_{k,l}}\right)$. It follows,

$$P_k\left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right)^{-1} = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\frac{H_k\left(\overline{Z_k^{(1)}}\right) - \varepsilon_k H_k^{-1/\varepsilon_k} N_k^{-\zeta_k} H_k^2\left(\overline{Z_k^{(2)}}\right)^2}{\frac{\varepsilon_k-1}{\varepsilon_k} - \frac{\varepsilon_k-1}{\varepsilon_k} N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}}\left(\overline{Z_k^{(2)}}\right)^2}$$

$$P_k\left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right)^{-1} = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} H_k\frac{\varepsilon_k}{\varepsilon_k-1}\frac{\left(\overline{Z_k^{(1)}}\right) - \varepsilon_k H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}} N_k^{-\zeta_k}\left(\overline{Z_k^{(2)}}\right)^2}{1 - N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}}\left(\overline{Z_k^{(2)}}\right)^2}$$

which yields

$$1 = P_k^{-1}\left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right)\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} H_k\frac{\varepsilon_k}{\varepsilon_k-1}\frac{\left(\overline{Z_k^{(1)}}\right) - \varepsilon_k H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}} N_k^{-\zeta_k}\left(\overline{Z_k^{(2)}}\right)^2}{1 - N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}}\left(\overline{Z_k^{(2)}}\right)^2}$$

Note that

$$\begin{aligned} &P_k^{-1}\left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right)\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\frac{\varepsilon_k}{\varepsilon_k-1}H_k \\ &= N_k^{-\zeta_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{(1-\varepsilon_k)} P_k^{\varepsilon_k-1}\left(\frac{w}{1-\alpha_k}\right)^{(1-\varepsilon_k)\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{(1-\varepsilon_k)\gamma_k\alpha_k}\left(\prod_{l=1}^N P_l^{(1-\varepsilon_k)\omega_{k,l}}\right) \\ &= N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}} \end{aligned}$$

Let us define $X_k = N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}}$, thus to solve for the price we can solve the following equation in $X_k$:

$$1 = X_k\frac{\left(\overline{Z_k^{(1)}}\right) - \varepsilon_k X_k\left(\overline{Z_k^{(2)}}\right)^2}{1 - X_k^2\left(\overline{Z_k^{(2)}}\right)^2}$$

which is equivalent to

$$(\varepsilon_k - 1)\left(\overline{Z_k^{(2)}}\right)^2 X_k^2 - \left(\overline{Z_k^{(1)}}\right) X_k + 1 = 0$$

For ease of notation, let us note $A = \left(\overline{Z_k^{(1)}}\right)$ and $B = \left(\overline{Z_k^{(2)}}\right)^2$. This equation has two positive solutions on the real axis if $A^2 - 4(\varepsilon_k - 1)B > 0$. Let us assume that. Note that solving for the monopolitic case is equivalent to solve the above equation for $B=0$ which yeilds $X_k = \frac{1}{A}$. For $B>0$, the two solutions are

$$X_1 = \frac{A - \sqrt{A^2 - 4(\varepsilon_k - 1)B}}{2(\varepsilon_k - 1)B} \qquad \text{and} \qquad X_2 = \frac{A + \sqrt{A^2 - 4(\varepsilon_k - 1)B}}{2(\varepsilon_k - 1)B}$$

Note that when $B \to 0$, $X_1 \to \frac{1}{A}$ whereas $X_2 \to +\infty$. To ensure continuity, we select the former over the latter solution. The admissible solution is thus

$$X_k = \frac{\left(\overline{Z_k^{(1)}}\right) - \sqrt{\left(\overline{Z_k^{(1)}}\right)^2 - 4(\varepsilon_k - 1)\left(\overline{Z_k^{(2)}}\right)^2}}{2(\varepsilon_k - 1)\left(\overline{Z_k^{(2)}}\right)^2}$$

Since,

$$X_k = N_k^{-\zeta_k \varepsilon_k} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{(1-\varepsilon_k)} P_k^{\varepsilon_k - 1} \left(\frac{w}{1-\alpha_k}\right)^{(1-\varepsilon_k)\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{(1-\varepsilon_k)\gamma_k\alpha_k} \left(\prod_{l=1}^{N} P_l^{(1-\varepsilon_k)\omega_{k,l}}\right)$$

$$X_k^{\frac{1}{\varepsilon_k - 1}} = N_k^{-\zeta_k \frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{-1} P_k \left(\frac{w}{1-\alpha_k}\right)^{-\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{-\gamma_k\alpha_k} \left(\prod_{l=1}^{N} P_l^{-\omega_{k,l}}\right)$$

$$P_k = X_k^{\frac{1}{\varepsilon_k - 1}} N_k^{\zeta_k \frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right) \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} \left(\prod_{l=1}^{N} P_l^{\omega_{k,l}}\right)$$

$$\log P_k = \log\left(X_k^{\frac{1}{\varepsilon_k - 1}} N_k^{\zeta_k \frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right) \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\right) + \sum_{l=1}^{N} \omega_{k,l} P_l$$

In matrix form,

$$(I - \Omega)\log P = \left\{\log\left(X_k^{\frac{1}{\varepsilon_k - 1}} N_k^{\varphi_k \frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right) \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\right)\right\}_k$$

$$\log P = (I - \Omega)^{-1}\left\{\log\left(\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} N_k^{\varphi_k \frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right) X_k^{\frac{1}{\varepsilon_k - 1}}\right)\right\}_k$$

To find the expression for the markup, let us note that

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k - 1}{\varepsilon_k}} \left(\overline{Z_k^{(2)}}\right)^2\right) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - X_k^2 B\right)$$

using the fact that $BX_k^2 = \frac{AX_k - 1}{\varepsilon_k - 1}$

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \frac{AX_k - 1}{\varepsilon_k - 1}\right) = \left(\frac{\varepsilon_k - AX_k}{\varepsilon_k}\right)$$

which yields the results.

Finally to compute sector level productivity, note that

$$
\begin{aligned}
Z_k^{\gamma_k(\alpha_k-1)} &= N_k^{\zeta_k} H_k^{\frac{1}{\varepsilon_k}} \left( N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}} A - \varepsilon_k N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}} B \right) \\
&= N_k^{\zeta_k \frac{\varepsilon_k}{\varepsilon_k-1}} X_k^{\frac{1}{\varepsilon_k-1}} \left( X_k A - \varepsilon_k X_k^2 B \right) \\
&= N_k^{\zeta_k \frac{\varepsilon_k}{\varepsilon_k-1}} X_k^{\frac{1}{\varepsilon_k-1}} \left( X_k A - \frac{\varepsilon_k}{\varepsilon_k-1}(AX_k-1) \right) \\
&= N_k^{\zeta_k \frac{\varepsilon_k}{\varepsilon_k-1}} X_k^{\frac{1}{\varepsilon_k-1}} \frac{1}{\varepsilon_k-1} \left( \varepsilon_k - AX_k \right)
\end{aligned}
$$

which yeidls the results.

*Bertrand Competition*

From lemma 1,

$$
\mu_k^{-1} = \frac{\varepsilon_k-1}{\varepsilon_k} - \frac{1}{\varepsilon_k}\frac{\varepsilon_k-1}{\varepsilon_k} N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}} \left(\overline{Z_k^{(2)}}\right)^2
$$

and from lemma 2,

$$
Z_k^{\gamma_k(\alpha_k-1)} = H_k \left(\overline{Z_k^{(1)}}\right) - H_k^{-1/\varepsilon_k} N_k^{-\zeta_k} H_k^2 \left(\overline{Z_k^{(2)}}\right)^2
$$

where $H_k = N_k^{-\zeta_k\varepsilon_k} \left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{-\varepsilon_k} P_k^{\varepsilon_k} \left(\frac{w}{1-\alpha_k}\right)^{-\varepsilon_k\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{-\varepsilon_k\gamma_k\alpha_k} \left(\prod_{l=1}^N P_l^{-\varepsilon_k\omega_{k,l}}\right)$. It follows,

$$
P_k \left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right)^{-1} = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} \frac{H_k \left(\overline{Z_k^{(1)}}\right) - H_k^{-1/\varepsilon_k} N_k^{-\zeta_k} H_k^2 \left(\overline{Z_k^{(2)}}\right)^2}{\frac{\varepsilon_k-1}{\varepsilon_k} - \frac{1}{\varepsilon_k}\frac{\varepsilon_k-1}{\varepsilon_k} N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}} \left(\overline{Z_k^{(2)}}\right)^2}
$$

$$
P_k \left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right)^{-1} = \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} H_k \frac{\varepsilon_k}{\varepsilon_k-1} \frac{\left(\overline{Z_k^{(1)}}\right) - H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}} N_k^{-\zeta_k} \left(\overline{Z_k^{(2)}}\right)^2}{1 - \frac{1}{\varepsilon_k} N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}} \left(\overline{Z_k^{(2)}}\right)^2}
$$

which yields

$$
1 = P_k^{-1} \left(\prod_{k=1}^N P_l^{\omega_{k,l}}\right) \left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)} \left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k} H_k \frac{\varepsilon_k}{\varepsilon_k-1} \frac{\left(\overline{Z_k^{(1)}}\right) - H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}} N_k^{-\zeta_k} \left(\overline{Z_k^{(2)}}\right)^2}{1 - \frac{1}{\varepsilon_k} N_k^{-2\zeta_k} H_k^{2\frac{\varepsilon_k-1}{\varepsilon_k}} \left(\overline{Z_k^{(2)}}\right)^2}
$$

Note that

$$
\begin{aligned}
&P_k^{-1}\left(\prod_{k=1}^{N} P_l^{\omega_{k,l}}\right)\left(\frac{w}{1-\alpha_k}\right)^{\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{\gamma_k\alpha_k}\frac{\varepsilon_k}{\varepsilon_k-1}H_k \\
&= N_k^{-\zeta_k\varepsilon_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{(1-\varepsilon_k)} P_k^{\varepsilon_k-1}\left(\frac{w}{1-\alpha_k}\right)^{(1-\varepsilon_k)\gamma_k(1-\alpha_k)}\left(\frac{r}{\alpha_k}\right)^{(1-\varepsilon_k)\gamma_k\alpha_k}\left(\prod_{l=1}^{N} P_l^{(1-\varepsilon_k)\omega_{k,l}}\right) \\
&= N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}}
\end{aligned}
$$

Let us define $X_k = N_k^{-\zeta_k} H_k^{\frac{\varepsilon_k-1}{\varepsilon_k}}$, thus to solve for the price we can solve the following equation in $X_k$:

$$
1 = X_k \frac{\left(\overline{Z_k^{(1)}}\right) - X_k\left(\overline{Z_k^{(2)}}\right)^2}{1-\frac{1}{\varepsilon_k}X_k^2\left(\overline{Z_k^{(2)}}\right)^2}
$$

which is equivalent to

$$
\left(1-\frac{1}{\varepsilon_k}\right)\left(\overline{Z_k^{(2)}}\right)^2 X_k^2 - \left(\overline{Z_k^{(1)}}\right) X_k + 1 = 0
$$

For ease of notation, let us note $A = \left(\overline{Z_k^{(1)}}\right)$ and $B = \left(\overline{Z_k^{(2)}}\right)^2$. This equation has two positive solutions on the real axis if $A^2 - 4\left(1-\frac{1}{\varepsilon_k}\right)B > 0$. Let us assume that. Note that solving for the monopolitic case is equivalent to solve the above equation for $B = 0$ which yeilds $X_k = \frac{1}{A}$. For $B > 0$, the two solutions are

$$
X_1 = \frac{A - \sqrt{A^2 - 4\left(1-\frac{1}{\varepsilon_k}\right)B}}{2\left(1-\frac{1}{\varepsilon_k}\right)B} \qquad \text{and} \qquad X_2 = \frac{A + \sqrt{A^2 - 4\left(1-\frac{1}{\varepsilon_k}\right)B}}{2\left(1-\frac{1}{\varepsilon_k}\right)B}
$$

Note that when $B \to 0$, $X_1 \to \frac{1}{A}$ whereas $X_2 \to +\infty$. To ensure continuity, we select the former over the latter solution. The admissible solution is thus

$$
X_k = \frac{\left(\overline{Z_k^{(1)}}\right) - \sqrt{\left(\overline{Z_k^{(1)}}\right)^2 - 4\left(1-\frac{1}{\varepsilon_k}\right)\left(\overline{Z_k^{(2)}}\right)^2}}{2\left(1-\frac{1}{\varepsilon_k}\right)\left(\overline{Z_k^{(2)}}\right)^2}
$$

The rest of the proof is similar to the Cournot case. □

# C3. Sector Dynamics

## C3.1. Distribution Dynamics

**Proof of Proposition** 3..6. Sector $k$'s Productivity Distribution Dynamics

For $n > 0$, thanks to assumption 2 we have

$$g_{t+1,n}^{(k)} = f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}$$

where $f_{k,t+1}^{n',n}$ is the number of firms in state $n'$ at $t+1$ that were in state $n$ at time $t$. Given assumption 2 the $3 \times 1$ vector $f_{k,t+1}^{\cdot,n} = (f_{k,t+1}^{n-1,n}, f_{k,t+1}^{n,n}, f_{k,t+1}^{n+1,n})'$ follow a multinomial distribution with number of trial $\mu_{t,n}^{(k)}$ and event probabilities $(a_k, b_k, c_k)'$. It follows that $\mathbb{E}_t\left[f_{k,t+1}^{\cdot,n}\right] = \mu_{t,n}^{(k)}(a_k, b_k, c_k)'$ and $\mathbb{C}ov_t\left[f_{k,t+1}^{\cdot,n}\right] = \mu_{t,n}^{(k)}\Sigma$ with

$$\Sigma = \begin{pmatrix} a(1-a) & -ab & -ac \\ -ab & b(1-b) & -bc \\ -ac & -bc & c(1-c) \end{pmatrix}$$

Note that $f_{k,t+1}^{\cdot,n}$ are idependent across $n$ and thus

$$\mathbb{E}_t\left[g_{t+1,n}^{(k)}\right] = \mathbb{E}_t\left[f_{k,t+1}^{n,n-1}\right] + \mathbb{E}_t\left[f_{k,t+1}^{n,n}\right] + \mathbb{E}_t\left[f_{k,t+1}^{n,n+1}\right] = ag_{t,n+1}^{(k)} + bg_{t,n}^{(k)} + cg_{t,n-1}^{(k)}$$
$$\mathbb{V}ar_t\left[g_{t+1,n}^{(k)}\right] = \mathbb{V}ar_t\left[f_{k,t+1}^{n,n-1}\right] + \mathbb{V}ar_t\left[f_{k,t+1}^{n,n}\right] + \mathbb{V}ar_t\left[f_{k,t+1}^{n,n+1}\right] = a(1-a)g_{t,n+1}^{(k)} + b(1-b)g_{t,n}^{(k)} + c(1-c)g_{t,n-1}^{(k)}$$

To complete the proof let us look at the covariance structure.

$$\mathbb{C}ov_t\left[g_{t+1,n}^{(k)}; g_{t+1,n'}^{(k)}\right] = \mathbb{C}ov_t\left[f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}; f_{k,t+1}^{n',n'-1} + f_{k,t+1}^{n',n'} + f_{k,t+1}^{n',n'+1}\right] = 0 \text{ if } |n-n'| > 2$$

since the $f_{k,t+1}^{\cdot,n}$ are idependent across $n$. For $n' = n+1$, we have:

$$\begin{aligned}
\mathbb{C}ov_t\left[g_{t+1,n}^{(k)}; g_{t+1,n+1}^{(k)}\right] &= \mathbb{C}ov_t\left[f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}; f_{k,t+1}^{n+1,n} + f_{k,t+1}^{n+1,n+1} + f_{k,t+1}^{n+1,n+2}\right] \\
&= \mathbb{C}ov_t\left[f_{k,t+1}^{n,n}; f_{k,t+1}^{n+1,n}\right] + \mathbb{C}ov_t\left[f_{k,t+1}^{n,n+1}; f_{k,t+1}^{n+1,n+1}\right] \\
&= -bcg_{t,n}^{(k)} - abg_{t,n+1}^{(k)}
\end{aligned}$$

using the fact that $\mathbb{C}ov_t\left[f_{k,t+1}^{\cdot,n}\right] = \mu_{t,n}^{(k)}\Sigma$ for all $n > 0$. The same reasoning apply for $n' = n+2$.

For $n = 0$, because of assumption 2 we have

$$g_{t+1,0}^{(k)} = f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1}$$

Given assumption 2 the $2 \times 1$ vector $f_{k,t+1}^{\cdot,0} = (f_{k,t+1}^{0,0}, f_{k,t+1}^{1,0})'$ follow a multinomial distribution with number of trial $g_{t,0}^{(k)}$ and event probabilities $(a_k + b_k, c_k)'$. The same reasoning apply than for $n > 0$.

For $n = M$, because of assumption 2 we have

$$g_{t+1,M}^{(k)} = f_{k,t+1}^{M,M-1} + f_{k,t+1}^{M,M}$$

Given assumption 2 the $2 \times 1$ vector $f_{k,t+1}^{\cdot,M} = (f_{k,t+1}^{M-1,M}, f_{k,t+1}^{M,M})'$ follow a multinomial distribution with number of trial $g_{t,M}^{(k)}$ and event probabilities $(a_k, c_k+b_k)'$. The same reasoning apply than for $n > 0$.

Gathering the results yields that in matrix form:

$$g_{t+1}^{(k)} = (\mathcal{P}^{(k)})' g_t^{(k)} + \epsilon_t^{(k)}$$

where $\epsilon_t^{(k)}$ is the $M \times 1$ vector of $\epsilon_{t,n}^{(k)}$.
□

**Proof of Proposition** 3..7. Sector $k$'s Productivity Stationary Distribution

Let us drop the $(k)$ superscript and subscript to simplify notation. The stationary distribution is a sequence that solve the following system:

$$\begin{array}{ll} (BC1) & g_0 = (a+b)g_0 + a g_1 \\ (BC2) & g_M = c g_{M-1} + (b+c) g_M \\ (EH) & g_n = a g_{n+1} + b g_n + c g_{n-1} \end{array}$$

Let us solve for the general solution of $(EH)$. This equation is a second order linear difference equation eqaivalent to $0 = a g_{n+1} + (b-1) g_n + c g_{n-1} = a g_{n+1} - (a+c) g_n + c g_{n-1}$, with an associated second order plynomial $aX^2 - (a+c)X + c = 0$ which have roots $1$ and $\frac{c}{a}$. The general solution of $(EH)$ is thus $g_n = K_1 + K_2 \left(\frac{c}{a}\right)^n$ where $K_1$ and $K_2$ are constant to solve for.

Let us substitute this general solution in the equation $(BC1)$, it yields

$$K_1 + K_2 = (a+b)(K_1 + K_2) + aK_1 + aK_2 \frac{c}{a} = (2a+b)K1 + (a+b+c)K2$$

since $a+b+c = 1$, $(BC1)$ implies $K_1 = (2a+b)K_1$. Since $a < c$ and $a+b+c = 1$, then $2a + b \neq 1$ and thus $K_1 = 0$. The general solution of this sytem is then $g_n = K_2 \left(\frac{c}{a}\right)^n$. It is trivial to see that $(BC2)$ is satisfied by this general solution. Since $n = \frac{\log \varphi^n}{\log \varphi}$, thus $\left(\frac{c}{a}\right)^n = \exp\left(-s \log \frac{a}{c}\right) = \exp\left(-\frac{\log \varphi^n}{\log \varphi} \log \frac{a}{c}\right) = (\varphi^n)^{-\delta}$ with $\delta = \frac{\log \frac{a}{c}}{\log \varphi}$. It follows that $g_n = K_2 (\varphi^n)^{-\delta}$

To solve for $K_2$, let us use the fact that $g_n$ has to sum to $N_k$.

$$N_k = \sum_{n=0}^{M} g_n = K_2 \sum_{n=0}^{M} \left(\varphi^{-\delta}\right)^n = K_2 \frac{1 - \left(\varphi^{-\delta}\right)^{M+1}}{1 - \varphi^{-\delta}}$$

since $\varphi^{-\delta} < 1$. It follows that $K_2 = N_k \frac{(1-\varphi^{-\delta})}{1-(\varphi^{-\delta})^{M+1}}$ and $g_n^{(k)} = N_k \frac{(1-\varphi^{-\delta})}{1-(\varphi^{-\delta})^{M+1}} (\varphi^n)^{-\delta}$. □

## C3.2. Dynamics of Moments

Let us define $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z_t(k,i)^\xi$ the $\xi$th moment of the productivity distribution within sector $k$ at time $t$. Note that since productivity evolves on the discrete state space $\Phi_k = \{1, \varphi_k, \cdots, \varphi_k^n, \cdots, \varphi_k^{M_k}\}$, we can rewrite $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z_t(k,i)^\xi = \sum_{i=1}^{N_k} \varphi_k^{\xi n_{t,k,i}}$ where $n_{t,k,i}$ is such that the firm $i$ in sector $k$ has a producitivity level

$\varphi^{n_{t,k,i}}$ at time $t$. It follows that $MZ_{t,k}(\xi) = \sum_{n=0}^{M_k}(\varphi_k^n)^\xi g_{t,n}^{(k)}$ by instead of summing over firms $i$, summing over productivity level $\varphi_k^n$. Below, I am showing two lemmas that totally described the dynamics of the moments $MZ_{t,k}(\xi)$ for any $\xi$. With these results in hand I am then characterizing the dynamics of the two moments of interest: $\overline{Z_{t,k}^{(1)}}$ and $\Delta_{t,k}$.

**Lemma 3** (Dynamics of Moments of the Productivity Distribution): *Under assumption 2, the $\xi$th moment of the productivity distribution within sector $k$, $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z(k,i)^\xi$, satisfies*

$$MZ_{t+1,k}(\xi) = \rho_k(\xi)MZ_{t,k}(\xi) + O_{t,k}^M(\xi) + \sigma_{t,k}(\xi)\varepsilon_t$$
$$\sigma_{t,k}(\xi)^2 = \varrho_k(\xi)MZ_{t,k}(2\xi) + O_{t,k}^\sigma(\xi)$$

*where $\varepsilon_t$ is an iid (across $t$ and $k$) random variable following a $\mathcal{N}(0,1)$, $\rho_k(\xi) = a_k\varphi^{-\xi} + b_k + c_k\varphi^\xi$, and $\varrho_k(\xi) = a_k\varphi^{-2\xi} + b_k + c_k\varphi^{2\xi} - \rho_k(\xi)^2$.*

**Proof of Lemma 3.** Dynamics of Moments of the Productivity Distribution

Note first that

$$MZ_{t+1,k}(\xi) = \sum_{i=1}^{N_k} Z_{t+1}(k,i)^\xi = \sum_{i=1}^{N_k} \varphi_k^{\xi n_{t+1,k,i}} = \sum_{n=0}^{M_k}(\varphi_k^n)^\xi g_{t+1,n}^{(k)}$$

where $g_{t+1,n}^{(k)}$ is a stochastic as shown in proposition 3..6. In the proof of this proposition we have shown that for $n > 0$

$$g_{t+1,n}^{(k)} = f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}$$

where $f_{k,t+1}^{n',n}$ is the number of firms in state $n'$ at $t+1$ that were in state $n$ at time $t$. Given assumption 2 the $3 \times 1$ vector $f_{k,t+1}^{\cdot,n} = (f_{k,t+1}^{n-1,n}, f_{k,t+1}^{n,n}, f_{k,t+1}^{n+1,n})'$ follow a multinomial distribution with number of trial $g_{t,n}^{(k)}$ and event probabilities $(a_k, b_k, c_k)'$. In other words,

$$f_{k,t+1}^{\cdot,n} = \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(\mu_{t,n}^{(k)}, \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}\right)$$

Severini (2005) (p377 example 12.7) show that a multinomial distribution can be approximate (i.e converge in distribution) by a multivariate normal distribution:

$$\frac{1}{\sqrt{g_{t,n}^{(k)}}}\left(f_{k,t+1}^{\cdot,n} - g_{t,n}^{(k)}\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}\right) \underset{g_{t,n}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z \rightsquigarrow \mathcal{N}(0,\Sigma)$$

where

$$\Sigma = \begin{pmatrix} a(1-a) & -ab & -ac \\ -ab & b(1-b) & -bc \\ -ac & -bc & c(1-c) \end{pmatrix}$$

For $n = 0$, we have

$$g_{t+1,0}^{(k)} = f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1}$$

Given assumption 2 the $2 \times 1$ vector $f_{k,t+1}^{\cdot,0} = (f_{k,t+1}^{0,0}, f_{k,t+1}^{1,0})'$ follow a multinomial distribution with number of trial $g_{t,0}^{(k)}$ and event probabilities $(a_k + b_k, c_k)'$. Using the same result in Severini (2005),

$$\frac{1}{\sqrt{g_{t,0}^{(k)}}} \left( f_{k,t+1}^{\cdot,0} - g_{t,0}^{(k)} \left( \begin{smallmatrix} a_k + b_k \\ c_k \end{smallmatrix} \right) \right) \xrightarrow[g_{t,0}^{(k)} \to \infty]{\mathcal{D}} Z \rightsquigarrow \mathcal{N}(0, \Sigma_0)$$

where

$$\Sigma_0 = \begin{pmatrix} c(1-c) & -c(1-c) \\ -c(1-c) & c(1-c) \end{pmatrix}$$

For $n = M$, we have

$$g_{t+1,0}^{(k)} = f_{k,t+1}^{M,M} + f_{k,t+1}^{M,M-1}$$

Given assumption 2 the $2 \times 1$ vector $f_{k,t+1}^{\cdot,M} = (f_{k,t+1}^{M-1,M}, f_{k,t+1}^{M,M})'$ follow a multinomial distribution with number of trial $g_{t,M}^{(k)}$ and event probabilities $(a_k, b_k + c_k)'$. Using the same result in Severini (2005),

$$\frac{1}{\sqrt{g_{t,0}^{(k)}}} \left( f_{k,t+1}^{\cdot,M} - g_{t,M}^{(k)} \left( \begin{smallmatrix} a_k \\ b_k + c_k \end{smallmatrix} \right) \right) \xrightarrow[g_{t,M}^{(k)} \to \infty]{\mathcal{D}} Z \rightsquigarrow \mathcal{N}(0, \Sigma_0)$$

where

$$\Sigma_M = \begin{pmatrix} a(1-a) & -a(1-a) \\ -a(1-a) & a(1-a) \end{pmatrix}$$

Let us keep this results in mind and let us go back to (I drop the subscript $k$ to keep the notation parcimonious)

$$
\begin{aligned}
MZ_{t+1,k}(\xi) &= \sum_{n=0}^{M} (\varphi^n)^\xi G^{(k)}_{t+1,n} = g^{(k)}_{t+1,0} + \sum_{n=1}^{M-1} (\varphi^n)^\xi g^{(k)}_{t+1,n} + (\varphi^M)^\xi g^{(k)}_{t+1,M} \\
&= f^{0,0}_{k,t+1} + f^{0,1}_{k,t+1} + \sum_{n=1}^{M-1} (\varphi^n)^\xi \left( f^{n,n-1}_{k,t+1} + f^{n,n}_{k,t+1} + f^{n,n+1}_{k,t+1} \right) + (\varphi^M)^\xi \left( f^{M,M-1}_{k,t+1} + f^{M,M}_{k,t+1} \right) \\
&= f^{0,0}_{k,t+1} + f^{0,1}_{k,t+1} + \sum_{n=1}^{M-1} (\varphi^\xi)^n f^{n,n-1}_{k,t+1} + \sum_{n=1}^{M-1} (\varphi^\xi)^n f^{n,n}_{k,t+1} + \sum_{n=1}^{M-1} (\varphi^\xi)^n f^{n,n+1}_{k,t+1} + (\varphi^M)^\xi \left( f^{M,M-1}_{k,t+1} + f^{M,M}_{k,t+1} \right) \\
&= f^{0,0}_{k,t+1} + f^{0,1}_{k,t+1} + \sum_{n=0}^{M-2} (\varphi^\xi)^{n+1} f^{n+1,n}_{k,t+1} + \sum_{n=1}^{M-1} (\varphi^\xi)^n f^{n,n}_{k,t+1} + \sum_{n=2}^{M} (\varphi^\xi)^{n-1} f^{n-1,n}_{k,t+1} + (\varphi^M)^\xi \left( f^{M,M-1}_{k,t+1} + f^{M,M}_{k,t+1} \right. \\
&= f^{0,0}_{k,t+1} + \varphi^\xi f^{1,0}_{k,t+1} + \sum_{n=1}^{M-1} (\varphi^\xi)^n \left( \varphi^\xi f^{n+1,n}_{k,t+1} + f^{n,n}_{k,t+1} + \varphi^{-\xi} + f^{n-1,n}_{k,t+1} \right) + (\varphi^\xi)^M \left( f^{M,M}_{k,t+1} + \varphi^{-\xi} f^{M-1,M}_{k,t+1} \right) \\
&= \begin{pmatrix} 1 \\ \varphi^\xi \end{pmatrix}' \begin{pmatrix} f^{0,0}_{k,t+1} \\ f^{1,0}_{k,t+1} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^\xi)^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}' \begin{pmatrix} f^{n-1,n}_{k,t+1} \\ f^{n,n}_{k,t+1} \\ f^{n+1,n}_{k,t+1} \end{pmatrix} + (\varphi^\xi)^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f^{M-1,M}_{k,t+1} \\ f^{M,M}_{k,t+1} \end{pmatrix} \\
&= \left( \rho_{k,0} g^{(k)}_{t,0} + \sqrt{\varrho_{k,0} g^{(k)}_{t,0}} \varepsilon_{t+1,0} \right) + \sum_{n=1}^{M-1} (\varphi^\xi)^n \left( \rho_k g^{(k)}_{t,n} + \sqrt{\varrho_k g^{(k)}_{t,n}} \varepsilon_{t+1,n} \right) \dots \\
&\quad \dots + (\varphi^\xi)^M \left( \rho_{k,M} g^{(k)}_{t,M} + \sqrt{\varrho_{k,M} g^{(k)}_{t,n}} \varepsilon_{t+1,M} \right)
\end{aligned}
$$

Since $\begin{pmatrix} f^{n-1,n}_{k,t+1} \\ f^{n,n}_{k,t+1} \\ f^{n+1,n}_{k,t+1} \end{pmatrix} \approx Z \rightsquigarrow \mathcal{N}\left( \mu^{(k)}_{t,n} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}, \mu^{(k)}_{t,n} \Sigma \right)$ it follows that $\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}' \begin{pmatrix} f^{n-1,n}_{k,t+1} \\ f^{n,n}_{k,t+1} \\ f^{n+1,n}_{k,t+1} \end{pmatrix} \approx$ $\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}' Z \rightsquigarrow \mathcal{N}\left( \mu^{(k)}_{t,n} \begin{pmatrix} x^{-\xi} \\ 1 \\ x^\xi \end{pmatrix}' \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}, \mu^{(k)}_{t,n} \begin{pmatrix} x^{-\xi} \\ 1 \\ x^\xi \end{pmatrix}' \Sigma \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix} \right) = \mathcal{N}\left( \mu^{(k)}_{t,n} \rho_k, \mu^{(k)}_{t,n} \varrho_k \right)$. where $\rho_k = a_k \varphi^{-\xi} + b_k + c_k \varphi^\xi$ and $\varrho_k = a_k \varphi^{-2\xi} + b_k + c_k \varphi^{2\xi} - \rho_k^2$. The same reasoning apply for $n = M$ with $\rho_{k,M} = \rho_k + c(1 - \varphi^\xi) := \rho_k + \widetilde{\rho}_{k,M}$ and $\varrho_{k,M} = \varrho_k - c(1-c)(1 - x^{2\xi}) - 2cb(1 - \varphi^\xi) - 2ca(1 - \varphi^\xi) := \varrho_k + \widetilde{\varrho}_{k,M}$. The same reasoning apply for $n = 0$ with $\rho_{k,0} = \rho_k + a(1 - \varphi^{-\xi}) := \rho_k + \widetilde{\rho}_{k,0}$ and $\varrho_{k,0} = \varrho_k - a(1-a)(1 - x^{-2\xi}) - 2ab(1 - \varphi^{-\xi}) - 2ac(1 - \varphi^{-\xi}) := \varrho_k + \widetilde{\varrho}_{k,0}$. From this it follows that

$$
\begin{aligned}
MZ_{t+1,k}(\xi) &= \left( \widetilde{\rho}_{k,0} g^{(k)}_{t,0} \right) + \rho_k \sum_{n=0}^{M} (\varphi^\xi)^n g^{(k)}_{t,n} + (\varphi^\xi)^M \left( \widetilde{\rho}_{k,M} g^{(k)}_{t,M} \right) + \sigma_{t,k}(\xi) \varepsilon_{t+1} \\
&= \rho_k(\xi) MZ_{t,k}(\xi) + O^M_{t,k}(\xi) + \sigma_{t,k}(\xi) \varepsilon_{t+1}
\end{aligned}
$$

Where $O^M_{t,k}(\xi) = \widetilde{\rho}_{k,0} g^{(k)}_{t,0} + (\varphi^\xi)^M \widetilde{\rho}_{k,M} g^{(k)}_{t,M}$ .Since the $\varepsilon_{t+1,n}$ are independent across

$n$, the variance of $\sigma_{t,k}(\xi)\varepsilon_t$ is the sum of the variances of $\sqrt{\varrho_k g_{t,n}^{(k)}}\varepsilon_{t+1,n}$ i.e

$$\begin{aligned}
\sigma_{t,k}(\xi)^2 &= \varrho_{k,0} g_{t,0}^{(k)} + \sum_{n=1}^{M-1} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \varrho_{k,M} g_{t,n}^{(k)} \\
&= \left(\varrho_k + \widetilde{\varrho}_{k,0}\right) g_{t,0}^{(k)} + \sum_{n=1}^{M-1} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \left(\varrho_k + \widetilde{\varrho}_{k,M}\right) g_{t,n}^{(k)} \\
&= \widetilde{\varrho}_{k,0} g_{t,0}^{(k)} + \sum_{n=0}^{M} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \widetilde{\varrho}_{k,M} g_{t,n}^{(k)} \\
&= \varrho_k(\xi) MZ_{t,k}(2\xi) + O_{t,k}^{\sigma}(\xi)
\end{aligned}$$

where $O_{t,k}^{\sigma}(\xi) = \widetilde{\varrho}_{k,0} g_{t,0}^{(k)} + (\varphi^{2\xi})^M \widetilde{\varrho}_{k,M} g_{t,n}^{(k)}$. Moreover, $\varepsilon_{t+1}$ follows a standard normal distribution since the $\varepsilon_{t+1,n}$ are also normaly distributed. □

**Lemma 4** (Covariance of Moments of the Productivity Distribution): *Under assumption 2, the covariance between the $\xi$th moment and the $\xi'$th moment of the productivity distribution within sector $k$ is given by*

$$\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] = \overline{\varrho}_k(\xi,\xi') MZ_{t,k}(\xi'+\xi) + O_{t,k}^{C}(\xi,\xi')$$

*where $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z(k,i)^{\xi}$ and $\overline{\varrho}_k(\xi,\xi') = a(1-a)\varphi^{-(\xi+\xi')} + b(1-b) + c(1-c)\varphi^{\xi+\xi'} - ab(\varphi^{-\xi} + \varphi^{-\xi'}) - ac(\varphi^{-(\xi-\xi')}\varphi^{\xi-\xi'}) - bc(\varphi^{\xi} + \varphi^{\xi'})$.*

**Proof of Lemma 4**. Covariance of Moments of the Productivity Distribution

In the proof of 3, we had

$$MZ_{t+1,k}(\xi) = \begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi})^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi})^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}$$

Thus

$$
\begin{aligned}
&\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] = \\
&\mathbb{C}ov_t\Bigg[\begin{pmatrix}1\\ \varphi^{\xi}\end{pmatrix}'\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^{\xi})^n\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^{\xi}\end{pmatrix}'\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix}+(\varphi^{\xi})^M\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix};\begin{pmatrix}1\\ \varphi^{\xi'}\end{pmatrix}'\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^{\xi'})^n\begin{pmatrix}\varphi^{-\xi'}\\ 1\\ \varphi^{\xi'}\end{pmatrix}'\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix}+(\\
=&\mathbb{C}ov_t\left[\begin{pmatrix}1\\ \varphi^{\xi}\end{pmatrix}'\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix};\begin{pmatrix}1\\ \varphi^{\xi'}\end{pmatrix}'\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix}\right]+\sum_{n=1}^{M-1}\sum_{n'=1}^{M-1}(\varphi^{\xi})^n(\varphi^{\xi'})^{n'}\mathbb{C}ov_t\left[\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^{\xi}\end{pmatrix}'\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix};\begin{pmatrix}\varphi^{-\xi'}\\ 1\\ \varphi^{\xi'}\end{pmatrix}'\begin{pmatrix}f^{n'-1,n'}_{k,t+1}\\ f^{n',n'}_{k,t+1}\\ f^{n'+1,n'}_{k,t+1}\end{pmatrix}\right]+\dots \\
&\quad\dots+(\varphi^{\xi+\xi'})^M\mathbb{C}ov_t\left[\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix};\begin{pmatrix}\varphi^{-\xi'}\\ 1\end{pmatrix}'\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix}\right] \\
=&\mathbb{C}ov_t\left[\begin{pmatrix}1\\ \varphi^{\xi}\end{pmatrix}'\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix};\begin{pmatrix}1\\ \varphi^{\xi'}\end{pmatrix}'\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix}\right]+\sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n\mathbb{C}ov_t\left[\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^{\xi}\end{pmatrix}'\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix};\begin{pmatrix}\varphi^{-\xi'}\\ 1\\ \varphi^{\xi'}\end{pmatrix}'\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix}\right]\dots \\
&\quad\dots+(\varphi^{\xi+\xi'})^M\mathbb{C}ov_t\left[\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix};\begin{pmatrix}\varphi^{-\xi'}\\ 1\end{pmatrix}'\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix}\right]
\end{aligned}
$$

where at the second line, we use the fact that $f^{\cdot,0}_{k,t+1}$ and $f^{\cdot,M}_{k,t+1}$ are independent of the $f^{\cdot,n}_{k,t+1}$ for any $0 < n < M$ and in the third line that $f^{\cdot,n}_{k,t+1}$ are independent across $n$. Using the fact that $\mathbb{C}ov[A'X, B'Y] = A'\mathbb{C}ov[X,Y]B$ for vectors $A$ and $B$ and random vectors $X$ and $Y$ of appropriate size, we have

$$
\begin{aligned}
&\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] = \\
&\begin{pmatrix}1\\ \varphi^{\xi}\end{pmatrix}'\mathbb{C}ov_t\left[\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix};\begin{pmatrix}f^{0,0}_{k,t+1}\\ f^{1,0}_{k,t+1}\end{pmatrix}\right]\begin{pmatrix}1\\ \varphi^{\xi'}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^{\xi}\end{pmatrix}'\mathbb{C}ov_t\left[\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix};\begin{pmatrix}f^{n-1,n}_{k,t+1}\\ f^{n,n}_{k,t+1}\\ f^{n+1,n}_{k,t+1}\end{pmatrix}\right]\begin{pmatrix}\varphi^{-\xi'}\\ 1\\ \varphi^{\xi'}\end{pmatrix}\dots \\
&\quad\dots+(\varphi^{\xi+\xi'})^M\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\mathbb{C}ov_t\left[\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix};\begin{pmatrix}f^{M-1,M}_{k,t+1}\\ f^{M,M}_{k,t+1}\end{pmatrix}\right]\begin{pmatrix}\varphi^{-\xi'}\\ 1\end{pmatrix}
\end{aligned}
$$

Using the definition of $\Sigma$, $\Sigma_0$ and $\Sigma_M$ yields

$$
\begin{aligned}
&\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] = \\
&g^{(k)}_{t,0}\begin{pmatrix}1\\ \varphi^{\xi}\end{pmatrix}'\Sigma_0\begin{pmatrix}1\\ \varphi^{\xi'}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n g^{(k)}_{t,n}\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^{\xi}\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi'}\\ 1\\ \varphi^{\xi'}\end{pmatrix}+(\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\Sigma_M\begin{pmatrix}\varphi^{-\xi'}\\ 1\end{pmatrix}
\end{aligned}
$$

To complete the proof, let us just note that

$$
\begin{aligned}
\begin{pmatrix}\varphi^{-\xi}\\ 1\\ \varphi^{\xi}\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi'}\\ 1\\ \varphi^{\xi'}\end{pmatrix} &= a(1-a)\varphi^{-(\xi+\xi')}+b(1-b)+c(1-c)\varphi^{\xi+\xi'}-ab(\varphi^{-\xi}+\varphi^{-\xi'})-ac(\varphi^{-(\xi-\xi')}\varphi^{\xi-\xi'})-bc(\varphi^{\xi}+\varphi^{\xi'}) \\
\begin{pmatrix}1\\ \varphi^{\xi}\end{pmatrix}'\Sigma_0\begin{pmatrix}1\\ \varphi^{\xi'}\end{pmatrix} &= c(1-c)(1-\varphi^{\xi'}-\varphi^{\xi}+\varphi^{\xi+\xi'}) \\
\begin{pmatrix}\varphi^{-\xi}\\ 1\end{pmatrix}'\Sigma_M\begin{pmatrix}\varphi^{-\xi'}\\ 1\end{pmatrix} &= a(1-a)(1-\varphi^{-\xi'}-\varphi^{-\xi}+\varphi^{-(\xi+\xi')})
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&O^C_{t,k}(\xi,\xi') \\
&\quad = g^{(k)}_{t,0}\left(\left(\begin{smallmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{smallmatrix}\right)'\Sigma\left(\begin{smallmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{smallmatrix}\right)-\left(\begin{smallmatrix}1\\\varphi^{\xi}\end{smallmatrix}\right)'\Sigma_0\left(\begin{smallmatrix}1\\\varphi^{\xi'}\end{smallmatrix}\right)\right)+(\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\left(\left(\begin{smallmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{smallmatrix}\right)'\Sigma\left(\begin{smallmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{smallmatrix}\right)-\left(\begin{smallmatrix}\varphi^{-\xi}\\1\end{smallmatrix}\right)'\Sigma_M\left(\begin{smallmatrix}\varphi^{-\xi'}\\1\end{smallmatrix}\right)\right) \\
&\quad = g^{(k)}_{t,0}\overline{\varrho_{k,0}}+(\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\overline{\varrho_{k,M}}
\end{aligned}
$$

□

**Proof of Proposition 3..8**. Dynamics of $\overline{Z^{(1)}_{t,k}}$ and $\Delta_{t,k}$

Using lemma 3 and the fact that $\left(\overline{Z^{(n)}_{t,k}}\right)^n = MZ_{t,k}\big(n(\varepsilon_k-1)\gamma_k(1-\alpha_k)\big)$, we have

$$
\begin{aligned}
\frac{\overline{Z^{(1)}_{t+1,k}}}{\overline{Z^{(n)}_{t,k}}} &= \rho^{(1)}_k + \frac{O^{M,(1)}_{t,k}}{\overline{Z^{(n)}_{t,k}}} + \frac{\sigma^{(1)}_{t,k}}{\overline{Z^{(n)}_{t,k}}}\varepsilon^{(1)}_{t+1} \\
\left(\frac{\sigma^{(1)}_{t,k}}{\overline{Z^{(n)}_{t,k}}}\right)^2 &= \varrho^{(1)}_k\Delta_{t,k} + \frac{O^{\sigma,(1)}_{t,k}}{\left(\overline{Z^{(n)}_{t,k}}\right)^2}
\end{aligned}
$$

Using a similar reasonning, we have

$$
\begin{aligned}
\Delta_{t+1,k} &= \left(\frac{\overline{Z^{(2)}_{t+1,k}}}{\overline{Z^{(1)}_{t+1,k}}}\right)^2 \\
&= \rho^{(2)}_k\left(\frac{\overline{Z^{(2)}_{t,k}}}{\overline{Z^{(1)}_{t,k}}}\right)^2\left(\frac{\overline{Z^{(1)}_{t,k}}}{\overline{Z^{(1)}_{t+1,k}}}\right)^2+\left(\frac{\overline{Z^{(1)}_{t,k}}}{\overline{Z^{(1)}_{t+1,k}}}\right)^2\frac{O^{M,(2)}_{t,k}}{\left(\overline{Z^{(1)}_{t,k}}\right)^2}+\frac{\sigma^{(2)}_{t,k}}{\left(\overline{Z^{(2)}_{t,k}}\right)^2}\left(\frac{\overline{Z^{(2)}_{t,k}}}{\overline{Z^{(1)}_{t,k}}}\right)^2\left(\frac{\overline{Z^{(1)}_{t,k}}}{\overline{Z^{(1)}_{t+1,k}}}\right)^2\varepsilon^{(2)}_{t+1}
\end{aligned}
$$

with

$$
\frac{\sigma^{(2)}_{t,k}}{\left(\overline{Z^{(2)}_{t,k}}\right)^2} = \varrho^{(2)}_k\left(\frac{\overline{Z^{(4)}_{t,k}}}{\overline{Z^{(2)}_{t,k}}}\right)^4 + \frac{O^{\sigma,(2)}_{t,k}}{\left(\overline{Z^{(2)}_{t,k}}\right)^2}
$$

Finally, it can be shown that

$$\mathbb{C}ov_t\left[\varepsilon^{(1)}_{t+1};\varepsilon^{(2)}_{t+1}\right] = \frac{\varrho^{(1,2)}_k\left(\overline{Z^{(3)}_{t,k}}\right)^3 + O^{C,(1,2)}_{t,k}}{\sigma^{(1)}_{t,k}\sigma^{(2)}_{t,k}}$$
$$= \frac{\frac{\varrho^{(1,2)}_k\left(\overline{Z^{(3)}_{t,k}}\right)^3}{\left(\overline{Z^{(1)}_{t,k}}\right)^2\left(\overline{Z^{(2)}_{t,k}}\right)^2} + \frac{O^{C,(1,2)}_{t,k}}{\left(\overline{Z^{(1)}_{t,k}}\right)^2\left(\overline{Z^{(2)}_{t,k}}\right)^2}}{\left(\varrho^{(1)}_k\Delta_{t,k} + \frac{O^{\sigma,(1)}_{t,k}}{\left(\overline{Z^{(n)}_{t,k}}\right)^2}\right)\left(\varrho^{(2)}_k\left(\frac{\overline{Z^{(4)}_{t,k}}}{\overline{Z^{(2)}_{t,k}}}\right)^4 + \frac{O^{\sigma,(2)}_{t,k}}{\left(\overline{Z^{(2)}_{t,k}}\right)^2}\right)}$$
$$= \frac{\frac{\varrho^{(1,2)}_k\left(\overline{Z^{(3)}_{t,k}}\right)^3}{\left(\overline{Z^{(1)}_{t,k}}\right)^2\left(\overline{Z^{(2)}_{t,k}}\right)^2} + \frac{O^{C,(1,2)}_{t,k}}{\left(\overline{Z^{(1)}_{t,k}}\right)^2\left(\overline{Z^{(2)}_{t,k}}\right)^2}}{\left(\varrho^{(1)}_k\Delta_{t,k}\right)\left(\varrho^{(2)}_k\left(\frac{\overline{Z^{(4)}_{t,k}}}{\overline{Z^{(2)}_{t,k}}}\right)^4\right)}$$

This complete the proof.□

## C4. Closing the Model

**Proof of Proposition 4..1.** Factor's Market Clearing Conditions

The labor market clearing condition is

$$\begin{aligned}
L &= \sum_{k=1}^{N}\sum_{i=1}^{N_k} L(k,i) = \sum_{k=1}^{N}\sum_{i=1}^{N}\gamma_k(1-\alpha_k)\left(\frac{w}{\lambda(k,i)}\right)^{-1} y(k,i)\\
&= w^{-1}\sum_{k=1}^{N}\gamma_k(1-\alpha_k)\sum_{i=1}^{N}\lambda(k,i)y(k,i) = w^{-1}\sum_{k=1}^{N}\gamma_k(1-\alpha_k)\lambda_k Y_k\\
&= w^{-1}\sum_{k=1}^{N}\gamma_k(1-\alpha_k)\mu_k^{-1}P_kY_k\\
L &= w^{-1}\sum_{k=1}^{N}(1-\alpha_k)\gamma_k\mu_k^{-1}\left(\widetilde{\beta_k}P^CC + \widetilde{\nu_k}P^II\right)
\end{aligned}$$

where we use at the second line propostion 2..2: $L(k,i) = \gamma_k(1-\alpha_k)\left(\frac{w}{\lambda(k,i)}\right)^{-1} y(k,i)$, the definition of $\lambda_k$ and the fact that $\lambda_k = \mu_k^{-1}P_k$ at the third line and the proposition 3..3 at the fourth line.

Similarly, the capital market clearing condition wirtes

$$K = r^{-1}\sum_{k=1}^{N}\gamma_k\alpha_k\mu_k^{-1}\left(\widetilde{\beta_k}P^CC + \widetilde{\nu_k}P^II\right)$$

□

## C5. A Special Case

**Proof of Proposition 5..2.** No Capital Case

*Solving for the equilibrium wage*

Without loss of generality, let us normalized the composite consumption good to $1$. It follows that $0 = \log 1 = \log P^C = \beta' \{\log P\}_k$. Using proposition 3..5, we have

$$0 = \beta'(I-\Omega)^{-1}\left\{\log\left(w^{\gamma_k}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2}\right)\right)^{\frac{-1}{\varepsilon_k-1}}\right)\right\}_k$$

$$0 = \beta'(I-\Omega)^{-1}\left\{\{\gamma_k\}_k \log w + \log\left(\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2}\right)\right)^{\frac{-1}{\varepsilon_k-1}}\right)\right\}_k$$

$$\beta'(I-\Omega)^{-1}\{\gamma_k\}_k \log w = -\beta'(I-\Omega)^{-1}\left\{\log\left(\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2}\right)\right)^{\frac{-1}{\varepsilon_k-1}}\right)\right\}_k$$

Note that $\beta'(I-\Omega)^{-1}\{\gamma_k\}_k = \beta'\mathbb{I} = 1$ since the row of $\Omega$ sum to $\{1-\gamma_k\}_k$[10] and since the row of $\beta'$ sum to one. This yields the result

$$\log w = -\beta'(I-\Omega)^{-1}\left\{\log\left(\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2}\right)\right)^{\frac{-1}{\varepsilon_k-1}}\right)\right\}_k$$

*Solving for aggregate consumption*

Using the houshold labor supply condition, we have that $wL = w^{\frac{\chi}{\chi-1}} C^{\frac{-\eta}{\chi-1}}$. By proposition 5..1, we have $C = \frac{wL}{1-\widetilde{\beta}'\left(\frac{\mu-1}{\mu}\right)}$ where we use the fact that $\alpha_k = 0$ and

[10] Which implies $\Omega\mathbb{I} = \{1-\gamma_k\}_k$ and thus $(I-\Omega)\mathbb{I} = \{\gamma_k\}_k$ which implies $\mathbb{I} = (I-\Omega)^{-1}\{\gamma_k\}_k$.

$P^C = 1$. Combining these two together yields

$$C = \frac{w^{\frac{\chi}{\chi+\eta-1}}}{\left(1 - \widetilde{\beta}' \left(\frac{\mu-1}{\mu}\right)\right)^{\frac{\chi-1}{\chi+\eta-1}}}$$

taking logs

$$\log C = \frac{\chi}{\chi+\eta-1} \log w - \frac{\chi-1}{\chi+\eta-1} \log \left(1 - \widetilde{\beta}' \left(\frac{\mu-1}{\mu}\right)\right)$$

substituing the expression for the wage yields

$$\log C = \frac{-\chi}{\chi+\eta-1} \beta'(I-\Omega)^{-1} \left\{ \log \left( \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right) \left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k - 1}} \left( f_k \left( \frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2} \right) \right)^{\frac{-1}{\varepsilon_k - 1}} \right) \right\}_k$$
$$- \frac{\chi-1}{\chi+\eta-1} \log \left(1 - \widetilde{\beta}' \left(\frac{\mu-1}{\mu}\right)\right)$$

where $\widetilde{\beta}' = \beta'(I - \mu^{-1}\Omega)^{-1}$ with $\mu^{-1} = diag\{\mu_k^{-1}\}_k$, and $\left(\frac{\mu-1}{\mu}\right) = \left\{\frac{\mu_k - 1}{\mu_k}\right\}_k = \{1 - \mu_k^{-1}\}_k$.

Note that

$$\mu_k^{-1} = 1 - \frac{1}{\varepsilon_k} f_k \left( \frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2} \right)$$
$$1 - \mu_k^{-1} = \frac{1}{\varepsilon_k} f_k \left( \frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2} \right)$$

thus

$$\log C = \frac{-\chi}{\chi+\eta-1}\beta'(I-\Omega)^{-1}\left\{\log\left(\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)\left(\overline{Z_k^{(1)}}\right)^{\frac{-1}{\varepsilon_k-1}}\left(f_k\left(\frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2}\right)\right)^{\frac{-1}{\varepsilon_k-1}}\right)\right\}_k$$
$$-\frac{\chi-1}{\chi+\eta-1}\log\left(1-\widetilde{\beta}'\left\{\frac{1}{\varepsilon_k}f_k\left(\frac{\overline{Z_k^{(2)}}}{\left(\overline{Z_k^{(1)}}\right)^2}\right)\right\}_k\right)$$

□

**Proof of Corollary** 3..2. Concentration and Centrality

First, I am showing an intermediate results. Let us comput the derivative of $\beta'(I-S\Omega)^{-1}$ with respect of $S_k$ where $S = diag\left(\{S_k\}_k\right)$ is a diagonal matrix.

$$\begin{aligned}\frac{d\beta'(I-S\Omega)^{-1}}{dS_k} &= -\beta'(I-S\Omega)^{-1}\frac{d(I-S\Omega)}{dS_i}(I-S\Omega)^{-1}\\ &= \beta'(I-S\Omega)^{-1}\frac{dS}{dS_i}\Omega(I-S\Omega)^{-1}\\ &= \beta'(I-S\Omega)^{-1}\frac{dS}{dS_i}S^{-1}S\Omega(I-S\Omega)^{-1}\\ &= \beta'(I-S\Omega)^{-1}\frac{d\log S}{dS_i}S\Omega(I-S\Omega)^{-1}\\ &= \beta'(I-S\Omega)^{-1}\frac{d\log S}{dS_i}\left((I-S\Omega)^{-1}-I\right)\end{aligned}$$

Let us then apply this to the vector of sectors' supplier centrality $\widetilde{\beta} = \beta'(I-\mu^{-1}\Omega)^{-1}$ where $\mu^{-1} = diag\left(\{\mu_k^{-1}\}_k\right) = diag\left(\{1-\frac{f_k}{\varepsilon_k}\}_k\right)$ is a diagonal matrix. Applying the above equation yields

$$\begin{aligned}\frac{d\widetilde{\beta}'}{df_k} &= \beta'(I-\mu^{-1}\Omega)^{-1}\frac{d\log\mu^{-1}}{df_k}\left((I-\mu^{-1}\Omega)^{-1}-I\right)\\ &= \widetilde{\beta}'\frac{d\log\mu^{-1}}{df_k}\left(\Psi^{(s)}-I\right)\end{aligned}$$

where $\Psi^{(s)} = (I-\mu^{-1}\Omega)^{-1}$. Note that

$$\frac{d\log\mu^{-1}}{df_k} = \frac{d}{df_k}\left[diag\left(\{\log(1-\frac{f_k}{\varepsilon_k})\}_k\right)\right] = e_k e_k'\frac{-1}{\varepsilon_k\mu_k^{-1}} = -\frac{e_k e_k'}{\varepsilon_k-f_k}$$

where $e_k' = (0,\cdots,0,1,0,\cdots 0)$ is the $k$th base vector of $\mathbb{R}^N$ thus $e_k e_k'$ is the diagonal matrix full of zeros except the $k$th element of the diagonal which is equal to one. It follows that:

$$\frac{d\widetilde{\beta}'}{df_k} = -\widetilde{\beta}' \frac{e_k e_k'}{\varepsilon_k - f_k} \left(\Psi^{(s)} - I\right) = -\frac{\widetilde{\beta}_k}{\varepsilon_k - f_k} e_k' \left(\Psi^{(s)} - I\right)$$

Taking that vector expression at column $l$ yields

$$\frac{d\widetilde{\beta}_l}{df_k} = -\frac{\widetilde{\beta}_k}{\varepsilon_k - f_k} \left(\Psi^{(s)}_{k,l} - \mathbb{I}_{k,l}\right)$$

The same reasonning applies for $\widetilde{\nu}$. □

**Proof of Corollary** 5..3. Concentration and Centrality

*Wage Volatility*

The total derivative of the equilibrium wage is

$$d\log w = \sum_{k=1}^{N} \left( \frac{\partial \log w}{\partial \log \overline{Z^{(1)}}_k} d\log \overline{Z^{(1)}}_k + \frac{\partial \log w}{\partial \log \Delta_k} d\log \Delta_k \right)$$

which is also, after substituing the partial derivative of $w$ equal to

$$d\log w = \sum_{k=1}^{N} \left( \frac{\overline{\beta}_k}{\varepsilon_k - 1} d\log \overline{Z^{(1)}}_k - \frac{\overline{\beta}_k e_k}{\varepsilon_k - 1} d\log \Delta_k \right)$$

where $e_k = \frac{d\log f_k(\Delta_k)}{d\log \Delta_k}$ is the elasticity of the distortion $f_k$ w.r.t to $\Delta_k$. It follows that at the first order

$$\log\left(\frac{w_{t+1}}{w_t}\right) \approx \sum_{k=1}^{N} \left( \frac{\overline{\beta}_k}{\varepsilon_k - 1} \log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right) - \frac{\overline{\beta}_k e_{t,k}}{\varepsilon_k - 1} \log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right) \right)$$

Taking the conditional variance

$$\mathbb{V}ar_t\left[\log\left(\frac{w_{t+1}}{w_t}\right)\right] \approx \sum_{k=1}^{N} \left(\frac{\overline{\beta}_k}{\varepsilon_k - 1}\right)^2 \mathbb{V}ar_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)\right] + \left(\frac{\overline{\beta}_k e_{t,k}}{\varepsilon_k - 1}\right)^2 \mathbb{V}ar_t\left[\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right] \dots$$

$$\dots - 2\left(\frac{\overline{\beta}_k}{\varepsilon_k - 1}\right)\left(\frac{\overline{\beta}_k e_{t,k}}{\varepsilon_k - 1}\right) \mathbb{C}ov_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right); \log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$

since the and $\overline{Z^{(1)}}_{t+1,k}$ the $\Delta_{t+1,k}$ are independent across $k$. Using the proposition

3..8, one can show that (at the first order)

$$\mathbb{V}ar_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)\right] = \varrho_k^{(1)}\Delta_{t,k} + o_{t,k}^{\sigma,(1)}$$
$$\mathbb{V}ar_t\left[\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right] = 4\left(\varrho_k^{(1)}\Delta_{t,k} + o_{t,k}^{\sigma,(1)}\right) + \varrho_k^{(2)}\kappa_{t,k} + o_{t,k}^{\sigma,(2)} - 4\left(\overline{\varrho}_k Skew_{t,k} + o_{t,k}^{C,(2)}\right)$$
$$\mathbb{C}ov_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right);\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right] = \overline{\varrho}_k Skew_{t,k} + o_{t,k}^{C,(2)} - 2\left(\varrho_k^{(1)}\Delta_{t,k} + o_{t,k}^{\sigma,(1)}\right)$$

It follows after substituing

$$\mathbb{V}ar_t\left[\log\left(\frac{w_{t+1}}{w_t}\right)\right] \approx \sum_{k=1}^{N}\left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2\left(\mathbb{V}ar_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)\right] + e_{t,k}^2\mathbb{V}ar_t\left[\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]\dots\right.$$
$$\left.\dots -2e_{t,k}\mathbb{C}ov_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right);\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]\right)$$
$$\approx \sum_{k=1}^{N}\left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2\left(\varrho_k^{(1)}\Delta_{t,k} + o_{t,k}^{\sigma,(1)} + e_{t,k}^2\left(4\left(\varrho_k^{(1)}\Delta_{t,k} + o_{t,k}^{\sigma,(1)}\right) + \varrho_k^{(2)}\kappa_{t,k} + o_{t,k}^{\sigma,(2)} - 4\left(\overline{\varrho}_k Skew_{t,k} + o_{t,k}^{C,(2)}\right)\right)\dots\right.$$
$$\left.\dots - 2e_{t,k}\left(\overline{\varrho}_k Skew_{t,k} + o_{t,k}^{C,(2)} - 2\left(\varrho_k^{(1)}\Delta_{t,k} + o_{t,k}^{\sigma,(1)}\right)\right)\right)$$

Taking the derivative with respect to $\varrho_k^{(1)}$ yields

$$\breve{\beta}_{t,k}^{w} = \left(\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2\Delta_{t,k}\left(4e_{t,k}^2 + 4e_{t,k} + 1\right)$$

*Consumption Volatility*
The total derivative of the aggregate consumption is

$$d\log C = \sum_{k=1}^{N}\left(\frac{\partial\log C}{\partial\log\overline{Z^{(1)}}_k}d\log\overline{Z^{(1)}}_k + \frac{\partial\log C}{\partial\log\Delta_k}d\log\Delta_k\right)$$

which is also, after substituing the partial derivative of $C$ equal to

$$d\log C = \sum_{k=1}^{N}\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}d\log\overline{Z^{(1)}}_k + \dots\right.$$
$$\left.\dots + \left(\frac{-\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}e_k + \frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)e_k\right)d\log\Delta_k\right)$$

where $\widetilde{\varepsilon_k}^{-1} = \sum_{l=1}^{N}\Psi_{k,l}^{(s)}\frac{\mu_l-1}{\mu_l}$ and where $\widetilde{\varepsilon}^{-1} = (I-\mu^{-1}\Omega)^{-1}\left\{\frac{f_k(\Delta_k)}{\varepsilon_k}\right\}_k$ is the elasticity centrality. It follows that at the first order

$$\log\left(\frac{C_{t+1}}{C_t}\right)\approx\sum_{k=1}^{N}\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)+\dots\right.$$
$$\left.\dots+\left(\frac{-\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}+\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)\right)e_k\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right)$$

Taking the conditional variance

$$\mathbb{V}ar_t\left[\log\left(\frac{C_{t+1}}{C_t}\right)\right]\approx\sum_{k=1}^{N}\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2\mathbb{V}ar_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)\right]+\dots$$
$$\dots+\left(\frac{-\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}+\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)\right)^2e_k^2\mathbb{V}ar_t\left[\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$
$$\dots+2\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)\left(\frac{-\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}+\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)\right)e_k\mathbb{C}ov_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right);\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$

since the and $\overline{Z^{(1)}}_{t+1,k}$ the $\Delta_{t+1,k}$ are independent across $k$. Let us introduce some notation to keep this computation simple. Let us defined $A=\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)$ and $B=\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)$, the above expression thus becomes

$$\mathbb{V}ar_t\left[\log\left(\frac{C_{t+1}}{C_t}\right)\right]\approx\sum_{k=1}^{N}A^2\mathbb{V}ar_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)\right]+(B-A)^2e_k^2\mathbb{V}ar_t\left[\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$
$$\dots+2A(-A+B)e_k\mathbb{C}ov_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right);\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$
$$\approx\sum_{k=1}^{N}A^2\mathbb{V}ar_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right)\right]$$
$$+\left(A^2+B^2-2AB\right)e_k^2\mathbb{V}ar_t\left[\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$
$$\dots+2\left(AB-A^2\right)e_k\mathbb{C}ov_t\left[\log\left(\frac{\overline{Z^{(1)}}_{t+1,k}}{\overline{Z^{(1)}}_{t,k}}\right);\log\left(\frac{\Delta_{t+1,k}}{\Delta_{t,k}}\right)\right]$$
$$\approx\sum_{k=1}^{N}A^2\left(\varrho_k^{(1)}\Delta_{t,k}+o_{t,k}^{\sigma,(1)}\right)$$
$$+\left(A^2+B^2-2AB\right)e_k^2\left(4\left(\varrho_k^{(1)}\Delta_{t,k}+o_{t,k}^{\sigma,(1)}\right)+\varrho_k^{(2)}\kappa_{t,k}+o_{t,k}^{\sigma,(2)}-4\left(\overline{\varrho}_kSkew_{t,k}+o_{t,k}^{C,(2)}\right)\right)$$
$$\dots+2\left(AB-A^2\right)e_k\left(\overline{\varrho}_kSkew_{t,k}+o_{t,k}^{C,(2)}-2\left(\varrho_k^{(1)}\Delta_{t,k}+o_{t,k}^{\sigma,(1)}\right)\right)$$

Taking the derivative with respect to $\varrho_k^{(1)}$ yields

$$\breve{\beta}_{t,k}^{C}=\Delta_{t,k}\left(A^2(1+4e_k^2+4e_k)+4B^2e_k^2-4AB\left(2e_k+1\right)e_k\right)$$

which after substituing the expression of $A$ and $B$:

$$\breve{\beta}^{C}_{t,k} = \Delta_{t,k}\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)^2\left(1+4e_k^2+4e_k\right)\dots$$

$$\dots + 4\Delta_{t,k}\left(\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)\right)^2 e_k^2$$

$$\dots - 4\Delta_{t,k}\left(\frac{\chi}{\chi+\eta-1}\frac{\overline{\beta}_k}{\varepsilon_k-1}\right)\left(\frac{\chi-1}{\chi+\eta-1}\left(\frac{Pro}{wL}\right)\left(\frac{pro_k}{Pro}\right)\mu_k\left(1-\widetilde{\varepsilon_k}\right)\right)\left(2e_k+1\right)e_k$$

□