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Second-order approximation of dynamic models with time-varying risk

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ABSTRACT

This paper provides first and second-order approximation methods for the solution of non-linear dynamic stochastic models in which the exogenous state variables follow conditionally linear stochastic processes displaying time-varying risk. The first-order approximation is consistent with a conditionally linear model in which risk is still time-varying but has no distinct role – separated from the primitive stochastic disturbances – in influencing the endogenous variables. The second-order approximation of the solution, instead, is sufficient to get this role. Moreover, risk premia, evaluated using only a first-order approximation of the solution, will be also time varying.

1. Introduction

In the last decade, there has been an increasing interest among researchers and policymakers in developing dynamic general equilibrium models to study business cycle properties of macroeconomic variables and to conduct policy analysis. This research agenda has been accompanied by parallel developments in solution methods and estimation techniques aimed at handling different challenges that richer models pose to economists. For example, second-order approximation techniques have been proposed by Schmitt-Grohé and Uribe (2004) and Benigno and Woodford (2012) to address welfare comparisons across policy regimes while Bayesian analysis has been developed for estimating dynamic general equilibrium models (An and Schorfheide, 2007).

In this work, we propose a solution method for non-linear dynamic stochastic models in which the exogenous stochastic processes display time-varying risk. While the use of models with time-varying risk is quite popular in finance, only recently there has been considerable attention on the role and the effects that risk or uncertainty and their variations over time have on macroeconomic variables.[1] Our solution method is based on appropriately defined first and second-order

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[1] Bloom (2009) examines the effects of an increase in uncertainty on investment and hiring decisions by firms, Bloom et al. (2009) extend a canonical real business cycle model to study the impact of change in the variance to productivity innovation on economic activity while Fernandez-Villaverde et al.

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approximations of the equilibrium conditions which can be effective in studying how time-variation in the exogenous risk influences the equilibrium allocation in standard macroeconomic models. This is in contrast with other solution methods, recently proposed, relying on third-order approximations as in Fernandez-Villaverde et al. (2011a,b).[2]

We consider a class of non-linear dynamic stochastic models in which the exogenous state variables follow conditionally linear stochastic processes where either variances or standard deviations of the primitive shocks are time-varying and modelled through stochastic linear processes. We show that a first-order approximation of the solution can be consistent with a conditionally linear solution in which the process for the exogenous state variables is not approximated and still displays time-varying volatility. Indeed, whether the exogenous state process is approximated or not does not affect the other coefficients of the linear approximation nor the dimension of the relevant endogenous state variables.[3]

There are three clear advantages of following a conditionally linear approximation instead of a fully linear approximation. First, the approximated linear solution would still display a role for time-varying risk in affecting the evolution of the endogenous variables of the model.[4] However, this is not a "distinct and direct" role, since we cannot disentangle the primitive shocks from the shocks to their variance or standard deviation: if shocks are zero, risk does not directly influence the endogenous variables. Second, the fact that stochastic volatility enters the first-order approximation, although not disjointly, has important implications also for higher-order approximations. In particular, we show that a second-order approximation of the policy rules is sufficient to imply a "distinct and direct" role for time-varying volatility in affecting the endogenous variables, whereas with other approaches a more computationally demanding third-order approximation is needed. Third, within a second-order approximation of the model, a conditionally linear approximation, where volatility is still time-varying, can be sufficient to characterize time variation in covariances and therefore in risk premia, whereas a standard linear approximation would only deliver constant risk premia.

Our approach is related to Justiniano and Primiceri (2008) since their *partially nonlinear* approximation, at the level of the first-order approximation of the solution, is consistent with our proposed conditionally linear approximation when the exogenous state variables follow conditionally linear processes. We also provide a second-order approximation of the solution to characterize a distinct role for exogenous risk in affecting the endogenous variables. In particular we consider two models of time-varying volatility, one with a stochastic linear process for the standard deviation of the primitive shocks and another with a linear process for the variance.[5] The latter model is indeed also more parsimonious in the second-order approximation.

Under these two alternative classes of processes for the standard deviation or the variance, our contribution can be also read as a generalization of the second-order approximation methods of Schmitt-Grohé and Uribe (2004), Kim et al. (2008) and Gomme and Klein (2011) to the case in which the exogenous state variables follow heteroskedastic processes. Under more general approaches to model time-varying exogenous uncertainty, recent works by Fernandez-Villaverde et al. (2011a, b) have provided approximation methods relying on third-order approximations to capture the role of exogenous time-varying uncertainty on the endogenous variables. We compare the results between our approach and theirs, using models in which time-varying exogenous uncertainty falls under the two classes of processes considered in this work. In general, our second-order approximation is a better approximation than a standard second-order approximation. Moreover, the direct relationship between variation in exogenous uncertainty and endogenous variables that we find with our method is the same as the one that can be obtained through a standard third-order approximation.

Finally, there are other contributions which have characterized how time-varying risk affects endogenous variables. But in these cases, as in Rudebusch and Swanson (2012), exogenous state variables follow homoskedastic processes as in Schmitt-Grohé and Uribe (2004) and time-varying endogenous (not exogenous) risk affects the endogenous variables only in a third-order approximation. Amisano and Tristani (2009) is instead a more related work since they analyze models where volatility is subject to discrete switching-regime changes and show that the time-varying volatility can affect the second-order approximation.

The structure of this work is the following. Section 2 present a simple example in which the main idea is conveyed and compared with the standard approximation method. Section 3 presents first and second-order approximations in a model in which the exogenous state variables have time-varying linear process for the conditional variance. Section 4 applies our methods to the benchmark neoclassical growth model and evaluate its accuracy. Section 5 concludes.

2. A simple example

Before presenting our solution method in a general form, we write down a simple example to show how it works also in comparison with the standard approximation theory discussed in the literature (Fernandez-Villaverde et al. 2011a,b).

(*footnote continued*)
(2011a) show how changes in the volatility of the foreign real interest rate are an important mechanism in explaining the behavior of output, consumption and investment in emerging market economies.

[2] Bloom et al. (2009), following Krusell and Smith (1998), use instead a value function iteration approach which is more computationally demanding and difficult to implement even in small scale dynamic general equilibrium models.

[3] We follow here the insights of Justiniano and Primiceri (2008) which indeed define a *partially nonlinear* approximation.

[4] This role has been particular relevant for Justiniano and Primiceri (2008) to deliver a model that can be estimated parsimoniously in order to investigate which sources of risk have contributed the most to the fall in macroeconomic volatility associated with the US Great Moderation.

[5] Justiniano and Primiceri (2008) model the log of the standard deviation as a stochastic linear process.

To illustrate it in a sharp way, we consider the simplest asset-price model implied by a one-good endowment economy in a representative-agent utility-maximizing framework. In this model the real interest rate, R_t, is determined by the standard Euler Equation:

$$1 = \beta R_t E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right\} \tag{1}$$

where E_t is the rational expectations operator; β, with $0<\beta<1$, is the subjective discount factor; C_t is consumption and γ measures risk aversion, with $\gamma>0$. In equilibrium, the growth rate of the logarithm of consumption is equal to the growth rate of the logarithm of output. Here we assume that the log-output growth follows the process:

$$x_{t+1} = c_{t+1} - c_t = \rho(c_t - \overline{c}) + u_t \epsilon_{c,t+1} \tag{2}$$

where $z \equiv \log Z$ for every variable Z; $\overline{c}$ is the deterministic level of log consumption; ρ is such that $\rho \leq 0$; $\epsilon_{c,t+1}$ is a white-noise process. Time-varying uncertainty is modelled with a process for the conditional variance of x_{t+1}, denoted by u_t^2, which follows:

$$u_t^2 = (1-\lambda)\overline{u}^2 + \lambda u_{t-1}^2 + \sigma_v^2 \epsilon_{v,t+1} \tag{3}$$

where $\overline{u}^2$ is the steady-state level of the variance; λ is such that $0<\lambda<1$; the shock $\epsilon_{v,t+1}$ is a white-noise process and $\sigma_v>0$ is a scalar. A linear specification, as in (3), is critical for applying our method. Later we extend the analysis also to a linear process for the standard deviation.

We note that the process (2) is linear in the composite shock $\xi_{t+1} \equiv u_t \epsilon_{c,t+1}$ and its conditional expectation is also linear. In fact

$$E_t x_{t+1} = \rho(c_t - \overline{c})$$

Given (2) and (3), the real interest rate is determined in equilibrium through (1).

Under the assumption of a log-normal distribution of the primitive shocks, there is a closed-form solution. Defining $r_t = \ln R_t + \ln \beta$, we get

$$r_t = \gamma E_t x_{t+1} - \frac{\rho^2}{2} Var_t x_{t+1}$$

which can be expressed as a function of the state variables using (2) and (3)

$$r_t = \rho\gamma(c_t - \overline{c}) - \frac{\gamma^2}{2} u_t^2.$$

In this solution the real rate is a function of the two state variables of the model and in particular of the time-varying variance of the shock.

Log-normality is critical to obtain a closed-form solution. But, in general, closed-form solutions are not available and approximation methods can be used. These methods require the shocks $\epsilon_{c,t+1}$ and $\epsilon_{u,t+1}$ to be appropriately bounded in probability or to have bounded support.

The deterministic steady state around which the approximation is taken follows when $\sigma=0$ and $\overline{u}=0$, implying $r_t = \overline{r} = 0$ and $c_t = \overline{c}$. This is the common starting point for both the method presented in the literature and our approach.

Under the standard approximation theory discussed in the literature, a first-order approximation of Eq. (1) around the deterministic steady state implies

$$r_t = \gamma E_t x_{t+1} + \mathcal{O}(\|r_t\|^2, \|x_t\|^2, \|r_t x_t\|), \tag{4}$$

where the term $\mathcal{O}(\cdot)$ collects the remainders of the approximation which are at least of a second order in the norms of the variables in the expansion.

Moreover, as discussed among others by Fernandez-Villaverde et al. (2011a,b), a first-order approximation of the process (2) is also taken

$$x_{t+1} = c_{t+1} - c_t = \rho(c_t - \overline{c}) + \overline{u}\epsilon_{c,t+1} + \mathcal{O}(\|u_t \epsilon_{c,t+1}\|), \tag{5}$$

where indeed the composite term $u_t \epsilon_{c,t+1}$ is approximated. Using the above process into (4), one obtains the first-order approximated solution for the real rate

$$r_t = \rho\gamma(c_t - \overline{c}) + \mathcal{O}(\|r_t\|^2, \|x_t\|^2, \|r_t x_t\|, \|u_t \epsilon_{c,t+1}\|). \tag{6}$$

Our method relies on the same first-order approximation (4) but, as a difference, it does not approximate the process (2) since it takes into account that the latter is already linear in the composite shock $\xi_{t+1} = u_t \epsilon_{c,t+1}$, and that moreover it is a conditionally-linear process. Indeed, when we use (2) into (4), the real rate will be the same function of the state as in (6).

In a first-order approximation, endogenous variables are the same linear function of the state variables under both methods. But there are three important differences: (1) our first-order approximation is not in the class of linear approximations but in the class of conditionally linear approximations; (2) under the standard method the primitive shocks are homoskedastic while under our method, since both (2) and (3) hold, they will be heteroskedastic; (3) under the standard approach, the remainder will be of order at least

$$\mathcal{O}(\|\overline{u}\epsilon_{c,t+1}\|^2, \|u_t\epsilon_{c,t+1}\|) = \mathcal{O}(\overline{u}^2\|\epsilon_{c,t+1}\|^2, \|\xi_{t+1}\|),$$

while under our approach they will be of order at least $\mathcal{O}(\|\xi_{t+1}\|^2)$.

Finally, note that under both methods the impulse response of the real rate with respect to a shock to the variance, $\epsilon_{u,t+1}$, is zero. Therefore there is not a distinct and direct role for time-varying volatility in affecting the endogenous variables of the model.

We now consider the second-order approximation of (1) which implies, under both methods, that

$$r_t = \gamma E_t x_{t+1} - \frac{\gamma^2}{2} Var_t x_{t+1} + \mathcal{O}(\|r_t\|^3, \|x_t\|^3, \|r_t^2 x_t\|, \ldots). \tag{7}$$

Under the standard approach, as in Fernandez-Villaverde et al. (2011a,b), a second-order approximation of (2) is also taken, which in this case recovers the original process for x_{t+1}

$$\begin{aligned} x_{t+1} &= \rho(c_t - \overline{c}) + \overline{u}\epsilon_{c,t+1} + (u_t - \overline{u})\epsilon_{c,t+1} \\ &= \rho(c_t - \overline{c}) + u_t\epsilon_{c,t+1}. \end{aligned} \tag{8}$$

At this stage both methods are consistent with (7), (2) and (3).

However, the differences arise when we express r_t as a function of the state variables. Under the standard approach we obtain:

$$r_t = \rho\gamma(c_t - \overline{c}) - \frac{\gamma^2}{2}\overline{u}^2 + \mathcal{O}(\|r_t\|^3, \|x_t\|^3, \|r_t^2 x_t\|, \ldots) \tag{9}$$

since to evaluate the variance one has to use the first-order approximation of the process (2), i.e. Eq. (5).

Under our approach, coherently with our first-order approximation, the process (2) can be used and the solution will be of the form:

$$r_t = \rho\gamma(c_t - \overline{c}) - \frac{\gamma^2}{2}u_t^2 + \mathcal{O}(\|r_t\|^3, \|x_t\|^3, \|r_t^2 x_t\|, \ldots) \tag{10}$$

where the remainder will be at least of order $\mathcal{O}(\|\xi_{t+1}\|^3)$. Under the standard approach, instead, the remainder will be of order at least $\mathcal{O}(\|\xi_{t+1}\|^2, \overline{u}^3\|\epsilon_{c,t+1}\|^3)$.

The important result is that now our second-order approximation displays a distinct role for volatility in influencing the real interest rate, similar to the log-normal exact solution, whereas under the standard approach the impulse response function to a volatility shock is still zero. To get a distinct role for volatility in affecting the real interest rate, under the standard method, a third-order approximation of (1) is needed:

$$r_t = \gamma E_t x_{t+1} - \frac{\gamma^2}{2} Var_t x_{t+1} + \frac{\gamma^3}{6} E_t x_{t+1}^3 - \frac{\gamma^3}{2} E_t x_{t+1} E_t x_{t+1}^2 + \frac{\gamma^3}{3}(E_t x_{t+1})^3 + \mathcal{O}(\|\cdot\|^4). \tag{11}$$

A third-order approximation of (8) would still deliver (8) and therefore be consistent with (2). According to this process, it follows that

$$E_t x_{t+1} = \rho(c_t - \overline{c})$$

$$E_t x_{t+1}^2 = \rho^2(c_t - \overline{c})^2 + \overline{u}^2$$

$$E_t x_{t+1}^3 = \rho^3(c_t - \overline{c})^3 + 3\rho(c_t - \overline{c})\overline{u}^2.$$

Substituting the previous expressions into (11), we obtain that the third-order terms cancel-out so that (11) can be rewritten as

$$r_t = \gamma E_t x_{t+1} - \frac{\gamma^2}{2} Var_t x_{t+1} + \mathcal{O}(\|\cdot\|^4).$$

However, now, the variance $Var_t x_{t+1}$ can be evaluated using the second-order approximation of the process (2), i.e. the process itself. Therefore, the real rate can be expressed as a function of the state variables as follows:

$$r_t = \rho\gamma(c_t - \overline{c}) - \frac{\gamma^2}{2}u_t^2 + \mathcal{O}(\|\cdot\|^4), \tag{12}$$

where now time-varying volatility has a distinct role in affecting the real rate.

Some observations follow from this example. By comparing the standard approximation method with ours, we note that our first-order approximation is in general a better approximation than a standard first-order approximation. In this special case, our second-order approximation method delivers the same solution as the standard third-order approximation and, as such, it is clearly better than the standard second-order approximation, since its third-order approximation is at least an improvement upon it.

With more general models, it is not necessarily the case that our second-order approximation coincides with the standard third-order approximation, but it will be the case that our first and second-order approximations are better approximations than their standard counterparts. Moreover, even under more general models, it will always be the case that the impulse responses of the endogenous variables with respect to the shock to volatility coincide using our second-order approximation method and a standard third-order approximation.

3. The general model

We consider the following general model which encompasses a wide variety of dynamic stochastic models:

$$E_t\{f(y_{t+1}, x_{t+1}, y_t, x_t)\} = \mathbf{0}, \tag{13}$$

where $E_t\{\cdot\}$ denotes the mathematical expectations operator conditional on the information available at date t and $f(\cdot)$ is a vector, of size n, of functions. The vector y_t, of non-predetermined variables, is of size $n_y \times 1$ while the vector x_t of state variables is of size $n_x \times 1$, with $n_y + n_x = n$. In particular, the vector x_t can be partitioned into a vector of endogenous state variables k_t and a vector of exogenous predetermined variables z_t of size $n_z \times 1$, as follows:

$$x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}.$$

The vector z_t follows the exogenous stochastic process given by

$$z_{t+1} = \Lambda_z z_t + Z\xi_{t+1} \tag{14}$$

where Z and Λ_z are matrices of order $n_z \times n_z$. The vector ξ_{t+1} is also of dimension $n_z \times 1$ and is given by

$$\xi_{t+1} = U_t \varepsilon_{z,t+1} \tag{15}$$

where $\varepsilon_{z,t+1}$ is a $n_z \times 1$ vector of innovations, which are assumed to have a bounded support and to be independently and identically distributed with mean zero and variance/covariance matrix I_z, where I_z is an identity matrix of dimension $n_z \times n_z$; U_t is a diagonal matrix of dimension $n_z \times n_z$ whose elements on the main diagonal are collected into vector u_t, of dimension $n_z \times 1$.

Our solution method applies to two alternative ways of modelling time-varying volatility. Under the first class, u_t^2, the vector containing the squared value of each element of u_t, follows the process:

$$u_{t+1}^2 = \sigma_z^2 (I_z - \Lambda_u)\overline{u}^2 + \Lambda_u u_t^2 + \sigma_v^2 V \varepsilon_{v,t+1} \tag{16}$$

where $\overline{u}$ is a vector of dimension $n_z \times 1$ with $\overline{u}^2$ being a vector of the same dimension whose elements are each the square of the respective element of $\overline{u}$; V and Λ_u are matrices of order $n_z \times n_z$, $\varepsilon_{v,t+1}$ is a vector of innovation of dimension $n_z \times 1$ which are assumed to have a bounded support and to be independently and identically distributed with mean zero and variance/covariance matrix I_z; σ_v and σ_z are scalars with $\sigma_v, \sigma_z \geq 0$. We further assume that the initial condition on u_t^2 is such that $u_{t_0-1}^2 = \sigma_z^2 \overline{u}^2$.

Under the second class of processes considered in this work, we assume that u_t follows the exogenous stochastic linear process given by

$$u_{t+1} = \sigma_z (I_z - \Lambda_u)\overline{u} + \Lambda_u u_t + \sigma_v V \varepsilon_{v,t+1} \tag{17}$$

where V and Λ_u are matrices of order $n_z \times n_z$, $\varepsilon_{v,t+1}$ is a $n_z \times 1$ vector of innovations which again are assumed to have a bounded support and to be independently and identically distributed with mean zero and variance/covariance matrix I_z; $\overline{u}$ is a vector of dimension $n_z \times 1$ while σ_z and σ_v are scalars with σ_z, $\sigma_v \geq 0$. The initial condition on the process for u_t is such that $u_{t_0-1} = \sigma_z \overline{u}$.

Given Eqs. (15) and (16) or (17), the model generalizes the framework of Schmitt-Grohé and Uribe (2004) to a case in which the volatility is time varying and stochastic. In particular, the process for the exogenous state variable (14) is conditionally linear where each element of the vector u_t captures the conditional standard deviation of each element of the stochastic disturbance ξ_{t+1}; such variances or standard deviations are allowed to vary over time in a stochastic way following the autoregressive process described by Eq. (16) or (17). The model boils down to the framework of Schmitt-Grohé and Uribe (2004) under the assumptions $\sigma_v = 0$ and $\overline{u}_i = 1$ for all $i = 1, \ldots, n_z$, since in this case:

$$\xi_{t+1} = \sigma_z \varepsilon_{z,t+1}.$$

We make three important remarks on the above structure which are important to define the class of models of interest in this work. First: the vector u_t is not a distinct argument of the set of equilibrium conditions (13) with respect to what is already captured by the state vector x_t. Second: accordingly, Eq. (16) or (17) are kept separate from the equilibrium conditions that constitute the model, and therefore are not included in Eq. (13).[6] Third: the vector of exogenous state variables z_t follows a conditionally linear process given by (14).

The restrictions due to modelling time-varying volatility as in (16) or (17) limits the generality of our solution method. However, on the other side, the modelling of exogenous time-varying volatility as in our analysis is quite common in the finance literature (see for example Engle, 2001; Bansal and Yaron, 2004). Moreover, the model described in (13) is quite

[6] These remarks are further clarified by the application of Section 4 (see Eq. (41) and footnote 12).

general. For more general class of processes modelling time-varying volatility, standard perturbation approaches can be used.[7]

3.1. Solution

Given the above defined model and structure of the stochastic processes, a solution of (13) takes the form:

$$y_t = g(x_t, u_t, \sigma_z, \sigma_v) \tag{18}$$

$$x_{t+1} = h(x_t, u_t, \sigma_z, \sigma_v) + \overline{h}_\xi \xi_{t+1} \tag{19}$$

for generic functions $g(\cdot)$ and $h(\cdot)$ where $\overline{h}_\xi$ is a known $n_x \times n_z$ matrix:

$$\overline{h}_\xi \equiv \begin{bmatrix} \mathbf{0} \\ Z \end{bmatrix}.$$

We are interested in a second-order approximation of (18) and (19) around a deterministic steady state in which $\sigma_z = \sigma_v = 0$ and $u_t = \sigma_z \overline{u} = \mathbf{0}$. In this deterministic steady state $x_t = \overline{x}$ and $y_t = \overline{y}$ satisfy

$$\overline{y} = g(\overline{x}, \mathbf{0}, 0, 0)$$
$$\overline{x} = h(\overline{x}, \mathbf{0}, 0, 0)$$

or, equivalently

$$f(\overline{y}, \overline{x}, \overline{y}, \overline{x}) = \mathbf{0}.$$

The key insight of our approximation method is that $\overline{h}_\xi$ is a known matrix and that the solution given by (18) and (19) is already linear in the combined shock ξ_{t+1}. Our approximation strategy is to seek approximations for the unknown functions $g(\cdot)$ and $h(\cdot)$ but not for the term $\overline{h}_\xi \xi_{t+1}$ — which is instead perfectly known — in a way that the first and second-order approximations will be still linear in the joint disturbance $\overline{h}_\xi \xi_{t+1}$.

In what follows, we present our method for the case in which the variance of the stochastic disturbance follows the linear process (16). We leave to the appendix the case in which the standard deviation of the exogenous disturbances is linear, as in (17).

3.2. First-order approximation

First, we characterize a first-order approximation of (18) and (19) in which we approximate to a first order the functions $g(\cdot)$ and $h(\cdot)$ while we keep the linearity of the solution with respect to the composite shock ξ_{t+1}. This first-order approximation belongs to the class of conditionally linear approximations. We guess and verify that this approximation takes the form:

$$\tilde{y}_t = \overline{g}_x \tilde{x}_t \tag{20}$$

$$\tilde{x}_{t+1} = \overline{h}_x \tilde{x}_t + \overline{h}_\xi \xi_{t+1} \tag{21}$$

where $\tilde{y}_t \equiv y_t - \overline{y}$, $\tilde{x}_t \equiv x_t - \overline{x}$ and $\overline{g}_x$ and $\overline{h}_x$ are the Jacobian matrices of the functions $g(\cdot)$ and $h(\cdot)$ with respect to x, of size $n_y \times n_x$ and $n_x \times n_x$, respectively, and evaluated at the steady state. To verify this guess, we take a first-order approximation of (13), obtaining

$$D\overline{f}_{\hat{y}} \cdot E_t \tilde{y}_{t+1} + D\overline{f}_{\hat{x}} \cdot E_t \tilde{x}_{t+1} + D\overline{f}_y \cdot \tilde{y}_t + D\overline{f}_x \cdot \tilde{x}_t = 0 \tag{22}$$

where $D\overline{f}_{\hat{y}}$, $D\overline{f}_{\hat{x}}$, $D\overline{f}_y$, and $D\overline{f}_x$ are matrices containing the respective gradients of the vector of functions $f(\cdot)$ taken with respect to the arguments of the function and evaluated at the above-defined steady state. In particular hats denote the gradient with respect to time $t+1$ vectors: $\hat{y}$ stands for y_{t+1} and $\hat{x}$ for x_{t+1}. It is important to note that the first-order approximation described in (22) is the same as under other approaches.

To verify our guess, we plug (20) and (21) into (22) noting that $E_t \xi_{t+1} = 0$. It follows that the matrices $\overline{g}_x$ and $\overline{h}_x$ have to satisfy the following set of $n \times n_x$ conditions:

$$D\overline{f}_{\hat{y}} \overline{g}_x \overline{h}_x + D\overline{f}_y \overline{g}_x + D\overline{f}_{\hat{x}} \overline{h}_x + D\overline{f}_x = \mathbf{0}. \tag{23}$$

The above set of conditions can be solved using standard algorithms. Indeed, it corresponds to that of Schmitt-Grohé and Uribe (2004) in the case in which the volatility is non stochastic: the matrices $\overline{g}_x$ and $\overline{h}_x$ are the same as in their framework.

[7] Fernandez-Villaverde et al. (2011a,b) and Justiniano and Primiceri (2008) model a linear process for the log of the standard deviations to assure that variances remain always positive. This is not necessary in our case since we are assuming a bounded support for the shock $\varepsilon_{v,t}$ which is needed anyway, for the goodness of the approximation. Given that out of the steady state σ_z is positive, it is always possible to find an appropriate lower bound on $\varepsilon_{v,t}$ such that u_t remains bounded above zero.

However, the overall solution given by (20) and (21) does not correspond to their solution since the driving stochastic disturbance is still a non-linear process, which is described by (15). In particular, (20), (21) together with (15) and (17) represent the best conditionally linear solution of (22) given that the exogenous state variables follow (14)–(17) and given that the vector u_t does not enter the set of Eq. (13) nor their arguments. Notice first that (22) just imposes restrictions on the linear approximations of the functions $g(\cdot)$ and $h(\cdot)$ of (18) and (19). Since $E_t\xi_{t+1}=0$, the approximations (20) and (21) are conditionally linear. Moreover since $\overline{h}_\xi$ is known, the best approximation of the term $\overline{h}_\xi\xi_{t+1}$, in Eq. (19), is just the term itself, which is what appears in (21).

In the standard perturbation approach, as in Fernandez-Villaverde et al. (2011a,b), the term ξ_{t+1} is also linearized and therefore a linear approximation of the exogenous state variables takes the form:

$$\tilde{x}_{t+1}=\overline{h}_x\tilde{x}_t+\overline{h}_\xi\overline{U}\sigma_z\varepsilon_{z,t+1}, \tag{24}$$

in which $\overline{U}$ is the diagonal matrix containing the vector $\overline{u}$ on its diagonal. Solution (24) is now in the form of a linear multivariate autoregressive process, but it is not the best conditionally linear approximation of (19). In our approximation (20) and (21) together with (15) and the linear process (17) are all that is needed to characterize the conditionally linear approximation. In Fernandez-Villaverde et al. (2011a,b), it suffices instead to consider (20), (21) and (24) where time-varying volatility does not play any role.[8]

We will show that there are several advantages implied by our conditionally linear approximation. In our case, for example, first-order approximations will retain a role for stochastic volatility, as in Justiniano and Primiceri (2008), although not a distinct role, since risk enters only jointly with the structural shock. In contrast, the first-order approximation of Fernandez-Villaverde et al. (2011a,b) will lose any role for time-varying risk. Importantly, this difference between our and their linear approximation will be also reflected in the second-order approximation and especially in the role that time-varying volatility plays in it. A further advantage of our approach, indeed, is that time-varying volatility will play a "distinct and direct" role in a second-order approximation whereas in Fernandez-Villaverde et al. (2011a,b) a third-order approximation is needed. With "distinct and direct" role, we mean that the impulse response functions of the variables of interest with respect to the primitive volatility shock $\varepsilon_{v,t+1}$ can be in general different from zero.[9] As a consequence, a very appealing implication of our method is that risk premia evaluated using first-order approximations will be time-varying, in contrast to the constant risk premia implied by the framework of Fernandez-Villaverde et al. (2011a,b). In their context, higher-order approximations would be needed to characterize time-varying risk premia.

We conclude this section by noting that a complete linear approximation to (18) and (19) can be represented as

$$\tilde{y}_t=\overline{g}_x\tilde{x}_t+\overline{g}_u u_t+\overline{g}_z\sigma_z+\overline{g}_v\sigma_v$$

$$\tilde{x}_{t+1}=\overline{h}_x\tilde{x}_t+\overline{h}_u u_t+\overline{h}_z\sigma_z+\overline{h}_v\sigma_v+\overline{h}_\xi\xi_{t+1}.$$

However, plugging the above equations into (22) shows that $\overline{g}_u$, $\overline{g}_z$, $\overline{g}_v$, $\overline{h}_u$, $\overline{h}_z$, $\overline{h}_v$ are all zero matrices.

3.3. Second-order approximation

In this section, we characterize a second-order approximation of the solutions (18) and (19). We guess and verify that it takes the form:

$$\tilde{y}_t=\overline{g}_x\tilde{x}_t+\frac{1}{2}(I_y\otimes\tilde{x}'_t)\overline{g}_{xx}\tilde{x}_t+\frac{1}{2}\overline{g}_{uu}u_t^2+\frac{1}{2}\overline{g}_{zz}\sigma_z^2, \tag{25}$$

$$\tilde{x}_{t+1}=\overline{h}_x\tilde{x}_t+\frac{1}{2}(I_x\otimes\tilde{x}'_t)\overline{h}_{xx}\tilde{x}_t+\frac{1}{2}\overline{h}_{uu}u_t^2+\frac{1}{2}\overline{h}_{zz}\sigma_z^2+\overline{h}_\xi\xi_{t+1}, \tag{26}$$

where $\otimes$ denotes the Kronecker product, and $\overline{g}_{xx}$, $\overline{g}_{uu}$, $\overline{g}_{zz}$, $\overline{h}_{xx}$, $\overline{h}_{uu}$, $\overline{h}_{zz}$ are conformable matrices, corresponding to the Magnus-Neudecker Hessian matrices of functions $\overline{g}$ and $\overline{h}$ with respect to the arguments in the indexes.[10] Specifically, $\overline{g}_{xx}$ is defined as

$$\overline{g}_{xx}=\frac{\partial^2 g(x,u,\sigma_z,\sigma_v)}{\partial x\partial x'}=D_x\,g[(D_x g(\overline{x},\mathbf{0},0,0))'],$$

evaluated at the steady state, and consists of n_y vertically stacked symmetric $n_x\times n_x$ matrices ($\overline{g}_{xx}$ is therefore of size $n_y\cdot n_x\times n_x$). The matrix $\overline{h}_{xx}$ is of size $n_x\cdot n_x\times n_x$, $\overline{h}_{zz}$ and $\overline{g}_{zz}$ are of size $n_y\times 1$ and $n_x\times 1$, respectively, while $\overline{g}_{uu}$ and $\overline{h}_{uu}$ are matrices of order $n_y\times n_z$ and $n_x\times n_z$, respectively.[11]

[8] See Justiniano and Primiceri (2008) for further arguments to justify what they call a "*partially nonlinear*" approximation in the same model of Fernandez-Villaverde et al. (2011a,b).

[9] Accordingly, since in our first-order approximation there is no distinct role for volatility in affecting the endogenous variables, the impulse response of any variable with respect to a volatility shock is always zero.

[10] See Magnus and Neudecker (1999). Table 1 in Appendix presents the dimensions of all the matrices involved in the first and second-order approximations.

[11] Notice that the expansion with respect to σ_v^2 is zero up to second-order terms.

To evaluate this guess, we take a second-order approximation of (13), to get

$$0 = E_t\Big\{D\overline{f}^i_{\hat{y}} \cdot \tilde{y}_{t+1} + D\overline{f}^i_{\hat{x}} \cdot \tilde{x}_{t+1} + D\overline{f}^i_{y} \cdot \tilde{y}_t + D\overline{f}^i_{x} \cdot \tilde{x}_t + \frac{1}{2}\tilde{y}'_{t+1} \cdot D\overline{f}^i_{\hat{y}\hat{y}} \cdot \tilde{y}_{t+1}$$
$$+\tilde{x}'_{t+1} \cdot D\overline{f}^i_{\hat{y}\hat{x}} \cdot \tilde{y}_{t+1} + \tilde{y}'_t \cdot D\overline{f}^i_{\hat{y}y} \cdot \tilde{y}_{t+1} + \tilde{x}'_t \cdot D\overline{f}^i_{\hat{y}x} \cdot \tilde{y}_{t+1}$$
$$+\frac{1}{2}\tilde{x}'_{t+1} \cdot D\overline{f}^i_{\hat{x}\hat{x}} \cdot \tilde{x}_{t+1} + \tilde{y}'_t \cdot D\overline{f}^i_{\hat{x}y} \cdot \tilde{x}_{t+1} + \tilde{x}'_t \cdot D\overline{f}^i_{\hat{x}x} \cdot \tilde{x}_{t+1}$$
$$+\frac{1}{2}\tilde{y}'_t \cdot D\overline{f}^i_{yy} \cdot \tilde{y}_t + \tilde{x}'_t \cdot D\overline{f}^i_{yx} \cdot \tilde{y}_t + \frac{1}{2}\tilde{x}'_t \cdot D\overline{f}^i_{xx} \cdot \tilde{x}_t\Big\}, \tag{27}$$

for each $i = 1,\ldots,n$ and where f^i denotes the i-component of the vector f. This can be written in a more compact form as

$$\mathbf{0} = E_t\left\{D\overline{f}\begin{bmatrix}\tilde{y}_{t+1}\\ \tilde{x}_{t+1}\\ \tilde{y}_t\\ \tilde{x}_t\end{bmatrix} + \frac{1}{2}\begin{bmatrix}I_n\otimes\tilde{y}_{t+1}\\ I_n\otimes\tilde{x}_{t+1}\\ I_n\otimes\tilde{y}_t\\ I_n\otimes\tilde{x}_t\end{bmatrix}'H\overline{f}\begin{bmatrix}\tilde{y}_{t+1}\\ \tilde{x}_{t+1}\\ \tilde{y}_t\\ \tilde{x}_t\end{bmatrix}\right\}, \tag{28}$$

where $D\overline{f} \equiv [D\overline{f}_{\hat{y}}\ \ D\overline{f}_{\hat{x}}\ \ D\overline{f}_{y}\ \ D\overline{f}_{x}]$ denotes the $n \times 2n$ Jacobian matrix of function f, and $H\overline{f}$ the corresponding $2n^2 \times 2n$ Magnus–Neudecker Hessian matrix, evaluated at the steady state:

$$H\overline{f} = D\,\mathrm{vec}[(D\overline{f})'].$$

We now evaluate the second-order expansion (28), using Eqs. (20) and (21) to evaluate the second-order terms, taking into account (16), and the second-order guess solutions (25) and (26) to evaluate the first-order terms, taking moreover into account the restrictions implied by (23).

We obtain:

$$\mathbf{0} = \frac{1}{2}E_t\{D\overline{f}_{\hat{y}}[(\overline{g}_x\otimes\tilde{x}'_t)\overline{h}_{xx}\tilde{x}_t + \overline{g}_x\overline{h}_{uu}u_t^2 + \overline{g}_x\overline{h}_{zz}\sigma_z^2 + \overline{g}_{zz}\sigma_z^2$$
$$+[I_y\otimes(\overline{h}_x\tilde{x}_t + \overline{h}_\xi\xi_{t+1})']\overline{g}_{xx}(\overline{h}_x\tilde{x}_t + \overline{h}_\xi\xi_{t+1}) + \sigma_z^2\overline{g}_{uu}(I_z-\Lambda_u)\overline{u}^2 + \overline{g}_{uu}\Lambda_u u_t^2]$$
$$+D\overline{f}_{\hat{x}}[(I_x\otimes\tilde{x}'_t)\overline{h}_{xx}x_t + \overline{h}_{uu}u_t^2 + \overline{h}_{zz}\sigma_z^2] + D\overline{f}_y[(I_y\otimes\tilde{x}'_t)\overline{g}_{xx}x_t + \overline{g}_{uu}u_t^2 + \overline{g}_{zz}\sigma_z^2]$$
$$+[I_n\otimes(\overline{g}_x\overline{h}_x\tilde{x}_t + \overline{g}_x\overline{h}_\xi\xi_{t+1})']H\overline{f}_{\hat{y}} \cdot \tilde{w}_{t+1} + [I_n\otimes(\overline{h}_x\tilde{x}_t + \overline{h}_\xi\xi_{t+1})']H\overline{f}_{\hat{x}} \cdot \tilde{w}_{t+1}$$
$$+(I_n\otimes\tilde{x}'_t\overline{g}'_x)H\overline{f}_y \cdot \tilde{w}_{t+1} + (I_n\otimes\tilde{x}'_t)H\overline{f}_x \cdot \tilde{w}_{t+1}\}, \tag{29}$$

where $\tilde{w}_{t+1} \equiv [\tilde{y}'_{t+1}\ \ \tilde{x}'_{t+1}\ \ \tilde{y}'_t\ \ \tilde{x}'_t]'$ is a $2n \times 1$ vector and $H\overline{f}_{\hat{y}}$, $H\overline{f}_{\hat{x}}$, $H\overline{f}_y$, and $H\overline{f}_x$ are the Magnus-Neudecker Hessian matrices of the vector of functions $f(\cdot)$ taken with respect to the arguments of the function and evaluated at the above-defined steady state, such that

$$H\overline{f} = \begin{bmatrix}H\overline{f}_{\hat{y}}\\ H\overline{f}_{\hat{x}}\\ H\overline{f}_y\\ H\overline{f}_x\end{bmatrix}.$$

Specifically, $H\overline{f}_{\hat{y}}$ is defined as

$$H\overline{f}_{\hat{y}} = D\,\mathrm{vec}[(D\overline{f}_{\hat{y}})'],$$

and analogously for the other terms. Moreover, Eqs. (20) and (21) imply

$$\tilde{w}_{t+1} = \overline{M}_x\tilde{x}_t + \overline{M}_\xi\xi_{t+1}, \tag{30}$$

where M_x and M_ξ are matrices of order $2n \times n_x$ and $2n \times n_z$, respectively, defined by

$$\overline{M}_x \equiv \begin{bmatrix}\overline{g}_x\overline{h}_x\\ \overline{h}_x\\ \overline{g}_x\\ I_x\end{bmatrix}, \quad \overline{M}_\xi \equiv \begin{bmatrix}\overline{g}_x\overline{h}_\xi\\ \overline{h}_\xi\\ \mathbf{0}_{(n_y\times n_z)}\\ \mathbf{0}_{(n_x\times n_z)}\end{bmatrix}. \tag{31}$$

From Eq. (58), and using (30), we can collect the quadratic terms in the vector $\tilde{x}_t$, to obtain

$$\mathbf{0} = \frac{1}{2}E_t\{(D\overline{f}_{\hat{y}} \cdot \overline{g}_x\otimes\tilde{x}'_t)\overline{h}_{xx}\tilde{x}_t + (D\overline{f}_{\hat{y}}\otimes\tilde{x}'_t\overline{h}'_x)\overline{g}_{xx}\overline{h}_x\tilde{x}_t + (D\overline{f}_{\hat{x}}\otimes\tilde{x}'_t)\overline{h}_{xx}x_t$$
$$+(D\overline{f}_y\otimes\tilde{x}'_t)\overline{g}_{xx}x_t + (I_n\otimes\tilde{x}'_t\overline{h}'_x\overline{g}'_x)H\overline{f}_{\hat{y}} \cdot \overline{M}_x\tilde{x}_t + (I_n\otimes\tilde{x}'_t\overline{h}'_x)H\overline{f}_{\hat{x}} \cdot \overline{M}_x\tilde{x}_t$$
$$+(I_n\otimes\tilde{x}'_t\overline{g}'_x)H\overline{f}_y \cdot \overline{M}_x\tilde{x}_t + (I_n\otimes\tilde{x}'_t)H\overline{f}_x \cdot \overline{M}_x\tilde{x}_t\}. \tag{32}$$

Following Gomme and Klein (2011), given a generic $n \cdot m \times m$ matrix A consisting of n square matrices A_i stacked vertically, with $i = 1, \ldots, n$, we define trm(A) as the $n \times 1$ vector of traces of the n matrices A_i:

$$\text{trm}(A) = [\text{tr}(A_1)\ \ \text{tr}(A_2)\ \ \ldots\ \ \text{tr}(A_n)]'.$$

We can use the above operator to show that moment condition (32) implies the following set of $n \cdot n_x \times n_x$ equations:

$$\begin{aligned}\mathbf{0} = &(D\overline{f}_{\hat{y}} \cdot \overline{g}_x \otimes I_x)\overline{h}_{xx} + (D\overline{f}_{\hat{y}} \otimes \overline{h}'_x)\overline{g}_{xx}\overline{h}_x + (D\overline{f}_{\hat{x}} \otimes I_x)\overline{h}_{xx} \\ &+ (D\overline{f}_y \otimes I_x)\overline{g}_{xx} + (I_n \otimes \overline{h}'_x \overline{g}'_x)H\overline{f}_{\hat{y}} \cdot \overline{M}_x + (I_n \otimes \overline{h}'_x)H\overline{f}_{\hat{x}} \cdot \overline{M}_x \\ &+ (I_n \otimes \overline{g}'_x)H\overline{f}_y \cdot \overline{M}_x + H\overline{f}_x \cdot \overline{M}_x,\end{aligned} \tag{33}$$

which can be solved for the unknown matrices $\overline{g}_{xx}$ and $\overline{h}_{xx}$, given $\overline{h}_x$, $\overline{g}_x$, $D\overline{f}$ and $H\overline{f}$.

We can collect the remaining terms:

$$\begin{aligned}\mathbf{0} = E_t\{&D\overline{f}_{\hat{y}}[\overline{g}_x \overline{h}_{uu} u_t^2 + \overline{g}_x \overline{h}_{zz}\sigma_z^2 + \overline{g}_{zz}\sigma_z^2 + (I_y \otimes \xi'_{t+1}\overline{h}'_\xi)\overline{g}_{xx}\overline{h}_\xi \xi_{t+1} \\ &+ \sigma_z^2 \overline{g}_{uu}(I_z - \Lambda_u)\overline{u}^2 + \overline{g}_{uu}\Lambda_u u_t^2] + D\overline{f}_{\hat{x}}[\overline{h}_{uu}u_t^2 + \overline{h}_{zz}\sigma_z^2] + D\overline{f}_y[\overline{g}_{uu}u_t^2 + \overline{g}_{zz}\sigma_z^2] \\ &+ (I_n \otimes \xi'_{t+1}\overline{h}'_\xi \overline{g}'_x)H\overline{f}_{\hat{y}} \cdot \overline{M}_\xi \xi_{t+1} + (I_n \otimes \xi'_{t+1}\overline{h}'_\xi)H\overline{f}_{\hat{x}} \cdot \overline{M}_\xi \xi_{t+1}\}.\end{aligned} \tag{34}$$

Given a generic $n \cdot m \times m$ matrix A consisting of n square matrices A_i stacked vertically, with $i = 1, \ldots, n$, we define dgv(A) as the $m \times n$ matrix that stacks horizontally the main diagonals of each of the $m \times m$ matrices A_i:

$$\text{dgv}(A) = [\text{diagv}(A_1)\ \ \text{diagv}(A_2)\ \ \ldots\ \ \text{diagv}(A_n)],$$

where diagv(A_i) is an $m \times 1$ vector collecting the elements on the main diagonal of A_i. We can use the above operator, together with the matrix trace operator defined above, to show, for generic and conformable matrices A and B:

$$\begin{aligned}E_t\{(I \otimes \xi'_{t+1}A')BA\xi_{t+1}\} &= E_t\{\text{trm}[(I \otimes \xi'_{t+1}A')BA\xi_{t+1}]\} \\ &= \text{trm}[(I \otimes A')BAE_t\{\xi_{t+1}\xi'_{t+1}\}] = \text{trm}[(I \otimes A')BAU_tU'_t] = \text{dgv}[(I \otimes A')BA]'u_t^2.\end{aligned}$$

Using the above to express the quadratic terms in ξ_{t+1} in Eq. (58) in terms of u_t^2, we collect the latter to obtain the following system of $n \times n_z$ conditions:

$$\begin{aligned}\mathbf{0} = &(D\overline{f}_{\hat{y}} \cdot \overline{g}_x + D\overline{f}_{\hat{x}})\overline{h}_{uu} + D\overline{f}_{\hat{y}} \cdot \overline{g}_{uu}\Lambda_u + D\overline{f}_y \cdot \overline{g}_{uu} \\ &+ \text{dgv}[(D\overline{f}_{\hat{y}} \otimes \overline{h}'_\xi)\overline{g}_{xx}\overline{h}_\xi + (I_n \otimes \overline{h}'_\xi \overline{g}'_x)H\overline{f}_{\hat{y}} \cdot \overline{M}_\xi + (I_n \otimes \overline{h}'_\xi)H\overline{f}_{\hat{x}} \cdot \overline{M}_\xi]',\end{aligned} \tag{35}$$

which can be solved for matrices $\overline{h}_{uu}$ and $\overline{g}_{uu}$.

Finally, we can collect the terms in σ_z^2 from Eq. (58), to show that matrices $\overline{h}_{zz}$ and $\overline{g}_{zz}$ solve the following system of $n \times 1$ equations:

$$\mathbf{0} = (D\overline{f}_{\hat{y}}\overline{g}_x + D\overline{f}_{\hat{x}})\overline{h}_{zz} + (D\overline{f}_{\hat{y}} + D\overline{f}_y)\overline{g}_{zz} + D\overline{f}_{\hat{y}}\overline{g}_{uu}(I_z - \Lambda_u)\overline{u}^2. \tag{36}$$

4. Application: the neoclassical growth model

To apply our method to a simple example, we consider the standard neoclassical growth model, as in Schmitt-Grohé and Uribe (2004). We denote consumption with C_t and the capital stock at the beginning of period t with K_t. The parameters β, δ, γ and α represent (respectively) the subjective discount factor, the depreciation rate of capital, relative risk aversion and the return to scale of capital in the production function. The equilibrium conditions of the model are given by

$$K_{t+1} - e^{a_t}K_t^\alpha - (1-\delta)K_t + C_t = 0 \tag{37}$$

$$E_t\left\{\beta[\alpha e^{a_{t+1}}K_{t+1}^{\alpha-1} + (1-\delta)]\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right\} - 1 = 0 \tag{38}$$

$$a_{t+1} = \rho a_t + u_t \varepsilon_{a,t+1} \tag{39}$$

$\forall t \geq 0$, given K_0 and $a_0 = 0$; where a_t denotes the log of the productivity shock. In particular, the innovation $\varepsilon_{a,t+1}$ to the log-productivity process (39) is identically and independently distributed process with mean zero and unitary variance; u_t captures the time-varying conditional standard deviation of a_{t+1} and ρ is a parameter, with $0 \leq \rho < 1$. We model the square of u_t, i.e. the conditional variance of a_{t+1}, as an exogenous stochastic linear process:

$$u_{t+1}^2 = (1-\lambda)\sigma_a^2\overline{u}^2 + \lambda u_t^2 + \sigma_v^2 \varepsilon_{v,t+1} \tag{40}$$

with initial condition $u_0^2 = \sigma_a^2\overline{u}^2$ where λ is a coefficient such that $0 \leq \lambda < 1$, while σ_a and σ_v are non-negative scalars; $\overline{u}$ is a positive parameter and the innovation $\varepsilon_{a,t+1}$ is identically and independently distributed process with mean zero and unitary variance. Notice that since $E_t(u_t\varepsilon_{a,t+1}) = 0$, the log-productivity process (39) is a conditionally linear stochastic process.

We can cast this model in the general notation of Section 2. Defining $c_t \equiv \ln C_t$, $k_t \equiv \ln K_t$, we can write $y_t = [c_t]$ and $x_t = [k_t, a_t]$ and therefore[12]

$$E_t\{f(y_{t+1}, y_t, x_{t+1}, x_t)\} = E_t \begin{bmatrix} \beta[\alpha e^{a_{t+1}+(\alpha-1)k_{t+1}} + (1-\delta)]e^{-\gamma c_{t+1}} - e^{-\gamma c_t} \\ e^{k_{t+1}} - e^{a_t+\alpha k_t} - (1-\delta)e^{k_t} + e^{c_t} \\ a_{t+1} - \rho a_t \end{bmatrix} = 0. \tag{41}$$

According to (18) and (19), a solution to (41) takes the form:

$$c_t = g(k_t, a_t, u_t, \sigma_a, \sigma_v) \tag{42}$$

$$k_{t+1} = h(k_t, a_t, u_t, \sigma_a, \sigma_v) \tag{43}$$

$$a_{t+1} = \rho a_t + \xi_{t+1}$$

with $\xi_{t+1} \equiv u_t \varepsilon_{a,t+1}$ and where the square of u_t follows (40).

In the non-stochastic steady-state, in which $\sigma_a = \sigma_v = 0$ and $f(\overline{y}, \overline{y}, \overline{x}, \overline{x}) = 0$, the following system is used to solve for $\overline{K}$ and $\overline{C}$:

$$\delta\overline{K} - \overline{K}^{\alpha} + \overline{C} = 0,$$

$$\beta[\alpha\overline{K}^{\alpha-1} + (1-\delta)] = 1.$$

Using the calibration of Schmitt-Grohé and Uribe (2004), i.e. $\beta = 0.95$, $\delta = 1$, $\alpha = 0.3$, $\rho = 0$, $\gamma = 2$, we obtain:

$$\overline{K} = 0.1664, \quad \overline{C} = 0.4175.$$

According to (20) and (21), a first-order approximation of (42) and (43) takes the form:

$$\tilde{c}_t = g_k \tilde{k}_t + g_a a_t \tag{44}$$

$$\tilde{k}_{t+1} = h_k \tilde{k}_t + h_a a_t \tag{45}$$

where we have defined $\tilde{c}_t \equiv \ln C_t - \ln \overline{C}$, $\tilde{k} \equiv \ln K_t - \ln \overline{K}$ and the coefficients g_k, g_a, h_k and h_a coincide with those of Schmitt-Grohé and Uribe (2004):

$$g_k = 0.2525, \quad g_a = 0.8417$$
$$h_k = 0.4191, \quad h_a = 1.3970.$$

However, there is an important difference between our approximation and that of Schmitt-Grohé and Uribe (2004). In our case, a_t follows the conditionally linear and heteroskedastic process (39), in which the conditional variance is modelled as in (40). In their framework, instead, shocks are homoskedastic and a_t follows the following linear process:

$$a_{t+1} = \rho a_t + \sigma_a \overline{u} \varepsilon_{a,t+1}. \tag{46}$$

In Fernandez-Villaverde et al. (2011a,b) the original stochastic process for the exogenous state variables is heteroskedastic, but a linear approximation of this process would be consistent with (46) in which risk is no longer time-varying. Instead, in our first-order approximation stochastic volatility still matters and will be particularly relevant when estimating the model.

However, as mentioned in Section 3.2, in our first-order approximation risk does not play a "distinct and direct" role. To see this point, we discuss the impulse response functions. Defining the impulse response of a generic variable x_t at time $t+j$ with respect to the shock ε_t as

$$I(x_{t+j}|\varepsilon_t) = \frac{\partial(E_t x_{t+j} - E_{t-1} x_{t+j})}{\partial \varepsilon_t},$$

we obtain that the impulse response with respect to the shock $\varepsilon_{a,t}$ is given by

$$I(\tilde{c}_{t+j}|\varepsilon_{a,t}) = g_k I(\tilde{k}_{t+j}|\varepsilon_{a,t}) + g_a I(a_{t+j}|\varepsilon_{a,t})$$

$$I(\tilde{k}_{t+j+1}|\varepsilon_{a,t}) = h_k I(\tilde{k}_{t+j}|\varepsilon_{a,t}) + h_a I(a_{t+j}|\varepsilon_{a,t})$$

for each $j \ge 0$ with $I(\tilde{k}_t|\varepsilon_{a,t}) = 0$ where

$$I(a_{t+j+1}|\varepsilon_{a,t}) = \rho I(a_{t+j}|\varepsilon_{a,t})$$

for each $j \ge 0$ and

$$I(a_t|\varepsilon_{a,t}) = \sigma_a \overline{u}.$$

[12] Notice that u_t does not enter system (41), and, accordingly, neither does Eq. (40), as remarked in Section 3.

The impulse response with respect to the shock $\varepsilon_{a,t}$ will not be affected by the fact that shocks are heteroskedastic or not and therefore will coincide with those of Schmitt-Grohé and Uribe (2004). However, even if we compute the impulse response with respect to risk, i.e. with respect to the shock $\varepsilon_{v,t}$, this will be zero at all times: $I(\tilde{c}_{t+j}|\varepsilon_{v,t})=0$ and $I(\tilde{k}_{t+j}|\varepsilon_{v,t})=0$ for each $j\geq 0$. Therefore risk will not play a distinct and separate role in affecting the variables of interest even in our first-order approximation. To get this role, we need to go to a second-order approximation.

Following (25) and (26), the second-order approximation will be of the form:

$$\tilde{c}_t = g_k\tilde{k}_t + g_a a_t + \frac{1}{2}g_{uu}u_t^2 + \frac{1}{2}g_{kk}\tilde{k}_t^2 + \frac{1}{2}g_{aa}a_t^2 + g_{ka}a_t\tilde{k}_t + \frac{1}{2}g_{\sigma\sigma}\sigma_a^2$$

$$\tilde{k}_{t+1} = h_k\tilde{k}_t + h_a a_t + \frac{1}{2}h_{uu}u_t^2 + \frac{1}{2}h_{kk}\tilde{k}_t^2 + \frac{1}{2}h_{aa}a_t^2 + h_{ka}a_t\tilde{k}_t + \frac{1}{2}h_{\sigma\sigma}\sigma_a^2$$

where again a_t follows (39) and u_t^2 follows (40). To compute the numerical values for the remaining coefficients, we consider the calibration adopted by Schmitt-Grohé and Uribe (2004) for the structural parameters, and $\sigma_a=\sigma_v=\overline{u}=1$ and $\lambda=0.5$ for the parameters entering Eq. (40) and governing the dynamics of stochastic volatility. This calibration implies

$$g_{uu}=-0.1444,\quad g_{kk}=-0.0051,\quad g_{aa}=-0.0569,\quad g_{ka}=-0.0171,\quad g_{\sigma\sigma}=-0.0478,$$
$$h_{uu}=0.3622,\quad h_{kk}=-0.0070,\quad h_{aa}=-0.0778,\quad h_{ka}=-0.0233,\quad h_{\sigma\sigma}=0.1199.$$

It is also clear that second-order-approximation impulse response function with respect to the shock $\varepsilon_{a,t}$ will not be affected by the fact that shocks are heteroskedastic or not and therefore will correspond to those of Schmitt-Grohé and Uribe (2004). Instead, now there is a distinct role for risk to affect the variables of interest. Indeed, the impulse responses with respect to the shock $\varepsilon_{v,t}$ will be of the form:

$$I(\tilde{c}_{t+j}|\varepsilon_{v,t}) = g_k I(\tilde{k}_{t+j}|\varepsilon_{v,t}) + g_{uu}I(u_{t+j}^2|\varepsilon_{v,t})$$

$$I(\tilde{k}_{t+j+1}|\varepsilon_{v,t}) = h_k I(\tilde{k}_{t+j}|\varepsilon_{v,t}) + h_{uu}I(u_{t+j}^2|\varepsilon_{v,t})$$

for each $j\geq 0$ with $I(\tilde{k}_t|\varepsilon_{v,t})=0$ where

$$I(u_{t+1+j}^2|\varepsilon_{v,t}) = \lambda I(u_{t+j}^2|\varepsilon_{v,t})$$

for each $j\geq 0$ and

$$I(u_t^2|\varepsilon_{v,t}) = \sigma_v^2.$$

Obviously, in Schmitt-Grohé and Uribe (2004) there is no role at all for time-varying volatility while in Fernandez-Villaverde et al. (2011a,b) there will not be a distinct role and therefore impulse responses with respect to $\varepsilon_{v,t}$ will be zero. To get this role, they have to go to higher-order approximations.

In Fig. 1 we show the impulse response of consumption and capital to 1% change in risk to productivity shock. The impact response of consumption and investment depends on the relative strength of two opposite forces. On the one hand, higher volatility tends to increase the supply of saving for future production and therefore for precautionary reasons.[13] On the other hand, higher volatility increases the expected excess return on capital reducing its appeal as an asset to accumulate. Under our parametrization, in particular with $\delta=1$, the precautionary-saving effect dominates and on impact consumption decreases while investment rises.[14] In the following periods because of capital accumulation, production and consumption increase above their steady state levels as long as agents still accumulate capital above steady state.

As we have already discussed, another important advantage of our approach with respect to Schmitt-Grohé and Uribe (2004) and Fernandez-Villaverde et al. (2011a,b) is that risk-premia evaluated using first-order approximation will be time-varying. To see this, let r_{t+1} be the risk-free real rate, and define $r_{k,t+1}$ as the return on capital from period t to period $t+1$:

$$r_{k,t+1} = \alpha e^{a_{t+1}+(\alpha-1)k_{t+1}} + (1-\delta).$$

Using the above, we can show that in a second-order approximation the expected excess return of capital is given by

$$E_t(\tilde{r}_{k,t+1}-\tilde{r}_{t+1}) + \frac{1}{2}var_t(\tilde{r}_{k,t+1}) = \gamma\, cov_t(\tilde{r}_{k,t+1},\Delta\tilde{c}_{t+1})$$

where $\tilde{r}_{k,t+1}$ and $\tilde{r}_{t+1}$ denote the log deviation from steady state of the real return on capital and the risk-free rate, respectively. The right-hand side measures the risk premium which is time varying

$$\begin{aligned} cov_t(\tilde{r}_{k,t+1},\Delta\tilde{c}_{t+1}) &= \phi\, cov_t(a_{t+1}+(\alpha-1)\tilde{k}_{t+1}, g_k\tilde{k}_{t+1}+g_a a_{t+1}) \\ &= \phi\{E_t[g_a a_{t+1}^2 + ((\alpha-1)g_a+g_k)\tilde{k}_{t+1}a_{t+1} + (\alpha-1)g_k\tilde{k}_{t+1}^2] \\ &\quad -E_t[a_{t+1}+(\alpha-1)\tilde{k}_{t+1}]E_t[g_k\tilde{k}_{t+1}+g_a a_{t+1}]\} = \phi g_a u_t^2 \end{aligned}$$

[13] This channel is stronger when the depreciation is larger, and is clearly dominant with full depreciation.

[14] When $\gamma=\delta=1$, saving is always a constant fraction of income and therefore risk does not have a distinct role.

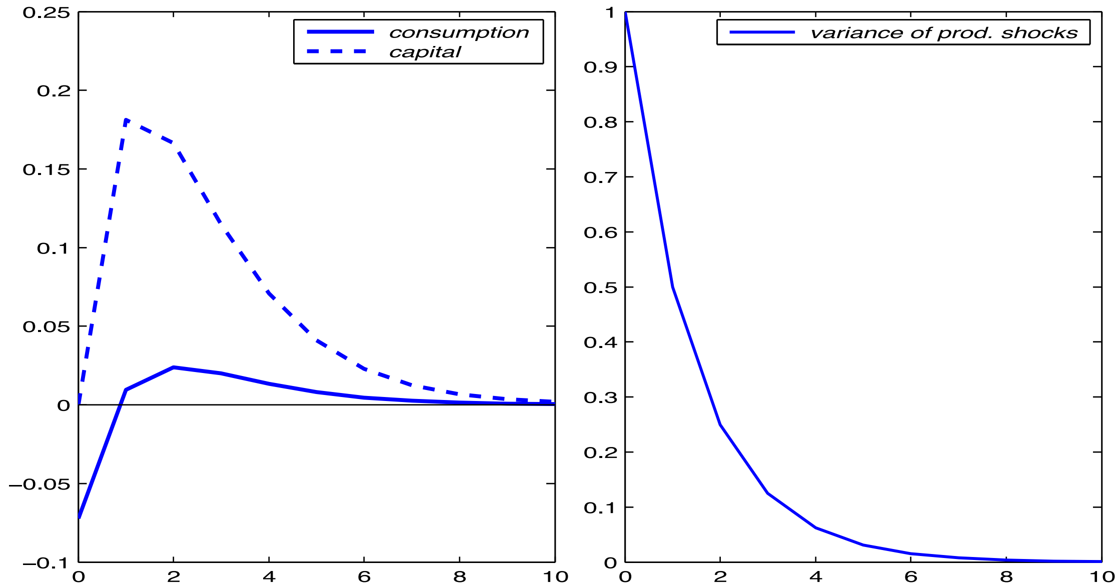

Fig. 1. Dynamic response of consumption and capital to a 1% innovation to the variance of productivity shocks. Percentage points.

depending on the time-variation of the variance u_t^2, where $\phi \equiv 1-\beta(1-\delta)$. In Schmitt-Grohé and Uribe (2004) and Fernandez-Villaverde et al. (2011a,b), this risk premium, computed using a first-order approximation, will be constant.

4.1. *Evaluating the accuracy of the approximation.*

In this section we evaluate the accuracy of our conditionally linear approximation and compare it with the standard approach.[15] We do this under two perspectives: by using the implied impulse-response functions and Euler Equation Errors.

We first show the implications of the two approaches for the implied impulse-response functions. We showed in Section 2 that, for the simple example explored therein, a second-order approximation based on our method delivers the same solution as a third-order approximation based on the standard approach, and therefore identical impulse-response functions. For the case of the neoclassical growth model analyzed in this Section, however, the approximated solutions are different, between our second-order approximation and the standard third-order one, as the latter would feature additional cubic terms like $\tilde{k}_t^3$, a_t^3 and the related cross products. However, as discussed already in Section 2, in this case (as in the general case) the key implication of our method is that the impulse-response functions to a volatility shock that are implied by our *second*-order approximation are the same as those implied by the standard *third*-order approximation. This is shown in Fig. 2, which displays on the left-hand panel the IRF to a 1% volatility shock implied by the standard third-order approximation, and on the right-hand panel the same IRF implied by our second-order approximation (replicating the one displayed in Fig. 1). The two IRFs are identical.

A second way to evaluate the accuracy of a given approximation method is to compare the approximated policy functions with the true ones. For an asset pricing model admitting closed-form solutions, Collard and Juillard (2001) perform the accuracy check by measuring the average deviation of the approximated decision rules from the true ones directly. When the true policy functions are not available, as in our application, an indirect way to compare decision rules is proposed by Judd (1992), and relies upon the use of Euler Equations to measure the intertemporal error that agents would make when using the approximated decision rules instead of the true ones when forming expectations.[16]

In particular, given the Euler Equation (38), we can check the accuracy of the approximation by computing the residual implied by this equilibrium condition, which would be zero if evaluated using the true decision rules, when evaluated using the approximated policy functions instead. This residual should in principle be lower when the order of approximation is higher. To have a scale-free measure easily interpretable from an economic point of view, Judd (1992) suggests to normalize

[15] To compute third-order approximated solutions to the neoclassical growth model, we use the MATLAB codes used and discussed in Andreasen (2010)—publicly available at http://sites.google.com/site/mandreasendk/home-1.

[16] See, among others, Aruoba et al. (2006) for an extensive use of this approach to compare alternative solution methods for a prototypical DSGE model.

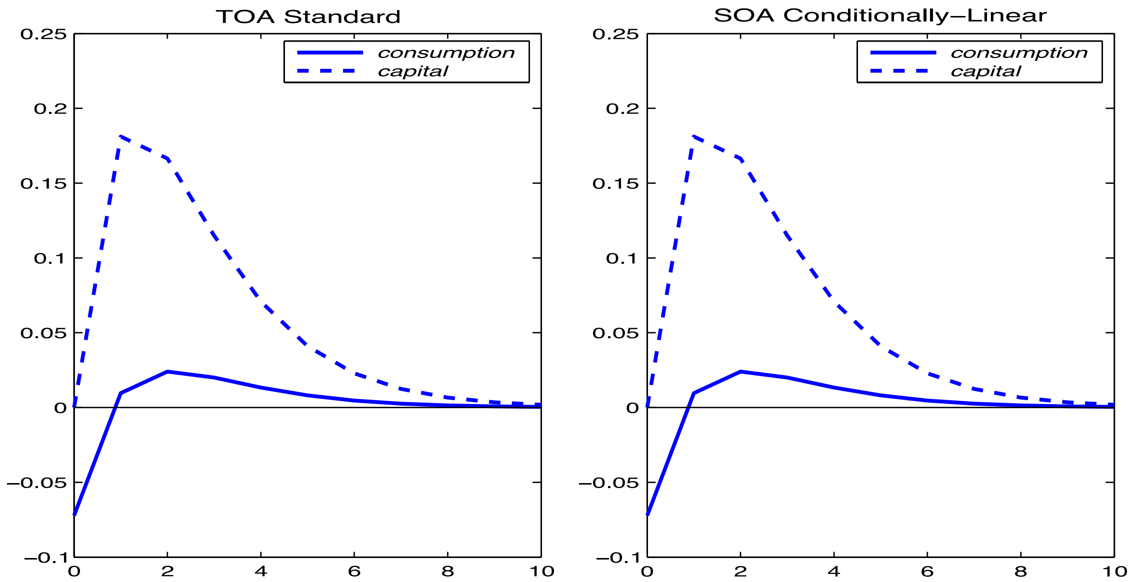

Fig. 2. Dynamic response of consumption and capital to a 1% innovation to the variance of productivity shocks. Percentage points. Left Panel: Third-Order Approximation (TOA), Standard Approach. Right Panel: Second-Order Approximation (SOA), Conditionally-Linear Approach.

such residual by the level of current consumption, and compute the $\log_{10}$:

$$EEE(k_t, a_t, u_t, \sigma_a, \sigma_v) = \log_{10}\left[1 - \frac{(\beta E_t\{C_{t+1}^{-\gamma}[\alpha e^{a_{t+1}} K_{t+1}^{\alpha-1} + (1-\delta)]\})^{-1/\gamma}}{C_t}\right]. \tag{47}$$

The above residual is evaluated using alternative approximations of the policy functions (42) and (43) and Eq. (39) in the place of C_t, C_{t+1}, K_{t+1} and a_{t+1}. The economic interpretation is simple: a $EEE = -6$ would mean that if agents form expectations using the approximated policy functions, instead of the true ones, they would make an intertemporal error of 1 $ for every 10^6 $ worth of consumption.

Notice that the conditional expectation in Eq. (47) deals with two different kinds of uncertainty: the uncertainty about the future *level* of productivity, affecting the future return to capital and parameterized by σ_a, and the uncertainty about the future *volatility* of productivity, affecting the future level of consumption, and parameterized by σ_v. In order to evaluate the impact of these two kinds of uncertainty on the accuracy of the approximation, Fig. 3 displays two sets of EEE: the ones on the top panels are evaluated with respect to *both* uncertainties, i.e. with the conditional expectation computed with respect to the joint distribution of $\varepsilon_{a,t+1}$ and $\varepsilon_{v,t+1}$; the residuals in the bottom panels, instead, are evaluated only with respect the uncertainty about the *conditional variance*, by setting the level shock at its unconditional mean ($\varepsilon_{a,t+1} = 0$) and computing the conditional expectation only with respect to the distribution of $\varepsilon_{v,t+1}$. The values for σ_a and σ_v are calibrated at .007 and .00175, respectively: the former is the familiar standard deviation usually reported for the Solow Residual for the US economy, and the latter implies substantial fluctuations in volatility while still ensuring positive realizations for the variances, given the linear process (40).[17] The integral involved by the conditional expectation was evaluated using 20-nodes Gauss–Hermite quadratures (two-dimensional for the top panels and one-dimensional for the bottom ones).

Each panel reports the Euler Equation residuals for different initial levels of capital K_t. The panels on the left reports EEE evaluated using a standard perturbation approach, up to third order, while the panels on the right display the EEE evaluated using our method.

The top-left panel shows the familiar result that higher-order approximations under the standard approach improve upon lower-order ones, implying smaller approximation errors.[18] When the stock of capital is around its steady-state level, the Euler Equation Error associated with a second-order approximation under the standard approach is about −9.5, while the one associated with a third-order approximation reaches about −11.

[17] See Fernandez-Villaverde and Rubio-Ramirez (2011b) for an analogous calibration.

[18] Notice that the specific calibration that we study, and in particular full depreciation ($\delta = 1$), implies that the residual from the log-linear approximation is independent of the current level of capital, as the Euler Equation (38) is exactly log-linear under certainty equivalence. This further implies that sufficiently far away from the steady state the log-linear approximation is actually better than the second-order and eventually also the third-order approximations.

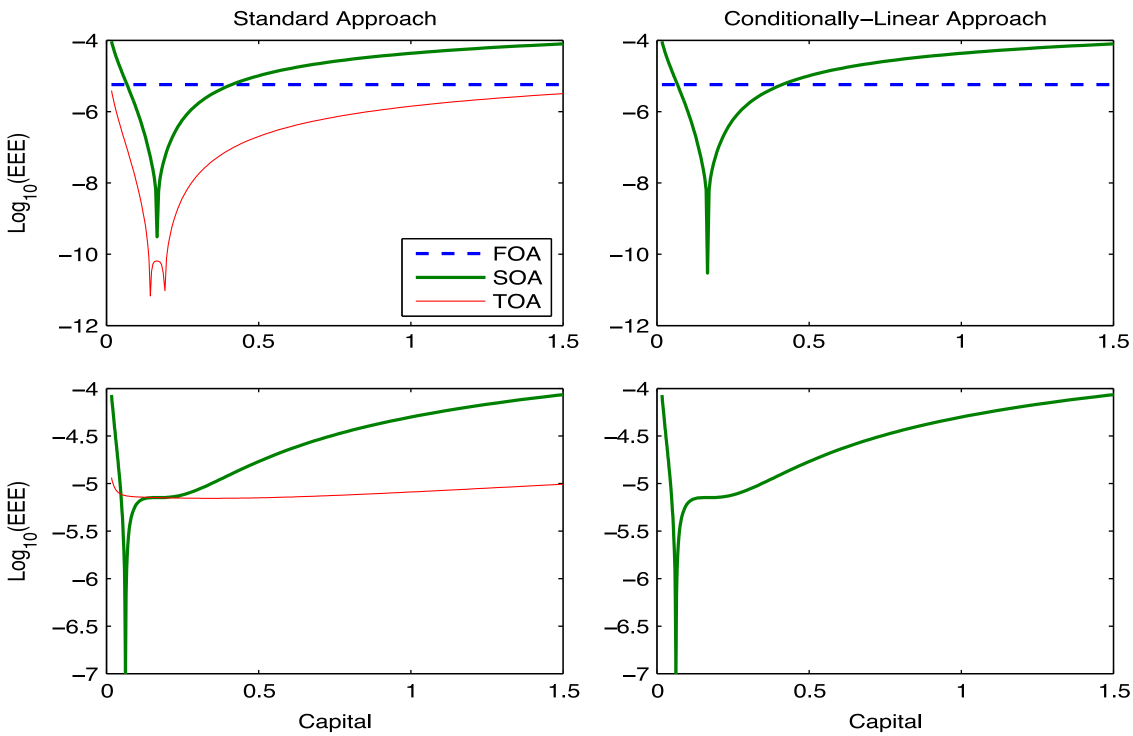

Fig. 3. Euler Equation Errors. FOA: First-Order Approximation. SOA: Second-Order Approximation. TOA: Third-Order Approximation. Top panels: level shocks only. Bottom panels: volatility shocks only. Left panels: standard perturbation approach. Right panels: Our method. Steady-State capital is about 0.1664.

The top-right panel shows the main implication of our conditionally linear approach: the Euler Equation residuals associated with first- and second-order approximations are essentially the same as those obtained under the standard approach. In particular, our second-order approximation does not perform worse than its counterpart under the standard approach and even marginally improves upon it around the steady-state level of capital, reaching a lowest value of about −10.6. The bottom panels, which perform the same computations with respect to the distribution of the volatility shock only, deliver the same result.[19] These results confirm the analytical findings discussed in the simple example of Section 2.

This analysis shows that a second-order approximation under our method, which is at least as accurate as a standard second-order approximation (as shown by Fig. 3), has the important advantage, over a standard second-order approximation, of displaying a "distinct and direct" role of stochastic volatility, which can be obtained under the standard approach only by going to a third-order approximation (as shown in Fig. 2).

5. Conclusion

Recent models used in macroeconomics examine the role of stochastic volatility for the equilibrium allocation. To solve these models, researchers have appealed to global solutions or high-order approximation techniques. Global-solution techniques suffer from the 'curse of dimensionality', since the number of state variables limits their computational efficiency. Commonly used approximation techniques require third-order expansion of the equilibrium conditions in order to display a distinct role for stochastic volatility.

In this paper, we propose a first and second-order approximation method to study the role of time-varying exogenous risk in discrete-time dynamic stochastic models which encompass standard dynamic general equilibrium models with rational expectations. In our framework, an important assumption is that the exogenous state variables follow a conditionally linear stochastic process in which either the variance or the standard deviation of the primitive shocks are modelled through a stochastic linear process. In this way, we generalize the framework and the method developed by Schmitt-Grohé and Uribe (2004), Kim et al. (2008) and Gomme and Klein (2011) to the case in which the exogenous state variables follow an heteroskedastic process.

[19] The residuals associated with a first-order approximation are again independent of the level of capital and constant at a level of about −12.5, for both the standard and our methods. They are not shown in the figure for the sake of readability.

Table 1
Matrices used in the derivations.

Symbol	Description	Dimension
n	Number of all variables: $n = n_y + n_x$	1
n_y	Number of non-predetermined variables	1
n_x	Number of predetermined variables	1
n_z	Number of exogenous state variables	1
$D\overline{f}$	Jacobian matrix of function f	$n \times 2 \cdot n$
$D\overline{f}_{\hat{y}}$	Jacobian matrix of function f with respect to y_{t+1}	$n \times n_y$
$D\overline{f}_{\hat{x}}$	Jacobian matrix of function f with respect to x_{t+1}	$n \times n_x$
$D\overline{f}_y$	Jacobian matrix of function f with respect to y_t	$n \times n_y$
$D\overline{f}_x$	Jacobian matrix of function f with respect to x_t	$n \times n_x$
$H\overline{f}$	Magnus-Neudecker Hessian matrix of function f	$2 \cdot n^2 \times 2 \cdot n$
$H\overline{f}_{\hat{y}}$	MN Hessian matrix with respect to y_{t+1}	$n \cdot n_y \times 2 \cdot n$
$H\overline{f}_{\hat{x}}$	MN Hessian matrix with respect to x_{t+1}	$n \cdot n_x \times 2 \cdot n$
$H\overline{f}_y$	MN Hessian matrix with respect to y_t	$n \cdot n_y \times 2 \cdot n$
$H\overline{f}_x$	MN Hessian matrix with respect to x_t	$n \cdot n_x \times 2 \cdot n$
$\overline{g}_x$	First-order coefficient matrix, x_t on y_t	$n_y \times n_x$
$\overline{g}_{xx}$	Second-order coefficient matrix, x_t on y_t	$n_y \cdot n_x \times n_x$
$\overline{g}_{uu}$	Second-order coefficient matrix, u_t^2 on y_t (specif. I)	$n_y \times n_z$
$\overline{g}_{uu}$	Second-order coefficient matrix, u_t on y_t (specif. II)	$n_y \cdot n_z \times n_z$
$\overline{g}_{zz}$	Second-order coefficient matrix, σ_z on y_t	$n_y \times 1$
$\overline{g}_{vv}$	Second-order coefficient matrix, σ_v on y_t (specif. II)	$n_y \times 1$
$\overline{g}_{zu}$	Second-order coefficient matrix, $\sigma_z u_t$ on y_t (specif. II)	$n_y \times n_z$
$\overline{h}_x$	First-order coefficient matrix, x_t on x_{t+1}	$n_x \times n_x$
$\overline{h}_{xx}$	Second-order coefficient matrix, x_t on x_{t+1}	$n_x \cdot n_x \times n_x$
$\overline{h}_{uu}$	Second-order coefficient matrix, u_t^2 on x_{t+1} (specif. I)	$n_x \times n_z$
$\overline{h}_{uu}$	Second-order coefficient matrix, u_t on x_{t+1} (specif. II)	$n_x \cdot n_z \times n_z$
$\overline{h}_{zz}$	Second-order coefficient matrix, σ_z on x_{t+1}	$n_x \times 1$
$\overline{h}_{vv}$	Second-order coefficient matrix, σ_v on x_{t+1} (specif. II)	$n_x \times 1$
$\overline{h}_{zu}$	Second-order coefficient matrix, $\sigma_z u_t$ on x_{t+1} (specif. II)	$n_x \times n_z$

Note: Specification I: linear process for u_t^2. Specification II: linear process for u_t.

The main contribution of our paper is to show that first and second-order approximations of the solution are sufficient to capture most of the relevant elements needed to study the impact of exogenous uncertainty in standard macroeconomic models. There a7re three main advantages of following our method. First, a first-order approximation falls in the broader class of conditionally linear approximations displaying a role for time-varying volatility, although not a distinct one. Second, given that a first-order approximation retains a role for stochastic volatility, the second-order approximation of the solution implies that the time-varying volatility of primitive shocks can directly affect the endogenous variables. Third, it follows from the previous results that risk-premia evaluated using first-order approximations will be time-varying. All these advantages translate into a more parsimonious model that is more easily tractable for estimation purposes.

In addition to characterizing the second-order approximation of the solution when shocks are conditionally linear, the paper offers a set of MATLAB codes designed to compute the coefficients of the first and second-order approximations and provides a simple example to illustrate the applicability of the method.[20] In fact, our method can be applied easily to several macroeconomic models ranging from real business cycle models, to monetary models and also to asset-pricing or finance models. In Benigno et al. (2012), we employ this method to analyze how risk and monetary policy interact to determine prices, exchange rates and asset prices in an open-economy model.

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[20] The set of codes is available under the webpage of the authors.

Appendix

In this Appendix we present our second-order approximation method under the assumption that time-varying volatility is modelled with the process (17). It is straightforward to show that a first-order approximation of this model is identical to that of Section 3.2 in the main text except that now (17) replaces (16).

Instead of Eqs. (25) and (26), a second order approximation will be of the form:

$$\tilde{y}_t = \overline{g}_x\tilde{x}_t + \frac{1}{2}(I_y \otimes \tilde{x}'_t)\overline{g}_{xx}\tilde{x}_t + \frac{1}{2}(I_y \otimes u'_t)\overline{g}_{uu}u_t + \frac{1}{2}\overline{g}_{vv}\sigma_v^2 + \frac{1}{2}\overline{g}_{zz}\sigma_z^2 + \overline{g}_{zu}\sigma_z u_t \tag{48}$$

$$\tilde{x}_{t+1} = \overline{h}_x\tilde{x}_t + \frac{1}{2}(I_x \otimes \tilde{x}'_t)\overline{h}_{xx}\tilde{x}_t + \frac{1}{2}(I_x \otimes u'_t)\overline{h}_{uu}u_t + \frac{1}{2}\overline{h}_{vv}\sigma_v^2 + \frac{1}{2}\overline{h}_{zz}\sigma_z^2 + \overline{h}_{zu}\sigma_z u_t + \overline{h}_\xi\xi_{t+1} \tag{49}$$

where I_y and I_x are identity matrices of order $n_y \times n_y$ and $n_x \times n_x$, respectively, and $\overline{g}_{xx}$, $\overline{g}_{uu}$, $\overline{g}_{zz}$, $\overline{g}_{vv}$, $\overline{g}_{zu}$, $\overline{h}_{xx}$, $\overline{h}_{uu}$, $\overline{h}_{zz}$, $\overline{h}_{vv}$, $\overline{h}_{zu}$ are conformable matrices, corresponding to the Magnus-Neudecker Hessian matrices of functions $\overline{g}$ and $\overline{h}$ with respect to the arguments in the indexes.

To evaluate this guess, we take a second-order approximation of (13), to get

$$\begin{aligned}0 = E_t\{&D\overline{f}^i_{\hat{y}} \cdot \tilde{y}_{t+1} + D\overline{f}^i_{\hat{x}} \cdot \tilde{x}_{t+1} + D\overline{f}^i_y \cdot \tilde{y}_t + D\overline{f}^i_x \cdot \tilde{x}_t + \frac{1}{2}\tilde{y}'_{t+1} \cdot D\overline{f}^i_{\hat{y}\hat{y}} \cdot \tilde{y}_{t+1}\\ &+\tilde{x}'_{t+1} \cdot D\overline{f}^i_{\hat{y}\hat{x}} \cdot \tilde{y}_{t+1} + \tilde{y}'_t \cdot D\overline{f}^i_{\hat{y}y} \cdot \tilde{y}_{t+1} + \tilde{x}'_t \cdot D\overline{f}^i_{\hat{y}x} \cdot \tilde{y}_{t+1}\\ &+\frac{1}{2}\tilde{x}'_{t+1} \cdot D\overline{f}^i_{\hat{x}\hat{x}} \cdot \tilde{x}_{t+1} + \tilde{y}'_t \cdot D\overline{f}^i_{\hat{x}y} \cdot \tilde{x}_{t+1} + \tilde{x}'_t \cdot D\overline{f}^i_{\hat{x}x} \cdot \tilde{x}_{t+1}\\ &+\frac{1}{2}\tilde{y}'_t \cdot D\overline{f}^i_{yy} \cdot \tilde{y}_t + \tilde{x}'_t \cdot D\overline{f}^i_{yx} \cdot \tilde{y}_t + \frac{1}{2}\tilde{x}'_t \cdot D\overline{f}^i_{xx} \cdot \tilde{x}_t\},\end{aligned} \tag{50}$$

for each $i = 1, \ldots, n$ and where f^i denotes the i-component of the vector f.

We use Eqs. (20) and (21) into (28) to evaluate the second-order terms and (48) and (49) to evaluate the first-order terms, taking into account the restrictions (23).

Making use of $E_t\xi_{t+1} = \mathbf{0}$, we obtain:

$$\begin{aligned}\mathbf{0} = \frac{1}{2}E_t\{&D\overline{f}_{\hat{y}}[(\overline{g}_x \otimes \tilde{x}'_t)\overline{h}_{xx}\tilde{x}_t + (\overline{g}_x \otimes u'_t)\overline{h}_{uu}u_t + \overline{g}_x\overline{h}_{zz}\sigma_z^2 + \overline{g}_x\overline{h}_{vv}\sigma_v^2 + 2\overline{g}_x\overline{h}_{zu}\sigma_z u_t\\ &+(I_y \otimes u'_{t+1})\overline{g}_{uu}u_{t+1} + [I_y \otimes (\overline{h}_x\tilde{x}_t + \overline{h}_\xi\xi_{t+1})']\overline{g}_{xx}(\overline{h}_x\tilde{x}_t + \overline{h}_\xi\xi_{t+1}) + \overline{g}_{zz}\sigma_z^2\\ &+\overline{g}_{vv}\sigma_v^2 + 2\overline{g}_{zu}\sigma_z u_{t+1}] + D\overline{f}_{\hat{x}}[(I_x \otimes \tilde{x}'_t)\overline{h}_{xx}x_t + (I_x \otimes u'_t)\overline{h}_{uu}u_t + \overline{h}_{zz}\sigma_z^2\\ &+\overline{h}_{vv}\sigma_v^2 + 2\overline{h}_{zu}\sigma_z u_t] + D\overline{f}_y[(I_y \otimes \tilde{x}'_t)\overline{g}_{xx}x_t + (I_y \otimes u'_t)\overline{g}_{uu}u_t + \overline{g}_{zz}\sigma_z^2\\ &+\overline{g}_{vv}\sigma_v^2 + 2\overline{g}_{zu}\sigma_z u_t] + [I_n \otimes (\overline{g}_x\overline{h}_x\tilde{x}_t + \overline{g}_x\overline{h}_\xi\xi_{t+1})']H\overline{f}_{\hat{y}} \cdot \tilde{w}_{t+1}\\ &+[I_n \otimes (\overline{h}_x\tilde{x}_t + \overline{h}_\xi\xi_{t+1})']H\overline{f}_{\hat{x}} \cdot \tilde{w}_{t+1} + (I_n \otimes \tilde{x}'_t\overline{g}'_x)H\overline{f}_y \cdot \tilde{w}_{t+1}\\ &+(I_n \otimes \tilde{x}'_t)H\overline{f}_x \cdot \tilde{w}_{t+1}\}.\end{aligned} \tag{51}$$

Note, first, that the matrices $\overline{h}_{xx}$ and $\overline{g}_{xx}$ solve the same set of equations as in (33).

We can then collect the remaining terms and obtain:

$$\begin{aligned}\mathbf{0} = \frac{1}{2}E_t\{&D\overline{f}_{\hat{y}}[(\overline{g}_x \otimes u'_t)\overline{h}_{uu}u_t + \overline{g}_x\overline{h}_{zz}\sigma_z^2 + \overline{g}_x\overline{h}_{vv}\sigma_v^2 + 2\overline{g}_x\overline{h}_{zu}\sigma_z u_t + (I_y \otimes u'_{t+1})\overline{g}_{uu}u_{t+1}\\ &+(I_y \otimes \xi'_{t+1}\overline{h}'_\xi)\overline{g}_{xx}\overline{h}_\xi\xi_{t+1} + \overline{g}_{zz}\sigma_z^2 + \overline{g}_{vv}\sigma_v^2 + 2\overline{g}_{zu}\sigma_z u_{t+1}]\\ &+D\overline{f}_{\hat{x}}[(I_x \otimes u'_t)\overline{h}_{uu}u_t + \overline{h}_{zz}\sigma_z^2 + \overline{h}_{vv}\sigma_v^2 + 2\overline{h}_{zu}\sigma_z u_t]\\ &+D\overline{f}_y[(I_y \otimes u'_t)\overline{g}_{uu}u_t + \overline{g}_{zz}\sigma_z^2 + \overline{g}_{vv}\sigma_v^2 + 2\overline{g}_{zu}\sigma_z u_t]\\ &+(I_n \otimes \xi'_{t+1}\overline{h}'_\xi\overline{g}'_x)H\overline{f}_{\hat{y}} \cdot \overline{M}_\xi\xi_{t+1} + (I_n \otimes \xi'_{t+1}\overline{h}'_\xi)H\overline{f}_{\hat{x}} \cdot \overline{M}_\xi\xi_{t+1}\}.\end{aligned} \tag{52}$$

Recall that for generic and conformable matrices A and B:

$$\begin{aligned}E_t\{(I \otimes \xi'_{t+1}A')BA\xi_{t+1}\} &= E_t\{\text{trm}[(I \otimes \xi'_{t+1}A')BA\xi_{t+1}]\}\\ &= \text{trm}[(I \otimes A')BAE_t\{\xi_{t+1}\xi'_{t+1}\}] = \text{trm}[(I \otimes A')BAU_tU'_t],\end{aligned}$$

where "trm" is the matrix trace operator defined earlier, and in the last equality we used $E_t(\xi_{t+1}\xi'_{t+1}) = U_tU'_t$, as implied by Eq. (15).

Given a generic square matrix A, of order m, we define diagm(A) as the diagonal matrix whose main diagonal is that of matrix A. Given a generic $n \cdot m \times m$ matrix B consisting of n square matrices B_i stacked vertically, with $i = 1, \ldots, n$, we define dgm(B) as the $n \cdot m \times m$ matrix that stacks vertically the $m \times m$ diagonal matrices diagm(B_i):

$$\text{dgm}(B) = [\text{diagm}(B_1)\ \ \text{diagm}(B_2)\ \ \ldots\ \ \text{diagm}(B_n)]'.$$

Moreover, since U_t is a diagonal matrix whose vector on the main diagonal is u_t, the following also holds

$$\text{trm}[(I \otimes A')BAU_tU'_t] = \text{trm}\{\text{dgm}[(I \otimes A')BA] \cdot u_tu'_t\},$$

from which we can conclude:

$$E_t\{(I \otimes \xi'_{t+1} A')BA\xi_{t+1}\} = \mathrm{trm}\{\mathrm{dgm}[(I \otimes A')BA] \cdot u_t u'_t\}. \tag{53}$$

Recall the definition of the process for the standard deviations:

$$u_{t+1} = \sigma_z (I_z - \Lambda_u)\overline{u} + \Lambda_u u_t + \sigma_v V \varepsilon_{v,t+1}.$$

We can use the above definition to write the quadratic term in u_{t+1} in Eq. (52) as:

$$\begin{aligned} E_t\{(D\overline{f}_{\hat{y}} \otimes u'_{t+1})\overline{g}_{uu} u_{t+1}\} &= E_t\{\sigma_z^2 [D\overline{f}_{\hat{y}} \otimes \overline{u}'(I_z - \Lambda_u)']\overline{g}_{uu}(I_z - \Lambda_u)\overline{u} \\ &+ (D\overline{f}_{\hat{y}} \otimes u'_t \Lambda'_u)\overline{g}_{uu}\Lambda_u u_t + \sigma_v^2 (D\overline{f}_{\hat{y}} \otimes \varepsilon'_{v,t+1} V')\overline{g}_{uu} V \varepsilon_{v,t+1} \\ &+ 2\sigma_z [D\overline{f}_{\hat{y}} \otimes \overline{u}'(I_z - \Lambda_u)']\overline{g}_{uu}\Lambda_u u_t\}. \end{aligned} \tag{54}$$

Using the above to collect all second-order terms in u_t from Eq. (52), considering Eq. (53) and exploiting the operators "trm" and "dgm", we obtain the following system of $n \cdot n_z \times n_z$ equations:

$$\begin{aligned} \mathbf{0} &= (D\overline{f}_{\hat{y}} \cdot \overline{g}_x \otimes I_z)\overline{h}_{uu} + (D\overline{f}_{\hat{y}} \otimes \Lambda'_u)\overline{g}_{uu}\Lambda_u \\ &+ (D\overline{f}_{\hat{x}} \otimes I_z)\overline{h}_{uu} + (D\overline{f}_y \otimes I_z)\overline{g}_{uu} + \mathrm{dgm}[(D\overline{f}_{\hat{y}} \otimes \overline{h}'_\xi)\overline{g}_{xx}\overline{h}_\xi \\ &+ (I_n \otimes \overline{h}'_\xi \overline{g}'_x)H\overline{f}_{\hat{y}} \cdot \overline{M}_\xi + (I_n \otimes \overline{h}'_\xi)H\overline{f}_{\hat{x}} \cdot \overline{M}_\xi], \end{aligned} \tag{55}$$

which can be solved for matrices $\overline{g}_{uu}$ and $\overline{h}_{uu}$, given $\overline{h}_x$, $\overline{g}_x$, $\overline{h}_{xx}$, $\overline{g}_{xx}$, $D\overline{f}$ and $H\overline{f}$. Notice that $\overline{h}_{uu}$ and $\overline{g}_{uu}$ will therefore consist of n_x and n_y, respectively, vertically stacked matrices of dimensions $n_z \times n_z$ which will be diagonal matrices.

We can further collect terms in $\sigma_z u_t$ from Eq. (52), considering Eq. (54) and using the trm operator, to obtain a set of $n \times n_z$ equations:

$$\begin{aligned} \mathbf{0} &= (D\overline{f}_{\hat{y}} \cdot \overline{g}_x + D\overline{f}_{\hat{x}})\overline{h}_{zu} + D\overline{f}_{\hat{y}} \cdot \overline{g}_{zu}\Lambda_u + D\overline{f}_y \cdot \overline{g}_{zu} \\ &+ [D\overline{f}_{\hat{y}} \otimes \overline{u}'(I_z - \Lambda_u)']\overline{g}_{uu}\Lambda_u, \end{aligned} \tag{56}$$

which can be solved for the unknown matrices $\overline{g}_{zu}$ and $\overline{h}_{zu}$, given $\overline{g}_x$, $\overline{g}_{uu}$, and $D\overline{f}$.

Similarly, we can collect the terms in σ_z^2 obtaining a set of $n \times 1$ equations

$$\begin{aligned} \mathbf{0} &= (D\overline{f}_{\hat{y}} \cdot \overline{g}_x + D\overline{f}_{\hat{x}})\overline{h}_{zz} + (D\overline{f}_{\hat{y}} + D\overline{f}_y)\overline{g}_{zz} \\ &+ 2D\overline{f}_{\hat{y}} \cdot \overline{g}_{zu}(I_z - \Lambda_u)\overline{u} + [D\overline{f}_{\hat{y}} \otimes \overline{u}'(I_z - \Lambda_u)']\overline{g}_{uu}(I_z - \Lambda_u)\overline{u}, \end{aligned} \tag{57}$$

which can be solved for $\overline{g}_{zz}$ and $\overline{h}_{zz}$, given $\overline{g}_x$, $\overline{g}_{zu}$, $\overline{g}_{uu}$, and $D\overline{f}$.

Finally, we can collect the terms in σ_v^2 obtaining a set of $n \times 1$ equations:

$$\mathbf{0} = (D\overline{f}_{\hat{y}} \cdot \overline{g}_x + D\overline{f}_{\hat{x}})\overline{h}_{vv} + (D\overline{f}_{\hat{y}} + D\overline{f}_y)\overline{g}_{vv} + \mathrm{trm}[(D\overline{f}_{\hat{y}} \otimes V')\overline{g}_{uu} V], \tag{58}$$

which deliver $\overline{g}_{vv}$ and $\overline{h}_{vv}$, given $\overline{g}_x$, $\overline{g}_{uu}$, and $D\overline{f}$.

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