Aggregate Fluctuations in Economies with Private Information

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May, 2019

Abstract: This paper introduces a general method for computing equilibria with heterogeneous agents and aggregate shocks that is particularly suitable for economies with private information. It then applies the method to two examples: a Mirlees economy and an Hopenhayn-Nicolini economy. After providing a sharp analytical characterization for the Mirlees economy with logarithmic preferences, the paper finds that the method reproduces those analytical results perfectly. It also finds that, even under more general preferences, the private information turns out to be completely irrelevant for the aggregate dynamics of the Mirlees economy. In contrast, it plays a crucial role in the Hopenhayn-Nicolini economy.

Keywords: Adverse selection, moral hazard, risk sharing, business cycles, private information, social insurance, optimal contracts, computational methods, heterogeneous agents.

1 Introduction

This paper introduces a general method for computing recursive equilibria of economies with both idiosyncratic and aggregate shocks and applies it to economies with private information. Economies with private information are particularly difficult to solve because optimal promised values are contingent on the realization of the aggregate shocks. This makes one of the endogenous state variables, the distribution of agents across individual states, not only infinite dimensional but state-contingent. The computational

*A previous version of this paper was circulated as "Adverse Selection, Risk Sharing and Business Cycle Fluctuations". I thank V.V. Chari, Chris Phelan, Venky Venkateswaran, three anonymous referees, and participants at various seminars and conferences for useful comments. The views expressed here do not necessarily reflect those of the Federal Reserve Bank of Chicago or the Federal Reserve System. Address: Federal Reserve Bank of Chicago, Research Department, 230 South LaSalle Street, Chicago, IL 60604. E-mail: mveracierto@frbchi.org. Phone: (312) 322-5695.

method can handle this case without difficulty. Additionally, it has three features that make it particularly attractive as a general computational method: 1) it keeps track of the full distribution of agents across individual states as a state variable, 2) it handles irregular shapes for this distribution, and 3) it incorporates the distribution's exact law of motion.

My basic strategy for the computational method is to parametrize individual decision rules as spline approximations and to keep long histories of the spline coefficients as state variables. Starting from the deterministic steady-state distribution, I use the history of decision rules implied by the spline coefficients to obtain the current distribution of individuals across individual states. I do this by performing Monte Carlo simulations on a large panel of agents. All individual first-order conditions and aggregate feasibility constraints are then linearized with respect to the history of spline coefficients.[2] The resulting linear model is then solved using standard methods. I show that a simple transformation can be applied to this solution in order to handle the case of contingent endogenous state variables.

After describing the computational method, I use it to solve the mechanism design problems for two examples of private information economies. These economies are of interest on their own and illustrate very different types of interaction between private information and aggregate dynamics. The first one merges two basic benchmarks in the macroeconomics and public finance literatures: a standard real business cycle (RBC) model and a Mirleesian economy. In this economy, risk-averse agents value consumption and leisure and receive idiosyncratic shocks to their value of leisure. These shocks, which are i.i.d. over time and across individuals, are assumed to be private information. The production technology is standard. Output, which can be consumed or invested, is produced using capital and labor. The production function is subject to aggregate productivity shocks that follow an AR(1) process.

A social planner designs dynamic contracts for the agents in this economy. Following the literature, a dynamic contract is given a standard recursive formulation where a promised value to the agent describes its state. Given the current state, the contract specifies current consumption, current hours worked, and next-period state-contingent promised values as a function of the value of leisure reported by the agent. Since the model has a large number of agents and the shocks to the value of leisure are idiosyncratic, the social planner needs to keep track of the whole distribution of promised values across individuals as a state variable. Given this distribution, the aggregate stock of capital, and the aggregate productivity level, the social planner seeks to maximize the present discounted utility of agents subject to incentive

[2] Since the method heavily relies on Monte Carlo simulations, sampling errors may be a concern for linearizing the model. For this reason, it is important to simulate a large panel of individuals. In this paper, I work with panels of about 10 million individuals. In order to do this, I heavily rely on GPU computing.

compatibility, promise keeping, and aggregate resource feasibility constraints.

This Mirlees economy turns out to provide an ideal test case for the computational method because for the case of logarithmic preferences (a benchmark case in the RBC literature), I am able to provide a sharp analytical characterization of the solution to the mechanism design problem. In particular, I show that the utility of consumption, utility of leisure, and next-period promised values are all linear, strictly increasing functions of the current promised value. The slopes of these functions are all independent of the reported value of leisure and, while the utilities of consumption and leisure have a common slope less than one, the slope of next-period promised values is equal to one (as a consequence, promised values follow a random walk). Over the business cycle, all of these functions shift vertically while keeping constant the differences across reported values of leisure. In turn, the distributions of promised values and log-consumption levels shift horizontally over the business cycle while maintaining their shapes. While consumption inequality is constant, the dispersion of the distribution of log-hours worked is countercyclical. In terms of aggregate dynamics, I find a striking irrelevance result: The business cycle fluctuations of all macroeconomic variables (i.e, aggregate output, consumption, investment, hours worked, and capital) are exactly the same under private information as under full information. Once the information frictions are dealt with in an optimal way, they have no implications for the aggregate dynamics of the economy.

I find that the computational method passes the test perfectly well: Under logarithmic preferences, it recovers all the analytical results described above. Since the computational method does not rely on the particular functional form of the utility function, this provides significant evidence about its accuracy. Having established this, I then use the method to analyze more general preferences. However, I obtain the same basic irrelevance result for all the CRRA preferences that I consider: The stationary behavior of all macroeconomic variables in the economy with private information is numerically indistinguishable from the same economy with full information. This is true even though the cross-sectional distributions of promised values, instead of shifting horizontally over time, now changes its shape.

The Mirlees RBC economy provides a valuable test case scenario for the computational method, represents an interesting theoretical benchmark, and illustrates a case in which there is no interaction between the private information and aggregate dynamics. An unappealing feature of this economy, however, is that the i.i.d. structure of its idiosyncratic shocks is highly unrealistic and precludes giving any empirical content to it.[3] The second example considered differs from the Mirlees economy in that it has a much more realistic structure of idiosyncratic uncertainty, and in that it illustrates a case in which

[3]As I describe later on, introducing persistent shocks to the Mirlees economy would be extremely costly from a computational point of view.

the information frictions play an important role for aggregate dynamics.

The second economy merges a RBC model with an important benchmark in the optimal unemployment insurance literature: the Hopenhayn and Nicolini (1997) model. In this economy, all the production is done in a central island. Agents become exogenously separated from the production island and in order to get back to it, they need to search. The probability of arriving at the production island depends on the search intensity of the agent, which is private information. While the search intensity of agents is not observable, their location (either inside or outside the production island) is and, therefore, recursive contracts can be made contingent on this information. Agents are risk-averse, value consumption, and dislike having to search.

I calibrate this Hopenhayn-Nicolini RBC model to U.S. observations and compare its optimal aggregate dynamics under full and private information. Contrary to the Mirlees economy, this model captures a realistic amount of idiosyncratic risk since it is calibrated to reproduce the average duration of employment and non-employment observed in U.S. data. Also contrary to the Mirlees economy, I find that the presence of information frictions has important effects on its aggregate dynamics. In terms of steady-state dynamics, I find that the private information reduces aggregate employment, capital, investment, and output by 19% and creates a significant amount of consumption inequality: The cross-sectional standard deviation of log consumption goes from zero to 0.26. In terms of impulse responses to an aggregate productivity shock, the private information reduces the peak response of aggregate employment by 20% and increases its half-life from 3.5 years to 6.5 years. The reason why the private information matters so much for aggregate dynamics is that the social planner faces a nontrivial trade-off between aggregate employment and insurance provision: Given that the separation rate is exogenous, the planner can only increase aggregate employment by inducing agents to increase their search intensity. However, since their search intensity is private information, this can only be done by increasing the differences between the promised values of becoming employed and the promised values of continuing to be unemployed (thus reducing the amount of insurance provision). Given this trade-off, the social planner chooses to generate a lower aggregate employment level at the deterministic steady state and to respond less to aggregate productivity shocks at business cycle frequencies.

In short, the paper introduces a general computational method and illustrates its applicability with two examples that provide a clear economic message: that the importance of private information for aggregate dynamics critically depends on the exact nature of the information frictions considered.

The paper is organized as follows. Section 2 discusses the related literature. Section 3 presents the general computational method. Section 4 presents the Mirlees economy. Section 5 presents the Hopenhayn-Nicolini economy. Finally, Section 6 concludes the paper. All proofs are provided in an accompanying Technical Appendix.

2 Related Literature

This paper is closely related to a vast literature on computational methods, but it has salient differences.[4] The seminal papers by Krusell et al. (1998) and Den Haan (1996) summarize the cross-sectional distribution with a small set of moments. In contrast, the method in this paper keeps track of the whole distribution. Den Haan (1997) and Algan et al. (2008) also keep track of the whole distribution but parametrize the distribution with a flexible exponential polynomial form, allowing them to solve the model using quadrature and projection techniques. For many applications this may be an accurate and convenient approach, but for economies with odd-shaped distributions (such as the Hopenhayn-Nicolini RBC model in this paper), it may not be. The method in this paper is able to handle odd shapes for the cross-sectional distribution as long as it is generated by smooth individual decision rules. In addition to projection methods, the literature has explored perturbation methods, which are essentially local approximation methods around a deterministic steady state. Early versions include Campbell (1998), Dotsey et al. (1999), and Veracierto (2002) – the last two in the context of (S,s) economies.[5] Perhaps the most widely known perturbation method is Reiter (2009), which is closely related to Campbell (1998).[6] Instead of parametrizing the cross-sectional distribution as a polynomial, Reiter (2009) keeps a finite histogram of the distribution as a state variable. While the perturbation method allows him to greatly reduce the coarseness of the histogram, a limitation of Reiter's method is that the law of motion for the distribution needs to be approximated, and this can be a highly non-linear mapping. Instead, my method here embodies the exact law of motion for the distribution. Winberry (2018) introduces a very interesting perturbation method which, similarly to Algan et al. (2008), parametrizes the distribution

[4]See Algan et al. (2014) for a survey of computational methods.

[5]The method in this paper is actually a generalization of the approach used in Veracierto (2002). In that paper, histories of past decision rules were also used as state variables. However, under (S,s) adjustments, lower and upper adjustment thresholds could be used to parametrize the complete individual decision rules. In this paper, decision rules are smooth functions and therefore parametrized as spline approximations. Also, in Veracierto (2002) the (S,s) adjustments together with a finite number of idiosyncratic shocks led to a finite support for the distribution of agents and, therefore, to a finite dimensional aggregate state. Here, the support of the distribution of agents is infinite.

[6]The recent method in Ahn et al. (2018) is essentially an adaptation of Reiter's method to continuous time. Other perturbation methods in the literature include Preston and Roca (2007) and Mertens and Judd (2018), both of which perturb a deterministic steady state with no aggregate or idiosyncratic shocks. In contrast, the method in this paper perturbs a deterministic steady state with no aggregate shocks but positive idiosyncratic uncertainty.

with a flexible exponential polynomial form. The perturbation method allows him to carry a polynomial of large order as a state variable (or, equivalently, a large number of moments), which greatly improves the description of the distribution. However, his method also relies on an approximation for the law of motion of the cross-sectional distribution. Furthermore, it is important to emphasize that none of the papers cited above addresses the case of state-contingent cross-sectional distributions, which is a key feature of economies with private information. This paper does address this case.

The Mirlees RBC economy I consider in this paper is closely related to previous work in the social insurance and dynamic public finance literatures (e.g, Atkeson and Lucas (1992), Green (1987), Golosov et al. (2007), Farhi and Werning (2012)). However, interactions with aggregate fluctuations have been mostly neglected in the literature. Notable exceptions are Phelan (1994), da Costa and Luz (2018), Werning (2007), and Scheuer (2013). Phelan (1994) considered a production economy without capital, hidden actions, i.i.d. aggregate shocks, and unobservable i.i.d. idiosyncratic shocks. Under assumptions of CARA preferences and agents facing a constant probability of dying, he characterized the model analytically and found two main results: that the cross-sectional distribution of consumption levels depends on the entire history of aggregate shocks, and that there is a well defined long-run distribution over cross-sectional consumption distributions. The Mirlees RBC model in this paper differs from Phelan (1994), not only because it has hidden types instead of hidden actions, but because it has CRRA preferences and a neoclassical production function with capital and persistent aggregate shocks. In terms of results, an apparent similarity between the papers is that even in my model with logarithmic preferences, the cross-sectional distributions of consumption and leisure depends on the entire history of aggregate shocks. However, this is only due to the presence of capital. Without it, the cross-sectional distribution would depend only on the current realization of aggregate productivity.

In fact, this lack of memory in the case of no capital and logarithmic preferences has already been shown by da Costa and Luz (2018). In that paper, da Costa and Luz consider a finite horizon version of Phelan's economy in which agents have CRRA preferences and live as long as the economy. Contrary to Phelan (1994), their cross-sectional distribution of consumption becomes degenerate as the time horizon of the economy becomes large. Interestingly, da Costa and Luz find that when log preferences are used, the cross-sectional distribution of consumption does not depend on the entire history of aggregate shocks but only on the current realization. However, when the elasticity of intertemporal substitution is different from one, the cross-sectional distribution of consumption has memory of the past history. Relative to da Costa and Luz (2018), a major contribution of the analysis of the logarithmic Mirlees economy in this paper is that, in addition to an economy with capital and persistent aggregate shocks, I am able to provide a tight analytical characterization of the optimal contracts and an irrelevance result of the information frictions for aggregate dynamics. Da Costa and Luz focus on the dependence of the cross-sectional distribution on past aggregate shocks and they provide no comparisons of aggregate dynamics under full and private information. For preferences different from the logarithmic case, I am able to compute solutions for infinite horizon economies.

Werning (2007) considered an RBC Mirlees economy with different permanent types of agents, in which the types are private information. Assuming separable utility functions, he provided a sharp characterization of the optimal savings and labor wedges over the business cycle. In particular, he showed that savings wedges are always zero in the cross-section and over the business cycle. In contrast, labor wedges are positive in the cross-section and, if the distribution of labor productivity is fixed across types, constant over time. The RBC Mirlees economy in this paper differs from his in that the source of the private information is not permanent types but idiosyncratic i.i.d. shocks that change over time. Consequently, instead of having incentive compatibility constraints only at time zero, here they must hold at every time period and history of idiosyncratic and aggregate shocks. In addition to this difference in environments, Werning focused on characterizing optimal wedges and not on the effects of the private information on aggregate dynamics.

Scheuer (2013) considered a static economy with different types of agents subject to idiosyncratic and aggregate shocks. Individual output levels depend on the realizations of the idiosyncratic and aggregate shocks, probability distributions over idiosyncratic shocks depend on individual effort levels and on the aggregate shock, and preferences depend on consumption and effort levels. All these dependencies differ across agent types. While the agent types are public information, effort levels are hidden. Scheuer shows that in a constrained efficient allocation, the ratios of expected inverse marginal utilities between different aggregate shocks must be equalized across the different types of agents. The rest of the paper is devoted to implementing the efficient allocation as a competitive equilibrium with transfers and taxes on financial markets. In addition to corresponding to a dynamic economy with hidden types instead of a static economy with hidden actions, the optimal allocation of my Mirlees RBC model is not characterized by Scheuer's intratemporal condition because the underlying economy has ex-ante identical agents instead of heterogeneous types. I don't address the issue of implementability, but focus instead on the consequences of private information for aggregate dynamics and on characterizing the optimal amount of inequality over the business cycle (issues not considered by Scheuer).[7]

The irrelevance result in my RBC Mirlees economy is related to others in the literature. Krueger and

[7] In principle, the implementation with non-linear taxes in Albanesi and Sleet (2006) could be extended to the stochastic optimal allocation of my Mirlees economy.

Lustig (2010) considered an incomplete markets endowment economy with idiosyncratic and aggregate shocks. The economy has a Lucas tree that yields a fraction of an aggregate stochastic endowment, and a continuum of agents that receive idiosyncratic shocks to their shares on the non-tree part of the aggregate endowment. Agents cannot insure against their idiosyncratic shocks: They can only trade in a risk-free bond and on the Lucas tree, subject to solvency constraints. Krueger and Lustig show that if preferences are CRRA, the aggregate endowment follows a random walk, and the distribution of idiosyncratic endowment shares is independent of the aggregate endowment shock, then there is no trade in the bonds market and only the stock market operates. Moreover, the cross-sectional distributions of wealth and consumption are independent of the aggregate shocks, and the absence of insurance markets is completely irrelevant for the aggregate risk premium. On the surface, these results are closely related to the irrelevance result for the Mirlees economy in this paper.[8] However, while Krueger and Lustig consider an endowment economy, I consider a production economy. Thus, while the incomplete markets in Krueger and Lustig (2010) cannot affect aggregate dynamics by assumption, I am able to address the effects of information frictions on aggregate dynamics. Furthermore, the structure of equilibria with incomplete markets is very different from those of constrained-efficient allocations under private information, in which incentive compatibility constraints must be satisfied.

This difference is most clearly seen when comparing this paper with Werning (2015). Most of Werning's paper focuses on the demand side of a deterministic Bewley-Huggett-Aiyagari incomplete markets model with a fixed outside asset, and shows that under certain conditions, aggregate consumption and interest rates are related by the Euler equation of a representative agent. However, this representative agent does not correspond to the one obtained under complete markets (in particular his discount factor depends on the amount of idiosyncratic uncertainty while the complete markets representative agent does not). As a result, aggregate consumption levels differ under incomplete and complete markets.[9] By comparison, in this paper I provide conditions under which aggregate allocations are identical under private or full information.[10] In the last section of his paper, Werning introduces capital accumulation

[8]However, my irrelevance result does not require aggregate consumption to follow a random walk, only log preferences.

[9]However, if the amount of idiosyncratic uncertainty is constant over time, Werning argues that the responses of aggregate consumption to changes in interest rates are the same under incomplete and complete markets. This is a potentially useful result that could greatly simplify the analysis of aggregate dynamics in different contexts.

[10]The aggregate allocations under private and full information coincide with those of a **common** representative agent. If the social planner uses a different social discount factor than the private discount factor, then this representative agent has time varying discount factors (even though the amount of idiosyncratic uncertainty is constant over time). However, this is completely unimportant for the irrelevance result.

and aggregate shocks, and provides a full irrelevance result for an RBC economy that is closely related to the one in this paper. In his economy agents value consumption, dislike working, and receive idiosyncratic shocks to their labor productivity. Agents can save in capital but cannot borrow. There are spot markets for labor and capital that are used by firms as inputs to a production function, subject to aggregate productivity shocks. For this economy, Werning shows that if agents value consumption according to log preferences, their disutility of labor supply is isoelastic, the depreciation rate of capital is equal to one, and the production function is Cobb-Douglas, then the aggregate dynamics of capital and labor are identical to their counterparts under complete markets. In this equilibrium, aggregate hours worked are constant over time. Moreover, if the initial distribution of wealth is at an invariant steady state, the cross-sectional distributions of consumption and hours worked are also constant over time. In contrast, the irrelevance result in my paper is obtained under any neoclassical production function and depreciation rate of capital; and it holds even though aggregate hours worked and the cross-sectional distribution of hours worked fluctuate over time. The only requirement is that preferences be logarithmic with respect to consumption and leisure. The sharp differences between the conditions needed to obtain the irrelevance results in Werning (2015) and in this paper point to the fundamentally different structures of equilibria with incomplete markets and of constrained-efficient allocation under private information. Neither irrelevance result reduces to the other.

The work that is most closely related to the irrelevance result in my paper is Farhi and Werning (2012). Fahri and Werning consider a very similar Mirleesian economy, except that it has no aggregate productivity shock, idiosyncratic shocks are persistent, and the social planner is only allowed to optimize with respect to the consumption allocations (labor allocations are taken to be beyond the planner's control). Starting from the steady state of a Bewley economy, Fahri and Werning perform the dynamic public finance experiment of evaluating the welfare gains associated with moving to an optimal consumption plan. They show that when preferences are logarithmic in consumption, along the transitionary dynamics of the model all aggregate variables behave exactly the same as in the representative agent of the full information case. Thus, my irrelevance result in this paper can be seen as extending Fahri and Werning's result to allow the social planner to optimize with respect to labor as well as consumption and to do so in an environment subject to aggregate uncertainty.[11]

Finally, the Hopenhayn-Nicolini model in this paper is related to a vast literature on optimal unem-

[11]Contrary to Farhi and Werning (2012), the transitionary dynamics in my Mirlees RBC economy coincide with those of a representative agent economy only if the agent's preferences shift over time in a particular way. This is due to the overlapping generations structure (introduced to obtain a stationary distribution of agents) and only happens if the social planner discounts the future with a discount rate that is different from the agents'.

ployment insurance. The model is tightly related to Hopenhayn and Nicolini (2009), since it is essentially the same model but embodied in an RBC context.[12] However, similarly to Hopenhayn and Nicolini (1997), Werning (2002), and Shimer and Werning (2008), Hopenhayn and Nicolini (2009) analyzes optimal UI insurance abstracting from business cycle considerations. Kroft and Notowidigdo (2011), Sanchez (2008) and Williams and Li (2015) all consider business cycle effects but perform their analysis within a principal-agent setup. In contrast, I consider an economy-wide social planning problem under aggregate uncertainty that not only takes into account the individual incentive constraints, but the aggregate feasibility constraints. While there is a large general equilibrium literature analyzing optimal UI provision in a deterministic environment, the analysis in business cycle settings is more limited. Recent examples include Landais et al. (2018), Jung and Kuester (2015), and Mitman and Rabinovich (2015) in Mortensen-Pissarides frameworks and Boostani et al. (2017) in a directed search environment. However, all of these papers impose exogenous restrictions to the financial markets that agents have available (e.g, exogenous borrowing limits, hand-to-mouth workers, etc.) and to the policy instruments that the government has available (e.g, homogeneous UI benefits, infinite duration of UI benefits, constant probability of UI termination, etc.). In contrast, the Hopenhayn-Nicolini RBC model in this paper analyzes the true social optimum for the environment considered, in which the only restrictions to risk sharing arise because of the presence of private information. Thus, it provides a useful benchmark to which equilibria with exogenous restrictions to risk sharing and policy instruments could be compared.

3 A general computational method

This section describes a general method for computing equilibria of economies with heterogeneous agents and aggregate shocks. Although the method will be applied later on to economies with asymmetric information, this section makes it clear that it is applicable to a much wider variety of settings.[13]

The basic framework is as follows. The economy is populated by individual decision makers that solve

[12]However, instead of having indivisible search decision, search intensities in my model are continuous. Also, instead of characterizing optimal allocations only at low promised values, I characterize them over the whole support of the distribution (although my characterization is numerical instead of analytical).

[13]For example, it could be applied to economies without state-contingent distributions such as Bewley-Huggett-Aiyagari.

maximization problems of the following form at every time period $t \geq 1$:[14]

$$v_{ht}(a, x_1, x_2) = \max_{u_{h1,t+1}, u_{h2t}} \left\{ E_t \left[\sum_s R_h(s, a, x_1, x_2, \left[u_{h1,t+1}\left(s, a'\right)\right]_{a'}, \left[u_{h2t}\left(s, a'\right)\right]_{a'}, z_t, p_t, p_{t+1}) \psi_s \right] + \right.$$

$$\left. E_t \left[\sum_s \sum_{a'} \beta_h(a, a', z_t, p_t, p_{t+1}) v_{h,t+1}\left(a', x_1'\left(s, a'\right), x_2'\left(s, a'\right)\right) \pi_h\left[a, a', u_{h1,t+1}\left(s, a'\right), u_{2ht}\left(s, a'\right)\right] \psi_s \right] \right\} \tag{3.1}$$

subject to

$$x_1'\left(s, a'\right) = G_{h1}\left(a, x_1, x_2, s, a', u_{h1,t+1}\left(s, a'\right)\right), \tag{3.2}$$

$$x_2'\left(s, a'\right) = G_{h2}\left(a, x_1, x_2, s, a', u_{h2t}\left(s, a'\right)\right), \tag{3.3}$$

$$0 \leq E_t\left[C_h\left(a, x_1, x_2, \left[u_{h1,t+1}\left(s, a'\right)\right]_{s,a'}, \left[u_{h2t}\left(s, a'\right)\right]_{s,a'}, z_t, p_t, p_{t+1}\right)\right], \tag{3.4}$$

where h is the permanent type of the individual (e.g, being a household or a firm), a is a vector of individual states that take a finite number of values (e.g, persistent idiosyncratic shocks), z_t is a vector of aggregate shocks, x_1 is a vector of individual state variables whose values are contingent on the realizations of a and z_t, x_2 is a vector of individual state variables whose values are contingent on the realization of a but independent of z_t, s is a vector of i.i.d. idiosyncratic shocks with distribution ψ, $u_{h1,t+1}$ is a vector of (a', z_{t+1})-contingent decision variables, u_{h2t} is a vector of (a')-contingent decision variables, p_t is a vector of equilibrium prices (whose stochastic process is taken as given by the individual), G_{h1} and G_{h2} define the laws of motion for x_1 and x_2, respectively, C_h is a vector valued function defining constraints on $u_{h1,t+1}$ and u_{h2t}, β_h is a function that describes the discounting of future payoffs (allowing for idiosyncratic and/or aggregate preference shocks, as well as discounting using market prices), and π_h describes the transition probabilities for a (potentially affected by the individual's decisions).[15] While a and s take a finite number of values, all other variables take real values.[16] The solution to this sequence

[14]In what follows I use the convention that a variable is dated t if its value becomes known when the date-t aggregate shocks are realized. If the dating of a variable x is clear from the context, I avoid dating it explicitly and its next period value will be denoted by x'. In particular, I avoid dating the arguments of individual value functions and decision rules.

[15]While the dependence of $u_{h1,t+1}$ or u_{h2t} on a' is not critical, the dependence of $u_{h1,t+1}$ on z_{t+1} is what distinguishes it from u_{h2t}. Any decision variable that is not contingent on z_{t+1} is assumed to be included in u_{h2t}. The same assumptions apply to x_1 and x_2. The presence of individual state and decision variables that depend on the realization of the aggregate shocks plays a crucial role in economies with private information.

[16]The reason I introduce the i.i.d. shocks s explicitly instead of subsuming them in the vector a is because of the restrictions across realizations of s that equation (3.4) allows for. These cross-restrictions play a crucial role in certain economies with private information (representing incentive compatibility constraints).

of maximization problems is a stochastic process $\{v_{ht}, u_{h1,t+1}, u_{h2t}\}_{t=1}^{\infty}$ of functions over (a, x_1, x_2). The permanent type h implicitly defines the space in which (a, x_1, x_2) lie.[17]. There is a finite number of different permanent types in the economy.

The distribution of h-type agents across individual states (a, x_1, x_2) at the beginning of period t is described by a measure μ_{ht}. The law of motion for μ_{ht} is given by the following equation:

$$
\begin{aligned}
\mu_{h,t+1}\left(\{a'\} \times \mathcal{X}_1 \times \mathcal{X}_2\right) &= \phi_h\left(\{a'\} \times \mathcal{X}_1 \times \mathcal{X}_2\right) \\
&+ \sum_s \left(\int_{\mathcal{B}} \pi_h \left[a, a', u_{h1,t+1}\left(a, x_1, x_2, s, a'\right), u_{h2t}\left(a, x_1, x_2, s, a'\right)\right] d\mu_{ht} \right) \psi_s,
\end{aligned} \tag{3.5}
$$

for every a' and Borel sets $\mathcal{X}_1$ and $\mathcal{X}_2$, where

$$
\begin{aligned}
\mathcal{B} &= \left\{(a, x_1, x_2) : G_{h1}\left(a, x_1, x_2, s, a', u_{h1,t+1}\left(a, x_1, x_2, s, a'\right)\right) \in \mathcal{X}_1 \right. \\
&\quad \left. \text{and } G_{h2}\left(a, x_1, x_2, s, a', u_{h2t}\left(a, x_1, x_2, s, a'\right)\right) \in \mathcal{X}_2 \right\}.
\end{aligned} \tag{3.6}
$$

and the initial distribution μ_{h1} is given. The measure ϕ_h describes an exogenous endowment of new agents (e.g, to accommodate exogenous entry of firms in a firm dynamics context or newborns in a households life cycle context), while the second term describes the endogenous evolution of the distribution. Observe that since $u_{h1,t+1}$ is contingent on the realization of z_{t+1}, the same is true for $\mu_{h,t+1}$. I assume that μ_{h1}, ϕ_h, and π_h are such that the total number of h-type agents $\int \mu_{ht}$ is constant over time and equal to Γ_h, independent of the stochastic process $\{u_{h1,t+1}, u_{h2t}\}_{t=1}^{\infty}$.

In what follows, it will be useful to differentiate the h-type of agents that are infinitely lived and for which the maximization problem (3.1)-(3.4) is independent of a and s. Henceforth, all variables corresponding to such "representative" types of agents will be denoted with a subscript r, while the h subscript will be reserved for heterogeneous types. An important characteristic of representative types of agents is that the measure μ_{rt} describing their distribution across individual states will have mass at a single point $(x_{r1t}, x_{r2,t-1})$. Therefore, it will be convenient to replace μ_{rt} with that single point and replace the law of motion (3.5)-(3.6) with

$$
x_{r1,t+1} = G_{r1}\left(x_{r1,t}, x_{r2,t-1}, u_{r1,t+1}\left(x_{r1,t}, x_{r2,t-1}\right)\right), \tag{3.7}
$$

$$
x_{r2t} = G_{r2}\left(x_{r1t}, x_{r2,t-1}, u_{r2t}\left(x_{r1,t}, x_{r2,t-1}\right)\right), \tag{3.8}
$$

where the initial $(x_{r1,1}, x_{r2,0})$ is given.

[17] I avoid introducing a subscript h for these variables in order to simplify notation. However, the context will always make clear the permanent type h that they correspond to.

The stochastic price process $\{p_t\}_{t=1}^{\infty}$, taken as given in the maximization problems (3.1)-(3.4), is an equilibrium process if for every $t \geq 1$,

$$Q\left(z_t, \left[\sum_s \left(\int M_h(a, x_1, x_2, \left[u_{h2t}\left(a, x_1, x_2, s, a'\right)\right]_{a'}) d\mu_{ht}\right) \psi_s\right]_h, \left[x_{r1t}, x_{r2,t-1}, u_{r2t}\left(x_{r1t}, x_{r2,t-1}\right)\right]_r\right) = 0, \tag{3.9}$$

where Q is a vector valued function (of the same dimensionality as p_t) describing aggregate feasibility and/or market clearing conditions, M_h is a vector valued function that determines which moments of μ_{ht} are arguments of Q, and $(x_{r1t}, x_{r2,t-1}, u_{r2t})$ are the states and decision functions of the r-type of representative agents. Observe that the z_{t+1}-contingent decision variables $u_{h1,t+1}$ and $u_{r1,t+1}$ do not enter Q.

The vector of aggregate shocks z_t follows an AR(1) process $z_{t+1} = N z_t + \varepsilon_{t+1}$, where $E_t\left[\varepsilon_{t+1}\right] = 0$ and z_1 is given.

The high dimensionality of the equilibrium objects makes computing equilibria for this type of setting a nontrivial task. My approach will be to replace these objects with a finite set of numbers that approximate them arbitrarily well. Moreover, the finite representation will be chosen in such a way that the law of motion corresponding to equations (3.5)-(3.6) will be a linear mapping. All first-order conditions and aggregate feasibility constraints will then be linearized with respect to the variables in the finite representation (at their deterministic steady state values), delivering a linear rational expectations model that can be solved using almost standard methods.[18] Since under the chosen finite representation the law of motion corresponding to equations (3.5)-(3.6) is already linear, this method has the advantage that the linearization does not introduce any further approximation errors to it: The method not only keeps track of the distributions μ_{ht} arbitrarily well over all of their supports, but also uses their exact laws of motion. Since the method performs a linearization at the deterministic steady state equilibrium values of all variables, it requires computing these values as a first step.

3.1 Computing the deterministic steady state

While computing a deterministic steady state for this type of model is standard, this section describes in detail a specific algorithm that serves to introduce objects and notation that will be needed later on. Throughout the section, I assume that a deterministic steady state equilibrium exists.

In order to compute a steady state, I start by making z_t identical to zero and fixing the price vector at some value p. For each r-type of representative agent, the vector of time invariant state and decision vari-

[18] The "almost" qualifier will be clarified later on.

ables $(x_{1r}, x_{2r}, u_{1r}, u_{2r})$ can then be directly obtained from the first-order conditions of the corresponding maximization problem.

I find it convenient to solve the maximization problems given by equations (3.1)-(3.4) using spline approximations and value function iterations.[19] To start, I restrict each component of the vector of endogenous individual state variables (x_1, x_2) for each h-type agent to lie in a closed interval and define a set of grid points in it that includes the extremes.[20] The cartesian product of all these sets of grid points defines a finite set of grid points for (x_1, x_2), which is described by a vector $(\bar{x}_{1j}, \bar{x}_{2j})_{j=1}^{J_h}$. Given the value function v_h from the previous iteration, which is used to evaluate (x_1', x_2') (possibly outside the grid points), the maximization problem in equations (3.1)-(3.4) is solved for at the grid points $(\bar{x}_{1j}, \bar{x}_{2j})_{j=1}^{J_h}$. Once, the vectors of new values $\bar{v}_h = \left[v_h\left(a, \bar{x}_{1j}, \bar{x}_{2j}\right)\right]_{a,j}$, $\bar{u}_{h1} = \left[u_{h1}\left(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a'\right)\right]_{a,j,s,a'}$, and $\bar{u}_{h2} = \left[u_{h2}\left(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a'\right)\right]_{a,j,s,a'}$ are obtained, I extend their values to the full domain of (x_1, x_2) using splines. These value function iterations continue until $\bar{v}_h$ converges. Observe that the solution obtained depends on the price vector p, which has been fixed.

For heterogenous agents, the steady state version of equations (3.5)-(3.6) describes the recursion that the time invariant distribution μ_h has to satisfy. This equation corresponds to the case of a continuum of agents. However, I find it convenient to perform the recursion in the case of a large but finite number of agents. In particular, consider a large but finite number I_h of h-type agents and endow them with some individual states (a, x_1, x_2). Using the functions u_{h1} and u_{h2} already obtained, I simulate the evolution of the individual states of these I_h agents for a large number of periods T. To be precise, if an h-type agent i has the individual state (a, x_1, x_2) at the beginning of the current period, then the individual state (a', x_1', x_2') at the beginning of the following period is randomly determined as follows:

$$\begin{aligned}
&\text{(i)} \quad \text{with probability } \pi_h\left[a, a', u_{h1}\left(a, x_1, x_2, s, a'\right), u_{h2}\left(a, x_1, x_2, s, a'\right)\right]\psi_s, \text{ it is given by} \qquad (3.10)\\
&\qquad \left[a', G_{h1}\left(a, x_1, x_2, s, a', u_{h1}\left(a, x_1, x_2, s, a'\right)\right), G_{h2}\left(a, x_1, x_2, s, a', u_{h2}\left(a, x_1, x_2, s, a'\right)\right)\right],\\
&\text{(ii)} \quad \text{with probability } 1 - \sum_{s,a'}\pi\left[a, a', u_{h1}\left(a, x_1, x_2, s, a'\right), u_{h2}\left(a, x_1, x_2, s, a'\right)\right]\psi_s \text{ it is determined by } \phi_h.
\end{aligned}$$

Observe that the transition in (ii) takes place when the individual dies and is replaced by a newborn whose initial state is unrelated to the state of the predecessor.

---

[19]For representative agents with state contingent state variables x_{1r}, it will be important to follow the procedure described in this paragraph as well since the steady state objects described here will be needed later on.

[20]When restricting each of these variables to lie in a closed interval, one should modify the steady state maximization problem (3.1)-(3.4) to incorporate the corresponding constraints on x_1' and x_2'. The use of splines is what requires each component of (x_1, x_2) to lie in a closed interval.

Simulating the I_h agents and their descendants for T periods using the law of motion in (3.10), I obtain a realized distribution $\left(a^i, x_1^i, x_2^i\right)_{i=1}^{I_h}$ of individual states across the I_h agents. Doing this for every h-type, the aggregate feasibility conditions can then be computed as

$$Q\left(0, \left[\sum_s \left(\Gamma_h \frac{1}{I_h}\sum_{i=1}^{I_h} M_h(a^i, x_1^i, x_2^i, \left[u_{h2}\left(a^i, x_1^i, x_2^i, s, a'\right)\right]_{a'})\right)\psi_s\right]_h, \left[x_{r1}, x_{r2}, u_{r2}\right]_r\right) = 0. \tag{3.11}$$

Observe that by the law of large numbers, equation (3.11) will become an arbitrarily good approximation of equation (3.9) as all I_h and T tend to infinity.

If equation (3.11) is not satisfied, the price vector p must be changed until it is. This represents a standard root finding problem.

3.2 Computing the stationary stochastic solution

As I already mentioned, computing the stationary stochastic solution requires linearizing the first-order conditions to the maximization problems given by equations (3.1)-(3.4), the laws of motion (3.5)-(3.6), the laws of motion (3.7)-(3.8), and the aggregate feasibility conditions given by equation (3.9) with respect to a convenient set of variables. It is important to point out from the outset that the method does not handle occasionally binding constraints: It assumes that each component in equation (3.4) either always holds with equality or always holds with strict inequality.

In order to illustrate some of the issues involved in the linearization of the first-order conditions, I will use equation (3.1) as an example since it represents the most complex type.[21] The first issue is the existence of a continuum of equations (3.1), since (x_1, x_2) take a continuum of values. I solve this "curse of dimensionality" by considering the equation only at the grid points $(\bar{x}_{1j}, \bar{x}_{2j})_{j=1}^{J_h}$ that were used in the computation of the deterministic steady state. Another issue is that each of this finite number of equations depends on the infinite dimensional object $v_{h,t+1}$, since it is a function of (x_1', x_2'), and I need to evaluate these variables outside the grid points. In this case, I solve the "curse of dimensionality" by considering that $v_{h,t+1}$ is a spline approximation and, therefore, is completely determined by the vector $\bar{v}_{h,t+1} = \left[v_{h,t+1}\left(a, \bar{x}_{1j}, \bar{x}_{2j}\right)\right]_{a,j}$, i.e, by the value of the function at the grid points. Consequently, after substituting equations (3.2)-(3.3) into equation (3.1) and linearizing at the corresponding steady state

[21] Equation (3.1) enters the set of first order conditions if the transition probabilities π_h depend on $u_{h1,t+1}$ or u_{h2t}. In this case, the level of v_{ht} enters the first order conditions and the definitional equation (3.1) must be included. If π_h does not depend on $u_{h1,t+1}$ or u_{h2t}, only the derivatives of v_{ht} enter the first order conditions. However, the issues discussed here in the context of equation (3.1) apply to other first-order conditions, including the definitional equation for the derivatives of v_{ht}. For reasons I will explain in Section 3.3, it is important to write first order conditions using the derivatives of the value function and not as second order stochastic difference equations.

values, I am left with the following finite set of equations:

$$0 = E_t \left\{ \mathcal{L}_h^v \left(\bar{v}_{h,t}, \bar{u}_{h1,t+1}, \bar{u}_{h2t}, z_t, p_t, p_{t+1}, \bar{v}_{h,t+1} \right) \right\}, \tag{3.12}$$

where $\bar{u}_{h1,t+1} = \left[u_{h1,t+1} \left(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a' \right) \right]_{a,j,s,a'}$, $\bar{u}_{h2t} = \left[u_{h2t} \left(a, \bar{x}_{1j}, \bar{x}_{2j}, s, a' \right) \right]_{a,j,s,a'}$ and $\mathcal{L}_h^v$ is a vector valued linear function with the same dimensionality as $\bar{v}_{ht}$.

Particular attention should be given to the first-order conditions corresponding to grid points $(a, \bar{x}_{1j}, \bar{x}_{2j})$ for which the deterministic steady state choice of some component of $x_1' \left(s, a' \right)$ or $x_2' \left(s, a' \right)$ hits one of the extremes imposed by the use of spline approximations.[22] At these grid points, the maximization problem (3.1)-(3.4) should be modified by imposing the constraint that the corresponding component of equation (3.2) or (3.3) must evaluate to the corresponding extreme. The first-order conditions used at these grid points should be those of the modified problem. A consequence of this is that if the optimal choice of some component of $x_1' \left(s, a' \right)$ or $x_2' \left(s, a' \right)$ hits an extreme in the steady state solution, it will always hit it in the stochastic solution. This will certainly distort the stochastic decision rules close to the extremes, so in practice one should choose these extremes far enough that the invariant distribution μ_h puts little mass close to them (minimizing the relevance of these distortions).

Linearizing the aggregate feasibility conditions described by equation (3.9) presents more complicated issues because of their dependence on the integrals $\left[\int M_h d\mu_{ht} \right]_h$. To make progress, these integrals must be represented with a convenient finite set of variables. To do this, I follow a strategy that is closely related to the one used in Section 3.1 for computing statistics under the invariant distributions. In particular, for each heterogenous type of agent h, consider the same large but finite number of agents I_h used in that section and endow them with the same realized distribution of individual states $\left(a^i, x_1^i, x_2^i \right)_{i=1}^{I_h}$ that was obtained when computing the steady state. Now, assume that these agents populated the economy N time periods ago and consider the history $\{ u_{h1,t+1-n}, u_{h2,t-n} \}_{n=1}^{N}$ of decision rules that were realized during the last N periods (where t is considered to be the current period). Since these decision rules are spline approximations, this history can be represented by the finite list of values $\{ \bar{u}_{h1,t+1-n}, \bar{u}_{h2,t-n} \}_{n=1}^{N}$. Using this history of decision rules, I can simulate the evolution of individual states for the I_h agents and their descendants during the last N time periods to update the distribution of individual states from the initial $\left(a^i, x_1^i, x_2^i \right)_{i=1}^{I_h}$ to a current distribution $\left(a_t^i, x_{1t}^i, x_{2,t-1}^i \right)_{i=1}^{I_h}$. In particular, I can initialize the distribution of individual states at the beginning of period $t-N$ as $\left(a_{t-N}^i, x_{1,t-N}^i, x_{2,t-N-1}^i \right) = \left(a^i, x_1^i, x_2^i \right)$, for $i = 1, ..., I_h$. Given a distribution of individual states $\left[\left(a_{t-n}^i, x_{1,t-n}^i, x_{2,t-n-1}^i \right) \right]_{i=1}^{I_h}$ at period $t-n$, the individual state $\left(a_{t-n+1}^i, x_{1,t-n+1}^i, x_{2,t-n}^i \right)$ of each agent i at period $t-n+1$ is randomly determined as

[22] See footnote 19.

follows:

(i) with probability $\pi_h\left[a^i_{t-n}, a', u^i_{h1,t+1-n}(s,a'), u^i_{h2,t-n}(s,a')\right]\psi_s$, it is given by $\left(a', G^i_{h1,t+1-n}(s,a'),\right.$ $\left.G^i_{h2,t-n}(s,a')\right)$, where $\left(u^i_{h1,t+1-n}(s,a'), u^i_{h2,t-n}(s,a'), G^i_{h1,t+1-n}(s,a'), G^i_{h2,t-n}(a')\right)$ are the values of $(u_{h1,t+1-n}, u_{h2,t-n}, G_{h1}, G_{h2})$ evaluated at $\left(a^i_{t-n}, a', x^i_{1,t-n}, x^i_{2,t-n-1}, s, a'\right)$,

(ii) with probability $1-\sum_{s,a'}\pi_h\left[a^i_{t-n}, a', u^i_{h1,t+1-n}(s,a'), u^i_{h2,t-n}(s,a')\right]\psi_s$, it is determined by ϕ_h.

Proceeding recursively for $n = N, N-1, ..., 1$, I obtain a realized distribution $\left(a^i_t, x^i_{1t}, x^i_{2,t-1}\right)^{I_h}_{i=1}$ at the beginning of period t. This distribution can be used to compute statistics under the distribution μ_{ht}. In particular, having followed the above procedure for each h-type of heterogeneous agents, I can rewrite equation (3.9) as

$$\begin{aligned} 0 \;=\; & Q\left(z_t, \left[\sum_s\left(\Gamma_h\frac{1}{I_h}\sum_{i=1}^{I_h} M_h(a^i_t, x^i_{1t}, x^i_{2,t-1}, \left[u_{h2t}\left(a^i_t, x^i_{1t}, x^i_{2,t-1}, s, a'\right)\right]_{a'})\right)\psi_s\right]_h,\right. \\ & \left., \left[x_{r1t}, x_{r2,t-1}, u_{r2t}\left(x_{r1t}, x_{r2,t-1}\right)\right]_r\right) \end{aligned} \tag{3.13}$$

Since u_{h2t} and u_{r2t} are spline approximations, they can also be summarized by their values at the grid points $\bar{u}_{h2t}$ and $\bar{u}_{r2t}$.[23] As a consequence, equation (3.13) can be linearized at the deterministic steady state values to get the following finite set of equations:

$$0 = \mathcal{L}^Q\left(z_t, \left[\left\{\bar{u}_{h1,t+1-n}\right\}^N_{n=1}, \left\{\bar{u}_{h2,t-n}\right\}^N_{n=0}\right]_h, \left[x_{r1t}, x_{r2,t-1}, \bar{u}_{r2t}\right]_r\right) \tag{3.14}$$

where $\mathcal{L}^Q$ is a vector valued linear function.[24]

My approach of representing the distribution μ_{ht} with a finite history of values greatly simplifies the description of the law of motion in equations (3.5)-(3.6). In fact, updating the distribution μ_{ht} is merely reduced to updating those histories. In particular, the date-$(t+1)$ histories can be obtained from the

[23]For simplicity, I assume here that all representative agents have state-contingent states x_{1r}. However, for representative agents with no state-contingent states, instead of writing equation (3.13) in terms of their decision rules u_{r2t}, it is often more convenient to write it directly in terms of the values of their type-2 decision variables at date t. Consequently, for this type of representative agents, $\bar{u}_{r2t}$ in equations (3.14) and (3.18) is not a vector of spline coefficients but a vector of values for type-2 decision variables.

[24]Taking numerical derivatives of equation (3.13) with respect to each spline coefficient in the list $\left[\left\{\bar{u}_{h1,t+1-n}\right\}^N_{n=1}, \left\{\bar{u}_{h2,t-n}\right\}^N_{n=0}\right]_h$ requires simulating I_h agents over N periods. Therefore, obtaining the linear function $\mathcal{L}^Q$ requires performing a large number of Monte Carlo simulations. While this seems a daunting task, it is easily parallelizable. Thus, using massively parallel computer systems can play an important role in reducing computing times and keeping the task manageable.

date-t histories and the current values $\bar{u}_{h1,t+1}$ and $\bar{u}_{h2t}$ using the following equations:

$$\bar{u}_{h1,(t+1)-n} = \bar{u}_{h1,t-(n-1)} \tag{3.15}$$

$$\bar{u}_{h2,(t+1)-n} = \bar{u}_{h2,t-(n-1)}, \tag{3.16}$$

for $n = 1, ..., N$. Observe that the law of motion described by equations (3.15)-(3.16) is already linear, so no further linearization is needed. Also observe that the variables that are N periods old in the date-t history are dropped from the date-$(t+1)$ history. Thus, the law of motion described by these equations introduces a truncation. However, introducing a life cycle structure to the h-type of heterogenous agents will make the consequences of this truncation negligible. The reason is that the truncation only affects agents surviving for N consecutive periods and, given sufficiently small survival probabilities and/or a sufficiently large N, there will be very few of these agents. Apart from this negligible truncation, there are no further approximations errors in the representation of the law of motion given by equations (3.5)-(3.6) – a crucial benefit of using the computational method described in this paper.

Since all $u_{r1t,t+1}$ and u_{r2t} are also spline approximations they are summarized by their values at the grid points $\bar{u}_{r1t,t+1}$ and $\bar{u}_{r2t}$. The laws of motion (3.7)-(3.8) can then be linearized to obtain

$$0 = \mathcal{L}^{G_{r1}}\left(x_{r1,t+1}, x_{r1t}, x_{r2,t-1}, \bar{u}_{r1t,t+1}\right), \tag{3.17}$$

$$0 = \mathcal{L}^{G_{r2}}\left(x_{r2t}, x_{r1t}, x_{r2,t-1}, \bar{u}_{r2t}\right), \tag{3.18}$$

where $\mathcal{L}^{G_{r1}}$ and $\mathcal{L}^{G_{r2}}$ are vector valued linear functions of the same dimensionality as $x_{r1,t+1}$ and x_{r2t}, respectively.

Once all equations have been linearized, I am left with a stochastic linear rational expectations model with a non-standard feature – namely, that some of the decision variables during the current period and some of the endogenous states during the next period are contingent on the realization of the aggregate shocks during the next period. Fortunately, this difficulty can be handled easily. The reason is that the stochastic state contingent solution that I seek can be easily constructed from the solution to the deterministic version of the model, and this version has a standard structure that can be solved using well known methods. In what follows, I describe the linear stochastic model in detail and show how to perform this transformation.

3.3 Solving the linearized model

Define the following vectors:

$$x_t^1 = \left[\left[\{\Delta\bar{u}_{h1,t+1-n}\}_{n=1}^N\right]_h, [\Delta x_{r1t}]_r\right], \tag{3.19}$$

$$x_{t-1}^2 = \left[\left[\{\Delta\bar{u}_{h2,t-n}\}_{n=1}^N\right]_h, [\Delta x_{r2,t-1}]_r\right], \tag{3.20}$$

$$y_{t+1}^1 = \left[[\Delta\bar{u}_{h1,t+1}]_h, [\Delta\bar{u}_{r1,t+1}]_r\right], \tag{3.21}$$

$$y_t^2 = \left[\left[\Delta\bar{v}_{ht}, \Delta\left(\frac{\partial\bar{v}_{ht}}{\partial x}\right), \Delta\bar{q}_{ht}, \Delta\bar{u}_{h2t}\right]_h, \left[\Delta\bar{v}_{rt}, \Delta\left(\frac{\partial\bar{v}_{rt}}{\partial x}\right), \Delta\bar{q}_{rt}, \Delta\bar{u}_{r2t}\right]_r, \Delta p_t\right], \tag{3.22}$$

where Δ represents deviations from steady state values. $\partial\bar{v}_{ht}/\partial x$ and $\bar{q}_{ht}$ are the derivatives of v_{ht} and the Lagrange multipliers of constraints (3.4), respectively, evaluated at the grid points of the h-type of heterogeneous agents. $\partial\bar{v}_{rt}/\partial x$ and $\bar{q}_{rt}$ are similar objects but for the r-type of representative agents. The linearized model can then be written as

$$0 = B_{11}x_t^1 + B_{12}x_{t-1}^2 + C_{12}y_t^2 + D_1 z_t, \tag{3.23}$$

$$0 = A_{21}x_{t+1}^1 + B_{21}x_t^1 + B_{22}x_{t-1}^2 + C_{21}y_{t+1}^1, \tag{3.24}$$

$$0 = A_{32}x_t^2 + B_{31}x_t^1 + B_{32}x_{t-1}^2 + C_{32}y_t^2, \tag{3.25}$$

$$0 = H_{41}x_t^1 + H_{42}x_{t-1}^2 + J_{42}y_{t+1}^2 + K_{41}y_{t+1}^1 + K_{42}y_t^2 + M_4 z_t, \tag{3.26}$$

$$\begin{aligned} 0 &= E_t\left\{F_{52}x_{t+1}^2 + G_{52}x_t^2 + H_{51}x_t^1 + H_{52}x_{t-1}^2 + J_{52}y_{t+1}^2 + K_{51}y_{t+1}^1,\right. \\ &\quad \left. +K_{52}y_t^2 + L_5 z_{t+1} + M_5 z_t\right\} \end{aligned} \tag{3.27}$$

$$z_{t+1} = N z_t + \varepsilon_{t+1}, \tag{3.28}$$

where (3.23) represents the aggregate feasibility constraints (equation 3.14), (3.24) is the law of motion for x_t^1 (equations 3.15 and 3.17), (3.25) is the law of motion for x_{t-1}^2 (equations 3.16 and 3.18), (3.26) is the first-order conditions for $u_{h1,t+1}$ and $u_{r1,t+1}$ evaluated at the grid points, and (3.27) represents the constraints (3.4), the first-order conditions for u_{h2t} and u_{r2t}, the definitions of $\bar{v}_{ht}$ and $\bar{v}_{rt}$ (e.g, equation 3.12), and the envelope conditions for $\partial\bar{v}_{ht}/\partial x$ and $\partial\bar{v}_{rt}/\partial x$, all evaluated at the grid points.[25] I seek a recursive solution to equations (3.23)-(3.28) of the following form:

$$x_{t+1}^1 = \Omega_{11}x_t^1 + \Omega_{12}x_{t-1}^2 + \Psi_1 z_t + \Theta_1 z_{t+1}, \tag{3.29}$$

$$x_t^2 = \Omega_{21}x_t^1 + \Omega_{22}x_{t-1}^2 + \Psi_2 z_t, \tag{3.30}$$

$$y_{t+1}^1 = \Phi_{11}x_t^1 + \Phi_{12}x_{t-1}^2 + \Gamma_1 z_t + \Lambda_1 z_{t+1}, \tag{3.31}$$

$$y_t^2 = \Phi_{21}x_t^1 + \Phi_{22}x_{t-1}^2 + \Gamma_2 z_t. \tag{3.32}$$

[25] Actually, only the constraints in (3.4) that hold with equality are included in the system of equations. Also, only the Lagrange multipliers of these constraints are included in $\bar{q}_{ht}$ and $\bar{q}_{rt}$ in equation 3.22.

My strategy will be to construct it from the recursive solution to the deterministic version of equations (3.23)-(3.28), in which ε_{t+1} is set to zero and the expectations operator is dropped.[26] This deterministic version has identical structure as the system analyzed in Uhlig (1999) and can be solved using identical methods.[27] Its solution has the following form:

$$x^1_{t+1} = P_{11}x^1_t + P_{12}x^2_{t-1} + Q_1 z_t, \tag{3.33}$$

$$x^2_t = P_{21}x^1_t + P_{22}x^2_{t-1} + Q_2 z_t, \tag{3.34}$$

$$y^1_{t+1} = R_{11}x^1_t + R_{12}x^2_{t-1} + S_1 z_t, \tag{3.35}$$

$$y^2_t = R_{21}x^1_t + R_{22}x^2_{t-1} + S_2 z_t. \tag{3.36}$$

Proposition 1 *Let (3.33)-(3.36) be the solution to the deterministic version of equations (3.23)-(3.28). Define* $\Omega_{11} = P_{11}$, $\Omega_{12} = P_{12}$, $\Omega_{21} = P_{21}$, $\Omega_{22} = P_{22}$, $\Psi_2 = Q_2$, $\Phi_{11} = R_{11}$, $\Phi_{12} = R_{12}$, $\Phi_{21} = R_{21}$, $\Phi_{22} = R_{22}$, $\Gamma_2 = S_2$, *and*

$$\Theta_1 = \Upsilon A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}S_2, \tag{3.37}$$

$$\Psi_1 = \Upsilon\left[A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{22}Q_2 + A_{21}^{-1}C_{21}K_{41}^{-1}K_{42}S_2 + A_{21}^{-1}C_{21}K_{41}^{-1}M_4\right], \tag{3.38}$$

$$\Lambda_1 = -K_{41}^{-1}J_{42}R_{21}\Theta_1 - K_{41}^{-1}J_{42}S_2, \tag{3.39}$$

$$\Gamma_1 = -K_{41}^{-1}J_{42}R_{21}\Psi_1 - K_{41}^{-1}J_{42}R_{22}Q_2 - K_{41}^{-1}K_{42}S_2 - K_{41}^{-1}M_4, \tag{3.40}$$

where

$$\Upsilon = \left[I - A_{21}^{-1}C_{21}K_{41}^{-1}J_{42}R_{21}\right]^{-1} \tag{3.41}$$

Then, (3.29)-(3.32) solves the stochastic system (3.23)-(3.28).

Proof. *The solution is verified using algebraic manipulations and the law of iterated expectations.*[28] ∎

The next section will apply the general computational method described so far to a Mirlees economy. This economy happens to be an ideal test case scenario for the method's accuracy because under logarithmic preferences, its business cycle fluctuations can be characterized analytically. In this section, I will

[26]For this strategy to work it is important to write the first order conditions for the heterogeneous agents in equations (3.26)-(3.27) using the derivatives of the value functions and not as second order difference equations. For representative agents with no state contingent state variables it is often more convenient to write their first order conditions as second order difference equations. It is only for this reason that x^2_t, x^2_{t+1} and z_{t+1} are included in equation (3.27).

[27]In fact, I use the same notation as Uhlig (1999), page 38, to facilitate comparisons. The only difference is that the variables here written as x^1_t and y^1_{t+1} are there written as x^1_{t-1} and y^1_t. However, in a deterministic context this difference is immaterial (it can be considered a simple notational issue).

[28]See Technical Appendix 7 for a complete proof.

show that, while the computational method does not exploit the structure of the logarithmic preferences in any way, it recovers those analytical results accurately.

4 A Mirlees economy

The economy is populated by a unit measure of agents subject to stochastic lifetimes. Whenever an agent dies they are immediately replaced by a newborn, leaving the aggregate population level constant over time.[29] The preferences of an individual born at date $T \geq 0$ are given by[30]

$$E_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} \sigma^{t-T} \left[u\left(c_t\right) + \alpha_t n\left(1 - h_t\right) \right] \right\}, \tag{4.1}$$

where σ is the survival probability, $0 < \beta < 1$ is the discount factor, $\alpha_t \in \{\bar{\alpha}_1, ..., \bar{\alpha}_S\}$ is the idiosyncratic value of leisure, and u and n are continuously differentiable, strictly increasing and strictly concave utility functions. Realizations of α_t are assumed to be i.i.d. both across individuals and across time. The probability that $\alpha_t = \bar{\alpha}_s$ is given by ψ_s. A key assumption is that α_t is private information of the individual.

Output, which can be consumed or invested, is produced with the following production function:

$$Y_t = e^{z_t} F(K_{t-1}, H_t), \tag{4.2}$$

where Y_t is output, z_t is aggregate productivity, K_{t-1} is capital, H_t is hours worked, and F is a neoclassical production function. The aggregate productivity level z_t follows a standard AR(1) process given by:

$$z_{t+1} = \rho z_t + \varepsilon_{t+1}, \tag{4.3}$$

where $0 < \rho < 1$, and ε_{t+1} is normally distributed with mean zero and standard deviation σ_ε. The initial z_0 is given.

Capital is accumulated using a standard linear technology given by

$$K_t = \left(1 - \delta\right) K_{t-1} + I_t, \tag{4.4}$$

where I_t is gross investment and $0 < \delta < 1$. The initial K_{-1} is given.

[29] As in Phelan (1994), the stochastic lifetime guarantees that there will be a stationary distribution of agents across individual states.

[30] Observe that I am deviating from Section 3 in that the initial time period is 0 instead of 1. The reason for doing this will be explained below.

In what follows, I will describe the mechanism design problem for this economy. To do this, it will be convenient to distinguish between two types of agents: young and old. A young agent is one that has been born at the beginning of the current period. An old agent is one that has been born in some previous period. The social planner must decide recursive plans for both types of agents. The state of a recursive plan is the value (i.e, discounted expected utility) that the agent is entitled to at the beginning of the period. Given this promised value, the recursive plan specifies the current utility of consumption, the current utility of leisure, and next-period promised values as functions of the value of leisure currently reported by the agent. The social planner is fully committed to the recursive plans they choose and agents have no outside opportunities available.

A key difference between the young and the old is in terms of promised values. Since during the previous period the social planner has already decided on some recursive plan for a currently old agent, the planner is restricted to delivering the corresponding promised value during the current period. In contrast, the social planner is free to deliver any value to a currently young agent since this is the first period they are alive. Reflecting this difference, I will specify the individual state of an old agent to be their promised value v and their current value of leisure s (henceforth, I will refer to the value of leisure $\bar{\alpha}_s$ by its subindex s). At date t, their current utility of consumption, utility of leisure, and next-period promised value are denoted by $u_{ost}(v)$, $n_{ost}(v)$ and $w_{os,t+1}(v)$, respectively, where $w_{os,t+1}(v)$ is a random variable contingent on the realization of z_{t+1}. In turn, the individual state of a young agent is solely given by their current value of leisure s. At date t, the agent's current utility of consumption, utility of leisure, and next-period promised value are denoted by u_{yst}, n_{yst} and $w_{ys,t+1}$ respectively, where $w_{ys,t+1}$ is also contingent on the realization of z_{t+1}.

The social planner seeks to maximize the weighted sum of the welfare levels of current and future generations of agents, subject to individual incentive compatibility and promise keeping constraints, as well as aggregate feasibility constraints. To map the social planner's problem into the structure described in Section 3, it will be convenient to decompose it into a sequence of sub-planning problems, instead of describing it as an economy-wide planning problem.[31] In each period, there are two sub-planning problems: one sub-planning problem concerned with providing insurance and incentives to individuals, and another sub-planning problem concerned with making production and investment decisions. In these sub-planning problems, the joint stochastic process for the shadow price of labor (in terms of the consumption good), q_t, and the shadow price of the consumption good (in utiles), λ_t, are taken as given. The solutions to these sequences of sub-planning problems correspond to those of the economy-wide

[31] Technical Appendix 8 describes the economy-wide planning problem in detail.

planning problem if certain side conditions are satisfied.

The sub-planning problems for individuals differ depending on whether the individual is young or old. For $t \geq 1$, the sub-planning problem for old individuals is as follows:

$$P_{ot}(v) = \max \sum_s \psi_s \left\{ q_t h(n_{ost}) - c(u_{ost}) + \theta\sigma E_t \left[\frac{\lambda_{t+1}}{\lambda_t} P_{o,t+1}(w_{os,t+1}) \right] \right\} \tag{4.5}$$

subject to

$$u_{ost} + \bar{\alpha}_s n_{ost} + \beta\sigma E_t [w_{os,t+1}] \geq u_{ojt} + \bar{\alpha}_s n_{ojt} + \beta\sigma E_t [w_{oj,t+1}], \text{ for every } (s,j), \tag{4.6}$$

$$v = \sum_s \{u_{ost} + \bar{\alpha}_s n_{ost} + \beta\sigma E_t [w_{os,t+1}]\} \psi_s, \tag{4.7}$$

where $h(n)$ are the hours worked implied by the utility of leisure n (i.e. $h(n) = 1 - n^{-1}(n)$), and $c(u)$ is the consumption level implied by the utility of consumption u (i.e. $c(u) = u^{-1}(u)$). Observe that the current "social profits" in equation (4.5) are given by the social value of the hours worked by the old agent, net of the consumption goods that are transferred to them. Also observe that the sub-planner discounts the future social profits of the old individual using the social discount factor θ, the survival probability σ, and the stochastic social discount factor λ_{t+1}/λ_t. The social discount rate θ is the Pareto weight of the next-period generation of young agents relative to the Pareto weight of the current generation of young agents.[32] I assume that $\beta\sigma < \theta < 1$. Equation (4.6) is the incentive compatibility constraint. It states that the expected value to the individual of truthfully reporting the value of leisure s must be at least as large as the expected value to the individual of misreporting any other value of leisure j. Equation (4.7) is the promise-keeping constraint. It states that the social sub-planner must deliver the expected value v that was promised at the beginning of the period.

For $t \geq 1$, the sub-planning problem for young individuals is as follows:

$$P_{yt} = \max \sum_s \psi_s \left\{ \frac{u_{yst} + \bar{\alpha}_s n_{yst} + \beta\sigma E_t [w_{ys,t+1}]}{\lambda_t} + q_t h(n_{yst}) - c(u_{yst}) + \theta\sigma E_t \left[\frac{\lambda_{t+1}}{\lambda_t} P_{o,t+1}(w_{ys,t+1}) \right] \right\} \tag{4.8}$$

subject to

$$u_{yst} + \bar{\alpha}_s n_{yst} + \beta\sigma E_t [w_{ys,t+1}] \geq u_{yjt} + \bar{\alpha}_s n_{yjt} + \beta\sigma E_t [w_{yj,t+1}], \text{ for every } (s,j). \tag{4.9}$$

Observe that in this case the social surplus is given by the expected lifetime utility level of the young agent (in current consumption units), plus the expected social value of the hours worked by the agent, net of the expected consumption goods transferred to them. Since, conditional on surviving the young agent becomes old after one period, the function used to evaluate next-period continuation values is $P_{o,t+1}$.

[32] I assume that relative Pareto weights are constant across generations.

For $t \geq 1$, the sub-planning problems for production decisions is

$$P_{pt}(K) = \max \left\{ e^{z_t} F(K, H_t) - q_t H_t - I_t + \theta E_t \left[\frac{\lambda_{t+1}}{\lambda_t} P_{p,t+1}\left((1-\delta) K + I_t\right) \right] \right\}. \tag{4.10}$$

Observe that the social surplus generated by this planning problem is given by output net of the value of the labor input and the value of investment.

The economy-wide distribution of old agents across promised values v at the beginning of period $t \geq 1$ is given by a measure μ_t, while the number of young agents is constant over time and given by $1 - \sigma$. Given the stochastic sequence of decision rules $\left\{ [u_{ost}, n_{ost}, w_{os,t+1}, u_{yst}, n_{yst}, w_{ys,t+1}]_s \right\}_{t=1}^{\infty}$ that solves the corresponding sub-planning problems for individuals, the law of motion for μ_t is given as follows:

$$\mu_{t+1}(B) = \sigma \sum_s \int_{\{v:\ w_{os,t+1}(v) \in B\}} \psi_s d\mu_t + (1-\sigma)\, \sigma \sum_{s:\ w_{ys,t+1} \in B} \psi_s, \tag{4.11}$$

for every Borel set B and $t \geq 1$. Equation (4.11) states that the number of old agents that have a promised value in the Borel set B at the beginning of the following period is given by the sum of two terms. The first term sums all currently old agents that receive a next-period promised value in the set B and do not die. The second term does the same for all currently young agents.

The economy-wide stock of capital at the beginning of period t is equal to K_{t-1}. Given the stochastic sequence of decision rules $\{H_t, I_t\}_{t=1}^{\infty}$ that solve the corresponding sub-planning production problems, K_t follows a stochastic process given by

$$K_t = (1-\delta) K_{t-1} + I_t(K_{t-1}), \tag{4.12}$$

for $t \geq 1$.

Given the values for K_0 and μ_1 determined by the optimal choices made at $t = 0$ and the realized value z_1, the side conditions that the stochastic shadow prices $\{q_t, \lambda_t\}_{t=1}^{\infty}$ need to satisfy for $t \geq 1$ are the following:

$$(1-\sigma) \sum_s c(u_{yst})\, \psi_s + \int \sum_s c(u_{ost}(v))\, \psi_s d\mu_t + I_t(K_{t-1}) = e^{z_t} F\left[K_{t-1}, H_t(K_{t-1})\right], \tag{4.13}$$

and

$$H_t(K_{t-1}) = (1-\sigma) \sum_s h(n_{yst})\, \psi_s + \int \sum_s h(n_{ost}(v))\, \psi_s d\mu_t. \tag{4.14}$$

Equation (4.13) describes the aggregate feasibility constraint for the consumption good. It states that the total consumption of young and old agents, plus aggregate investment cannot exceed aggregate output. Equation (4.14) is the aggregate labor feasibility constraint. It states that the input of hours into the production function cannot exceed the total hours worked by young and old agents.

Given values for K_0, μ_1 and z_1, the continuation optimal plan starting at $t = 1$ and characterized by equations (4.5)-(4.14) has the general structure described in Section 3.[33] As a consequence, the deterministic steady state optimal allocation can be computed as in Section 3.1 and the stationary stochastic optimal allocation can be computed as in Section 3.2. However, for the analysis that follows it will be necessary to complete the description of the optimal plan starting at $t = 0$. Date 0 is special because it has no ongoing recursive plans in place for which promised values must be delivered. Thus, at date 0 all agents must be treated as young, there is no sub-planning problem for old individuals, the sub-planning problem for young individuals is given by equations (4.8)-(4.9) and the sub-planning problem for production decisions is given by equation (4.10). The side conditions that the shadow prices (q_0, λ_0) must satisfy in order to obtain the economy-wide social optimum are

$$\sum_s c\left(u_{ys0}\right)\psi_s + I_0\left(K_{-1}\right) = e^{z_0}F\left[K_{-1}, H_0\left(K_{-1}\right)\right], \tag{4.15}$$

and

$$H_0\left(K_{-1}\right) = \sum_s h\left(n_{ys0}\right)\psi_s, \tag{4.16}$$

which are analogous to equations (4.13)-(4.14) but reflect the fact that all agents at $t = 0$ are young. In turn, μ_1 is determined by

$$\mu_1\left(B\right) = \sigma \sum_{s:\ w_{ys1}\in B} \psi_s \tag{4.17}$$

and K_1 is determined by equation (4.12).

4.1 Logarithmic preferences

This section provides a sharp characterization of the optimal allocation when preferences are logarithmic. To do this, it will be useful to introduce the following non-stationary representative agent planning problem:

$$\max E_0 \left\{ \sum_{t=0}^{\infty} \phi_t \theta^t \left[u\left(C_t\right) + \bar{\alpha} n\left(1 - H_t\right)\right] \right\} \tag{4.18}$$

subject to:

$$C_t + K_t - (1-\delta) K_{t-1} \le e^{z_t} F\left(K_{t-1}, H_t\right), \tag{4.19}$$

where $\phi_t > 0$ is a deterministic preference shifter with positive limit, $\bar{\alpha} = \sum_s \bar{\alpha}_s \psi_s$ is the expected idiosyncratic value of leisure, and (z_0, K_{-1}) is taken as given. The following Lemma links the optimal allocation of the Mirlees economy to the solution of this representative agent planning problem.

[33] Technical Appendix 9 describes in detail how to map this Mirlees economy into the general structure of Section 3.

Lemma 2 *Suppose that u and n are logarithmic. Define $\phi = \{\phi_t\}_{t=0}^{\infty}$ as follows*

$$\phi_t = \begin{cases} 1, \text{ for } t = 0 \\ (1-\sigma) + \rho_t, \text{ for } t \geq 1, \end{cases} \tag{4.20}$$

where $\{\rho_t\}_{t=1}^{\infty}$ is given by

$$\rho_t = \begin{cases} \frac{\beta\sigma}{\theta}, \text{ for } t = 1 \\ \frac{\beta\sigma}{\theta}\rho_{t-1} + (1-\sigma)\frac{\beta\sigma}{\theta}, \text{ for } t \geq 2, \end{cases} \tag{4.21}$$

Then, the optimal aggregate allocation of the economy with ***private information*** *is identical to the optimal allocation of the representative agent economy with preference shifters ϕ.*

The proof of this Lemma, which is given in Technical Appendix 11, shows in detail why the logarithmic preferences give rise to this exact aggregation result.[34] The basic reason is that the inverse Euler equations, which characterize an optimal allocation under private information, become linear when preferences are logarithmic. This allows me to integrate the inverse Euler equations across all individuals and obtain a relation between aggregate variables that reproduces the (direct) Euler equations of the representative agent planning problem. The preference shifters ϕ are needed for obtaining the aggregation result only because the social planner is allowed to discount the welfare of future generations at a different rate than private agents discount future utility. In fact, if the relative Pareto weight θ is the same as the private discount factor β, we see from equation (4.21) that $\rho_t = \sigma$ for all $t \geq 1$ and from equation (4.20) that $\alpha_t = 1$ for all $t \geq 0$. That is, in this case, the Mirleesian planner chooses the same aggregate allocation as if they were deciding the optimal plan for a representative agent economy with stationary preferences.

Observe that the optimal allocation of the full information economy can be obtained by dropping the incentive compatibility constraints (4.6) and (4.9) from the corresponding optimization problems. Since none of these incentive constraints are used in the proof of Lemma 2, I have a second important result.

Lemma 3 *Suppose that u and n are logarithmic. Define $\phi = \{\phi_t\}_{t=0}^{\infty}$ as in equation (4.20). Then, the optimal aggregate allocation of the economy with* ***full information*** *is identical to the optimal allocation of the representative agent economy with preference shifters ϕ.*

In addition, since the optimal aggregate allocations of the economy with private information and the economy with full information are equal to the same object, I have the following Corollary.

Corollary 4 *Suppose that u and n are logarithmic. Then, the optimal aggregate allocation of the economy with private information is identical to the optimal aggregate allocation of the economy with full information.*

[34]The proof of Lemma 2 is constructive.

This Corollary provides a strong irrelevance result: Under logarithmic preferences, the information frictions play no role for aggregate dynamics. The information frictions affect individual allocations since agents are not fully insured and suffer from the lack of insurance. However, this has no effect on aggregate variables.

Observe that independently of the value of θ, from equation (4.21) we verify that ρ_t converges to a positive value and, therefore, that ϕ_t converges to a positive value as well.[35] Since it is well known that the solution to the representative agent economy with stationary preferences (constant ϕ_t) converges to a stationary stochastic process, I can say the same about the aggregate optimal allocations of the economies with private and full information (using Lemmas 2 and 3). Thus, I have the following result.

Corollary 5 *Suppose that u and n are logarithmic. Then, the aggregate optimal allocations of the economies with private and full information converge to a stationary stochastic process. Moreover, this stationary process is the one associated with a representative agent economy with stationary preferences (zero preference shifters).*

This Corollary provides a natural test for the accuracy of the computational method introduced in Section 3, since it is straightforward to compute the stationary stochastic solution to the representative agent problem and compare the results. However, such a test would only consider aggregate variables. It would be desirable to test the accuracy of the computed individual allocations as well. The purpose of the following Lemma is to provide a strong characterization of the optimal individual allocation rules so that it that can be used for such a test.

Lemma 6 *Suppose that u and n are logarithmic. Define $\Delta \ln \lambda_t = \ln \lambda_t - \ln \lambda^*$ and $\Delta \ln q_t = \ln q_t - \ln q^*$, where λ^* and q^* are the deterministic steady state values of λ_t and q_t, respectively. Then, the stationary solution to the private information planning problem satisfies that for every s,*

$$u_{yst} = u^*_{ys} - \Delta \ln \lambda_t \tag{4.22}$$

$$n_{yst} = n^*_{ys} - \Delta \ln \lambda_t - \Delta \ln q_t \tag{4.23}$$

$$w_{ys,t+1} = w^*_{ys} - \frac{1}{b}\left(\Delta \ln \lambda_{t+1} + \Delta \pi_{t+1}\right) \tag{4.24}$$

$$u_{ost}(v) = u^*_{os} + bv + \Delta \pi_t \tag{4.25}$$

$$n_{ost}(v) = n^*_{os} + bv + \Delta \pi_t - \Delta \ln q_t \tag{4.26}$$

$$w_{os,t+1}(v) = w^*_{os} + v - \frac{1}{b}\left(\Delta \ln \lambda_{t+1} + \Delta \pi_{t+1} - \Delta \ln \lambda_t - \Delta \pi_t\right) \tag{4.27}$$

[35] Recall that θ was assumed to be greater than $\beta\sigma$.

where $0 < b = \frac{1-\beta\sigma}{1+\bar{\alpha}} < 1$ *and* $\Delta\pi_t$ *is given by*

$$\Delta\pi_t = -\beta\sigma\Delta\ln\lambda_t + (1-\beta\sigma)\sum_{k=1}^{\infty}(\beta\sigma)^k E_t\left[\Delta\ln\lambda_{t+k}\right] + b\bar{\alpha}\sum_{k=0}^{\infty}(\beta\sigma)^k E_t\left[\Delta\ln q_{t+k}\right].$$

Proof: *These functional forms satisfy all constraints and first-order conditions.*[36]■

Equations (4.22)-(4.24) indicate that for young agents the utility of consumption, the utility of leisure, and next-period promised values shift over the business cycle by amounts that are independent of the reported type. Equation (4.25) states that all $u_{ost}(v)$ are linear parallel functions that shift vertically over the business cycle by amounts that are independent of the reported type. While equations (4.26) and (4.27) show that the same is true for the utility of leisure and next-period promised values, the slopes of all $w_{os,t+1}(v)$ are equal to one. Thus, promised values follow a random walk process with innovations that depend on the realization of the idiosyncratic and aggregate shocks.[37]

The individual allocation rules described in Lemma 6 have strong implications for the cyclical behavior of inequality. This behavior is described by the following Lemma and its Corollary.

Lemma 7 *Let* μ_t, φ_t *and* ζ_t *be the cross-sectional distributions of promised values* v, *consumption utilities* u, *and leisure utilities* n, *respectively. Let* μ^*, φ^* *and* ζ^* *be their deterministic steady state values. Then for every real numbers* a_1 *and* a_2, *with* $a_1 < a_2$,

$$\mu_t\left[\left(a_1 - \frac{\Delta\ln\lambda_t + \Delta\pi_t}{b}, a_2 - \frac{\Delta\ln\lambda_t + \Delta\pi_t}{b}\right)\right] = \mu^*\left[(a_1, a_2)\right], \tag{4.28}$$

$$\varphi_t\left[(a_1 - \Delta\ln\lambda_t, a_2 - \Delta\ln\lambda_t)\right] = \varphi^*\left[(a_1, a_2)\right], \tag{4.29}$$

$$\zeta_t\left[(a_1 - \Delta\ln\lambda_t - \Delta\ln q_t, a_2 - \Delta\ln\lambda_t - \Delta\ln q_t)\right] = \zeta^*\left[(a_1, a_2)\right]. \tag{4.30}$$

Proof: *Follows from equations (4.22)-(4.27) and the law of motion (4.11).*[38]■

Since this Lemma implies that the distributions μ_t, φ_t, and ζ_t are mere horizontal translations of their deterministic steady state values, I have the following Corollary.

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[36]See Technical Appendix 12 for a complete proof.

[37]Even with no aggregate fluctuations, promised values follow a random walk. However, contrary to Atkeson and Lucas (1992), an immizerizing result is not obtained because of the stochastic lifetimes. As people die and are replaced by young agents, there is enough "reversion to the mean" in promised values that an invariant distribution is obtained (see Phelan 1994). The immizerizing result actually applies within each cohort of agents: Within each cohort the distribution of promised values spreads out over time.

[38]See Technical Appendix 13 for a complete proof.

Corollary 8 *The dispersions of the cross-sectional distributions of promised values and log-consumption levels are constant over the business cycle, while the dispersion of the cross-sectional distribution of log-hours worked is countercyclical.*

The first part of this Corollary follows directly from the fact that μ_t and φ_t do not change their shape and that the utility of consumption is logarithmic. The second part of the Corollary holds because the distribution ζ_t shifts to the left during a boom.[39] Since log-hours worked are related to the utilities of leisure according to $\ln(h) = \ln(1 - e^n)$ and this is a strictly decreasing and strictly concave function, when the distribution of utilities of leisure shifts to the left, the dispersion of the distribution of log-hours worked decreases.

4.2 Testing the computational method

Now that I have provided an analytical characterization of the business cycles of the economy with logarithmic preferences, in this section I test the computational method introduced in Section 3 by evaluating to what extent it is able to recover those analytical results.[40] This is an important test to perform since nothing in the computational method exploits the structure of the logarithmic preferences. To proceed, I first select numerical values for the model parameters. Whenever possible I choose parameter values that are standard in the RBC literature. However, I do not claim empirical content for the model since, being i.i.d., the shocks to the value of leisure are highly unrealistic. In order to bring the model to the data, persistent shocks would be needed. The problem is that computing a solution for this case would be extremely costly.[41] In order to simplify computations, I select the model time period to be one year and assume the idiosyncratic preference shock takes only two values: $\bar{\alpha}_L$ and $\bar{\alpha}_H$, with $\bar{\alpha}_L < \bar{\alpha}_H$. The production function is assumed to have a standard Cobb-Douglas form $F(K, H) = K^{\gamma} H^{1-\gamma}$.

[39] This follows from Corollary 5 and the fact that in a representative agent economy with logarithmic preferences, aggregate leisure is countercyclical.

[40] Technical Appendix 14 provides details on how to construct the linearized system (3.23)-(3.28) for the economy of this Section.

[41] In order to keep the recursive structure required by the computational method of Section 3, one would have to follow the approach introduced by Fernandes and Phelan (2000). Under persistent shocks, the state of the recursive contracts is given by a vector of threat-keeping values, one for each possible value of the idiosyncratic shock. Thus, the dimensionality of the individual allocation rules and, therefore, the computational complexity, grow exponentially with the number of possible values for the idiosyncratic shock. On the contrary, under i.i.d. shocks, the state of a recursive contract is always a single promised value, independent of the number of possible values for the idiosyncratic shock.

Following the RBC literature, I select a labor share $1-\gamma$ of 0.64, a depreciation rate δ of 0.10, a private discount factor β of 0.96, a persistence of aggregate productivity ρ of 0.95, and a variance of the innovations to aggregate productivity σ_{ε}^2 equal to 4×0.007^2. The social discount factor θ is chosen to be the same as the private discount factor β. The values of leisure $\bar{\alpha}_L$ and $\bar{\alpha}_H$ are chosen to satisfy two criteria: Aggregate hours worked H equal 0.31 (a standard target in the RBC literature) and the hours worked by old agents with the high value of leisure and the highest possible promised value $n_{oH}(v_{\max})$ be a small but positive number. The rationale for this second criterion is that I want to maximize the relevance of the information frictions while keeping an internal solution for hours worked. The resulting values for $\bar{\alpha}_L$ and $\bar{\alpha}_H$ are 1.164 and 2.218, respectively. The probability of drawing a high value of leisure ψ_H is chosen to maximize the standard deviation of the invariant distribution of promised values. It turns out that a value of $\psi_H = 0.50$ achieves this. In terms of the life-cycle structure, I choose $\sigma = 0.975$ to generate an expected lifespan of 40 years.

While the above parameters are structural, there are a number of computational parameters to be determined. The number of grid points in the spline approximations J, the total number of agents simulated I, the length of the simulations for computing the invariant distribution T, and the length of the histories kept as state variables when computing the business cycles N are all chosen to be as large as possible, while keeping the computational task manageable and results being robust to non-trivial changes in their values. Their chosen values are 20, 2^{23}, 1000, and 273, respectively.[42] It turns out that under these computational parameters, the linearized system described in Section 3.3 has about $12,000$ variables (a large system indeed).

Finally, the lower and upper bounds for the range of possible promised values $v_{\min}$ and $v_{\max}$ were chosen so that the fraction of agents in the intervals $[v_1, v_2]$ and $[v_{J-1}, v_J]$ are each less than 0.1%. Thus, truncating the range of possible values at $v_{\min}$ and $v_{\max}$ should not play an important role in the results. The chosen values for $v_{\min}$ and $v_{\max}$ are -35.0 and -16.3, respectively.

Before turning to the business cycle results, I illustrate different features of the model at its deterministic steady state. Figure 1.A shows the invariant distribution of promised values across the $J-1$ intervals $[v_j, v_{j+1}]_{j=1}^{J-1}$, defined by the grid points of the spline approximations. While it is hard to see at this coarseness level, the distribution is approximately symmetrical. More importantly, we see that the invariant distribution puts very little mass at extreme values. In consequence, in what follows I will report allocation rules only between the 7th and 15th ranges of the histogram. The reason is not only

[42] Given the value selected for the survival probability σ, less than 0.1% of individuals survive more than N periods. Thus, the truncation imposed by keeping track of a finite history of decision rules introduces a very small approximation error.

that there are too few agents at the tails of the distribution for them to matter, but also that being close to the artificial bounds $v_{\min}$ and $v_{\max}$ greatly distorts the shape of the allocation rules.

While not apparent in Figure 1.A, the invariant distribution of promised values generates too little heterogeneity. The standard deviations of the cross-sectional distribution of log-consumption levels and log-hours worked are 0.04 and 0.35, respectively. This compares with values of 0.50 and 0.82 reported by Heathcote et al. (2010) for 1981 (the year of lowest consumption heterogeneity in their sample).[43] The reason for the small amount of heterogeneity is that there is no persistence in the idiosyncratic shocks: The only way that the model can generate large deviations from the mean is through long streams of repeated bad shocks or good shocks, and these are unlikely to happen. Unsurprisingly, an unrealistic idiosyncratic shock process generates an unrealistic amount of cross-sectional heterogeneity.[44]

Figure 1.B reports the utility of consumption for old agents $u_{oL}(v)$ and $u_{oH}(v)$ across promised values v, as well as those of young agents u_{yL} and u_{yH} (which are independent of v). We see that, in all cases the utility of consumption is higher when the value of leisure is low. Both u_{oL} and u_{oH} are strictly increasing in the promised value v, are linear (with slope less than one), and are parallel to each other. Moreover, the vertical difference between u_{oL} and u_{oH} is the same as between u_{yL} and u_{yH}. Figure 1.C reports the utility of leisure for old agents $n_{oL}(v)$ and $n_{oH}(v)$ across promised values v, as well as those of young agent n_{yL} and n_{yH}. In all cases leisure is lower when the value of leisure is low. Both n_{oL} and n_{oH} are strictly increasing in the promised value v, are linear (with slope less than one), and are parallel to each other. Moreover, the vertical difference between n_{oL} and n_{oH} is the same as between n_{yL} and n_{yH}. In turn, Figure 1.D reports the next-period promised values for old agents $w_{oL}(v)$ and $w_{oH}(v)$ across promised values v, as well as those of young agent w_{yL} and w_{yH}. We see that in all cases next-period promised values are higher when the value of leisure is low. Both w_{oL} and w_{oL} are strictly increasing in the promised value v, are linear (with slope equal to one), and are parallel to each other. We also see that the vertical difference between w_{oL} and w_{oH} is the same as between w_{yL} and w_{yH}. Observe that Figure 1 verifies the linear functional forms given by the steady state versions of equations (4.25)-(4.27). Also, the economics in Figure 1 is quite intuitive: When an agent (young or old) reports a high value of leisure, the planner allows them to enjoy more leisure but, in compensation, they receive less consumption and are promised worse treatment in the future.

The discussion of business cycle dynamics that follows will center on the analysis of the impulse

[43] See their Figures 10 and 13.

[44] For this reason, there is no point in reporting other features of the cross section, such as optimal labor and capital wedges.

responses of different variables to a one standard deviation increase in aggregate productivity. Figure 2.A shows the impulse responses of the utility of consumption of young agents u_{yL} and u_{yH}. We see that both impulse responses are identical and that their shape qualitatively resembles one for aggregate consumption in a standard RBC model. Figure 2.B shows the impulse response of the utility of consumption of old agents with a low value of leisure $u_{oL}(v)$, at each of the eleven grid points $(v_j)_{j=6}^{16}$. While the figure shows eleven impulse responses, only one of them is actually seen because they happen to overlap perfectly. This means that, in response to the aggregate productivity shock, the function u_{oL} depicted in Figure 1.B shifts vertically over time in a parallel way. Figure 2.C, which does the same for u_{oH}, is identical to Figure 2.B. Thus, u_{oH} also shifts vertically over time in a parallel way and its increments are the same as those of u_{oL}. Figure 3 is analogous to Figure 2, except that it depicts the behavior of the utility of leisure. Figure 3.A shows that the impulse responses of n_{yL} and n_{yH} are identical and that they resemble the response of leisure in a standard RBC model, while Figures 3.B and 3.C indicate identical vertical parallel shifts of the functions n_{oL} and n_{oH} in response to the aggregate productivity shock. Turning to promised values, Figure 4.A shows that the impulse responses of w_{yL} and w_{yH} coincide. In turn, Figures 4.B and 4.C show that w_{oL} and w_{oH} shift vertically in a parallel way by identical amounts in response to an aggregate productivity shock. Thus, taken together, Figures 2-4 reproduce the analytical results of Lemma 6.

Figure 5.A shows the impulse responses of the cross-sectional standard deviations of promised values, log-consumption, and log-hours worked. We see that in response to a positive aggregate productivity shock, the standard deviations of promised values and log-consumption remain flat, while the standard deviation of log-hours worked decreases. Thus, Figure 5.A reproduces the analytical results of Corollary 8.

Finally, Figure 5.B shows the impulse responses of aggregate output Y, aggregate consumption C, aggregate investment I, aggregate hours worked H, and aggregate capital K in the benchmark economy with private information. Figure 5.C reports the impulse responses for the same variables but for the representative agent economy, revealing that both sets of impulse responses are identical.[45] Thus, Figures 5.B and 5.C reproduce the analytical result of Corollary 5.

I have verified that while the computational method was not designed to exploit any of the properties of the logarithmic case, it is able to exactly reproduce the analytical results derived for this case. Passing this test so successfully suggests that the computational method introduced in this paper could be quite

[45]The solution for the representative agent economy is obtained using a separate and much simpler code. The associated linear system has only 8 variables (compared to the about $12,000$ variables of the benchmark economy).

useful as a general method for computing aggregate fluctuations of economies with heterogeneous agents.

4.3 Extension to other preferences

To complete this section, I report results for preferences of the following form:

$$E_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} \sigma^{t-T} \left[\frac{c_t^{1-\varphi} - 1}{1-\varphi} + \alpha_t \frac{(1-h_t)^{1-\pi} - 1}{1-\pi} \right] \right\},$$

where $\varphi \neq 1$ and $\pi \neq 1$. Since I don't have analytical results for this more general functional form, I rely on the computational method for evaluating business cycle fluctuations in this case.

Without recalibrating other parameters, I considered different values for φ and π but in all cases I obtained similar results. For concreteness, I report results here for unit deviations from the $\varphi = 1$ and $\pi = 1$ case. For each of these cases, Table 1 reports the deterministic steady state values of all macroeconomic variables for the economies with private information and full information. We see that in each parametrization, all variables are nearly identical in both information scenarios.

In order to streamline the analysis of business cycle dynamics, I consider the $\varphi = 2$ and $\pi = 2$ as a representative case. Figure 6.A reports that, contrary to the log-log case, the cross-sectional distribution of promised values now follows some non-trivial dynamics: Instead of being constant, the standard deviation of promised values decreases significantly in response to a positive aggregate productivity shock. Despite this, the information frictions remain irrelevant for aggregate dynamics. Figure 6.B reports the impulse responses of all macroeconomic variables in the economy with private information while Figure 6.C does the same for the economy with full information. We see that both sets of impulse responses are identical. Thus, similar to the log-log case, the stationary behavior of the aggregate variables of the economy is not affected by the presence of information frictions.

Is this irrelevance result under CRRA preferences a purely numerical result, or is it part of a more general theoretical result that I have been unable to prove?[46] While the latter is a logical possibility, I strongly believe in the former. The reason for this is based on my proof of Lemma 2. The proof shows that, under logarithmic preferences, the inverse Euler equations that characterize the optimal allocation under private information become linear. This allows me to integrate the inverse Euler equations across all individuals and obtain a relation between aggegate variables that coincides with the direct Euler equation of the representative agent. When preference are CRRA, Jensen's inequality breaks the integration of the inverse Euler equations into an elemental relation between aggregate variables. I believe that the reason

[46]In either case, it is important to point out that the irrelevance result obtained here is not some artifact of the computational method, since it was already obtained when I compared deterministic steady states.

for my numerical irrelevance result under CRRA preferences is that, under i.i.d. shocks, the amount of cross-sectional heterogeneity is so small that Jensen's inequality becomes negligible.

5 An Hopenhayn-Nicolini economy

The Mirlees economy of Section 4 provided a valuable test for the computational method introduced in Section 3, and illustrated an interesting benchmark scenario in which there are no interactions between private information and aggregate dynamics. An unappealing feature of that economy, however, was that the i.i.d. structure of its idiosyncratic shocks precluded giving any empirical content to the model economy. This section considers a second example that differs from the Mirlees economy in that it has a much more realistic structure of idiosyncratic uncertainty, and in that it illustrates a scenario in which the information frictions play an important role for aggregate dynamics. The economy considered is an RBC model in which all production is done in a central island. Agents get exogenously separated from the production island and to get back to it, they need to search. The probability of arriving at the production island depends on the search intensity of the agent, which is private information. This hidden action creates a moral hazard problem similar to Hopenhayn and Nicolini (2009). While the search intensity of agents is not observable, their location (either inside or outside the production island) is and, therefore, recursive contracts can be made contingent on this information.

The economy is populated by a continuum of agents with stochastic lifetimes. The preferences of an agent born at date T are given by

$$E_T \left\{ \sum_{t=T}^{\infty} \beta^{t-T} \sigma^{t-T} \left[u\left(c_t\right) - s_t \right] \right\}, \tag{5.1}$$

where σ is the survival probability, β is the discount factor, and s_t is the private search intensity.

The production function is the same as in equation (4.2), except that H_t is now interpreted as employment instead of hours worked. The aggregate productivity shock follows the same AR(1) process given by equation (4.3), and capital is accumulated as in equation (4.4). The search technology is such that the probability of arriving at the production island at the beginning of the following period is given by some increasing and concave function $\eta\left(s_t\right)$. The exogenous separation rate is ϕ.

In what follows, agents located inside the production island are called employed (e) and agents located outside are called unemployed (u). Similarly to Section 4, I group agents into two sets, young (y) and old (o), and decompose the economy-wide planning problem (which seeks to maximize the weighted sum of the welfare levels of the current and future generations) into a sequence of sub-planning problems and side conditions. The sub-planning problems for production decisions are the same as in equation (4.10).

The sub-planning problems for individuals differ depending on the age and employment status of the agent. The date t sub-planning problem for employed old individuals is the following:

$$P_{oet}(v) = \max\left\{q_t - c(u_{oet}) + \theta\sigma E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\left(\phi P_{ou,t+1}(w_{oeu,t+1}) + (1-\phi) P_{oe,t+1}(w_{oee,t+1})\right)\right]\right\} \tag{5.2}$$

subject to

$$v = u_{oet} + \beta\sigma E_t\left[\phi w_{oeu,t+1} + (1-\phi) w_{oee,t+1}\right], \tag{5.3}$$

where u_{oet} is the utility of consumption, $w_{oee,t+1}$ is the next-period promised value in the event of continuing employed, $w_{oeu,t+1}$ is the next-period promised value in the event of becoming unemployed, $c(u) = u^{-1}(u)$, q_t is the shadow price of labor, λ_t is the shadow price of the consumption good, and θ is the Pareto weight of the next-period generation relative to the current generation of young agents. Observe that the current social value of an employed old worker is the value of their labor input, net of the consumption goods transferred to them. Equation (5.3) is the promise-keeping constraint.

The date t sub-planning problem for unemployed old individuals is

$$P_{out}(v) = -c(u_{out}) + \theta\sigma E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\left(\eta(s_{ot}) P_{oe,t+1}(w_{oue,t+1}) + [1-\eta(s_{ot})] P_{ou,t+1}(w_{ouu,t+1})\right)\right], \tag{5.4}$$

subject to

$$v = u_{out} - s_{ot} + \beta\sigma E_t\left[\eta(s_{ot}) w_{oue,t+1} + [1-\eta(s_{ot})] w_{ouu,t+1}\right], \tag{5.5}$$

$$\eta'(s_{ot}) \beta\sigma E_t\left[w_{oue,t+1} - w_{ouu,t+1}\right] - 1 \leq 0, \quad = 0 \text{ if } s_{ot} > 0, \tag{5.6}$$

where u_{out} is the utility of consumption, s_{ot} is the search intensity, $w_{ouu,t+1}$ is the next-period promised value in the event of continuing unemployed, and $w_{oue,t+1}$ is the next-period promised value in the event of becoming employed. Observe that the current social value of an old unemployed agent is simply the cost of the consumption goods transferred to them. Equation (5.5) is the promise-keeping constraint. Equation (5.6) is the first-order condition for the individual's optimal choice of search intensity. Since the search intensity is not observable, the agent chooses it to maximize their private gains. In an interior solution, this is attained by equating the marginal private benefits (given by the marginal impact on the hazard rate times the expected gains of becoming employed) to the marginal private cost (which is equal to one, given that s_{ot} enters linearly in the individual's utility function 5.1).[47]

[47] Corner solutions could be ruled out by assuming, for example, that $\eta(s) = s^{\alpha}$, $0 < \alpha < 1$. However, it turns out that the functional form is not able to generate large fluctuations in aggregate employment. The functional form used later on leads to corner solutions in the search intensity.

Since I assume that all agents are born unemployed, there is only one type of sub-planning problem for young agents. It is given by

$$P_{yt} = \max\left\{\frac{u_{yt} - s_{yt} + \beta\sigma E_t\left[\eta\left(s_{yt}\right) w_{ye,t+1} + \left[1-\eta\left(s_{yt}\right)\right] w_{yu,t+1}\right]}{\lambda_t} - c\left(u_{yt}\right) + \right.$$

$$\left. + \theta\sigma E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\left[\eta\left(s_{yt}\right) P_{oe,t+1}\left(w_{ye,t+1}\right) + \left[1-\eta\left(s_{yt}\right)\right] P_{ou,t+1}\left(w_{yu,t+1}\right)\right]\right]\right\}, \tag{5.7}$$

subject to

$$\eta'\left(s_{yt}\right)\beta\sigma E_t\left[w_{ye,t+1} - w_{yu,t+1}\right] = 1 \tag{5.8}$$

where u_{yt} is the utility of consumption, s_{yt} is the search intensity, $w_{yu,t+1}$ is the next-period promised value in the event of continuing unemployed, and $w_{ye,t+1}$ is the next-period promised value in the event of becoming employed. Observe that the present value of the lifetime utility of the young agent directly enters the current social value, since the economy-wide social planner seeks to maximize the weighted sum of the welfare levels of current and future generations. The consumption goods transferred to the young agent are substracted from the current social value, since the transfer tightens the aggregate consumption feasibility constraint of the economy-wide social planner. Equation (5.8) is similar to equation (5.6), except that it assumes an interior solution for s_{yt} (as will always be the case in equilibrium).

The economy-wide distribution of employed old agents across promised values is denoted by μ_{et}. The similar object for unemployed old agents is denoted by μ_{ut}. The law of motion for μ_{et} is given by:

$$\begin{aligned}\mu_{e,t+1}\left(\mathcal{B}\right) &= \left(1-\sigma\right)\sigma\eta\left(s_{yt}\right)\mathcal{I}\left[w_{ye,t+1}\in\mathcal{B}\right] + \sigma\int_{\{v:w_{oue,t+1}(v)\in\mathcal{B}\}}\eta\left(s_{ot}\left(v\right)\right)d\mu_{ut} \\ &\quad +\sigma\int_{\{v:w_{oee,t+1}(v)\in\mathcal{B}\}}\left(1-\phi\right)d\mu_{et},\end{aligned} \tag{5.9}$$

for every Borel set $\mathcal{B}$, where $\mathcal{I}$ is an indicator function that takes a value of one if its argument is true and zero, otherwise. This equation states that the number of employed old agents that have a promised value in the set $\mathcal{B}$ at the beginning of the following period is given by the sum of three terms. The first term includes all young agents that do not die, find the production island, and get a promised value in the set $\mathcal{B}$. The second term includes all unemployed old agents that do not die, find the production island, and get a promised value in the set $\mathcal{B}$. The third term includes all employed old agents that do not die, do not get separated from the production island, and get a promised value in the set $\mathcal{B}$. The law of motion for μ_{ut}, which is given by

$$\begin{aligned}\mu_{u,t+1}\left(\mathcal{B}\right) &= \left(1-\sigma\right)\sigma\left[1-\eta\left(s_{yt}\right)\right]\mathcal{I}\left[w_{yu,t+1}\in\mathcal{B}\right] + \sigma\int_{\{v:w_{ouu,t+1}(v)\in\mathcal{B}\}}\left[1-\eta\left(s_{ot}\left(v\right)\right)\right]d\mu_{ut} \\ &\quad +\sigma\int_{\{v:w_{oeu,t+1}(v)\in\mathcal{B}\}}\phi d\mu_{et},\end{aligned} \tag{5.10}$$

is similarly interpreted.

Given initial values for K_0, μ_{e1} and μ_{u1}, the side conditions that the stochastic process $\{q_t, \lambda_t\}_{t=1}^{\infty}$ must satisfy are

$$(1-\sigma)\, c\,(u_{yt}) + \int c\,(u_{out}\,(v))\, d\mu_{ut} + \int c\,(u_{oet}\,(v))\, d\mu_{et} + I_t\,(K_{t-1}) = e^{z_t} F\,(K_{t-1}, H_t) \tag{5.11}$$

and

$$H_t = \int d\mu_{et}. \tag{5.12}$$

Equations (5.11) and (5.12) are the feasibility conditions for consumption and labor, respectively.[48]

5.1 Quantitative results

This section evaluates the aggregate effects of the information frictions in a parametrized version of the model. In particular, I compare the steady-state and business cycle dynamics of the model described above with its full-information counterpart, which is obtained by removing the incentive compatibility constraints (5.6) and (5.8).[49] To do this, I set the time period to one year, make the utility function u logarithmic, give the production function the same Cobb-Douglas form as in Section 4.2, assume the search technology to be $\eta\,(s_t) = D\,(1 - e^{-\tau s_t})$, and choose parameter values as described next. The technological parameters γ, δ, ρ and σ_ε^2, the discount factors β and θ, and the survival probability σ are all set to be exactly the same values as in Section 4.2. The only parameters that are specific to the model of this section are the employment separation rate ϕ and the search technology parameters D and τ. To determine ϕ I turn to Krusell et al. (2017) who measured a quarterly employment-to-employment transition rate using CPS data, that implied an average duration of an employment spell equal to 8.9 years.[50] Reproducing this observation requires setting $\phi = 0.112$. In turn, the parameters D and τ are chosen to satisfy two criteria: that aggregate employment under full information be equal to 0.60

[48] Contrary to Section 4, I do not specify the $t = 0$ planning problems because here I am only interested in computing the stationary solution with the method described in Section 3. Taking the initial conditions at $t = 1$ as given suffices for this.

[49] Technical Appendix 15 shows how to map these economies into the structure of Section 3. In principle, equation (5.6) poses some difficulty because, for some range of promised values, it may be an occasionally binding constraint. I avoid this problem by imposing that the constraint binds for some promised value if and only if it binds at the deterministic steady state. This should provide a good approximation for aggregate dynamics because the contribution to aggregate hiring of agents close to the zero-search threshold is negligible. Therefore, it should make no difference if over the business cycle, some of these agents fluctuate between zero search and an epsilon amount, or never search.

[50] Observe that average employment spells are much longer than average job spells, since many workers experience job-to-job transitions without going through non-employment.

(the ratio of employment to the working age population in U.S. data), and that the peak value of the impulse response of aggregate employment to an aggregate productivity shock be as large as possible (since models with search frictions tend to have difficulties generating large employment fluctuations). The selected values of D and τ turn out to be 8.0 and 0.016, respectively. Observe that since the model is calibrated to generate the observed average duration of employment and non-employment, it embodies the empirically correct amount of idiosyncratic employment risk.[51]

Before turning to the business cycle results, I describe the deterministic steady-state properties of the benchmark economy. I start with the green line in Figure 7.A, which depicts the job-finding probability for old unemployed agents $\eta(s_o)$ across promised values v. Not surprisingly, this function is decreasing in v. That is, agents with higher promised values are required to search less. What is interesting is that there exists a threshold promised value above which the job-finding probability becomes zero (marked as a vertical dotted line). The orange line in Figure 7.A shows the job-finding probability $\eta(s_y)$ for young agents. We see that agents start their lives with a high job-finding probability.

Figure 7.B depicts next-period promised values for unemployed agents as a function of v. The green line is the 45-degree line, while the vertical dotted line marks the zero-search threshold promised value. In turn, the yellow line describes w_{ouu}, while the blue line describes w_{oue}. We see that w_{ouu} coincides with the 45-degree line above the zero-search threshold. Thus, if some agent enters unemployment with a promised value larger than that threshold, they remain unemployed forever and their promised value never changes (effectively retiring at that promised value). However, the w_{ouu} function remains uniformly below the 45-degree line for promised values below the non-search threshold. Thus, an unemployed agent's promised value decreases during their unemployment spell. Moreover, the punishment for remaining unemployed increases during the unemployment spell: The vertical difference between the 45-degree line and w_{ouu} increases with lower values of v. However, when an unemployed agent finds employment, the agent gets a significant reward in terms of next-period promised value (the vertical difference between w_{oue} and the 45-degree line). Since w_{oue} is parallel to the 45-degree line, this reward is the same at all promised values below the zero-search threshold. Observe that this reward is needed even when the agents put a tiny amount of search (i.e, when their promised value v is an epsilon below the zero-search threshold). The reason for this is that the search technology chosen has a finite slope at $s = 0$ and, therefore, the incentive compatibility constraint (5.6) requires a positive reward for inducing the agent to do even an infinitesimal amount of search. Also observe that, since no one with a promised value to the right of the zero-search threshold makes a transition to employment, the portion of the w_{oue} function to the right

[51] It is in this sense that the model of this section is more realistic than the Mirlees economy of Section 4.

of that threshold is completely meaningless. Figure 7.B also displays the next-period promised values of young agents in the event that they become employed w_{ye} or continue to be unemployed w_{yu}. We see that both are quite low.

Figure 7.C depicts next-period promised values for employed agents as a function of v. The green line again represents the 45-degree line, and the vertical dotted line marks the zero-search threshold. The blue line describes w_{oee}, while the yellow line describes w_{oeu}. We see that w_{ee} coincides with the 45-degree line at all values of v. That is, the promised values of employed agents do not change while they remain employed, which is quite intuitive since employed agents face no incentive problems. Observe that w_{oeu} also coincides with the 45-degree line but to the right of the zero-search threshold. That is, employed agents with promised values higher than the threshold do not see their promised values change when they become unemployed. This is also quite intuitive, since these agents will never search again (effectively retiring). However, we see that w_{oeu} remains below the 45-degree line to the left of the zero-search threshold, so in this range employed agents get punished when they become unemployed. In fact we see that the punishment, given by the vertical difference between the 45-degree line and w_{oeu}, increases with lower values of v. This is also intuitive since, besides what may happen with their consumption levels at the time that they become unemployed (an issue that will be addressed below), their search intensities (which reduce their welfare levels) increase with lower values of v.

Figure 8.A shows the log of consumption as a function of v. The vertical dotted line once again represents the zero-search threshold, while the blue line shows u_{oe} and the yellow line u_{ou}. The interesting feature of this figure is that at the right of the zero-search threshold, u_{oe} and u_{ou} coincide: The consumption level in that region is identical for employed and unemployed agents. Thus, when an employed agent retires, their consumption level remains exactly the same as when last employed. Not surprisingly, to the left of the zero-search threshold, u_{ou} must be higher than u_{oe} since, in order to obtain the same promised value v, the unemployed agents must be compensated with higher consumption for their positive search effort.

More interesting is to analyze the consumption changes that take place as agents transition between the different employment states, since this informs us about the amount of insurance provided. This is shown in Figure 8.B. The red line shows the consumption changes that take place when an employed agent continues to be employed in the next period. We see that their consumption levels do not change. This is quite obvious, since these agents experience no idiosyncratic shock to their employment state and since being employed entails no incentive problems, there is no reason to change their consumption levels. Much more interesting is the blue line, which shows the consumption changes that take place when an employed agent become unemployed in the following period. Again, we see that their consumption levels are not affected. This is actually quite intuitive: Since the idiosyncratic shocks that determine an employment-to-unemployment transition are completely exogenous and current search decisions are not affected by current consumption levels, there is no reason not to fully insure agents against those shocks. The gray line shows the consumption changes that take place when an unemployed agent continues to be unemployed the following period. For promised values larger than the zero-search threshold, we see that there are no consumption changes: Agents receive a constant consumption stream during their retirement. However, to the left of the zero-search threshold, we see that consumption drops if an unemployed agent continues to be unemployed. The reason is that in order to induce the agent to search, the planner needs to punish them in case that they continue to be unemployed (and reward them in case they become employed) and does this partially by reducing the agent's consumption level (recall that the agents are also induced to increase their search intensity if they continue to be unemployed). Observe that for promised values close to the zero-search threshold, the consumption change is small but it becomes significant at lower promised values. Thus, if an agent remains unemployed for a long period, the accumulated consumption loss can become quite significant. As a counterpart to this, the yellow line shows that when an unemployed agent finds employment, their consumption level increases abruptly: The increase is always larger than 18% and exceeds 40% for promised values close to the zero-search threshold. Observe that the portion of the yellow line to the right of the zero-search threshold is meaningless since no unemployed agent in that range makes a transition to employment.

Figures 8.C and 8.D report the implied invariant distributions μ_e and μ_u, respectively. We have already encountered the least upper bound for the support of those distributions: In Figure 7.B the value of the w_{oue} function at the zero-search threshold gives the highest promised value that an unemployed agent can possibly get by becoming employed. Hence, it is the least upper bound for the support of μ_e. Since we know from Figure 7.C that an employed agent with that promised value does not get punished when they become unemployed and retire, this value is also a least upper bound for the support of μ_u. While not shown in Figure 7.A, there exists a promised value at which the job-finding probability becomes one. This promised value is then a lower bound for the support of μ_u, and the value of w_{ue} evaluated at that promised value is a lower bound for the support of μ_e. However, Figures 8.C and 8.D show that there is no mass at such low promised values. The reason is that promised values drift down slowly while unemployment lasts (as shown by the vertical distance between the 45-degree line and w_{ouu} in Figure 7.B), and job-finding probabilities increase sharply as the promised value decreases, escaping unemployment very quickly already at the lowest promised values shown in the figures. Observe the large mass that μ_e has at $w_{ye} = -17.5$. This is because most young agents find employment right away, every new generation gets the same w_{ye}, and they accumulate at that value given the long average duration of employment. In contrast, the distribution μ_u does not get a large mass near $w_{yu} = -24.6$ because young agents find employment very quickly and, thus, they don't accumulate near that value. Also observe the rather odd shapes for μ_e and μ_u. The reason for this is that promised values do not change while employed, that the average duration of employment is high, that promised values drift down slowly during the relatively few unemployment episodes that agents experience during their lifetimes, that when they get reemployed their promised values jump by the same large amount, that agents get absorbed into retirement or death, and that newborns always start their second period of their lives either at w_{ye} or w_{yu}. Thus, there is not enough mixing in the distributions μ_e and μ_u.

Having characterized the deterministic steady state of the economy with information frictions, I now evaluate the effects of those information frictions on aggregate dynamics. I start by comparing the steady-states of the economy with information frictions and of its full-information counterpart. This is done in Table 2. Contrary to the Mirlees economy of Section 4, the information frictions have huge effects on the steady state dynamics of the Hopenhayn-Nicolini economy: They reduce aggregate employment, capital, investment, consumption, and output by 18.9%. The intuition for this result is straightforward. Given the constant separation rate, the only way that the planner can generate a high aggregate employment level is by inducing agents to search more intensively. However, the only way that the planner can do this is by increasing the difference between the promised values of becoming employed and the promised values of continuing unemployed. Thus, the social planner needs to hit agents with their insurance in order to do this. Since agents are risk-averse and the social planner cares for the welfare of agents, the planner decides to generate a lower aggregate employment level (and with it, lower capital, investment, and output). Despite this, the planner needs to tolerate very large consequences of the information frictions for the amount of consumption heterogeneity: While everybody consumes the same amount in the economy with full information (since they are fully insured), the standard deviation of log consumption becomes 26% in the economy with private information.

I now turn to the business cycle effects of the information frictions.[52] Figures 9.A through 9.D show impulse responses of aggregate employment, output, consumption, and investment, respectively, to a one standard deviation increase in aggregate productivity. The red lines report impulse responses for the economy with private information, while the blue lines report them for the full-information economy. Figure 9.A shows that the information frictions have significant effects on aggregate employment dynamics. The information frictions reduce the peak employment response by 20% and increases the half-

[52] Computing business cycle dynamics for this economy is much more involved than for the Mirlees economy of Section 4. Now histories of spline coefficients must be kept track for two distributions: μ_e and μ_u.

life of this response from 3.5 years to 6.5 years. Thus, the information frictions reduce the variability of aggregate employment and make it substantially more persistent. The intuition for why this is the case is similar to the previous paragraph. When a positive aggregate productivity shocks hits the economy, the planner becomes desperate to increase aggregate employment quickly for the standard reasons in an RBC economy. However, since the planner can only do this by increasing the difference between w_e and w_u and hitting agents with the insurance that they receive, the planner decides to increase aggregate employment more slowly but to keep it at a higher level for longer (in order to still reap the benefits of the aggregate productivity increase). Figure 9.B reports similar qualitative results for aggregate output (the quantitative effects are smaller since, this being an RBC model, the output dynamics are largely determined by the direct effects of aggregate productivity). The same can be said about aggregate consumption in Figure 9.C. However, the effects on aggregate investment in Figure 9.D are somewhat different: While the information frictions make investment more persistent, they also make it a touch more responsive. The reason for this is to provide workers with more capital as they remain employed for a longer period.

To complete the analysis, Figure 9.E reports the impulse response of the cross-sectional standard deviation of log consumption levels in the economy with private information (this statistic is always zero in the full-information economy). We see that the amount of inequality decreases on impact but that it subsequently reverses the initial response and increases. The reason why inequality decreases on impact is that aggregate employment is predetermined at the time that the aggregate shock hits the economy. Thus, there is no cost for the planner to reduce the amount of inequality at that point in anticipation of the increase in inequality that they will have to generate in subsequent periods in order to induce agents to search more intensively. However, the effects are rather small in magnitude: The peak response of the cross-sectional standard deviation of log consumption is barely above 0.4%.

6 Conclusions

In this paper, I introduced a general method for computing equilibria of economies with heterogeneous agents and aggregate shocks that is particularly suitable for economies with private information, and applied it to two examples: a Mirlees RBC economy and an Hopenhayn-Nicolini RBC economy. These economies illustrated very different types of interaction between private information and aggregate dynamics.

The Mirlees RBC economy had the particular advantage of lending itself to a sharp analytical characterization when preferences were logarithmic. Since the computational method doesn't use any of the structure of the logarithmic preferences, this provided an ideal test case scenario for it. The method passed the test with flying colors, reproducing all the analytical results. Besides serving as a test for the computational method, the analytical characterization provided a very interesting theoretical result: Under logarithmic preferences, the aggregate fluctuations of the economy are exactly the same under private or full information. This irrelevance result is related to others in the literature but, in the context of a Mirlees economy with endogenous labor supply and aggregate shocks, it is completely novel. For CRRA preferences, numerical results indicate that the irrelevance of the information frictions for aggregate dynamics still holds. However, the cross-sectional distribution of promised values, instead of being constant as in the logarithmic case, now changes its shape.

The Mirlees RBC economy was very interesting in illustrating a case in which the private information did not matter for aggregate dynamics. However, it had the unappealing feature that its i.i.d. shocks made it highly unrealistic. The second example considered, the Hopenhayn-Nicolini RBC model, had a much more realistic structure of idiosyncratic uncertainty and illustrated a case in which the information frictions played an important role for aggregate dynamics. Comparing the aggregate dynamics of this economy under private and full information now indicated that the level and volatility of aggregate employment becomes substantially smaller in the private information case. The basic reason for this is that, under private information, higher aggregate employment can only be obtained at the expense of lower insurance.

As a caveat to the computational method introduced in this paper, I would like to point out that, while it should prove useful and accurate for many applications, it has two related disadvantages. The first one is that is extremely slow. This should not be a problem when calibrating the steady-state of a model, since the computational method needs to be applied only once to obtain its aggregate fluctuations (once parameter values have been determined). However, it would be a problem in using it for estimating a model with formal econometric methods. The second disadvantage is that the dimensionality of the method grows exponentially with the number of endogenous state variables of the individuals. Therefore, at the current time it is extremely costly to apply the method to economies with more than one endogenous individual state variable. It is exactly for this reason that I was not able to analyze a much more realistic version of my Mirlees RBC economy with persistent idiosyncratic shocks, which is left for future research.

Also observe that, for both the Mirlees and the Hopenhayn-Nicolini RBC economies, the constrained-efficient aggregate fluctuations under private information were compared with their full information counterparts. This isolated the importance of the private information for aggregate dynamics. However, it would be extremely interesting to compare them with versions of the model with realistic financial markets and public policy in order to see how far from their socially optimum fluctuations actual economies may be. One could also compare them to the aggregate fluctuations obtained under optimal policy instruments restricted to belong to a certain class, to see how close to achieving constrained-efficient outcomes those policy instruments may be. I also leave these questions for future research.

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Table 1

Moral hazard economy: Steady state macroeconomic variables

(π, φ)	Information	Y	C	I	H	K
$(1,1)$	Private	0.52381	0.39070	0.13311	0.30999	1.3311
	Full	0.52381	0.39070	0.13311	0.31074	1.3311
$(1,2)$	Private	0.42990	0.32065	0.10925	0.25441	1.0924
	Full	0.42995	0.32069	0.10926	0.25444	1.0926
$(2,1)$	Private	0.75151	0.56054	0.19097	0.44474	1.9097
	Full	0.75168	0.56066	0.19101	0.44483	1.9101
$(2,2)$	Private	0.64229	0.47907	0.16322	0.38010	1.6322
	Full	0.64261	0.47931	0.16330	0.38029	1.6330

Table 2

Hopenhayn-Nicolini economy: Steady state effects

Information	Y	C	I	H	K	$\sigma(\ln c)$
Private	81.1	81.1	81.1	81.1	81.1	0.26
Full	100	100	100	100	100	0.0

Figure 1: Mirlees economy - Deterministic steady state

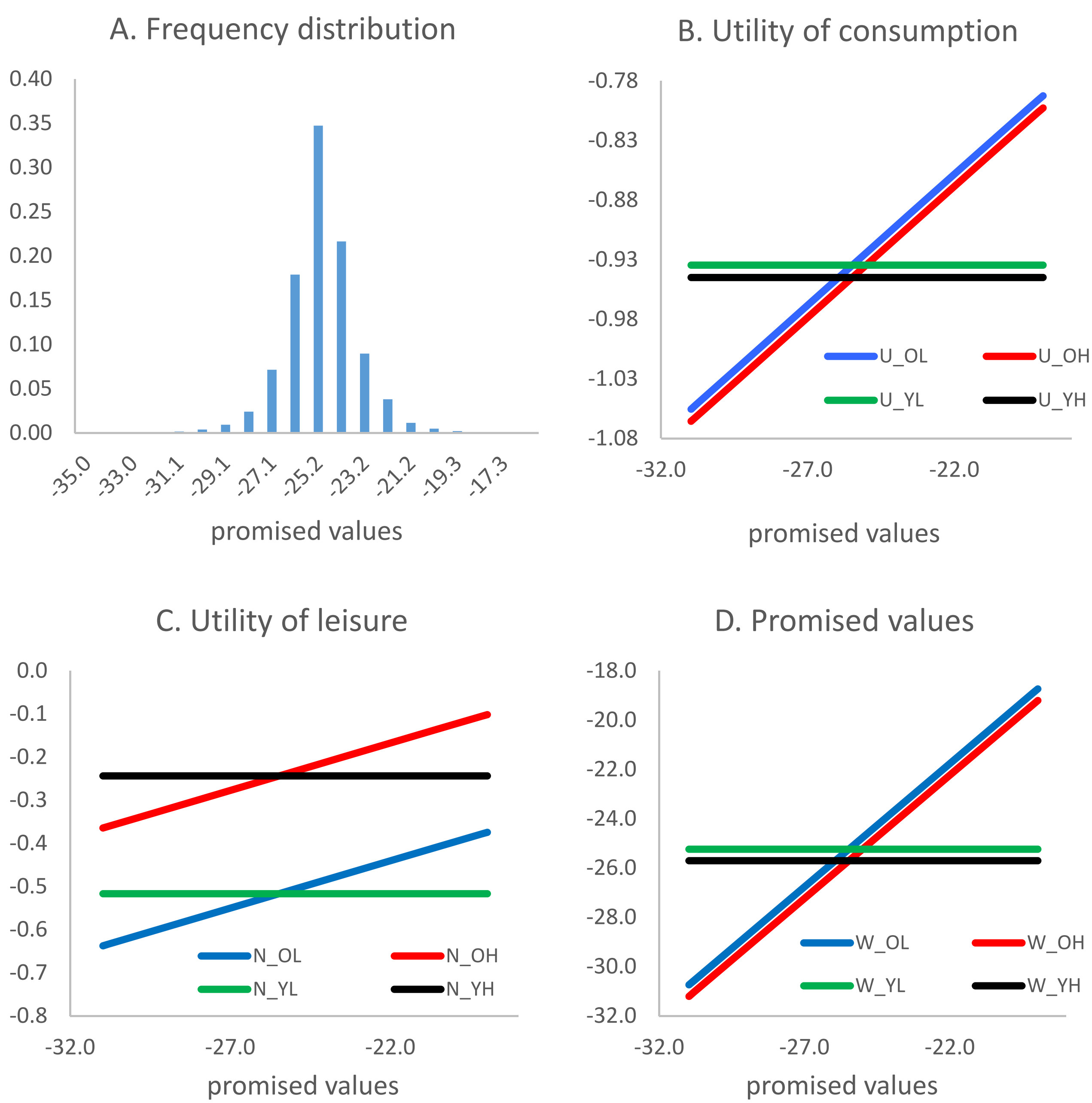

Figure 2: Mirlees economy - Impulse responses for consumption utilities

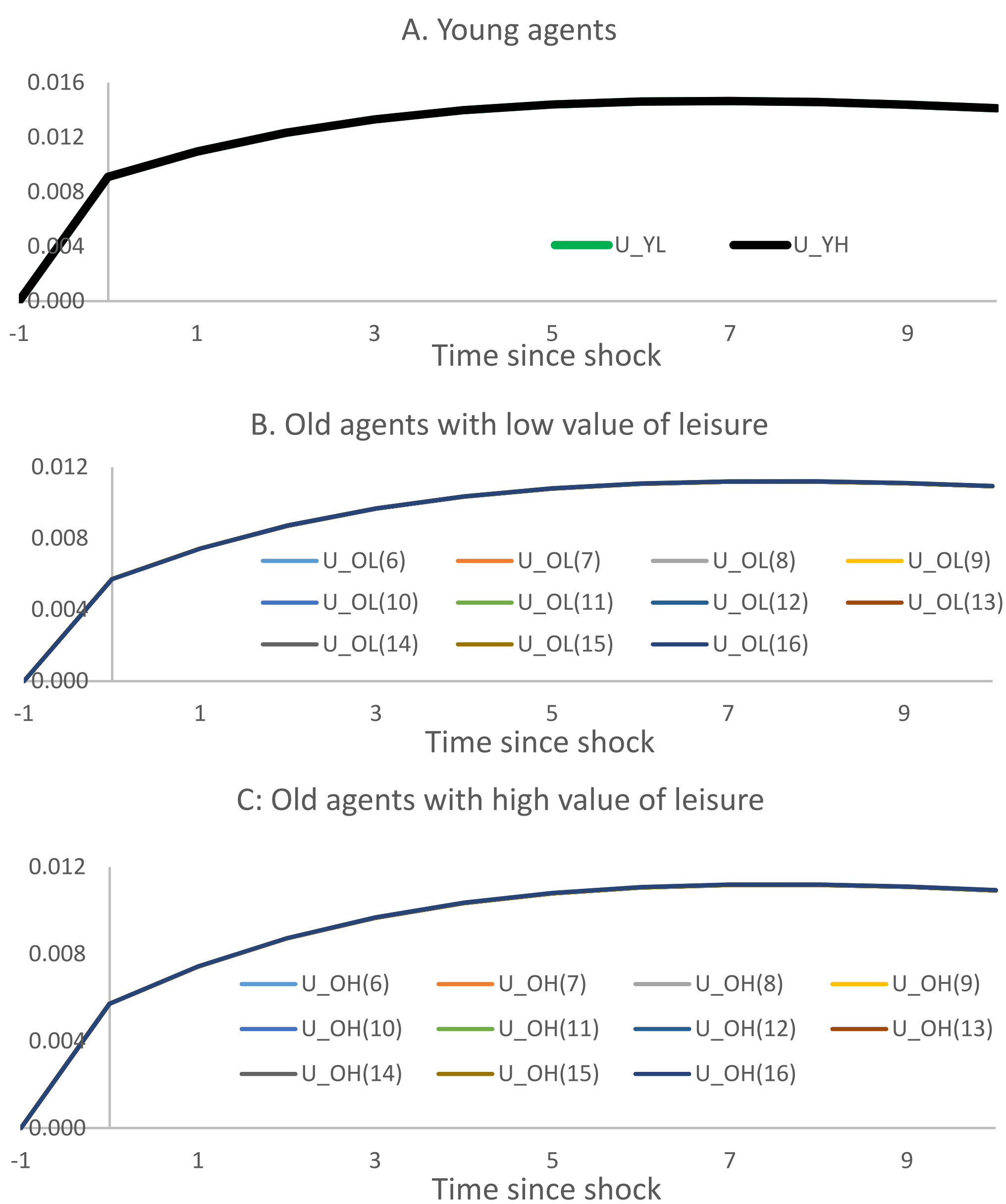

Figure 3: Mirlees economy - Impulse responses for leisure utilities

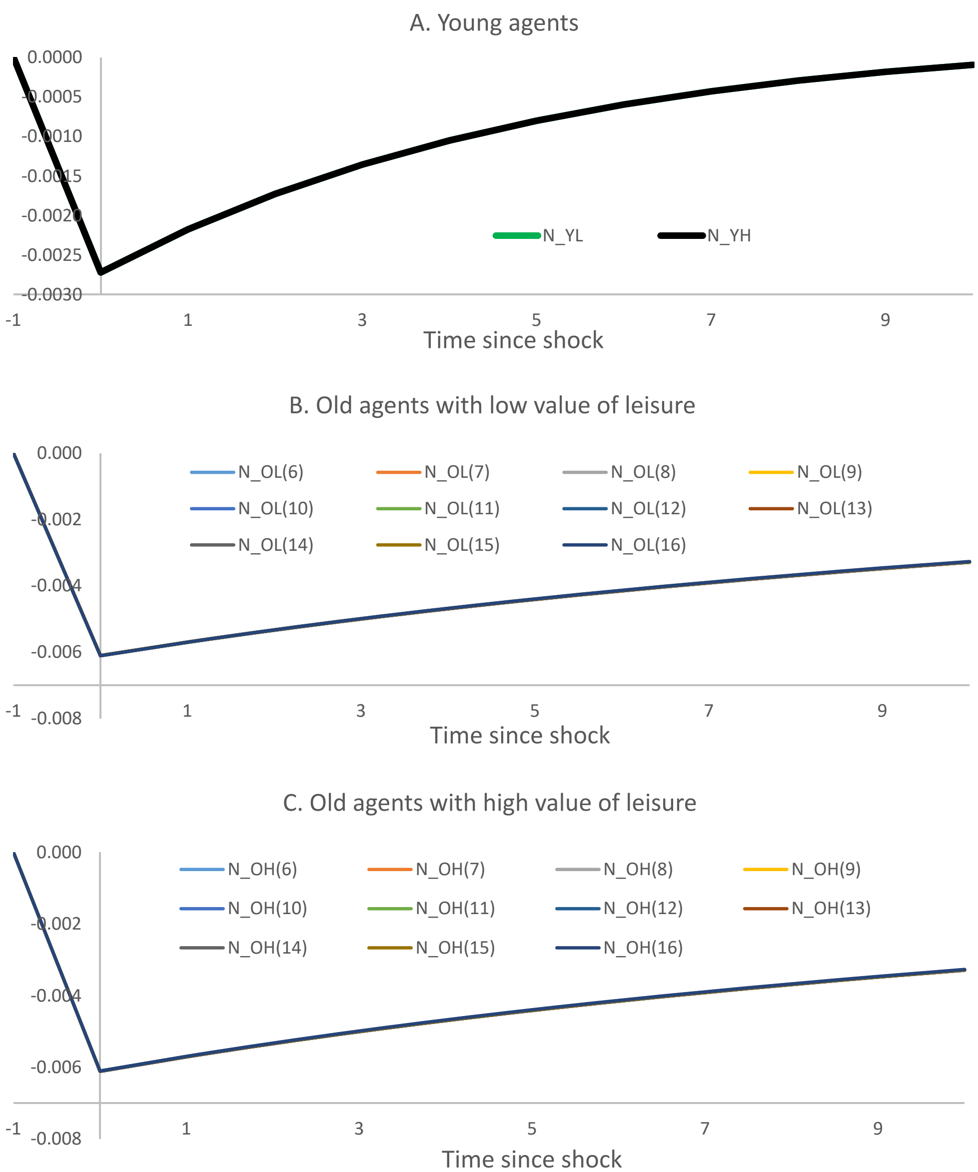

Figure 4: Mirlees economy - Impulse responses for promised values

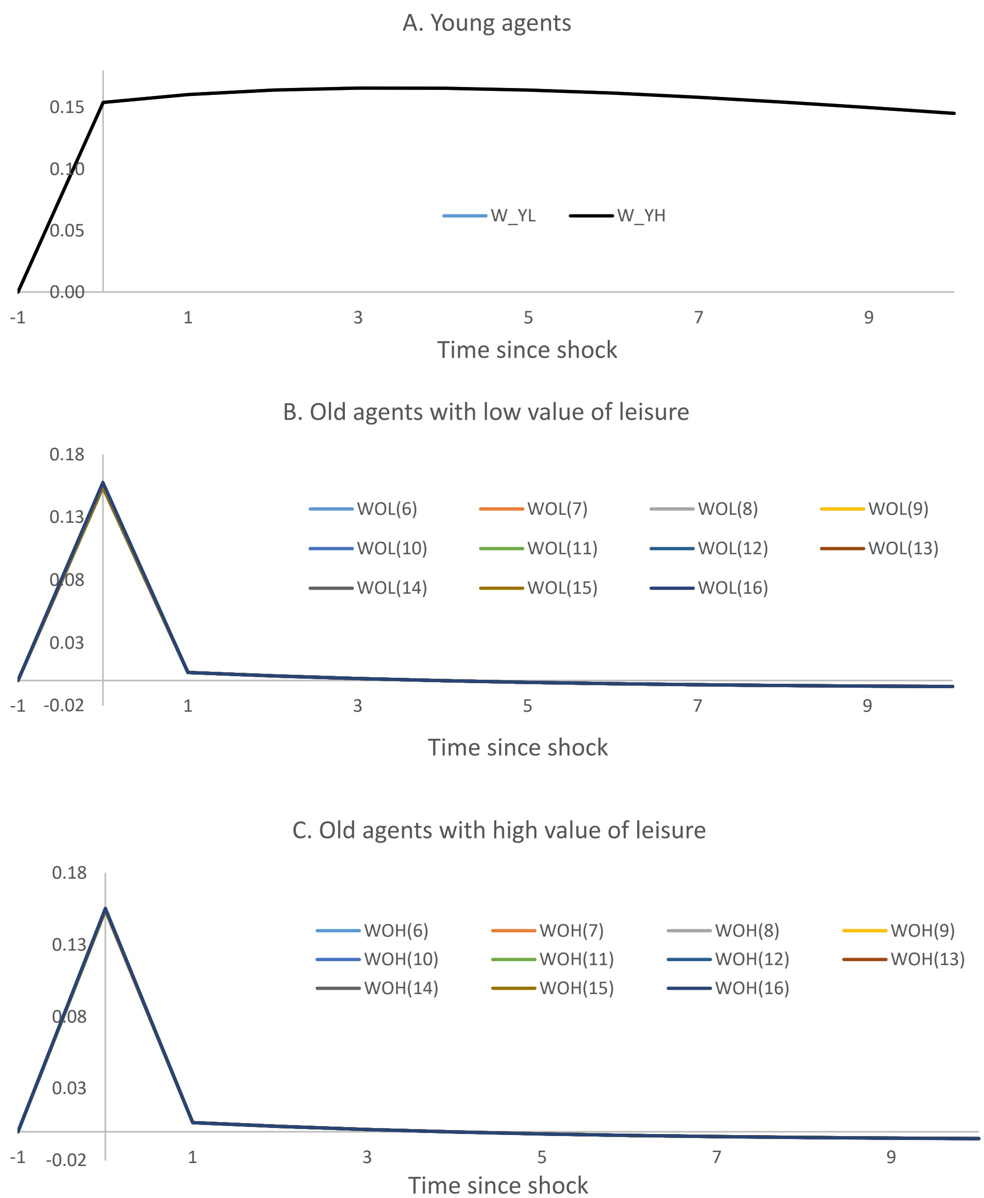

Figure 5: Mirlees economy - Cross-sectional distributions and macro variables

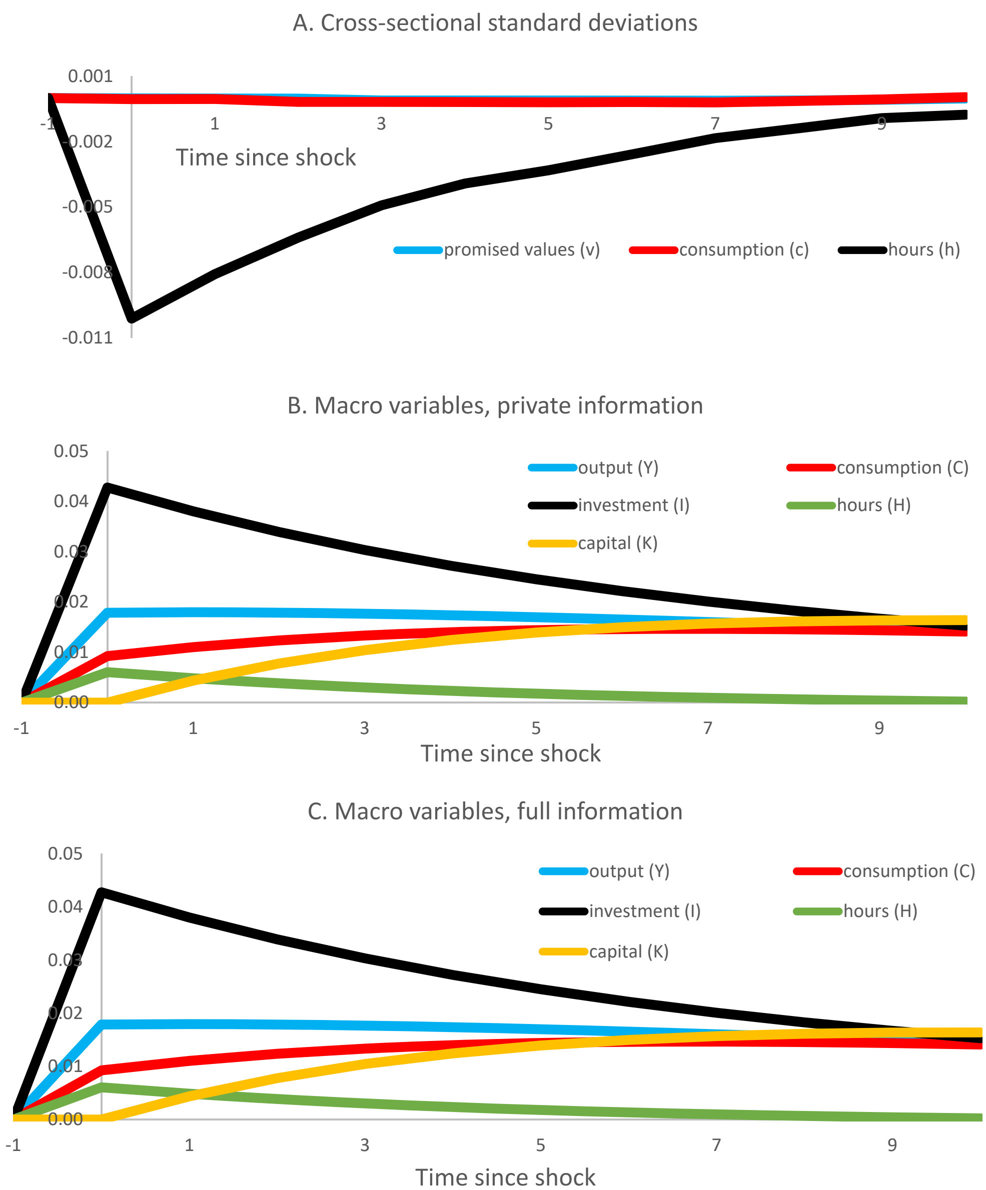

Figure 6: Mirlees economy - Cross-sectional distributions and macro variables

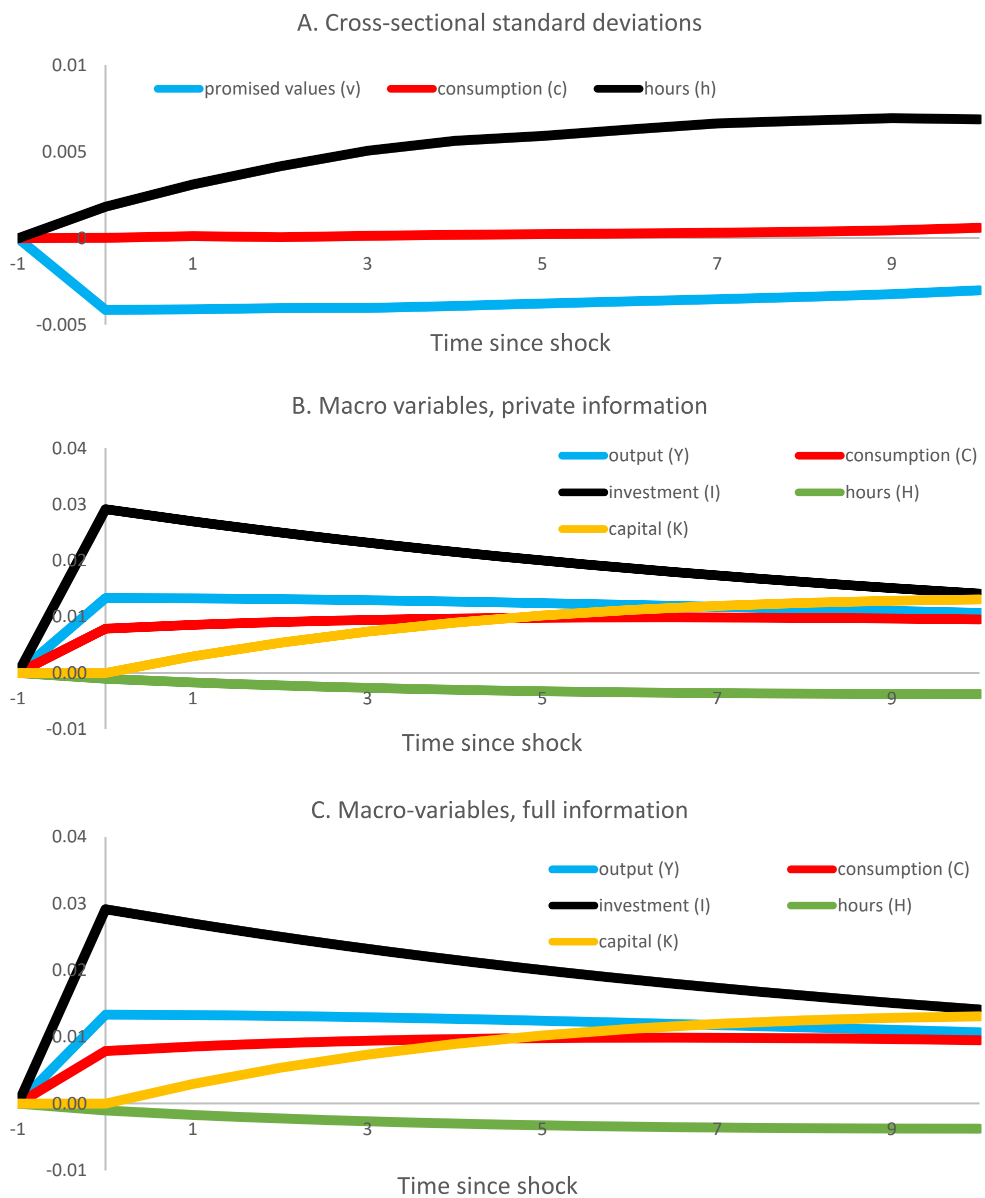

Figure 7: Hopenhayn-Nicolini economy - Job finding probabilities and promised values

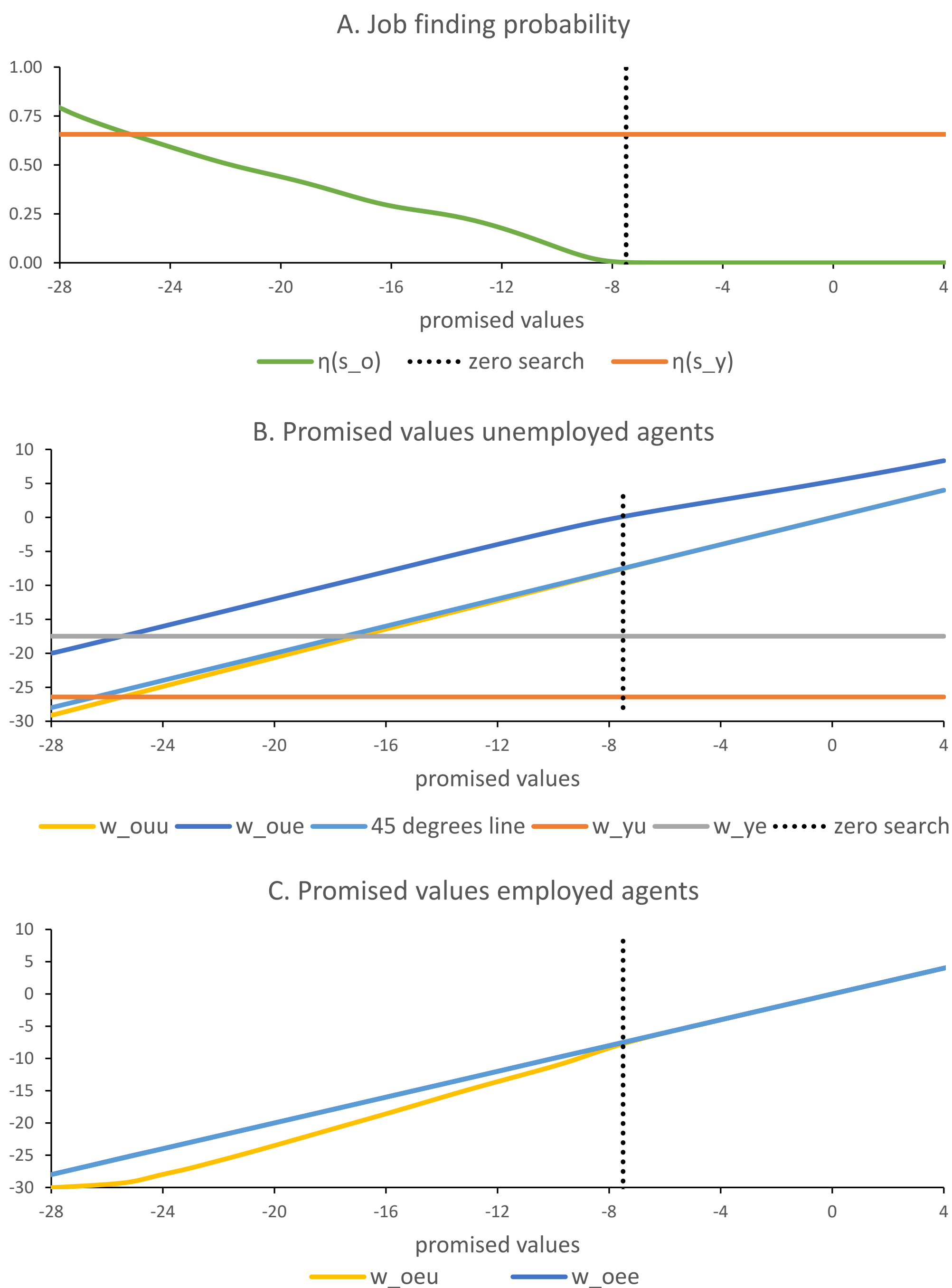

Figure 8: Hopenhayn-Nicolini economy - Consumption and cross-sectional distributions

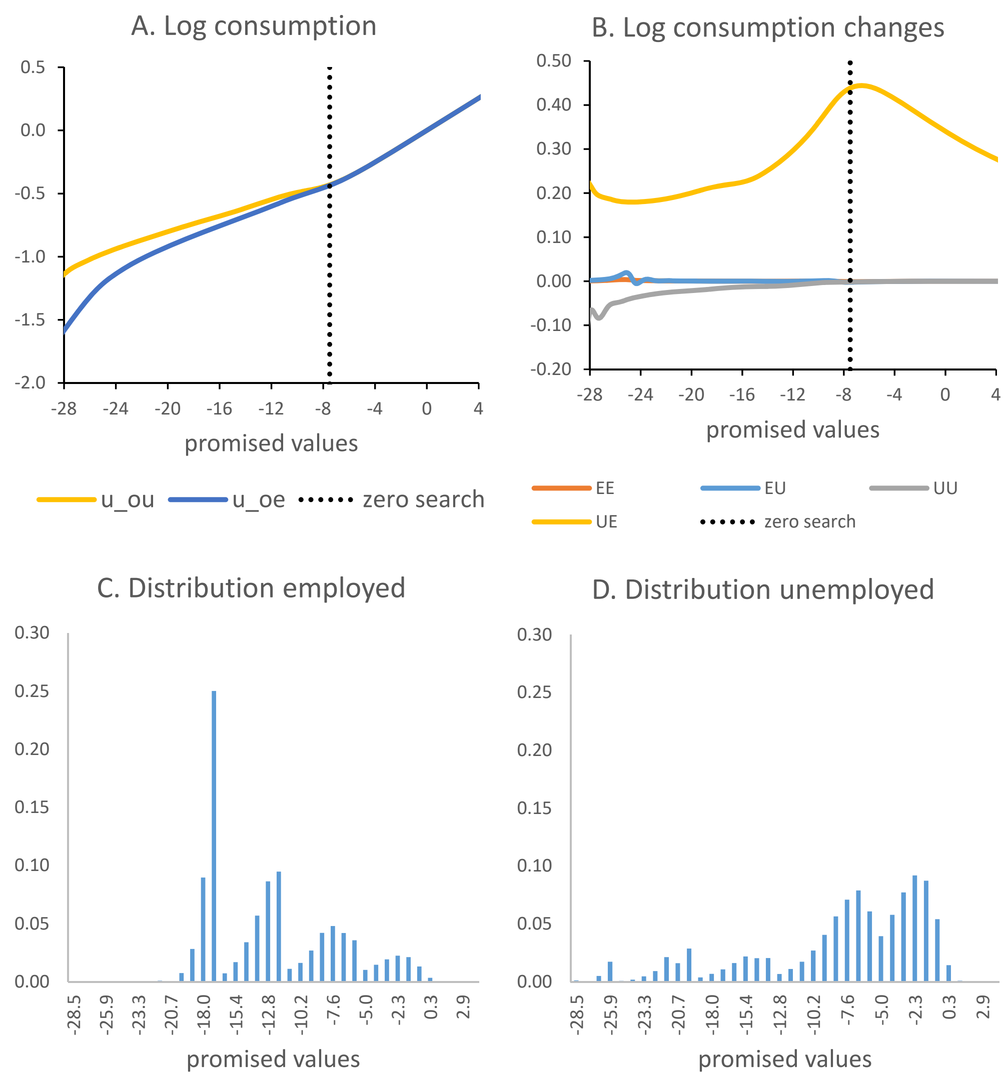

Figure 9: Hopenhayn-Nicolini economy - Impulse responses for macro variables and inequality