Dynamic Asset-Backed Security Design

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Abstract

We study a dynamic problem of the design and sale of securities backed by a long-lived collateral asset. Issuers are privately informed about the quality of the asset, and raise capital by securitizing it to fund a productive technology. Issuers can pledge not only the current period payoff from the assets, but also the future resale price. There is a dynamic feedback loop between the future asset price and today's issuers' decision where both adverse selection and the productivity level determine the liquidity of the securities. Multiple dynamic - liquid and illiquid - equilibria might arise when only equity contracts can be issued. We characterize the optimal security design and demonstrate that it involves short-term liquid collateralized debt, or short-term repo. It eliminates the multiple equilibria fragility and improves social welfare relative to the illiquid equity equilibrium. When repo contract is not flexible, repo runs might occur. Comparative statics generate rich dynamic properties of haircuts and interest rates for repos and repo runs in relation to productivity, adverse selection and contract riskiness.

Keyword: Liquidity; Security Design; Financial Fragility; Repo; Haircut; Repo Runs; Portfolio Repo

JEL classification: G10, G01

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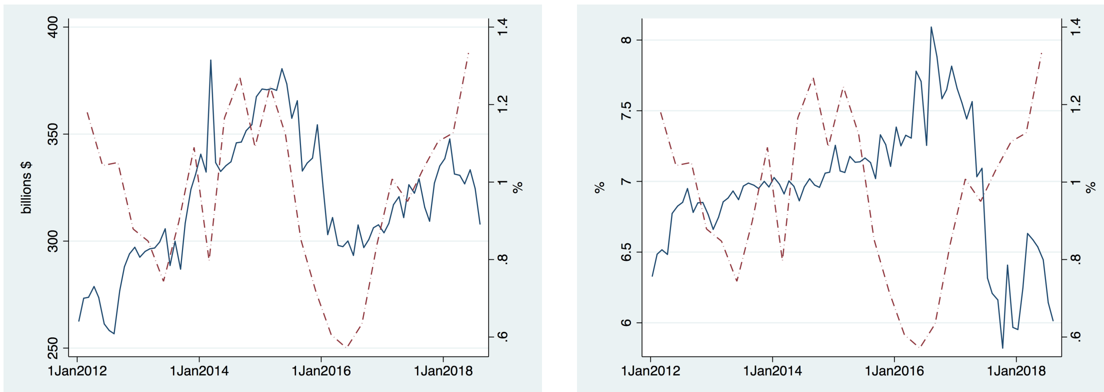
1 Introduction

In this paper, we study provision of liquidity to productive borrowers who face pledgeability constraints and adverse selection frictions when obtaining funding. Borrowers can ameliorate the impact of these constraints and frictions by borrowing against a long-lived (collateral) asset. One contribution of the paper is to provide a general theoretical framework to model the endogenous formation of funding liquidity in a dynamic environment. Built upon this general framework, we further analyze the consequences of dynamic security design for improving liquidity and social welfare.

Understanding the properties and the fragility of asset-backed liquidity is of systemic importance to the global economy. Since early 2000, we have observed the meteoric rise of asset price associated with productivity boom and its subsequent collapse. Asset-backed liquidity also experienced a similar pattern: there is a proliferation of short-term asset-backed borrowing facilities including repos and asset-backed commercial papers (ABCPs) before the Great Recession, followed by subsequent runs on these facilities leading to the ultimate demise of the ABCP market. Even today, asset-backed liquidity including repo financing remains a crucial source of short-term funding for financial institutions. Globally the repo transaction has an outstanding notional amount of $12 trillion around mid-2016 (CGFS, 2017). A survey of major European financial institutions finds that Euro repo market activities grow 19.4% from 2016 to 2017 (ICMA, 2018). If the repo market were ever under stress again, it would pose a great threat to the stability of international financial system.

Besides the fragility, asset-backed markets have many puzzling features. For example, a casual investigation of the recent data on US tri-party repo activities shows that the US GDP growth rate is related to repo transaction volume and haircut in different ways. Graphs in Figure 1 compare the US GDP growth rate with volume and value-weighted haircut for tri-party repo transactions backed by illiquid collaterals (that is, collaterals other than treasury and agency bonds) since 2012. We observe that since 2016, repo transaction volume is increasing while repo haircut is decreasing in the GDP growth. Both point out that repo liquidity is increasing. However, during the previous GDP growth spur between 2014 and 2016, repo transaction volume is up but repo haircut is not lowered. These casual observations indicate that real economy growth might have a complex relationship with repo liquidity. Indeed, with productivity and adverse selection as key ingredients, our dynamic model shows that the amount of repo transaction is increasing in productivity and decreasing in adverse selection while haircut is increasing in both productivity and adverse selection. Therefore, although adverse selection is lower during the growth peak, the rapid increase in productivity might contribute to the increase in haircut from 2014 to 2016, according to our theory.



(a) Repo Transaction Volume

(b) Value-Weighted Repo Haircuts

Figure 1: US GDP Growth Rate and Repo: Solids lines are US GDP growth rate (3-month rolling average). Dash-dotted lines are tri-party repo transaction volume and value-weighted haircut (3-month rolling average) for collaterals other than treasury and agency bonds respectively.

In our model, entrepreneurs who have access to a productive technology demand funding liquidity. These potential borrowers face typical frictions in the funding market. They may not be able to pledge the future cashflow due to either non-verifiability, un-observability, lack of commitment, or any other reasons. In this case, pledging collateral assets helps to obtain the liquidity necessary for the production.[1] However, the quality of collateral assets is often subject to adverse selection due to the incentive of borrowers to tamper the collateral quality for more funds. For example, historically, borrowers have incentive to debase metallic coins, ie., reduce the metallic content of the coins below the coin's face value, or use counterfeited coins to obtain more funding. In the recent time, collateral quality is subject to questioning because of the possibility that borrowers might pledge it multiple times.

More specifically, in our model, to take advantage of the productive technology, entrepreneurs issue securities that are backed by the cashflow from a long-lived collateral asset. By selling these asset-backed securities, investors raise liquidity to purchase the necessary inputs for production. We consider a dynamic setting where the quality of the collateral asset (captured by the distribution of its dividend payoff) varies period by period. Collateral is of either high or low quality, where the dividend distribution of the high quality collateral first-order stochastically dominates that of the low quality.[2] The borrowers

[1]Examples of collateral assets include coins made of precious metals, tangible assets such as real estate, silk garments, cattle, productive equipment, and the financial assets such as treasury securities in the recent times.

[2]To focus on the role of the asset as collateral, we assume that the asset is not an input in the production process and the dividend process is exogenous.

are privately informed about the current period quality at the beginning of each period. That is, there is adverse selection about the quality of the collateral asset between borrowers and lenders at the beginning of each period before any borrowing and production takes place.

We then explore the implications of optimal security design for liquidity provision in this dynamic environment with adverse selection. We begin with a benchmark case where borrowers are restricted to selling asset-backed *equity* for liquidity.[3] Given this limitation, the economy might exhibit fragility in terms of possible multiple (dynamic stationary) equilibria for the same underlying technology and beliefs. The logic behind the multiple equilibria in the benchmark case is based on a dynamic feedback loop between the future resale price and current borrowers' action: a high (low) anticipated future resale price for the collateral asset allows borrowers to exchange the asset-backed equity claims for more (less) capital in the current period to engage in productive technology. Borrowers with low-quality assets always sell equity claim backed by the low-quality asset to raise funding today. However, borrowers with high quality collateral are attracted to sell equity claims in exchange for today's funding for production only if productivity and resale price are high enough. When borrowers with high quality collateral pool with the low quality ones, the average quality of the security pool is higher, lowering the adverse selection endogenously, and therefore, justifying the high future asset price. The opposite is true for a low (anticipated) future resale price. The asset prices are self-fulfilling in this dynamic environment.

The dynamic feedback loop leads to three possible equilibrium regions in this economy. There is a 'separating' region where productivity is low and adverse selection is severe. In this region, high-quality borrowers choose to retain their asset-backed equity claims. Since only low quality borrowers are selling equity claims and engaging in production, the equity price today is indeed low, the economic output is limited, and the asset resale price in turn is depressed. There is also a 'pooling' region where productivity is high and adverse selection is mild. In this region, both types borrow against their equity claims to employ the productive technology, the equity price is high, the output is large and the asset price is, in turn, booming. For the intermediate values of productivity and adverse selection, there are multiple equilibria where both separating and pooling equilibria coexist.

Interestingly, asset price in this economy is more than the sum of the discounted future dividends because the collateral asset commands a liquidity price premium. The borrowers pledge the asset's dividend flows and resale price to overcome pledgability constraints and raise funding for production. The liquidity premium reflects a technology multiplier since the funds are more valuable when the

[3] The benchmark case can be relevant for under-developed financial markets where sophisticated securities are not available and borrowers can only pledge the entire cash flow from the collateral asset.

technology is more productive. This connection between productivity and liquidity premium might seem *counter-intuitive* because the long-lived asset is not a direct input in the production technology per se and serves only as a collateral to obtain funding liquidity. This theoretical finding about the technology multiplier might speak to the meteoric rise of asset price during the productivity boom we observed during the mid 2000s.

Next we turn to optimal security design. It is well understood in the literature that in a static economy optimal security design improves liquidity. In a dynamic economy, we demonstrate that optimal security design also eliminates the multiple equilibria fragility. To our knowledge this new role for security design has not been discovered till now. To state this result more explicitly, let us first describe our notion of liquid vs. illiquid security. We call a security liquid if both borrower types sells it. A liquid security commands a higher price so more funding can be raised by borrowers to scale up production. We call a security illiquid if only the low type sells it. An illiquid security has a lower price so less funding can be raised by low type borrowers to scale up production.

Our main result on optimal security design shows that there is a unique stationary dynamic security design equilibrium where the optimal design involves a *short-term liquid collateralized* debt tranche, ie., short-term repo, and the residual *illiquid* equity tranche.[4]

In the optimal security design, the issuer chooses the face value of the repo debt as large as possible in order to raise the maximum amount of liquidity. As the face value increases, the repo debt tranche incorporates more of the high dividend states. If the face value is too high, the high quality borrowers, who know these states are likely, might prefer to retain the debt tranche rather than pooling with the low quality borrowers to get a discount price for these states. Hence the security design pushes the face value of the repo debt up to the point where the high quality borrowers are indifferent between selling versus retaining the asset-backed debt tranche.[5] A key point is that the repo tranche always incorporates the resale price of the collateral. As the collateral price increases, selling the repo tranche becomes more attractive to the high quality type, allowing the security designer to increase the face value of the repo.

The dynamic security design equilibrium Pareto dominates the separating equilibria in the equity-only benchmark case and selects the pooling equilibrium in the multiple equilibria range. To see why, suppose a repo debt backed only by the future resale price is introduced. Since this debt is free of adverse selection problem, both borrower types will issue it to take advantage of the productive technology. Since

[4]In the model, dividend is independently distributed over time, so that the adverse selection problem only exists at the beginning of each period.

[5]Selling the debt tranche generates value through the technology multiplier. However, selling it is less attractive for the high type as he must pool with the low type and accept a lower price.

the technology multiplier is engaged in valuing the liquid debt, the collateral asset price rises. The higher asset price will allow the borrowers to increase the face value of repo debt by incorporating some of the high dividend states. The face value will increase until the high quality borrowers become indifferent between selling the liquid repo debt versus retaining it. In the separating equilibrium region of the equity-only benchmark case, this process leads to a liquid repo debt tranche that is traded by both types and improves the welfare of the borrowers. In the multiple equilibria region it selects the pooling equilibrium – that is, issuers sell the entire equity-like "pass-through" debt. In this unique security design equilibrium, both liquidity and production output are higher than the eliminated separating equilibrium.

The optimal security design in our model speaks to the empirical observation that most of the repo funding extended by money market mutual funds and security lenders is collateralized with Treasury or Agency-backed securities (Krishnamurthy, Nagel, and Orlov, 2014). Our setup on repo contracts backed by common collaterals captures that fact. The terms of the repo contract such as interest rates and haircut as well as the collateral asset price are endogenously determined in our model. Comparative analysis on these equilibrium variables allow us to generate some unique predictions and testable implications. For example, our model predicts that the magnitude of repo haircut has two components: the productivity of the borrower's technology, and the value of the equity tranche relative to the value of the collateral. The first component arises because borrowers, who price the collateral asset, value the liquidity service the asset provides, while lenders, who price the loan, does not value the liquidity service. It reflects heterogeneous valuation over the collateral assets among agents. This component relates to the difference in opinion literature on leverage starting with Geanakoplos and Zame (2002), Geanakoplos (2003), Fostel and Geanakoplos (2012), and Simsek (2013). The second component arises because of information friction (and/or adverse selection). This component has been emphasized by Dang, Gorton, and Holmström (2011) and Gorton and Ordonez (2014). Interestingly, the repo rate in our model is free of the adverse selection risk since repo debt is liquid and both high and low quality borrowers participate in this market. Nevertheless, repo debt is risky and repo rate is determined by the default risk of the repo contacts (which is related to the face value of the repo contract) and the demand for funding liquidity (which is related to the productivity). Our model also generates predictions on commonly used portfolio repos, which are repo contracts backed by a portfolio of collateral assets. It predicts that when the fraction of safe asset in the collateral pool increases, repo contract terms improve since the level of adverse selection is lowered. Not only our theory offers a new perspective on how adverse selection affects of the terms of short-term repo contracts, it also has implications on how asset-backed liquidity variables such as the volume, asset prices, asset price liquidity premium, haircuts/leverage, interest rates, liquidity premium, vary over productivity boom-bust cycle. These implications are based on two examples of the theory: one is to model the asset quality, ie., dividend payoff using a two-point distribution (high probability payoff for high quality borrowers and vice versa) where we are able to derive a closed form solution; the other is to model the asset quality and the productivity as Markov processes where numerical results are derived. For example, we find that when adverse selection is severe, repo haircut is more sensitive to asset quality than repo rate, generating a testable hypothesis. A numerical example shows an amplification of productivity shocks: a percentage increase in productivity leads the asset price to increase by more than 10 percentage points, generating a finding worthy of a further calibration exercise fitting the macro data.

In an extension of the paper, we demonstrate how repo runs might occur in this economy. This hinges on whether borrowers have the flexibility to adjust the security design at the beginning of each period. In the main part of the paper, we show that there is a unique equilibrium when borrowers has this flexibility. This implies that the over-night repo market could be robust to run. In practice, repo contract terms may not be updated daily because of associated administrative costs or simply inattention. When there is rigidity of repo contract, that is, the face value of the repo contract does not get updated with a certain probability at the beginning of each period, a run equilibrium might emerge and the liquidity of the repo market may deteriorate. Repo runs in our setup are dynamic feedback runs and hence distinct from bank runs as in Diamond and Dybvig (1983), the type of repo runs due to repo market microstructure features, liquidity need of the lenders as well as the capital position of the borrowers as in Martin, Skeie, and Von Thadden (2014) or the collateral crisis due to the endogenous information production studied in Gorton and Ordonez (2014). Repo runs in our model might unfold slowly and are marked by two stages. The initial stage is when the liquid security design equilibrium switches to the sunspot equilibrium where the pessimistic belief of an upcoming sunspot (associated with worsening fundamentals) is triggered. The second stage is when a sunspot actually hits the economy. When the economy enters a sunspot equilibrium, haircut of the repo contract immediately increases, because investors anticipates that the repo contract may be illiquid when a sunspot hits the economy. At the same time, the asset price and the repo volume are also lowered. When the sunspot hits, they decrease further, while the haircut increases further. The drop in repo liquidity is severe when the sunspot hits, because the repo backed by high quality collateral stops circulating entirely. Our model of runs also maps out the recovery process of asset-backed liquidity. When the contract terms are updated, the update restores investors' sentiment about the liquidity of this market, the price and the volume recover partially, to the levels right after equilibrium switch. The fluctuation driven by sunspots may take place repeatedly, until the sunspot equilibrium cannot be sustained as an equilibrium. It is possible that the asset-backed liquidity evaporates as in the ABCP market if adverse selection becomes so severe that only low quality borrowers are in the market.

Related literature. The seminal work of Akerlof (1970) started the literature on lemons market to study the impact of adverse selection on trade volume and efficiency. There is a long lineage of security design literature including Leland and Pyle (1977); Myers and Majluf (1984); DeMarzo and Duffie (1995); and DeMarzo and Duffie (1999) that examine informed sellers' incentive to issue optimal security to signal asset quality. For example, in DeMarzo and Duffie (1999), a closely related paper, retaining equity is a signal for quality. By comparison, ours is a competitive screen model and securities are designed to screen issuer types. Therefore implications are different. In ours, both borrower types issue debt while only the poor quality type issues equity. Moreover, extending the static setup to a dynamic environment allows us to discover that security design helps to mitigate adverse selection problem not only by increasing the amount of liquidity but also by eliminating fragility.

Our result that both borrower types issue debt and debt is liquid is reminiscent of the finding in Gorton and Pennacchi (1990)where they find that low-information-intensity (debt-like) securities protect sellers from the risk of selling only high-quality assets when trading with an informed buyer. Boot and Thakor (1993) also find that the optimal security design is implementable by a liquidity debt contract and an equity contract, and others. However, the motivation is to stimulate information production using information sensitive securities. This literature has now progressed to incorporate endogenous asymmetric information in optimal security design problem such as Yang (Forthcoming); Dang, Gorton, and Holmström (2013); and Farhi and Tirole (2015). That information friction affects moneyness of an asset has also been studied by Lester, Postlewaite, and Wright (2012) and Li, Rocheteau, and Weill (2012).

There has also emerged a literature on heterogeneous information and security design such as Ellis, Piccione, and Zhang (2017). Under diverse beliefs, however, there is no fragility under dynamic environment. There will be speculative premium under diverse beliefs but it is difficult to investigate financial fragility unless exogenous changes in beliefs are introduced. With adverse selection as in our model, the changes in market liquidity or "beliefs" can be endogenous.

By studying optimal collateral-backed security design and funding liquidity, our paper is also related to a long line of collateral literature in money and macroeconomics starting with the seminar work of Kiyotaki and Moore (1997) and recent studies on the prevalence of the use of repo contracts in funding financial institutions such as Geanakoplos and Zame (2002), Geanakoplos (2003), Fostel and Geanakoplos

(2012), Simsek (2013), and Gottardi, Maurin, and Monnet (2017). Increasingly attempts are made to incorporate financial frictions in the macroeconomic models or studying macroeconomic implication of financial friction such as collaterals to understand the boom and bust cycles. The recent papers include but are not limited to Gorton and Ordonez (2014); Kuong (2017); Parlatore (Forthcoming); and Miao and Wang (2018). Kurlat (2013) and Bigio (2015) study financial frictions that arise endogenously from adverse selection in a dynamic production economy.

Our paper is also closely to Plantin (2009), Chiu and Koeppl (2016), Donaldson and Piacentino (2017); and Asriyan, Fuchs, and Green (2017), where multiple equilibria is dynamic in nature. Although Asriyan, Fuchs, and Green (2017) focuses on sentiment-driven multiple equilibria and differs from ours in setup and implication, the insight that asset price and liquid is closely linked is very close to ours. Our insight on the implication of security design on the dynamic fragility is important contribution to this literature.

2 The Model Setup

In this economy, there are two types of agents. One type has access to a technology to produce an intermediate good. This technology is constant-returns-to-scale and allows the agent to produce one unit of the intermediate good from one unit of labor. However, the intermediate good does not provide direct utility. The other type possesses a "productive" technology that produces a consumption good using the intermediate good through a constant returns-to-scale technology. This technology is productive because an input of one unit of intermediate good generates $z > 1$ units of consumption good, and we term it the z-technology. We call the agent who has the ability to produce the intermediate good the type I agent and the agent who possesses the z-technology the type O agent.[6] In addition, both types of agents have access to a "basic" technology to produce consumption goods. This basic technology produces one unit of consumption good using one unit of labor.[7]

The assumption on the technologies in the economy is made to capture the gain from trade between the two types of agents. Intermediate goods can be interpreted as any inputs to the z-technology such as capital, equipments, or intermediate products. The O type agents (eg., entrepreneurs) would like to borrow as much intermediate goods as possible from the I type agents (eg., investors or suppliers) to engage in the productive z-technology. However, agent O's promise to pay back is not enforceable. This

[6]Here, O stands for owner since, as we show later, agent O will own a productive asset and I stands for investor since agent I will invest in asset based securities.

[7]The framework of dynamic analysis is borrowed from Lagos and Wright (2005).

is one of the key frictions that the dynamic economic mechanisms in our model are aimed to overcome.

Timing. The economy is set in discrete time and lasts forever. Each period has three dates. At date 1, the intermediate good is produced by agent I. At date 2, consumption good is produced via the z-technology using the intermediate good and/or the basic technology using labor. At date 3, consumption takes place. Any leftover intermediate or consumption good perishes at the end of the period.

Utilities and discounting. An agent's utility in period t is given by $U_t(x, l) = x - l$ where x is the amount of consumption good consumed and l is the amount of labor supplied by the agent. There is no discounting between sub-periods. Agents discount periods at a rate β, with $0 < \beta < 1$.

Productive Asset and Asymmetric Information. There is a productive asset in the economy, which pays s units of dividend in terms of consumption good at date 3. The total supply of the asset is A. With probability λ, the dividend of the asset follows distribution $F_L \in \Delta[s_L, s_H]$, with $0 \leq s_L < s_H$. With probability $1 - \lambda$, it follows distribution $F_H \in \Delta[s_L, s_H]$. We assume that F_H first order stochastically dominates F_L. The quality, denoted by $Q \in \{H, L\}$, represented by λ ($Q = L$ with probability λ), is i.i.d. over time. More generally, asset quality could be persistent over time. We will consider that case in later sections.

As mentioned earlier, in a frictionless environment, since the returns to scale of the z-technology are $z > 1$, agent O would like to borrow unlimited amount of intermediate goods from the I agents at the beginning of the period (i.e., date 1) in order to produce unlimited amount of consumption goods at date 2. However, agent O's promise to pay back is not enforceable. The productive asset provides liquidity because it can be used as collateral to back up agent O 's promise to pay back. If agent O owns the asset, she can borrow intermediate goods from the I agents at date 1 using both the dividend and the resale value of the asset at date 3 as collateral. This is possible because if agent O does not fulfill her promise, I agents can seize the collateral asset.

However, the use of this collateral asset for liquidity service is limited by an additional friction in our economy, which is asymmetric information. We assume that the quality of the collateral asset is privately observed by agent O at the beginning of the period (i.e, at date 1 of each period). This introduces an adverse selection problem which plays a key role in our analysis. The assumption that agent O is better informed of the collateral asset's quality can be motivated or micro-founded in several ways. As demonstrated later, agent O would purchase all collateral assets in equilibrium because her need of liquidity to kick start the z-technology. Consequently, she has a stronger incentive to acquire information on the collateral asset. Empirically, one can also motivate the superior information advantage of the asset owner by the fact that the asset quality can be easily tempered with by the owners. Historically, when both silver and gold were used as collaterals, borrowers have incentive to debase these collaterals or use counterfeits, various technologies (eg., special weighting) were adopted to assess the purity of these collaterals. In the modern environment, the possibility of pledging collateral multiple times creates similar adverse selection problems.

In this asymmetric information environment, the asset provides only limited amount of liquidity since the amount that agent O can borrow is bounded by the expected dividend and the resale value of the asset. As we will show in our baseline case, when agent O has superior information about the asset quality, resulting adverse selection tightens the liquidity constraint. In this case, agents' expectations about the asset price can make the adverse selection problem more or less severe, leading to multiple equilibria.

In this environment with adverse selection, agent O can improve liquidity available at the beginning of the each period by optimally designing securities which are used to exchange for the intermediate goods at date 1 and deliver consumption good payments at the end of each period. A security, hence is a state-contingent promise at date 1 of consumption good payment at date 3. Denote the payoff from security j at state s to be $y^j(s)$. Because agent O cannot commit to pay, the security must be backed by the dividend and the ex-dividend price of the asset, denoted by ϕ_t. The set of all feasible asset backed securities at time t for a given price ϕ_t is $\mathcal{I}_t(\phi_t) \subseteq \{y : y(s) \leq s + \phi_t, \forall s \in [s_L, s_H]\}$. The set $\mathcal{I}_t(\phi_t)$ captures any potential exogenous restrictions on the set of feasible securities. One possible set, $\mathcal{I}_t(\phi_t) = \{y : y(s) = s + \phi_t, \forall s \in [s_L, s_H]\}$, consists of only a single "pass-through" security which promises the dividend and resale value of the collateral asset. A second possibility, $\mathcal{I}_t(\phi_t) = \{y : y(s) \text{ increasing in } s, y(s) \leq s + \phi_t, \forall s \in [s_L, s_H]\}$, is the set of all monotone securities backed by the collateral asset. The monotonicity restriction is motivated by realism since the payoff from any loan collateralized directly by the asset or any other collateralized loan is increasing in s.

A security design is a finite selection of securities that are backed by the asset.

Definition 1. Given the asset price ϕ_t, a *security design* consists of a finite set of securities $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$.

When $j \in \mathcal{J}_t(\phi_t)$, we say that security j is available.

Trading environment. There are two types of markets in this economy. After state is realized at the end of each period t (ie., at date 3), a *centralized* market for the collateral asset opens for trading. The asset price, denoted by ϕ_t, is determined in this centralized market.

In addition, at the beginning of each period t (ie., at date 1), there are *decentralized* markets for intermediate goods. Specifically, for each available security, there is a decentralized sub-market where agent O meets at least two randomly chosen I agents to trade asset-based securities in exchange for intermediate goods. We assume that agent Is simultaneously make price offers per unit of the security. Agent O then observes the price offers and decides the quantity of the security to allocate to each agent I. Since each unit of the security must be backed by one unit of the asset, the total quantity of the security sold by agent O must be less than or equal to the amount of asset owned by agent O in that period. If agent O decides to sell a positive amount of the security, she allocates the amount of the security that she would like to sell to the agent I who offers the higher price. If several agent Is are tied for the highest offer, agent O equally splits the amount that she would like to sell between them.

The following figure summarizes the time and events in this setup.

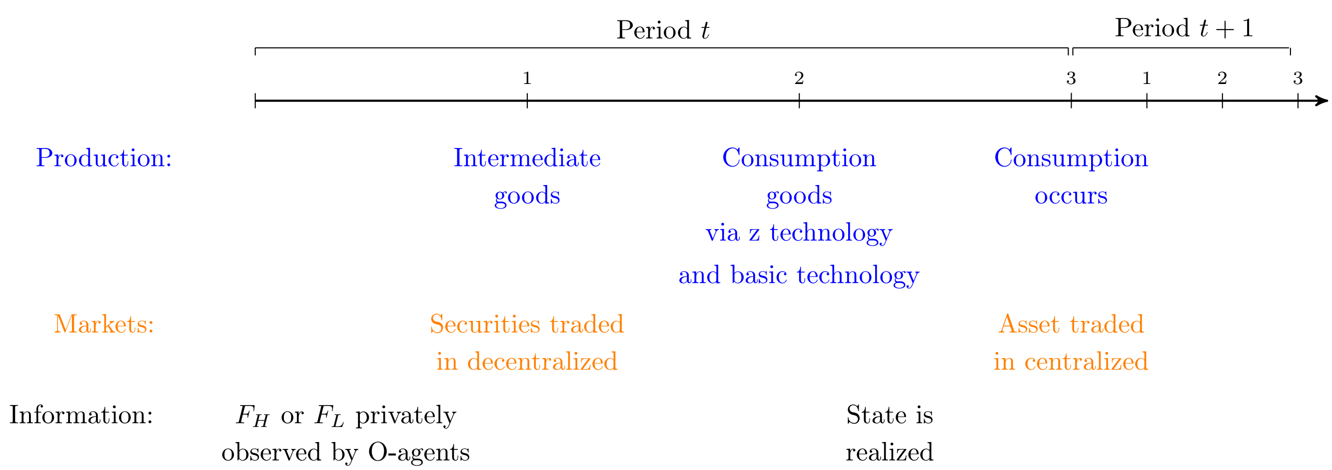


Figure 2: Timeline

3 Security Design Problem

3.1 Defining the security design problem

A few notations are in order before the definition. We denote the value function of agent O at date 1 of period t by $V_{o,t}(a, \mu_{o,t})$ and her value function at date 3 of period t by $W^s_{o,t}(c, a)$ at state s, where c is the amount of consumption goods, a is the amount of the asset that she brings into period t, and $\mu_{o,t}$ indicates the set of available securities. Similarly, we denote the value function of agent I in submarket for asset j by $V^j_{I,t}(a)$ and his value function in the last sub-period of period t to be $W^s_{I,t}(c, a)$. The security design problem is defined based on the characterization of these value functions.

Agent O's value function at date 3 of period t is solved by

$$W_{o,t}^{s}(c,a) = \max_{x,l,\tilde{a}\geq 0} x - l + \beta V_{o,t+1}(\tilde{a}), \tag{1}$$
$$s.t. x + \phi_t \tilde{a} = c + (s+\phi_t)\, a + l.$$

Likewise, Agent I's value function at date 3 of period t is solved by

$$W_{I,t}^{s}(c,a) = \max_{x,l,\tilde{a}\geq 0} x - l + \beta V_{I,t+1}(\tilde{a}), \tag{2}$$
$$s.t. x + \phi_t \tilde{a} = c + (s+\phi_t) a + l.$$

From (1) and (2), it is easy to see that $W_{o,t}^{s}$ and $W_{i,t}^{s}$ are linear in c and a since

$$W_{o,t}^{s}(c,a) = c + (s+\phi_t)\, a + W_{o,t}^{s}(0,0), \tag{3}$$
$$W_{I,t}^{s}(c,a) = c + (s+\phi_t)\, a + W_{i,t}^{s}(0,0). \tag{4}$$

Next, we characterize $V_{I,t}^{j}(a)$. We denote by $y_t^j(s)$ the payoff of asset j in state s and assume that that the high type values the security weakly more than the low type, i.e., $E_L y_t^j(s) \leq E_H y_t^j(s)$.[8] We denote by R_t^j the ratio of the expected value of the security under the low versus the high distribution, i.e., $R_t^j \equiv E_L y_t^j / E_H y_t^j$ As this ratio increases, the expected values of the asset under the low versus the high distribution become closer, and hence the adverse selection problem becomes less severe.

Recall that agent Is simultaneously make price offers per unit of the security, agent O observes the price offers and decides how much of the security to allocate to each agent I.[9] Hence, in principle, the value function depends on the offer that agent I makes in sub-market j.

Consider an arbitrary agent I who participates in submarket j and bids for security y_t^j at per-unit price q_t^j. If this is the highest bid, he receives $a_t^Q(q_t^j) \in [0,a]$ units of security j and pays $q_t^j a_t^Q(q_t^j)$ units of intermediate goods in return. Agent I's expected payoff from this bid is given by:

$$\begin{aligned} V_{I,t}^{j}(a) &= \int \lambda \left[-q_t^j a_t^L(q_t^j) + W_{I,t}^{s}(a_t^L(q_t^j) y_t^j(s), a)\right] dF_L(s) \\ &\quad + \int (1-\lambda)\left[-q_t^j a_t^H(q_t^j) + W_{I,t}^{s}(a_t^H(q_t^j) y_t^j(s), a)\right] dF_H(s) \\ &= \int \lambda \left[-q_t^j a_t^L(q_t^j) + a_t^L(q_t^j) y_t^j(s)\right] dF_L(s) + \int (1-\lambda)\left[-q_t^j a_t^H(q_t^j) + a_t^H(q_t^j) y_t^j(s)\right] dF_H(s) \qquad (5) \\ &\quad + \int W_{i,t}^{s}(0,a) d\left[\lambda F_L(s) + (1-\lambda) F_H(s)\right] \end{aligned}$$

[8]This assumption is automatically satisfied for monotone securities.

[9]In this formulation agent O has all the bargaining power, but this is not crucial for any of our results.

where the second equality is obtained by substituting (4) using the fact that $W^s_{i,t}$ are linear in c and a.[10] If q^j_t is not the highest bid, this agent's expected payoff is given only by the third term of the second equality since he is not allocated any of the security in sub-market j.

The winning bid q^j_t must satisfy two conditions. First, due to Bertrand competition I agents make zero surplus in expectation. This means that q^j_t must equal the expected value of a unit of the security given the expectation of I agents about the quantities that will be sold by the two types. Second, these expectations must be incentive compatible in the sense that if I agents anticipate that a given type of the O agent will sell a positive amount of the security at per-unit price q^j_t, that type must find it profitable to sell the security given the price. The next proposition shows that among the prices that satisfy the zero surplus and incentive compatibility conditions, Bertrand competition selects the highest one.

Proposition 1. *If $R^j_t > \zeta \equiv 1-(z-1)/\lambda z$, in submarket j the price of the security is $q^j_t = \lambda E_L y^j_t + (1-\lambda)E_H y^j_t$ and $a^L_t(q^j_t) = a^H_t(q^j_t) = a$. If $R^j_t < \zeta$ then the price of the security is $q^j_t = E_L y^j_t$ and $a^L_t(q^j_t) = a$ and $a^H_t(q^j_t) = 0$.*[11]

Proof. Let $\overline{q}^j = \lambda E_L y^j_t + (1-\lambda)E_H y^j_t$. Note that $z\overline{q}^j - E_H y^j_t \gtreqless 0$ iff $R^j_t \gtreqless \zeta$.

Consider the case $R^j_t > \zeta$. Suppose that the equilibrium price q^j_t is strictly less than $\overline{q}$. In this case an I agent can deviate and bid $\overline{q} - \epsilon$ where $\epsilon > 0$. For ϵ small enough, $z(\overline{q} - \epsilon) - E_H y^j_t > 0$. Hence at this price both types sell a units of the security and the deviation generates strictly positive surplus. This means that the equilibrium price must be at least $\overline{q}$. At price $\overline{q}$ or above both types will sell a units of the security, hence the only price that is consistent with zero profit condition is $q^j_t = \overline{q}$.

Now consider the case $R^j_t < \zeta$. In this case high type will sell the security only if q^j_t is sufficiently larger than $\overline{q}$. However, at prices above $\overline{q}$, I agents make negative profit. Hence equilibrium price must be below $\overline{q}$. If q^j_t is below $\left(E_L y^j_t\right)/z$ then neither type sells the security. In this case, one of the I agents can deviate and bid $E_L y^j_t - \epsilon$ where $\epsilon > 0$. For ϵ small enough, $z\left(E_L y^j_t - \epsilon\right) - E_L y^j_t > 0$ so the low type sells the security and the deviating agent makes strictly positive surplus. If q^j_t is at least $\left(E_L y^j_t\right)/z$ but less than $E_L y^j_t$ then the low type sells the security to the I agents who bid that price. In this case, one of the I agents who bids $E_L y^j_t$ or less can deviate and bid slightly above q^j_t. This agent then buys the

[10] If q^j_t ties with $k-1$ other highest bids, his expected payoff is as above except that $a^Q_t(q^j_t)$ terms are now divided by k.

[11] When $R^j_t = \zeta$ there are multiple equilibria. In particular both pooling and separating (and even semi-separating) equilibria are possible. To simplify exposition in this knife edge case we will select the pooling equilibrium. To see why there are multiple equilibria, suppose I agents bid $E_L y^j_t$, the low type sells a units and I agents make zero profit. Since $R^j_t = \zeta$, to attract the high type, an I agent must deviate to bidding at least $\lambda E_L y^j_t + (1-\lambda)E_H y^j_t$. But this deviation is not profitable since by deviating an I agent can not make positive surplus. Hence both $E_L y^j_t$ and $\lambda E_L y^j_t + (1-\lambda)E_H y^j_t$ can be sustained as equilibrium bids.

security alone and increases her surplus. At prices greater than equal to $E_L y_t^j$ (and below $\overline{q}$), the low type alone sells a units of the security. Hence the only price that is consistent with zero profit condition is $q_t^j = E_L y_t^j$. □

This proposition shows that when R_t^j is above the threshold ζ, the adverse selection problem is not too severe and both types sell a units of the security. In this case the security price is the pooling price $q_t^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. When R_t^j is below the threshold, the adverse selection problem is too severe and only the low type sells a units of the security. In this case the security price is the separating price $q_t^j = E_L y_t^j$.

Now we are ready to state the optimal security design problem. Agent O chooses security design $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$ to maximize

$$\begin{aligned} V_{o,t}(a) &= \lambda \int W_{o,t}^s \left(\sum_{j \in \mathcal{J}_t(\phi_t)} a_t^L(j) \left[z q_t^j - y_t^j(s) \right], a \right) dF_L(s) \\ &+ (1-\lambda) \int W_{o,t}^s \left(\sum_{j \in \mathcal{J}_t(\phi_t)} a_t^H(j) \left[z q_t^j - y_t^j(s) \right], a \right) dF_H(s) \\ &= \lambda \int \left\{ \sum_{j \in \mathcal{J}_t(\phi_t)} a_t^L(j) \left[z q_t^j - y_t^j(s) \right] + a\,(s+\phi_t) \right\} dF_L(s) \\ &+ (1-\lambda) \int \left\{ \sum_{j \in \mathcal{J}_t(\phi_t)} a_t^H(j) \left[z q_t^j - y_t^j(s) \right] + a\,(s+\phi_t) \right\} dF_H(s) + W_{o,t}^s(0,0), \end{aligned} \tag{6}$$

subject to

$$\sum_{j \in \mathcal{J}_t(\phi_t)} y_t^j(s) \le s + \phi_t, \forall s, \tag{7}$$

$$q_t^j = \begin{cases} \lambda E_L y_t^j + (1-\lambda) E_H y_t^j, & \text{if } R_t^j \ge \zeta, \\ E_L y_t^j, & \text{if } R_t^j < \zeta, \end{cases} \tag{8}$$

$$a_t^L(j) = a \text{ and } a_t^H(j) = \begin{cases} a, & \text{if } R_t^j \ge \zeta, \\ 0, & \text{if } R_t^j < \zeta. \end{cases} \tag{9}$$

The security design is done ex-ante, before Agent O learns the asset quality. At the security design stage, Agent O simply decides which sub-markets are open for trading but she cannot commit to trading in a given sub-market. The first constraint ensures that the security design is feasible in the sense that Agent

O should be able to fulfill her promises in every sub-market and in all states. The second and third constraints say that prices and quantities in the decentralized security markets must be the equilibrium outcomes characterized in Proposition 1.

We now state the equilibrium definition for the dynamic security design problem.

Definition 2. A dynamic stationary equilibrium consists of asset prices ϕ_t, security design $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$ and security prices q_t^j for each $j \in \mathcal{J}_t(\phi_t)$ such that (i) $\mathcal{J}_t(\phi_t)$ solves the security design problem (6), (ii) security price q_t^j satisfies equation (8) and (iii) ϕ_t solves the Euler equation given by:

$$\phi_t = \beta \left[z \left(\sum_{j \in P_t} q_t^j + \lambda \sum_{j \in \mathcal{J}_t(\phi_t) \setminus P_t} q_t^j \right) + (1-\lambda) \sum_{j \in \mathcal{J}_t(\phi_t) \setminus P_t} E_H y_t^j \right], \tag{10}$$

where $j \in P_t \subseteq \mathcal{J}_t(\phi_t)$ iff $R_t^j \geq \zeta$.

4 The Baseline: Fragility of the Dynamic Lemons Market

In this section, we consider the benchmark case where Agent O secures liquidity only by selling Agent I the collateral asset at the beginning of each period in exchange for intermediate goods as inputs for the z-technology. We demonstrate that this economy is fragile and exhibits dynamic multiplicity in prices. That is, we show that for a given price path there might be multiple equilibria in the decentralized markets.

For this benchmark case we use the notion of equilibrium in Definition 2 except that we take the collateral asset as the only available security. That is we set $\mathcal{I}_t(\phi_t) = \{y : y(s) = s + \phi_t, \forall s \in [s_L, s_H]\}$. Hence the optimization problem in (6) becomes trivial since there is only a single feasible security which is the asset itself, but in equilibrium (8) and (10) must still be satisfied. The payoff of the collateral asset in state s is $s + \phi_t$. Hence, by (8) the price of the collateral asset in the decentralized market is given by $q_t^P = \phi_t + \lambda E_L s + (1-\lambda) E_H s$ if $(E_L s + \phi_t)/(E_H s + \phi_t) \geq \zeta$ and $q_t^S = \phi_t + E_L s$ otherwise. Using (10) we obtain the price of the collateral asset in the centralized market as

$$\phi_t = \begin{cases} \beta z q_{t+1}^P, & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} \geq \zeta, \\ \beta \left[z \lambda q_{t+1}^S + (1-\lambda)(\phi_{t+1} + E_H s) \right], & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} < \zeta. \end{cases} \tag{11}$$

Plugging q_t^P and q_t^S into (11) we observe that a pooling equilibrium, in which both types of agent O sell the asset in the decentralized market for the intermediate goods, exists if and only if

$$\frac{E_L s + \phi^P}{E_H s + \phi^P} \geq \zeta, \tag{12}$$

where asset price in the pooling equilibrium is given by,

$$\begin{aligned}\phi^P &= \beta z(\phi^P + \lambda E_L s + (1-\lambda)E_H s), \\ \phi^P &= \frac{\beta z\left(\lambda E_L s + (1-\lambda)E_H s\right)}{1-\beta z}.\end{aligned} \tag{13}$$

Similarly, a separating equilibrium in which only the low type of agent O sells the asset in the decentralized market for the intermediate goods, exists if and only if

$$\frac{E_L s + \phi^S}{E_H s + \phi^S} < \zeta, \tag{14}$$

where asset price in the separating equilibrium is given by,

$$\begin{aligned}\phi^S &= \beta\left[\lambda z\left(\phi^S + E_L s\right) + (1-\lambda)\left(\phi^S + E_H s\right)\right], \\ \phi^S &= \frac{\beta\left[\lambda z E_L s + (1-\lambda)E_H s\right]}{1-\beta(\lambda z + 1 - \lambda)}.\end{aligned} \tag{15}$$

Note that the pooling price is always higher than the separating price:

$$\phi^P = \frac{\beta z\left[\lambda E_L s + (1-\lambda)E_H s\right]}{1-\beta z} > \phi^S = \frac{\beta\left[\lambda z E_L s + (1-\lambda)E_H s\right]}{1-\beta(\lambda z + 1 - \lambda)}.$$

Furthermore, the discounted value of future dividends is $\beta\left(\lambda E_L s + (1-\lambda)E_H s\right)/(1-\beta)$. It is easy to see that, since $z > 1$, the price for the asset is strictly higher than the discounted value of future dividends in both scenarios. The difference is justified by the collateral service provided by the asset. The pooling price is higher because the collateral service is more valuable in the pooling equilibrium of the decentralized market as both types use the collateral to purchase the intermediate goods. Moreover, when z is higher, there is more demand for collateral which justifies a higher asset price.

Since the pooling price is higher than the separating price, for the same underlying parameters, there maybe multiple price equilibria. That is, the separating price ϕ^S can be consistent with a separating equilibrium and the pooling price ϕ^P can be consistent with a pooling equilibrium in the decentralized market.

Corollary 1. *The condition for price multiplicity when agent O uses the productive assets as collateral for intermediate goods is*

$$1 - \frac{z-1}{z}\frac{1}{\lambda(1-\beta)} < \frac{E_L s}{E_H s} < 1 - \frac{z-1}{z}\frac{1}{\lambda\left(1-\beta+\beta(1-\lambda)(z-1)\right)}. \tag{16}$$

Proof. By Proposition 1 the condition for price multiple equilibria is,

$$\frac{E_L s + \phi^S}{E_H s + \phi^S} < \zeta \leq \frac{E_L s + \phi^P}{E_H s + \phi^P}.$$

Plugging for ϕ_S and ϕ_P we obtain the condition for multiplicity in 16. Hence for intermediate values of $E_L s / E_H s$ both price equilibria exist. $\square$

The existence of multiple price equilibria is due to a dynamic feedback loop. If agents anticipate the asset to be traded in a pooling equilibrium in the decentralized market, the asset price is high. In turn, when the price is high, the high type O agent is willing to pool. Conversely, if agents anticipate the asset to be traded in a separating equilibrium in the decentralized market, the asset price is low. In turn, when the price is low, the high type O agent keeps the asset. The beliefs are self-fulfilling. We show next security design helps to eliminate this type of fragility in the economy.

5 Dynamic Security Design with Monotone Securities

In this section we restriction attention to monotone securities and solve the security security design problem given in ((6)). We show that agent O can use security design to overcome the fragility of the price equilibrium that arises when agents can only trade the underlying asset.

5.1 Solving for Optimal Security Design

As a preliminary step, we first show that optimal security design involves at most two securities. One security is always liquid and the other one is illiquid.

Lemma 1. *If two securities y^j and y^k are both liquid (illiquid) then $y^j + y^k$ is also liquid (illiquid). Moreover, if a security design involves y^j and y^k, replacing the two securities by their combination $y^j + y^k$ is also a feasible security design and provides the same payoff to agent O. Hence, w.l.o.g. we can restrict attention to security design that involves at most two securities, a liquid and an illiquid one.*

According to 1, we focus on security design with at most one liquid and one illiquid tranches. Note that when a liquid and an illiquid security are combined the resulting security might be liquid (illiquid) and strictly improve (lower) agent O's payoff. In fact, whenever there are two tranches and the liquidity constraint is not binding for the liquid tranche, that is if $E_L y^j > \zeta E_H y^j$, it is always possible to combine part of the illiquid tranche with the liquid one and strictly improve agent O's payoff. We state this observation in the next lemma.

Lemma 2. *If a security design is optimal and involves a liquid and an illiquid tranche, the liquid tranche must satisfy the liquidity constraint with equality.*

Proof. From Lemma 1 we can restrict attention to security design with two tranches. Suppose a given security design involves two tranches where y^j is liquid which satisfies liquidity constraint strictly and y^k is illiquid. Now take $y^j + \epsilon y^k$ and $(1-\epsilon)\, y^k$. This is a feasible design. If ϵ is small enough $y^j + \epsilon y^k$ is still liquid, and the new design improves agent O's payoff. □

Additionally, the feasibility constraint must always hold with equality. To see this suppose there are two securities y_t^j and y_t^k where the former is liquid and the latter is illiquid and some of the dividend is not incorporated into either security, that is, $y_t^j(s) + y_t^k(s) < s + \phi$ for a positive measure of states. If the unused portion of the dividends is incorporated into the illiquid tranche then there are two possibilities. If the illiquid tranche becomes liquid, agent O's payoff increases since both types can now use the new security to borrow. If the illiquid tranche remains illiquid, it still increases agent O's payoff since the low type can borrow more and the high type is still not trading the illiquid portion. Given Lemma 1 and the fact that the feasibility constraint must be binding, the designer's problem can be simplified into choosing a liquid tranche $y(s)$ and an illiquid tranche $s + \phi - y(s)$. Computing the prices of the two tranches from (8) and plugging into (6) the optimal security design simplifies to:

$$\max_{y(s)} (z-1)\left[\lambda(E_L s + \phi) + (1-\lambda) E_H y(s)\right] \tag{17}$$

$$\begin{aligned} s.t.\, s + \phi - y(s) &\geq 0, \forall s, \\ E_L y(s) - \zeta E_H y(s) &\geq 0, \\ y(s) \text{ is weakly increasing on } &[s_L, s_H]. \end{aligned} \tag{18}$$

The first constraint above is the feasibility constraint and requires $y(s)$ to be backed by the underlying asset in every state. The second is the pooling constraint and guarantees that the high O type agent sells a units of security y.

Clearly the liquid tranche in an optimal security design must satisfy $y(s) \geq \phi$ for all $s \in [s_L, s_H]$. Following Ellis, Piccione, and Zhang (2017), we write the monotone security $y(s)$ as:

$$y(s) = \phi + s_L + \int_{s_L}^{s} x(j)dj,$$

where $x(j) \geq 0$ for all $j \in [s_L, s_H]$.[12] Let $\widetilde{F}_Q(s) = 1 - F_Q(s)$ for $Q \in \{L, H\}$ and $s \in [s_L, s_H]$. Then,

$$E_Q y(s) = \phi + s_L + \int_{s_L}^{s_H} \widetilde{F}_Q(j) x(j) dj.$$

[12] In our analysis we restrict attention to securities that can be written as the sum of an absolutely continuous increasing function and countably many jump points.

Hence, the optimal security design problem (17) is equivalent to the following:

$$\arg\max_{x} \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds, \tag{19}$$

$$s.t. \int_{s_L}^{s} x(j)dj \leq s - s_L, \forall s \in [s_L, s_H], \tag{20}$$

$$\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi \geq 0, \tag{21}$$

$$x(s) \geq 0, \forall s \in [s_L, s_H] \tag{22}$$

In the above problem, (20) corresponds to the feasibility constraint, (21) corresponds to the pooling constraint and (22) guarantees that the security is monotone.

The next proposition shows that, as long as $f_L(s)/f_H(s)$ is decreasing, the optimal liquid tranche is a debt contract with face value $\phi + D$.

Proposition 2. *Assume that* $\frac{f_L(s)}{f_H(s)}$ *is decreasing in* s*. The optimal security is a unique standard debt contract* y_D *such that*

$$y_D(s) = \phi + \min(s, D),$$

for some $D \in (s_L, s_H]$*.*

To prove this proposition we use the Lagrangian for the optimization problem (19) and proceed in three steps. First, we show that when the dividend is above a cutoff $x(s) = 0$ or equivalently y must be flat. Second, we show that feasibility constraint must be binding at s whenever $x(s) > 0$. In other words, $y(s) = \phi + s$ whenever the liquid security is increasing, thus it promises the resale price and all of the dividend in such states. Finally, we show that there is a unique cutoff below which $x(s) > 0$ and above which $x(s) = 0$. The proof also shows that the optimal security cannot have jump points. Together, these steps imply that the optimal liquid security must be a debt contract.

5.2 Characterizing the Optimal Liquid Security

Suppose the designer anticipates that the liquid security will be liquid as long as $E_L y / E_H y \geq \zeta$. In this case, the prices of equity and liquid debt are

$$q_E = \int_{D}^{s_H} \widetilde{F}_L(s)ds,$$

$$q_D = \phi + s_L + \lambda \int_{s_L}^{D} \widetilde{F}_L(s)ds + (1-\lambda) \int_{s_L}^{D} \widetilde{F}_H(s)ds$$

Using Proposition 2 and plugging for ζ, the optimization problem in (19) and the associated constraints (20)-(22) can be simplified as

$$\max_{D\in[s_L,s_H]} \int_{s_L}^{D} \widetilde{F}_H(s)ds \tag{23}$$

subject to

$$(z-1)\left[\phi + s_L + \lambda\int_{s_L}^{D}\widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{D}\widetilde{F}_H(s)ds\right] \geq \lambda\int_{s_L}^{D}\left[\widetilde{F}_H(s)-\widetilde{F}_L(s)\right]ds, \tag{24}$$

where the constraint is the condition for the both O types to pool and issue the liquid debt.

To complete the characterization of optimal liquid security we solve for the equilibrium given in Definition 2. Note that the equilibrium boils down to solving (23) to find the optimal debt level $D \in (s_L, s_H]$ given the asset price ϕ and determining the asset price ϕ in the centralized market through the corresponding Euler equation:

$$\phi = \beta\left\{z\left(q_D + \lambda q_E\right) + (1-\lambda)\int_{D}^{s_H}\widetilde{F}_H(s)ds\right\}. \tag{25}$$

Proposition 3. *Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. If $E_L s/E_H s < 1-(z-1)/(z\lambda(1-\beta))$, there is a unique equilibrium where the face value of the debt $D \in (s_L, s_H)$ and the asset price ϕ are solutions to the following two equations:*

$$\phi = \frac{z}{z-1}\lambda\int_{s_L}^{D}\left[\widetilde{F}_H(s)-\widetilde{F}_L(s)\right]ds - \int_{s_L}^{D}\widetilde{F}_H(s)ds - s_L \tag{26}$$

$$\phi = \frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{D}^{s_H}\widetilde{F}_H(s)ds\right\} \tag{27}$$

Otherwise, there is a unique equilibrium where $D = s_H$ and $\phi = \frac{\beta}{1-\beta z}z\left[\lambda E_L s + (1-\lambda)E_H s\right]$. Moreover, in the former case the equilibrium of the security design problem strictly Pareto dominates all equilibria of the case where only the asset can be used as collateral. In the latter case, the equilibrium of the security design problem strictly Pareto dominates the separating equilibrium of the case where only the asset can be used as collateral and replicates the pooling equilibrium.

The formal proof of the proposition is in the Appendix. We provide an intuitive discussion of this result and the economics mechanism behind it in the next subsection. The following corollary follows immediately from Proposition 3.

Corollary 2. *Under the welfare improving security design equilibrium, there is non-trivial tranching when $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$.*

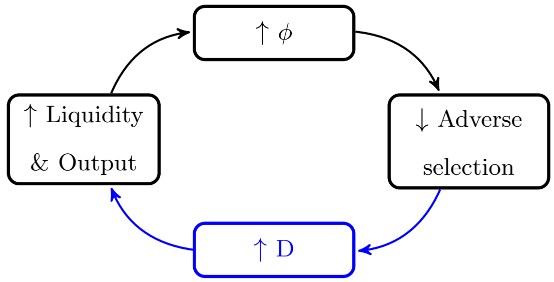


Figure 3: Asset Price ϕ and Liquid Debt Face Value $\phi + D$

Note that this condition is the same condition for the left boundary of multiple equilibria region in (16) indicating that security design improves the liquidity of the unique separating regime when only equity is allowed to be traded.

5.3 Discussion of the Unique Equilibrium under Optimal Security Design

In this section we compare the results from Section 4 and the optimal security design problem of Section 5.2 and discuss the underlying economic mechanism. There are two important differences with the dynamic lemons market when the optimal security design is introduced: 1) there is non-trivial welfare improving tranching in the separating equilibrium region; and 2) the pooling equilibrium is selected as the unique equilibrium in the multiple equilibria region.

Figure 3 illustrates the feedback loop between the asset price and the face value of debt that leads to these differences. As the face value of the liquid debt $\phi + D$ increases, more of the dividend states are pledged as collateral, more funds are raised for the productive sector and the real output goes up. This in turn leads to an increase in the collateral asset price ϕ which is incorporated into the face value of debt alleviating the adverse selection problem and allowing even more dividend to be pledged as collateral.

To understand this mechanism, we revisit the equilibrium construction in the optimal security design equilibrium. Suppose that the security designer sells a liquid debt tranche with a face value $\phi + D$ and an illiquid equity tranche. Note that the liquid debt tranche incorporates the resale price of the asset in the face value since both types of debt issuers' promise to return ϕ to the creditors is credible. Recall that the asset price ϕ in the centralized market, after substituting for the prices of the debt and equity tranches q_D and q_E, is given by the Euler equation:

$$\phi = \frac{\beta}{1 - \beta z} \left\{ z \left[\lambda E_L s + (1 - \lambda) E_H s \right] - (1 - \lambda)(z - 1) \int_D^{s_H} \widetilde{F}_H(s) ds \right\}, \tag{28}$$

where it is immediate that ϕ is increasing in D.

For any D let $\phi(D)$ be the asset price in the centralized market satisfying (28). Let $\underline{\phi} = \phi(s_L)$ and $\phi^P = \phi(s_H)$. Recall from Section 4 that ϕ^S is the asset price when only the low type sells the asset and high type retains both the resale price and the dividend. In contrast, the asset price calculation in (28) assumes that both types of borrowers sell (liquid) debt claims backed by the future resale price at the minimal as collateral. As a result, $\underline{\phi} > \phi^S$. On the other hand, $\phi(s_H)$ is the same as the pooling price ϕ^P. To see this, note that ϕ^P is calculated assuming that both types use the resale price and the entire dividend of the asset as collateral which is equivalent to setting the face value of the liquid debt contract to $\phi^P + s_H$. The solid line in Figure 4 depicts the function $\phi(D)$.[13]

Next consider the designer's choice of D as a function of the asset price ϕ. Optimal security design chooses D as large as possible making sure that the debt tranche is liquid. As D increases, the debt tranche incorporates more of the high dividend states. If D is too high, the high type, who knows that those states are likely, might prefer to retain the debt tranche rather than pooling with the low type. Hence, the security design can push up D to the point where the high type is indifferent between selling or retaining the debt. As the asset price increases, selling the debt tranche becomes more attractive to the high type, allowing the security designer to increase D. Recall $D(\phi)+\phi$ be the optimal face value of debt given the asset price ϕ.[14] The dash dotted line in Figure 4 depicts the function $D(\phi)$ for the case $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$.[15] The figure illustrates that no matter how low the asset price is, as long as tranching is feasible, optimal security design involves a debt tranche that incorporates some dividend. That is, $D\left(\underline{\phi}\right) > s_L$. This is a robust feature of security design that holds regardless of underlying parameters. Also note that in the region depicted, adverse selection is severe, and even when the asset price is as high as possible, high type prefers to retain the equity tranche. That is, $D\left(\phi^P\right) < s_H$.

Using these two curves, $\phi(D)$ and $D(\phi)$ we can find the equilibrium values of the face value of debt and the asset price, (D^*, ϕ^*). The equilibrium is where the two curves intersect, ie, when $\phi^* = \phi(D^*)$ and $D^* = D(\phi^*)$. As Figure 4 shows, when $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$, the equilibrium face value of debt $D^* \in (s_L, s_H)$. This explains the first difference in the results of the two sections.

Perhaps more interesting is the case when $E_L s/E_H s > 1-(1-\zeta)/(1-\beta)$ given in Figure 5 where

[13] Note that ϕ is strictly increasing for $D \in [s_L, s_H)$, $\partial\phi/\partial D$ is decreasing and is zero at $D = s_H$.

[14] $D(\phi)$ is constructed as the unique solution to the following equation for a given ϕ

$$\mathcal{T}_\phi(D) = (z-1)\left[\phi + s_L + \int_{s_L}^{D} \widetilde{F}_H(s)ds\right] - z\lambda \int_{s_L}^{D} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds = 0$$

whenever there is a solution in $[s_L, s_H]$. If there is no solution, ie, if $\mathcal{T}(s_H) > 0$, then $D(\phi) = s_H$.

[15] Recall that this is the left boundary of multiple equilibria region in 16. In this region adverse selection leads to a unique separating equilibrium without security design.

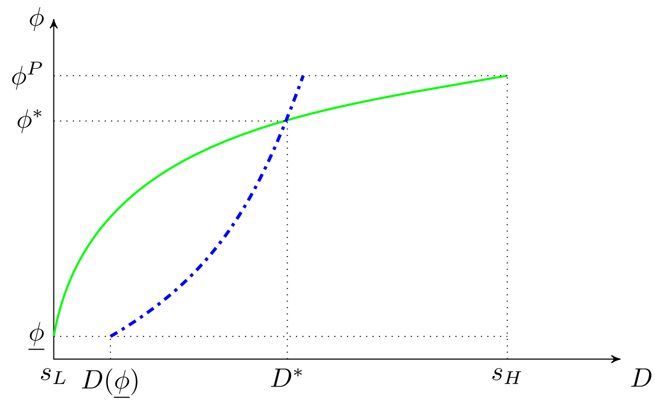


Figure 4: $\phi(D)$ and $D(\phi)$ when $E_L s/E_H s < 1 - (z-1)/(s\lambda(1-\beta))$.

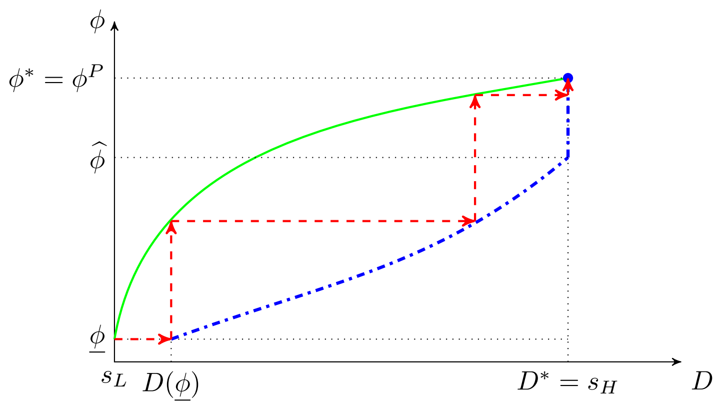


Figure 5: $\phi(D)$ and $D(\phi)$ when $E_L s/E_H s > 1 - (z-1)/(s\lambda(1-\beta))$.

the second difference arises. In this case, adverse selection is less severe and $D(\phi)$ function is shifted to the right as the same asset price can sustain a higher face value of the liquid debt. When the asset price is above a threshold denoted by $\widehat{\phi}$, optimal security design sets the face value of debt to s_H which is captured by the vertical part of $D(\phi)$ function. The two curves intersect only at the upper right corner, $(s_H, \overline{\phi})$. As a result, there is a unique equilibrium for the security design problem and it involves setting the face value of debt at $D^* = s_H$.

The scenario depicted in Figure 5 may seem surprising since, as we illustrated in Section 4, without the possibility of security design there are multiple equilibria in part of this region. With security design, we obtain a unique equilibrium in which Agent O sells the entire "pass-through" debt tranche in a pooling equilibrium. To understand this, note that without security design the high type decides among only two options: whether to use the resale price and dividend of the asset as collateral versus to retain both parts. The outcome depends on the asset price. In the good equilibrium $\phi = \phi^P$ and the high type sells the asset. In the bad equilibrium, $\phi = \phi^S$ and the high type retains the asset. The bad equilibrium cannot survive with security design because even if the asset price were ϕ^S, the optimal security design would be able to improve this separating equilibrium by creating a liquid debt tranche with the face value ϕ^S, which in turns would increase the asset price above ϕ^S. Both graphs in Figures 4 and 5 in fact show that the equilibrium asset price in the optimal security design equilibrium is not less than $\underline{\phi} = \phi(s_L) > \phi^S$ (since the face value of the liquid debt is never below $\underline{\phi} + s_L$). Given the increase in the asset price to $\underline{\phi}$ from ϕ^S, the high type's participation constraint is relaxed, this leads to the optimal security design to incorporate more of the dividend into the debt tranche (that is $D > s_L$). A higher D will increase the asset price ϕ and so on. This unravelling process is illustrated in Figure 5 with the dashed arrows. As the figure shows when the asset price is $\underline{\phi}$, face value of the debt rises to $\underline{\phi} + D\left(\underline{\phi}\right)$. When the face value of the debt increases to $\underline{\phi} + D\left(\underline{\phi}\right)$, the asset price further increases, and so on. The process ends when price rises to ϕ^P.

The uniqueness of equilibrium does not depend on the restriction to issuing monotone securities. The following proposition shows that if we allow Arrow securities against the dividend payment and always pledge the resale value of the asset with the liquid securities, there always exists a unique equilibrium.

Proposition 4. *Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal security design under Arrow securities has two tranches, the liquid tranche $y_{1t}(s)$ and illiquid tranche $y_{2t}(s)$.*

$$y_{1t}(s) = \phi + s_L + (s - s_L)\mathbb{I}(s \leq D),$$
$$y_{2t}(s) = (s - s_L)\mathbb{I}(s > D).$$

If $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$, there is a unique equilibrium for the optimal design, where $D \in (s_L, s_H)$ and the asset price ϕ are solutions to the following two equations:

$$\phi = \frac{\int_{s_L}^{D} sdF_H(s) - z\int_{s_L}^{D} sdF_\lambda(s)}{z-1} \tag{29}$$

$$\phi = \frac{\beta}{1-\beta z}\left[z\int sdF_\lambda(s) - (z-1)(1-\lambda)\int_D^{s_H} sdF_H(s)\right] \tag{30}$$

where $F_\lambda(s) = \lambda F_L(s) + (1-\lambda)F_H(s)$. Otherwise, there is a unique equilibrium where $D = s_H$ and $\phi = \frac{\beta}{1-\beta z}z\left[\lambda E_L s + (1-\lambda)E_H s\right]$.

As in Proposition 3, in the former case, the equilibrium of the security design problem strictly Pareto dominates all equilibria of the case where only the asset can be used as collateral. In the latter case, the equilibrium of the security design problem strictly Pareto dominates the separating equilibrium of the case where only the asset can be used as collateral and replicates the pooling equilibrium.

5.4 Short-Term Repo Contract as an Implementation of the Optimal Security Design

The optimal security design is implemented by a one-period repo contract, which is liquid, an equity-like contract, which is illiquid. The terms of the contract are endogenous. Therefore, our theory offers a perspective on how adverse selection affects of the terms of short-term repo contracts backed by long-term assets. The face value of the repo contract is

$$\phi + D.$$

The expected value of the repo contract for the lender is

$$q_D = \phi + s_L + \int_{s_L}^{D} \left[\lambda \tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)\right] ds.$$

The value of collateral underlying the repo contract in the beginning of a period to the productive borrowers is

$$\phi/\beta = z\phi + z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{D}^{s_H} \tilde{F}_H(s) ds$$

The last term reflects the loss of value from the illiquid equity tranche.

We are now ready to state the terms of the repo contract, including repo rate, R, and haircut, h. The definition of repo rate is straightforward:

$$R = \frac{\text{face value}}{\text{expected loan value}} - 1 = \frac{D + \phi - q_D}{q_D}. \tag{31}$$

When the expected quality of the debt contract is low relative to the face value, the repo rate is high. The asset quality might have two opposing effects on repo rate. When asset quality worsens (improves), expected loan value is lower (higher), leading to a high (low) repo rate. At the same time, the face value of the debt might be adjusted down (up), implying a lower (higher) likelihood of default and resulting in a lower (higher) repo rate.

The definition of repo haircut in our model is:

$$
\begin{aligned}
h = 1 - \frac{\text{expected loan value}}{\text{collateral value}} &= \frac{(z-1)q_D + \lambda z q_E + (1-\lambda)\int_D^{s_H} \tilde{F}_H(s)ds}{\phi/\beta} \\
&\approx \underbrace{(z-1)}_{\text{productivity}} + \underbrace{\frac{\int_D^{s_H}\left[\lambda \tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)\right]ds}{\phi/\beta}}_{\text{equity/collateral}}, \text{ if } z \text{ is close to 1.}
\end{aligned} \tag{32}
$$

It is immediate that repo haircut has two components: the productivity of the borrower's technology, and the value of the equity tranche relative to the value of the collateral. The first component arises because borrowers, who price the collateral asset, value the liquidity service the asset provides, while lenders, who price the loan, does not value the liquidity service. The term $z-1$ is the net value of the liquidity service. It reflects heterogeneous valuation over the collateral assets between lenders and borrowers in our model. This component is similar to the difference-in-opinion explanation of haircut in Geanakoplos (2003) and Fostel and Geanakoplos (2012). The second component arises because of information friction. It reflects how adverse selection affects the repo contract. This component has been emphasized by Dang, Gorton, and Holmström (2011) and Gorton and Ordonez (2014).

6 Equilibrium Properties

In this section, we will look at properties of the optimal security with two examples.

Example 1: two point distribution. Suppose that the high quality asset pays one dividend with probability π_H and the low quality asset pays the dividend with probability π_L, $0 < \pi_L < \pi_H < 1$. In this case, we obtain closed form solutions for the face value of the debt, $D + \phi$, and the asset price, ϕ, as follows:

$$
\begin{aligned}
D &= \frac{1}{\frac{z}{z-1}\lambda(\pi_H - \pi_L) - \pi_H}\phi, \\
\phi &= \frac{\beta\left[z\lambda\pi_L + (1-\lambda)\pi_H\right]}{1 - \beta z - \beta\frac{(1-\lambda)(z-1)\pi_H}{\frac{z}{z-1}\lambda(\pi_H-\pi_L)-\pi_H}}.
\end{aligned}
$$

Given these expressions, we can clearly delineate the effect of asset quality in this economy. In this particular example, we can demonstrate that the effect is monotone: $\frac{dD}{d\lambda} < 0$ and $\frac{d\phi}{d\lambda} < 0$. Moreover, the effect of productivity z can also be easily assessed. Because $\frac{\partial D}{\partial z} > 0$, more debt is created in good times. The sensitivity of asset price to productivity is amplified by the endogenous security design.

The terms of the repo contract can also be expressed in closed form and allow us to examine the determinants of repo rates and haircuts in this particular example. The repo rate is expressed as

$$R = \left[\frac{1-\pi_H}{\lambda(\pi_H - \pi_L)} + 1\right](z-1)\,. \tag{33}$$

It is immediate that repo rate is increasing in the productivity of technology z which measures the demand for liquidity from the productive borrowers. It is also clear that in this particular example, repo rate is increasing in λ. That is, a worsening (improving) asset quality leads to a lower (higher) repo rate, indicating that the face value of the repo debt drops (increases) significantly to eliminate (incorporate) risk states. To give another perspective on how repo rate is related to the riskiness of the cashflow and information frictions, we rewrite the above expression using the incentive constraint of the high quality seller of the repo contract, which is

$$zq^D = \pi_H D + \phi,$$

and obtain

$$R = \frac{\phi + D}{q^D} - 1 = \underbrace{\frac{\phi + D}{\phi + \pi_H D}}_{\text{Cashflow Riskiness}} \underbrace{z}_{\text{Productivity}} - 1. \tag{34}$$

Taking repo debt face value $\phi + D$ as given, (34) implies that the interest rate depends on the riskiness of the high quality assets directly. The degree of information friction plays an indirect role through debt face value. In fact, if the high quality asset pays dividend for sure, (33) implies that the repo rate R is $z-1$. In this extreme case the repo rate is insensitive to changes in asset quality and driven purely by the productivity from the productive borrowers, which measures their liquidity demand. This example illustrates that in our model, repo rates capture more the demand for liquidity and cashflow riskiness of the repo contract and less so asset quality. This is due to the nature of security design: both high and low quality borrowers participate in the repo market and repo debts are free from adverse selection.

The repo haircut in this example can be expressed as

$$h = 1 - \beta + \frac{\beta}{1 - \frac{\lambda(\pi_H - \pi_L)}{(z-1)[(1-\lambda)\pi_H + \lambda\pi_L]}}. \tag{35}$$

Suppose z and β are close to 1, from (35) we then have

$$h \simeq \underbrace{(z-1)}_{\text{Productivity}} \left[1 - \underbrace{\frac{\pi_H}{\lambda\left(\pi_H - \pi_L\right)}}_{\text{Information Friction}}\right] + 1 - \beta.$$

It demonstrates again the two components in repo haircut highlighted in equation 32: one is related to the liquidity services of the collateral due to the technology productivity z and the other is related to the ratio of equity tranche over the collateral asset which is pinned down by the information friction $\lambda(\pi_H - \pi_L)$. In this particular example, $\frac{\partial h}{\partial \lambda} = \frac{\pi_H}{\lambda^2(\pi_H - \pi_L)}(z-1) > 0$. That is, as the asset quality deteriorates, haircut monotonically increases. Furthermore, haircut is also increasing in the quality difference between high and low type $\pi_H - \pi_L$, a measure of severity of adverse selection. This example shows again that the haircut of a repo contract is a robust indicator of information frictions over the asset quality, reflecting the magnitude of adverse selection, while the interest rate reflects the cashflow riskiness of the repo contract.

Example 2: Markov process for asset quality and project productivity In this example, we introduce Markov processes for asset quality and project productivity. Assume that the aggregate state x follows a Markov process, and parameters such as asset quality λ and productivity z are functions of the state: where $x \sim G(\cdot)$,

$$x_{t+1} \begin{cases} = x_t, & \text{with probability } \rho, \\ \sim G(x), & \text{with probability } 1-\rho. \end{cases}$$

We characterize the stationary Markov equilibrium. Given the state x, denote the end-of-period value of asset price if tomorrow's state is x to be ϕ_x and the face value of the liquid debt contract $E_x\phi + D_x$, where $E_x\phi \equiv \rho\phi_x + (1-\rho)E\phi$, $E\phi \equiv \int \phi_x dG(s)$. From optimal securitization decisions, summarized by equation (A.1),

$$\begin{aligned} E_x\phi &= \rho\phi_x + (1-\rho)E\phi \\ &= \frac{z_x}{z_x - 1}\lambda_x \int_{s_L}^{D_x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{D_x} \widetilde{F}_H(s) ds - s_L, \end{aligned}$$

we have

$$\phi_x = E\phi + \frac{1}{\rho}\left\{\frac{z_x}{z_x - 1}\lambda_x \int_{s_L}^{D_\lambda} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{D_x} \widetilde{F}_H(s) ds - s_L - E\phi\right\}. \tag{36}$$

From the Euler equation at the end of the period, after the quality of the asset in the period is revealed,

$$\phi_x = \frac{\beta}{1 - \beta\rho z_x}\left\{z_x\left[\lambda_x E_L s + (1-\lambda_x) E_H s + (1-\rho)E\phi\right] - (1-\lambda)(z_x - 1)\int_{D_x}^{s_H} \widetilde{F}_H(s) ds\right\} \tag{37}$$

(36) and (37) solve jointly (D_x, ϕ_x) for all states.

Suppose $z_x = z$ and $\lambda_x = x \in [0,1]$. Then, with the process, the quality distribution is persistent over time but may change with probability $1-\rho$. When the quality distribution changes, the distribution

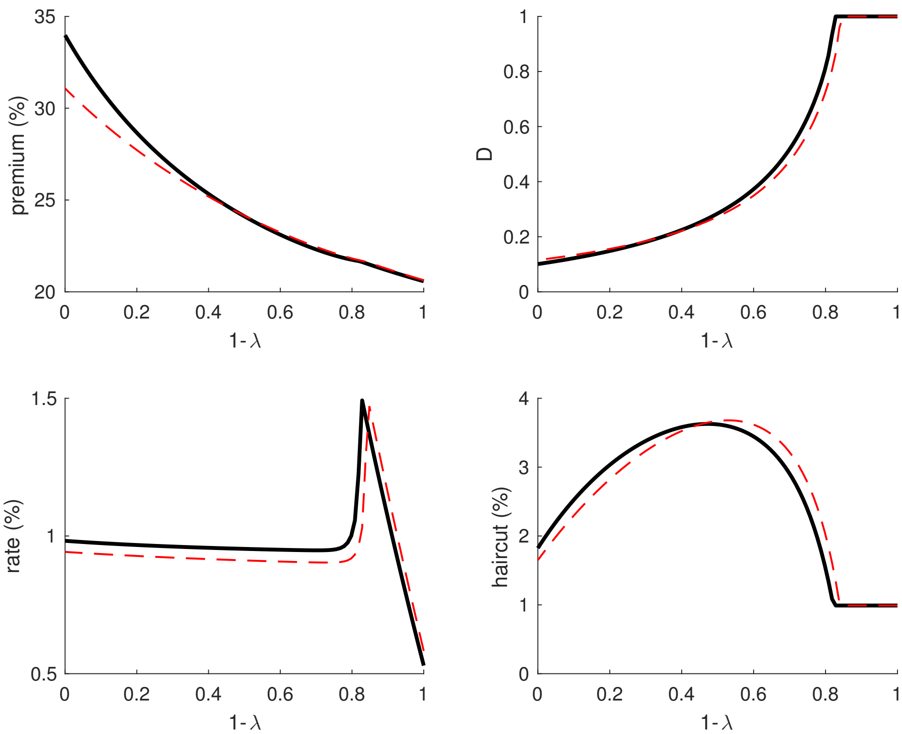


Figure 6: Asset quality, asset price and terms of the repo contract. The parameter for the numerical examples are as follows: high quality asset dividend follows a Beta distribution with $(a, b) = (10, 1)$ and low quality asset dividend follows a Beta distribution with $(a, b) = (0.1, 1)$, $\lambda \sim U[0, 1]$, $\beta = .95$, $z = 1.01$. The solid lines are drawn with $\rho = .95$ and the dashed lines with $\rho = .90$.

parameter λ will be drawn from distribution G. We focus on the stationary Markov equilibrium. Security design and asset price depends on the quality distribution λ. Figure 6 illustrates how the collateral value, face value, interest rate and haircut of the repo contract respond to shocks to quality distribution.

The upper-left side subfigure show the liquid premium in the asset price, defined as difference between the actual asset price, ϕ_x, and the value of the asset without providing liquidity services, denoted φ_x, as a percentage of φ_x.

$$\varphi_x = \frac{\beta\left[(1-\rho)E\varphi + \lambda_x E_L s + (1-\lambda_x)E_H s\right]}{1-\beta\rho}, \ E\varphi = \int \varphi_x dG(x).$$

It reflects the liquidity value of the asset. When λ increases, both D and φ decrease so that the value of liquidity services provided by a collateral decreases. But because both high and low quality assets provides some liquidity service, the liquidity value of collateral decreases more slowly than φ_x. This is why the liquidity premium decreases in $1 - \lambda$. This shows the liquidity gain from security design is higher for assets of lower average quality.

Both the haircut (bottom right subfigure) and the repo rate (bottom left subfigure) change non-monotonically in λ. For haircut, this is because the value of the equity tranche is non-monotonic. When the asset quality is on average good (high $1-\lambda$), the information friction is small enough that there is no need to tranche the cash flow. In this case, no illiquid equity tranche is created. Then, according to (32), the haircut then only reflects heterogeneous valuation over liquidity between the borrower and lenders. When the asset quality is on average bad, the repo tranche is also very likely to default and the value of the equity tranche is small in that case. When the asset quality is in the intermediate range, the adverse selection is severe and hence the ratio of equity tranche to the asset is high, resulting in large haircut.

The interest rate on the repo contract, when there is a non-trivial equity tranche, is for the most part decreasing in asset quality, reflecting the declining default probability and loss from default. The uptick in the repo rate reflects the opposing effect of changing asset quality mentioned previously when discussing equation (31): the face value of the debt might increase faster and incorporate more risky dividend states relative to the expected value of repo debt as asset quality improves.

We observe that in this example when the illiquidity induced by adverse selection is strong (high λ), haircut is very sensitive to changes in λ while the repo rate barely responds. This is qualitatively consistent with empirical observations during the repo runs where there were rare changes in repo rates but haircut skyrocketed.

The red dash lines correspond to lower persistence of the Markov process. When the quality distribution is less persistent, the collateral value is less responsive to the current productivity. When the high quality state is less persistent, adverse selection becomes more severe in that state, face value and repo rate decrease and haircut increases in that state.

Alternatively, suppose $\lambda_x = \lambda$ and $z_x = (1-x)z_L + xz_H$. Figure 7 illustrates that when firms are productive, the asset price is high, more repo contracts are issued, repo rate increases and repo haircut decreases.

Notice that a percentage point increase in productivity leads the collateral asset to increase in value by more than 10 percentage point. This amplification of productivity shocks reflects the dynamic feedback loop between the future collateral value and liquidity of the current market. Future collateral value increases in future productivity. This reduces adverse selection in the current market, further increasing the collateral value. As productivity increases, the face value of the repo contract backed by a collateral increases, its interest rate increases and its haircut decreases. As before, the red dash lines correspond

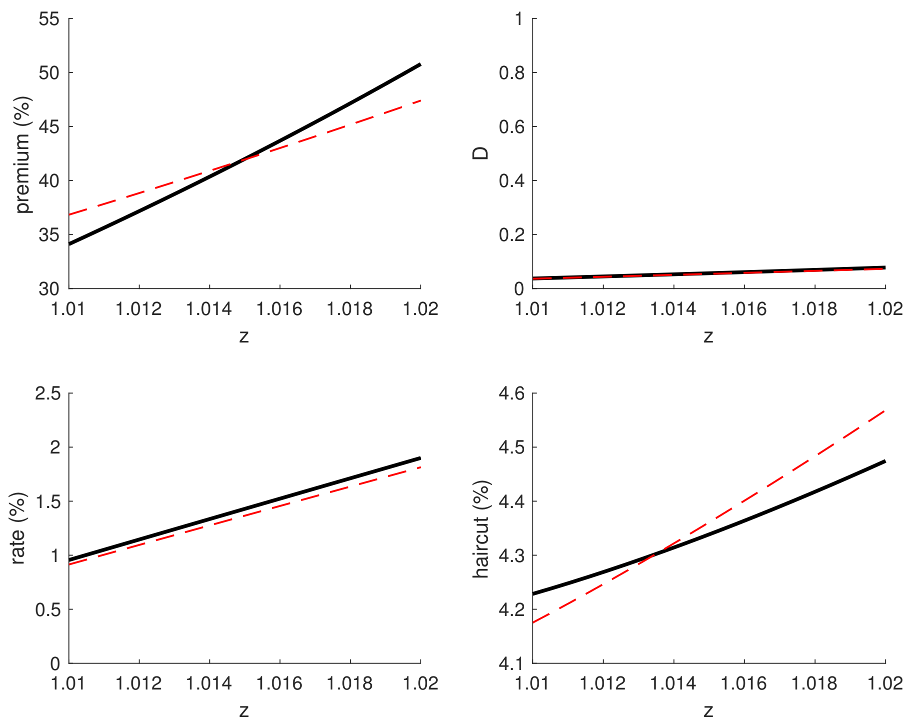


Figure 7: Productivity, asset price and terms of the repo contract. The parameter for the numerical examples are as follows: high quality asset dividend follows a Beta distribution with $(a, b) = (10, 1)$ and low quality asset dividend follows a Beta distribution with $(a, b) = (0.1, 1)$, $\lambda = 0.99$, $\beta = .95$, $z \sim U[1.01, 1.02]$. The solid lines are drawn with $\rho = .95$ and the dashed lines with $\rho = .90$.

to lower persistence of the Markov process.

Example 3: Portfolio Repo to improve asset quality. In this example, we illustrate the observation that pooling safe assets with the collateral asset that suffers from information frictions can improve the liquidity of the repo market. To derive analytical results, we use the two-point distribution used in Example 1. Denote the fraction of safe assets in the asset pool to be ω. To keep the example tractable, we assume that the asset pool pays 0 or 1 with probability. Given ω and the quality of the collateral Q, the probability that the pool pays 1 is $\omega + (1-\omega)\pi_Q$. When there is a non-trivial debt tranche in the optimal design, we can show that both asset price and the debt threshold of the asset pool are increasing in ω.

$$D = \frac{1}{(1-\omega)\left[\frac{z}{z-1}\lambda\left(\pi_H - \pi_L\right) - \pi_H\right] - \omega}\phi,$$

$$\phi = \frac{\beta\left[\omega\left(z\lambda + (1-\lambda)\right) + (1-\omega)\left(z\lambda\pi_L + (1-\lambda)\pi_H\right)\right]}{1 - \beta z - \beta\frac{(1-\lambda)(z-1)[\omega+(1-\omega)\pi_H]}{(1-\omega)\left[\frac{z}{z-1}\lambda(\pi_H-\pi_L)-\pi_H\right]-\omega}}.$$

This implies that the portfolio repo improves liquidity of the collateral asset.

The liquidity improvement also shows up in the term of the repo contract. Let the interest and haircut of the pool be R^ω and h^ω. R^0 and h^0

$$R^\omega = R^0 = \left[\frac{1-\pi_H}{\lambda(\pi_H - \pi_L)} + 1\right](z-1) \tag{38}$$

That the interest rate does not respond to ω echoes what we learned in Example 1, that the interest rate does not respond to information frictions.

The haircut

$$h^\omega = 1 - \beta + \frac{\beta}{1 - \frac{\lambda(1-\omega)(\pi_H-\pi_L)}{(z-1)[\omega+(1-\omega)((1-\lambda)\pi_H+\lambda\pi_L)]}}.$$

Suppose z and β are close to 1

$$h^\omega \simeq 1 - \beta + (z-1)\left[1 - \frac{\frac{\omega}{1-\omega} + \pi_H}{\lambda\left(\pi_H - \pi_L\right)}\right] = h^0 - \frac{\omega(z-1)}{(1-\omega)\lambda(\pi_H - \pi_L)}. \tag{39}$$

The haircut decreases in ω implying that pooling the collateral asset with safe assets reduces information friction. This theoretical finding corresponds with the empirical finding in Julliard et al. (2018) where they find the haircuts of repo contracts backed by a portfolio including AAA rated assets receives (statistically significant) 0.9% to 1.15% lower haircut compared with repo contracts without any AAA rated assets, controlling for counter-party and collateral characteristics.

7 Contract Rigidity and Repo Run

As we show in Section 5, optimal security design eliminates fragility and improves welfare. However, the construction relies on the assumption that borrowers have the flexibility to design securities and change the terms of the contracts at the beginning of each period. For example, our results imply that when borrowers are able to update the terms of over-night repo contracts each day then the repo market is robust to run. In practice, repo contract terms may not be updated daily because of associated administrative costs or simply inattention. Next, we show that the induced rigidity may be a crucial source of fragility.

Suppose that the security design, represented by the debt threshold D, is not updated at the beginning of each period. Instead, it remains the same with probability γ in each period. Suppose that when the design is not updated, repo market receives sunspot. Denote the probability that the design is not updated (or is rigid) and the market receives a sunspot by χ where $1 \geq \gamma \geq \chi > 0$. As we show when the design is rigid and the market receives a sunspot, the liquidity of the repo market may deteriorate. We call a dynamic stationary equilibrium in which the asset price drops after a sunspot a repo run equilibrium. Let the market price of the asset with rigid design and sunspot by ϕ^S and the price when the design is flexible by ϕ. We assume that the sunspot of illiquidity arrives at the end of a period, after securities are traded.

The formal definition of a repo run equilibrium requires a slight modification to Definition 2 since in a repo run equilibrium there are two possible continuation prices ϕ and ϕ^S.

Definition 3. A dynamic stationary sunspot equilibrium with rigid contracts given χ and γ consists of asset prices ϕ, ϕ^S, $E\phi = \phi(1-\chi) + \phi^S\chi$, security design $\mathcal{J}(E\phi) \subseteq \mathcal{I}(E\phi)$ and security prices q^j for each $j \in \mathcal{J}(E\phi)$ such that (i) $\mathcal{J}(E\phi)$ solves the security design problem (6) given the expected asset value $E\phi$, (ii) security price q_t^j satisfies equation (8), and (iii) ϕ_t solves the Euler equation given by:

$$\phi = \beta \left[z \left(\sum_{j \in P} q^j + \lambda \sum_{j \in \mathcal{J}(E\phi) \setminus P} q^j \right) + (1-\lambda) \sum_{j \in \mathcal{J}(E\phi) \setminus P} E_H y^j \right], \tag{40}$$

$$\phi^S = \beta \left\{ z\lambda E_L s + (1-\lambda) E_H s + z \left[\gamma \phi^S + (1-\gamma)\phi \right] \right\}, \tag{41}$$

where $j \in P \subseteq \mathcal{J}(E\phi)$ iff $R^j \geq \zeta$.

Using the modified definition and following similar steps leading to Proposition 3 give the incentive constraint of owners of high quality collateral which is analogous to (26):

$$\phi(1-\chi) + \phi^S\chi = \frac{z}{z-1}\lambda \int_{s_L}^{D} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s) \right] ds - \int_{s_L}^{D} \widetilde{F}_H(s) ds - s_L, \tag{42}$$

We also have two Euler equations for the two prices depending on whether the economy receives a sunspot or not, equation (40) and (41). Equation (40) can be simplified as

$$\phi = \beta \left\{ z \left[\lambda E_L s + (1-\lambda) E_H s \right] - (1-\lambda)(z-1) \int_D^{s_H} \widetilde{F}_H(s) ds + z \left[\chi \phi^S + (1-\chi)\phi \right] \right\}, \tag{43}$$

The following proposition shows that in addition to the equilibrium characterized in Proposition 3, there also exists a sunspot equilibrium where the debt tranche is illiquid when the security design is not updated and the investors receive a pessimistic sunspot.

Proposition 5. *In an environment with rigid design, there exists a cutoff $\mathit{\Gamma}(\gamma, \chi)$ which is increasing in γ and χ with*

$$\mathit{\Gamma}(\gamma, \chi) > 1 - \frac{z-1}{z\lambda(1-\beta)}$$

such that whenever

$$\frac{E_L s}{E_H s} < \mathit{\Gamma}(\gamma, \chi), \tag{44}$$

there exists a unique repo run equilibrium satisfying equation (42), (43) and (41). In the repo run equilibrium the debt tranche is illiquid when the security design is not updated and the investors receive a sunspot. In this equilibrium,

(i) The asset price is higher before the run, i.e., $\phi > \phi^S$.

(ii) When (44) holds debt threshold D is strictly lower with repo run than without.

(iii) Debt threshold with repo run is decreasing in probability of rigidity γ and probability of sunspot given rigidity χ.

(iv) As χ approaches one the unique equilibrium approaches the illiquid equilibrium when only equity is available as collateral asset.

(v) Compared with the equilibrium without repo run, the asset price ϕ in the repo run equilibrium is lower.

(vi) When investors receive a sunspot, the interest rate of the repo contract increases, and its haircut response is indeterminate.

(vii) Welfare in the run equilibrium is Pareto dominated by the equilibrium without run.

The fact that there exists a non trivial sunspot equilibrium for $\frac{E_L s}{E_H s} \in (1 - \frac{z-1}{z\lambda(1-\beta)}, \mathit{\Gamma}(\gamma, \chi))$ implies that the sunspot equilibrium may arise in the parameter region, where there exists a liquid passthrough security in the liquid equilibrium.

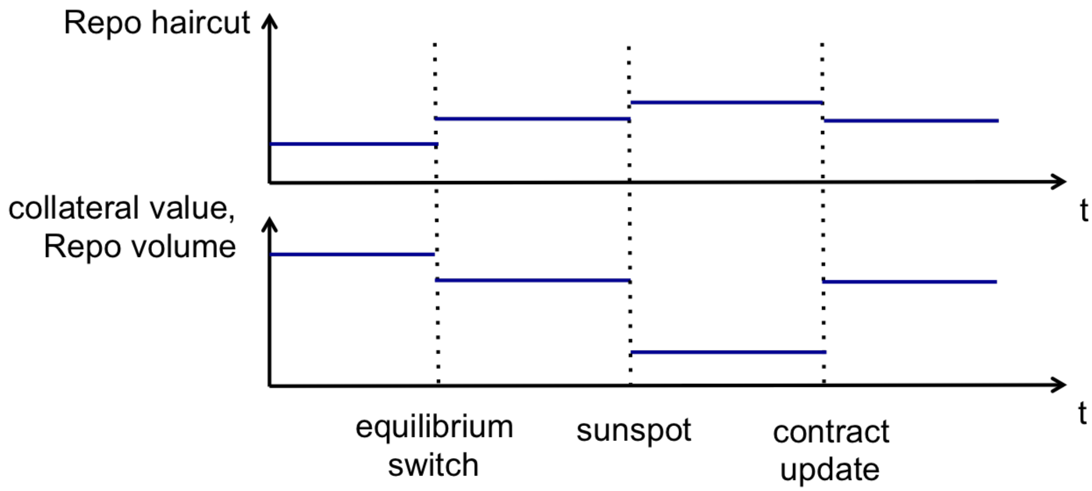


Figure 8: Dynamics of Repo run.

The terms of the repo contract when investors receive a pessimistic sunspot are rigid in the sense that the book value of the repo contract, $D+\phi$, does not change. But because the asset price decreases to a lower level, ϕ^S, the effective default threshold increases from D to $D^S \equiv \min(s_H, D+\phi-\phi^S)$. By claim (i), $\phi > \phi^S$. So, $D^S > D$ as long as $D < s_H$. When investors receive the sunspot, the repo contract also becomes illiquid. Only owners with low quality assets trade the asset. In this scenario, denote the effective interest rate and haircut to be R^S and h^S respectively and the price of security j q_j^S.

$$R^S = \frac{D+\phi-q_D^S}{q_D^S}, \tag{45}$$

$$h^S = 1 - \frac{(z-1)q_D^S}{\phi^S/\beta}, \tag{46}$$

where

$$q_D^S = \phi^S + s_L + \int_{s_L}^{D^S} \tilde{F}_L(s)ds.$$

In the appendix, we show $q_D^S < q_D$. Then, repo rate increases, $R^S > R$. The response of haircut is indeterminate because both $\phi^S < \phi$ and $q_D^S < q_D$.

The dynamics in a typical repo run is illustrated in Figure 8. In the figure before the moment marked by the equilibrium switch the economy is a "good" equilibrium in which even when the security is rigid agents expect that the repo tranche will be liquid and the asset price will be high. After the switch a repo run typically takes two stages. First, the equilibrium switches to the sunspot equilibrium described in

Proposition 5. Once the economy enters a sunspot equilibrium, haircut of the repo contract immediately increases, because investors anticipate that the repo contract will be illiquid when a sunspot hits the economy. At the same time, the asset price and the repo volume decrease. When the sunspot actually hits the economy, asset price and the repo volume decrease further. The repo rate increases further while the repo haircut may also increase. This occurs despite that the face value of repo debt remains unchanged due the contract rigidity. The drop in repo volume and the asset price is higher when the sunspot hits, because the repo backed by high quality collateral stops circulating entirely. When the contract terms are updated, the update restores investors' sentiment about the liquidity of the repo market, the price and the volume recover partially, to the levels right after equilibrium switch. The fluctuation driven by sunspots may take place repeatedly as long as the economy remains in the sunspot equilibrium.

Notice that the equilibrium switch can be triggered either by a switch of self-fulfilling beliefs from the equilibrium without repo run to an equilibrium with repo run, or by a small shift in the fundamental. Suppose the fundamental of the economy, represented by asset quality λ or productivity z, is initially such that condition (44) does not hold. As the fundamental deteriorates and condition (44) holds, even if the change in fundamentals is very small, a sunspot equilibrium might emerge, leading to a discontinuous drop in market liquidity and the asset price.

8 Conclusion

Our paper studies optimal security design in a dynamic lemons setup. We show that the implementation of optimal security design involves liquid short-term repo contracts. Because optimal security design helps coordinate investors' inter-temporal decisions, the dynamic lemons market under optimal security design is robust to multiple-equilibrium fragility induced by inter-temporal mis-coordination. We derive dynamic equilibrium properties of repo rates, haircuts and volume in relation to real productivity and adverse selection levels. We also show that repo run may emerge when there is rigidity in security design. At the aggregate, our paper provides some insights on the liquidity formation of the financial market over the productivity cycle and the asset quality cycle.

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A Appendix

A.1 Proof of Lemma 1

Proof. If two securities, y^j and y^k, are both liquid, $E_L y^j \geq \zeta E_H y^j$ and $E_L y^k \geq \zeta E_H y^k$. Then combining the two security retains liquidity. Similarly, combining two illiquid securities results in an illiquid security. To see the second statement, first note that replacing the two securities with their combination is clearly feasible. In addition, when y^j, y^k and $y^j + y^k$ all trade in a pooling (separating) equilibrium, q^{jk}, the price of $y^j + y^k$, is the sum of q^j and q^k, the prices of y^a and y^b. Now consider the liquid case. Ignoring the irrelevant terms, Agent O's payoff when the two securities are separate is:

$$\lambda \int \left\{ a \left[zq^j - y^j(s) \right] + a \left[zq^k - y^k(s) \right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a \left[zq^j - y^j(s) \right] + a \left[zq^k - y^k(s) \right] \right\} dF_H(s)$$

and when they are combined is:

$$\lambda \int \left\{ a \left[zq^{jk} - \left(y^j(s) + y^k(s) \right) \right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a \left[zq^{jk} - \left(y^j(s) + y^k(s) \right) \right] \right\} dF_H(s).$$

Since $q^{jk} = q^j + q^k$, when the liquid securities are combined agent O's payoff is unchanged.

Next consider the illiquid case. Once again ignoring the irrelevant terms, Agent O's payoff when the two securities are separate is:

$$\lambda \int \left\{ a \left[zq^j - y^j(s) \right] + a \left[zq^k - y^k(s) \right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a y^j(s) + a y^k(s) \right\} dF_H(s)$$

and when they are combined is:

$$\lambda \int \left\{ a \left[zq^{jk} - \left(y^j(s) + y^k(s) \right) \right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a \left(y^j(s) + y^k(s) \right) \right\} dF_H(s).$$

Once again, when the illiquid securities are combined agent O's payoff is unchanged. □

A.2 Proof of Proposition 2

Proof. First note that the feasible set is compact, convex and nonempty so the optimization problem must have a solution. Moreover, since the objective function is bounded above, the solution must be finite. The Lagrangian of the optimization problem is

$$\begin{aligned} \mathcal{L}(x; \gamma, \mu, \mu_x) = & \int_{s_L}^{s_H} \widetilde{F}_H(s) x(s) ds + \int_{s_L}^{s_H} \gamma(s) \left[s - s_L - \int_{s_L}^{s} x(j) dj \right] ds \\ & + \mu \left\{ \int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] x(s) ds + (1-\zeta)\phi \right\} + \int_{s_L}^{s_H} \mu_x(s) x(s) ds. \end{aligned}$$

Note that for any feasible x and for $\gamma \geq 0$, $\mu \geq 0$ and $\mu_x \geq 0$ we have

$$\mathcal{L}(x;\gamma,\mu,\mu_x) \geq \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds.$$

Let $\mathcal{L}(\gamma,\mu,\mu_x) = \max_x \mathcal{L}(x;\gamma,\mu,\mu_x)$. Let $\mathcal{L}^* = \min_{\gamma\geq 0,\mu\geq 0,\mu_x\geq 0} \mathcal{L}(\gamma,\mu,\mu_x)$. Note that $\mathcal{L}^*$ is the value of the original optimization problem. We can rewrite $\mathcal{L}(x;\gamma,\mu,\mu_x)$ as

$$\begin{aligned}\mathcal{L}(x;\gamma,\mu,\mu_x) = \quad & \int_{s_L}^{s_H} \left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] - \int_s^{s_H}\gamma(j)dj + \mu_x(s)\right\} x(s)ds \\ & +\mu(1-\zeta)\phi + \int_{s_L}^{s_H}\left(\int_s^{s_H}\gamma(j)dj\right)ds.\end{aligned}$$

Let $\eta(s) = \int_s^{s_H}\gamma(j)dj$. We can rewrite the problem as:

$$\begin{aligned}\mathcal{L}(x;\eta,\mu,\mu_x) = \quad & \int_{s_L}^{s_H} \left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] - \eta(s) + \mu_x(s)\right\} x(s)ds \\ & +\mu(1-\zeta)\phi + \int_{s_L}^{s_H}\eta(s)\,ds.\end{aligned}$$

Now note that the quantity inside the curly brackets must be zero or otherwise the value of the optimization problem would be infinite. Consider the following dual problem of the optimization problem,

$$\begin{aligned}\min_{\mu\geq 0} \quad \min_{\eta\geq 0,\mu_x\geq 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H}\eta(s)\,ds \\ s.t. \quad & \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] - \eta(s) + \mu_x(s) = 0.\end{aligned}$$

Note that the value of this problem is $\mathcal{L}^*$. Let $H_\mu(s) = \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]$. We can rewrite the above problem one more time as:

$$\begin{aligned}\min_{\mu\geq 0} \quad \min_{y\geq 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H}\eta(s)\,ds \\ s.t. \quad & \eta(s) \geq H_\mu(s),\end{aligned}$$

and the constraint that $\eta(s)$ is a decreasing function in s. Note, $h_\mu(s) \equiv \frac{\partial H_\mu(s)}{\partial s} = -f_H(s)\left[1 + \mu\left(\frac{f_L(s)}{f_H(s)} - \zeta\right)\right]$. Clearly $H_\mu(s_L) > 0$ and $H_\mu(s_H) = 0$. Since $\mu > 0$ we must have $h_\mu(s_L) < 0$. To see this suppose $h_\mu(s_L) \geq 0$. Then it must be the case that $1 + \mu\left(\frac{f_L(s)}{f_H(s)} - \zeta\right) \leq 0$. Since $\frac{f_L(s)}{f_H(s)}$ is decreasing, this implies that $h_\mu(s) > 0$ for all $s \in (s_L, s_H]$ contradicting that $H_\mu(s_H) = 0$.

Since $\frac{f_L(s)}{f_H(s)}$ is decreasing in s one of the following must be true:

(i) There exists a unique cutoff $\widehat{s}_\mu \in (s_L, s_H)$ such that $h_\mu(s) < 0$ for $s < \widehat{s}_\mu$ and $h_\mu(s) > 0$ for $s > \widehat{s}_\mu$,

(ii) $h_\mu(s) < 0$ for all $s \in (s_L, s_H)$.

In case (i) the function $H_\mu(s)$ crosses from positive to negative once, eventually increasing to zero at s_H. In case (ii) $H_\mu(s)$ decreases to zero at s_H. Let $s_\mu^* \in (s_L, s_H)$ be the unique s for which $H_\mu(s) = 0$ if it exists, otherwise let $s_\mu^* = s_H$.

Note that for given $\mu \geq 0$ optimal η_μ is given by:

$$\eta_\mu(s) = \begin{cases} H_\mu(s) & \text{if} \quad s \leq s_\mu^* \\ 0 & \text{if} \quad s > s_\mu^* \end{cases}.$$

Plugging this into the minimization problem we get:

$$\min_{\mu \geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{s_\mu^*} \left(\widetilde{F}_H(s) + \mu \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] \right) ds.$$

The first order condition for this problem is:

$$(1-\zeta)\phi + \int_{s_L}^{s_\mu^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] ds + \frac{\partial s_\mu^*}{\partial \mu} H_\mu\left(s_\mu^*\right) \geq 0$$

Because $H_\mu\left(s_\mu^*\right) = 0$,

$$(1-\zeta)\phi + \int_{s_L}^{s_\mu^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] ds \geq 0$$

with complementary slackness.

Let $s^* \in (s_L, s_H]$ be the unique s for which

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] ds = 0$$

if it exists. If

$$(1-\zeta)\phi + \int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] ds > 0$$

for all $s \in [s_L, s_H]$, then $s^* = s_H$.

If $s^* < s_H$ then $\mu > 0$, $s_\mu^* = s^*$, and

$$\mathcal{L}^* = \mu(1-\zeta)\phi + \int_{s_L}^{s^*} \left(\widetilde{F}_H(s) + \mu \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] \right) ds = \int_{s_L}^{s^*} \widetilde{F}_H(s) ds.$$

If $s^* = s_H$ then $\mu = 0$, $s_\mu^* = s_H$, and

$$\mathcal{L}^* = \int_{s_L}^{s_H} \widetilde{F}_H(s) ds.$$

To complete the proof note that $x(s) = 1$ for $s \in [s_L, s^*]$ and $x(s) = 0$ for $s \in [s^*, s_H]$ achieves the value $\mathcal{L}^*$ and it is feasible, and must be optimal for the original problem. □

A.3 Proof of Proposition 3

Proof. Observe that to maximize (23) agent O must set D as large as possible subject to satisfying the constraint (24). We first show that either there is a unique D that satisfies (24) with equality, or (24) is not binding. Let

$$
\begin{aligned}
\mathcal{T}(x) &\equiv (z-1)\left[\phi + s_L + \lambda \int_{s_L}^{x} \widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{x} \widetilde{F}_H(s)ds\right] - \lambda \int_{s_L}^{x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds \\
&= (z-1)\left[\phi + s_L + \int_{s_L}^{x} \widetilde{F}_H(s)ds\right] - z\lambda \int_{s_L}^{x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds.
\end{aligned}
$$

Observe that,

$$
\begin{aligned}
&\mathcal{T}(s_L) = (z-1)\left(\phi + s_L\right) > 0, \qquad && \mathcal{T}'(x) = (z-1)\widetilde{F}_H(x) - z\lambda\left[\widetilde{F}_H(x) - \widetilde{F}_L(x)\right], \\
&\mathcal{T}'(s_L) = z - 1 > 0, && \mathcal{T}'(s_H) = 0,
\end{aligned}
$$

$$
\mathcal{T}''(x) = -(z-1)f_H(x) + z\lambda\left[f_H(x) - f_L(x)\right] = f_H(x)\left[z(\lambda-1) + 1 - z\lambda\frac{f_L(x)}{f_H(x)}\right].
$$

When $\frac{f_L(x)}{f_H(x)}$ is monotonically decreasing in s, $\mathcal{T}(x)$ is quasi-concave with $\mathcal{T}(s_L) > 0$. So, there is either a unique D that satisfies $\mathcal{T}(D) = 0$ or $\mathcal{T}(x) > 0$ for all $x \in [s_L, s_H]$.

Case (i): Constraint (24) is binding. In this case the face value of the debt contract that solves the security design problem is given by:

$$
\phi = \frac{z}{z-1}\lambda \int_{s_L}^{D} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{D} \widetilde{F}_H(s)ds - s_L. \tag{A.1}
$$

In addition, the asset price ϕ satisfies (25). Substituting for q_D and q_E we rewrite (25) as:

$$
\phi = \frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{D}^{s_H} \widetilde{F}_H(s)ds\right\}. \tag{A.2}
$$

Substituting ϕ in (A.1) using (A.2), the equilibrium can be solved by a single equation of D, $\Gamma(D) = 0$, where

$$
\begin{aligned}
\Gamma(D) = &\frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{D}^{s_H} \widetilde{F}_H(s)ds\right\} \\
&- \frac{z}{z-1}\lambda \int_{s_L}^{D} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds + \int_{s_L}^{D} \widetilde{F}_H(s)ds + s_L
\end{aligned}
$$

Observe that:

$$
\begin{aligned}
\Gamma'(D) &= \frac{\beta}{1-\beta z}(1-\lambda)(z-1)\widetilde{F}_H(D) - \frac{z}{z-1}\lambda\left[\widetilde{F}_H(D) - \widetilde{F}_L(D)\right] + \widetilde{F}_H(D) \\
&= \left[\frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right]\widetilde{F}_H(D) + \frac{z}{z-1}\lambda\widetilde{F}_L(D). \\
\Gamma''(D) &= -\left[\frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right] f_H(D) - \frac{z}{z-1}\lambda f_L(D) \\
&= f_H(D)\left\{\frac{z}{z-1}\lambda\left[1 - \frac{f_L(D)}{f_H(D)}\right] - \frac{\beta}{1-\beta z}(1-\lambda)(z-1) - 1\right\} \\
\Gamma(s_L) &= s_L\left[1 + \frac{\beta}{1-\beta z}(1-\lambda)(z-1)\right] + \frac{\beta}{1-\beta z}\left[z\lambda E_L s + (1-\lambda)E_H s\right] > 0 \\
\Gamma'(s_L) &= \frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 > 0 \\
\Gamma'(s_H) &= 0.
\end{aligned}
$$

Once again $\Gamma(s)$ is quasi-concave if $\frac{f_L(D)}{f_H(D)}$ is monotonically decreasing in D. Because $\Gamma(s_L) > 0$, there is a unique equilibrium. The constraint (24) is binding iff $\Gamma(s_H) < 0$. We rewrite $\Gamma(s_H)$ as:

$$
\begin{aligned}
\Gamma(s_H) &= \frac{\beta z}{1-\beta z}\left[\lambda E_L s + (1-\lambda)E_H s\right] - \frac{z}{z-1}\lambda\int_{s_L}^{s_H}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right]ds + \int_{s_L}^{s_H}\widetilde{F}_H(s)ds + s_L \\
&= \frac{E_H s}{(1-\beta z)(z-1)}\left[\lambda z(1-\beta)\left(\frac{E_L s}{E_H s} - 1\right) + z - 1\right].
\end{aligned}
$$

Hence, $\Gamma(s_H) < 0$ iff

$$
\frac{E_L s}{E_H s} < 1 - \frac{z-1}{z\lambda(1-\beta)}.
$$

Case (ii): Constraint (24) is not binding. □

A.4 Proof of Proposition 4

Claim 1. Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal securities are

$$
\begin{aligned}
y_{1t}(s) &= \phi + s_L + (s - s_L)\mathbb{I}(s \leq D), \\
y_{2t}(s) &= (s - s_L)\mathbb{I}(s > D).
\end{aligned}
$$

for some $D \in (s_L, s_H]$.

Proof. The maximization

$$\arg\max_{x,m} \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds, \tag{A.3}$$

$$s.t. \int_{s_L}^{s} x(j)dj \le s - s_L, \forall s \in [s_L, s_H], \tag{A.4}$$

$$\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi \ge 0, \tag{A.5}$$

$$\int_{s_L}^{s} x(j)dj \ge 0, \forall s \in [s_L, s_H] \tag{A.6}$$

First note that the feasible set is compact, convex and nonempty so the optimization problem must have a solution. Moreover, since the objective function is bounded above, the solution must be finite. The Lagrangian of the optimization problem is

$$\begin{aligned}\mathcal{L}(x;\gamma,\mu,\nu) = &\int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds + \int_{s_L}^{s_H} \gamma(s)\left[s - s_L - \int_{s_L}^{s} x(j)dj\right] ds \\ &+ \mu\left\{\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi\right\} + \int_{s_L}^{s_H} \nu(s)\left[\int_{s_L}^{s} x(j)dj\right] ds.\end{aligned}$$

Note that for any feasible x and for $\gamma \ge 0$, $\mu \ge 0$ and $\nu \ge 0$ we have

$$\mathcal{L}(x;\gamma,\mu,\nu) \ge \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds.$$

Let $\mathcal{L}(\gamma,\mu,\nu) = \max_x \mathcal{L}(x;\gamma,\mu,\nu)$. Let $\mathcal{L}^* = \min_{\gamma\ge 0,\mu\ge 0,\mu_x\ge 0} \mathcal{L}(\gamma,\mu,\nu)$. Note that $\mathcal{L}^*$ is the value of the original optimization problem. We can rewrite $\mathcal{L}(x;\gamma,\mu,\nu)$ as

$$\begin{aligned}\mathcal{L} = &\int_{s_L}^{s_H} \left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] + \int_{s}^{s_H} [\nu(j) - \gamma(j)]\, dj\right\} x(s)ds \\ &+ \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \left(\int_{s}^{s_H} \gamma(j)dj\right) ds\end{aligned}$$

Now note that the quantity inside the curly brackets must be zero or otherwise the value of the optimization problem would be infinite. Consider the following dual problem of the optimization problem,

$$\begin{aligned}\min_{\mu\ge 0} \quad \min\nolimits_{\gamma\ge 0,\nu\ge 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H}\left(\int_{s}^{s_H} \gamma(j)dj\right) ds \\ s.t. \qquad & \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] + \int_{s}^{s_H} [\nu(j) - \gamma(j)]\, dj = 0.\end{aligned}$$

Note that the value of this problem is $\mathcal{L}^*$. Let $H_\mu(s) = \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]$. Let $\eta(s) = \int_s^{s_H} \gamma(j)dj$, $\xi(s) = \int_s^{s_H} \nu(j)dj$. We can rewrite the above problem one more time as:

$$\begin{aligned}\min_{\mu\ge 0} \quad \min\nolimits_{\eta,\xi\ge 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \eta(s)ds \\ s.t. \qquad & H_\mu(s) + \xi(s) - \eta(s) = 0.\end{aligned}$$

and the constraints that $\eta(s)$ and $\xi(s)$ are decreasing functions in s.

Note, $h_\mu(s) \equiv \frac{\partial H_\mu(s)}{\partial s} = -f_H(s)\left[1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right)\right]$. Clearly $H_\mu(s_L) > 0$ and $H_\mu(s_H) = 0$. Since $\mu > 0$ we must have $h_\mu(s_L) < 0$. To see this suppose $h_\mu(s_L) \geq 0$. Then it must be the case that $1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right) \leq 0$. Since $\frac{f_L(s)}{f_H(s)}$ is decreasing, this implies that $h_\mu(s) > 0$ for all $s \in (s_L, s_H]$ contradicting that $H_\mu(s_H) = 0$.

Since $\frac{f_L(s)}{f_H(s)}$ is decreasing in s one of the following must be true:

(i) There exists a unique cutoff $\widehat{s}_\mu \in (s_L, s_H)$ such that $h_\mu(s) < 0$ for $s < \widehat{s}_\mu$ and $h_\mu(s) > 0$ for $s > \widehat{s}_\mu$,

(ii) $h_\mu(s) < 0$ for all $s \in (s_L, s_H)$.

In case (i) the function $H_\mu(s)$ crosses from positive to negative once, eventually increasing to zero at s_H. In case (ii) $H_\mu(s)$ decreases to zero at s_H.

Note that for given $\mu \geq 0$ optimal η_μ and ξ_μ are given by:

$$\xi_\mu(s) = \begin{cases} -H_\mu(\hat{s}_\mu) & \text{if} \quad s \leq \hat{s}_\mu, \\ -H_\mu(s) & \text{if} \quad s > \hat{s}_\mu. \end{cases}$$

$$\eta_\mu(s) = \begin{cases} H_\mu(s) - H_\mu(\hat{s}_\mu) & \text{if} \quad s \leq \hat{s}_\mu, \\ 0 & \text{if} \quad s > \hat{s}_\mu. \end{cases}$$

This is because ξ_μ and η_μ must be decreasing in s. When $s > \hat{s}_\mu$, $H_\mu(s)$ is increasing. So it is feasible to let $\eta_\mu(s) = 0$ and $\xi_\mu(s) = -H_\mu(s)$ in this region. When $s < \hat{s}_\mu$, $H_\mu(s)$ is decreasing in s. The optimal η and ξ would be $\xi_\mu(s) = -H_\mu(\hat{s}_\mu)$ and $\eta_\mu(s) = H_\mu(s) - H_\mu(\hat{s}_\mu)$. Plugging this into the minimization problem we get:

$$\begin{aligned} &\min_{\mu\geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left(H_\mu(s) - H_\mu(\hat{s}_\mu)\right) ds \\ =&\min_{\mu\geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left\{\widetilde{F}_H(s) - \widetilde{F}_H(\hat{s}_\mu) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(\hat{s}_\mu) - \zeta\widetilde{F}_H(\hat{s}_\mu)\right)\right]\right\} ds \end{aligned}$$

The first order condition for this problem is:

$$\Gamma(\hat{s}_\mu) \equiv (1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(\hat{s}_\mu) - \zeta\widetilde{F}_H(\hat{s}_\mu)\right)\right] ds \geq 0$$

with complementary slackness.

$$\frac{\partial\Gamma(s^*)}{\partial s^*} = (s^* - s_L)f_H(s^*)\left(\frac{f_L(s^*)}{f_H(s^*)} - \zeta\right).$$

By definition of $\hat{s}_\mu$, $\frac{f_L(s^*)}{f_H(s^*)} - \zeta = -\frac{1}{\mu}$. So, $\frac{\partial \Gamma(s^*)}{\partial s^*} = -(s^* - s_L)\frac{f_H(s^*)}{\mu} < 0$. And $\Gamma(s_L) = (1-\zeta)\phi > 0$. Then, if there exists a solution for $\Gamma(s^*) = 0$, the solution is unique and it satisfies

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) - \left(\widetilde{F}_L(s^*) - \zeta \widetilde{F}_H(s^*)\right)\right] ds = 0$$

Otherwise,

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) - \left(\widetilde{F}_L(s^*) - \zeta \widetilde{F}_H(s^*)\right)\right] ds > 0$$

for all $s \in [s_L, s_H]$ and $s^* = s_H$.

If $s^* < s_H$ then $\mu > 0$, $\hat{s}_\mu = s^*$, and

$$\mathcal{L}^* = \int_{s_L}^{s^*} \left[\widetilde{F}_H(s) - \widetilde{F}_H(s^*)\right] ds.$$

If $s^* = s_H$ then $\mu = 0$, $s^*_\mu = s_H$, and

$$\mathcal{L}^* = \int_{s_L}^{s_H} \widetilde{F}_H(s) ds.$$

To complete the proof note that $\int_{s_L}^{s} x(j)dj = s - s_L$ for $s \in [s_L, s^*]$ and $\int_{s_L}^{s} x(j)dj = 0$ for $s \in [s^*, s_H]$ achieves the value $\mathcal{L}^*$ and it is feasible, and must be optimal for the original problem. □

Proof. Given Claim 1 that the optimal security design under Arrow securities has two tranches, the liquid tranche $y_{1t}(s)$ and illiquid tranche $y_{2t}(s)$.

$$\begin{aligned} y_{1t}(s) &= \phi + s_L + (s - s_L)\mathbb{I}(s \le D), \\ y_{2t}(s) &= (s - s_L)\mathbb{I}(s > D). \end{aligned}$$

The equilibrium is solved by the following two equations, representing the incentive constraint of an owner with high quality collateral and the Euler equation for the asset price. In the incentive constraint,

$$z\left[\phi + s_L + \int_{s_L}^{D} (s - s_L)\, dF_\lambda(s)\right] - \left[\phi + s_L + \int_{s_L}^{D} (s - s_L)\, dF_H(s)\right] \ge 0$$

One can easily verify that the left hand side of the incentive constraint is decreasing in D as long as the monotone likelihood ratio assumption holds. This confirms the conjecture of the optimal security design.

The Euler equation for the asset price is

$$\phi = \frac{\beta}{1 - \beta z}\left[z\left(s_L + \int (s - s_L) dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{D}^{s_H} (s - s_L)\, dF_H(s)\right]$$

The equilibrium value of D is determined by

$$0 = \Gamma(D) = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s-s_L)dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_D^{s_H}(s-s_L)\,dF_H(s)\right]$$
$$-\frac{s_L + \int_{s_L}^{D}(s-s_L)dF_H(s) - z\left[s_L + \int_{s_L}^{D}(s-s_L)dF_\lambda(s)\right]}{z-1}$$

$$\Gamma(s_L) = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s-s_L)dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{s_L}^{s_H}(s-s_L)\,dF_H(s)\right] + s_L > 0$$

$$\begin{aligned}\Gamma'(D) &= \frac{\beta(z-1)(1-\lambda)Df_H(D)}{1-\beta z} - \frac{Df_H(D) - zD\left[\lambda f_L(D) + (1-\lambda)f_H(D)\right]}{z-1}\\ &= Df_H(D)\left\{\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1 - z\left[\lambda f_L(D)/f_H(D) + (1-\lambda)\right]}{z-1}\right\}\\ &= Df_H(D)\left\{\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z(1-\lambda)}{z-1} + \frac{z\lambda f_L(D)/f_H(D)}{z-1}\right\}\end{aligned}$$

If $\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z(1-\lambda)}{z-1} < 0$, there exists a unique D^* such that $\Gamma'(D) > 0$ if and only if $D < D^*$. There exists a most one solution for the equation $\Gamma(D) = 0$.

$$\begin{aligned}\Gamma(s_H) &= \frac{\beta z}{1-\beta z}\left[(1-\lambda)\mathbb{E}_H s + \lambda \mathbb{E}_L s\right] - \frac{\left[1-(1-\lambda)z\right]\mathbb{E}_H s - z\lambda\mathbb{E}_L s}{z-1}\\ &= \mathbb{E}_H s\left[\frac{\beta z}{1-\beta z}(1-\lambda) - \frac{1-(1-\lambda)z}{z-1} + \left(\frac{\beta z}{1-\beta z}\lambda + \frac{z\lambda}{z-1}\right)\frac{\mathbb{E}_L s}{\mathbb{E}_H s}\right].\end{aligned}$$

The condition for there to be a unique $D \in (s_L, s_H)$ in equilibrium is

$$\begin{aligned}\frac{\mathbb{E}_L s}{\mathbb{E}_H s} &\le -\frac{\left(\frac{\beta z}{1-\beta z} + \frac{z}{z-1}\right)(1-\lambda) - \frac{1}{z-1}}{\left(\frac{\beta z}{1-\beta z} + \frac{z}{z-1}\right)\lambda}\\ &= 1 - \frac{z-1}{\lambda z\,(1-\beta)}.\end{aligned}$$

□

A.5 Proof of Proposition 5

Proof. Let $C_1(D) = z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_D^{s_H}\widetilde{F}_H(s)ds$ and $C_2 = z\lambda E_L s + (1-\lambda)E_H s$. Note that $C_1(D) > C_2$ for $D \in (s_L, s_H]$. From equations (43) and (41) we get:

$$\begin{aligned}\phi^S = &\quad \tfrac{\beta}{1-\beta z}\left[\tfrac{1-(1-\chi)\beta z}{1-\beta(\gamma-\chi)z}C_2 + \tfrac{\beta z(1-\gamma)}{1-\beta(\gamma-\chi)z}C_1(D)\right]\\ \phi = &\quad \tfrac{\beta}{1-\beta z}\left[\tfrac{\beta z\chi}{1-\beta(\gamma-\chi)z}C_2 + \tfrac{1-\beta z\gamma}{1-\beta(\gamma-\chi)z}C_1(D)\right]\end{aligned}$$

From the above equations it is immediate that $\phi > \phi^S$ proving (i). Letting

$$\Lambda = \frac{((1-\chi) - \beta(\gamma-\chi)z)}{1-\beta(\gamma-\chi)z}$$

we get

$$\begin{aligned}\phi(1-\chi)+\phi^S\chi =& \frac{\beta}{1-\beta z}\left[(1-\Lambda)C_2 + \Lambda C_1(D)\right] \\ =& \frac{\beta}{1-\beta z}\left[z\lambda E_L s + ((1-\Lambda)+\Lambda z)(1-\lambda)E_H s - \Lambda(1-\lambda)(z-1)\int_D^{s_H}\widetilde{F}_H(s)ds\right].\end{aligned}$$

Substituting into (42) we see that equilibrium can be solved by a single equation of D, $\Gamma(D)=0$, where

$$\begin{aligned}\Gamma(D) = \quad & \tfrac{\beta}{1-\beta z}\left[z\lambda E_L s + ((1-\Lambda)+\Lambda z)(1-\lambda)E_H s - \Lambda(1-\lambda)(z-1)\int_D^{s_H}\widetilde{F}_H(s)ds\right] \\ & -\tfrac{z}{z-1}\lambda\int_{s_L}^{D}\left[\widetilde{F}_H(s)-\widetilde{F}_L(s)\right]ds + \int_{s_L}^{D}\widetilde{F}_H(s)ds + s_L\end{aligned}$$

Observe that:

$$\begin{aligned}\Gamma'(D) &= \frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)\widetilde{F}_H(D) - \frac{z}{z-1}\lambda\left[\widetilde{F}_H(D)-\widetilde{F}_L(D)\right] + \widetilde{F}_H(D) \\ &= \left[\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right]\widetilde{F}_H(D) + \frac{z}{z-1}\lambda\widetilde{F}_L(D). \\ \Gamma''(D) &= -\left[\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)+1-\frac{z}{z-1}\lambda\right]f_H(D) - \frac{z}{z-1}\lambda f_L(D) \\ &= f_H(D)\left\{\frac{z}{z-1}\lambda\left[1-\frac{f_L(D)}{f_H(D)}\right] - \frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)-1\right\} \\ \Gamma(s_L) &= s_L\left[1+\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)\right] + \frac{\beta}{1-\beta z}\left[z\lambda E_L s + (1-\lambda)E_H s\right] > 0 \\ \Gamma'(s_L) &= \left[\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)+1\right] > 0 \\ \Gamma'(s_H) &= 0.\end{aligned}$$

$\Gamma(s)$ is quasi-concave if $\frac{f_L(D)}{f_H(D)}$ is monotonically decreasing in D. Because $\Gamma(s_L) > 0$, there is a unique equilibrium. In this unique equilibrium $D < s_H$ iff $\Gamma(s_H) < 0$. We write $\Gamma(s_H)$ as:

$$\begin{aligned}\Gamma(s_H) = \quad & \tfrac{\beta}{1-\beta z}\left[z\lambda E_L s + ((1-\Lambda)+\Lambda z)(1-\lambda)E_H s\right] \\ & -\tfrac{z}{z-1}\lambda\int_{s_L}^{s_H}\left[\widetilde{F}_H(s)-\widetilde{F}_L(s)\right]ds + \int_{s_L}^{s_H}\widetilde{F}_H(s)ds + s_L\end{aligned}$$

or

$$\begin{aligned}\Gamma(s_H) = \frac{E_H s}{(1-\beta z)(z-1)} \quad & \left[(\beta(z-1)((1-\Lambda)+\Lambda z)(1-\lambda) - z(1-\beta z)\lambda + (1-\beta z)(z-1))\right. \\ & \left. + (1-\beta)\lambda z\tfrac{E_L s}{E_H s}\right]\end{aligned}$$

Hence $\Gamma(s_H) < 0$ iff

$$\frac{E_L s}{E_H s} < \frac{(1-\beta z)(1-z(1-\lambda)) - \beta(z-1)((1-\Lambda)+\Lambda z)(1-\lambda)}{(1-\beta)\lambda z} \equiv \mathit{\Gamma}(\gamma,\chi).$$

It is easy to see that $\mathit{\Gamma}(\gamma,\chi) > 1-(z-1)/(z\lambda(1-\beta))$ iff $z>1$. Note that:

$$\frac{\partial\Gamma}{\partial\Lambda} = \quad \tfrac{\beta(z-1)(1-\lambda)}{1-\beta z}\left[E_H s - \textstyle\int_D^{s_H} \widetilde{F}_H(s)ds\right] \quad > 0.$$

This means that as Λ increases, Γ shifts up. As a result when (44), Γ crosses zero at a higher value, implying that the face value D is increasing in Λ.

Since no repo run equilibrium corresponds to $\Lambda = 1$, we see that debt threshold D is strictly lower with repo run than without (proving (ii)). As χ approaches one, Λ approaches zero and the unique equilibrium approaches the illiquid equilibrium when only equity is available as collateral asset (proving iv). Moreover,

$$\frac{\partial\Lambda}{\partial\gamma} = \frac{-\beta z\chi}{(1-\beta(\gamma-\chi)z)^2} < 0$$

and

$$\frac{\partial\Lambda}{\partial\chi} = \frac{\beta\gamma z - 1}{(1-\beta(\gamma-\chi)z)^2} < 0$$

Hence if probability of rigidity or sunspot increases, D decreases (proving (iii)). Claims (v) and (vii) are immediate.

For claim (vi), notice that

$$\begin{aligned} q_D^S &= \phi^S + s_L + \int_{s_L}^{D^S} \tilde{F}_L(s)ds \\ &= \phi + D - D^S + s_L + \int_{s_L}^{D^S} \tilde{F}_L(s)ds \\ &= \phi + D - \int_{s_L}^{D^S} F_L(s)ds. \end{aligned}$$

Because

$$\begin{aligned} q_D &= \phi + s_L + \int_{s_L}^{D} \mathbb{E}\tilde{F}_Q(s)ds \\ &= \phi + D - \int_{s_L}^{D} \mathbb{E}F_Q(s)ds, \end{aligned}$$

where $\mathbb{E}\tilde{F}_Q(s) = \lambda\tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)$, $\mathbb{E}F_Q(s) = \lambda F_L(s) + (1-\lambda)F_H(s)$. Because

$$\int_{s_L}^{D} \mathbb{E}F_Q(s)ds < \int_{s_L}^{D} F_L(s)ds < \int_{s_L}^{D^S} F_L(s)ds,$$

we have $q_D^S < q_D$.

$$\begin{aligned}
\frac{q_D}{\phi} &= \frac{\phi + s_L + \int_{s_L}^{D} \mathbb{E}\tilde{F}_Q(s)ds}{\phi} \\
&= 1 + \frac{s_L}{\phi} + \frac{\int_{s_L}^{D} \mathbb{E}\tilde{F}_Q(s)ds}{\phi}.
\end{aligned}$$

$$\begin{aligned}
\frac{q_D^S}{\phi^S} &= \frac{\phi + D - \int_{s_L}^{D^S} F_L(s)ds}{\phi + D - D^S} \\
&\approx 1 + \frac{D^S}{\phi} - \frac{\int_{s_L}^{D^S} F_L(s)ds}{\phi} \\
&= 1 + \frac{s_L}{\phi} + \frac{\int_{s_L}^{D^S} \tilde{F}_L(s)ds}{\phi}
\end{aligned}$$

Because $\tilde{F}_L(s) < \mathbb{E}\tilde{F}_Q(s)$ but $D^S > D$, haircut could either increase or decrease. $\square$