Representation of I(1) autoregressive Hilbertian processes

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Abstract

We extend the Granger-Johansen representation theory for I(1) vector autoregressive processes to accommodate processes that take values in an arbitrary complex separable Hilbert space. This more general setting is of central relevance for statistical applications involving functional time series. We obtain necessary and sufficient conditions for the existence of I(1) solutions to a given autoregressive law of motion generalizing the Johansen I(1) condition, and a characterization of such solutions. To accomplish this we obtain necessary and sufficient conditions for a pole in the inverse of a holomorphic index-zero Fredholm operator pencil to be simple, and a formula for its residue. In the case of first order autoregressive dynamics with a unit root our results take a particularly simple form, with the residue associated with the simple pole at one proportional to a Riesz projection.

1 Introduction

Results on the existence and representation of I(1) solutions to vector autoregressive laws of motion are among the most important and subtle contributions of econometricians to time series analysis, yet also among the most widely misunderstood. The best known such result is the so-called Granger representation theorem, which first appeared in an unpublished UC San Diego working paper of Granger (1983). In this paper, Granger, having recently introduced the concept of cointegration (Granger, 1981) sought to connect statistical models of time series based on linear process representations to regression based models more commonly employed in econometrics. The main result of Granger (1983) first emerged in published form in Granger (1986) without proof, but more prominently in the widely cited 1987 *Econometrica* article by Engle and Granger, where it is labeled the "Granger representation theorem", with the exclusion of the first author presumably due to the paper having resulted from the merger of previous independent contributions.

The proof of the Granger representation theorem in Engle and Granger (1987) is incorrect. Moreover, the error can be traced back to the original working paper of Granger (1983). A regrettably belated counterexample to Lemma A1 of Engle and Granger (1987), which is also Theorem 1 of Granger (1983), may be found buried in a footnote of Johansen (2009). Johansen was familiar with Granger's work on representation theory at an early stage, authoring a closely related Johns Hopkins working paper in 1985 that was eventually published as Johansen (1988). More pertinently, Johansen (1991) provided what appears to be the first correct statement and proof of a modified version of the Granger representation theorem. This contribution did not merely correct a technical error of Granger; it reoriented attention toward a central issue: when does a given vector autoregressive law of motion admit an I(1) solution? The answer to this question is given by the Johansen I(1) condition, which is a necessary and sufficient condition on the autoregressive polynomial and its first derivative at one for a vector autoregressive law of motion to be satisfied by an I(1) process.

A relatively unknown paper of Schumacher (1991)—the sole published citation is Kuijper and Schumacher (1992)—contains a striking observation on the Johansen I(1) condition: it corresponds to a necessary and sufficient condition for the inverse of a holomorphic matrix pencil to have a simple pole at a given point in the complex plane. Various authors later rediscovered and exploited this insight. In particular, Faliva and Zoia (2002, 2009, 2011) have used it to provide a systematic reworking of Granger-Johansen representation theory through the lens of analytic function theory. A nice aspect of this approach is that it extends naturally to the development of I(d) representation theory with integral $d \geq 2$: just as the Johansen I(1) condition can be reformulated as a necessary and sufficient condition for a simple pole, analogous I(d) conditions can be reformulated from necessary and sufficient conditions for poles of order d. Franchi and Paruolo (2016) have recently taken precisely this approach to develop a general I(d) representation theory.

In this paper we extend the Granger-Johansen representation theory for I(1) vector autoregressive processes to accommodate processes that take values in an arbitrary complex separable Hilbert space. This more general setting is of central relevance for statistical applications involving functional time series (Hörmann and Kokoszka, 2012). In the spirit of Faliva and Zoia, we commence by obtaining a suitable extension of the analytic function theory underlying the Granger-Johansen theory. Specifically, we obtain necessary and sufficient conditions for a pole in the inverse of a holomorphic index-zero Fredholm operator pencil to be simple, and a formula for its residue. We then apply this result to obtain necessary and sufficient conditions for the existence of I(1) solutions to a given autoregressive law of motion in a complex separable Hilbert space generalizing the Johansen I(1) condition, and a characterization of such solutions.

Some preliminary results on extending the Granger-Johansen representation theory to a Hilbert space context have been provided by Beare and Seo (2016). In particular, Theorem 4.2 in that paper deals with the special case where we have a first order autoregressive law of motion and a compact self-adjoint autoregressive operator. This case is simpler to deal with because the spectral theorem may be brought to bear upon the problem. The more general case dealt with here is significantly more involved.

We structure the remainder of the paper as follows. Section 2 sets the scene with notation, definitions and some essential mathematics. Section 3 contains our results providing necessary and sufficient conditions for a simple pole in the inverse of a holomorphic index-zero Fredholm operator pencil. Section 4 contains our extension of the Granger-Johansen I(1) representation theory to a Hilbert space context. We conclude with some examples in Section 5.

2 Preliminaries

2.1 Notation

Let H denote a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. At times we will require H to be separable. Given a set $G \subseteq H$, let $G^\perp$ denote the orthogonal complement to G, and let $\operatorname{cl} G$ denote the closure of G. Let $\mathcal{L}_H$ denote the Banach space of continuous linear operators from H to H with operator norm $\|A\|_{\mathcal{L}_H} = \sup_{\|x\|\leq 1} \|A(x)\|$. Let $A^* \in \mathcal{L}_H$ denote the adjoint of an operator $A \in \mathcal{L}_H$. Let $\mathrm{id}_H \in \mathcal{L}_H$ denote the identity map on H. Given a vector subspace $V \subseteq H$, let $\mathrm{P}_V \in \mathcal{L}_H$ denote the orthogonal projection on V, and let $A\restriction_V$ denote the restriction of an operator $A \in \mathcal{L}_H$ to V. Given two vector subspaces $V, W \subseteq H$ with $V \cap W = \{0\}$, let $V \oplus W$ denote their direct sum. When we write $V \oplus W$ for vector subspaces $V, W \subseteq H$, it is implied that $V \cap W = \{0\}$.

2.2 Four fundamental subspaces

Given an operator $A \in \mathcal{L}_H$, we define four vector subspaces of H as follows.

$$\ker A = \{x \in H : A(x) = 0\} \tag{2.1}$$
$$\operatorname{coker} A = \{x \in H : A^*(x) = 0\} \tag{2.2}$$
$$\operatorname{ran} A = \{A(x) : x \in H\} \tag{2.3}$$
$$\operatorname{coran} A = \{A^*(x) : x \in H\}. \tag{2.4}$$

These four fundamental subspaces are called, respectively, the kernel, cokernel, range and corange of A. They are related to one another in the following way (see e.g. Conway, 1990, pp. 35–36):

$$\ker A = (\operatorname{coran} A)^\perp, \qquad \operatorname{coker} A = (\operatorname{ran} A)^\perp, \tag{2.5}$$
$$\operatorname{cl}\operatorname{ran} A = (\operatorname{coker} A)^\perp, \qquad \operatorname{cl}\operatorname{coran} A = (\ker A)^\perp. \tag{2.6}$$

We refer to this collection of four relations as the strong rank-nullity theorem.

2.3 Fredholm operators

An operator $A \in \mathcal{L}_H$ is said to be a Fredholm operator if (a) $\operatorname{ran} A$ and $\operatorname{coran} A$ are closed, and (b) $\ker A$ and $\operatorname{coker} A$ are finite dimensional. The index of a Fredholm operator A is the integer

$$\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A, \tag{2.7}$$

where dim indicates dimension. An index-zero Fredholm operator A satisfies what is known as the Fredholm alternative: either A is invertible, or $\dim \ker A > 0$. It can be shown that if $K \in \mathcal{L}_H$ is compact, then $\mathrm{id}_H + K$ is Fredholm of index zero. See Conway (1990, ch. XI) or Gohberg et al. (1990, ch. XI) for more on Fredholm operators.

2.4 Operator pencils

An operator pencil is a map $A : U \to \mathcal{L}_H$, where U is some open connected subset of $\mathbb{C}$. We say that an operator pencil A is holomorphic on an open connected set $D \subseteq U$ if, for each $z_0 \in D$, the limit

$$A^{(1)}(z_0) \coloneqq \lim_{z \to z_0} \frac{A(z) - A(z_0)}{z - z_0} \tag{2.8}$$

exists in the norm of $\mathcal{L}_H$. It can be shown (Gohberg et al., 1990, pp. 7–8) that holomorphicity on D in fact implies analyticity on D, meaning that, for every $z_0 \in D$, we may represent A on D in terms of a power series

$$A(z) = \sum_{k=0}^{\infty} (z - z_0)^k A_k, \quad z \in D, \tag{2.9}$$

where $A_0, A_1, \ldots$ is a sequence in $\mathcal{L}_H$ not depending on z.

The set of points $z \in U$ at which the operator $A(z)$ is noninvertible is called the spectrum of A, and denoted $\sigma(A)$. The spectrum is always a closed set, and if A is holomorphic on U, then $A(z)^{-1}$ depends holomorphically on $z \in U \setminus \sigma(A)$ (Markus, 2012, p. 56). A lot more can be said about $\sigma(A)$ and the behavior of $A(z)^{-1}$ if we assume that $A(z)$ is a Fredholm operator for every $z \in U$. In this case we have the following result, a proof of which may be found in Gohberg et al. (1990, pp. 203–204). It is a crucial input to our main results.

Analytic Fredholm Theorem. *Let $A : U \to \mathcal{L}_H$ be a holomorphic Fredholm operator pencil, and assume that $A(z)$ is invertible for some $z \in U$. Then $\sigma(A)$ is at most countable and has no accumulation point in U. Furthermore, for $z_0 \in \sigma(A)$ and $z \in U \setminus \sigma(A)$ sufficiently close to z_0, we have*

$$A(z)^{-1} = \sum_{k=-m}^{\infty} (z - z_0)^k N_k, \tag{2.10}$$

where $m \in \mathbb{N}$ and $N_{-m}, N_{-m+1}, \ldots$ is a sequence in $\mathcal{L}_H$ not depending on z. The operator N_0 is Fredholm of index zero and the operators $N_{-m}, \ldots, N_{-1}$ are finite rank.

The analytic Fredholm theorem tells us that $A(z)^{-1}$ is holomorphic except at a discrete set of points, which are poles. The technical term for this property of $A(z)^{-1}$ is meromorphicity. In the Laurent series given in (2.10), if we assume without loss of generality that $N_{-m} \neq 0$, then the integer m is the order of the pole of $A(z)^{-1}$ at z_0. A pole of order one is said to be simple, and in this case the corresponding residue is N_{-m}.

For further reading on operator pencils we suggest Gohberg et al. (1990) and Markus (2012).

2.5 Random elements of Hilbert space

In this subsection we require H to be separable. The concepts and notation introduced will not be used until Section 4.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random element of H is a Borel measurable map $Z : \Omega \to H$. Noting that $\|Z\|$ is a real valued random variable, we say that Z is integrable if $E\|Z\| < \infty$, and in this case there exists a unique element of H, denoted EZ, such that $E\langle Z, x\rangle = \langle EZ, x\rangle$ for all $x \in H$. We call EZ the expected value of Z.

Let L^2_H denote the Banach space of random elements Z of H (identifying random elements that are equal with probability one) that satisfy $E\|Z\|^2 < \infty$ and $EZ = 0$, equipped with the norm $\|Z\|_{L^2_H} = (E\|Z\|^2)^{1/2}$. For each $Z \in L^2_H$ it can be shown that $Z\langle z, Z\rangle$ is integrable for all $x \in H$. The operator $C_Z \in \mathcal{L}_H$ given by

$$C_Z(x) = E(Z\langle x, Z\rangle), \quad x \in H, \tag{2.11}$$

is called the covariance operator of Z. It is guaranteed to be positive semidefinite, compact and self-adjoint.

The monograph of Bosq (2000) provides a detailed treatment of time series taking values in a real Hilbert or Banach space. A complex Hilbert space setting was studied more recently by Cerovecki and Hörmann (2017).

3 Necessary and sufficient conditions for a simple pole

The following result provides necessary and sufficient conditions for a pole in the inverse of a holomorphic index-zero Fredholm operator pencil to be simple, and a formula for its residue. Some remarks follow the proof.

Theorem 3.1. *For an open connected set $U \subseteq \mathbb{C}$, let $A : U \to \mathcal{L}_H$ be a holomorphic index-zero Fredholm operator pencil. Suppose that $A(z)$ is not invertible at $z = z_0 \in U$ but is invertible at some other point in U. Then the following three conditions are equivalent.*

(1) *$A(z)^{-1}$ has a simple pole at $z = z_0$.*

(2) *The map $B : \ker A(z_0) \to \operatorname{coker} A(z_0)$ given by*

$$B(x) = \mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0)(x), \quad x \in \ker A(z_0), \tag{3.1}$$

is bijective.

(3) *$H = \operatorname{ran} A(z_0) \oplus A^{(1)}(z_0) \ker A(z_0)$.*

Under any of these conditions, the residue of $A(z)^{-1}$ at $z = z_0$ is the operator

$$H \ni x \mapsto B^{-1} \mathrm{P}_{\operatorname{coker} A(z_0)}(x) \in H. \tag{3.2}$$

Proof. The analytic Fredholm theorem implies that $A(z)^{-1}$ is holomorphic on a punctured neighborhood $D \subset U$ of z_0 with a pole at z_0, and for $z \in D$ admits the Laurent series expansion

$$A(z)^{-1} = \sum_{k=-m}^{\infty} N_k (z - z_0)^k, \tag{3.3}$$

where $m \in \mathbb{N}$ is the order of the pole at z_0, and $N_k \in \mathcal{L}_H$ for $k \geq -m$, with $N_{-m} \neq 0$. The operator pencil A is holomorphic on $D \cup \{z_0\}$ and thus for $z \in D$ admits the Taylor series expansion

$$A(z) = \sum_{k=0}^{\infty} \frac{1}{k!} A^{(k)}(z_0)(z - z_0)^k. \tag{3.4}$$

Combining (3.3) and (3.4) we obtain, for $z \in D$,

$$\mathrm{id}_H = \left(\sum_{k=-m}^{\infty} N_k (z - z_0)^k \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^{(k)}(z_0)(z - z_0)^k \right) \tag{3.5}$$

$$= \sum_{k=-m}^{\infty} \left(\sum_{j=0}^{m+k} \frac{1}{j!} N_{k-j} A^{(j)}(z_0) \right) (z - z_0)^k. \tag{3.6}$$

We will show that (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3), in that order. Suppose that condition (1) is false, meaning that $m > 1$. Then the coefficients of $(z-z_0)^{-m}$ and $(z-z_0)^{-m+1}$ in the expansion of the identity in (3.6) must be zero. That is,

$$N_{-m}A(z_0) = 0 \tag{3.7}$$

and

$$N_{-m+1}A(z_0) + N_{-m}A^{(1)}(z_0) = 0. \tag{3.8}$$

Equation (3.7) implies that $N_{-m} \operatorname{ran} A(z_0) = \{0\}$, while equation (3.8) implies that $N_{-m}A^{(1)}(z_0) \ker A(z_0) = \{0\}$. If the direct sum decomposition in condition (3) were valid, we could conclude that $N_{-m} = 0$; however, this is impossible since N_{-m} is the leading coefficient in the Laurent series (3.3), which is nonzero by construction. Thus if condition (3) is true then condition (1) must also be true: (3) $\Rightarrow$ (1).

We next show that (1) $\Rightarrow$ (2). Suppose that (1) is true, meaning that $m = 1$. The coefficients of $(z-z_0)^{-1}$ and $(z-z_0)^0$ in the expansion of the identity in (3.6) must be equal to 0 and id_H respectively. Since $m = 1$, this means that

$$N_{-1}A(z_0) = 0 \tag{3.9}$$

and

$$N_0 A(z_0) + N_{-1}A^{(1)}(z_0) = \mathrm{id}_H. \tag{3.10}$$

It is apparent from (3.10) that $N_{-1}A^{(1)}(z_0)\restriction_{\ker A(z_0)} = \mathrm{id}_H\restriction_{\ker A(z_0)}$. Consequently, applying the orthogonal projection decomposition $\mathrm{id}_H = \mathrm{P}_{\operatorname{ran} A(z_0)} + \mathrm{P}_{(\operatorname{ran} A(z_0))^\perp}$, we find that

$$\mathrm{id}_H\restriction_{\ker A(z_0)} = N_{-1}\mathrm{P}_{\operatorname{ran} A(z_0)}A^{(1)}(z_0)\restriction_{\ker A(z_0)} + N_{-1}\mathrm{P}_{(\operatorname{ran} A(z_0))^\perp}A^{(1)}(z_0)\restriction_{\ker A(z_0)}. \tag{3.11}$$

Equation (3.9) implies that $N_{-1}\mathrm{P}_{\operatorname{ran} A(z_0)} = 0$, while the strong rank-nullity theorem asserts that $(\operatorname{ran} A(z_0))^\perp = \operatorname{coker} A(z_0)$. Equation (3.11) thus reduces to

$$\mathrm{id}_H\restriction_{\ker A(z_0)} = N_{-1}\mathrm{P}_{\operatorname{coker} A(z_0)}A^{(1)}(z_0)\restriction_{\ker A(z_0)}. \tag{3.12}$$

This shows that N_{-1} is the left-inverse of $\mathrm{P}_{\operatorname{coker} A(z_0)}A^{(1)}(z_0)\restriction_{\ker A(z_0)}$, implying that $\mathrm{P}_{\operatorname{coker} A(z_0)}A^{(1)}(z_0)\restriction_{\ker A(z_0)}$ is injective. If we reduce the codomain of this injection to its range, the resulting bijection is the map B, provided that

$$\mathrm{P}_{\operatorname{coker} A(z_0)}A^{(1)}(z_0) \ker A(z_0) = \operatorname{coker} A(z_0). \tag{3.13}$$

To see why (3.13) is true, observe that $\mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0)\restriction_{\ker A(z_0)}$ is an isomorphism between the vector spaces $\ker A(z_0)$ and $\mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0) \ker A(z_0)$. Isomorphic vector spaces have the same dimension, so

$$\dim \mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0) \ker A(z_0) = \dim \ker A(z_0). \tag{3.14}$$

Since $A(z_0)$ is Fredholm of index zero, $\dim \ker A(z_0) = \dim \operatorname{coker} A(z_0) < \infty$. Thus we see that the vector spaces $\mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0) \ker A(z_0)$ and $\operatorname{coker} A(z_0)$ have the same finite dimension. The former vector space is a subset of the latter, so equality (3.13) holds. Thus we have shown that (1) $\Rightarrow$ (2).

We next show that (2) $\Rightarrow$ (3). Suppose that (2) is true. Establishing the direct sum decomposition in condition (3) amounts to showing that

$$H = \operatorname{ran} A(z_0) + A^{(1)}(z_0) \ker A(z_0) \tag{3.15}$$

and that

$$\operatorname{ran} A(z_0) \cap A^{(1)}(z_0) \ker A(z_0) = \{0\}. \tag{3.16}$$

Condition (2) implies that $\mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0) \ker A(z_0) = \operatorname{coker} A(z_0)$, and the strong rank-nullity theorem asserts that $\operatorname{coker} A(z_0) = (\operatorname{ran} A(z_0))^{\perp}$. Since $H = \operatorname{ran} A(z_0) + (\operatorname{ran} A(z_0))^{\perp}$, we therefore have

$$H = \operatorname{ran} A(z_0) + \mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0) \ker A(z_0). \tag{3.17}$$

The fact that $\mathrm{P}_{\operatorname{coker} A(z_0)}$ is an orthogonal projection operator means that every element of $\mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0) \ker A(z_0)$ can be written as the sum of an element of $A^{(1)}(z_0) \ker A(z_0)$ and an element of $(\operatorname{coker} A(z_0))^{\perp}$. But

$$(\operatorname{coker} A(z_0))^{\perp} = \operatorname{cl} \operatorname{ran} A(z_0) = \operatorname{ran} A(z_0), \tag{3.18}$$

with the first equality following from the strong rank-nullity theorem and the second equality following from the fact that $A(z_0)$ is Fredholm. Thus every element of H can be written as the sum of an element of $\operatorname{ran} A(z_0)$ and an element of $A^{(1)}(z_0) \ker A(z_0)$, and (3.15) is proved. To establish (3.16) we observe that any element $x \in \operatorname{ran} A(z_0) \cap A^{(1)}(z_0) \ker A(z_0)$ may be written as $x = A^{(1)}(z_0)(y)$ for some $y \in \ker A(z_0)$. Orthogonally projecting both sides of this equality on $(\operatorname{ran} A(z_0))^{\perp} = \operatorname{coker} A(z_0)$ gives $0 = \mathrm{P}_{\operatorname{coker} A(z_0)} A^{(1)}(z_0)(y)$. The bijectivity of B asserted by condition (2) thus requires us to have $y = 0$, implying that $x = 0$. Thus (3.16) is proved, and we have shown that (2) $\Rightarrow$ (3).

It remains only for us to show that when $A(z)^{-1}$ has a simple pole at $z = z_0$, its residue is of the stated form. The residue is the operator N_{-1} in the Laurent series in (3.3), with $m = 1$. Composing both sides of (3.12) with $B^{-1}\mathrm{P}_{\operatorname{coker} A(z_0)}$, we obtain

$$N_{-1}\mathrm{P}_{\operatorname{coker} A(z_0)} = \mathrm{id}_H\restriction_{\ker A(z_0)} B^{-1}\mathrm{P}_{\operatorname{coker} A(z_0)}. \tag{3.19}$$

The operator on the right-hand side of equality (3.19) is the claimed residue. We therefore need to show that $N_{-1}\mathrm{P}_{\operatorname{coker} A(z_0)} = N_{-1}$. This is accomplished by observing that $N_{-1}\mathrm{P}_{(\operatorname{coker} A(z_0))^\perp} = N_{-1}\mathrm{P}_{\operatorname{ran} A(z_0)} = 0$, with the first equality following from (3.18) and the second from (3.9). □

Remark 3.1. The closest results we have found to Theorem 3.1 in prior literature are those of Steinberg (1968) and Howland (1971). These authors worked in a more general Banach space setting, but also required that $A(z) = \mathrm{id}_H + K(z)$ for some compact operator $K(z)$, which is more restrictive than requiring $A(z)$ to be Fredholm of index zero. Steinberg (1968) established sufficient conditions for a simple pole, and Howland (1971) established the equivalence of conditions (1) and (3).

Remark 3.2. Our requirement that the Fredholm operator $A(z)$ be of index zero cannot be dispensed with, at least not for $z = z_0$, without making it impossible to satisfy condition (2). This is because bijectivity of B requires its domain and codomain to have the same dimension. However, our proof that condition (3) implies condition (1) does not use the index-zero property.

Remark 3.3. Condition (2) is the one that most closely resembles the condition used in the classical case $H = \mathbb{C}^n$ to establish I(1) representation theory. That condition, as given by Schumacher (1986, 1991), is as follows. Let $A(z)$ be an $n \times n$ matrix depending holomorphically on z in a neighborhood of z_0, nonsingular except at z_0, where it has rank $n - r$. Let α and β be $n \times r$ matrices of full column rank such that $\alpha' A(z_0) = 0$ and $A(z_0)\beta = 0$. Then $A(z)^{-1}$ has a simple pole at z_0 if and only if $\alpha' A^{(1)}(z_0)\beta$ is invertible. To relate this condition to our condition (2), observe that since $\mathbb{C}^r$, $\ker A(z_0)$ and $\operatorname{coker} A(z_0)$ are isomorphic we may view α as a linear map from $\operatorname{coker} A(z_0)$ to $\mathbb{C}^n$ and β as a linear map from $\ker A(z_0)$ to $\mathbb{C}^n$, and set $\alpha = \mathrm{id}_{\mathbb{C}^n}\restriction_{\operatorname{coker} A(z_0)}$ and $\beta = \mathrm{id}_{\mathbb{C}^n}\restriction_{\ker A(z_0)}$. Noting that α^* is the orthogonal projection operator $\mathrm{P}_{\operatorname{coker} A(z_0)}$ with codomain restricted to $\operatorname{coker} A(z_0)$, we find that $B = \alpha^* A^{(1)}(z_0)\beta$. Thus the bijectivity of B asserted in our condition (2) is equivalent to the invertibility of $\alpha' A^{(1)}(z_0)\beta$ asserted in the classical condition.

Condition (3) of Theorem 3.1, which first appears in Howland (1971), provides a purely geometric necessary and sufficient condition for a simple pole. In the special case where our operator pencil is not merely holomorphic and Fredholm of index zero but is in fact of the form $A(z) = \mathrm{id}_H - zK$ with $K \in \mathcal{L}_H$ compact, this geometric condition takes on a particularly simple form. Moreover, the direct sum decomposition it asserts serves to define an oblique projection that is a scalar multiple of the residue of our simple pole. The following corollary to Theorem 3.1 provides details.

Corollary 3.1. *Let $K \in \mathcal{L}_H$ be compact, and consider the operator pencil $A(z) = \mathrm{id}_H - zK$, $z \in \mathbb{C}$. If $A(z)$ is not invertible at $z = z_0 \in \mathbb{C}$ then the following two conditions are equivalent.*

(1) *$A(z)^{-1}$ has a simple pole at $z = z_0$.*

(2) *$H = \operatorname{ran} A(z_0) \oplus \ker A(z_0)$.*

Under either of these conditions, the residue of $A(z)^{-1}$ at $z = z_0$ is the projection on $\ker A(z_0)$ along $\operatorname{ran} A(z_0)$, scaled by $-z_0$.

Proof. The assumptions of Theorem 3.1 are satisfied by $A(z)$, so we need only show that $A^{(1)}(z_0) \ker A(z_0) = \ker A(z_0)$ and that, when our two equivalent conditions are satisfied, the operator defined in (3.2) corresponds to projection on $\ker A(z_0)$ along $\operatorname{ran} A(z_0)$ scaled by $-z_0$. The former claim follows from the fact that $A^{(1)}(z_0) = -K$ and that $K(x) = z_0^{-1}x$ for all $x \in \ker A(z_0)$. To see why the latter claim is true, it may be helpful to consult Figure 3.1, which provides a visual aid to the arguments that follow. First note that the operator B given in the statement of Theorem 3.1 reduces in this case to the map

$$\ker A(z_0) \ni x \mapsto -z_0^{-1} \mathrm{P}_{\operatorname{coker} A(z_0)}(x) \in \operatorname{coker} A(z_0). \tag{3.20}$$

The inverse operator B^{-1} therefore sends an element $x \in \operatorname{coker} A(z_0)$ to the point in $\ker A(z_0)$ whose orthogonal projection on $\operatorname{coker} A(z_0)$ is $-z_0 x$, which is uniquely defined due to the bijectivity of B established in Theorem 3.1. The action of the residue given in (3.2) upon any element $x \in H$ can therefore be decomposed as follows: we first orthogonally project x upon $\operatorname{coker} A(z_0)$, obtaining $y = \mathrm{P}_{\operatorname{coker} A(z_0)}(x)$; then we map y to the unique point in $\ker A(z_0)$ whose orthogonal projection upon $\operatorname{coker} A(z_0)$ is y; and finally we scale by $-z_0$. This is equivalent to projecting x on $\ker A(z_0)$ along the orthogonal complement to $\operatorname{coker} A(z_0)$, and then scaling by $-z_0$. Our claim about the residue of $A(z)^{-1}$ at $z = z_0$ now follows from (3.18). □

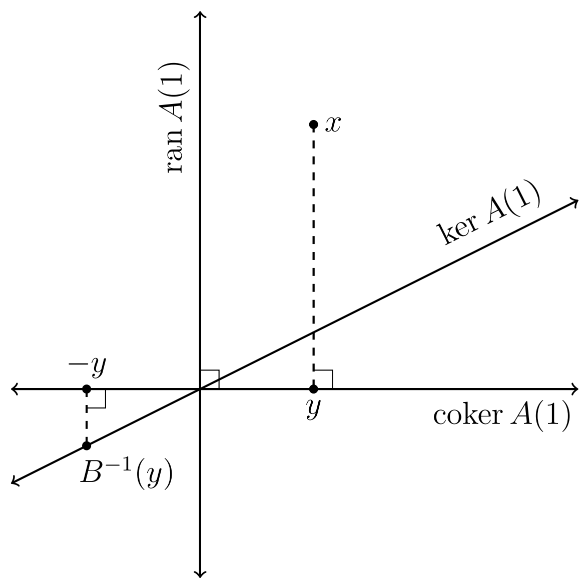

Figure 3.1: Visual aid to the proof of Corollary 3.1, with $z_0 = 1$.

Remark 3.4. The oblique projection appearing in Corollary 3.1 is in fact the Riesz projection for the eigenvalue $\sigma = z_0^{-1}$ of K. Said Riesz projection is defined (Gohberg et al., 1990, p. 9; Markus, 2012, pp. 11–12) by the contour integral

$$P_{K,\sigma} = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} (z\mathrm{id}_H - K)^{-1}\mathrm{d}z, \tag{3.21}$$

where Γ is a positively oriented smooth Jordan curve around σ separating it from zero and from any other eigenvalues of K, and where the integral of an $\mathcal{L}_H$-valued function should be understood in the sense of Bochner. It is known that $P_{K,\sigma}$ is a projection on the eigenspace associated with the eigenvalue σ, which in our case is $\ker A(z_0)$. Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth parametrization of Γ, and rewrite (3.21) as

$$P_{K,\sigma} = \frac{1}{2\pi\mathrm{i}} \int_0^1 (\gamma(t)\mathrm{id}_H - K)^{-1}\gamma'(t)\mathrm{d}t. \tag{3.22}$$

The image of Γ under the reciprocal transform $z \mapsto z^{-1}$, which we denote Γ', is a positively oriented smooth Jordan curve around z_0 separating it from any other poles of $A(z)^{-1}$ and from zero. It admits the parametrization $t \mapsto 1/\gamma(t) =: \delta(t)$. A little calculus shows that $\gamma'(t) = -\delta'(t)/\delta(t)^2$, and so from (3.22) we have

$$P_{K,\sigma} = \frac{-1}{2\pi\mathrm{i}} \int_0^1 \delta(t)^{-1}(\mathrm{id}_H - \delta(t)K)^{-1}\delta'(t)\mathrm{d}t = \frac{-1}{2\pi\mathrm{i}} \oint_{\Gamma'} z^{-1}A(z)^{-1}\mathrm{d}z. \quad (3.23)$$

The residue theorem therefore tells us that $P_{K,\sigma}$ is the negative of the residue of $z^{-1}A(z)^{-1}$ at $z = z_0$, implying that the residue of $A(z)^{-1}$ at $z = z_0$ is $-z_0 P_{K,\sigma}$. It now follows from Corollary 3.1 that when the direct sum decomposition $H = \operatorname{ran} A(z_0) \oplus \ker A(z_0)$ is satisfied, the Riesz projection $P_{K,\sigma}$ is the projection on $\ker A(z_0)$ along $\operatorname{ran} A(z_0)$.

4 Representation theory

In this section we require H to be separable.

4.1 I(1) sequences in L^2_H

We require a suitable extension of the notion of an I(1) process to the present Hilbert space context. As in Beare and Seo (2016), we say that a sequence $X = (X_t, t \geq 0)$ in L^2_H is I(1) if its first differences $\Delta X_t = X_t - X_{t-1}$ satisfy

$$\Delta X_t = \sum_{k=0}^{\infty} A_k(\varepsilon_{t-k}) \quad (4.1)$$

for all $t \geq 1$, where $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ is an iid sequence in L^2_H with positive definite covariance operator Σ, and $(A_k, k \geq 0)$ is a sequence in $\mathcal{L}_H$ satisfying $\sum_{k=0}^{\infty} k\|A_k\|_{\mathcal{L}_H} < \infty$ and $\sum_{k=0}^{\infty} A_k \neq 0$. That the series in (4.1) is convergent in L^2_H follows easily from the summability condition on the norms of the operator coefficients, which we refer to as 1-summability.

Given an I(1) sequence X in L^2_H, we define the cointegrating space of X to be the collection of all $x \in H$ such that the sequence of inner products $(\langle X_t, x\rangle, t \geq 0)$ is a stationary sequence of complex valued random variables. Clearly, the cointegrating space of an I(1) sequence is a vector subspace of H. Beare and Seo (2016, Prop. 3.1) showed that, given a sequence of differences $\Delta X = (\Delta X_t, t \geq 1)$ in L^2_H, the set of $x \in H$ such that $(\langle X_t, x\rangle, t \geq 0)$ can be made stationary by a suitable choice of the initial condition $X_0 \in L^2_H$ is precisely the kernel of the long run covariance operator of ΔX. That result was proved for a real Hilbert space H, but extends easily to the case of a complex Hilbert space.

4.2 Main results

Let $D_r \subset \mathbb{C}$ denote the open disk centered at zero with radius r. Our main result follows. It establishes necessary and sufficient conditions for an autoregressive Hilbertian law of motion to admit an I(1) solution.

Theorem 4.1. *Let $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ be an iid sequence in L^2_H with positive definite covariance operator $\Sigma \in \mathcal{L}_H$. Fix $p \in \mathbb{N}$, and let $\Phi_1, \ldots, \Phi_p \in \mathcal{L}_H$ be compact with $\Phi_p \neq 0$. Consider the operator pencil $A : \mathbb{C} \to \mathcal{L}_H$ given by $\Phi(z) = \mathrm{id}_H - \sum_{j=1}^{p} z^j \Phi_j$. Suppose that $\Phi(z)$ is noninvertible at $z = 1$ and invertible at every other z in the closed unit disk. Then $\Phi(z)$ is invertible on $D_{1+\eta} \setminus \{1\}$ for some $\eta > 0$, and $\Phi(z)^{-1}$ is holomorphic on $D_{1+\eta} \setminus \{1\}$, with a pole at $z = 1$. Moreover, the following four conditions are equivalent.*

(1) *$\Phi(z)^{-1}$ has a simple pole at $z = 1$.*

(2) *The map $\Xi : \ker \Phi(1) \to \operatorname{coker} \Phi(1)$ given by*

$$\Xi(x) = \mathrm{P}_{\operatorname{coker} \Phi(1)} \Phi^{(1)}(1)(x), \quad x \in \ker \Phi(1), \tag{4.2}$$

is bijective.

(3) *$H = \operatorname{ran} \Phi(1) \oplus \Phi^{(1)}(1) \ker \Phi(1)$.*

(4) *There exists an I(1) sequence $X = (X_t, t \geq 0)$ in L^2_H satisfying*

$$X_t = \sum_{j=1}^{p} \Phi_j(X_{t-j}) + \varepsilon_t \tag{4.3}$$

for all $t \geq p$, with cointegrating space equal to $\operatorname{coran} \Phi(1)$.

When our four equivalent conditions are true, an I(1) sequence satisfying the requirements of condition (4) may be constructed by choosing a suitable initial condition $X_0 \in L^2_H$ and then setting

$$\Delta X_t = \sum_{k=0}^{\infty} \Psi_k(\varepsilon_{t-k}) \tag{4.4}$$

for all $t \geq 1$, where $\Psi_k \in \mathcal{L}_H$ is defined by

$$\Psi_k = \frac{1}{k!} \Psi^{(k)}(0), \quad k \geq 0, \tag{4.5}$$

and $\Psi(z)$ is the holomorphic extension of $(1 - z)\Phi(z)^{-1}$ over one.

Proof. Compactness of the operators $\Phi_1, \ldots, \Phi_p$ makes $\Phi(z)$ Fredholm of index zero for all $z \in \mathbb{C}$. The analytic Fredholm theorem therefore implies that $\Phi(z)^{-1}$ is holomorphic on $D_{1+\eta} \setminus \{1\}$ for some $\eta > 0$, with a pole at one, and by applying Theorem 3.1 we deduce that conditions (1), (2) and (3) are equivalent, and that they imply that the residue at one is the operator

$$H \ni x \mapsto \Xi^{-1} \mathrm{P}_{\operatorname{coker} \Phi(1)}(x) \in H. \tag{4.6}$$

We will complete our demonstration of the equivalence of our four conditions by showing that (1) + (2) ⇒ (4) and that (4) ⇒ (2), commencing with the former implication. Under condition (1) we may define $\Psi : D_{1+\eta} \to \mathcal{L}_H$ to be the holomorphic extension of $(1-z)\Phi(z)^{-1}$ over one, and define Ψ_k as in (4.5). The fact that the Taylor series

$$\Psi(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \Psi^{(k)}(0) z^k \tag{4.7}$$

converges on $D_{1+\eta}$ implies (see e.g. Pollock, 1999, p. 79) that

$$\lim_{k \to \infty} (1+\delta)^k \|\Psi_k\|_{\mathcal{L}_H} = 0 \quad \text{for any } \delta \in (0, \eta). \tag{4.8}$$

For $k \geq 0$ define $\tilde{\Psi}_k = -\sum_{j=k+1}^{\infty} \Psi_j$, and note that (4.8) implies that

$$\lim_{k \to \infty} (1+\delta)^k \|\tilde{\Psi}_k\|_{\mathcal{L}_H} = 0 \quad \text{for any } \delta \in (0, \eta). \tag{4.9}$$

Let $\tilde{\Psi} : D_{1+\eta} \to \mathcal{L}_H$ be the operator pencil given by the power series

$$\tilde{\Psi}(z) = \sum_{k=0}^{\infty} z^k \tilde{\Psi}_k, \quad z \in D_{1+\eta}, \tag{4.10}$$

convergence of which is assured by (4.9). Let $\tilde{\Phi} : \mathbb{C} \to \mathcal{L}_H$ be the operator pencil that is identically equal to zero if $p = 1$, or is given by

$$\tilde{\Phi}(z) = \mathrm{id}_H - \sum_{j=1}^{p-1} z^j \tilde{\Phi}_j \tag{4.11}$$

if $p \geq 2$, where $\tilde{\Phi}_j = -\sum_{i=j+1}^{p} \Phi_i$ for $j = 1, \ldots, p-1$. It is straightforward to verify that the equalities

$$\Phi(z) = z\Phi(1) + (1-z)\tilde{\Phi}(z) \tag{4.12}$$

and

$$\Psi(z) = \Psi(1) + (1-z)\tilde{\Psi}(z) \tag{4.13}$$

are valid for all $z \in D_{1+\eta}$ by comparing the coefficients in power series expansions of either side of each equality (c.f. Johansen, 1995, Lem. 4.1). Using (4.12) and (4.13) and the fact that $\Phi(z)\Psi(z) = (1-z)\mathrm{id}_H$ due to the definition of Ψ, and in particular $\Phi(1)\Psi(1) = 0$, we obtain

$$(1-z)\mathrm{id}_H = z(1-z)\Phi(1)\tilde{\Psi}(z) + (1-z)\tilde{\Phi}(z)\Psi(z), \quad z \in D_{1+\eta}. \tag{4.14}$$

Rearranging gives

$$\tilde{\Phi}(z)\Psi(z) = -z\Phi(1)\tilde{\Psi}(z) + \mathrm{id}_H, \quad z \in D_{1+\eta}. \tag{4.15}$$

By comparing the coefficients in power series expansions of either side of equation (4.15) and noting that $\Psi_k = \tilde{\Psi}_k - \tilde{\Psi}_{k-1}$ for all $k \geq 1$, we see (c.f. Hansen, 2005, p. 26) that $\Psi_0 = \mathrm{id}_H$ and that

$$\tilde{\Psi}_k - \tilde{\Psi}_{k-1} = -\Phi(1)\tilde{\Psi}_{k-1} + \sum_{j=1}^{p-1} \tilde{\Phi}_j(\tilde{\Psi}_{k-j} - \tilde{\Psi}_{k-j-1}) \tag{4.16}$$

for all $k \geq 1$, where if $p = 1$ the sum over j is understood to be zero, and if $p \geq 2$ we define $\tilde{\Psi}_k = -\Psi(1)$ for $k = -p+1, \ldots, -1$.

We will demonstrate that (1) + (2) $\Rightarrow$ (4) by using the sequences of operators $(\Psi_k, k \geq 0)$ and $(\tilde{\Psi}_k, k \geq 0)$ in $\mathcal{L}_H$ to construct a suitable I(1) sequence X in L^2_H. The exponential decay of $\|\Psi_k\|_{\mathcal{L}_H}$ and of $\|\tilde{\Psi}_k\|_{\mathcal{L}_H}$ assured by (4.8) and (4.9) allows us to define stationary sequences $W = (W_t, t \in \mathbb{Z})$ and $\nu = (\nu_t, t \in \mathbb{Z})$ in L^2_H whose t^{th} members are given by

$$W_t = \sum_{k=0}^{\infty} \Psi_k(\varepsilon_{t-k}) \quad \text{and} \quad \nu_t = \sum_{k=0}^{\infty} \tilde{\Psi}_k(\varepsilon_{t-k}) \tag{4.17}$$

respectively. We now construct a sequence $X = (X_t, t \geq 0)$ in L^2_H by letting X_0 be an arbitrary element of L^2_H satisfying $\Phi(1)(X_0 - \nu_0) = 0$, and setting $X_t = X_0 + \sum_{s=1}^{t} W_s$ for $t \geq 1$. The sequence X is I(1) provided that the 1-summability condition $\sum_{k=0}^{\infty} k\|\Psi_k\|_{\mathcal{L}_H} < \infty$ and nondegeneracy condition $\sum_{k=0}^{\infty} \Psi_k \neq 0$ are satisfied. The former is immediate from the exponential decay of $\|\Psi_k\|_{\mathcal{L}_H}$. The latter is satisfied since having $\Psi(1) = \sum_{k=0}^{\infty} \Psi_k = 0$ would contradict condition (1). We conclude that the sequence X is I(1).

We next show that X satisfies (4.3) for $t \geq p$. From (4.16) we have

$$\Psi_k = -\Phi(1)\tilde{\Psi}_{k-1} + \sum_{j=1}^{p-1} \tilde{\Phi}_j \Psi_{k-j} \tag{4.18}$$

for all $k \geq 1$, where if $p = 1$ the sum over j is understood to be zero, and if $p \geq 3$ we define $\Psi_k = 0$ for $k = -p+2, \ldots, -1$. Using (4.18) and the equality $\Psi_0 = \mathrm{id}_H$ it is straightforward to demonstrate that

$$W_t = -\Phi(1)(\nu_{t-1}) + \sum_{j=1}^{p-1} \tilde{\Phi}_j(W_{t-j}) + \varepsilon_t \tag{4.19}$$

for all $t \in \mathbb{Z}$, with the sum over j understood to be zero if $p = 1$. Using the equalities $\Psi_0 = \mathrm{id}_H$, $\tilde{\Psi}_0 = \mathrm{id}_H - \Psi(1)$ and $\Psi_k = \tilde{\Psi}_k - \tilde{\Psi}_{k-1}$ for $k \geq 1$, it is also straightforward to demonstrate that

$$W_t = \Psi(1)(\varepsilon_t) + \nu_t - \nu_{t-1} \tag{4.20}$$

for all $t \in \mathbb{Z}$. Cumulating (4.20) over t shows that X satisfies

$$X_t = X_0 - \nu_0 + \Psi(1)(\xi_t) + \nu_t \tag{4.21}$$

for all $t \geq 1$, where $\xi_t = \sum_{s=1}^{t} \varepsilon_s$. Applying the operator $\Phi(1)$ to both sides of (4.21) and using the fact that $\Phi(1)(X_0 - \nu_0) = 0$ and that $\Phi(1)\Psi(1) = 0$ we obtain $\Phi(1)(X_t) = \Phi(1)(\nu_t)$ for all $t \geq 1$. From (4.19) we therefore have

$$\Delta X_t = -\Phi(1)(X_{t-1}) + \sum_{j=1}^{p-1} \tilde{\Phi}_j(\Delta X_{t-j}) + \varepsilon_t \quad \text{for all } t \geq p. \tag{4.22}$$

Adding X_{t-1} to both sides of this last equality and simplifying the algebra shows that X satisfies (4.3) for all $t \geq p$.

To complete our demonstration that $(1) + (2) \Rightarrow (4)$ it remains to verify that X has cointegrating space $\operatorname{coran}\Phi(1)$. To accomplish this it will be helpful to first show that $\operatorname{coran}\Phi(1) = \operatorname{coker}\Psi(1)$. Use the residue formula (4.6) to write

$$\Psi(1) = \mathrm{id}_H\restriction_{\ker\Phi(1)} \Xi^{-1} \mathrm{P}_{\operatorname{coker}\Phi(1)}. \tag{4.23}$$

Taking the adjoint of both sides of this equation and noting that, for any vector subspace $V \subseteq H$, the adjoint to $\mathrm{id}_H\restriction_V$ is the map $H \ni x \mapsto \mathrm{P}_V(x) \in V$, we obtain

$$\Psi(1)^* = \mathrm{id}_H\restriction_{\operatorname{coker}\Phi(1)} \Xi^{*-1} \mathrm{P}_{\ker\Phi(1)}. \tag{4.24}$$

We see immediately that $\Psi(1)^*(x) = 0$ for any $x \in (\ker \Phi(1))^\perp$. Further, since Ξ is bijective under condition (2), Ξ^{*-1} must also be bijective (see e.g. Conway, 1990, p. 32) and so $\Psi(1)^*(x) \neq 0$ for any nonzero $x \in \ker \Phi(1)$. It follows that

$$\operatorname{coker} \Psi(1) = (\ker \Phi(1))^\perp. \tag{4.25}$$

But $(\ker \Phi(1))^\perp = \operatorname{cl} \operatorname{coran} \Phi(1) = \operatorname{coran} \Phi(1)$, with the first equality following from the strong rank-nullity theorem and the second following from the fact that $\Phi(1)^*$ is Fredholm and therefore has closed range. Thus we have $\operatorname{coker} \Psi(1) = \operatorname{coran} \Phi(1)$.

We now verify that X has cointegrating space $\operatorname{coker} \Psi(1)$. Let x be an arbitrary element of H. Using (4.21), for $t \geq 1$ we may write

$$\langle X_t, x\rangle = \langle X_0 - \nu_0, x\rangle + \langle \Psi(1)(\xi_t), x\rangle + \langle \nu_t, x\rangle. \tag{4.26}$$

Suppose that $x \in \operatorname{coker} \Psi(1)$. Recalling that we chose the initial condition X_0 such that $\Phi(1)(X_0 - \nu_0) = 0$, and noting that (4.25) implies that $\operatorname{coker} \Psi(1) = (\ker \Phi(1))^\perp$, we have $\langle X_0 - \nu_0, x\rangle = 0$; moreover, for $t \geq 1$ we have $\langle \Psi(1)(\xi_t), x\rangle = \langle \xi_t, \Psi(1)^*(x)\rangle = 0$, and so in view of (4.26) we have $\langle X_t, x\rangle = \langle \nu_t, x\rangle$ for all $t \geq 0$. Thus $(\langle X_t, x\rangle, t \geq 0)$ inherits the stationarity of ν, meaning that x belongs to the cointegrating space of X. Next suppose that $x \notin \operatorname{coker} \Psi(1)$. Noting that ξ_t has covariance operator $t\Sigma$, we see that

$$\mathrm{E}|\langle \Psi(1)(\xi_t), x\rangle|^2 = t\langle \Sigma\Psi(1)^*(x), \Psi(1)^*(x)\rangle, \tag{4.27}$$

which increases without bound as t grows due to the fact that Σ is positive definite and $\Psi(1)^*(x) \neq 0$. Using (4.26) and the Minkowski inequality, for all $t \geq 1$ we may bound the square root of $\mathrm{E}|\langle \Psi(1)(\xi_t), x\rangle|^2$ by the (finite) quantity

$$\left(\mathrm{E}|\langle X_t, x\rangle|^2\right)^{1/2} + \left(\mathrm{E}|\langle X_0 - \nu_0, x\rangle|^2\right)^{1/2} + \left(\mathrm{E}|\langle \nu_t, x\rangle|^2\right)^{1/2}. \tag{4.28}$$

The second term in (4.28) does not depend on t, and the third term also does not depend on t due to the stationarity of ν. The first term in (4.28) therefore has to vary with t, because the sum of the three terms bounds the quantity $(\mathrm{E}|\langle \Psi(1)(\xi_t), x\rangle|^2)^{1/2}$, which becomes arbitrarily large as t grows. This rules out stationarity of $(\langle X_t, x\rangle, t \geq 0)$, and so we conclude that x does not belong to the cointegrating space of X. This shows that the cointegrating space of X is precisely $\operatorname{coker} \Psi(1)$, which we showed earlier is equal to $\operatorname{coran} \Phi(1)$. Our demonstration that $(1) + (2) \Rightarrow (4)$ is complete.

We next show that (4) $\Rightarrow$ (2). Under condition (4) there exists an I(1) sequence $X = (X_t, t \geq 1)$ in L^2_H satisfying (4.3) for all $t \geq p+1$, with cointegrating space given by $\operatorname{coran} \Phi(1)$. Let $\Upsilon : \mathbb{C} \to \mathcal{L}_H$ be the holomorphic (in fact, polynomial of degree $p-1$) operator pencil given by $\Upsilon(z) = (1-z)^{-1}(\Phi(z) - \Phi(1))$ for $z \neq 1$ and $\Upsilon(1) = -\Phi^{(1)}(1)$. Then for all $z \in \mathbb{C}$ we have

$$\Phi(z) = \Phi(1) + (1-z)\Upsilon(z). \tag{4.29}$$

Similarly, let $\tilde{\Upsilon} : \mathbb{C} \to \mathcal{L}_H$ be the operator pencil given by $\tilde{\Upsilon}(z) = (1-z)^{-1}(\Upsilon(z) - \Upsilon(1))$ for $z \neq 1$ and $\tilde{\Upsilon}(1) = -\Upsilon^{(1)}(1)$, so that for all $z \in \mathbb{C}$ we have

$$\Upsilon(z) = \Upsilon(1) + (1-z)\tilde{\Upsilon}(z). \tag{4.30}$$

Note that $\tilde{\Upsilon}(z)$ is a polynomial in z of degree $p-2$ if $p \geq 2$, and is identically zero if $p = 1$. A little algebra shows that in the former case we may write $\tilde{\Upsilon}(z) = \sum_{j=0}^{p-2} z^j \tilde{\Upsilon}_j$ with $\tilde{\Upsilon}_j = -\sum_{k=j+2}^{p} (k-j-1)\Phi_k$. Combining (4.29) with (4.30) and the equality $\Upsilon(1) = -\Phi^{(1)}(1)$ we obtain

$$\Phi(z) = \Phi(1) - (1-z)\Phi^{(1)}(1) + (1-z)^2\tilde{\Upsilon}(z) \tag{4.31}$$

for all $z \in \mathbb{C}$. It follows that

$$X_t - \sum_{j=1}^{p} \Phi_j(X_{t-j}) = \Phi(1)(X_t) - \Phi^{(1)}(1)(\Delta X_t) + \sum_{j=0}^{p-2} \tilde{\Upsilon}_j(\Delta^2 X_{t-j}) \tag{4.32}$$

for all $t \geq p+1$, where $\Delta^2 X_t = \Delta X_t - \Delta X_{t-1}$. To see why, observe that the coefficients of different lags of X_t on either side of (4.32) are the same as the coefficients of different powers of z on either side of (4.31). Since X satisfies the autoregressive law of motion (4.3) under condition (4), we now have

$$\Phi(1)(X_t) - \Phi^{(1)}(1)(\Delta X_t) + \sum_{j=0}^{p-2} \tilde{\Upsilon}_j(\Delta^2 X_{t-j}) = \varepsilon_t \tag{4.33}$$

for all $t \geq p+1$.

Suppose that Ξ is not bijective, so that condition (2) is not satisfied; we shall deduce a contradiction. If Ξ is not bijective then neither is Ξ^* (Conway, 1990, p. 32). Since $\Phi(1)$ is Fredholm, we know that Ξ^* is a linear map between finite dimensional vector spaces. It therefore satisfies the Fredholm alternative: either Ξ^* is invertible or there exists a nonzero $x_0 \in \operatorname{coker} \Phi(1)$ such that $\Xi^*(x_0) = 0$. The former alternative can be ruled out because Ξ^* is not bijective, so the latter must be true. In view of the form of Ξ given in condition (2) and using the fact that, for any vector subspace $V \subseteq H$, the adjoint to $\mathrm{id}_H\restriction_V$ is the map $H \ni x \mapsto \mathrm{P}_V(x) \in V$, we deduce that

$$\Xi^*(x) = \mathrm{P}_{\ker \Phi(1)} \Phi^{(1)}(1)(x) \quad \text{for all } x \in \operatorname{coker} \Phi(1). \tag{4.34}$$

For $\Xi^*(x_0) = 0$ to be true we thus require that $\Phi^{(1)}(1)^*(x_0) \in (\ker \Phi(1))^\perp$. Since $(\ker \Phi(1))^\perp = \operatorname{cl} \operatorname{coran} \Phi(1) = \operatorname{coran} \Phi(1)$ by the strong rank-nullity theorem and Fredholm property of $\Phi(1)$, this establishes the existence of a nonzero $x_0 \in H$ satisfying both

$$\Phi(1)^*(x_0) = 0 \quad \text{and} \quad \Phi^{(1)}(1)^*(x_0) \in \operatorname{coran} \Phi(1). \tag{4.35}$$

Taking the inner product of both sides of (4.33) with x_0 and using the first part of (4.35) we obtain

$$-\langle \Delta X_t, \Phi^{(1)}(1)^*(x_0)\rangle + \sum_{j=0}^{p-2} \langle \tilde{\Upsilon}_j(\Delta^2 X_t), x_0\rangle = \langle \varepsilon_t, x_0\rangle \tag{4.36}$$

for all $t \geq p+1$. Cumulating (4.36) over t yields

$$-\langle X_t - X_p, \Phi^{(1)}(1)^*(x_0)\rangle + \sum_{j=0}^{p-2} \langle \tilde{\Upsilon}_j(\Delta X_t - \Delta X_p), x_0\rangle = \langle \xi_t - \xi_p, x_0\rangle \tag{4.37}$$

for all $t \geq p+1$, where we recall that $\xi_t = \sum_{s=1}^t \varepsilon_s$. We rewrite this as

$$\begin{aligned} \langle \xi_t, x_0\rangle = \langle \xi_p + \Phi^{(1)}(1)(X_p), x_0\rangle - \sum_{j=0}^{p-2} \langle \tilde{\Upsilon}_j(\Delta X_p), x_0\rangle \\ - \langle X_t, \Phi^{(1)}(1)^*(x_0)\rangle + \sum_{j=0}^{p-2} \langle \tilde{\Upsilon} j(\Delta X_t), x_0\rangle. \end{aligned} \tag{4.38}$$

Using this equality and the Minkowski inequality, for all $t \geq p+1$ we may bound the square root of $\mathrm{E}|\langle \xi_t, x_0\rangle|^2$ by the (finite) quantity

$$\begin{aligned} \left(\mathrm{E}|\langle \xi_p + \Phi^{(1)}(1)(X_p), x_0\rangle|^2\right)^{1/2} + \sum_{j=0}^{p-2} \left(\mathrm{E}|\langle \tilde{\Upsilon}_j(\Delta X_p), x_0\rangle|^2\right)^{1/2} \\ + \left(\mathrm{E}|\langle X_t, \Phi^{(1)}(1)^*(x_0)\rangle|^2\right)^{1/2} + \sum_{j=0}^{p-2} \left(\mathrm{E}|\langle \tilde{\Upsilon}_j(\Delta X_t), x_0\rangle|^2\right)^{1/2}. \end{aligned} \tag{4.39}$$

The first and second terms in (4.39) do not depend on t. The third term in (4.39) does not depend on t because X has cointegrating space $\operatorname{coran}\Phi(1)$ under condition (4) and $\Phi^{(1)}(1)^*(x_0) \in \operatorname{coran}\Phi(1)$ from (4.35). The fourth term in (4.39) does not depend on t because ΔX is stationary. Thus our finite bound on the square root of $\mathrm{E}|\langle \xi_t, x_0\rangle|^2$ does not depend on t. But this is impossible because $\mathrm{E}|\langle \xi_t, x_0\rangle|^2 = t\langle \Sigma(x_0), x_0\rangle$, which increases without bound as t grows due to the fact that Σ is positive definite and x_0 is nonzero. We have thus arrived at our desired contradiction, and conclude that (4) ⇒ (2). This completes our proof of the equivalence of conditions (1) through (4).

The final part of our theorem, concerning the construction of an I(1) sequence satisfying the requirements of condition (4), was already proved in our demonstration that (1) + (2) ⇒ (4). □

Remark 4.1. The latter part of Theorem 4.1 refers to choosing a "suitable" initial condition $X_0 \in L^2_H$. From the proof of Theorem 4.1 it is apparent that we may choose any $X_0 \in L^2_H$ satisfying $\Phi(1)(X_0 - \nu_0) = 0$.

Remark 4.2. The process X with differences given by (4.4) is shown in the proof of Theorem 4.1 to satisfy a Hilbert space version of the Beveridge-Nelson decomposition and, given suitable initialization, the error-correction representation. The former is equation (4.21) and the latter equation (4.22).

Remark 4.3. Condition (2) of Theorem 4.1 is the one that most closely resembles the Johansen I(1) condition. The connection between the two should be apparent from Remark 3.3. Condition (3) of Theorem 4.1 provides a geometric alternative to the Johansen I(1) condition. In Section 5 we revisit several examples used by Johansen (1995) to illustrate the application of his I(1) condition, showing how our geometric condition is or is not satisfied in each case.

We saw earlier that Theorem 3.1, which provided necessary and sufficient conditions for the inverse of a holomorphic index-zero Fredholm operator pencil to have a simple pole, took on a particularly simple form when the operator pencil was of the form $A(z) = \mathrm{id}_H - zK$ with $K \in \mathcal{L}_H$ compact. It should come as no surprise that, for the same reason, a simpler version of Theorem 4.1 is available for the special case of a first order autoregressive law of motion. We state it as a corollary.

Corollary 4.1. *Let $\varepsilon = (\varepsilon_t, t \in \mathbb{Z})$ be an iid sequence in L^2_H with positive definite covariance operator $\Sigma \in \mathcal{L}_H$, let $\Phi \in \mathcal{L}_H$ be compact, and let*

$\Pi = \mathrm{id}_H - \Phi$. *Suppose that* Φ *has an eigenvalue of one, and that all other eigenvalues of* Φ *are inside the unit circle. Then the following two conditions are equivalent.*

(1) $H = \operatorname{ran}\Pi \oplus \ker\Pi$.

(2) *There exists an* $\mathrm{I}(1)$ *sequence* $X = (X_t, t \geq 0)$ *in* L^2_H *satisfying*

$$X_t = \Phi(X_{t-1}) + \varepsilon_t \tag{4.40}$$

for all $t \geq 1$, *with cointegrating space equal to* $\operatorname{coran}\Pi$.

When our two equivalent conditions are true, an $\mathrm{I}(1)$ *sequence satisfying the requirements of condition* (2) *may be constructed by choosing a suitable initial condition* $X_0 \in L^2_H$ *and then setting*

$$\Delta X_t = \sum_{k=0}^{\infty} \Psi_k(\varepsilon_{t-k}) \tag{4.41}$$

for all $t \geq 1$, *where we define* $\Psi_0 = \mathrm{id}_H$ *and* $\Psi_k = -\Pi\Phi^{k-1}$ *for* $k \geq 1$. *The* $\mathcal{L}_H$*-convergent series* $\sum_{k=0}^{\infty} \Psi_k$ *is the negative of the projection on* $\ker\Pi$ *along* $\operatorname{ran}\Pi$.

Proof. Let $\Phi(z) = \mathrm{id}_H - z\Phi$. For each $z \in \mathbb{C}$, $\Phi(z)$ is Fredholm of index zero and therefore satisfies the Fredholm alternative: either $\Phi(z)$ is invertible, or $\dim\ker\Phi(z) > 0$. This means that the points of noninvertibility of $\Phi(z)$ are precisely the reciprocals of the eigenvalues of Φ. Our condition on the eigenvalues of Φ therefore implies that $\Phi(z)$ is noninvertible at $z = 1$ and invertible at all other points in the closed unit disk. The assumptions of Theorem 4.1 are therefore satisfied, and we may deduce from it that our condition (2) is satisfied if and only if $\Phi(z)^{-1}$ has a simple pole at $z = 1$. It then follows from Corollary 3.1 that our conditions (1) and (2) are equivalent, and that when they are satisfied the residue of our simple pole is the negative of the projection on $\ker\Pi$ along $\operatorname{ran}\Pi$. The second part of our corollary now follows from the second part of Theorem 4.1, provided we can show that

$$\frac{1}{k!}\Psi^{(k)}(0) = \Pi\Phi^{k-1} \tag{4.42}$$

for all $k \geq 1$, where $\Psi(z)$ is the holomorphic extension of $(1 - z)\Phi(z)^{-1}$ over $z = 1$. To show this, we differentiate both sides of the identity $\Psi(z)\Phi(z) = (1 - z)\mathrm{id}_H$ to obtain

$$\Psi^{(1)}(z)(\mathrm{id}_H - z\Phi) = \Psi(z)\Phi - \mathrm{id}_H. \tag{4.43}$$

Setting $z = 0$ and noting that $\Psi(0) = \mathrm{id}_H$ shows that (4.42) is true for $k = 1$. Next, we differentiate both sides of (4.43) $k - 1$ times to obtain

$$\Psi^{(k)}(z)(\mathrm{id}_H - z\Phi) = k\Psi^{(k-1)}(z)\Phi, \tag{4.44}$$

and then set $z = 0$, yielding $\Psi^{(k)}(0) = k\Psi^{(k-1)}(0)\Phi$ for all $k \geq 2$. It follows that (4.42) is true for all $k \geq 1$. □

Remark 4.4. If Φ is self-adjoint then the condition $H = \operatorname{ran}\Pi \oplus \ker\Pi$ is automatically satisfied, because the strong rank-nullity theorem implies that $\ker\Pi = (\operatorname{ran}\Pi)^{\perp}$. This explains why Theorem 4.2 of Beare and Seo (2016), which dealt with the case of a first order autoregressive Hilbertian process with compact self-adjoint autoregressive operator, did not involve an I(1) condition.

5 Examples

In this section we revisit the three examples used in the fourth chapter of Johansen (1995), which concerns representation theory for I(1) processes. These examples all take place in the space $H = \mathbb{C}^2$. The main point is to illustrate the use of our direct sum condition for verifying that a given autoregressive law of motion is satisfied by an I(1) process.

Example 5.1 (Example 4.1 in Johansen, 1995)**.** Consider the first order autoregressive law of motion $X_t = \Phi X_{t-1} + \varepsilon_t$ with autoregressive coefficient matrix

$$\Phi = \begin{bmatrix} 1+\alpha & -\alpha \\ 0 & 1 \end{bmatrix}. \tag{5.1}$$

It is easy to check that Φ has eigenvalues 1 and $1 + \alpha$. Corollary 4.1 requires these eigenvalues to be equal to one or inside the unit circle, so following Johansen we set $\alpha \in (-2, 0)$, excluding $\alpha = 0$ to rule out the uninteresting case where Φ is the identity. Define

$$\Pi = \mathrm{id}_H - \Phi = \begin{bmatrix} -\alpha & \alpha \\ 0 & 0 \end{bmatrix}. \tag{5.2}$$

In Figure 5.1 we depict the four fundamental subspaces $\ker\Pi$, $\operatorname{ran}\Pi$, $\operatorname{coker}\Pi$ and $\operatorname{coran}\Pi$. Since Π has real elements, we display these subspaces as subsets of $\mathbb{R}^2$, though of course they are in fact subsets of $\mathbb{C}^2$. We do the same in

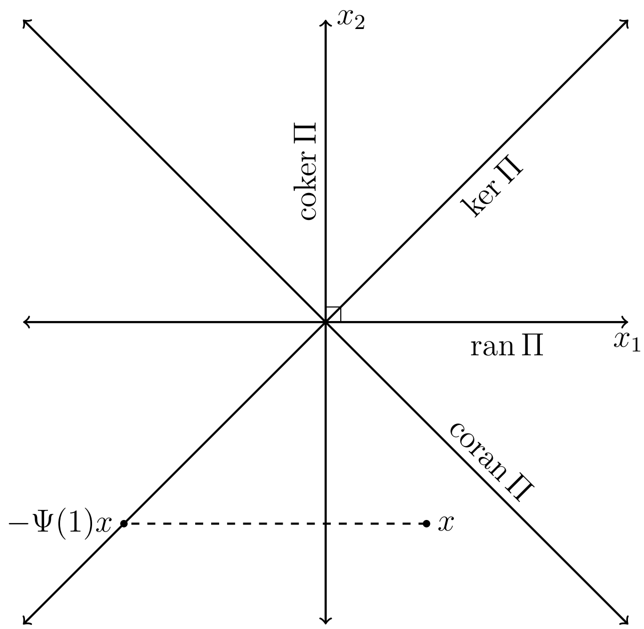

Figure 5.1: Visual aid to Example 5.1.

Examples 5.2 and 5.3 below. The kernel of Π is the linear span of the vector $(1,1)$, the range of Π is the linear span of the vector $(1,0)$, and the cokernel and corange of Π are the respective orthogonal complements of the range and kernel. According to Corollary 4.1, a necessary and sufficient condition for the existence of an I(1) solution to this autoregressive law of motion is that $H = \operatorname{ran}\Pi \oplus \ker\Pi$. It is clear from Figure 5.1 that this condition is indeed satisfied. Corollary 4.1 also gives us the exact form of $\Psi(1)$: it is the negative of the projection on $\ker\Pi$ along $\operatorname{ran}\Pi$; that is,

$$\Psi(1) = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}. \tag{5.3}$$

In Figure 5.1 we illustrate the action of $-\Psi(1)$ upon $x = (1,-2)$.

Example 5.2 (Example 4.2 in Johansen, 1995)**.** Consider the first order autoregressive law of motion $X_t = \Phi X_{t-1} + \varepsilon_t$ with autoregressive coefficient matrix

$$\Phi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{5.4}$$

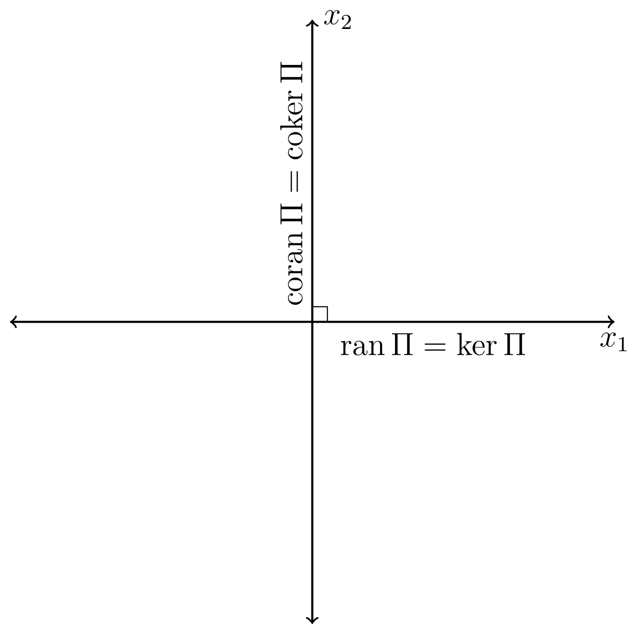

Figure 5.2: Visual aid to Example 5.2.

It is easy to check that 1 is the only eigenvalue of Φ. This satisfies the requirement of Corollary 4.1 that all eigenvalues of Φ are equal to one or are inside the unit circle. In Figure 5.2 we depict the kernel, range, cokernel and corange of the matrix $\Pi = \mathrm{id}_H - \Phi$. There is not a whole lot to see, because the kernel and range both coincide with the horizontal axis, and the cokernel and corange both coincide with the vertical axis. It is apparent that the direct sum condition $H = \operatorname{ran}\Pi \oplus \ker\Pi$ is violated, because $\operatorname{ran}\Pi + \ker\Pi$ is equal to the linear span of the vector $(1, 0)$, and $\operatorname{ran}\Pi \cap \ker\Pi \neq \{0\}$. Corollary 4.1 therefore tells us that there does not exist an I(1) solution to this autoregressive law of motion. In fact, Johansen (1995, p. 47) notes that the law generates I(2) processes.

Example 5.3 (Example 4.3 in Johansen, 1995)**.** Consider the second order autoregressive law of motion $X_t = \Phi_1 X_{t-1} + \Phi_2 X_{t-2} + \varepsilon_t$ with autoregressive coefficient matrices

$$\Phi_1 = \begin{bmatrix} \frac{5}{4} & \gamma - \frac{1}{4} \\ -\frac{1}{4} & \frac{5}{4} \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0 & -\gamma \\ 0 & 0 \end{bmatrix}. \tag{5.5}$$

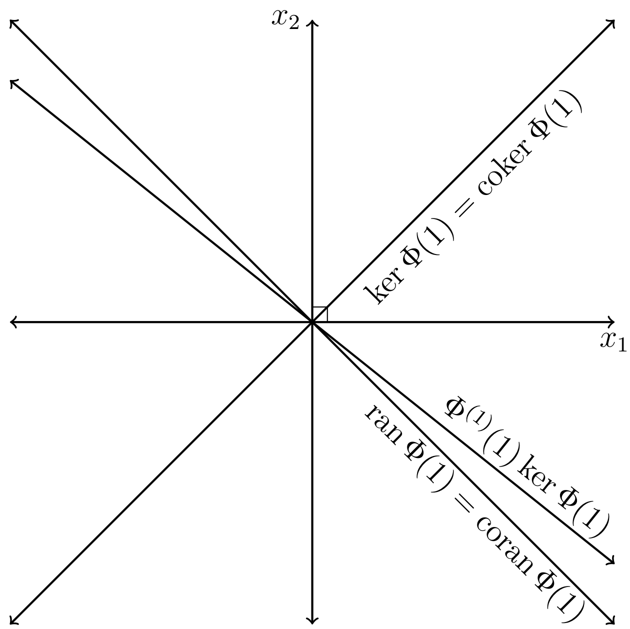

Figure 5.3: Visual aid to Example 5.3, with $\gamma = \frac{9}{4}$.

If $\gamma = 0$ then we have a first order law of motion with the autoregressive operator having eigenvalues 1 and $3/2$. This case falls outside the scope of our results because there is an eigenvalue outside the unit circle, so suppose that $\gamma \neq 0$. Consider the matrix pencil

$$\Phi(z) = \mathrm{id}_H - z\Phi_1 - z^2\Phi_2 = \begin{bmatrix} 1 - \frac{5}{4}z & -(\gamma - \frac{1}{4})z + \gamma z^2 \\ \frac{1}{4}z & 1 - \frac{5}{4}z \end{bmatrix}. \tag{5.6}$$

It is straightforward to show that the spectrum of this matrix pencil consists of two or three isolated points: 1, and the one or two solutions to the quadratic equation

$$\frac{1}{4}\gamma z^2 - \frac{3}{2}z + 1 = 0. \tag{5.7}$$

Suppose that these solutions are equal to one or outside the unit circle, so that Theorem 4.1 may be applied. In Figure 5.3 we depict the kernel, range, cokernel and corange of $\Phi(1)$. The kernel and cokernel coincide, as do the range and corange. This is because $\Phi(1)$ is self-adjoint. Differentiating (5.6) we obtain

$$\Phi^{(1)}(z) = \begin{bmatrix} -\frac{5}{4} & -\gamma + \frac{1}{4} + 2\gamma z \\ \frac{1}{4} & -\frac{5}{4} \end{bmatrix}. \tag{5.8}$$

It is apparent that the vector subspace $\Phi^{(1)}(1)\ker\Phi(1)$ depends on the parameter γ. Following Johansen, we set $\gamma = 9/4$, and note that in this case (5.7) has the unique solution $z = 4/3$, which is outside the unit circle as required. In Figure 5.3 we depict the subspace $\Phi^{(1)}(1)\ker\Phi(1)$ that obtains when $\gamma = 9/4$. It is a line through the origin with slope $-4/5$. We see that $\operatorname{ran}\Phi(1)$ and $\Phi^{(1)}(1)\ker\Phi(1)$ are linearly independent subspaces whose direct sum is H. Thus condition (3) of Theorem 4.1 is satisfied, and we know that there exists an I(1) process satisfying our second order autoregressive law of motion. Note however that if we reduced γ from 9/4 to 2, this would have the effect in Figure 5.3 of rotating the line $\Phi^{(1)}(1)\ker\Phi(1)$ clockwise about the origin so that it coincides with the line $\operatorname{ran}\Phi(1)$, leading to a failure of our direct sum condition for an I(1) solution.

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