Asset Pricing with Dynamically Inconsistent Agents

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Abstract

This paper develops a framework to study general equilibrium implications for an economy in which agents are allowed to have dynamically inconsistent time and risk preferences. This framework accommodates, but is not limited to, the following settings: (1) non-exponential discounting; (2) horizon dependent risk aversion; (3) current state dependent risk aversion. In these models preferences over future outcomes change over time and thus the Bellman optimality principle does not hold. In the spirit of Strotz (1955) I take a game theoretic approach to the solution of agent's problem.

The main result of the paper is an explicit characterization of the equilibrium within a general setting, including the state price density, market price of risk, the interest rate, the return volatility and the equity premium. The state price density admits decomposition into two parts: a standard component, equal to the intertemporal marginal rate of substitution, and an adjustment component. Similar decompositions hold for the market price of risk and the equilibrium interest rate. The adjustment terms reflect the conflicting preferences between the agent today and his "future selves". I illustrate the results derived for the general model in a number of concrete applications. I show that an economy with dynamically inconsistent agents can produce stochastic stock market volatility, counter-cyclical behavior of expected returns, and a downward-sloping term structure of equity.

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1 Introduction

The standard asset pricing framework assumes that economic agents have dynamically consistent preferences, in the sense that the Bellman optimality principle holds. In a multiperiod setting this means that a plan for some future period deemed optimal at an earlier point in time will remain optimal when that future period actually arrives. The optimal plan of action is then independent of the initial point and we can use standard optimization methods, like dynamic programming, to find the optimal decision strategies. Within expected utility theory dynamically consistent preferences over a consumption profile $\{c_s\}_{s=t}^T$ are typically described by the following functional form:

$$E_t\left[\int_t^T \delta(s-t)U(c_s, X_s)ds\right] \tag{1}$$

with the discounting function defined as

$$\delta(s-t) = e^{-\rho(s-t)}.$$

This incorporates the case of state-dependent utility when utility of consumption at a future point in time s depends on the value of some state process at time s, X_s. Models with external habit formation (Abel (1990), Campbell and Cochrane (1999), Chan and Kogan (2002)) are examples of state-dependent utility, where X is the time varying subsistence level of the agent. State dependent utility is also considered in the production-based asset pricing model of Cox et al. (1985) and in the more recent work of Berrada et al. (2013).

It is important to recognize that by assuming dynamically consistent preferences as in (1) we make certain assumptions about the time discounting and risk attitudes of the agents. To achieve dynamic consistency preferences need to be restricted along the following dimensions:

1. The decision maker's discount function δ is only allowed to be exponential. Strotz (1955) in his seminal paper pointed out that every choice of discounting function, apart from the exponential case, will lead to a dynamically inconsistent problem. By assuming exponential discounting, the model assumes that the discount rate, $-\delta'(s-t)/\delta(s-t)$, is constant and equal to ρ. Under this assumption deterministic trade-offs between today and tomorrow are to be treated in the same way as deterministic trade-offs between 100 and 101 days from now. I refer to this assumption as *delay independence in time preferences*, because the rate of time preference is restricted to be independent of the length of delay until the outcomes occur, $s-t$.

2. Risk aversion, as captured by the curvature of the utility function U in (1), is not allowed to change as a function of temporal distance to risky events. I will refer to this assumption as *delay independence in risk preferences*. Risky gambles taking place in the near future are not allowed to give rise to higher risk aversion than more distant ones. If we allow for risk preferences to explicitly depend on the distance to risky events the instantaneous utility function U would depend explicitly on $s - t$. For example, in the model of power utility, risk aversion γ would be a function of $s - t$. Such preferences are no longer dynamically consistent. As Eisenbach and Schmalz (2014) point out delay dependence of risk aversion is conceptually orthogonal to the preference for the timing of the resolution of uncertainty, as introduced in Kreps and Porteus (1978).

3. Dynamic consistency requires that future preferences are not allowed to depend on the current state of the decision maker. This applies to both time and risk preferences. I will refer to this assumption as *independence of the current state*. When standing at time t and making decisions about the future point in time s the preference ordering of the decision maker is allowed depend on the future state, X_s, but is not allowed to reflect the current state, X_t. The latter is referred to by Loewenstein et al. (2003) as "projection bias" since immediate preferences at the time the decision is made are projected onto points in the future.

The main focus of this paper is in relaxing assumptions underlying time-consistency and incorporating a general model of dynamically inconsistent preferences in a consumption-based asset pricing framework.

Behavioral evidence Evidence on revealed risk tolerance and patience which I review below calls these assumptions into question. Facing decisions about an uncertain future people tend to let their immediate environment influence their preferences. Tastes appear to be changing over time.

Consider first the assumption of delay independence (points 1 and 2 in the list above). Dynamically consistent preferences entail delay independence, meaning that time and risk preferences over rewards received with delay are restricted to be independent of when that delay occurs. Contrary to this assumption, evidence on revealed patience (presented in Thaler (1981), Benzion et al. (1989), Loewenstein and Prelec (1992), Ainslie and Haslam (1992), Frederick et al. (2002)) suggests that decision makers behave more patiently with respect to long-horizon trade-offs and more impatiently to short-horizon ones. Moreover, a growing body of evidence (Sagristano et al. (2002), Noussair and Wu (2006), Abdellaoui et al. (2011), Epper and Fehr-Duda (2012) and Eisenbach and Schmalz (2014)) indicates that preferences tend to exhibit delay dependence in the risk dimension as well. According to these findings, risk tolerance changes as a function of the temporal distance to risky events, with risky gambles that take place immediately giving rise to higher risk aversion than more distant ones.

Let us now turn to the assumption that preferences are independent of the current state (point 3 in the list above). Contrary to this assumption, it is natural to think of the immediate environment of the decision maker as shaping the preferences. Loewenstein et al. (2003) review evidence from a variety of domains that supports this view. As an intuitive example, Loewenstein (2005) discusses a situation that is familiar to academics: to accept or decline a seminar invitation to a distant university. The decision often depends on immediate preferences. Feeling rested and energetic at the moment is likely to result in accepting the offer, whereas being jetlagged in another country would prompt declining another long-distance trip, even if it is to take place much further in time. In many other situations like these, when choices faced are more complex and more important the immediate preferences of the decision maker do influence decisions.

Methodology and results In this paper, I allow for time and risk preferences to change over time. That is, preferences can be dynamically inconsistent in both the time and risk dimensions. The goal is to study the effect on asset prices in competitive markets. To my knowledge, this work is the first to obtain, within a general environment, an explicit characterization of market equilibrium with dynamically inconsistent preferences. Towards this, I consider a representative investor whose preferences are allowed to vary systematically as a function of the time until the outcomes and as a function of the present (stochastic) state.

In the spirit of the consistent planning approach of Strotz (1955) an agent's consumption and portfolio strategies are determined by the outcome of an intrapersonal game with players being consecutive incarnations of the same agent. I use the extended dynamic programming results of Björk and Murgoci (2014b) to obtain a system of equations for the determination of the agent's value function. I combine this extended recursive relation with natural market clearing conditions so that the characterization of equilibrium prices can be provided. In a dynamically consistent setting, this is easily done using the martingale approach (often referred to as the Cox et al. (1985) methodology). In the present setting, however, we do not have access to the martingale approach and the characterization of the equilibrium becomes much more involved. The increased complexity is due to the fact that the system of recursions obtained from the extended dynamic programming results of Björk and Murgoci (2014b) involves *endogenous* objects, while the goal is to solve for prices in terms of *exogenously* given data.

The main result of the paper is an explicit characterization of the equilibrium quantities within a general setting, including the state price density, the market price of risk, the interest rate, the return volatility and the equity premium. The state price density admits decomposition into two parts: a standard component, equal to the intertemporal marginal rate of substitution, and an adjustment component. I show that we can interpret the adjustment component in the state price density as compounding disagreements between the valuations of consumption streams by an agent's different selves. The market price of risk and the equilibrium interest rate can be decomposed, in a similar fashion, into a standard part that is obtained in the time consistent model and an extra part that arises when preferences are time inconsistent. These adjustment terms reflect the disagreement between the agent's preferences today and the future preferences which change with time.

I use the results obtained in the general setting to investigate a number of concrete applications.

1. Non-exponential discounting. This is a class of models that allows the time preferences of the agent to exhibit delay dependence. In this paper I recover the results of Luttmer and Mariotti (2003) as a special case of the general model I consider. While Luttmer and Mariotti (2003) resort to a continuous time approximation of their discrete time results for the power utility case, I work directly in continuous time and present explicit expressions of the asset pricing quantities for a generic utility function. In this model the market price of risk does not depend on the shape of the discounting function, but the risk free rate does. The risk premia on risky assets remain unaffected by the non-exponential discounting function unless the volatility of the aggregate endowment is stochastic.

2. Horizon dependent risk aversion. In this setting the risk tolerance is allowed to change as a function of the delay until the risky events. The interesting outcome of this model is that with an endowment process given by the geometric Brownian motion the model generates excess volatility and a stochastic price-dividend ratio. This is in contrast to the deterministic price dividend ratio and the return volatility equal to endowment volatility obtained in the standard time consistent setting when risk aversion is constant. Moreover, given that the agent exhibits higher risk tolerance for rewards materializing in the distant future, the model can reproduce the downward-sloping term structure of equity premia documented in van Binsbergen et al. (2012).

3. Current state dependent risk aversion. In this model the risk preferences of the agent depend on the immediate stochastic state of the economy. If the economy is

in recession, the representative agent is more risk averse and "projects" this high immediate risk aversion on the trade-offs that take place not only immediately but also at future points in time. This is in line with the recent evidence on counter-cyclical risk aversion documented by Cohn et al. (2013) in an experiment with finance professionals. In the time consistent version of this model as pointed out by Gordon and St-Amour (2004) counter-cyclical risk aversion does not help to resolve the equity premium puzzle, but on the contrary, lowers the equity premium if aggregate endowment is not lower than one. I show that the conclusions are different in a dynamically inconsistent model. A model with risk aversion dependent on the current state generates excess volatility, higher market price of risk and a counter-cyclical risk premium.

Related literature This paper relates to the literature that relaxes the assumptions of dynamically consistent preferences. Evidence of revealed preferences that violate these assumptions has been increasingly attracting economists' attention. However, the focus has largely been on allowing delay dependence in time preferences. The findings on revealed patience increasing with delay have triggered a large literature on non-exponential discounting or so called hyperbolic preferences. Harris and Laibson (2001) and Krusell and Smith (2003) study the effects of non-exponential discounting in partial equilibrium with a focus on consumption and savings behavior. My focus, as well as my methodology, differ from these earlier contributions. While these papers study the partial equilibrium effects of non-exponential discounting, I examine a general equilibrium setting. Moreover, I consider a model that is far more general and is not limited to the non-exponential discounting model. My analysis of dynamically inconsistent preferences in this general setting has been made possible thanks to the recent results in Björk and Murgoci (2014a) and Björk and Murgoci (2014b) who undertake a rigorous study of dynamically inconsistent control problems. In these two papers the authors derive an extension of standard dynamic programming methods that can be used in a wide range of applications. In this paper I apply the continuous time theory of Björk and Murgoci (2014b) in a general equilibrium setting.

The paper is related closely to the works by Luttmer and Mariotti (2003), Eisenbach and Schmalz (2014), and Andries et al. (2014). Luttmer and Mariotti (2003) study the general equilibrium effects of non-exponential discounting on asset prices. I benefited greatly from the insights and discussions that Luttmer and Mariotti (2003) provide. The general setting I study nests non-exponential discounting as a special case which allows me to recover and generalize their results. Eisenbach and Schmalz (2014) present experimental evidence on delay dependence in risk preferences and discuss potential origins of these preferences. Eisenbach and Schmalz (2014) take an important step by pointing out that preferences can be dynamically inconsistent not only in the time dimension as captured by non-exponential discounting. In an independent and recent paper Andries et al. (2014) study the dynamic model of asset pricing with horizon dependent risk aversion introduced in Eisenbach and Schmalz (2014) with recursive utility. I study horizon dependent risk aversion in the expected utility setting and therefore the recursive utility model of Andries et al. (2014) is not a special case of this paper. The main focus of Andries et al. (2014) is on the implications for the term structure of risk premia, while I present other implications on price dividend ratio and return volatility. Moreover, in contrast to these works, the setting of this paper provides characterization of equilibrium when preferences depend on the immediate state of the decision maker at the time of the decision.

Outline of the paper The remainder of the paper is organized as follows. Section 2 presents a simplified one-dimensional version of the model. The main reason for considering a one-dimensional case is for simplicity of exposition. In Section 3 I discuss the main asset pricing results for the one-dimensional case. Section 4 presents the general multi-dimensional version of the model and summarizes the key results in the general case. In Section 5 I apply the general methodology to models with delay dependence in time preferences (non-exponential discounting) and risk preferences, as well as models with dependence on the current state. Appendix A contains necessary technical results, Appendix B presents a benchmark CRRA-lognormal economy, and Appendix C contains all proofs.

2 A simple economy with dynamically inconsistent preferences

In this section, I describe the economy under study for a simple one-dimensional version of the model, while a more general setup is discussed in Section 4. In this economy there is a single non-storable good available for consumption that follows an exogenously specified process. The novelty of the paper lies in the specification of the representative agent's preferences. I relax the standard assumptions of delay and present state independence, thus allowing the preferences to be dynamically inconsistent.

2.1 Setup

I consider a continuous-time, complete markets Markovian economy on the time span $[0, T]$ for some horizon T. Uncertainty is represented by a probability space supporting a one-dimensional Wiener process w. In what follows all random processes are assumed to be adapted with respect to the usual augmentation of the filtration generated by this Wiener process. I also assume that all processes satisfy the regularity conditions for stochastic integrals to be well defined.

There is a single perishable consumption good, and the aggregate dividend or endowment process is denoted by e. The interpretation is that owning a right to the endowment stream provides the owner with $e_t dt$ units of consumption good over the interval $[t, t+dt]$. I assume that e is given exogenously by the stochastic process

$$de_t = e_t \left[\mu_e dt + \sigma_e dw_t\right], \tag{2}$$

where μ_e and σ_e are constants with $\mu_e > \sigma_e^2/2$ and $\sigma_e > 0$.

Moreover, let X_t be an exogenous state variable, which can be related to the general state of economy. It is given by a one-dimensional factor process that evolves according to

$$dX_t = \mu_{X,t} dt + \sigma_{X,t} dw_t, \tag{3}$$

and where $\mu_{X,t} = \mu_X(e_t, X_t)$ and $\sigma_{X,t} = \sigma_X(e_t, X_t)$ with deterministic drift and volatility functions μ_X and σ_X. I will use μ_X, σ_X as shorthand for the coefficients in equation (3). In this simple setting the Wiener process w is the same in (2) and (3). This will be relaxed in Section 4.

The financial market is assumed to be complete, that is, trading in the available assets can perfectly hedge changes in the stochastic investment opportunity set. This can be achieved assuming that there are two continuously traded securities: a risky asset in positive supply of one unit and a locally risk free asset in zero net supply. The risky asset, the stock, is a claim to the aggregate endowment (2). The stock price is denoted by S_t and evolves according to

$$dS_t = S_t \left[\mu_{S,t} dt + \sigma_{S,t} dw_t\right]. \tag{4}$$

for some drift $\mu_{S,t}$ and some volatility $\sigma_{S,t}$ that are to be determined endogenously in the market equilibrium. The price of the locally risk free asset is B_t and evolves according to

$$dB_t = B_t r_t dt, \tag{5}$$

for some short rate process r which is to be determined in equilibrium.

To sum up the discussion up to this point, the *exogenously* given objects in the model are the endowment process and the factor process given by (2)-(3). The *endogenous* objects are the price processes in (4)-(5), or, equivalently, the short rate r_t and $\mu_{S,t}$, $\sigma_{S,t}$. The endogenous processes are to be determined in the market equilibrium. I look for a Markovian equilibrium where asset prices are functions of exogenous state variables $(t, e, X) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}$.

The representative agent has financial wealth W_t at time t denominated in units of the consumption good, investing a u_t fraction of wealth in the risky asset and the rest in the risk free asset. The agent also consumes a non-negative amount $c_t dt$ ($c_t \geq 0$) in the period $[t, t + dt]$. The increase in financial wealth over $[t, t + dt]$ is

$$dW_t = \underbrace{W_t r_t dt}_{\text{risk free return}} + \underbrace{W_t u_t \left(\frac{dS_t + e_t dt}{S_t} - r_t \right) dt}_{\text{risky excess return}} - \underbrace{c_t dt}_{\text{consumption}}, \tag{6}$$

where the first part is the risk free return, and the second term capture the excess return (capital gains and dividends) from investing in the risky asset, and the last term captures consumption expenditures. The agent's initial wealth is $W_0 = S_0$, that is at time 0 the agent is endowed with a claim to the aggregate endowment.

I consider feedback control laws, i.e., the controls are of the form $u_t = \mathbf{u}(t, W_t, e_t, X_t)$ where the control law $\mathbf{u}$ is a vector valued deterministic function of time, wealth, aggregate dividend and the underlying state. Similarly, the consumption strategy is of the form $c_t = \mathbf{c}(t, W_t, e_t, X_t)$.

2.2 Preferences

In the standard framework with state-dependent utility the objective of the agent is that of maximizing expected lifetime utility modeled by

$$E_t \left[\int_t^T U(s, c_s, X_s) ds \right]. \tag{7}$$

Note, that if discounting function is exponential, i.e. is of the form $\delta(s - t) = e^{-\rho(s-t)}$, we can factor out $e^{\rho t}$ and convert the problem of maximizing (1) into a standard time consistent problem of maximizing (7). This model allows the utility of a consumption level at a future point in time s to depend on the value of the factor at time s, X_s. Models with external habit formation (Abel (1990), Campbell and Cochrane (1999), Chan and Kogan (2002)) are examples of state-dependent utility, where the factor process X is the time varying subsistence level of the representative agent. State dependent utility as in (7) is also considered in the production-based asset pricing model of Cox et al. (1985) and in the more recent work of Berrada et al. (2013).

A standard way of attacking the problem of maximizing (7) is to apply the martingale approach (often referred to as Cox and Huang (1989) approach) or to use dynamic programming. These approaches are applicable because in a model where an agent's reward is defined by (7) we have access to the Bellman optimality principle. It says that if a control is optimal on the time interval $[t, T]$, then it is also optimal on the sub-interval $[s, T]$ for every s with $t \leq s \leq T$.

In this paper I consider the expected lifetime utility of the representative agent at time t, employing a consumption and investment policy $(\mathbf{u}, \mathbf{c})$, given by

$$J(t, W_t, e_t, X_t; u, c) \equiv E_t \left[\int_t^T U(s, c_s, X_t) ds \right], \tag{8}$$

where X_t is the time t value of the state process X. Comparing these preferences to the standard model of state dependent utility in (7) we see that the instantaneous utility function in (8) depends on the factor process X **evaluated at time** t. I.e., X is evaluated at the date of today, as opposed to in (7), where X is evaluated at time s, i.e. at the time of consumption. The objective is to maximize the expected lifetime utility subject to the dynamic budget constraint (6), no-bankruptcy constraint $W_t \geq 0$, and the nonegativity constraint for consumption $c_t \geq 0$. This, however, will lead to a dynamically inconsistent problem because as the factor process X changes, the preferences change accordingly. With (8) as the continuation utility of the agent the current value of the state X_t enters the utility function, and the Bellman optimality principle does not hold.

One way to think about these preferences is that the agent has different preference orderings at different points in time. When standing at time t it is the environment at time t that shapes the agent's preferences. As time evolves, the tastes change. Viewing the agent as a collection of different "selves", this implies that future "selves" will have different preferences than the present "self". Consequently, the optimality concept will differ for different incarnations of the agent at different times.

Specifying the preferences of the agent as in (8) allows us to consider time and risk preferences that evaluate deterministic and risky trade-offs based on the distance in time from today (delay dependence). Delay dependence can be captured if $X_t \equiv t$. Moreover, the agent's preferences are allowed to depend on the immediate stochastic state of the agent (state dependence).

Special cases of expected lifetime utility in (8) include the following examples.

Example 1 Non-exponential discounting models with continuation utility given by

$$E_t \left[\int_t^T \delta(s - t) U(c_s) ds \right]. \tag{9}$$

In this case $X_t = t$ and the discounting function δ is not restricted to be of the exponential form $\delta(\tau) = e^{-\rho\tau}$, where $\tau = s - t$ is the length of delay until time s arrives. To capture higher impatience toward short-run trade-offs relative to the long-run ones the discounting function can be specified as decreasing with the delay.

Example 2 Delay dependence in risk aversion:

$$E_t \left[\int_t^T e^{-\rho s} U(s - t, c_s) ds \right] \tag{10}$$

In this case $X_t = t$ and time discounting is exponential. However the risk preferences are allowed to be delay dependent. For example, let the the instantaneous utility be the power utility

$$U(\tau, c) = \frac{c^{1-\gamma(\tau)}}{1 - \gamma(\tau)}$$

with risk aversion $\gamma(\tau)$ decreasing with the delay, $\tau = s - t$. This is in line with the evidence that gambles taking place closer in time generate higher risk aversion than those happening much later.

Example 3 Current state dependence in risk preferences:

$$E_t \left[\int_t^T e^{-\rho s} U(c_s, X_t) ds \right],$$

with, for instance, a power utility function

$$U(c, x) = \frac{c^{1-\gamma(x)}}{1 - \gamma(x)},$$

where the relative risk aversion is allowed to depend on the current state of the economy. In this setting we can also study projection bias in habit formation

$$E_t \left[\int_t^T e^{-\rho s} U(c_s - X_t) ds \right],$$

where X is the habit process capturing the current level of well-being of the agent.

Example 4 Current state dependence in time preferences:

$$E_t \left[\int_t^T e^{-\delta(X_t) s} U(c_s) ds \right],$$

where the discount rate is independent of the delay but is a function of the current state of the economy.

2.3 Intrapersonal equilibrium

With the continuation utility of the representative agent dependent on the value of the state at time t as in (8) the usual optimality concept does not apply. The objective is to maximize (8) but the agent today and his "future selves" have conflicting preferences. One approach to the solution of the agent's control problem is the pre-commitment approach. It consists in solving the problem of maximizing utility at the initial time point and assuming that the agent can commit to this initial plan of action. An alternative approach is the game theoretic approach in the spirit of Strotz (1955). The idea is to view the agent's consumption and portfolio choice problem as a game with the players being the agent and his future selves and look for the Markov perfect equilibrium.

In this paper, I take the game theoretic approach and use the continuous time theory of time inconsistent stochastic control developed in Björk and Murgoci (2014b). Conceptually, I will take the agent's choices to be the outcomes of an intrapersonal game with distinct players representing the agent's preferences at different points in time. Consider a game, where we have one player for each point in time t, "player t". For each t, player t chooses a pair of control functions $\mathbf{u}(t,\cdot), \mathbf{c}(t,\cdot)$ which determine the investment and consumption decisions at time t. Putting together the control functions for all the players we have a pair of feedback control laws $\mathbf{u}, \mathbf{c}$. Intuitively, we would like to say that a pair of strategies $\hat{\mathbf{u}}, \hat{\mathbf{c}}$ is a Markov perfect equilibrium if, for each t, it has the following property: if for each $s > t$ player s chooses controls $\hat{\mathbf{u}}(s,\cdot), \hat{\mathbf{c}}(s,\cdot)$, then it is optimal for player t to choose $\hat{\mathbf{u}}(t,\cdot), \hat{\mathbf{c}}(t,\cdot)$. In discrete time this is a perfectly valid definition, but in continuous time it needs to be made more precise. Following Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), and Björk and Murgoci (2014b) I make the following formal definition of the interpersonal equilibrium concept.

Definition 1. *Consider a control law pair* $(\hat{\mathbf{u}}, \hat{\mathbf{c}})$ *and choose arbitrary* u *and* $c > 0$. *Define another pair of control laws* $(\mathbf{u}_h, \mathbf{c}_h)$ *by*

$$(\mathbf{u}_h(t,\cdot), \mathbf{c}_h(t,\cdot)) = \begin{cases} (u, c) & \text{for} \quad t \le s \le t+h, \\ (\hat{\mathbf{u}}_h(t,\cdot), \hat{\mathbf{c}}_h(t,\cdot)) & \text{for} \quad t+h \le s \le T, \end{cases}$$

The pair of control laws $\hat{\mathbf{u}}, \hat{\mathbf{c}}$ *constitute an intrapersonal equilibrium control if*

$$\lim_{h \to 0} \frac{J(\cdot; \hat{\mathbf{u}}, \hat{\mathbf{c}}) - J(\cdot; \mathbf{u}_h, \mathbf{c}_h)}{h} \ge 0 \quad \text{for all } u, c.$$

The intrapersonal equilibrium value function V is defined as

$$V(t, W, e, x) = J(t, W, e, x; \hat{\mathbf{u}}, \hat{\mathbf{c}}).$$

If the utility function depends on the state process in the dynamically consistent manner, that is the lifetime utility is given by (7), the above definition of intrapersonal equilibrium will coincide with the definition of an optimal strategy and we have the standard HJB equation

$$\sup_{u,c} \{(\mathcal{A}^{u,c}) V(t, W, e, x) + U(t, c, x)\} = 0, \tag{11}$$

where the differential operator $\mathcal{A}^{u,c}$ is the infinitesimal generator of the state variables (as defined in (A.3) in Appendix A) under the strategy (u, c). In more concrete terms

$$(\mathcal{A}^{u,c}) V(t, W, e, x) = E_{t,W,e,x} [dV_t] \frac{1}{dt},$$

this means that $(\mathcal{A}^{u,c}) V$ is the drift term in the Ito formula applied to the value function and it captures the expected change in the continuation value over the interval $[t, t + dt]$ if the strategy (u, c) is employed.

In the present setting, when deciding on a control action at time t we explicitly take into account that at future times the agent will have a different objective. I apply the general theory of Björk and Murgoci (2014b) to formulate the extension of the standard Hamilton-Jacobi-Bellman (HJB) equation for determination of the equilibrium value function V. Loosely speaking, the agent cannot commit to future consumption choices, instead he takes into account that as preferences change he will be re-optimizing in the future. The agent's problem can be thought then as a backward recursion problem. This idea will be reflected in an extension of the standard HJB equation (11) to a system of equations as stated in the following result.

Proposition 1. *The value function, $V(t, W, e, x)$, for the interpersonal equilibrium satisfies the following recursive relation (omitting the arguments):*

$$\sup_{u,c} \left\{ (\mathcal{A}^{u,c} V) + U - \left[\mu_X f_z + \frac{1}{2}\sigma_X^2 (f_{zz} + 2 f_{xz}) + \sigma_X \sigma_S u W f_{Wz} + \sigma_X \sigma_e e f_{ez} \right] \right\} = 0,$$

with the terminal condition $V(T, W, e, x) = 0$ and where the partial derivatives of V and f should be evaluated at (t, W, e, x) and (t, W, e, x, x), respectively.
Moreover, for every $z \in \mathbb{R}$ the function $(t, W, e, x) \mapsto f^z(t, W, e, x) \equiv f(t, W, e, x, z)$ is

defined by

$$\begin{aligned}\left(\mathcal{A}^{\hat{u},\hat{c}} f^z\right)(t, W, e, x) + U(t, \hat{c}, z) &= 0 \\ f^z(T, W, e, x) &= 0\end{aligned}$$

and has the following probabilistic interpretation

$$f(t, W_t, e_t, X_t, z) = E_t\left[\int_t^T U(s, \hat{c}_s, z) ds\right].$$

Remark 1. In the Proposition 1 it is important that the infinitesimal operator $\mathcal{A}^{u,c}$ (as defined in (A.3) in Appendix A) only operates on variables within parenthesis. Thus in the expression $\mathcal{A}^{\hat{u},\hat{c}} f^z$, the operator only acts on the variables (t, W, e, x) within the parenthesis, and does not act on the upper case index z, which is viewed as a fixed parameter.

Discussion of the extended HJB result The extended HJB is a system of deterministic recursive equations for the simultaneous determination of functions V and f. The extended HJB consists of a standard part, $(\mathcal{A}^{u,c}V) + U$, plus a new part that is captured by the extra terms in the V-equation involving the function f. To solve the V-equation we need to know f but f is determined by the optimal strategies $(\hat{\mathbf{u}}, \hat{\mathbf{c}})$, which in their turn are determined by the sup-part of the V-equation. Such a fixed point character of the problem is not unexpected since we are looking for a Markov perfect equilibrium of an intrapersonal game.

The additional terms in the HJB equation, loosely speaking, account for the incentives to deviate as time evolves and preferences change. More precisely, they correspond to the difference between the expected continuation value of deviating from the equilibrium strategy for a "small" time interval h from the perspective of the time t consumer, when compared to time $t + h$ consumer

$$\lim_{h\downarrow 0} \frac{1}{h}\left\{E_{t,W,e,x}\left[\int_{t+h}^T U(s, \hat{c}_s, x) ds\right] - E_{t,W,e,x}\left[\int_{t+h}^T U\left(s, \hat{c}_s, X_{t+h}\right) ds\right]\right\}.$$

If preferences are time consistent the preferences of the time t consumer are the same as of the time $t + h$ consumer, and thus this difference is zero. In this case the extended HJB simplifies to the standard HJB equation.

The Verification Theorem from Björk and Murgoci (2014b) will allow to treat the solution to the extended HJB system as the solution to the intrapersonal game.

Theorem 1. *Assume that (V, f) is a solution of the extended system in Proposition 1,* *and that the control law pair* $\hat{\mathbf{u}}, \hat{\mathbf{c}}$ *realizes the supremum in the first equation of the system. Then* $\hat{\mathbf{u}}, \hat{\mathbf{c}}$ *is an equilibrium control law, and* V *is the corresponding value function.*

2.4 Market equilibrium

In the present setting, we have two equilibrium concepts: **intrapersonal equilibrium** and **market equilibrium**. The concept of intrapersonal equilibrium replaces the standard individual optimality concept. Market equilibrium is then defined by requiring intrapersonal equilibrium (instead of individual optimality) and market clearing. Natural market clearing conditions are that the representative agent invests all the wealth in the stock market and consumes all dividends generated there.

Definition 2. *A market equilibrium is a set of price processes* (B_t, S_t)*, or equivalently processes* $(r_t, \mu_{S,t}, \sigma_{S,t})$*, and consumption and trading strategies* $(\mathbf{c}, \mathbf{u})$ *such that the following holds.*

(i) Given the prices, the consumption plan and the trading strategy constitute an intrapersonal equilibrium control law;

(ii) Financial markets clear: $u_S = 1$, $1 - u_S = 0$.

(iii) The consumption good market clears: $c_t = e_t$.

Let me make a remark about condition *(iii)*. The motivation for this goods market clearing condition comes from the fact that the agent is assumed to have enough money at time 0 to buy the right to the aggregate endowment process e, that is $W_0 = S_0$. Together with condition *(ii)* this means that in equilibrium we will have $W_t = S_t$ and $c_t = e_t$ at all times. When using the dynamic programming approach, however, I need to study the extended HJB for all possible combinations of W_t, e_t and S_t, and not only $W_t = S_t$. To include the case $W_t \neq S_t$ I study the extended HJB with condition for goods market clearing formulated as

$$c_t = \frac{W_t}{S_t} \cdot e_t. \tag{12}$$

This says that if the agent's wealth at time t is equal to W_t and all this is invested in the stock market, then the dividend rate is $W_t/S_t \cdot e_t$. I will refer to (12) as extended equilibrium condition. Using (12) in the wealth dynamics gives the equilibrium W dynamics as follows

$$dW_t = \mu_{S,t} W_t dt + W_t \sigma_{S,t} dw_t.$$

Comparing to the S dynamics we see that $W_t = S_t \cdot W_0/S_0$. Together the initial condition of the underlying economic model, $W_0 = S_0$, this implies goods market clearing in the economy under study, namely, condition *(iii)* in Definition 2, $c_t = e_t$.

3 Asset pricing results

I now turn to the asset pricing implications in the simple version of the model outlined in Section 2. In equilibrium, absence of arbitrage together with market completeness guarantees the existence of a unique stochastic discount factor, or state-price density, M. The stochastic discount factor (SDF) is defined by the property that the price of an asset paying $\mathcal{Y}$ at some future date τ is

$$P(t;\mathcal{Y}) = \frac{1}{M_t} E_t \left[M_\tau \mathcal{Y} \right]. \tag{13}$$

M is often referred to as "state pricing process" or "state price density". In a finite sample space setting, the stochastic discount factor corresponds to an Arrow-Debreu state price system. From general arbitrage theory we know that the evolution of the stochastic discount factor is

$$dM_t = -r_t M_t dt - \lambda_t M_t dw_t, \tag{14}$$

where λ is the market price of risk. The economic interpretation λ is standard, it is the risk premium per unit of volatility. From the dynamics of the stochastic discount factor in (14) we see the riskless interest rate, r, is the (negative of) the local rate of return and the market price of risk, λ, is the (negative of) volatility of M.

$$r_t = -\frac{1}{dt} E_t \left[\frac{dM_t}{M_t} \right], \quad \lambda_t = -\sqrt{\frac{1}{dt} \left(\frac{dM_t}{M_t} \right)^2}.$$

The ultimate goal is to characterize the stochastic discount factor in terms of exogenously given objects: the aggregate endowment process, e, the factor processes, X, and the utility function of the representative agent. Once we have an expression for the equilibrium SDF the equilibrium prices follow in a straightforward manner from (13). Namely, the price of a claim to an arbitrary dividend stream $D = \{D_s\}_{s=t}^T$ is the integral of future dividend flow, discounted using the equilibrium stochastic discount factor

$$P(t;D) = \frac{1}{M_t} E_t \left[\int_t^T M_s D_s ds \right].$$

Similarly we can price claims that pay a dividend at a specific point in time, so called zero-coupon dividend claims, as

$$P(t; D_\tau) = \frac{1}{M_t} E_t \left[M_\tau D_\tau \right].$$

The claim to a dividend stream can be though of as a portfolio of such zero-coupon dividend claims (strips) with different maturities.

In a dynamically consistent setting, the characterization of the equilibrium SDF is easily obtained using the martingale approach (often referred to as Cox et al. (1985) methodology). In the present setting, there is no martingale approach and the characterization of the equilibrium becomes more complicated. I use the extended HJB recursion as stated in Proposition 1 and impose market equilibrium condition to obtain the equilibrium HBJ equation. The difficulty is that this equilibrium HJB is written in terms of endogenous objects (like the value function V, the function f and the stock volatility σ_S) and we either need to compute or eliminate them. This is why, unlike in the standard setting I do not start with the expression for the equilibrium SDF immediately, but rather build up the results and state the SDF characterization at the final step.

The roadmap for the subsequent sections is as follows. I start by providing an explicit solution for the equilibrium market price of risk. Then I characterize the equilibrium stock prices and the interest rate. Finally, having obtained these I state the main result for the equilibrium stochastic discount factor in the general setting of the model outlined in Section 2.

3.1 The market price of risk

Combining Proposition 1 with the equilibrium definition leads to the following explicit characterization of the market price of risk process.

Proposition 2. *For every $z \in \mathbb{R}$, let the function G be defined as*

$$G(t, e, z) = E_{t,e} \left[\int_t^T U_c(s, e_s, z) e_s ds \right]. \tag{15}$$

Then, in equilibrium, the market price of risk (MPR) admits the following decomposition

$$\lambda_t = \underbrace{\lambda_t^e + \lambda_t^x}_{\text{consistent MPR}} + \underbrace{\lambda_t^z}_{\text{adjustment}},$$

with

$$\lambda_t^e = -\frac{U_{cc}}{U_c} e_t \sigma_e, \quad \lambda_t^x = -\frac{U_{cx}}{U_c} \sigma_X, \quad \lambda_t^z = \frac{G_z}{G} \sigma_X,$$

where with partial derivatives of utility function U and function G should be evaluated at the point (t, e_t, X_t).

Intuitively, one can think of the market price of risk as reflecting the motives of the representative agent to invest in risky asset market. In our model there are two such motives, to hedge consumption risk, and to hedge the risk of preferences changing over time driven by shocks to the factor process X.

Let us look closer at the intuition behind the result for the market price of risk in Proposition 2. The first part, $\lambda^e + \lambda^x$, corresponds to a "time consistent" market price of risk in the model where representative agent's preferences are given by (7). It consists of two terms coming from consumption risk and preference shocks, respectively. The first term, λ^e depends on the aggregate endowment risk as captured by σ_e. The multiplier of σ_e is the relative risk aversion of the representative agent, which is positive. This is intuitive, there is a positive relation between the volatility of aggregate consumption and the equilibrium market price of risk. The second term, λ^x, depends on risk in state variables, σ_X. If $U_{cx}(t, c, x) \neq 0$ the state variable enters the agent's preferences, and thus shocks to X are preference shocks that contribute to the market price of risk. The multiplier of σ_X is the sensitivity of investor's marginal utility to the state variable X. As can be seen from Proposition 2, if $U_{cx}(t, c, x) > 0$ dynamically consistent state dependent utility results in a lower the market price of risk, working against a resolution of the equity premium puzzle.

The "time inconsistent" adjustment of the part of the market price of risk, λ^z, similarly to λ^x depends on the volatility of the state variable that enters the utility function of the agent. However, in the "time inconsistent" component λ^z, the multiplier of σ_X is not contemporaneous sensitivity of marginal utility to changes in the state but the expectation of the future sensitivities evaluated given the state today. Using the definition of function G in (15), the adjustment term can be written as

$$\lambda^z = \frac{\partial}{\partial z} \log G(t, e_t, X_t) \sigma_X. \tag{16}$$

The intuition behind this this adjustment term will be discussed in Section 3.5.

If $U_{cx}(t, c, x) > 0$ the adjustment term λ^z, that comes from dynamically inconsistent preferences, will lead to a higher market price of risk, thus helping with resolution of the equity premium puzzle. In general, as a direct consequence of Proposition 2 we have the following corollary.

Corollary 1. *λ^x and λ^z have opposite signs.*

Furthermore, if the state variable is affecting marginal utility ($U_{cx}(t, c, x) \neq 0$), but it is locally deterministic ($\sigma_X = 0$), then there is no effect on the market price of risk. This observation implies that in the models with delay dependence, in either time preferences as in (9) or in risk preferences as in (10) the market price of risk process is not affected by the dynamic inconsistency of preferences. I stress this result in the following corollary.

Corollary 2. *If the factor process X that enters instantaneous utility function in (8) is locally deterministic, $\sigma_X = 0$, then the market price of risk is not affected by the fact that preferences of the representative agent are dynamically inconsistent.*

3.2 The equilibrium stock price

In this section I study the behavior of the stock price, which is the claim to the aggregate endowment (aggregate dividend) process e.

Proposition 3. *The equilibrium stock price is given by*

$$S_t = e_t \cdot \Gamma_t, \tag{17}$$

where the price-dividend ratio Γ is given by $\Gamma_t = \Gamma(t, e_t, X_t)$ with

$$\Gamma(t, e, x) = \frac{G(t, e, x)}{U_c(t, e, x)e} \tag{18}$$

and the function $G(t, e, z)$ is defined in (15).

We see from (17) that the stock price is proportional to the aggregate endowment e_t and the price-dividend ratio Γ_t which is the function of the exogenous state variables in the economy. Recall, that in equilibrium we have we have that the stock price equals the representative agent's financial wealth, $W_t = S_t$, at all times, and consumption equals the aggregate endowment, $c_t = e_t$. This means that the price-dividend ratio for the claim on aggregate endowment is equal to the equilibrium wealth-consumption ratio. Hence, I will be using these two terms interchangeably when referring to Γ.

To obtain some intuition we can relate the result in Proposition 3 to the standard outcome in the dynamically consistent economy. Consider, for instance, logarithmic preferences over consumption

$$E_t \left[\int_t^T e^{-\rho s} \ln(c_s) ds \right].$$

For this model, according to the definition (18) we recover the standard result

$$\Gamma_t = \int_t^T e^{-\rho(s-t)} ds = -\frac{1}{\rho} \left[e^{-\rho(T-t)} - 1 \right],$$

and if the horizon of the model is infinite the price-dividend ratio is constant, $\Gamma = 1/\rho$, which is a known conclusion.

Remark 2. In the present setting of dynamically inconsistent preferences, according to the Proposition 3 the price-dividend ratio for the stock is equal to

$$\Gamma_t = E_t \left[\int_t^T \frac{U_c(s, e_s, X_t)}{U_c(t, e_t, X_t)} \frac{e_s}{e_t} ds \right] \tag{19}$$

On the other hand we know that the claim to the aggregate endowment stream e, can be priced using the stochastic discount factor. Thus Γ must satisfy

$$\Gamma_t = E_t \left[\int_t^T \frac{M_s}{M_t} \frac{e_s}{e_t} ds \right]. \tag{20}$$

Comparing (19) and (20) it is tempting to conclude that M_s/M_t equals the ratio of marginal utilities, $U_c(s, e_s, X_t)/U_c(t, e_t, X_t)$ with the state process at the future date s evaluated at its today's value, X_t. This is, however, *not* the case. This can be easily seen from the following argument. If

$$\frac{M_s}{M_t} = \frac{U_c(s, e_s, X_t)}{U_c(t, e_t, X_t)}, \quad \text{for} \quad s > t.$$

then (using the fact that $M_0 = 1$) we conclude that

$$M_t = \frac{U_c(t, e_t, X_0)}{U_c(0, e_0, X_0)}, \quad \text{for all} \quad t.$$

However, taking the ratio of M_s and M_t we see that

$$\frac{M_s}{M_t} = \frac{U_c(s, e_s, X_0)}{U_c(t, e_t, X_0)} \neq \frac{U_c(s, e_s, X_t)}{U_c(t, e_t, X_t)},$$

and thus our conjecture was not correct. As will be shown later the stochastic discount factor in the model with dynamically inconsistent preferences is not equal to the intertemporal marginal rate of substitution of the representative investor. I also emphasize that the simple representation (19) only holds for the claim to the aggregate endowment, and does not hold for any other asset.

3.3 The equity premium and volatility of stock market

From the expression of the stock price given in Proposition 3 it is straightforward to obtain the dynamics of the stock market. Applying Ito formula and using the standard martingale condition for S leads to the following proposition.

Proposition 4. *The volatility of stock market returns, $\sigma_{S,t}$, is given by*

$$\sigma_{S,t} = \sigma_e + \sigma_{\Gamma,t}$$

where σ_e is given exogenously in (2) and $\sigma_{\Gamma,t} = \sigma_\Gamma(t, e_t, X_t)$ with

$$\sigma_\Gamma(t,e,x) = \frac{\Gamma_e(t,e,x)}{\Gamma(t,e,x)} e\sigma_e + \frac{\Gamma_x(t,e,x)}{\Gamma(t,e,x)} \sigma_X.$$

The equity risk premium is

$$\frac{1}{dt} E_t \left[\frac{dS_t + e_t dt}{S_t} \right] - r_t = \lambda_t \sigma_{S,t},$$

where λ is a process for the market price of risk and can be computed as stated in Proposition 2.

The stock price volatility depends on the background consumption risk, which is exogenous and captured by the volatility of aggregate endowment, and the volatility of the wealth-consumption ratio. The latter is the endogenous, as Mele (2007) calls it, price-induced, component of return volatility.

3.4 The equilibrium short rate

This section contains results for the equilibrium interest rate at which the market for riskless borrowing and lending clears.

Proposition 5. *In equilibrium, the riskless rate of interest admits the following decomposition*

$$r_t = \underbrace{r_t^e \; + \; r_t^x}_{\text{consistent r}} + \underbrace{r_t^z}_{\text{adjustment}},$$

with

$$\begin{aligned} r_t^e &= -\frac{U_{ct}}{U_c} - \mu_e \frac{U_{cc}}{U_c} - \frac{1}{2}\sigma_e^2 e_t^2 \frac{U_{ccc}}{U_c}, \\ r_t^x &= -\mu_X \frac{U_{cx}}{U_c} - \frac{1}{2}\sigma_X^2 \frac{U_{cxx}}{U_c} - \sigma_e e_t \sigma_X \frac{U_{ccx}}{U_c}, \end{aligned}$$

where partial derivatives of U should be evaluated at the point (t, e_t, X_t), and

$$r_t^z = \mu_X \frac{G_z}{G} - \frac{1}{2}\sigma_X^2 \frac{G_{zz}}{G} + \sigma_X^2 \frac{\partial}{\partial z}\frac{G_z}{G} + \sigma_e e \sigma_X \frac{\partial}{\partial e}\frac{G_z}{G} - \lambda_t^z \left(\lambda_t^e + \lambda_t^x\right)$$

where the function $G(t, e, z)$ is defined as in (15) and its derivatives are to be evaluated at the point (t, e_t, X_t).

Similarly to the market price of risk the risk free rate equals to the risk free rate one obtains in the equilibrium with dynamically consistent state dependent preferences, as in (7), plus an adjustment term. r_t^e relates the equilibrium interest rate to time preference, expected endowment growth rate μ_e and the variance rate σ_e^2 of aggregate endowment growth over the next instant. These are the standard terms contributing to the risk free rate in the absence of state-dependent preferences. They capture three savings motives: patience, intertemporal substitution and precautionary savings motive. With state-dependence, preferences respond shocks in the underlying state X, and we have three additional terms in the r_t^x.

The "time inconsistent" adjustment part of the risk free rate is r_t^z. To gain some intuition for this term let us simplify things and consider the case when the factor process X is locally deterministic, that is $\sigma_X = 0$, then the adjustment term r_t^z becomes

$$r_t^z = \mu_X \frac{\partial}{\partial z} \log G(t, e_t, X_t).$$

Thus, in this case, the adjustment term of the risk free rate depends on the rate of change of the factor process multiplied by the sensitivity of the equilibrium wealth-consumption ratio to this change. If the dynamics of the factor process X also has a stochastic part, we obtain additional terms in the risk free rate expression that capture the factor risk and the interaction (covariance) between the shocks to the factor process and the sensitivity of the wealth-consumption ratio.

3.5 The stochastic discount factor

Having derived the expression for the market price of risk and the equilibrium short rate we can now state the result about the stochastic discount factor in this model. Before I characterize the stochastic discount factor in the economy with dynamically inconsistent preferences, I first discuss the case with standard time consistent state dependent preferences as in (7). Recall the following standard result from the literature (see Munk (2013)).

Proposition 6. *In an economy with a time consistent state dependent utility function, the stochastic discount factor is given by*

$$M_t = \frac{U_c(t, e_t, X_t)}{U_c(0, e_0, X_0)}.$$

The next proposition establishes the main result about the stochastic discount factor in the economy with dynamically inconsistent preferences.

Proposition 7. *In the economy with dynamically inconsistent preferences the equilibrium stochastic discount factor is given by*

$$M_t = \frac{U_c(t, e_t, X_t)}{U_c(0, e_0, X_0)} \cdot M_t^z,$$

where M^z evolves according to

$$dM_t^z = -M_t^z \left[r_t^z dt + \lambda_t^z d\tilde{w}_t \right],$$

with λ_t^z and r_t^z being the "time inconsistent" components of the market price of risk and the equilibrium short rate from Propositions 2 and 5. Moreover, $\tilde{w}$ is a $\tilde{Q}$-Wiener process with measure $\tilde{Q}$ defined by $d\tilde{Q}/dP = \tilde{L}_t$, on $\mathcal{F}_t$, with the likelihood process $\tilde{L}$ given by

$$d\tilde{L}_t = -\tilde{L}_t(\lambda_t^e + \lambda_t^x) dw_t.$$

Intuition behind the stochastic discount factor Comparing Propositions 6 and 7 we see that the equilibrium stochastic discount factor M is a product of two factors. The first of these is the "standard" SDF from Proposition 6. The second factor is M^z, which accounts for dynamically inconsistent preferences. In order to understand the intuition behind the adjustment term M^z in the stochastic discount factor we argue as follows.

First, using the expressions for the market price of risk and the equilibrium short rate from Propositions 2 and 5 together with the definition of the infinitesimal operator (A.3) it can be shown that (see Appendix C for the derivation)

$$M_t^z = \exp\left\{ -\left(\int_0^t \alpha_s^z ds + \lambda_s^z dw_s \right) \right\} \tag{21}$$

where $\alpha_t^z = \alpha^z(t, e, x)$ with

$$\alpha^z(t, e, x) = \mathcal{A} \log G(t, e, x) - \mathcal{A} \log G^x(t, e), \tag{22}$$

and from Propositions 2 we recall that $\lambda_t^z = \lambda^z(t, e, x)$ is the "time inconsistent" component of the market price of risk given by

$$\lambda^z(t, e, x) = \frac{\partial}{\partial z} \log G(t, e, x) \sigma_X, \tag{23}$$

where G is defined in (15).

Secondly, looking closer at (21) with α^z and λ^z given as (22) and (23) we see that M^z can be informally written as

$$M_t^z = \exp\left\{ -\left(\int_0^t \log G(s+ds, e_{s+ds}, X_{s+ds}) - \log G(s+ds, e_{s+ds}, X_s) \right) \right\}.$$

Discretizing the time interval from 0 to t into N periods of "small" length h and writing the integral as a sum we get

$$M_t^z = \exp\left\{ \sum_{n=0}^{N-h} \log \frac{G(nh+h, e_{nh+h}, X_{nh})}{G(nh+h, e_{nh+h}, X_{nh+h})} \right\},$$

which we can also represent as a product

$$M_t^z = \prod_{n=0}^{N-h} \frac{G(nh+h, e_{nh+h}, X_{nh})}{G(nh+h, e_{nh+h}, X_{nh+h})} \tag{24}$$

Finally, using the definition of G we know that for every $s = nh$

$$\frac{G(s+h, e_{s+h}, X_s)}{G(s+h, e_{s+h}, X_{s+h})} = \frac{E_{s+h}\left[\int_{s+h}^T U_c(v, e_v, X_s) e_v dv\right]}{E_{s+h}\left[\int_{s+h}^T U_c(v, e_v, X_{s+h}) e_v dv\right]}.$$

Let us look closer at this ratio. The numerator is the expected value of the consumption stream starting at $s+h$ discounted by the marginal utility of the time s consumer, which I will loosely speaking refer to as "valuation" of this consumption stream by agent s. The denominator is the expected value of the same consumption stream discounted by the marginal utility of the time $s+h$ consumer, a "valuation" of the same consumption stream by agent $s+h$. Thus the ratio as captures the marginal preference changes or disagreement between the time s consumer and time $s+h$ consumer and we can think of the adjustment term M_t^z as compounding disagreements between the valuations of consumption streams starting from 0 to t by agent's different "selves".

A discrete time stochastic discount factor In discrete time the structure of the continuous result is the same, the equilibrium stochastic discount factor is a product of two terms, the "time consistent" SDF in the corresponding state dependent utility model, and the adjustment term accounting for the fact that "future selves" of the agent have different preference than the "current self".

$$M_t = \frac{U_c(t, e_t, X_t)}{U_c(0, e_0, X_0)} \cdot M_t^z,$$

with the one-period stochastic discount factor given by

$$m_{t+1} = \frac{U_c(t+1, e_{t+1}, X_{t+1})}{U_c(t, e_t, X_t)} \cdot m_{t+1}^z.$$

The adjustment term M_t^z in discrete time is obtained by setting the length of the period in (24) to one

$$M_t^z = \prod_{s=0}^{t-1} \frac{G(s+1, e_{s+1}, X_s)}{G(s+1, e_{s+1}, X_{s+1})}, \quad t > 0. \tag{25}$$

For the non-exponential discounting model, using the discrete time version of definition of function G in (15), we have the following expression for the adjustment term in the one period stochastic discount factor

$$m_{t+1}^z = \frac{E_{t+1}\left[\sum_{s=t+1}^{T} \delta(s-t) U'(e_s) e_s\right]}{E_{t+1}\left[\sum_{s=t+1}^{T} \delta(s-(t+1)) U'(e_s) e_s\right]},$$

or equivalently

$$m_{t+1}^z = \frac{E_{t+1}\left[\sum_{n=0}^{T-(t+1)} \delta(n+1) U'(e_{t+1+n}) e_{t+1+n}\right]}{E_{t+1}\left[\sum_{n=0}^{T-(t+1)} \delta(n) U'(e_{t+1+n}) e_{t+1+n}\right]}, \tag{26}$$

where we change a variable in the summation using $s = t + 1 + n$.

If we further specialize to the power utility case (26) reduces to the result in the discrete time model of Luttmer and Mariotti (2003).

Quasi-hyperbolic discounting In the dynamically consistent model when the discounting function $\delta(n)$ is equal to δ^n, we have the usual one period stochastic discount factor equal to

$$m_{t+1} = \delta \frac{U'(e_{t+1})}{U'(e_t)}. \tag{27}$$

With "quasi-hyperbolic discounting" considered in Phelps and Pollak (1968), Laibson (1997), and Krusell and Smith (2003) the discounting function is given by $\delta(0) = 1$ and

$\delta(n) = \beta\delta^n$ for $n \geq 1$. In this model the one period stochastic discount factor is

$$m_{t+1} = \left(\delta + \frac{\delta(\beta - 1)}{\Gamma_{t+1}} \right) \frac{U'(e_{t+1})}{U'(e_t)}, \tag{28}$$

where Γ_{t+1} is the discrete time equivalent of the equilibrium wealth consumption ratio defined in Proposition 3

$$\Gamma_{t+1} = E_{t+1} \left[\sum_{n=0}^{T-(t+1)} \delta(n) \frac{U'(e_{t+1+n})}{U'(e_{t+1})} \frac{e_{t+1+n}}{e_{t+1}} \right].$$

We see that with discount factors $1, \beta\delta, \beta\delta^2, ...$, the expression for the one period stochastic discount factor simplifies considerably comparing to the general non-exponential discounting model because of the special setting in which only the difference between agents "incarnations" today and tomorrow matters, after that things repeat. Note that if $\beta = 1$ we are back in the standard exponential discounting case and (28) reduces to (27). However, if $\beta < 1$ then the tomorrow's "self" is over-consuming from the point of view of today's "self". Therefore the rate as which the current "self" is willing to substitute consumption at time $t + 1$ for consumption at time t is reduced. Indeed, for $\beta < 1$ the one period stochastic discount factor in quasi-hyperbolic discounting model given in (28) is lower than in the standard exponential discounting model (27).

4 Generalization: a multidimensional version of the model

The analysis presented so far has been in the context of a one dimensional factor process X and a one dimensional Wiener process w. In this setting the endowment process was assumed to follow a geometric Brownian motion. The goal of this section is to demonstrate how the results generalize in the setting with a d-dimensional Wiener process and an n-dimensional factor process. Moreover, endowment is not restricted to have the simple geometric Brownian motion dynamics.

4.1 Setup

The uncertainty in the general version of the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ supporting a d-dimensional Brownian motion. There is a single consumption good, and the aggregate dividend or endowment process is denoted by e. I assume that e is given exogenously by the stochastic process

$$de_t = e_t \left[\mu_{e,t} dt + \sigma_{e,t} dw_t \right], \tag{29}$$

where $\mu_{e,t} = \mu_e(e_t, X_t)$ and $\sigma_{e,t} = \sigma_e(e_t, X_t)$ with a scalar valued drift function μ_e and a row vector valued volatility function σ_e. $X_t \in \mathbb{R}^n$ is an exogenously given vector of state variables that evolves according to

$$dX_t = \mu_{X,t}dt + \sigma_{X,t}dw_t, \tag{30}$$

and where $\mu_{X,t} = \mu_X(e_t, X_t)$ and $\sigma_{X,t} = \sigma_X(e_t, X_t)$ with a vector valued drift function μ_X and a matrix valued volatility function σ_X taking values in $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^d$ and are time-independent if the economy has an infinite horizon. I will use $\mu_e, \sigma_e, \mu_X, \sigma_X$ as a shorthand for coefficients in equations (29)–30).

An example of the set-up is the case when we have a stochastic opportunity set, that is μ_e and σ_e are driven by the factor process. One of the components of the factor process can be deterministic and equal to running time and another (stochastic) state variable:

$$dX_t = \begin{pmatrix} dX_{1,t} \\ dX_{2,t} \end{pmatrix} = \begin{pmatrix} dt \\ dX_{2,t} \end{pmatrix},$$

where X_2 does not need to be perfectly correlated with the aggregate endowment.

Financial investment opportunities consist of $d+1$ continuously traded securities: a dividend-paying stock in unit net supply, a locally risk free savings account in zero net supply and $d-1$ derivative assets in zero net supply. The stock is a claim to the aggregate endowment process (29). In this setup the market is complete as trading in the available assets can perfectly hedge changes in the stochastic investment opportunity set. I look for Markovian equilibria in which prices for the locally risk free asset, B, the risky stock, S, and the vector of financial derivative prices, F, follow processes

$$dB_t = B_t r_t dt \tag{31}$$

$$dS_t = S_t \left[\mu_{S,t}dt + \sigma_{S,t}dw_t\right] \tag{32}$$

$$dF_t = F_t \left[\mu_{F,t}dt + \sigma_{F,t}dw_t\right]. \tag{33}$$

To introduce some more compact notation let the vector of mean returns on risky assets and the volatility matrix be denoted by

$$\mu_t = \begin{pmatrix} \mu_{S,t} + \frac{e_t}{S_t} \\ \mu_{F,t} \end{pmatrix}, \quad \sigma_t = \begin{pmatrix} -\sigma_{S,t}- \\ -\sigma_{F,t}- \end{pmatrix}.$$

The volatility matrix σ is assumed to be invertible P–a.s. for all t. Processes r_t, μ_t, σ_t are to be determined endogenously in the market equilibrium.

The representative agent has financial wealth W_t at time t denominated in units of the consumption good, investing $u_t = (u_{St}, u_{F,t})^\top$ fractions of wealth in the risky assets and the rest in the risk free asset. The agent also consumes a nonnegative amount $c_t dt$ ($c_t \geq 0$) in the period $[t, t+dt]$. The increase in financial wealth over $[t, t+dt]$ is

$$dW_t = (W_t r_t - c_t)\, dt + W_t u_t^\top (\mu_t - r_t \mathbb{1}) dt + W_t u_t^\top \sigma_t dw_t \tag{34}$$

where the first part is the risk free return less consumption expenditures, and the remaining terms capture the excess return from investing in the risky assets. We consider feedback control laws, i.e., the controls are of the form $u_t = \mathbf{u}(t, W_t, e_t, X_t)$ where the control law $\mathbf{u}$ is a vector valued deterministic function of time, wealth and underlying state. Similarly, the consumption strategy is of the form $c_t = \mathbf{c}(t, W_t, e_t, X_t)$. The agent's initial wealth is $W_0 = S_0$, that is at time 0 the agent is endowed with a unit of stock.

The expected lifetime utility of the representative agent at time t, given a consumption and investment policy $(\mathbf{u}, \mathbf{c})$, is given by

$$J(t, W_t, X_t; \mathbf{u}, \mathbf{c}) \equiv E_t \left[\int_t^T U(s, c_s, X_t) ds \right], \tag{35}$$

where X_t is the time t value of the state process X.

4.2 Results

I now show how the results derived in the earlier part of the paper extend to the multi-dimensional case. I start with the multi-dimensional version of the extended HJB equation.

Proposition 8. *The value function, $V(t, W, e, x)$, for the interpersonal equilibrium satisfies the following recursive relation (omitting the arguments):*

$$\sup_{u,c} \Big\{ (\mathcal{A}^{u,c} V) + U - \Big[\sum_i^n \mu_{X_i} \frac{\partial f}{\partial z_i} + \frac{1}{2} \sum_{i,j}^n \sigma_{X_i} \sigma_{X_j}^\top \frac{\partial^2 f}{\partial z_i \partial z_j}$$
$$+ \sum_{i,j}^n \sigma_{X_i} \sigma_{X_j}^\top \frac{\partial^2 f}{\partial x_i \partial z_j} + \sum_i^n \sigma_{X_i} \sigma^\top u W \frac{\partial^2 f}{\partial W \partial z_i} + \sum_i^n \sigma_{X_i} \sigma_e^\top e \frac{\partial^2 f}{\partial e \partial z_i} \Big] \Big\} = 0,$$

with the terminal condition $V(T, W, e, x) = 0$ and where the partial derivatives of V and f should be evaluated at (t, W, e, x) and (t, W, e, x, x), respectively.
Moreover, for every $z \in \mathbb{R}^n$ the function $(t, W, e, x) \mapsto f^z(t, W, e, x) \equiv f(t, W, e, x, z)$ is

defined by

$$\left(\mathcal{A}^{\hat{u},\hat{c}} f^z\right)(t, W, e, x) + U(t, \hat{c}, z) = 0$$
$$f^z(T, W, e, x) = 0$$

and has the following probabilistic interpretation

$$f^z(t, W_t, e_t, X_t) = E_t\left[\int_t^T U(s, \hat{c}_s, z)ds\right].$$

The intuition behind the extra terms in the HJB equation is the same as in the one-dimensional case. They correspond to the difference between the expected continuation value of deviating from the equilibrium strategy for a "small" time interval h from the perspective of the time-t consumer, when compared to time-$t+h$ consumer

$$\lim_{h\downarrow 0}\frac{1}{h}\left\{E_{t,W,e,x}\left[\int_{t+h}^T U(s, \hat{c}_s, x)ds\right] - E_{t,W,e,x}\left[\int_{t+h}^T U\left(s, \hat{c}_s, X_{t+h}\right)ds\right]\right\}.$$

If preferences are time consistent the preferences of the time-t consumer are the same as of the time-$t+h$ consumer, and thus this difference is zero and we have the standard HJB equation

$$\sup_{u,c}\left\{(\mathcal{A}^{u,c}V) + U\right\} = 0.$$

The next proposition contains the main result about the stochastic discount factor in the multi-dimensional dynamically inconsistent economy.

Proposition 9. *The equilibrium stochastic discount factor (SDF) is given by*

$$M_t = \frac{U_c(t, e_t, X_t)}{U_c(0, e_0, X_0)} \cdot \exp\left\{-\left(\int_0^t \alpha_s ds + \beta_s dw_s\right)\right\},$$

where $\alpha_t = \alpha(t, e_t, X_t)$ *and* $\beta_t = \beta(t, e_t, X_t)$ *with*

$$\alpha(t, e, x) = \mathcal{A}\log G(t, e, x, x) - \mathcal{A}\log G^x(t, e, x)$$
$$\beta(t, e, x) = \frac{G_z(t, e, x, x)}{G(t, e, x, x)}\sigma_X$$

with the function $G : [0, T] \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *defined as*

$$G(t, e, x, z) = E_{t,e,x}\left[\int_t^T U_c(s, e_s, z)e_s ds\right].$$

The intuition behind this result is the same as in the on-dimensional case. The stochastic discount factor is equal to the ratio of marginal utilities times an adjustment factor that accounts for disagreements between the valuations of consumption streams starting from 0 to t by agent's different "selves".

5 Applications

This section contains a number of examples that apply the results of the general model discussed in Sections 3-4. Once I specialize to a particular case the results derived in the general framework simplify considerably and I obtain explicit solutions to these problems and provide further insights.

5.1 Non-exponential discounting

In this section, I consider the case when preferences of the representative agent are given by

$$E_t \left[\int_t^T \delta(s-t) U(c_s) ds \right], \tag{36}$$

where $\delta : [0, T] \to \mathbb{R}^+$ is not restricted to be exponential. Natural requirements are that $\delta(0) = 1$ and $\delta(\infty) = 0$. Using results of the multidimensional version of the model we can consider a more general endowment dynamics

$$de_t = e_t \left[\mu_{e,t} dt + \sigma_{e,t} dw_t \right],$$

where drift and volatility are not restricted to be constant, but can be driven by a stochastic factor process X. In this setting X does not enter utility, but influences the endowment dynamics.

If discounting function is exponential, i.e. is of the form $\delta(s-t) = e^{-\rho(s-t)}$, we can factor out $e^{\rho t}$ and convert the problem into a standard time consistent problem. With exponential discounting the rate of time-preference, $-\delta'(\tau)/\delta(\tau)$, is constant regardless of the delay. This means that trade-offs between today and tomorrow are treated the same as trade-offs between day 100 and day 101.

To allow for the instantaneous discount rate to change, for example, decline as the delay until outcomes materialize increases, discounting function needs to be non-exponential. In this case, the agent's preferences are dynamically inconsistent, since utility function (36) will explicitly depend on today's date t. It is known since Strotz (1955) that all forms of non-exponential time preferences introduce dynamic inconsistency.

The next proposition summarizes the main asset pricing results in the non-exponential discounting model, without specializing either to a particular utility function or endowment dynamics. This proposition generalizes the results in Luttmer and Mariotti (2003), where the result is obtained as a continuous-time approximation for the power utility case. Directly applying the result obtained in Sections 3-4 I can state the following.

Proposition 10. *In the equilibrium with non-exponential discounting the following holds.*

1. *The market price of risk is equal to*

$$\lambda_t = -\frac{U''(e_t)}{U'(e_t)} e_t \sigma_{e,t}.$$

2. *The equilibrium risk free rate is*

$$r_t = \frac{\Phi_t}{\Gamma_t} - \mu_{e,t} e_t \frac{U''(e_t)}{U'(e_t)} - \frac{1}{2}\sigma_{e,t}^2 e_t^2 \frac{U'''(e_t)}{U'(e_t)},$$

where $\Gamma_t = \Gamma(t, e_t, X_t)$ is the equilibrium wealth consumption ratio given by

$$\Gamma(t, e_t, X_t) = E_t\left[\int_t^T \delta(s-t)\frac{U'(e_s)}{U'(e_t)}\frac{e_s}{e_t} ds\right]$$

and $\Phi_t = \Phi(t, e_t, X_t)$ is given by

$$\Phi(t, e_t, X_t) = -E_t\left[\int_t^T \delta'(s-t)\frac{U'(e_s)}{U'(e_t)}\frac{e_s}{e_t} ds\right].$$

3. *The stochastic discount factor is*

$$M_t = \exp\left\{\int_0^t -\frac{\Phi(s, e_s, X_s)}{\Gamma(s, e_s, X_s)} ds\right\}\frac{U'(e_t)}{U'(e_0)}.$$

As expected from the result in Corollary 2 the market price of risk is not affected by non-exponential discounting. It is equal to the product of the relative risk aversion coefficient of the representative investor and the volatility of the aggregate endowment. However, the equilibrium risk free rate and thus the stochastic discount factor are affected.

Let us discuss the equilibrium interest rate in this model. In the expression for the risk free rate the second and the third terms are standard. They reflect willingness to save due to intertemporal substitution and precautionary savings motives, respectively. Besides these two savings motives the inclination to save increases with patience. This is captured by the first term, Φ_t/Γ_t. Luttmer and Mariotti (2003) refer to it as the effective subjective discount rate. When discounting function is of the exponential form, that is

$\delta(s-t) = e^{-\rho(s-t)}$ with a constant discount rate ρ, the expected lifetime utility given in (36) becomes

$$E_t\left[\int_t^T e^{-\rho(s-t)}U(c_s)ds\right] \tag{37}$$

and the expression for Φ simplifies considerably to

$$\Phi(t, e_t, X_t) = \rho\Gamma(t, e_t, X_t).$$

Hence we recover the standard risk free rate expression where impatience is captured by the constant discount rate ρ

$$r_t = \rho - \mu_{e,t}e_t\frac{U''(e_t)}{U'(e_t)} - \frac{1}{2}\sigma_{e,t}^2 e_t^2\frac{U'''(e_t)}{U'(e_t)}.$$

As a result, with exponential discounting the stochastic discount factor expression stated in Proposition 10 reduces to the standard result

$$M_t = e^{-\rho t}\frac{U'(e_t)}{U'(e_0)},$$

which says that the stochastic discount factor is equal to the intertemporal marginal rate of substitution of the representative investor.

Consider instead a non-exponential discounting function, for example, the *hyperbolic* discounting function discussed in Luttmer and Mariotti (2003)

$$\delta(\tau) = (1+\alpha\tau)^{-\beta/\alpha}e^{-\rho\tau},$$

which combines the exponential discounting part with the discount function proposed by Loewenstein and Prelec (1992). Figure 1 plots this discounting function together with the standard non-exponential discounting.

Insert Figure 1 here

The discount rate is

$$-\frac{\delta'(\tau)}{\delta(\tau)} = \rho + \frac{\beta}{1+\alpha\tau},$$

and it converges to $\rho + \beta$ as the delay τ goes to zero. For this discounting function we have

$$\frac{\Phi_t}{\Gamma_t} = \rho + \beta \frac{E_t\left[\int_t^T \frac{\delta(s-t)}{1+\alpha(s-t)} U'(e_s) e_s ds\right]}{E_t\left[\int_t^T \delta(s-t) U'(e_s) e_s ds\right]} > \rho.$$

Setting $\beta = 0$ recovers the standard exponential discount function, and we then have $\Phi_t/\Gamma_t = \rho$.

Power utility and GBM endowment To compare the results to a well-known CRRA-lognoromal benchmark (Appendix B) I further specialize within the framework of a non-exponential discounting model (36). I assume that the instantaneous utility is of the power form

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

and that μ_e and σ_e in the endowment process are constant so that aggregate endowment is given by the geometric Brownian motion. This model is referred to as CRRA-lognormal model since the utility function exhibits constant relative risk aversion equal to γ and

$$\ln(e_t) = \ln(e_0) + (\mu_e - \frac{1}{2}\sigma_e^2)t + \sigma_e w_t.$$

With these assumption the equilibrium wealth-consumption ratio is a deterministic function of time

$$\Gamma(t) = \int_t^T \delta(s-t) e^{(1-\gamma)(\mu_e - \frac{1}{2}\gamma\sigma_e^2)(s-t)} ds. \tag{38}$$

From Proposition 4 we know that Γ being deterministic implies that return volatility will be equal to volatility of the aggregate endowment, because $\sigma_\Gamma = 0$. Moreover, since the market price of risk is not affected by non-exponential discounting, in this setting it will be constant as well and equal to $\gamma\sigma_e$ (see Appendix B). These two observations imply that the equity premium in this model is constant. Furthermore, we know that the volatility of a return on the dividend strip is the sum of volatility on the aggregate endowment and the volatility of the dividend strip price-dividend ratio. Since Γ is a time integral of price-dividend ratios on dividend strips, we conclude that those are also deterministic. Hence, in this model the term structure of risk premia is not affected by non-exponential discounting and is constant as in the benchmark case (see Appendix B).

5.2 Horizon-dependent risk aversion

In this section, I specialize to the case when the preferences of the representative agent are given by

$$E_t\left[\int_t^T e^{-\rho s}U(s-t,c_s)ds\right]. \tag{39}$$

In this model the risk preferences, as captured by $U(\tau, c)$, are varying as a function of $\tau = s - t$, i.e. the distance in time from the present. This would allow risk tolerance to change as a function of the delay until the outcomes. For example, to have a higher risk tolerance (lower risk aversion) for rewards materializing in a distant future. Time discounting is assumed to be exponential in order to study the effect coming solely from delay dependence in risk preferences. Using the results from the general model it is of course possible to allow for delay dependence in both time and risk preferences.

Using results of the multidimensional version of the model we can consider a more general endowment dynamics

$$de_t = e_t\left[\mu_{e,t}dt + \sigma_{e,t}dw_t\right],$$

where drift and volatility are not restricted to be constant, but can be driven by a stochastic factor process X. In this setting X does not enter utility, but influences the endowment dynamics.

The next proposition summarizes the main results of this model.

Proposition 11. *In the equilibrium with delay dependence in risk preferences the following holds.*

1. *The market price of risk is equal to*

$$\lambda_t = -\frac{U_{cc}(0,e_t)}{U_c(0,e_t)}e_t\sigma_{e,t}.$$

2. *The equilibrium risk free rate is*

$$r_t = \underbrace{\rho - \mu_{e,t}e_t\frac{U_{cc}(0,e_t)}{U_c(0,e_t)} - \frac{1}{2}\sigma_{e,t}^2 e_t^2\frac{U_{ccc}(0,e_t)}{U_c(0,e_t)}}_{\text{without delay dependence in risk preferences}} + \frac{\Psi_t}{\Gamma_t}$$

with $\Gamma_t = \Gamma(t, e_t, X_t)$ being the equilibrium wealth consumption ratio, or, equiva-

lently, price dividend ratio for the stock,

$$\Gamma(t, e_t, X_t) = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{U_c(s-t, e_s)}{U_c(0, e_t)} \frac{e_s}{e_t} ds \right],$$

and $\Psi_t = \Psi(t, e_t, X_t)$ *given by*

$$\Psi(t, e_t, X_t) = -E_t \left[\int_t^T e^{-\rho(s-t)} \frac{U_{c\tau}(s-t, e_s)}{U_c(0, e_t)} \frac{e_s}{e_t} ds \right].$$

3. *The stochastic discount factor is*

$$M_t = \exp\left\{ \int_0^t -\left(\rho + \frac{\Psi(s, e_s, X_s)}{\Gamma(s, e_s, X_s)} \right) ds \right\} \frac{U_c(0, e_t)}{U_c(0, e_0)}.$$

In order to get intuition for the result in Proposition 11 one can think of the instantaneous utility of the form

$$U(\tau, c) = \frac{c^{1-\gamma(\tau)}}{1-\gamma(\tau)} \tag{40}$$

with $\gamma(\tau)$ decreasing in the delay, τ, to capture risk aversion that decreases for outcomes further in time. Then we see that the utility function given by $U(0, c)$ can be thought of as the standard power utility with constant relative risk aversion equal to $\gamma = \gamma(0)$.

Consistent with the conclusion in Corollary 2 Proposition 11 says that the market price of risk is not affected by the delay dependence in risk aversion. It is equal to the product of immediate relative risk aversion of the representative investor and the volatility of aggregate endowment, $\sigma_{e,t}$. For the utility as in (40) the expression for the market price of risk simplifies to $\gamma(0)\sigma_{e,t}$.

If the instantaneous utility is independent of the delay, $U_{c\tau} = 0$ and hence also $\Psi = 0$. Then the expression for the risk free rate from Proposition 11 reduces to standard result in a canonical consumption based asset pricing model with risk aversion over all outcomes being constant and equal to the immediate risk aversion, $\gamma(0)$

$$r_t = \rho + \gamma(0)\mu_{e,t} - \frac{1}{2}\gamma(0)(1+\gamma(0))\sigma_{e,t}^2.$$

If $U_{c\tau} \neq 0$, the risk free rate has an additional term Ψ_t/Γ_t and the stochastic discount factor is a product of a standard time consistent discount factor and an adjustment term that account for dynamically inconsistent preferences. If $U_{c\tau} > 0$ then $\Psi_t/\Gamma_t < 0$ and the risk free rate is lower than in the standard model.

GBM endowment, excess volatility and term structure of equity To exemplify Proposition 11 and compare the results to the CRRA-lognormal benchmark (Appendix B) I consider that the instantaneous utility given by (40) and further assume that μ_e and σ_e in the endowment process are constant so that aggregate endowment is given by the geometric Brownian motion.

With these assumptions the equilibrium wealth-consumption ratio is (see Appendix C for the derivation)

$$\Gamma_t = \Gamma(t, e_t) = \int_t^T e^{\xi(s-t)(s-t)} e_t^{\gamma(0)-\gamma(s-t)} ds, \tag{41}$$

where $\xi(\tau)$ is given by

$$\xi(\tau) = -\rho + (1 - \gamma(\tau))(\mu_e - \frac{1}{2}\gamma(\tau)\sigma_e^2).$$

If $\gamma(\tau)$ is constant and equal to γ, expression for wealth-consumption ratio in (41) reduces to a standard result in a CRRA-lognormal framework, which is a deterministic wealth-consumption ratio in (A.8). However, if $\gamma(0) \neq \gamma(s-t)$ we see from (41) that the equilibrium wealth-consumption ratio in this model is stochastic. This means that the volatility of equilibrium wealth-consumption ratio is non-zero. As can be seen from Proposition 4 this implies that volatility of the stock is not equal to volatility of the aggregate endowment which is a standard result in a CRRA-lognormal framework, but is in fact higher and moreover stochastic. This result is important and I stress it in the following proposition.

Proposition 12. *In the model with aggregate endowment following a geometric Brownian motion and instantaneous utility given by (40) the following holds.*

1. *The volatility of the stock market return is higher than the volatility of the aggregate endowment and is given by*

$$\sigma_{S,t} = \sigma_e + \sigma_{\Gamma,t}$$

where $\sigma_{\Gamma,t} = \sigma_\Gamma(t, e_t) \neq 0$ if $\gamma(\tau)$ is not constant in τ.

2. *The equity risk premium is given by*

$$\frac{1}{dt} E_t \left[\frac{dS_t + e_t dt}{S_t} \right] = \gamma(0)\sigma_e^2 + \gamma(0)\sigma_e\sigma_{\Gamma,t}.$$

Moreover, the fact that the price-dividend ratio for the claim to aggregate endowment is stochastic gives the hope for a non-trivial term structure of equity. Using the results about the stochastic discount factor we can price claims that pay the value of endowment process at a specific point in time, so called zero-coupon endowment claims. The claim to the aggregate endowment (aggregate dividend) stream can be though of as a portfolio of such zero-coupon endowment claims (strips) with different maturities. Looking at the expected excess returns on the dividend strips, or alternatively the equity yields defined as $y(t,\tau) = 1/\tau \ln\left(e_t/S(t;e_\tau)\right)$ as a function of maturity we can make conclusions about the term structure of equity. As Figure 2 illustrates, the model with delay dependence generates a decreasing term structure of equity.

Insert Figure 2 here

5.3 Risk aversion dependent on the current state

In this Section I consider the model in which risk preferences of the agent are dependent on the immediate (stochastic) state of the economy. Let the continuation value for the investor be given by

$$E_t\left[\int_t^T e^{-\rho s} U(c_s, X_t) ds\right], \tag{42}$$

and we can specialize, for example, to power utility of the form

$$U(c_s, X_t) = \frac{c_s^{1-\gamma(X_t)}}{1-\gamma(X_t)}, \tag{43}$$

where the risk aversion γ is a function of the immediate state of the decision maker.

That is, when standing at time t and making decisions about the future the preference ordering of the decision maker reflects the current state he finds himself in. For example, if the economy is in a recession the representative agent is more risk averse and "projects" this high immediate risk aversion on the trade-offs that take place not only immediately but also at future points in time. Since lifetime utility depends on the state variable X at time t such preferences are dynamically inconsistent. As the state of the economy will change so will the preferences. Once the recession is over and the economy is in a good state the representative agent in this model will become more risk tolerant and will "project" this risk tolerance onto points in the future.

Such model is in line with the recent evidence on counter-cyclical risk aversion coming from an experiment conducted and analyzed by Cohn et al. (2013). The authors primed financial professionals with either a boom or a bust scenario and then studied their risk attitude in subsequent investment tasks. The results of the experiment showed that having been primed with a financial bust resulted in a more risk averse behavior.

GBM endowment and counter-cyclical variation in risk premium Let the aggregate dividend (endowment) e follow a geometric Brownian motion

$$de_t = e_t \left[\mu_e dt + \sigma_e dw_t\right],$$

where w is a standard one-dimensional Wiener process.

In this setting, we also need to make an assumption about the state variable X that affects the agent's risk preferences. The idea is that X_t should be related to the general state of the economy, a variable that would summarize the business cycle conditions. Let us define a standard of living, or habit process, h, as weighted geometric average of past realizations of the aggregate endowment

$$\log(h_t) = \lambda \int_0^t e^{-\lambda(t-s)} \log(e_s) ds. \tag{44}$$

Then $X = \log(e/h)$ represents relative (log) endowment

$$dX_t = \underbrace{-\lambda(X_t - \bar{X})}_{\mu_X} dt + \underbrace{\sigma_e}_{\sigma_X} dw_t, \tag{45}$$

where $\bar{X} = (\mu_e - \frac{1}{2}\sigma_e^2)/\lambda$ is the long run mean. This state variable is used in Chan and Kogan (2002). High values of X can be interpreted as corresponding to good times, whereas low values of X can be thought of as capturing bad times.

Using the result from Proposition 3 the equilibrium wealth-consumption ratio can be explicitly computed to equal (see Appendix C for the derivation)

$$\Gamma_t = \Gamma(t, X_t) = \frac{1}{\xi(X_t)} \left(e^{\xi(X_t)(T-t)} - 1\right), \tag{46}$$

where $\xi : \mathbb{R} \to \mathbb{R}$ is given by

$$\xi(x) = -\rho + (1 - \gamma(x)) \left(\mu_e - \frac{1}{2}\sigma_e^2 \gamma(x)\right). \tag{47}$$

Moreover, if the time horizon is infinite, then the equilibrium wealth-consumption ratio simplifies to

$$\Gamma(X_t) = -\frac{1}{\xi(X_t)}$$

and thus ξ in (47) can be interpreted as the (negative of) the consumption-wealth ratio, or, equivalently, the dividend-price ratio for the claim to aggregate endowment. In this setting we have the following result about the volatility of the stock returns and the market price of risk in this model.

Proposition 13. *In the infinite horizon version of the model with aggregate endowment following a geometric Brownian motion and instantaneous utility given by (43) the following holds.*

1. *The volatility of equity return is $\sigma_{S,t} = \sigma_S(X_t)$ with*

$$\sigma(X_t) = \sigma_e + \frac{\Gamma'(X_t)}{\Gamma(X_t)}\sigma_X.$$

2. *The market price of risk is given by $\lambda_{S,t} = \lambda_S(X_t)$*

$$\begin{aligned}\lambda(X_t) &= \underbrace{\gamma(X_t)\sigma_e}_{\lambda_t^e} + \underbrace{\ln(e_t)\gamma'(X_t)\sigma_X}_{\lambda^x} + \underbrace{\left[-\ln(e_t)\gamma'(X_t) + \frac{\Gamma(X_t)}{\Gamma(X_t)}\right]}_{\lambda^z} \\ &= \gamma(X_t)\sigma_e + \frac{\Gamma'(X_t)}{\Gamma(X_t)}\sigma_X.\end{aligned}$$

The first conclusion from Proposition 13 is that return volatility is higher than the volatility of the aggregate endowment, σ_e, if the risk aversion is countercyclical, $\gamma'(x) < 0$. Secondly, as in the general model the market price of risk can be decomposed into three parts. The first two parts we obtain in the case of time consistent state dependence, that is in the model with instantaneous utility given by

$$U(c_s, X_s) = \frac{c_s^{1-\gamma(X_s)}}{1-\gamma(X_s)}.$$

Note that for endowment values greater than one we have $\ln(e) > 0$ and thus with $\gamma'(x) < 0$ the second term, λ^x is negative. This means that dynamically consistent state dependent utility with counter cyclical risk aversion results in a lower market price of risk, working against the resolution of the equity premium puzzle. However, together with the adjustment term, λ^z, that accounts for dynamically inconsistent preferences the market price of risk is higher. The equity premium for the stock return is given by

$$\frac{1}{dt}E_t\left[\frac{dS_t + e_t dt}{S_t}\right] = \lambda_t \sigma_{S,t}$$

and given results in Proposition 13 we can conclude that in this model we have a countercyclical risk premium. Plots in Figure 3 illustrate these results.

Insert Figure 3 here

6 Conclusions

In this paper, I relax the assumptions underlying time-consistency in a way that incorporates the growing evidence of research in psychology and behavioral economics. I take a game theoretic approach in the spirit of Strotz (1955) and incorporate a general model of dynamically inconsistent preferences in a consumption-based asset pricing framework.

The first main contribution of this paper is that I provide an explicit characterization of the stochastic discount factor in a dynamically inconsistent model which nests the standard time consistent result. The adjustment term in the stochastic discount factor is intuitive, it accounts for the fact that preferences change with time and reflects the different preferences of the agent today as opposed to his "future selves".

The second main contribution is that I show that in a standard endowment economy dynamically inconsistent preferences can help explain stylized asset pricing facts about stock market volatility, the cyclical behavior of expected returns, and the term structure of risk premia.

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Appendix

A Technical results

Lemma A.1 (Multidimensional Ito's lemma)**.** Let $F \in C^{1,2}([0,T] \times \mathbb{R}^n; \mathbb{R})$.

I will use $F_x(t,x)$ to denote the row vector $\left[\frac{\partial F}{\partial x_i}(t,x)\right]_{i=1,..n}$ and F_{xx} to stand for the matrix $\left[\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x)\right]_{i,j}$ and $F_t(t,x) = \frac{\partial F}{\partial t}(t,x)$.

Consider an n-dimensional Markov process $X = (X_1, ..., X_n)^\top$ satisfying an SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dw_t, \tag{A.1}$$

where w is a d-dimensional Wiener process, μ and σ are a vector valued drift function and a matrix valued volatility function of $(t,x) \in [0,T] \times \mathbb{R}^n$ taking values in $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^d$, respectively.

Let $Y_t = F(t, X_t)$, where X_t satisfies (A.1). Then

$$dY_t = \left\{ F_t(t,X_t) + F_x(t,X_t)\mu_t + \frac{1}{2}\text{tr}\left[\sigma_t\sigma_t^\top F_{xx}(t,X_t)\right]\right\} dt + F_x(t,X_t)\sigma_t dw_t. \tag{A.2}$$

We can give a more concise version of this formula by introducing the notation $\mathcal{A}$ for the operator defined on $C^{1,2}([0,T] \times \mathbb{R}^n; \mathbb{R})$, and depending not the coefficients μ and σ of the process for X. The infinitesimal operator $\mathcal{A}$ of process X that satisfies (A.1) transforms an arbitrary twice continuously differentiable function $F(t,x)$ as follows:

$$(\mathcal{A}F)(t,x) = F_t(t,x) + F_x(t,x)\mu(t,x) + \frac{1}{2}\text{tr}\left[\sigma\sigma^\top F_{xx}(t,x)\right]. \tag{A.3}$$

We thus see that $(\mathcal{A}F)(t,X_t)$ is the dt part in the Ito formula (A.2). So that $E_t[dF(t,X_t)] = (\mathcal{A}F)(t,X_t)dt$. Using this notation (A.2) becomes

$$dY_t = (\mathcal{A}F)(t,X_t)dt + F_x(t,X_t)\sigma_t dw_t.$$

As the notational convention we have that operator $\mathcal{A}$ operates only on the variables within parentheses, while the upper case indices are treated as constant parameters.

Lemma A.2 (Feynman-Kac)**.** Let X be the solution to (A.1). Then function F defined as

$$F(t,x) = E_{t,x}\left[\int_t^T K(s,X_s)ds\right]$$

admits the following PDE representation

$$(\mathcal{A}F)(t,x) + K(t,x) = 0, \tag{A.4}$$
$$F(T,x) = 0. \tag{A.5}$$

B Benchmark economy

Consider the representative agent economy with the lifetime utility given by

$$E_t \left[\int_t^T e^{-\rho(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right].$$

The stochastic discount factor in this model is given by

$$M_t = e^{-\rho t} \left(\frac{e_t}{e_0} \right)^{-\gamma}.$$

Moreover, let the aggregate endowment be given by a geometric Brownian motion

$$de_t = e_t \left[\mu_e dt + \sigma_e dw_t \right],$$

where μ_e and σ_e are constants and w is a one-dimensional standard Brownian motion.

Proposition B.1. The asset pricing implication of the benchmark model are as follows.

1. The risk free rate is constant, $r_t = r$, and equal to

$$r = \rho + \gamma\mu_e - \frac{1}{2}\gamma(1+\gamma)\sigma_e^2 \tag{A.6}$$

2. The market price of risk is constant, $\lambda_t = \lambda$, and equal to

$$\lambda = \gamma\sigma_e \tag{A.7}$$

3. The price-dividend ratio for a claim to the aggregate endowment is deterministic and equal to

$$\Gamma_t = \frac{1}{\xi} \left(e^{\xi(T-t)} - 1 \right), \tag{A.8}$$

   where

$$\xi = -\rho + (1-\gamma)(\mu_e - \frac{1}{2}\gamma\sigma_e^2).$$

   Moreover, for $T = \infty$ the price-dividend ratio is constant and equal to $\Gamma = -1/\xi$.

4. The return volatility is constant, $\sigma_{S,t} = \sigma_S$, and equal to the volatility of the aggregate endowment

$$\sigma_S = \sigma_e \tag{A.9}$$

5. The equity premium is constant and given by

$$\frac{1}{dt}E_t\left[\frac{dS_t + e_t dt}{S_t}\right] - r_t = \gamma\sigma_e^2. \tag{A.10}$$

6. The price dividend-ratio for a dividend strip with maturity n is

$$\Gamma_t^n = e^{\xi(n-t)}. \tag{A.11}$$

7. The volatility of a dividend strip with maturity n is equal to volatility of endowment.

C Proofs

Proof of Proposition 1. Using the results in Björk and Murgoci (2014b) the extended HJB system for our problem is

$$\sup_{u,c}\{(\mathcal{A}^{u,c}V)(t,W,e,x)+U(t,c,x)-(\mathcal{A}^{u,c}f)(t,W,e,x,x)+(\mathcal{A}^{u,c}f^{x})(t,W,e,x)\}=0$$
$$\left(\mathcal{A}^{\hat{\mathbf{u}},\hat{\mathbf{c}}}f^{z}\right)(t,W,e,x)+U(t,\hat{\mathbf{c}}(t,W,e,x),z)=0,$$
$$f^{z}(T,W,e,x)=0$$
$$V(T,W,e,x)=0,$$

where $\mathcal{A}^{u,c}$ denotes the infinitesimal (Dynkin) operator as defined in Lemma A.3. In order not to get lost in the details I write the wealth dynamics as

$$dW_t=\mu_W dt+\sigma_W dw_t,$$

where μ_W stands for $\mu_W(t,W_t,e_t,X_t,u_t,c_t)$, similarly σ_W denotes $\sigma_W(t,W_t,e_t,X_t,u_t)$. We then have $(\mathcal{A}^{u,c}V)(t,W,e,x)$ given by

$$\begin{aligned}\mathcal{A}^{u,c}V=&V_t+\mu_W V_W+\frac{1}{2}\sigma_W^2 V_{WW}+\mu_e e V_e+\frac{1}{2}\sigma_e^2 e^2 V_{ee}+\sigma_W\sigma_e e V_{We}\\&+\mu_X V_x+\frac{1}{2}\sigma_X^2 V_{xx}+\sigma_X\sigma_W V_{Wx}+\sigma_X\sigma_e e V_{ex}.\end{aligned}$$

Furthermore, $(\mathcal{A}^{u,c}f^{z})(t,W,e,x)$ is

$$\begin{aligned}\mathcal{A}^{u,c}f^{z}=&f_t+\mu_W f_W+\frac{1}{2}\sigma_W^2 f_{WW}+\mu_e e f_e+\frac{1}{2}\sigma_e^2 e^2 f_{ee}+\sigma_W\sigma_e e f_{We}\\&+\mu_X f_x+\frac{1}{2}\sigma_X^2 f_{xx}+\sigma_X\sigma_W f_{Wx}+\sigma_X\sigma_e e f_{ex},\end{aligned}$$

and $(\mathcal{A}^{u,c}f)(t,W,e,x,x)$ is

$$\begin{aligned}\mathcal{A}^{u,c}f=&f_t+\mu_W f_W+\frac{1}{2}\sigma_W^2 f_{WW}+\mu_e e f_e+\frac{1}{2}\sigma_e^2 e^2 f_{ee}+\sigma_W\sigma_e e f_{We}\\&+\mu_X f_x+\frac{1}{2}\sigma_X^2 f_{xx}+\sigma_X\sigma_W f_{Wx}+\sigma_X\sigma_e e f_{ex}\\&+\mu_X f_z+\frac{1}{2}\sigma_X^2 f_{zz}+\sigma_X^2 f_{xz}+\sigma_X\sigma_W f_{Wz}+\sigma_X\sigma_e e f_{ez}.\end{aligned}$$

Hence,

$$\begin{aligned}&-(\mathcal{A}^{u,c}f)(t,W,e,x,x)+(\mathcal{A}^{u,c}f^{x})(t,W,e,x)\\&=-\left[\mu_X f_z+\frac{1}{2}\sigma_X^2 f_{zz}+\sigma_X^2 f_{xz}+\sigma_X\sigma_W f_{Wz}+\sigma_X\sigma_e e f_{ez}\right],\end{aligned}\tag{A.12}$$

where derivatives of function f on the right hand side of (A.12) should be evaluated as (t,W,e,x,x).

Recalling that in our case we have $\sigma_W = Wu\sigma_S$ and using (A.12) in the extended HJB proves Proposition 1.

Proof of Proposition 2. The V-equation of the extended HJB system reads

$$\sup_{u,c}\{\mathcal{A}^{u,c}V + U - \left[\mu_X f_z + \frac{1}{2}\sigma_X^2 f_{zz} + \sigma_X^2 f_{xz} + \sigma_X\sigma_S uWf_{Wz} + \sigma_X\sigma_e e f_{ez}\right] = 0$$

where (and in the subsequent discussion) partial derivatives of V are evaluated at (t, W, e, x) and partial derivatives of f are evaluated at (t, W, e, x, x). The first order condition for the interior optimum are

$$\begin{aligned}
\frac{\partial}{\partial c}: &\qquad U_c = V_W \\
\frac{\partial}{\partial u}: &\qquad 0 = V_W W(\mu_S + \frac{e}{S} - r) + W^2 V_{WW}\sigma_S^2 u + WV_{We}e\sigma_e\sigma_S \\
&\qquad\quad + WV_{Wx}\sigma_X\sigma_S - Wf_{Wz}\sigma_X\sigma_S.
\end{aligned}$$

Combining the equilibrium condition, $u = 1$, with these first order conditions I obtain

$$\mu_S + \frac{e}{S} - r = -W\frac{V_{WW}}{V_W}\sigma_S^2 - \frac{V_{We}}{V_W}e\sigma_e\sigma_S - \frac{V_{Wx}}{V_W}\sigma_X\sigma_S + \frac{f_{Wz}}{V_W}\sigma_X\sigma_S.$$

From arbitrage theory we know that market price of risk $\lambda \in \mathbb{R}$ is a process such that

$$\mu_{S,t} + \frac{e_t}{S_t} - r_t = \lambda_t\sigma_{S,t}.$$

Hence, I can identify the market price of risk λ as

$$\lambda = -W\frac{V_{WW}}{V_W}\sigma_S - \frac{V_{We}}{V_W}e\sigma_e - \frac{V_{Wx}}{V_W}\sigma_X + \frac{f_{Wz}}{V_W}\sigma_X. \tag{A.13}$$

However, (A.13) involves endogenous objects that depend on V and f. The goal is to obtain the expression for λ in terms of exogenous objects: the utility function U and the exogenously given processes e and X. For this I use the first order condition for consumption together with extended equilibrium condition (12)

$$U_c(t, \frac{W}{S(t,e,x)}e, x) = V_W(t, W, e, x).$$

We then have

$$
\begin{aligned}
V_{WW}(t,W,e,x) &= U_{cc}(t,\frac{W}{S(t,e,x)}e,x)\frac{e}{S(t,e,x)},\\
V_{We}(t,W,e,x) &= U_{cc}(t,\frac{W}{S(t,e,x)}e,x)W\frac{S(t,e,x)-eS_e(t,e,x)}{S^2(t,e,x)},\\
V_{Wx}(t,W,e,x) &= U_{cc}(t,\frac{W}{S(t,e,x)}e,x)We\frac{-S_x(t,e,x)}{S^2(t,e,x)}+U_{cx}(t,\frac{W}{S(t,e,x)}e,x)
\end{aligned}
$$

Using the fact that from Ito we have

$$
\sigma_S(t,e,x)=\frac{S_e(t,e,x)}{S(t,e,x)}e\sigma_e+\frac{S_x(t,e,x)}{S(t,e,x)}\sigma_X
$$

together with the fact that in equilibrium in our model $W=S(t,e,x)$ allows to rewrite the first two terms in the expression for the market price of risk as follows

$$
-W\frac{V_{WW}}{V_W}\sigma_S-\frac{V_{We}}{V_W}e\sigma_e-\frac{V_{Wx}}{V_W}\sigma_X=-\frac{U_{cc}(t,e,x)}{U_c(t,e,x)}e\sigma_e-\frac{U_{cx}(t,e,x)}{U_c(t,e,x)}\sigma_X.
$$

To represent the last term in the expression for the market price of risk in terms of exogenous quantities we recall the probabilistic interpretation of function $f(t,W,e,x,z)$

$$
f(t,W,e,x,z)=E_{t,W,e,x}\left[\int_t^T U(s,\hat{\mathbf{c}}(s,W_s^{\hat{\mathbf{u}},\hat{\mathbf{c}}},X_s),z)ds\right].
$$

Using the extended equilibrium condition (12) we can write this as

$$
\begin{aligned}
f(t,W,e,x,z) &= E_{t,W,e,x}\left[\int_t^T U(s,\frac{W_s}{S(s,e_s,X_s)}e_s,z)ds\right]\\
&= E_{t,e,x}\left[\int_t^T U(s,\frac{W}{S(t,e,x)}e_s,z)ds\right].
\end{aligned}
$$

Thus we have

$$
f_{Wz}(t,W,e,x,z)=E_{t,e,x}\left[\int_t^T U_{cx}(s,\frac{W}{S(t,e,x)}e_s,z)\frac{1}{S(t,e,x)}e_sds\right].
$$

Also from the definition of f in Proposition 1 we know that

$$
V(t,W,e,x)=f(t,W,e,x,x).
$$

From this it follows that

$$
V_W(t,W,e,x)=f_W(t,W,e,x,x)=E_{t,x}\left[\int_t^T U_c(s,\frac{W}{S(t,e,x)}e_s,x)\frac{1}{S(t,e,x)}e_sds\right].
$$

Using the equilibrium condition for the original equilibrium in our model, $W = S(t,e,x)$, we can therefore conclude that

$$\frac{f_{Wz}(t,W,e,x,x)}{V_W(t,W,e,x,x)} = \frac{G_z(t,e,x)}{G(t,e,x)},$$

with

$$G(t,e,z) = E_{t,e}\left[\int_t^T U_c(s,e_s,z)e_s ds\right].$$

Putting all the parts of the expression for λ together we arrive at the final result stated in Proposition 2.

Proof of Proposition 3. Consider the ratio

$$\frac{f_W(t,W,e,x,z)\cdot S(t,e,x)}{U_c(t,\hat{c},x)\cdot e} \tag{A.14}$$

From the definition of f in Proposition 1 we know that

$$V(t,W,e,x) = f(t,W,e,x,x)$$

and hence we can write the first order condition for consumption as

$$U_c(t,\hat{c},x) = f_W(t,W,e,x,x).$$

Therefore the ratio in (A.14) is the price-dividend ratio for the claim on aggregate endowment, $S(t,e,x)/e$. The aim now it to obtain the expression for this ratio in terms of exogenous quantities. For this we recall the probabilistic interpretation of function $f(t,W,e,x,z)$

$$f(t,W,e,x,z) = E_{t,W,e,x}\left[\int_t^T U(s,\hat{\mathbf{c}}(s,W_s^{\hat{\mathbf{u}},\hat{\mathbf{c}}},X_s),z)ds\right].$$

and use the extended equilibrium condition (12) to re-write the above expression as

$$\begin{aligned} f(t,W,e,x,z) &= E_{t,W,e,x}\left[\int_t^T U(s,\frac{W_s}{S(s,e_s,X_s)}e_s,z)ds\right] \\ &= E_{t,e,x}\left[\int_t^T U(s,\frac{W}{S(t,e,x)}e_s,z)ds\right]. \end{aligned}$$

Thus we have

$$f_W(t,W,e,x,z) = E_{t,e,x}\left[\int_t^T U_c(s,\frac{W}{S(t,e,x)}e_s,z)\frac{1}{S(t,e,x)}e_s ds\right]$$

Using this together with the fact that for the original equilibrium in our model $W = S(t, e, x)$ the price-dividend ration (or equivalently the equilibrium wealth-consumption ratio) in (A.14) can be written as

$$E_{t,e,x}\left[\int_t^T \frac{U_c(s, e_s, x)}{U_c(t, e, x)} \frac{e_s}{e} ds\right]$$

and this proves the result.

Proof of Proposition 4. The equilibrium stock price is given by $S_t = e_t \Gamma_t$. From the fact that the equilibrium wealth-consumption ratio $\Gamma_t = \Gamma(t, e, x)$ with

$$\Gamma(t, e, x) = E_{t,e,x}\left[\int_t^T \frac{U_c(s, e_s, x)}{U_c(t, e, x)} \frac{e_s}{e} ds\right]$$

we see that the equilibrium wealth-consumption ratio Γ evolves as a Markov diffusion process

$$d\Gamma_t = \Gamma_t \left[\mu_{\Gamma,t} dt + \sigma_{\Gamma,t} dw_t\right],$$

where $\mu_{\Gamma,t} = \mu_\Gamma(t, e_t, X_t)$ and $\sigma_{\Gamma,t} = \sigma_\Gamma(t, e_t, X_t)$ with

$$\mu_\Gamma(t, e, x) = \frac{\mathcal{A}\Gamma(t, e, x)}{\Gamma(t, e, x)}, \quad \sigma_\Gamma(t, e, x) = \frac{\Gamma_e(t, e, x)}{\Gamma(t, e, x)} e\sigma_e + \frac{\Gamma_x(t, e, x)}{\Gamma(t, e, x)} \sigma_X.$$

Moreover, the endowment process is given exogenously as

$$de_t = e_t \left[\mu_{e,t} dt + \sigma_{e,t} dw_t\right].$$

The Ito formula gives

$$dS_t = e_t d\Gamma_t + \Gamma_t de_t + de_t d\Gamma_t$$

and using the dynamics of e and Γ it follows that

$$dS_t = S_t \left[\mu_{S,t} dt + \sigma_{S,t} dw_t\right]$$

with

$$\mu_{S,t} = \mu_{e,t} + \mu_{\Gamma,t} + \sigma_{e,t}\sigma_{\Gamma,t}, \quad \sigma_{S,t} = \sigma_{e,t} + \sigma_{\Gamma,t}.$$

Proof of Proposition 5. From the arbitrage theory we know

$$\mu_{S,t} + \frac{e_t}{S_t} - r_t = \lambda_t \sigma_{S,t}.$$

Here we know that $\mu_{S,t} = \mu_S(t, e_t, X_t)$, $\sigma_{S,t} = \sigma_S(t, e_t, X_t)$ and $\lambda_t = \lambda(t, e_t, X_t)$ with

$$\begin{aligned}
\mu_S(t,e,x) &= \mu_e(t,e,x) + \frac{\mathcal{A}\Gamma(t,e,x)}{\Gamma(t,e,x)} + \frac{\Gamma_e(t,e,x)}{\Gamma(t,e,x)} e\sigma_e^2(t,e,x) + \frac{\Gamma_x(t,e,x)}{\Gamma(t,e,x)}\sigma_X(t,e,x)\sigma_e(t,e,x),\\
\sigma_S(t,e,x) &= \sigma_e(t,e,x) + \frac{\Gamma_e(t,e,x)}{\Gamma(t,e,x)} e\sigma_e(t,e,x) + \frac{\Gamma_x(t,e,x)}{\Gamma(t,e,x)}\sigma_X(t,e,x),\\
\lambda(t,e,x) &= -\frac{U_{cc}(t,e,x)}{U_c(t,e,x)} e\sigma_e(t,e,x) - \frac{U_{cx}(t,e,x)}{U_c(t,e,x)}\sigma_X(t,e,x) + \frac{G_z(t,e,x,x)}{G(t,e,x,x)}\sigma_X(t,e,x).
\end{aligned}$$

Moreover $e/S(t,e,x) = 1/\Gamma(t,e,x)$ which we have derived in the previous results. Using this the expression for the risk free rate can be written as (omitting the arguments)

$$\begin{aligned}
r =& \frac{\Gamma_t}{\Gamma} + \frac{1}{\Gamma} + \mu_e e\left[\frac{1}{e} + \frac{\Gamma_e}{\Gamma}\right] + \mu_X \frac{\Gamma_x}{\Gamma} + \sigma_e^2 e^2 \left[\frac{1}{2}\frac{\Gamma_{ee}}{\Gamma} + \frac{1}{e}\frac{\Gamma_e}{\Gamma} + \frac{U_{cc}}{U_c}\left(\frac{1}{e} + \frac{\Gamma_e}{\Gamma}\right)\right]\\
&+ \sigma_X^2 \left[\frac{1}{2}\frac{\Gamma_{xx}}{\Gamma} + \left(\frac{U_{cx}}{U_c} - \frac{G_z}{G}\right)\frac{\Gamma_x}{\Gamma}\right]\\
&+ \sigma_e e \sigma_X \left[\frac{\Gamma_{ex}}{\Gamma} + \frac{1}{e}\frac{\Gamma_x}{\Gamma} + \frac{U_{cc}}{U_c}\frac{\Gamma_x}{\Gamma} + \left(\frac{U_{cx}}{U_c} - \frac{G_z}{G}\right)\left(\frac{1}{e} + \frac{\Gamma_e}{\Gamma}\right)\right]
\end{aligned}$$

where Γ is the equilibrium price-dividend ratio and is given by

$$\Gamma(t,e,x) = G(t,e,x)\frac{1}{U_c(t,e,x)e},$$

where $G(t,e,z)$ is

$$G(t,e,z) = E_{t,e,x}\left[\int_t^T U_c(s,e_s,z)e_s ds\right].$$

We will now rewrite the expression for the risk free rate using the definition of Γ in terms of G and later make use of a version of Feynman-Kac representation result for G. Firstly, we can rewrite the first term as

$$\frac{\Gamma_t(t,e,x)}{\Gamma(t,e,x)} + \frac{1}{\Gamma(t,e,x)} = \frac{G_t(t,ex)}{G(t,e,x)} - \frac{U_{ct}(t,e,x)}{U_c(t,e,x)} + \frac{U_c(t,e,x)e}{G(t,e,x)}.$$

Secondly, the term multiplying $\mu_e e$ can be written as

$$\frac{1}{e} + \frac{\Gamma_e(t,e,x)}{\Gamma(t,e,x)} = \frac{G_e(t,e,x)}{G(t,e,x)} - \frac{U_{cc}(t,e,x)}{U_c(t,e,x)}.$$

Next, consider the term multiplying μ_X. We have that

$$\frac{\Gamma_x(t,e,x)}{\Gamma(t,e,x)} = \frac{G_z(t,e,x)}{G(t,e,x)} - \frac{U_{cx}(t,e,x)}{U_c(t,e,x)}.$$

For the term multiplying $\sigma_e^2 e^2$ we get

$$\frac{1}{2}\frac{\Gamma_{ee}(t,e,x)}{\Gamma(t,e,x)} + \frac{1}{e}\frac{\Gamma_e(t,e,x)}{\Gamma(t,e,x)} + \frac{U_{cc}(t,e,x)}{U_c(t,e,x)}\left(\frac{1}{e} + \frac{\Gamma_e(t,e,x)}{\Gamma(t,e,x)}\right)$$
$$= \frac{1}{2}\frac{G_{ee}(t,e,x)}{G(t,e,x)} - \frac{1}{2}\frac{U_{ccc}(t,e,x)}{U_c(t,e,x)}.$$

As for the term multiplying σ_X^2 we have

$$\frac{1}{2}\frac{G_{zz}(t,e,x)}{G(t,e,x)} - \frac{1}{2}\frac{U_{cxx}(t,e,x)}{U_c(t,e,x)} - \frac{G_z(t,e,x)}{G(t,e,x)}\frac{G_z(t,e,x)}{G(t,e,x)} + \frac{G_z(t,e,x)}{G(t,e,x)}\frac{U_{cx}(t,e,x)}{U_c(t,e,x)}.$$

Finally, the last term multiplying $\sigma_e e \sigma_X$ can be rewritten as

$$\frac{G_{ez}(t,e,x)}{G(t,e,x)} - \frac{U_{ccx}(t,e,x)}{U_c(t,e,x)} + \frac{G_z(t,e,x)}{G(t,e,x)}\left(\frac{U_{cc}(t,e,x)}{U_c(t,e,x)} - \frac{G_e(t,e,x)}{G(t,e,x)}\right)$$

Putting everything together and rewriting we get

$$\begin{aligned}
r = \; & \frac{U_c(t,e,x)e}{G(t,e,x)} + \frac{G_t(t,e,x)}{G(t,e,x)} + \mu_e e\frac{G_e(t,e,x)}{G(t,e,x)} + \frac{1}{2}\sigma_e^2 e^2\frac{G_{ee}(t,e,x)}{G(t,e,x)} \\
& - \frac{U_{ct}(t,e,x)}{U_c(t,e,x)} - \mu_e e\frac{U_{cc}(t,e,x)}{U_c(t,e,x)} - \frac{1}{2}\frac{U_{ccc}(t,e,x)}{U_c(t,e,x)}\sigma_e^2 e^2 \\
& - \mu_X\frac{U_{cx}(t,e,x)}{U_c(t,e,x)} - \frac{1}{2}\sigma_X^2\frac{U_{cxx}(t,e,x)}{U_c(t,e,x)} - \sigma_e e\sigma_X\frac{U_{ccx}(t,e,x)}{U_c(t,e,x)} \\
& + \mu_X\frac{G_z(t,e,x)}{G(t,e,x)} + \frac{1}{2}\sigma_X^2\left(\frac{G_{zz}(t,e,x)}{G(t,e,x)} - 2\frac{G_z(t,e,x)}{G(t,e,x)}\frac{G_z(t,e,x)}{G(t,e,x)}\right) \\
& + \sigma_e e\sigma_X\left(\frac{G_{ez}(t,e,x)}{G(t,e,x)} - \frac{G_z(t,e,x)}{G(t,e,x)}\frac{G_e(t,e,x)}{G(t,e,x)}\right) \\
& + \frac{G_z(t,e,x)}{G(t,e,x)}\sigma_X\left(\frac{U_{cx}(t,e,x)}{U_c(t,e,x)}\sigma_X + \frac{U_{cc}(t,e,x)}{U_c(t,e,x)}\sigma_e e\right)
\end{aligned}$$

I now make use of the Feynman-Kac representation result for $G(t,e,z)$. Namely, we have that

$$G_t(t,e,z) + \mu_e e G_e(t,e,z) + \frac{1}{2}\sigma_e^2 e^2 G_{ee}(t,e,z) + U_c(t,e,z)e = 0.$$

This allows to conclude that the first row of the r expression above is equal to zero. The next two rows in the expression are standard components of the risk free rate in the state dependent unity model. Namely,

$$r^e(t,e,x) = -\frac{U_{ct}(t,e,x)}{U_c(t,e,x)} - \mu_e e\frac{U_{cc}(t,e,x)}{U_c(t,e,x)} - \frac{1}{2}\sigma_e^2 e^2\frac{U_{ccc}(t,e,x)}{U_c(t,e,x)}$$

and

$$r^x(t,e,x) = -\mu_X\frac{U_{cx}(t,e,x)}{U_c(t,e,x)} - \frac{1}{2}\sigma_X^2\frac{U_{cxx}(t,e,x)}{U_c(t,e,x)} - \sigma_e e\sigma_X\frac{U_{ccx}(t,e,x)}{U_c(t,e,x)}.$$

The remaining part we can write as

$$r^z(t,e,x) = \mu_X \frac{G_z(t,e,x)}{G(t,e,x)} - \frac{1}{2}\sigma_X^2 \frac{G_{zz}(t,e,x)}{G(t,e,x)} + \sigma_X^2 \frac{\partial}{\partial z}\frac{G_z(t,e,x)}{G(t,e,x)} + \sigma_e e \sigma_X \frac{\partial}{\partial e}\frac{G_z(t,e,x)}{G(t,e,x)}$$
$$- \lambda^z(t,e,x)\left(\lambda^e(t,e,x) + \lambda^x(t,e,x)\right)$$

This gives the result stated in the proposition.

Proof of Proposition 7. Using Ito formula we know that

$$dU_c(t,e_t,X_t) = \mathcal{A}U_c(t,e_t,X_t)dt + \left(\sigma_e e_t U_{cc}(t,e_t,X_t) + \sigma_X U_{cx}(t,e_t,X_t)\right)dw_t,$$

where using our results from Propositions 2 and 5

$$\mathcal{A}U_c(t,e_t,X_t) = -\left(r_t^e + r_t^x\right), \quad \sigma_e e_t U_{cc}(t,e_t,X_t) + \sigma_X U_{cx}(t,e_t,X_t) = -\left(\lambda_t^e + \lambda_t^x\right).$$

Consider the following dynamics of M^z

$$dM_t^z = -M_t^z\left[r_t^z dt + \lambda_t^z d\tilde{w}_t\right].$$

Moreover, let $\tilde{w}$ be a $\tilde{Q}$-Wiener process with measure $\tilde{Q}$ defined by $d\tilde{Q}/dP = \tilde{L}_t$, on $\mathcal{F}_t$, with the likelihood process $\tilde{L}$ given by

$$d\tilde{L}_t = -\tilde{L}_t(\lambda_t^e + \lambda_t^x)dw_t.$$

From Girsanov theorem we know that the P dynamics of M^z is

$$dM_t^z = -M_t^z\left[\left(r_t^z + \lambda_t^z(\lambda_t^e + \lambda_t^x)\right)dt + \lambda_t^z dw_t\right].$$

Looking at the dynamics of the product

$$M_t = \frac{U_c(t,e_t,X_t)}{U_c(0,e_0,X_0)} \cdot M_t^z$$

we obtain

$$dM_t = -M_t\Big[\underbrace{\left(r_t^e + r_t^x + r_t^z\right)}_{=r_t}dt + \underbrace{\left(\lambda_t^e + \lambda_t^x + \lambda_t^z\right)}_{\lambda_t}dw_t\Big].$$

Proof of Formula (21). From Proposition 7 we know that the dynamics of M^z is given by

$$dM_t^z = -M_t^z\left[r_t^z dt + \lambda_t^z d\tilde{w}_t\right],$$

where $\tilde{w}$ is a $\tilde{Q}$-Wiener process. From Girsanov theorem we know that the P-dynamics of M^z is

$$dM_t^z = -\left[r_t^z + \lambda_t^z(\lambda_t^e + \lambda_t^x)\right] M_t^z dt - \lambda_t^z M_t^z dw_t$$

and thus

$$M_t^z = \exp\left\{-\left(\int_0^t \alpha_s^z ds + \lambda_s^z dw_s\right)\right\}$$

with

$$\alpha_s^z = r_s^z + \lambda_s^z(\lambda_s^e + \lambda_s^x) + \frac{1}{2}(\lambda_s^z)^2.$$

Using the definitions of λ^z and r^z from Propositions 2 and 5 we can rewrite $\alpha^z(t, e, x)$ as follows

$$\begin{aligned}
\alpha^z(t,e,x) =& \mu_X \frac{G_z(t,e,x)}{G(t,e,x)} - \frac{1}{2}\sigma_X^2 \frac{G_{zz}(t,e,x)}{G(t,e,x)} + \sigma_X^2 \frac{\partial}{\partial z}\frac{G_z(t,e,x)}{G(t,e,x)} + \sigma_e e \sigma_X \frac{\partial}{\partial e}\frac{G_z(t,e,x)}{G(t,e,x)} \\
&+ \frac{1}{2}\sigma_X^2 \left(\frac{G_z(t,e,x)}{G(t,e,x)}\right)^2 \\
=& \mu_X \frac{G_z(t,e,x)}{G(t,e,x)} + \frac{1}{2}\sigma_X^2 \frac{\partial}{\partial z}\frac{G_z(t,e,x)}{G(t,e,x)} + \sigma_e e \sigma_X \frac{\partial}{\partial e}\frac{G_z(t,e,x)}{G(t,e,x)} \\
=& \mu_X \frac{\partial}{\partial z}\log G(t,e,x) + \frac{1}{2}\sigma_X^2 \frac{\partial^2}{\partial z^2}\log G(t,e,x) + \sigma_e e \sigma_X \frac{\partial}{\partial ze}\log G(t,e,x) \\
=& \mathcal{A}G\log(t,e,x) - \mathcal{A}\log G^x(t,e).
\end{aligned}$$

In the last step I used the infinitesimal operator $\mathcal{A}$ defined in (A.3) and the notational convention that $\mathcal{A}$ operates only on the variables within parentheses, while the upper case indices are treated as constant parameters.

Proof of Proposition 8. The proof is a multidimensional version of the proof for Proposition 1.

Proof of Proposition 9. The proof is a multidimensional version of the proofs for Propositions 2, 5 and 7.

Proof of Propositions 10-11. Direct application of results obtained in the general model.

Proof of Formula (41). In the case of the power utility

$$U(\tau, c) = \frac{c^{1-\gamma(\tau)}}{1-\gamma(\tau)}.$$

We have

$$U_c(\tau, c) = c^{-\gamma(\tau)}, \quad U_{c\tau}(\tau, c) = -\gamma'(\tau)c^{-\gamma(\tau)} \ln c.$$

The G function is then

$$G(t, e, z) = E_{t,e}\left[\int_t^T \exp(-\rho s) e_s^{1-\gamma(s-z)} ds\right]$$

We know that the price dividend ratio is

$$\begin{aligned}\Gamma(t, e) &= \frac{1}{U_c(t, e, t)e} G(t, e, t) \\ &= \frac{1}{\exp(-\rho t)e^{1-\gamma(0)}} E_{t,e}\left[\int_t^T \exp(-\rho s) e_s^{1-\gamma(s-t)} ds\right]\end{aligned}$$

We see that we need to compute

$$E_{t,e}\left[e_s^{1-\gamma(s-t)}\right].$$

If e follows a GBM then

$$e_s = e_t \exp\left\{(\mu_e - \frac{1}{2}\sigma_e^2)(s-t) + \sigma_e(W_s - W_t)\right\}$$

Taking the power $1 - \gamma(s-t)$

$$e_s^{1-\gamma(s-t)} = e_t^{1-\gamma(s-t)} \exp\left\{(1-\gamma(s-t))(\mu_e - \frac{1}{2}\sigma_e^2)(s-t) + (1-\gamma(s-t))\sigma_e(W_s - W_t)\right\}.$$

Finally, taking the time t expectation

$$E_t\left[e_s^{1-\gamma(s-t)}\right] = e_t^{1-\gamma(s-t)} \exp\left\{(1-\gamma(s-t))\left(\mu_e - \frac{1}{2}\gamma(s-t)\sigma_e^2\right)(s-t)\right\}.$$

Using this in our expression of the price dividend ratio we have

$$\Gamma(t, e) = \int_t^T e^{\gamma(0)-\gamma(s-t)} \exp\left\{\xi(s-t)(s-t)\right\} ds,$$

where $\xi(s-t)$ is given by

$$\xi(s-t) = -\rho + (1-\gamma(s-t))(\mu_e - \frac{1}{2}\gamma(s-t)\sigma_e^2).$$

Proof of Proposition 12. Follows from the general theory and Equation (41).

Proof of Formula (46). In the case with power utility we have $U_c(s, e_s, z) = \exp(-\rho s)e_s^{-\gamma(z)}$

$$G(t, e, z) = E_{t,e}\left[\int_t^T \exp(-\rho s)e_s^{1-\gamma(z)}ds\right].$$

If e follows a GBM we have

$$e_s = e_t \exp\left\{(\mu_e - \frac{1}{2}\sigma_e^2)(s-t) + \sigma_e(W_s - W_t)\right\}$$

and thus

$$e_s^{1-\gamma(z)} = e_t^{1-\gamma(z)} \exp\left\{(1-\gamma(z))(\mu_e - \frac{1}{2}\sigma_e^2)(s-t) + (1-\gamma(z))\sigma_e(W_s - W_t)\right\}.$$

Taking expectations

$$\begin{aligned} E_t\left[e_s^{1-\gamma(z)}\right] &= e_t^{1-\gamma(z)} \exp\left\{(1-\gamma(z))(\mu_e - \frac{1}{2}\sigma_e^2)(s-t) + (1-\gamma(z))^2\frac{1}{2}\sigma_e^2(s-t)\right\} \\ &= e_t^{1-\gamma(z)} \exp\left\{(1-\gamma(z))(\mu_e - \frac{1}{2}\sigma_e^2\gamma(z))(s-t)\right\}. \end{aligned}$$

This gives

$$G(t, e, z) = \exp(-\rho t)e^{1-\gamma(z)}\frac{\exp(\xi(z)(T-t)) - 1}{\xi(z)},$$

where

$$\xi(z) = -\rho + (1-\gamma(z))(\mu_e - \frac{1}{2}\sigma_e^2\gamma(z))$$

and recalling that the price dividend ratio is given by $G(t, e, x)/U_c(t, e, x)e$ we arrive at expression (46).

Proof of Proposition 13. The expression for volatility follows directly from Proposition 4 and (46). The market price of risk follows from Proposition 2. In order to derive the expression for the market price of risk we compute $G_z(t, e, x)/G(t, e, x)$.

$$\begin{aligned} G(t, e, z) &= -\exp(-\rho t)e^{1-\gamma(z)}\xi^{-1}(z) \\ G_z(t, e, z) &= \exp(-\rho t)e^{1-\gamma(z)}\left[\ln(e)\frac{\gamma'(z)}{\xi(z)} + \frac{\xi'(z)}{\xi^2(z)}\right] \end{aligned}$$

and therefore

$$G_z(t, e, x)/G(t, e, x) = -\ln(e)\gamma'(x) - \frac{\xi'(x)}{\xi(x)}.$$

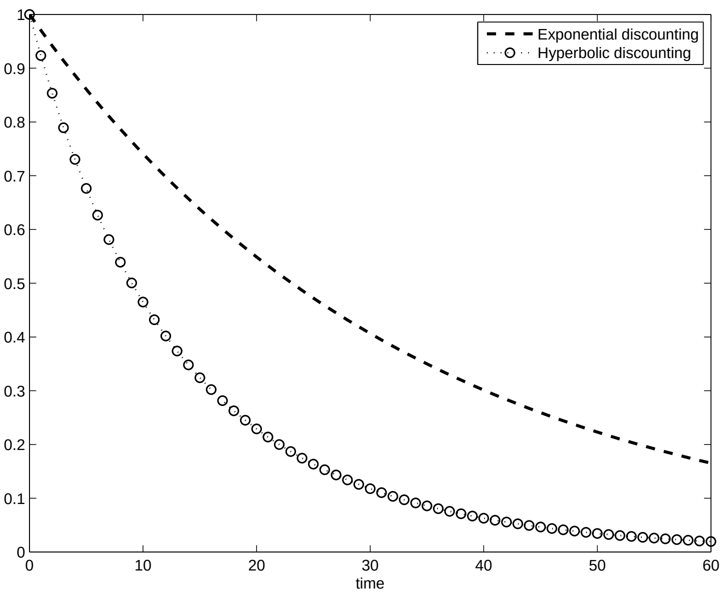

Figure 1: This figure plots the hyperbolic and exponential discounting functions. The parameter values are $\rho = 0.03$, $\alpha = 0.015$, $\beta = 0.05$.

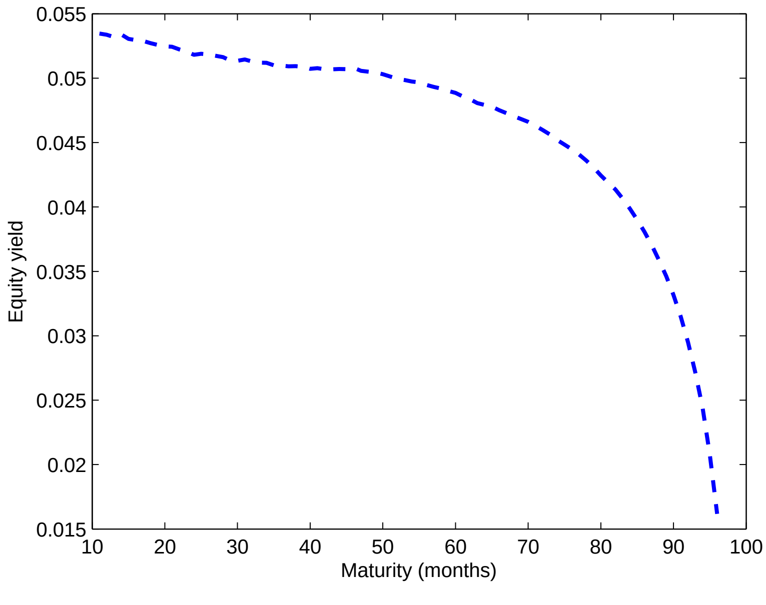

Figure 2: This figure plots the simulated equity yield $y(t, \tau)$ as a function of maturity. The parameter values are $\mu_e = 0.02$, $\sigma_e = 0.035$, $\rho = 0.053$. The risk aversion is assumed to be decreasing linearly with horizon, from 10 to 1.

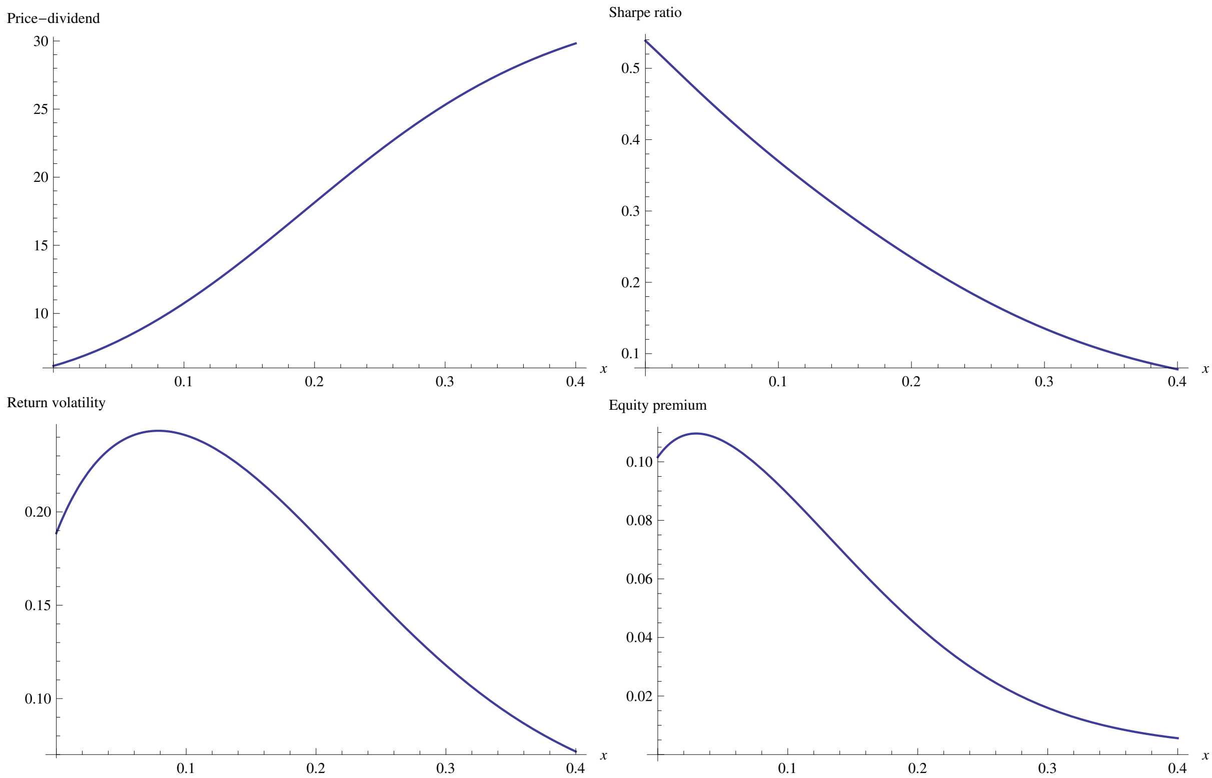

Figure 3: This figure plots the price dividend ratio, the Sharpe ratio and the conditional moments of returns for the risky asset against the state variable X. The parameter values are $\mu_e = 0.02$, $\sigma_e = 0.035$, $\lambda = 0.08$, $\rho = 0.053$. Under this choice of parameters $E[X] = 0.24$, $\sigma[X] = 0.08$. Risk aversion decreases exponentially from 11 to 1 as a function of x.