Robust Conditional Predictions in Dynamic Games: An Application to Sovereign Debt

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Abstract

We study the problem of an outside observer who, based on observations on a particular economic setting, tries to make an inference about the set of equilibria where the economy could be; our aim is to obtain robust predictions conditional on equilibrium history for dynamic policy games. Our main result is a characterization of "equilibrium consistent outcomes": outcomes of the game in a particular period, that are consistent with a subgame perfect equilibrium *conditional* on an *equilibrium* history. We focus on a model of sovereign debt as in Eaton and Gersovitz (1981); but, our methodology can be readily applied to other dynamic policy games as in models of capital taxation or monetary policy. For this model, we characterize the set of equilibrium consistent debt prices. While the upper bound is the price in the optimal Markov equilibria, the lowest equilibrium consistent price is both positive and backward looking. In our baseline case, where output is continuous or there is a public randomization device, the last opportunity cost (the amount of debt just repaid, the conditions under which that debt was repaid, and how much debt was issued) is a sufficient statistic of the set of equilibrium consistent prices. When income is discrete or there are no randomization devices available, the whole history could determine equilibrium consistent prices; whether or not a particular event in history matters for equilibrium consistent prices today depends on the likelihood and the length of that particular history. We finally link the set of equilibrium consistent outcomes with the predictions of a Bayesian econometrician who is uncertain about the prior over outcomes.

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1 Introduction

Following Kydland and Prescott (1977) the literature on optimal policy when the government lacks commitment has studied dynamic policy games in applications including optimal capital taxation (e.g. Chari and Kehoe (1990), Phelan and Stacchetti (2001)), optimal monetary policy (e.g. Ireland (1997), Chang (1998), Sleet (2001)) and sovereign debt (e.g. Calvo (1988), Eaton and Gersovitz (1981), Chari and Kehoe (1993a), Cole and Kehoe (2000)). This literature has helped us to understand how lack of commitment in government policies introduces restrictions on the set of outcomes that the government can achieve.

One of the challenges in studying dynamic policy games is equilibrium multiplicity. These settings feature multiple equilibria that usually imply different predictions over observable outcomes. So, our objective in this paper is to obtain robust (conditional) predictions in dynamic policy games; robust conditional predictions refer to predictions that hold across all equilibria, use histories of play to make an inference over observable outcomes, and only rely on the assumption that histories are generated by the path of some equilibrium. To obtain these predictions, our main contribution is to conceptually introduce and characterize "equilibrium consistent outcomes": outcomes of the game in a particular period that are consistent with a subgame perfect equilibrium, conditional on an observed history.

We focus on a model of sovereign debt as in Eaton and Gersovitz (1981). In this model, a small open economy is endowed with a stochastic stream of income and is inhabited by consumers who receive transfers from the government. To smooth consumption of the households, the government can borrow from international debt markets. The government is benevolent, in the sense it that wants to maximize the utility of the households, and has limited commitment in terms of repayment. In case it chooses to default on its debt, the punishment is permanent exclusion from the financial market.

Even though we focus on a model of sovereign debt as in Eaton and Gersovitz (1981) our results generalize to other dynamic policy games. One of the reasons is that the sovereign debt setting features time inconsistency in the same way as most dynamic macro models with lack of commitment; the agent forms an expectation regarding a government action and then the government takes an action. In this sense, our results can be readily generalized to other dynamic macro settings.

Our main result is the characterization of equilibrium consistent outcomes in a particular period (debt prices, debt policy, and default policy). Aided by this characterization, we obtain bounds over equilibrium consistent debt prices that are history dependent. While the highest equilibrium consistent price is the best Markov equilibria (as in Eaton and Gersovitz (1981)), the lowest equilibrium consistent price is both positive and backward looking. In our baseline case, where output is continuous or there is a public randomization device, the last opportunity cost (the amount of debt just repaid, the conditions under which that debt was repaid, and how much debt was issued) determines this lowest price, and hence is a sufficient statistic for the set of equilibrium consistent prices.

The set of equilibrium consistent debt prices that we characterize has intuitive comparative statistics. For example, if the country has just repaid a high amount of debt, or has repaid under harsh economic conditions (low output), the lowest equilibrium consistent price needs to be higher. The intuition is that this choice (repayment decision) would not have been rational if the price that the country was expecting was too low. An example of the applicability of our results is the probability of a crises conditional on the observed history. The Eaton and Gersovitz setting has been widely used to study sovereign defaults due to changes in fundamentals.[1] We show that in this setting, sovereign crises occur on the equilibrium path; but, more importantly, the severity of the crises that an outside observer can expect depends on the history of past actions. In this sense, there is a role for past actions in refining future outcomes.

The result that the last opportunity cost is a sufficient statistic for equilibrium consistent prices is in a way surprising. The reason is that the outside observer (or econometrician) is only using observations from the last period to make inferences, even though he has a whole history of data available. However, this result is a direct expression of Robustness: the outside observer needs to take into account that the expected payoff that rationalized a particular decision could have been realized only in histories that never occurred. When income is continuous, because any particular history has probability zero, it can always be the case that the expected payoff was promised for states that did not materialize[2].

To better understand the role of Robustness in determining equilibrium consistent outcomes, we then move to the case where there are no randomization devices available or output is discrete. In a two period example, we provide a simple argument for why we obtain the sufficiency result in the continous income case. Then, we characterize how history matters for the lowest equilibrium consistent price and show that, in principle, the whole history can affect equilibrium consistent prices in a particular period. The restrictions that past decisions impose on current prices (and policies) can be decomposed into two terms. The first term is the value that the government receives in the realized history times the probability of that history. The second term is the value of the best equilibrium, off the equilibrium path. This term is the difference between the value of the best equilibrium and the value of the best equilibrium times the probability of the observed history. So, the link beetween current prices and past decisions depends on the strength (probability) and the length (discounting) of the link between current and past periods.

Finally, as an interpretation of our results, we relate equilibrium consistent outcomes to Robust Bayesian Analysis. In Bayesian Analysis, the econometrician has a prior over fundamental parameters and after observing the data derives a posterior; in Robust Bayesian analysis the econometrician is uncertain about the prior and hence, after observing the data, derives a set of posteriors from the set of possible priors. In a dynamic policy game, given equilibrium multiplicity, the Bayesian econometrician has a prior over outcomes (mappings from exogenous variables into endogenous variables). In many contexts, there is no reason to favor one outcome over the other; there is uncertainty with respect to the prior. We show that the set of equilibrium consistent outcomes we characterize is the support of the posterior over outcomes of a Bayesian econometrician that only assume that the data was generated by a subgame perfect equilibrium; in other words, is agnostic with respect to the particular prior over equilbrium outcomes.

[1] There are some exceptions, where this setting has been used to study crises. See Chatterjee and Eyigungor (2012) Section 4. Our baseline case differs in the timing of the sunspots.

[2] This intuition was first introduced by Gul and Pearce (1996) to show that Forward induction has much less predictive power as a solution concept if there are sunspots.

Outline The paper is structured as follows. Section 2 covers a motivating example. Section 3 reviews the literature. Section 4 introduces the model and characterizes a Markov equilibrium. Section 5 characterizes equilibrium consistent outcomes. Section 6 discusses the characterization of equilibrium consistent outcomes for the case of discrete income. Section 7 links equilibrium consistent outcomes with the predictions of a Bayesian econometrician who is uncertain about the prior over outcomes. Section 8 characterizes equilibrium consistent outcomes in a model with savings and excusable defaults. Section 9 concludes.

2 Motivating Example

In this section, we provide a simple example that introduces the concept of equilibrium consistency and provides an intuition for the main propositions in the paper. Suppose that a government and an international lender play a "lending game". In the first stage, the government decides to repay some debt that it owes to the lender. If the government defaults it obtains utility of 2; if the government repays it plays a coordination game with the lenders. The government choose among three different debt levels b_L, b_M, b_H. The lender chooses among three debt prices q_L, q_M, q_H. The payoffs are depicted in figure $(a, b, c > 0)$ 2.

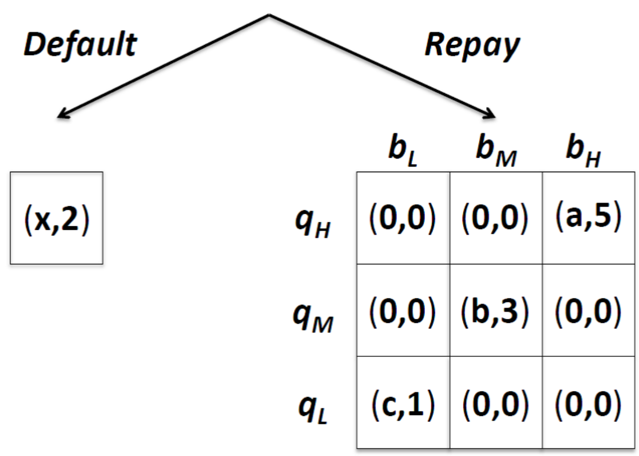

Figure 2.1: Lending game

SPE There are three subgame perfect equilibria (in pure strategies) in this game

$$\mathcal{E} = \{[(D, \text{If } P \text{ choose } b_L)(q_L)]\,[(P, b_M)(q_M)]\,[(P, b_H)(q_H)]\}$$

In the first equilibrium σ_1, the government defaults on its debt, and if it didn't default it would coordinate on the low equilibrium with the lender. Anticipating a payoff of 1, the government decides to default and obtains a payoff of 2. In the second equilibrium σ_2, the government repays its debt, anticipating that they will coordinate on the medium equilibrium. Finally, in the third equilibrium σ_3, the government repays its debt, anticipating that they will coordinate in the high equilibrium.

Equilibrium Consistency Suppose that an outside observer (she) just observed that the government has repaid debt. What are the outcomes that she thinks are plausible? If she does *not assume* that the history is on the path of an SPE, then any of the three Nash equilibria in the coordination game are plausible outcomes; they are subgame perfect in that subgame. The outcomes are

$$O(Pay\,|_{h(\neg\mathcal{E})}) = \{(q_L, b_L), (q_M, b_M), (q_H, b_H)\}$$

where $h(\neg\mathcal{E})$ denotes that the history is not on the path of a subgame perfect equilibrium. On the contrary, suppose now that she assumes that the history comes from equilibrium play; that it is on the path of a subgame perfect equilibrium. Then, the only two possible outcomes are

$$O(Pay\,|_{h(\mathcal{E})}) = \{(q_M, b_M), (q_H, b_H)\}$$

The reason is that there is no SPE where the government repays and the players coordinate on the low equilibrium *on the equilibrium path*. So, as outside observers, we use the fact that history is coming from an equilibrium to rule out outcomes that are not part of an equilibrium $\mathcal{E}$. For this game, $\{(q_M, b_M), (q_H, b_H)\}$ are *equilibrium consistent outcomes*, after observing repayment.

Note that in this simple game, equilibrium consistency is easily characterized. In most of the dynamic policy games characterizing equilibrium outcomes is a challenging task; the game is repeated, has intra-period dynamics, continuation payoffs need to be part of a continuation equilibrium, and action spaces are continuous.[3] The main result of the paper is the characterization of equilibrium consistent outcomes in a model of sovereign debt as in Eaton and Gersovitz (1981).

Robust Bayesian Analysis We just characterized the outcomes on which an outside observer places positive probability after observing an equilibrium history, and named them "equilbrium consistent outcomes." We now show how equilibrium consistent outcomes relate to Robust Bayesian Analysis. Suppose we are interested in a particular statistic from the game. In the paper, our result is on equilibrium consistent prices; but we can obtain *bounds* on any statistic of interest (debt levels, consumption, volatility of consumption, etc.). We first show how equilibrium consistency restricts the posteriors in Bayesian analysis. Then, we show how these posteriors depend on possibly uncertain priors, and how this yields bounds on the statistic of interest.

Suppose a Bayesian econometrician has a prior over SPE: equilibrium 1 is played with probability p_1, equilibrium 2 is played with probability p_2, and equilibrium 3 is played with probability p_3. After observing that the government has re-payed its debt, the probability

[3] Examples of SPE characterizations in dynamic policy games are Chari and Kehoe (1990), Atkeson (1991), Chari and Kehoe (1993b), Chari and Kehoe (1993a), Stokey (1991), Chang (1998) and Phelan and Stacchetti (2001). For a unified treatment see Ljungqvist and Sargent (2004).

of the medium and high outcomes will be

$$\frac{p_2}{1-p_1}, \frac{p_3}{1-p_1}$$

This implies a posterior over prices that is given by

$$\mathbb{E}^p\left(q \mid \text{Pay}\right) = \frac{p_2}{1-p_1} q_M + \frac{p_3}{1-p_1} q_H$$

Suppose that the econometrician is unsure about the prior; but *assumes* that he is observing an equilibrium path. Then, the set of prices that are consistent with his assumption is

$$[q_M, q_H] = \bigcup_{\mathbf{p}\in\Delta(\sigma_1,\sigma_2,\sigma_3)} \mathbb{E}^p\left(q \mid \text{Pay}\right)$$

where $\Delta\left(\sigma_1, \sigma_2, \sigma_3\right)$ is the simplex over strategies.[4] Section 7 formally proves the connection beetween equilibrium consistent outcomes and robust Bayesian analysis.

Trembles Note that trembles will not change the set of equilibrium consistent prices. Suppose now that the government trembles when it decides repayment or default; that is, with probability $\Pr\left(\text{mistake at (Pay,Default)}\right) = \delta \in (0,1)$. Then, the posterior probability of observing q_L is

$$\epsilon\left(\delta\right) = \frac{\delta p_1}{\delta p_1 + \left(1-\delta\right)\left(1-p_1\right)}$$

Denote by p_δ the posterior over outcomes. This implies that the posterior expectation on prices is given by

$$\mathbb{E}^{p_\delta}\left(q \mid \text{Pay}\right) = \epsilon\left(\delta\right) q_L + \left(1-\epsilon\left(\delta\right)\right)\left[\frac{p_2}{1-p_1} q_M + \frac{p_3}{1-p_1} q_H\right]$$

When the probability of the tremble approaches to zero

$$\lim_{\delta\to 0} \bigcup_{p_\delta} \mathbb{E}^{p_\delta}\left(q \mid \text{Pay}\right) = [q_M, q_H]$$

We also show this result in Section 7.

3 Literature Review

Our paper conceptually introduces and characterizes equilibrium consistent outcomes. In order to do so, we characterize equilibria in a dynamic game where there is a time inconsistency problem for the government. Our paper, in that sense, relates to the literature on credible government policies; the seminal contributions in that literature are Chari and

[4] Note that, in the case that a zero probability event is observed, there are no restrctions in the updating of the econometrician.

Kehoe (1990) and Stokey (1991)[5]. These two papers use the techniques developed in Abreu (1988) to characterize all subgame perfect equilibria with strategies that revert to the worst equilibrium after a deviation; in addition, they require that the strategies are consistent with a competitive equilibrium. Our characterization of implementable outcomes is in spirit related to those of Chari and Kehoe (1990) and Stokey (1991), but our focus is on characterizing outcomes in a particular period. So, using the techniques of Abreu (1988), our characterization relies on the best and worst equilibrium prices in a particular period.

Our paper relates to a strand of robustness literature that finds restrictions over observable outcomes that hold across all equilibria. The two papers more closely related to our work are Angeletos and Pavan (2013) and Bergemann and Morris (2013). The first paper obtains predictions that hold across every equilibrium in a global game with an endogenous information structure. The second paper obtains predictions that hold across every possible information structure in a class of coordination games. Our paper relates to them in that we obtain predictions that hold across all equilibria. In some sense, our results are weaker because our predictions are regarding the equilbrium set; on the other hand, our problem has the additional challenge of being a (repeated) dynamic complete information game. The latter is is precisely the root of weaker predictions. Still, we can make novel and intuitive predictions on how the set of equilibria is changing.

The literature of sovereign debt[6] has evolved in several directions. One direction, the quantitative literature on sovereign debt, focuses on a model where asset markets are incomplete and there is limited commitment for repayment, following Eaton and Gersovitz (1981), to study the quantitative properties of spreads, debt capacity, and business cycles. The aim of this strand of the literature is to account for the observed behavior of the data. The seminal contributions in this literature are Aguiar and Gopinath (2006) and Arellano (2008) which study economies with short term debt. Long term debt was introduced by Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2012). The quantitative literature of sovereign debt has already been successful in explaining the most salient features in the data.[7] Our paper shares with this literature the focus on a model along the lines of Eaton and Gersovitz (1981) but rather than charaterizing a particular equilibrium, it tries to study predictions regarding the set of equilibria.

Another direction of the literature focuses in equilibrium multiplicity, and in particular, in self fulfilling debt crises. The seminal contribution is Calvo (1988). Cole and Kehoe (2000) introduce self-fulfilling debt crises in a full-fledged dynamic model where the equilbrium selection mechanism is a sunspot that is realized simultaneously with output. Lorenzoni and Werning (2013) study equilbrium multiplicity in a dynamic version of Calvo (1988). Our paper studies multiplicity but in the Eaton and Gersovitz (1981) setting; the crucial difference beetween the setting in Calvo (1988) and the one Eaton and Gersovitz (1981) is that in the latter the government issues debt (with commitment) and then the price is realized. This implies that equilbrium multiplicity is coming from the multiplicity of beliefs

[5] Atkeson (1991) extends the approach to the case with a public state variable. Phelan and Stacchetti (2001) and Chang (1998) extend the approach to study models where individual agents hold stocks (capital and money respectively).

[6] For a review see Eaton and Fernandez (1995), Aguiar and Amador (2013a).

[7] Other examples in this literature are Yue (2010), Bai and Zhang (2012), Pouzo and Presno (2011), Borri and Verdelhan (2009), D Erasmo (2008), Bianchi, Hatchondo, and Martinez (2012).

regarding continuation equilbria. Other papers in this literature are Giavazzi and Pagano (1989) and Alesina, Prati, and Tabellini (1989).[8]

Our paper also relates to the literature on model uncertainty. The literature on mechanism design has studied robust implementation when there is uncertainty about the prior over unobserved characteristics of the agents, with the only assumption for the modeler being that there is common knowledge of rationality. The seminal paper in this literature is Bergemann and Morris (2005). Penta (2014) studies robust mechanism design in a dynamic setting. Xandri (2012) studies a dynamic policy game with strategic uncertainty where the agent can learn about the payoff and epistemic type of their opponents by observing past actions.

In a series of papers, Hansen and Sargent[9], and coauthors, study settings where the decision makers are uncertain about an exogenous stochastic process, following the tradition in Gilboa and Schmeidler (1989).[10] Applications include the cost of business cycles (Barillas, Hansen, and Sargent (2009)), optimal monetary policy (Orlik and Presno (2011)), sovereign default (Pouzo and Presno (2011)), and optimal fiscal policy (Karantounias (2011)).

4 Setup

Our model of sovereign debt follows Eaton and Gersovitz (1981). Time is discrete and denoted by $t \in \{0, 1, 2, ...\}$. A small open economy receives a stochastic stream of income denoted by y_t. Income follows an iid process[11] with cdf denoted by $F(\cdot)$. The government is benevolent and wants to maximize the utility of the households. It does so by trading bonds in the international bond market. The household evaluates consumption streams according to

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

The sovereign issues short term debt at a price q_t. Our convention is that $b_{t+1} \geq 0$ denotes debt. We assume that the government cannot save.[12] The budget constraint for the economy

[8]Another strand of the literature, a mechanism design approach, studies the risk sharing agreement between international debt holders and a sovereign with some primitive contracting frictions. Worrall (1990) studies an economy with limited commitment. Atkeson (1991) studies an economy with limited commitment and moral hazard and finds that capital outflows during bad times are a feature of the optimal contract. Dovis (2014) studies an economy with private information and limited commitment and finds that, the periods of autarky are part of the equilibrium path. Hopenhayn and Werning (2008) study firm financing when there is private information regarding the outside option of the firm and limited commitment to repayment and find that there is default along the equilibrium path. Aguiar and Amador (2013b) exploit this approach to study the optimal repayment of sovereign debt when there are bonds of different maturities.

[9]Hansen and Sargent (2008) provides a textbook treatment of the approach.

[10]Other papers that explore the axiomatic foundations of model uncertainty are Klibanoff, Marinacci, and Mukerji (2005), Maccheroni, Marinacci, and Rustichini (2006) and Strzalecki (2011).

[11]We will focus on the i.i.d. case to simplify the notation and stress that distory dependence is not comping from the predictability of income. Our results generalize to the case where output y_t follows Markovian stochastic process $F(y_{t+1} \mid y_t)$.

[12]The assumption that the government cannot save before default is for simplicity; in this case, autarky is a subgame perfect equilibrium. In Section 8 we characterize subgame perfect equilibria in the case that the government can also save and defaults are excusable. The assumption that the government cannot save

is then

$$c_t = y_t - b_t + q_t b_{t+1}$$

There is limited enforcement of debt. Therefore, the government will repay debts only if it is more convenient to do so. The only consequence of default is that the goverment will be in autarky forever.[13]

Lenders Investors are risk neutral and discount future payoffs at the rate r. They choose loans to maximize the expected profits

$$\phi = -q_t b_{t+1} + \frac{1-\delta_t}{1+r} b_{t+1}$$

where δ_t is the endogenous probability of default in $t+1$. The outside option of the investors is zero. Individual rationality and absence of arbitrage opportunities imply that the price of the bond is given by

$$q_t = \frac{1-\delta_t}{1+r}$$

Timing The timing of the game is as follows. In period t, the government enters with b_t bonds that needs to repay. Then income y_t is realized. The government has the option to default $d_t \in \{0, 1\}$. If the government does not default, the government runs an auction of face value b_{t+1}. Then, the price of the bond q_t is realized. Finally, consumption takes place, and is given by $c_t = y_t - b_t + q_t b_{t+1}$. If the government decides to default, consumption is equal to income. If the government has defaulted in the past, consumption is also equal to income $c_t = y_t$.[14]

4.1 Dynamic Game: Notation and Definitions

In this section we describe the basic notation for the dynamic game setup. We follow as closely as possible the notation in Mailath and Samuelson (2006).

Histories An income history is a vector $y^t = (y_0, y_1, \ldots, y_t)$ of all income realizations up to time t. A **history** is a vector $h^t = (h_0, h_1, ..., h_{t-1})$, where $h_t = (y_t, d_t, b_{t+1}, q_t)$ is the description of all realized values of income and actions, and $h = h'h''$ is the append operator. A **partial history** is an initial history h^t concatenated with a part of h_t. For example, $h = (h^t, y_t, d_t, b_{t+1})$ is a history where we have observed h^t, output y_t has been realized, the government decisions (d_t, b_{t+1}) have been made, but market price q_t has not yet been realized. We will denote these histories $h = h^{t+1}_-$. The set of all partial histories (initial and partial) is denoted by $\mathcal{H}$, and $\mathcal{H}_g \subset \mathcal{H}$ are those where the government has to make a decision; i.e., $h = (h^t, y_t)$. Likewise, $\mathcal{H}_m \subset \mathcal{H}$ is the set of partial histories where investors set prices; i.e., $h^{t+1}_- = (h^t, y_t, d_t, b_{t+1})$.

after default is to obtain equilibria with positive price of debt following Bulow and Rogoff (1988).

[13] This assumption is again for convenience, so the Markov equilibrium with state b_t, y_t is the best SPE. We discuss this in more detail in section 5.

[14] The literature of sovereign debt introduces exogenous default costs to obtain higher levels of debt capacity. If the only punishment is exculsion from the financial market, there is (numerically) no debt capacity. This is not crucial for our theoretical results.

Outcomes An **outcome path** is a sequence of measurable functions[15]

$$x = \left(d_t\left(y^t\right), b_{t+1}\left(y^t\right), q_t\left(y^t\right)\right)_{t\in\mathbb{N}}$$

The set of all outcomes is denoted by $\mathcal{X}$. Sometimes we will write $x = \left(d_t^x\left(y^t\right), b_{t+1}^x\left(y^t\right), q_t^x\left(y^t\right)\right)_{t\in\mathbb{N}}$ to make explicit that the default, bond policies and prices are the ones associated with the path x. An **outcome** x_t (the evaluation of a path at a particular period) is a description of the government's policy function and market pricing function at time t

$$x_t = \left(d_t^x\left(\cdot\right), b_{t+1}^x\left(\cdot\right), q_t^x\left(\cdot\right)\right)$$

where the functions in x_t are $d_t : Y \to \{0,1\}$, $b_{t+1} : Y \to \mathbb{R}_+$, and $q_t : Y \to \mathbb{R}_+$. Our focus will be on a **shifted outcome**, $x_{t-} \equiv \left(q_{t-1}, d_t\left(\cdot\right), b_{t+1}\left(\cdot\right)\right)$. The reason to do this is that the prices in q_{t-1} will only be a function of the next period default decision.

Strategies A **strategy profile** is a complete description of the behavior of both the government and the market, for any possible history. Formally, a strategy profile is defined as a pair of measurable functions $\sigma = (\sigma_g, q_m)$, where $\sigma_g : \mathcal{H}_g \to \{0,1\}\times\mathbb{R}_+$ and $q_m : \mathcal{H}_m \to \mathbb{R}_+$. The government decision will usually be written as

$$\sigma_g\left(h^t, y_t\right) = \left(d_t^{\sigma_g}\left(h^t, y_t\right), b_{t+1}^{\sigma_g}\left(h^t, y_t\right)\right)$$

so that $d_t^{\sigma_g}\left(\cdot\right)$ is the default decision for strategy σ_g and $b_{t+1}^{\sigma_g}\left(\cdot\right)$ its bond issuance decision. Σ_g is the set of all strategies for the government, and Σ_m is the set of market pricing strategies. $\Sigma = \Sigma_g \times \Sigma_m$ is the set of all strategy profiles. Given a history h^t, we define the continuation strategy induced by h^t as

$$\sigma_{|h^t}\left(h^s\right) = \sigma\left(h^t h^s\right)$$

Every strategy profile σ generates an outcome path $x := x\left(\sigma\right)$.[16] Given a set $S \subseteq \Sigma$ of strategy profiles, we denote $x\left(S\right) = \cup_{\sigma\in S} x\left(\sigma\right)$ for the set of outcome paths of profiles $\sigma \in S$.

Payoffs For any strategy profile σ, we define the continuation value function $V : \Sigma\times\mathcal{H}_g \to \mathbb{R}$ as

$$V\left(\sigma \mid h^t\right) = \mathbb{E}_t\left\{\sum_{s=t}^{\infty}\beta^s\left[d_s u\left(y_s - b_s + q_s b_{s+1}\right) + \left(1-d_s\right)u\left(y_s\right)\right]\right\}$$

where $\left(y_s, d_s, b_{s+1}, q_s\right)$ are on the path $x = x\left(\sigma_{|h^t}\right)$.[17]

Definition 4.1. A strategy profile $\sigma = (\sigma_g, q_m)$ constitutes a **subgame perfect equilibrium** (SPE) if and only if, for all partial histories $h^t \in \mathcal{H}_g$

$$V\left(\sigma \mid h^t\right) \geq V\left(\sigma_g', q_m \mid h^t\right) \text{ for all } \sigma_g' \in \Sigma_g \tag{4.1}$$

[15]For our baseline case, where after default the government is permanently in autarky, the functions have the restriction that bond issues and prices are not defined after a default has been observed: $b_{t+s+1}\left(y^t y^s\right) = q_{t+s}\left(y^t y^s\right) = \emptyset$ for all y^s and y^t such that $d_t\left(y^t\right) = 1$.

[16]It can be defined recursively as follows: at $t = 0$ jointly define $\left(d_0\left(y_0\right), b_1\left(y_0\right), q_1\left(y_0\right)\right) \equiv \left(d_0^{\sigma_g}\left(y_0\right), b_1^{\sigma_g}\left(y_0\right), q_m\left(y_0, b_1^{\sigma_g}\left(y_0\right)\right)\right)$ and $h^1 = \left(y_0, d_0\left(y_0\right), b_1\left(y_0\right), q_1\left(y_0\right)\right)$. For $t > 0$, we define $\left(d_t\left(y^t\right), b_{t+1}\left(y^t\right), q_t\left(y^t\right)\right) \equiv \left(d_0^{\sigma_g}\left(h^t, y_t\right), b_1^{\sigma_g}\left(h^t, y_t\right), q_m\left(h^t, y_t\right)\right)$ and $h^{t+1} = \left(h^t, y_t, d_t\left(y^t\right), b_{t+1}\left(y^t\right), q_t\left(y^t\right)\right)$

[17]The utility of a strategy profile that specifies negative consumption is $-\infty$.

and for all histories $h_{-}^{t+1} = (h^t, y_t, d_t, b_{t+1}) \in \mathcal{H}_m$

$$q_m\left(h_{-}^{t+1}\right) = \frac{1}{1+r}\int\left(1 - d^{\sigma_g}\left(h^{t+1}, y_{t+1}\right)\right) dF\left(y_{t+1} \mid y_t\right) \tag{4.2}$$

That is, the strategy of the government is optimal given the pricing strategy of the lenders $q_m(\cdot)$, and likewise $q_m(\cdot)$ is consistent with the default policy generated by σ_g. The set of all subgame perfect equilibria is denoted as $\mathcal{E} \subset \Sigma$.

Best and Worst SPE The best SPE is the Markov equilibrium that we characterize below and the worst SPE is autarky. This follows from our assumption of no savings ($b \geq 0$) and permanent exclusion after default. We focus on this case for two reasons. First, for clarity: to contrast the worst equilibrium price (that is zero, and only dependent on bonds issued) and the worst equilibrium consistent price (that is positive, and depends on the past history). Second, under these conditions computing equilibrium consistent prices uses as an input the Markov equilbrium that is characterized in the literature of sovereign debt as in Arellano (2008) and Aguiar and Gopinath (2006); the computation will involve solving numerically a simple equation with one unknown.[18] The best and worst equilibrium prices are defined as

$$\overline{q}(h_{-}^{t+1}) := \max_{\sigma\in\mathcal{E}} q_m\left(h_{-}^{t+1}\right)$$
$$\underline{q}(h_{-}^{t+1}) := \min_{\sigma\in\mathcal{E}} q_m\left(h_{-}^{t+1}\right)$$

4.2 A Markov Equilibrium

We characterize a Markov perfect equilibrium where the state variables are bonds and income. This case provides intuition on how a particular equilibrium operates, and for why we focus on a set of outcomes and not on each outcome individually. In addition, the prices and value functions are inputs in the characterization of equilbrium consistent outcomes.

Government Decisions The value of a government that has the option to default is given by

$$\overline{\mathbb{W}}(b) = \mathbb{E}_y\left[\max\left\{\overline{V}^{nd}(b,y), V^D(y)\right\}\right]$$

This is the expected value of the maximum between not defaulting $\overline{V}^{nd}(b,y)$ and the value of defaulting $V^D(y)$. The value of not defaulting is given by

$$\overline{V}^{nd}(b,y) = \max_{b'\geq 0} u(y - b + q(b')b') + \beta\overline{\mathbb{W}}(b') \tag{4.3}$$

That is, the government repays debt, obtains a capital inflow (outflow), and from the budget constraint consumption is given by $y - b + q(b')b'$; next period has the option to default b' bonds. The value of defaulting solves

$$V^d(y) = u(y) + \beta\mathbb{E}_{y'}V^d(y')$$

[18] In Section 8 we characterize Equilibrium Consistent Outcomes for the case in which there are savings or defaults are excusble.

and is just the value of consuming income forever. These value functions define a default set

$$D(b) = \left\{ y \in Y : \overline{V}^{nd}(b,y) < V^d(y) \right\}$$

Definition 4.2. A Recursive Equilibrium (with state b, y) is a: set of policy functions $(c(y,b), d(y,b), b'(y,b))$, a bond price function $q(b')$ and a default set $D(b)$ such that: $c(y,b)$ satisfies the resource constraint; taking as given $q(b')$ the government bond policy maximizes $\overline{V}^{nd}$; the bond price $q(b')$ is consistent with the default set

$$q(b') = \frac{1 - \int_{D(b')} dF(y')}{1+r}$$

Default During Bad Times Under our assumption of no reentry after default and i.i.d. income, Arellano (2008) showed that defaults occur during bad times. The intuition is as follows. Because income is i.i.d. and there is permanent exclusion, a country will never default if there are capital inflows available. If there are capital inflows available, the country can wait one more period, and default later. This strategy dominates defaulting today. In addition, because income is i.i.d., the maximum capital inflow (lowest capital outflow) that the country can experience is independent of income. If a country defaults in state y, it means that there are no capital inflows available; so there is a capital outflow for any continuation b'. So, for the lowest capital outflow, the value of continuation will decrease more (due to the concavity of the utility function) than the value of default as income decreases. So, if continuation is not preferred with high income, it will not be preferred with low income.

Quantitative Properties For the non-i.i.d. case and the case with reentry after default, default need not occur only during bad times. Also, if exclusion is the only punishment for default, debt capacity is too low. That is why the literature of sovereign debt imposes output costs from default. The starting point for this literature are the papers by Arellano (2008) and Aguiar and Gopinath (2006), which study a setting as in Eaton and Gersovitz (1981) with short term debt and probabilistic reentry after default. Arellano and Ramanarayanan (2012), Hatchondo and Martinez (2009), and Chatterjee and Eyigungor (2012), study models with long term debt. Chatterjee and Eyigungor (2012) introduce an asymmetric an alternative cost function and are successful in explaining debt capacity, level, and volatility of spreads. Our focus will not be on matching key moments in the data, but in obtaining predictions that hold across all equilibria, in a particular version of the models in the quantitative literature of sovereign debt; or in other words, in the Eaton and Gersovitz (1981) model (with no savings prior to default).

Focus on Set of Outcomes The following example provides further intuition how a particular equilbrium operates, and at the same time helps us to make the point why we focus on predictions about the set of equilibria, as opposed to Bergemann and Morris (2013) and Angeletos and Pavan (2013). A well known property of sovereign debt models is that higher debt implies a higher default probability. That is, if $b_1 \leq b_2$, then $\delta(b_1) \leq \delta(b_2)$. The argument is that, if the government wants to default with higher debt, it will want to default with lower debt. In math

$$\begin{aligned} V^d(y) &\geq u(y - b_1 + q(b')b') + \beta\overline{\mathbb{W}}(b') \\ &\geq u(y - b_2 + q(b')b') + \beta\overline{\mathbb{W}}(b') \\ &\geq V^d(y) \end{aligned}$$

But note that, when we examine the set of all subgame perfect equilibrium, this property will not hold across every equilibrium. The same level of debt b_1 can be associated with different continuation equilibria

$$u(y - b_1(h) + q(b'(h))b'(h)) + \beta w(h, b')$$

(with $h \in \mathcal{H}$ and $w(h, b')$ the value of the continuation equilibria in a different SPE) and therefore, different default probabilities.

Notation The autarky continuation (that corresponding to the worst equilibrium value) is

$$\mathbb{V}^d \equiv \int u(y')dF(y')$$

and the autarky utility (conditional on defaulting) is simply

$$V^d(y) \equiv u(y) + \beta\mathbb{V}^d \tag{4.4}$$

The continuation utility (conditional on not defaulting) of a choice b' given bonds (b, y) is

$$\overline{V}^{nd}(b, y, b') = u(y - b + b'\overline{\mathbf{q}}(b')b') + \beta\overline{\mathbb{W}}(b') \tag{4.5}$$

where $\overline{q}(b')$ is the bond price schedule under the best continuation equilibrium (the Markov equilibrium that we just characterized), if $y_t = y$ and the bonds to be paid tomorrow are $b_{t+1} = b'$. Recall that

$$\overline{V}^{nd}(b, y) = \max_{\tilde{b}\geq 0} \overline{V}^{nd}\left(b, y, \tilde{b}\right) \tag{4.6}$$

5 Equilibrium Consistent Outcomes

In this section we show the main result in our paper, a characterization of equilibrium consistent outcomes for the baseline case where income is a continuous random variable. Subsection 5.3, contains the characterization of equilibrium consistent outcomes. Aided with this characterization, in Subsection 5.4, we characterize equilbrium consistent prices and discuss comparative statistics with respect to history.

5.1 Equilibrium Consistency: Definitions

Definition 5.1. (***Consistency***) A history h is **consistent with** (or **generated by**) an outcome path $x \iff d_s = d_s^x(y^s)$, $b_{s+1} = b_{s+1}^x(y^s)$ and $q_s = q_s^x(y^s)$ for all $s < l(h)$ (where $l(h)$ is the length of the history).

If a history h is consistent with an outcome path x we denote it as $h \in \mathcal{H}(x)$. Intuitively, consistency of a history with an outcome means that, given the path of exogenous variables, the endogenous observed variables coincide with the ones that are generated by the outcome.

Definition 5.2. A history h is consistent with strategy profile $\sigma \iff h \in \mathcal{H}(x(\sigma))$.[19]

If a history h is consistent with a strategy σ we denote it as $h \in \mathcal{H}(\sigma)$. Intuitively, a history is consistent with a strategy if the history is consistent with the outcome that is generated by the strategy. Given a set $S \subseteq \Sigma$ of strategy profiles, we use $x(S) = \cup_{\sigma \in S} x(\sigma)$ to denote the set of outcome paths of profiles $\sigma \in S$. The inverse operator for $\mathcal{H}(\cdot)$ are respectively $X(\cdot)$ for the outcomes consistent with history h. We use $\Sigma(h)$ to denote the strategy profiles consistent with h. For a given set of strategy profiles $S \subseteq \Sigma$, we write $\mathcal{H}(S) = \bigcup_{\sigma \in S} \mathcal{H}(\sigma)$ as the set of **S−consistent** histories. When $S = \mathcal{E}$, we call $\mathcal{H}(\mathcal{E})$ the set of **equilibrium consistent histories.** The set of equilibria consistent with history h is defined as $\mathcal{E}_{|h} := \mathcal{E} \cap \Sigma(h)$.[20]

Definition 5.3. (**S− consistent outcomes**) An outcome path $x = (d_t(\cdot), b_{t+1}(\cdot), q_t(\cdot))_{t \in \mathbb{N}}$ is $S-$ **consistent with history** $h^t \iff \exists\, \sigma \in S \cap \Sigma(h^t)$ such that $x = x(\sigma)$. If $S = \mathcal{E}$ we say x is **equilbrium consistent with history** h^t, and we denote it as "$x \in x\left(\mathcal{E}_{|h}\right)$".

5.2 Motivation: Robust Bayesian Analysis

In Bayesian analysis, the econometrician has a prior over the set of fundamental parameters Θ;[21] here will be denoted by $\pi(\theta)$. In addition, because of equilibrium multiplicity, she also has a prior $p(x)$ over the set of outcomes $\mathcal{X}$.[22] Using data (in our case, in the form of a history) and these priors, she obtains a posterior. Suppose that she is interested in the (posterior) mean of a particular statistic $T(x, \theta)$. Conditional on the data, her prediction is

$$\mathbb{E}^{p,\pi}\left[T(x,\theta) \mid \text{data}\right]$$

There are many situations where the econometrician will not want to favor one equilbrium against another one; that is, there is uncertainty with respect to the prior $p \in \mathcal{P}$. Then, there is a whole range of posterior means of the statistic that is given (for a fixed θ, or a degenerate prior over Θ) by

$$\left[\min_p \mathbb{E}^p\left[T(\theta, x) \mid \text{data}\right], \max_p \mathbb{E}^p\left[T(\theta, x) \mid \text{data}\right]\right]$$

[19]Remember that each strategy σ generates an outcome path $x := x(\sigma)$. It can be defined recursively as follows: at $t = 0$ jointly define $(d_0(y_0), b_1(y_0), q_1(y_0)) \equiv \left(d_0^{\sigma_g}(y_0), b_1^{\sigma_g}(y_0), q_m\left(y_0, b_1^{\sigma_g}(y_0)\right)\right)$ and $h^1 = (y_0, d_0(y_0), b_1(y_0), q_1(y_0))$. For $t > 0$, we define $(d_t(y^t), b_{t+1}(y^t), q_t(y^t)) \equiv \left(d_0^{\sigma_g}(h^t, y_t), b_1^{\sigma_g}(h^t, y_t), q_m(h^t, y_t)\right)$ and $h^{t+1} = (h^t, y_t, d_t(y^t), b_{t+1}(y^t), q_t(y^t))$.

[20]This notation is useful to precisely formulate questions such as: "Is the observed history the outcome of some subgame perfect equilibria?" In our notation "$h \in \mathcal{H}(\mathbf{SPE})$".

[21]In our model, discount factor, parameters of the utility function, volatility of output, etc

[22]For example, Eaton and Gersovitz (1981) select a Markov equilibrium. Chatterjee and Eyigungor (2012) choose an equilibrium with arbitrary probability of crises in their study of optimal maturity of debt. These would be examples of degenerate priors; in other words, there is a particular equilibrium selection.

We will focus on priors over *equilibrium outcomes*, but we will be agnostic about the particular prior. In an application of our main result, we will characterize the set of equilibrium consistent (with history h) debt prices

$$\left[\min_{x\in x\left(\mathcal{E}_{|h}\right)} q_t^x,\ \max_{x\in x\left(\mathcal{E}_{|h}\right)} q_t^x\right]$$

This interval characterizes the support of the posterior over prices, when the only assumption is that the observed history is part of an SPE.

5.3 Equilibrium Consistency: Characterization

Suppose that we have observed so far an equilibrium consistent history $h_-^t = (h^{t-1}, y_{t-1}, d_{t-1}, b_t)$ (where price at time t has not yet been realized), and we want to characterize the set of *shifted* outcomes $x_{t-} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ consistent with this history[23]. Theorem 5.1 provides a full characterization of the set of equilibrium consistent outcomes $x_{t-}\left(\mathcal{E}_{|h_-^t}\right)$, showing that past history only matters through the opportunity cost of not defaulting at $t-1$, $u(y_{t-1}) - u(c_{t-1})$.

Proposition 5.1 (**Equilibrium Consistent Outcomes**)**.** *Suppose $h_-^t = (h^{t-1}, y_{t-1}, d_{t-1}, b_t)$ is an equilibrium consistent history, with no default so far. Then $x_{t-} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ is equilibrium consistent with $h_-^t \iff$ the following conditions hold:*

a. Price is consistent

$$q_{t-1} = \frac{1}{1+r^*}\left(1 - \int d_t(y_t)dF(y_t)\right) \tag{5.1}$$

b. IC government

$$(1-d(y_t))\left[u(y_t - b_t + \overline{q}(b_{t+1})b_{t+1}) + \beta\overline{\mathbb{W}}(b_{t+1})\right] + d(y_t)V^d(y_t) \geq V^d(y_t) \tag{5.2}$$

c. Promise keeping

$$\beta\left[\int_{d_t=0} \overline{V}^{nd}(b_t, y_t, b_{t+1}(y_t))\, dF(y_t) + \int_{d_t=1} V^d(y_t)\, dF(y_t)\right] \geq$$
$$[u(y_{t-1}) - u(y_{t-1} - b_{t-1} + q_{t-1}b_t)] + \beta\mathbb{V}^d \tag{5.3}$$

Proof. See Appendix. □

[23] An outcome in period t was given by $x_t = \left(d_t^x(\cdot), b_{t+1}^x(\cdot), q_t^x(\cdot)\right)$; the policies and prices of period t. x_{t-} has the policies of period t but the prices of period $t-1$. The focus in x_{t-} as opposed to x_t simplifies the characterization of equilbrium consistent otucomes.

If conditions (a) through (c) hold, we write simply

$$\left(q_{t-1}, d_t(\cdot), b_{t+1}(\cdot)\right) \in \mathbb{ECO}\left(b_{t-1}, y_{t-1}, b_t\right)$$

where $\mathbb{ECO}$ stands for "equilibrium consistent outcomes".

Conditions (5.1) and (5.2) in Proposition 5.1 provide a characterization of the set of SPE outcomes. Condition (5.1) states that the price q_{t-1} needs to be consistent with the default policy $d_t(\cdot)$. Condition (5.2) states that a policy $d_t(\cdot), b_{t+1}(\cdot)$ is implementable in an SPE if it is incentive compatible given that following the policy is rewarded with the best equilibrium and a deviation is punished with the worst equilibrium. The argument in the proof follows Abreu (1988)[24]. These two conditions are necessary and sufficient for an outcome to be part of an SPE.[25]

Equilibrium consistent outcomes are characterized by an additional condition, (5.3), which is the main contribution of this paper. This condition characterizes how past observed history (if *assumed* to be generated by an *equilibrium* strategy profile) introduces restrictions on the set of equilibrium consistent policies. In our setting, condition (5.3) will guarantee that the government's decision at $t-1$ of not defaulting was optimal. That is, on the path of some SPE profile $\hat{\sigma}$, the incentive compatibility constraint from government's utility maximization in $t-1$ is

$$u\left(c_{t-1}\right) + \beta V\left(\hat{\sigma} \mid h^t\right) \geq u\left(y_{t-1}\right) + \beta \mathbb{V}^d \tag{5.4}$$

where $V\left(\hat{\sigma} \mid h^t\right)$ is the continuation value of the equilibrium, as defined before. One interpretation of (5.4) is that the net present value (with respect to autarky) that the government must expect from not defaulting, must be greater (for the choice to have been done optimally) than the opportunity cost of not defaulting: $u\left(y_{t-1}\right) - u\left(c_{t-1}\right)$. This must be true for *any* SPE profile that could have generated h_{-}^t.

The intuition for why (5.3) is necessary for equilibrium consistency is as follows. Notice that the previous inequality also holds for the case the continuation equilibrium is actually the best continuation equilibrium. Therefore, for any equilibrium consistent policy $\left(d(\cdot), b'(\cdot)\right)$ it has to be the case that

$$V\left(\hat{\sigma} \mid h^t\right) = \int_{y_t : d_t(y_t)=0} V^d\left(y_t\right) dF\left(y_t\right) +$$

$$\int_{y_t : d(y_t)=1} \left[u\left(y_t - b_t + b'(y)\, \hat{q}_m\left(h^t, y_t, d_t, b_{t+1}(y_t)\right)\right) + \beta V\left(\hat{\sigma} \mid h^{t+1}\right)\right] dF\left(y_t\right)$$

$$\leq \int_{y_t : d_t(y_t)=0} V^d\left(y_t\right) dF\left(y_t\right) + \int_{y_t : d(y_t)=1} \overline{V}^{nd}\left(b_t, y_t, b_{t+1}\right) dF\left(y_t\right) \tag{5.5}$$

Equations (5.4) and (5.5) imply

$$\beta\left[\int_{d_t=0} \overline{V}^{nd}\left(b_t, y_t, b_{t+1}(y_t)\right) dF\left(y_t\right) + \int_{d_t=1} V^d\left(y_t\right) dF\left(y_t\right)\right] \geq$$

[24] This is the argument in Chari and Kehoe (1990) and Stokey (1991)

[25] Note that at any history (even on those *inconsistent* with equilibria) SPE policies are a function of only one state: the debt that the government has to pay at time t (b_t). There are two reasons for this. First, the stock of debt summarizes the physical environment. Second, the value of the worst equilibrium only depends on the realized income.

$$[u(y_{t-1}) - u(y_{t-1} - b_{t-1} + q_{t-1}b_t)] + \beta\underline{\mathbb{V}}^d \tag{5.6}$$

This is condition (5.3). So if the policies do not satisfy (5.3), they are not part of an SPE that generated the history h^t_-; in other words, there is no SPE consistent with h^t_- with policies $(d_t(\cdot), b_{t+1}(\cdot))$ for period t.

We also show that this condition is sufficient, so if $(d_t(\cdot), b_{t+1}(\cdot))$ satisfy conditions (5.1), (5.2), and (5.3), we can always find an SPE profile $\hat{\sigma}$ that would generate x_{t-} on its equilibrium path. This result is somewhat surprising, because even after a long history the sufficient statistics to forecast the outcome x_{t-} are (b_{t-1}, b_t, y_{t-1}). This is where robustness of the analyst (uncertainty about the equilibrium selection) is expressed. Because income y is a continuous random variable, any promises (in terms of expected utility) that rationalized past choices are "forgotten" each period; the reason is that the outside observer needs to take into account that promises *could* have been be realized in states that did not occur. So, effectively

$$\mathbb{ECO}\left(h^t_-\right) = \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t)$$

Finally, notice that even though the outside observer is using just a small fraction of the history, the set of equilbrium consistent outcomes exhibits history dependence beyond the one in SPE. As recently stated, the set of equilibrium consistent outcomes is a function of the Markovian state variable (b_t), but depends through condition (5.3) on b_{t-1}, y_{t-1} as well, so history matters through variables (b_{t-1}, y_{t-1}, b_t). Thus, there is a role for past actions to signal future behavior.[26]

5.4 Equilibrium Consistent Prices

Aided with the characterization of equilibrium consistent outcomes in Proposition 5.1, we will characterize the set of equilibrium debt prices that are consistent with the observed history $h^t_- = (h^{t-1}, y_{t-1}, d_{t-1}, b_t)$. That is

$$\overline{q}\left(h^t_-\right) = \max_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

$$\underline{q}\left(h^t_-\right) = \min_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q} \tag{5.7}$$

where

$$(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}(b_{t-1}, y_{t-1}, b_t)$$

Highest Equilibrium Consistent Price The highest equilibrium consistent price is the one of the Markov Equilibrium that we characterized in Section 4. Note that the expected value of the incentive compatibility constraint (5.2), is the value of the option to default $\mathbb{W}(b')$, in the Markov Equilibrium. The promise-keeping will be generically not binding for the best equilibrium (given that the country did not default). For these two reasons, the

[26]Notice that this role contrasts the dependence of the quantitative literature for sovereign debt that follows Eaton and Gersovitz (1981) as in Arellano (2008) and Aguiar and Gopinath (2006) where the fact that a country has just repaid a high quantity of debt, does not affect the future prices that will obtain.

best equilibrium consistent price is the one obtained with the default policy and bond policy that maximize the value of the option. So

$$\overline{q}\left(h_{-}^{t}\right)=\overline{\mathbf{q}}\left(b_{t}\right) \tag{5.8}$$

Lowest Equilibrium Consistent Price Our focus will be on the characterization of the lowest equilibrium consistent price. Note that the lowest SPE price is zero. This follows because default is implementable after any history if we do not take into account the promise keeping constraint (5.3). On the contrary, we will show that lowest equilibrium consistent price is positive, for every equilibrium history. Furthermore, because the set of equilibrium consistent outcomes after history h_{-}^{t} depends only on (b_{t-1}, y_{t-1}, b_t), it holds that

$$\underline{q}\left(h_{-}^{t}\right)=\underline{\mathbf{q}}\left(b_{t-1}, y_{t-1}, b_{t}\right) \tag{5.9}$$

From (5.8) and (5.9), the set of equilibrium consistent prices will be

$$q_{t}\in\left[\underline{\mathbf{q}}\left(b_{t-1}, y_{t-1}, b_{t}\right), \overline{\mathbf{q}}\left(b_{t}\right)\right] \tag{5.10}$$

Proposition 5.2 establishes the main result of this subsection: a full characterization of $\underline{\mathbf{q}}\left(b, y, b'\right)$ (we drop time subscripts) as a solution to a convex minimization program, which can be reduced to a one equation/one variable problem.

Proposition 5.2. *Suppose (b, y, b') are such that $\overline{V}^{nd}\left(b, y, b'\right) > V^{d}\left(y\right)$ (i.e., not defaulting was feasible under the best continuation equilibrium). Then there exists a constant $\gamma = \gamma\left(b, y, b'\right) \geq 0$ such that*

$$\underline{\mathbf{q}}\left(b, y, b'\right)=\frac{1-\int \underline{d}\left(y'\right) dF\left(y' \mid y\right)}{1+r}$$

where

$$\underline{d}\left(y'\right)=0 \iff \overline{V}^{nd}\left(b', y'\right) \geq V^{d}\left(y'\right)+\gamma \text{ for all } y' \in Y$$

and γ is the minimum solution to the equation:

$$\beta \int_{\Delta^{nd} \geq \gamma} \Delta^{nd} d\hat{F}\left(\Delta^{nd}\right)=u\left(y\right)-u\left(y-b+\frac{1-\hat{F}\left(\gamma \mid y\right)}{1+r} b'\right) \tag{5.11}$$

where $\Delta^{nd} \equiv \overline{V}^{nd}\left(b', y'\right)-V^{d}\left(y'\right)$ and $\hat{F}\left(\Delta^{nd}\right)$ its conditional cdf. If $dF\left(\cdot\right)$ is absolutely continuous, then γ is the unique solution to equation 5.11.

The proof is in the appendix. We provide a sketch of the argument. First, note that, by choosing the bond policy of the best equilibrium, all of the constraints imposed by equilibrium consistency are relaxed because the value of no default increases. So, finding the lowest ECO price will amount to finding the default policy that yields the lowest price and is consistent with equilibrium. Second, notice that the promise keeping constraint needs to be binding in the optimum. If not, the minimization problem has as its only constraint the incentive compatibility constraint, and the minimum price is zero (with a policy of default in every state). But, if the price is zero, the promise keeping constraint will not be satisfied. Third, notice that the incentive compatibility constraint will not be binding. Intuitively, imposing default is not costly in terms of incentives, and for the lowest equilibrium consistent price, we want to impose default in as many states as possible.

From these observations, note that the tradeoff of the default policy of the lowest price will be: imposing defaults in more states will lower the price at the expense of a tighter promise keeping constraint. This condition pins down the states where the government defaults; as many defaults as possible, but not so many that no default in the previous period was not worth the effort. This, implies that the policy is pinned down by

$$\underline{d}\left(y'\right)=0 \iff \overline{V}^{nd}\left(b',y'\right)\geq V^{d}\left(y'\right)+\gamma$$

where γ is a constant to be determined. This constant solves a single equation: is the minimum value such that the promise keeping holds with equality, with the optimal bond policy, evaluated at the best continuation

$$\beta\int_{\Delta^{nd}\geq\gamma}\Delta^{nd}d\hat{F}\left(\Delta^{nd}\mid y\right)=u\left(y\right)-u\left(y-b+\frac{1-\hat{F}\left(\gamma\mid y\right)}{1+r}b'\right) \tag{5.12}$$

Remark 5.1. Note that the best equilibrium default policy at t

$$d\left(y_t\right)=0 \iff \overline{V}^{nd}\left(b_t,y_t\right)\geq V^{d}\left(y_t\right)$$

On the contrary, the default policy of the lowest equilibrium consistent price is

$$d\left(y_t\right)=0 \iff \overline{V}^{nd}\left(b_t,y_t\right)\geq V^{d}\left(y_t\right)+\gamma$$

where γ is the constant that solves (5.12) and depends on (b_{t-1},y_{t-1},b_t). The default policy is shifted to create more defaults, to lower the price, but not so many that the promise-keeping was not satisfied (i.e., we cannot rationalize previous choices).

Remark 5.2. Notice that by focusing on equilibrium consistent outcomes uncovers a novel tension that is not present in SPE. At a particular history h_{-}^{t}, implementing default is not costly because it is always as good as the worst equilibrium. However, implementing default today lowers the prices that the government was expecting in the past and makes it harder to rationalize a particular history.

The next Corollary describes how the set of equilibirum consistent prices changes with the history of play.

Corollary 5.1. *Let $\underline{q}\left(b,y,b'\right)$ be the lowest $\mathbb{ECO}\left(b,y,b'\right)$ price. It holds that*

a. $\underline{q}\left(b,y,b'\right)$ is decreasing in b'

b. $\underline{q}\left(b,y,b'\right)$ *is increasing in* b.

c. For every equilibrium $\left(b,y,b'\right)$, $-b+b'\underline{q}\left(b,y,b'\right)\leq 0$

d. $\underline{q}$ *is decreasing in* y *, and so is the set* $Q=\left[\underline{q}\left(b,y,b'\right),\overline{q}\left(b'\right)\right]$

First, note that as in the best equilibrium, the lowest equilibrium consistent price is decreasing in the amount of debt issued b'. The intuition is that higher amounts of debt issued imply a more relaxed promise keeping constraint. In other words, the past choices of the government can be rationalized with a lower price. A similar intuition holds for b; if the country just repaid a high amount of debt (i.e., made an effort for repaying), past choices are rationalized by higher prices.

Second, note that if there is a positive capital inflow with the lowest equilibrium consistent price, it implies that

$$u(y) - u\left(y - b + b'\underline{q}(b, y, b')\right) < 0$$

Intuitively, the country is not making any effort in repaying the debt. Therefore, it need not be the case that the country was expecting high prices for debt in the next period. Mathematically, when there is a positive capital outflow with the lowest equilibrium consistent price, γ is infinite. This implies that $\frac{1-\hat{F}(\gamma)}{1+r^*} = \underline{q}(b, y, b') = 0$, which contradicts a positive capital inflow.

Finally, because there are no capital inflows with the lowest equilibrium consistent price, repaying debt at this price will become more costly as income is lower; this due to the concavity of the utility function.[27] Mathematically, because of concavity,

$$u(y) - u\left(y - b + b'\underline{q}(b, y, b')\right)$$

is[28] increasing as income decreases, and therefore, the promise keeping constraint tightens as income decreases. Note that, in the non i.i.d. case, this property will not hold, because, even though the burden of repayment is higher, the value of repayment in terms of the continuation value can be increasing.

5.5 Equilibrium Consistent Histories

We conclude the section with the construction of equilibrium consistent histories. So far we used as a condition that an observed history h^t was equilibrium consistent. For example, to obtain the characterization of equilibrium prices $\left[\underline{\mathbf{q}}(b_t, y_t, b_{t+1}), \overline{\mathbf{q}}(b_{t+1})\right]$, it is necessary to check that the observed history up to $t-1$ was equilibrium consistent to begin with. If not, then $\left[\underline{q}\left(h_-^{t+1}\right), \overline{q}\left(h_-^{t+1}\right)\right] = \emptyset$ and the whole endeavor is pointless. To complete the characterization in order for these to provide robust conditional predictions for equilibrium behavior, we show how equilibrium consistent histories are constructed from the initial history $h^0 = b_0$. First, start with the initial history $h^0 = b_0$, in which nothing has yet happened (except the initial bond level b_0). Clearly h^0 is an equilibrium consistent history. Take now a history $h^1 = (h^0, y_0, d_0, b_1, q_0)$ that follows history h^0 (with debt b_0). Here, we need to check the three conditions in Proposition 5.1 to find $\left(\hat{d}_0(\cdot), \hat{b}_1(\cdot), \hat{q}_0(\cdot)\right)$. Then h^1 will be equilibrium consistent as well. Following this logic, one can check equilibrium consistent

[27]This observation is used in the literature of sovereign debt. For example, to show that default occurs in bad times, as in Arellano (2008), or to show monotonicity of bond policies with respect to debt, as in Chatterjee and Eyigungor (2012).

[28]The change in this expression will depend on the sign of $u(y) - u\left(y - b + b'\frac{1-\hat{F}(\gamma)}{1+r^*}\right)$, that is positive due to the result of no capital inflows with the lowest equilbrium consistent price.

recursively: start by knowing h^0 is equilibrium consistent, show h^1 is equilibrium consistent using Proposition 5.1, then repeating the procedure for $s = 2$ and so on.

6 $\mathbb{ECO}$: Discrete Income

Our main result in the previous section was a characterization of equilibrium consistent outcomes. In this result, the fact that the last opportunity cost is a sufficient statistic for equilibrium consistent outcomes (and prices as a consequence) is somewhat surprising: the outside observer is only using observations from the last period to make an inference, even though she has a whole history of data available. However, as we will see below in a simple two period example, this result is a direct expression of Robustness: the econometrician needs to take into account that the expected payoff that rationalized a particular decision, could have been realized only in histories that did not occurr. When income is continuous, because any particular history has probability zero, it can be the case that the expected payoff was promised to states that did not materialized. We then show how history matters in the case of discrete income (or no randomization devices available).

6.1 Two Periods: Continuous Income

Suppose that we observe $(\overline{y}_0, b_0, q_0, b_1, \overline{y}_1, b_2)$. Denote $h^1 = \overline{y}_0, b_0, q_0, b_1$. We will show that

$$\underline{q}_1(\overline{y}_0, b_0, q_0, b_1, \overline{y}_1, b_2) = \underline{q}_1(b_1, \overline{y}_1, b_2)$$

when income is continuous. In order to do this, note that, because $\overline{y}_0, b_0, q_0, b_1$ is an equilibrium history, there is a continuation value function such that

$$\sum_i p(y_{1i}) V_0(y_{1i}, b_1) \geq \frac{1}{\beta} \left[u(\overline{y}_0) - u(\overline{y}_0 - b_0 + q_0 b_1) \right] + \underline{\mathbb{W}} \tag{6.1}$$

where $V_0(y_{1i}, b_1)$ is the continuation value function of a continuation equilibrium strategy after history $\overline{y}_0, b_0, q_0, b_1, y_{1i}$. Note also that $\overline{y}_1, b_2$ and the decision not to default are part of an SPE. This implies that the following constraint has to hold

$$u(\overline{y}_1 - b_1 + q_1(h^1, \overline{y}_1, b_2) b_2) + \beta \sum_i p(y_{2i}) V_1(y_{2i}, b_2) \geq u(\overline{y}_1) + \beta \underline{\mathbb{W}} \tag{6.2}$$

Note, also, it has to be the case that

$$V_0(\overline{y}_1, b_1) = (\overline{y}_1 - b_1 + q_1(h^1, \overline{y}_1, b_2) b_2) + \beta \sum_i p(y_{2i}) V_1(y_{2i}, b_2) \tag{6.3}$$

That is, on the equilbrium path, the promised continuation needs to coincide with the continuation that we observe in history $(\overline{y}_0, b_0, q_0, b_1, \overline{y}_1, b_2)$. Finally, it also needs to be that the case that

$$V_1(y_{2i}, b_2) \in [\underline{\mathbb{W}}, \overline{\mathbb{W}}(y_{2i}, b_2)] \tag{6.4}$$

$$V_1(y_{2i}, b_2) \in [\underline{\mathbb{W}}, \overline{\mathbb{W}}(y_{2i}, b_2)] \tag{6.5}$$

where we abuse notation slightly for the continuation value sets. Now, the lowest equilibrium consistent price solves

$$\min_{\{V_0(y_{1i},b_1)\}_{y_{1i}},\{V_1(y_{2i},b_2)\}_{y_{2i}}} q \tag{6.6}$$

subject to (6.1), (6.2), (6.3), (6.4), and (6.5). Our objective is to show that if $p(\overline{y}_1) = 0$, the constraint (6.1) is not binding, and therefore, the solution will not depend on $(\overline{y}_0, b_0, q_0)$. To solve (6.6), we want to relax the constraint (6.1) as much as possible. So we pick the continuation value function

$$V_0(y_{1i}, b_1) = \begin{cases} V_0(\overline{y}_1, b_1) & \text{for } y_{1i} = \overline{y}_1 \\ \overline{\mathbb{W}}(y_{1i}, b_1) & \text{for } y_{1i} \neq \overline{y}_1 \end{cases}$$

where $V_0(\overline{y}_1, b_1)$ is free at the moment. Because the histories $(\overline{y}_0, b_0, q_0, b_1, \text{for } y_{1i} \neq \overline{y}_1)$ are not realized, it could have been the case that the best continuation followed. The outside observer cannot neglect this possibility. Then, by adding and subtracting $p(\overline{y}_1)\overline{V}(\overline{y}_1, b_1)$, we can rewrite the left hand side of (6.1) as

$$\sum_i p(y_{1i})V_0(y_{1i}, b_1) = p(\overline{y}_1)\left[V_0(\overline{y}_1, b_1) - \overline{V}(\overline{y}_1, b_1)\right] + \sum_i p(y_{1i})\overline{\mathbb{W}}(y_{1i}, b_1) \tag{6.7}$$

Plugging (6.7) in (6.1)

$$p(\overline{y}_1)\left[V_0(\overline{y}_1, b_1) - \overline{V}(\overline{y}_1, b_1)\right] + \sum_i p(y_{1i})\overline{\mathbb{W}}(y_{1i}, b_1) \geq \frac{1}{\beta}\left[u(\overline{y}_0) - u(\overline{y}_0 - b_0 + q_0 b_1)\right] + \underline{\mathbb{W}} \tag{6.8}$$

where $\overline{V}(\overline{y}_1, b_1)$ is the value of not default in the best equilibrium when bonds are b_1 and income is $\overline{y}_1$. So, when income is continuous, $p(\overline{y}_1) = 0$. So, the constraint will not be binding if

$$u(\overline{y}_0 - b_0 + q_0 b_1) + \beta \sum_i p(y_{1i})\overline{\mathbb{W}}(y_{1i}, b_1) \geq u(\overline{y}_0) + \beta\underline{\mathbb{W}}$$

holds. And this holds because $\overline{y}_0, b_0, q_0, b_1$ is an SPE history where the government did not default.

6.2 Two Periods: Discrete Income

If income is discrete, then $b_1, \overline{y}_1, b_2$ will not be sufficient statistics to summarize history. The intuition is that the future policies affect previous decisions, because the particular realized history does not have probability zero. Define

$$\mathbf{oc}_0 = u(\overline{y}_0) - u(\overline{y}_0 - b_0 + q_0 b_1)$$

This is the opportunity cost of not defaulting. Rearrange (6.8), such that

$$V_0(\overline{y}_1, b_1) \geq \frac{1}{p(\overline{y}_1)}\left[\frac{1}{\beta}\mathbf{oc}_0 + \underline{\mathbb{W}} - \overline{\mathbb{W}}(b_1)\right] + \overline{V}(\overline{y}_1, b_1)$$

If this constraint binds, the lowest equilibrium consistent price is

$$\underline{q}_1(\overline{y}_0, b_0, q_0, b_1, \overline{y}_1, b_2)$$

with full history dependence.

When the Constraint is Binding? Whether it will bind or not, depends on the following. First, it depends on the past opportunity cost: if in the past, the government passed on default under very harsh circumstances, then the continuation value needs to be higher. Second, it depends on the strength of the link between current and past decision. If the government discounts more the future, or the history is less likely, then the constraint is less likely to be binding.

6.3 T Period History Dependence

We can extend the logic in the previous example to T periods. This case shows how the constraints in prices from past decisions, are weaker as the horizon increases. So, suppose that we observed the history up to

$$\overline{y}_0, b_0, q_0, b_1, ..., \overline{y}_T, b_{T+1}$$

We would like to solve for the minimum equilibrium consistent price. The problem is the following

$$\min_{\{V_0(y_{1i},b_1)\}_{y_{1i}},.....,\{V_T(y_{T+1i},b_{T+1})\}_{y_{T+1i}}} q_T$$

The constraints for consistency of the equilibrium decision are for $t = 0, ..., T-1$

$$\sum_i p(y_{t+1,i}) V_t(y_{t+1,i}, b_{t+1}) \geq \frac{1}{\beta}\left[u(\overline{y}_t) - u(\overline{y}_t - b_t + q_t b_{t+1})\right] + \underline{\mathbb{W}} \tag{6.9}$$

and for $t = T$

$$\sum_i p(y_{t+1,i}) V_t(y_{T+1,i}, b_{T+1}) \geq \frac{1}{\beta}\left[u(\overline{y}_T) - u(\overline{y}_T - b_T + q_T b_{T+1})\right] + \underline{\mathbb{W}} \tag{6.10}$$

The constraints for the values on the *realized history* are for $t = 1, ..., T$

$$V_{t-1}(\overline{y}_t, b_t) = u(\overline{y}_t - b_t + b_{t+1} q_t) + \sum_i p(y_{t+1,i}) V_t(y_{t+1,i}, b_{t+1}) \tag{6.11}$$

Thus, there are two types of constraints. First, constraints (6.9) concern the optimality of decision in $t = 0, ..., T-1$. Since, there are decisions from $t = 0$ to $t = T-1$, we have T of these constraints. The last price to appear is q_{T-1}. In addition, for equilibrium consistency, we need constraint 6.10. Note that in this constraint, the price we are solving for, q_T, appears directly. Second, we have a sequence of constraints on the realized history. There are $T-1$ of these constraints where observed prices enter, and one constraint where q_T appears directly.

The strategy to solve this program is the same as before. Off the equilibirum path choose the best continuation. On the equilibrium path, because of the constraint 6.11, once we choose a terminal continuation on the path, all the other continuation values are pinned down. Formally, we set

$$V_t(y_{t+1i}, b_{t+1}) = \begin{cases} V_t(\overline{y}_{t+1}, b_{t+1}) & \text{for } y_{t+1i} = \overline{y}_{t+1} \\ \overline{V}(\overline{y}_{t+1}, b_{t+1}) & \text{for } y_{t+1i} \neq \overline{y}_{t+1} \end{cases}$$

Note that after this observation the following holds

$$\sum_i p(y_{t+1i})V_t(y_{t+1i}, b_{t+1}) = p(\overline{y}_{t+1})\left[V_t(\overline{y}_{t+1}, b_{t+1}) - \overline{V}(\overline{y}_{t+1}, b_{t+1})\right] + \sum_i p(y_{t+1i})\overline{\mathbb{W}}(y_{t+1i}, b_{t+1})$$

This is very intuitive and will let us rewrite all of the inequalities. So, we can rewrite constraints in (6.9) as

$$p(\overline{y}_{t+1})\left[V_t(\overline{y}_{t+1}, b_{t+1}) - \overline{V}(\overline{y}_{t+1}, b_{t+1})\right] + \sum_i p(y_{t+1i})\overline{\mathbb{W}}(y_{t+1i}, b_{t+1}) \geq \frac{1}{\beta}\left[u(\overline{y}_t) - u(\overline{y}_t - b_t + q_t b_{t+1})\right]$$

for $t = 0, .., T-1$. We can rewrite this constraint as

$$\mathbb{P}_t(\text{Path})V_t(\text{Path}) + \mathbb{V}_t - \mathbb{P}_t(\text{Path})V_t(\text{Best}) \geq \frac{1}{\beta}\mathbf{oc}_t + \underline{\mathbb{W}}$$

There are three terms on the left hand side: the probability of the path times the value of the path, plus the value of the best equilibrium, minus the value of the best equilibrium on the path. Note that $V(\text{Path})$ will be completely pinned down by $V_{T-1}(\overline{y}_T, b_T)$. The other two terms are also functions of the history. How history determines q_T depends on the **length** and the **probability** of the path. Intuitively, if the path were very unlikely, the past places few restrictions; any possible future price that was going to be realized in the path, was very unlikely. So the equilibrium could have specified a low price, even with a high level of effort in the past, because the low price was unlikely. In addition, the length of the path matters. Choices made in the distant past will impose few restrictions in the future prices because these prices, and the utility associated with them, were realized very far into the future.

7 Equilibrium Consistency and RBA

In this section, we make a formal connection between equilibrium consistent outcomes and Robust Bayesian analysis. The main result is that, if the econometrician assumes the data generating process stems from a SPE of the game, then the set of equilibrium consistent outcomes essentially comprises the set of predictions a Bayesian econometrician can make, for *any* equilibrium Bayesian model (any prior over equilibrium outcomes).

7.1 Robust Bayesian Analysis

Based in the principles of Robust Bayesian statistics (see Berger et al. (1994)), we study the inferences that can be drawn from the observed data (a particular history h), which are *not* sensitive to the particular modeling assumptions (e.g., prior distribution chosen), across a given class of statistical models. Given that equilibrium multiplicity is a well-known problem of infinite horizon dynamic games, an econometrician must specify not only the physical environment for the economy, but also the equilibrium (or family of equilibria) on which they will focus their attention. Formally, the econometrician will try to draw inferences over:

a. **Fundamental parameters** $\theta \in \Theta$. These are parameters that fully describe the *physical environment of the economy* (examples are: The process for output $F(y_t)$, the utility function $u(c_t)$, discount factor $\beta \in (0,1)$, and international interest rate r.

b. **Endogenous parameters** $\alpha \in A$. These are parameters that given a physical description of the economy parametrize the stochastic process for the endogenous variables $\mathbf{x}(\alpha) = (d_t, b_{t+1}, q_t)$. These parameters comes from the equilibrium refinement (single valued or set valued) chosen by the econometrician.

For example, in the Eaton and Gersovitz (1981) setting it amounts to the following. The process for income $\{y_t\}_{t\in\mathbb{N}}$ can be an AR(1) process

$$\log y_t = \rho \log y_{t-1} + \epsilon_t$$

where y_t is output, and $\epsilon_t \sim_{i.i.d} N(0, \sigma_\epsilon^2)$. The utility function is $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$. Hence, the fundamental parameters in this economy are

$$\theta := \left(\rho, \sigma_\epsilon^2, \gamma, \beta, r^*\right)$$

The econometrician assumes that agents behave according to a particular rule, that relates exogenous variables with endogenous variables. The literature of sovereign debt focuses in the best perfect Markov equilibrium (with the restriction that after default there is a period in autarky). A special case is the equilibrium we covered in Section 4. So, $\mathbf{x}_\theta(\alpha) =$ best Markov equilibrium.

Bayesian vs Frequentist In a *frequentist approach*, parameters (θ, α) are estimated (by calibration or some other statistical procedure) to best fit the observed historic data. In this section we will focus on the *Bayesian approach* where the econometrician (or outside observer) has a prior distribution for the parameters (θ, α) and given data obtains a posterior. Our aim is to study inferences of Bayesian statistical models that hold any prior with support over equilibrium outcomes.

Definition 7.1. A **conditional model** $(m_\theta)_{\theta\in\Theta}$ is a family of triples

$$m_\theta = \{\mathcal{A}_\theta, (\alpha \to \mathbf{x}_\theta(\alpha) \in X), Q_\theta \in \Delta(A_\theta)\}$$

where $\mathcal{A}_\theta = (A_\theta, \Sigma_\theta)$ is the (measurable) space of *process parameters* $\alpha \in A_\theta$; $\alpha \to \mathbf{x}_\theta(\alpha)$ is the mapping that assigns to every parameter α a particular stochastic process $\mathbf{x}_\theta(\alpha)$ for the variables $(d_t, b_{t+1}, q_t)_{t\in\mathbb{N}}$ given an exogenous process for y_t; and $Q_\theta \in \Delta(A_\theta)$ is the $\Sigma_\theta-$measurable prior over $\alpha \in A_\theta$. w.l.o.g. we restrict attention to models where Q_θ is a full-support probability measure; i.e., $\operatorname{supp}(Q_\theta) = A_\theta$.[29] A conditional model m_θ is **parametric** if $A_\theta \subseteq \mathbb{R}^{k_m}$ (i.e., it has a finite-dimensional parameter space).

Definition 7.2. A ***Bayesian model*** (or *specification*) is a pair

$$m = \left\{(m_\theta)_{\theta\in\Theta}, p(\theta)\right\}$$

where $(m_\theta)_{\theta\in\Theta}$ is a conditional model and $p(\theta) \in \Delta(\Theta)$ is a prior over fundamental parameters.

[29]If this is not the case, the econometrician can work with an equivalent model, setting the parameter space to $\hat{A}_\theta = \operatorname{supp}(Q_\theta)$.

For the rest of this section, we will study Bayesian models conditional on a known fundamental parameter θ, fixing the physical environment (and hence dropping the dependence on θ). Once we condition on a particular value of the fundamental parameters, there is only uncertainty about the process followed by endogenous variables x. Given θ, one can map a particular model solely in terms of the probability distribution it implies over outcomes. Namely, given a conditional model $m = \{\mathcal{A}, \alpha \to \mathbf{x}(\alpha), Q \in \Delta(A)\}$, we can define the implied measure over outcomes as

$$\mathbf{Q}_m (B \subseteq \mathcal{X}) = Q\{\alpha \in A : \mathbf{x}(\alpha) \in B\}$$

We will refer to $\mathbf{Q}_m(\cdot)$ as m's **associated prior**.

Definition 7.3. m (with associated prior $\mathbf{Q}_m$) is a conditional **equilibrium model** if $\mathbf{Q}_m\left(x \in x\left(\mathcal{E}_{|h}\right)\right) = 1$; i.e., m assigns probability 1 to the process coming from a subgame perfect equilibrium profile.

The class of conditional equilibrium models is written as $\mathcal{M}_{\mathcal{E}}$. Also, given an equilibrium consistent history h, we write

$$\mathcal{M}_{\mathcal{E}}(h) = \left\{m : \mathbf{Q}_m\left(x\left(\mathcal{E}_{|h}\right)\right) = 1 \text{ and } \mathbf{Q}_m(\mathcal{X}(h)) > 0\right\}$$

i.e., the family of equilibrium models that assign positive probability to history h.[30]

7.2 Main Result

In the following proposition we will study the inferences a Bayesian econometrician makes conditional on a given fundamental parameter θ. It states the main result of this section, showing that the set of equilibrium consistent outcome paths $x\left(\mathcal{E}_{|h}\right)$ is essentially the union of all paths that have positive probability conditional on the observed history h, across all Bayesian equilibrium models.

Proposition 7.1. *Given an equilibrium consistent history $h \in \mathcal{H}(\mathcal{E})$*

a. The set of equilibrium consistent outcome paths satisfies:

$$x\left(\mathcal{E}_{|h}\right) = \{x \in X : \exists m \in \mathcal{M}_{\mathcal{E}}(h) \text{ and } \alpha \in supp\ (Q(\cdot \mid h)) \text{ such that } x = \mathbf{x}(\alpha)\} \tag{7.1}$$

b. For any measurable function $T : X \to \mathbb{R}$

$$\bigcup_{m \in \mathcal{M}_{\mathcal{E}}(h)} \int T(\mathbf{x}(\alpha))\, dQ(\alpha \mid h) = ch\ T\left(x\left(\mathcal{E}_{|h}\right)\right) \tag{7.2}$$

Restrictions on support First, note that (7.1) states that the outcomes that are equilibrium consistent after history h are the outcomes such that, there is a equilibirum conditional model that puts positive support on the parameters that maps into that outcome given the history. So, it formalizes the relation between a conditional equilibrium model and the set of equilibrium consistent outcomes given a history.

[30] A more general definition for which the results of the next section hold is to ask that for $\mathcal{X}(h)$ to be in the **support** of $\mathbf{Q}_m$.

Bounds on statistics Second, note that (7.2) can be rewritten it in terms of the associated prior over outcomes $\mathbf{Q}$

$$\bigcup_{m\in\mathcal{M}_{\mathcal{E}}(h)} \int T(x)\, d\mathbf{Q}_m (x \mid h) \subseteq \left[\inf_{x\in x\left(\mathcal{E}_{|h}\right)} T(x)\,, \sup_{x\in x\left(\mathcal{E}_{|h}\right)} T(x) \right]$$

with equality if ch $T\left(x\left(\mathcal{E}_{|h}\right)\right)$ is a closed set. Bayesian statisticians worry about the effect that the choice of the prior has for their inferences. To overcome this sensitivity, they choose a statistic T and report the interval of possible expected values of T under the posteriors in a family of priors $f \in \mathcal{F}$. For the case where $T\left(x\left(\mathcal{E}_{|h}\right)\right)$ is a compact set, we have that the set of all posterior expectations (conditional on h and θ) is identical to the interval $\left[\underline{T}(h), \overline{T}(h)\right]$, where $\underline{T}(h)$ and $\overline{T}(h)$ are, respectively, the minimum and maximum values of the set $\left\{T(x) : x \in x\left(\mathcal{E}_{|h}\right)\right\}$. The most important application of Proposition 7.1 is when we take y^t and $T_{y^t}(x) \equiv q_t^x\left(y^t\right)$. In this case, condition 7.2 helps us characterize the set of all expected values of bond prices q_t across all equilibrium Bayesian models as:

$$\bigcup_{m\in\mathcal{M}_{\mathcal{E}}(h)} \int q_t^{\mathbf{x}(\alpha)}\left(y^t\right) dQ(\alpha \mid h) = \left[\underline{q}_t, \overline{q}_t\right]$$

This is the interval characterized in Section 5.

7.3 Further Results

In this section we study models that are based on small perturbations on equilibrium profiles to formalize the intuition in the example of Section 2. Our focus will be on "$\epsilon-$ equilibrium models".

Definition 7.4. Model m is an $\epsilon-$**equilibrium model** if $\mathbf{Q}_m(x \in x(\mathcal{E})) \geq 1-\epsilon$.

In the example of Section 2, the government was trembling with probability δ in the decision to default. So, the models studied are $\epsilon-$ equilibria models with $\epsilon \equiv \delta p_1 / \left(\delta p_1 + (1-\delta)(1-p_1)\right)$. For a given (non-equilibrium) model m, we define

$$\mathbf{Q}_m^{\mathcal{E}}(B \subseteq \mathcal{X}) = \frac{\mathbf{Q}_m(B \cap x(\mathcal{E}))}{\mathbf{Q}_m(x(\mathcal{E}))}$$

as the equilibrium conditional prior. We will show that when $\epsilon \to 0$, the posterior moments calculated with $\epsilon-$ equilibrium models converge to the posterior means under their equilibrium conditional priors, and hence converge to elements in ch $T\left(x\left(\mathcal{E}_{|h}\right)\right)$.

Proposition 7.2. *Take an equilibrium history h and a family of models $(m_\epsilon)_{\epsilon\in(0,1)}$ (with a common parameter space) with associated priors $(\mathbf{Q}_\epsilon)_{\epsilon\in(0,1)}$ such that*

a. m_ϵ is an $\epsilon-$equilibrium model for all $\epsilon \in (0,1)$

b. There exist $\underline{p} > 0$ such that for all ϵ, $\mathbf{Q}_\epsilon(\mathcal{X}(h)) > \underline{p}$

Then, for any bounded and measurable function $T : X \to \mathbb{R}$ we have

$$\left|\int T(x)\, d\mathbf{Q}_{\epsilon}(x \mid h) - \int T(x)\, d\mathbf{Q}_{\epsilon}^{\mathcal{E}}(x \mid h)\right| \leq \epsilon \frac{(\overline{T} - \underline{T})}{\underline{p}} \tag{7.3}$$

where $\overline{T} = \sup T(x)$ *and* $\underline{T} = \inf T(x)$. *This implies that as* $\epsilon \to 0$

$$\left|\int T(x)\, d\mathbf{Q}_{\epsilon}(x \mid h) - \int T(x)\, d\mathbf{Q}_{\epsilon}^{\mathcal{E}}(x \mid h)\right| \to 0$$

Proof. See Appendix. □

Notice that for all $\epsilon > 0$, the prior $\mathbf{Q}_{\epsilon}^{\mathcal{E}}(\cdot)$ is an equilibrium prior, since by construction it assigns probability one to the set of equilibrium consistent outcomes. Proposition 7.1 then implies that

$$\int T(x)\, d\mathbf{Q}_{\epsilon}^{\mathcal{E}}(x \mid h) \in \text{ch}\; T\left(x\left(\mathcal{E}_{|h}\right)\right)$$

8 Extensions: Excusable Defaults and Savings

In this section we discuss how we characterize equilibrium consistent outcomes in a common setting for the literature of sovereign debt: we do not restrict that a default needs to be punished and, we allow for savings. This will break the connection beetween the best SPE and the Markov equilbrium that we characterized in Section 4, but autarky will still be the worst equilibrium. Given the best SPE values and prices, characterizing equilibrium consistent outcomes will follow the case in Section 5.

8.1 $\mathbb{ECO}$: Excusable Defaults

The setting where we do not impose that defaults *need* to be punished with financial exclusion is similar to the one in Atkeson (1991) and Worrall (1990)[31]. For the moment, assume that the government cannot save. The following proposition characterizes equilibrium consistent outcomes in this case.

Proposition 8.1 (ECO, excusable defaults). *Suppose* $h_{-}^{t} = (h^{t-1}, y_{t-1}, d_{t-1}, b_t)$ *is an equilibrium consistent history. Then* $x_{t-} = (q_{t-1}, d_t(\cdot), b_{t+1}(\cdot))$ *is an equilibrium consistent outcome at* $h_{-}^{t} \iff$ *the following conditions hold:*

a. Price is consistent

$$q_{t-1} = \frac{1}{1+r}\left(1 - \int d_t(y_t) dF(y_t)\right) \tag{8.1}$$

b. IC government

$$u(y_t - b_t(1 - d(y_t)) + \overline{q}^{E}(b_{t+1}(y_t))b_{t+1}(y_t)) + \beta \overline{\mathbb{W}}^{E}(b_{t+1}(y_t)) \geq V^{d}(y_t) \tag{8.2}$$

[31] In our case, we restrict the contract to be one where the face value can be chosen, but can either be defaulted or repaid in full.

c. Equilibrium consistency

$$\beta \int \left[u(y_t - b_t(1-d(y_t)) + \overline{q}^E(b_{t+1}(y_t))b_{t+1}(y_t)) + \beta \overline{\mathbb{W}}^E(b_{t+1}(y_t)) \right] dF(y_t) \geq$$

$$u(y_{t-1}) - u\left(y_{t-1} - b_{t-1}(1-d(y_{t-1})) + q_{t-1}b_{t+1}(y_t)\right) + \beta \underline{\mathbb{W}} \tag{8.3}$$

If conditions (a) through (c) hold, we write simply

$$(q_{t-1}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}_E(b_{t-1}, y_{t-1}, b_t)$$

where $\mathbb{ECO}$ stands for "equilibrium consistent outcomes" and the subscript E stands for the case of excusable defaults.

As in Section 5, conditions (8.1) and (8.2) characterize the set of SPE policies. The first condition (8.1) is again that the price has to be consistent with the default policy. The second condition (8.2) is the incentive compatibility for the government. The difference beetween (8.2) and the incentive compatibility of Proposition 5.1 that was given by

$$(1-d(y_t))\left[u(y_t - b_t + \overline{\mathbf{q}}(b_{t+1})b_{t+1}) + \beta \overline{\mathbb{W}}(b_{t+1})\right] + d(y_t)V^d(y_t) \geq V^d(y_t)$$

comes from the fact that defaults are not required to be punished. On the equilibrium path, defaults are excusable in the sense of Grossman and Huyck (1989);[32] off the equilibirum path they are punished with autarky, the worst equilibrium.

The intuition of condition (8.2) is similar to the incentive compatibility in Proposition 5.1 in Section 5. If in a history $h^t_- = (h^{t-1}, y_{t-1}, d_{t-1}, b_t)$ a default decision and bond issue decision wants to be implemented, it must be the case that it is weakly better than any deviation. Following Abreu (1988), any SPE can be implemented with strategies that impose the worst punishment in case of deviation; and, as in Proposition 5.1, we reward following the policy with the best equilibrium. This implies that d_t, b_{t+1} is implementable if

$$u(y_t - b_t(1-d_t) + \overline{q}(b_{t+1})b_{t+1}) + \beta \overline{W}(h^t_-, d_t, b_{t+1}) \geq$$

$$\max_{\tilde{d}, \tilde{b}'} u(y_t - b_t(1-\tilde{d}) + \underline{q}(\tilde{b}')\tilde{b}') + \beta \underline{W}(h^t_-, \tilde{d}, \tilde{b}') \tag{8.4}$$

where $\overline{W}, \underline{W}$ denote the best and worst continuation equilibria. The value of the best equilibrium is $\overline{\mathbb{W}}^E(b_{t+1})$. Because the worst equilibrium is autarky with a price of debt equal to zero ($\underline{q}(\tilde{b}') = 0$), the right hand side of (8.4) is equal to $V^d(y_t)$. Condition (8.2) follows. Again, conditions (8.1) and (8.2) are necessary and sufficient to characterize SPE outcomes.

Equilibrium consistent outcomes are characterized by an additional condition (8.3). The right hand side of (8.3) is the opportunity cost from not taking the best deviation last period. The left hand side specifies the expected value of the policy under the best equilibrium. The reason why conditions (8.1)-(8.3) are necessary and sufficient is the same as before.

[32] The reason why defaults are part of the equilbrium path is that they introduce stay contingency for the country and are also expected by the borrowers, so they will make zero profits on average.

The lowest equilibrium consistent price will solve

$$\underline{q}^E\left(h_-^{t+1}\right) = \min_{(\hat{q}, d_t(\cdot), b_{t+1}(\cdot))} \hat{q}$$

where

$$(\hat{q}, d_t(\cdot), b_{t+1}(\cdot)) \in \mathbb{ECO}_E(b_{t-1}, y_{t-1}, b_t)$$

The intuition of the solution to this program is similar the intuition that we had before. The bond policy will be the one of the best equilibrium, and the default policy will be tilted towards more defaults, but not so many that the previous choices cannot be rationalized. Again, the highest equilibrium consistent price will be $\overline{q}^E$, the best subgame perfect equilibrium price.

8.2 Best SPE

Note that the characterization of equilibrium consistent outcomes will use as input the best equilibrium price $\overline{q}^E(b_t)$ and the value function of the best equilibrium $\overline{\mathbb{W}}^E(b_{t+1}(y_t))$.[33]

Best Equilibrium Price Taking as given $\overline{\mathbb{W}}^E(b_{t+1}(y_t))$, the price function $\overline{q}^E$ solves the following functional equation

$$\overline{q}^E(b_t) = \max_{d(y_t), b'(y_t)} q$$

$$u(y_t - b_t(1 - d(y_t)) + \overline{q}(b_{t+1}(y_t))b_{t+1}(y_t)) + \beta\overline{\mathbb{W}}^E(b_{t+1}(y_t)) \geq V^d(y_t)$$

$$q = \frac{1}{1+r}\left(1 - \int d_t(y_t)dF(y_t)\right)$$

The default rule for each price will be denoted by $d(y_t; b_t)$. A solution to the operator is guaranteed due to the monotonicity of the operator and because the set of continuous and weakly decreasing functions endowed with the sup norm is a complete metric space.

Best Equilibrium Value Notice that we just obtained the best price taking as given the best equilibrium value for debt. Suppose now that we know the best price. The best equilibrium will be the equilibrium with highest expected value that meets the incentive compatibility and the price consistency constraint. It is given by

$$\overline{\mathbb{W}}^E(b_t) = \mathbb{E}_{y_t}\left[\overline{\mathbb{W}}^E(b_t, y_t)\right]$$

[33]Note that these ones will not be the ones of the Markov equilibrium that we characterized in Section 4. The reason is that now, the government is allowed to default, on the equilibrium path, without a punishment. A Markov equilibrium wih states b, y would imply that the government will default every debt that it has acquired. Therefore, there has a to be a price keeping constraint. An alternative approach is one as in Atkeson (1991) or Worrall (1990) that uses instead of b, y as a state variable, the funds that the government has after repayment, in our notation $y - (1 - d(y))b$. With this state variable, an approach as in Abreu, Pearce, and Stacchetti (1990) can be used to obtain the best equilibrium value and the policies.

where $\overline{\mathbb{W}}^E(b_t, y_t)$ solves

$$\overline{\mathbb{W}}(b_t, y_t) = \max_{d_t(y_t), b_{t+1}(y_t)} u(y_t - b_t(1 - d_t(y_t)) + \overline{q}^E(b_{t+1}(y_t))b_{t+1}(y_t)) + \beta \overline{\mathbb{W}}^E(b_{t+1}(y_t))$$

subject to

$$\overline{q}(b_t) = \frac{1}{1+r}\left(1 - \int d(y_t) dF(y_t)\right) \tag{8.5}$$

Note that, constraint (8.5), is the one that makes sure that the amount lent, will be defaulted with the best equilibrium default rule.

Algorithm The computation of the best price and best equilibrium suggests an algorithm for computation of the best equilibrium is as follows. Start with an initial guess of the best equilibrium price. Obtain the value of the best equilibrium. Iterate on prices. Iterate until convergence.

8.3 $\mathbb{ECO}$: Excusable Defaults and Savings

The most general characterization of SPE allows the government to save, and does not impose any exogenous punishment if it defaults. We can show that the worst equilibrium price for debt is zero. The reason is that SPE outcomes will be determined by the following conditions

$$q_{t-1} = \frac{1}{1+r}\left(1 - \int d_t(y_t) dF(y_t)\right)$$

$$u(y_t - b_t(1 - d(y_t)) + \overline{q}^{ES}(b_{t+1}(y_t))b_{t+1}(y_t)) + \beta \overline{\mathbb{W}}^{ES}(b_{t+1}(y_t)) \geq \max_{\tilde{d}, \tilde{b}'} u(y_t - b_t(1 - \tilde{d}) + \underline{q}(\tilde{b}')\tilde{b}') + \beta \underline{\mathbb{W}}(\tilde{b}')$$

Then, the worst SPE price for the case of savings and excusable defaults will be zero. So the characterization of equilibrium consistent outcomes is analogous to the one in Proposition 8.1 without the restriction that $b \geq 0$.

9 Conclusion and Discussion

Dynamic policy games have been extensively studied in macroeconomic theory to increase our understanding on how the outcomes that a government can achieve are restricted by its lack of commitment. One of the challenges in studying dynamic policy games is equilibrium multiplicity. Our paper acknowledges equilibrium multiplicity, and for this reason focuses on obtaining predictions that hold across all equilbria. To do this, we conceptually introduced and characterized equilibrium consistent outcomes. We did so under different settings, and we found that the assumption that a history was generated by the path of a subgame perfect equilibrium puts restrictions on current policies, and therefore on observables. In addition, we found intuitive conditions under which past decisions place restrictions on future policies; if the past decision occurred far away in time or in a history where the current history had low probability of occurrence, then it is less likely that a particular past decision influences current policies. In the extreme case that every particular history has probability zero, the restrictions of past decisions in current outcomes die out after one period. At first glance, this is surprising; but as we showed in the paper, this a direct consequence of robustness.

As we discussed in the text, equilbrium consistency is a general principle. Even though we focus on a model of sovereign debt that follows Eaton and Gersovitz (1981), our results generalize to other dynamic policy games. An example is the model of capital taxation as in Chari and Kehoe (1990). In that model, the entrepreneur invests and supplies labor, then the government taxes capital, and finally, the entrepreneur receives a payoff. The worst subgame perfect equilibrium is one where the government taxes all the capital. Note that, if the government has been consistently abstaining from taxing capital, then as outside observers we can rule out that the government will tax all capital. Past behavior, and the sole assumption of equilbrium, is giving information to the outside observer about future outcomes.

We think equilbrium consistency might have applications beyond policy games. The reason is that the sole assumption of equilibrium yields testable predictions. For example, the literature of risk sharing studies barriers to insurance and tries to test among different economics environments. Two environments that have received a lot of attention are Limited Commitment and Hidden Income. To test these two environments, a property of the efficient allocation with limited commitment is exploited: lagged consumption is a sufficient statistic of current consumption. If this hypothesis is rejected, then hidden income is favored in the data. However, the test is rejecting two hypotheses at the same time: efficiency and limited commitment. Our approach could, in principle, be suitable for a test that is tractable and robust to equilibrium multiplicity.

Over the course of the paper, we have been silent with respect to optimal policy. An avenue of future research is to relate equilibrium consistent outcomes and forward reasoning in dynamic games. Our conjecture is that, the set of equilibrium consistent outcomes will be intimately related with the set of outcomes if there is common knowledge of strong certainty of rationality. The reason is that, in the model of sovereign debt that we studied, the outside observer and the lenders have the same information set. Even in the motivating example, equlbrium consistent outcomes and outcomes when the solution concept is strong certainty of rationality are the same. In that case, our results have a different interpretation: the government is choosing the history to manage the expectations of the public.

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10 Appendix

Proof. (Theorem 5.1). (**Necessity, $\Longrightarrow$**) If $(d(\cdot), b'(\cdot))$ is SPE - consistent, there exists an SPE profile $\hat{\sigma}$ such that $h^t \in \mathcal{H}(\hat{\sigma})$ and

$$d(y_t) = d_t^{\hat{\sigma}}\left(h^t, y_t\right) \text{ and } b'(y) = b_{t+1}^{\hat{\sigma}}\left(h^t, y_t, d=0\right)$$

That is, there exists a SPE that generated the history h_-^t, specifies the contingent policy $d(\cdot), b'(\cdot)$ in period t, and satisfies conditions (5.1) to (5.3). Because $\hat{\sigma}$ is an SPE, using the results of Abreu, Pearce, and Stacchetti (1990) we know that if $d(y) = 0$ at $h^t = \left(h_-^t, q_{t-1}\right)$ then

$$u\left(y_t - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right)\right) + \beta W\left(\hat{\sigma} \mid h^{t+1}\right) \geq u(y_t) + \beta \mathbb{V}^d \tag{10.1}$$

By definition of best continuation values and prices

$$W\left(\hat{\sigma} \mid h^{t+1}\right) \leq \overline{\mathbb{W}}(b'(y_t)) \text{ and } q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right) \leq \overline{q}(b'(y_t)) \tag{10.2}$$

Because $b'(y_t) \geq 0$ (no savings assumption), and $u(\cdot)$ is strictly increasing, we can plug in (10.2) into (10.1) to conclude that

$$u\left(y_t - b + b'(y_t)\, \overline{q}\left(y_t, b'(y_t)\right)\right) + \beta \overline{\mathbb{W}}(b'(y_t)) \geq$$

$$u\left(y - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, d_t = 0, b'(y_t)\right)\right) + \beta W\left(\hat{\sigma} \mid h^{t+1}\right)$$

Proving condition (5.2). Further, since $\hat{\sigma}$ generated the observed history, past prices must be consistent with policy $(d(\cdot), b'(\cdot))$. Formally:

$$q_{t-1} = q_m^{\hat{\sigma}}\left(h^{t-1}, y_{t-1}, d_{t-1}, b_t\right) = \frac{1}{1+r^*}\left(1 - \int_{y_t \in Y} d^{\hat{\sigma}}\left(h^t, y_t\right) dF(y_t \mid y_{t-1})\right)$$

$$= \frac{1}{1+r}\left(1 - \int_{y_t \in Y} d(y_t)\, dF(y_t \mid y_{t-1})\right)$$

proving also condition (5.1). Condition (5.3) is the same as condition (5.2) but at $t-1$, using the usual promise keeping accounting. Formally, if $\hat{\sigma}$ is SPE and $h^t \in \mathcal{H}(\hat{\sigma})$ then the government's default and bond issue decision at $t-1$ was optimal given the observed expected prices

$$u\left(\underbrace{y_{t-1} - b_{t-1} + b_t q_{t-1}}_{=c_{t-1}}\right) + \beta W\left(\hat{\sigma} \mid h^t\right) \geq u(y_{t-1}) + \beta \mathbb{V}^d$$

Using the recursive formulation of $W(\cdot)$ we get the following inequality:

$$W\left(\hat{\sigma} \mid h^t\right) = \int_{y_t : d(y_t) = 0} \left[u\left(y_t - b_t + b'(y_t)\, q_m^{\hat{\sigma}}\left(h^t, y_t, d_t = 0, b'(y_t)\right)\right) + W\left(\hat{\sigma} \mid h^{t+1}\right)\right] dF(y_t)$$

$$+ \int_{y_t : d(y_t) = 1} \left[u(y_t) + \beta \mathbb{V}^d\right] dF(y_t)$$

$$\leq \int_{y_t:d(y_t)=0} \left[u\left(y_t - b_t + b'\left(y_t\right)\overline{q}\left(b'\left(y_t\right)\right)\right) + \overline{\mathbb{W}}\left(b'\left(y_t\right)\right)\right] dF\left(y_t\right)$$
$$+ \int_{y_t:d(y_t)=1} \left[u\left(y_t\right) + \beta\mathbb{V}^d\right] dF\left(y_t\right)$$

From the previous two inequalities, we show (5.3).

(**Sufficiency,** $\Longleftarrow$) We need to construct a strategy profile $\sigma \in SPE$ such that $h^t_- \in \mathcal{H}(\sigma)$ and $d(\cdot) = d^\sigma_t(h^t, \cdot)$ and $b'(\cdot) = b^\sigma_{t+1}(h^t, \cdot)$. Given that $h^t_- \in \mathcal{H}(SPE)$, we know there exists some SPE profile $\hat{\sigma} = (\hat{\sigma}_g, \hat{q}_m)$ that generated h^t_-. Let $\overline{\sigma}(b, y)$ be the best continuation SPE (associated with the best price $\overline{q}(\cdot)$) when $y_t = y$ and $b_{t+1} = b$. Let σ^{aut} be the strategy profile for autarky (associated with $q_m = 0$ for all continuation histories). Also, let $h^{t+1}(y_t) = (h^t, y_t, d(y_t), b'(y_t), \overline{q}(b'(y_t)))$ be the continuation history at $y_t = y$ and the policy $(d(\cdot), b'(\cdot))$ if the government faces the best possible prices. Define $(h^s, y_s) \prec h^t$ as the histories that precede h^t and are not equal to h^t. That is, if we truncate h^t to period s, we obtain h^s. Denote $(h^s, y_s) \not\prec h^t$ as the histories that do not precede h^t. The symbol $\preceq$ denotes, histories that precede and can be equal. Construct the following strategy profile $\sigma = (\sigma_g, q_m)$:

$$\sigma_g\left(h^s, y_s\right) = \begin{cases} \hat{\sigma}_g\left(h^s, y_s\right) & \text{for all } \left(h^s, y_s\right) \prec h^t \\ \sigma^{aut}\left(y_s\right) & \text{for all } s < t\text{and } \left(h^s, y_s\right) \not\prec h^t \\ d_t\left(h^t, y_t\right) = d\left(y_t\right) \text{ and } b_{t+1}\left(h^t, y_t\right) = b'\left(y_t\right) & \text{for } \left(h^t, y_t\right) \text{ for all } y_t \\ \overline{\sigma}_g\left(b_{s+1}, y_s\right)\left(h^s, y_s\right) & \text{for all } h^s \succeq h^{t+1}\left(y_t\right) \\ \sigma^{aut}\left(y_s\right) & \text{for all } s > t,\ h^s \not\succeq h^{t+1}\left(y_t\right) \end{cases}$$

and

$$q_m\left(h^s, y_s, d_s, b_{s+1}\right) = \begin{cases} \hat{q}_m\left(h^s, y_s, d_s, b_{s+1}\right) & \text{for all } \left(h^s, y_s\right) \prec h^t \\ 0 & \text{for all } s < t \text{ and } \left(h^s, y_s\right) \not\prec h^t \\ \overline{q}\left(b'\left(y_s\right)\right) & \text{for all } h^s \succeq \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right)\right) \\ 0 & \text{for all } h^s \not\succ \left(h^t, y_t, d\left(y_t\right), b'\left(y_t\right)\right) \end{cases}$$

By construction $h^t_- \in \mathcal{H}(\sigma)$. This is because, $\sigma = \hat{\sigma}_g$ for histories $(h^s, y_s) \preceq h^t$. Also, the strategy σ, prescribes the policy $(d(\cdot), b'(\cdot))$ on the equilibrium path. Now we need to show that the constructed strategy profile is indeed an SPE. For this, we will use the one deviation principle. See that for all histories with $s > t$ the continuation profile is an SPE (by construction); it prescribes the best continuation equilibrium, that is a SPE by definition. Now, we need to show that at h^t this is indeed an equilibrium. This comes from the second constraint, the incentive compatibility constraint

$$(1 - d(y_t))\left[u(y_t - b_t + \overline{q}(b_{t+1}((y_t)))b_{t+1}(y_t)) + \beta\overline{\mathbb{W}}(b_{t+1}((y_t)))\right]$$
$$+d(y_t)V^d(y_t) \geq V^d(y_t)$$

Note also that the default policy at $t-1$ was consistent with σ (and is an equilibrium) and that q_{t-1} is consistent with the policy $(d(\cdot), b'(\cdot))$. The promise keeping constraint

(5.3) translates into the exact incentive compatibility constraint for profile σ, showing that the default decision at $t-1$ was indeed optimal given profile σ. The "price keeping" (5.1) constraint also implies that q_{t-1} was consistent with policy $(d(\cdot), b'(\cdot))$. The final step in sufficiency is to show that, $s < t-1$ (that is $h^s \prec h^t$). Note that, because y is absolutely continuous, the particular y that is realized, has zero probability. So, the expected value of this new strategy is the same

$$W(\hat{\sigma} \mid h^s) = W\left(\sigma \mid h^t\right)$$

for all $h^s \prec h^t$ with $s < t-1$; the probability of the realization of h^t, is zero. All this together implies that σ is indeed an SPE and generates history h^t_- on the equilibrium path, proving the desired result. □

Proof. By Proposition 5.1, we can rewrite program (5.7) as,

$$\underline{q}(b, y, b') = \min_{q, d(\cdot) \in \{0,1\}^Y, b''(\cdot)} q$$

subject to

$$q = \frac{1 - \int d(y')\, dF(y' \mid y)}{1+r} \tag{10.3}$$

$$(1 - d(y'))\left(\overline{V}^{nd}(b', y', b''(y')) - V^d(y')\right) \geq 0 \tag{10.4}$$

and

$$\beta \int \left[d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b', y', b''(y'))\right] dF(y') - \beta \mathbb{V}^d \geq u(y) - u(y - b + b'q) \tag{10.5}$$

First, note that we can relax the constraint (10.4) and (10.5) by choosing

$$b''(y') = \underset{\hat{b} \geq 0}{\operatorname{argmax}}\, \overline{V}^{nd}\left(b', y', \hat{b}\right)$$

Second, define the set $R(b') = \left\{y' \in Y : \overline{V}^{nd}(b', y') \geq V^d(y')\right\}$ to be the set of income levels for which the government does not default, under the best continuation equilibrium. Note that, if $y' \notin R(b')$, it implies that no default is not equilibrium feasible for any continuation equilibrium (it comes from the fact that (10.4) is a necessary condition for no default). The minimization problem can now be written as

$$\underline{q}(b, y, b') = \min_{q, d(\cdot) \in \{0,1\}^Y} q$$

subject to

$$q = \frac{1 - \int d(y')\, dF(y' \mid y)}{1+r}$$

$$(1 - d(y'))\left[\overline{V}^{nd}(b', y') - V^d(y')\right] \geq 0 \text{ for all } y' \in R(b') \tag{10.6}$$

$$d(y') = 1 \text{ for all } y' \notin R(b') \tag{10.7}$$

$$\beta \int \left[d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b', y')\right] dF(y') - \beta \mathbb{V}^d \geq u(y) - u(y - b + b'q)$$

As a **preliminary step**, we need to show that this problem has a non-empty feasible set. For that, choose the default rule that makes all constraints be less binding: i.e. $d(y') = 0 \iff \overline{V}^{nd}(b', y') \geq V^d(y')$. This corresponds to the best equilibrium policy. If this policy is not feasible, then the feasible set is empty. Under this default policy, the one of the best equilibrium, the price q is equal to the best equilibrium price $q = \overline{\mathbf{q}}(b')$. The feasible set is non-empty if and only if

$$\beta \int \left[d(y') V^d(y') + (1 - d(y')) \overline{V}^{nd}(b', y')\right] dF(y') - \beta \mathbb{V}^d \geq u(y) - u(y - b + b'\overline{\mathbf{q}}(b'))$$

$$u(y - b + b'\overline{\mathbf{q}}(b')) + \beta \mathbb{W}(b') \geq u(y) + \beta \mathbb{V}^d \iff$$
$$\overline{V}^{nd}(b, y, b') \geq V^d(y)$$

where $\mathbb{W}(b')$ is the value of the option of defaulting b' bonds; this is the initial assumption of this proposition. Also, note that

$$\mathbb{V}^d = \int \left[d(y') V^d(y') + (1 - d(y'))V^d(y')\right] dF(y')$$

So, we can rewrite the promise keeping constraint as

$$\beta \int (1 - d(y')) \left[\overline{V}^{nd}(b', y') - V^d(y')\right] dF(y') \geq u(y) - u(y - b + b'q) \tag{10.8}$$

We focus on a **relaxed version of the problem.** We will allow the default rule to be $d(y') \in [0, 1]$ for all y'. Given the state variables (b, y, b') the relaxed problem is a convex minimization program in the space $(q, d(\cdot)) \in \left[0, \frac{1}{1+r}\right] \times \mathbb{D}(Y)$, where

$$\mathbb{D}(Y) \equiv \{d : Y \to [0, 1] \text{ such that } d(y') = 1 \text{ for all } y' \notin R(b')\}$$

is a convex set of default functions. Also, include the constraint for prices

$$q \geq \frac{1 - \int d(y') dF(y' \mid y)}{1 + r}$$

The intuition for this last constraint is that $d(y') = 1$ has to be feasible in the relaxed problem. The Lagrangian

$$\mathcal{L}(q, \delta(\cdot)) = q + \mu\left(-q + \frac{1 - \int d(y') dF(y' \mid y)}{1 + r}\right) +$$

$$\lambda\left(u(y) - u(y - b + b'q) - \beta \int (1 - d(y')) \left[\overline{V}^{nd}(b', y') - V^d(y')\right] dF(y')\right)$$

The optimal default rule $d(\cdot)$ must minimize the Lagrangian $\mathcal{L}$ given the multipliers (μ, λ) (where $\mu, \lambda \geq 0$). Notice that for $y' \in R(b')$ any $d \in [0, 1]$ is incentive constraint feasible, and

$$\frac{\partial \mathcal{L}}{\partial d(y')} = \left(-\frac{\mu}{1 + r} + \lambda\beta\left[\overline{V}^{nd}(b', y') - V^d(y')\right]\right) dF(y')$$

So, because it is a linear programming program, the solution is in the corners (and if it is not in the corners, it has the same value in the interior), then the values of y' such that the country does not default are given by

$$d\left(y'\right)=0 \iff \lambda\Delta^{nd}>\frac{\mu}{\beta\left(1+r\right)} \tag{10.9}$$

Note that $\lambda>0$ in the optimum. Suppose not; then $d\left(y'\right)=1$ for all $y'\in Y$ satisfies the IC and the price constraint. Then, the minimum price is

$$q\geq\frac{1-1}{1+r}$$

So, the minimizer will be zero, $q=0$. But, this will not meet the promise keeping constraint. Formally,

$$\beta\int V^{d}\left(y'\right)dF\left(y'\right)-\beta\mathbb{V}^{d}-u\left(y\right)+u\left(y-b\right)=$$

$$=\beta\left(\mathbb{V}^{d}-\mathbb{V}^{d}\right)+u\left(y-b\right)-u\left(y\right)=u\left(y-b\right)-u\left(y\right)<0$$

This implies $\lambda>0$. Note that, $\lambda>0$ implies that $\underline{\mathbf{q}}\left(b,y,b'\right)>0$. Define

$$\gamma\equiv\frac{\mu}{\lambda\beta\left(1+r\right)}$$

From (10.9)

$$d\left(y'\right)=0 \iff \Delta^{nd}\geq\gamma \iff \overline{V}^{nd}\left(b',y'\right)\geq V^{d}\left(y'\right)+\gamma$$

as we wanted to show. Aided with this characterization, from the promise keeping constraint we have an equation for γ as a function of the states

$$\beta\int_{\overline{V}^{nd}\left(b',y'\right)\geq V^{d}\left(y'\right)+\gamma}\left[\overline{V}^{nd}\left(b',y'\right)-V^{d}\left(y'\right)\right]dF\left(y'\right)=u\left(y\right)-u\left(y-b+b'q\right) \tag{10.10}$$

where

$$q=\frac{\Pr\left(\overline{V}^{nd}\left(b',y'\right)\geq V^{d}\left(y'\right)+\gamma\right)}{1+r} \tag{10.11}$$

Define

$$\Delta^{nd}(y'):=\overline{V}^{nd}\left(b',y'\right)-V^{d}\left(y'\right)$$

So,

$$q=\frac{\hat{F}\left(\Delta^{nd}(y')\geq\gamma\right)}{1+r}$$

where $\hat{F}$ is the probability distribution of $\Delta^{nd}(y')$. The **last step in the proof** involves showing that the solution is well defined. Define the function

$$G\left(\gamma\right)=\beta\int_{\Delta^{nd}\geq\gamma}\Delta^{nd}d\hat{F}\left(\Delta^{nd}\mid y\right)-u\left(y\right)+u\left(y-b+b'\frac{1-\hat{F}\left(\gamma\mid y\right)}{1+r}\right)$$

First, note that G is weakly decreasing in γ, that $G(0) > 0$ (from the assumption $\overline{V}^{nd}(b', y') - V^d(y') > 0$) and $\lim_{\gamma\to\infty} G(\gamma) = u(y-b) - u(y) < 0$. Second, note that G is right continuous in γ. These two observations imply that we can find a minimum $\gamma : G(\gamma) \geq 0$. If income is an absolutely continuous random variable, then $G(\cdot)$ is strictly decreasing and continuous, implying the existence of a unique γ such that $G(\gamma) = 0$. This determines the solution to the price minimization problem.

(**Proposition 7.1**) **Step 1**. Showing the first statement **(1)**. We first show if $x \in x\left(\mathcal{E}_{|h}\right)$ i.e. if x is equilibrium consistent at history h, we can construct an equilibirum model m and α in the conditional support such that $x = \mathbf{x}(\alpha)$. We construct it as follows: the possible values for the parameter α are $A = \{1\}$. The mapping is such that $\mathbf{x}(\alpha = 1) = x$. The measure Q is simply $Q(\alpha = 1) = 1$. Since $x \in x\left(\mathcal{E}_{|h}\right)$ we know there is an equilibrium σ that is consistent with x after h. Hence m_x is an equilibrium model. Also, according to our model $\Pr(x \in X(h)) = 1$, and hence $dQ(x \mid h) = \Pr(x \mid h) = 1 > 0$, finishing the proof. For the **converse**, take an equilibrium model $m \in \mathcal{M}_{\mathcal{E}}$ such that $\alpha \in \operatorname{supp}\ (Q(\cdot \mid h))$ such that $x = \mathbf{x}(\alpha)$. We will show that $x \in x\left(\mathcal{E}_{|h}\right)$. Using Bayes rule, the posterior distribution $Q(\alpha \mid h)$ after observing the history h

$$dQ(\alpha \mid h) = \begin{cases} \frac{dQ(\alpha)}{\int_{\hat{\alpha}: h \in \mathcal{H}(x_{\hat{\alpha}})} dQ(\hat{\alpha})} & \text{if } h \in \mathcal{H}(\mathbf{x}(\alpha)) \\ 0 & \text{if } h \notin \mathcal{H}(\mathbf{x}(\alpha)) \end{cases}$$

The prior was putting probability zero over non equilibrium outcomes, so the posterior has to be zero. This implies that $\alpha \in \operatorname{supp}(Q(\cdot \mid h)) \iff h \in \mathcal{H}(\mathbf{x}(\alpha)) = \mathcal{H}(\sigma_\alpha)$ for some $\sigma_\alpha \in \mathcal{E}_{|h}$ (since m is an equilibrium model). Therefore $x = x(\sigma_\alpha) \in x\left(\mathcal{E}_{|h}\right)$ finishing the proof.

Step 2. For **(2)**, first define $\underline{T} := \inf_{x \in x\left(\mathcal{E}_{|h}\right)} T(x)$ and $\overline{T} := \sup_{x \in x\left(\mathcal{E}_{|h}\right)} T(x)$. Take any equilibrium model m. Fix the history h. The expected value of $T(\cdot)$ under $Q(\cdot \mid h)$ is:

$$\mathbb{E}^{Q_\theta}\{T(x_\alpha) \mid h\} = \int_{\alpha \in A} T(\mathbf{x}(\alpha))\, dQ(\alpha \mid h) = \int_{\alpha \in \operatorname{supp}(Q(\cdot|h))} T(\mathbf{x}(\alpha))\, dQ(\alpha \mid h)$$

using in the second equality the definition of support, that was restricted without loss generality. Using equality 7.1 we know that for all $\alpha \in \operatorname{supp}(Q(\cdot \mid h))$ we have $\mathbf{x}(\alpha) \in x\left(\mathcal{E}_{|h}\right)$ and hence

$$\underline{T} \leq T(\mathbf{x}(\alpha)) \leq \overline{T} \text{ for all } \alpha \in \operatorname{supp}(Q(\cdot \mid h))$$

Each of the inequalities are strict unless $\underline{T} \in T\left(x\left(\mathcal{E}_{|h}\right)\right)$ and $\overline{T} \in T\left(x\left(\mathcal{E}_{|h}\right)\right)$ respectively, showing that $\mathbb{E}^{Q}\{T(\mathbf{x}(\alpha)) \mid h\} \in \left[\underline{T}, \overline{T}\right]$. We now need to show that it holds for every value in the convex hull. For any $\lambda \in \operatorname{ch} T\left(x\left(\mathcal{E}_{|h}\right)\right)$ there exist an equilibrium model m_λ such that $\mathbb{E}^{Q_\theta}\{T(\mathbf{x}(\alpha)) \mid h\} = \lambda$. First, suppose $\lambda \in \left(\underline{T}, \overline{T}\right)$. If $\lambda \in T\left(x\left(\mathcal{E}_{|h}\right)\right)$, we can specify model m as in the proof of **(1)** creating a model that assigns prob. 1 to $x : T(x) = \lambda$. If not, we know there exist equilibrium outcomes $x_1, x_2 \in x\left(\mathcal{E}_{|h}\right)$ and a number $\gamma \in (0,1)$ such that

$$\lambda = \gamma T(x_1) + (1-\gamma) T(x_2)$$

In this case, define m_λ with $A = \{1,2\}$, with mapping $\alpha = 1 \to x_1$ and $\alpha = 2 \to x_2$ and measure

$$Q^\lambda = \begin{cases} \alpha = 1 & \text{with prob. } \gamma \\ \alpha = 2 & \text{with prob. } 1-\gamma \end{cases}$$

is easy to check that $\mathbb{E}^{Q^\lambda}\left\{T\left(\mathbf{x}\left(\alpha\right)\right) \mid h\right\} = \lambda$. To finish the proof, we need to show the existence of such models on the cases when $\underline{T} \in \text{ch } T\left(x\left(\mathcal{E}_{|h}\right)\right)$ and $\overline{T} \in \text{ch } T\left(x\left(\mathcal{E}_{|h}\right)\right)$. In those cases, the construction from when $\lambda \in \left(\underline{T}, \overline{T}\right)$ applies. $\square$

Proof. (of **Proposition** 7.2) By Bayes rule:

$$\mathbf{Q}_\epsilon\left(B \mid h\right) \equiv \frac{\mathbf{Q}_\epsilon\left(B \cap \mathcal{X}\left(h\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}$$

which obviously implies that $\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right) \mid h\right) = 1$. Thus, to calculate $\mathbb{E}^{\mathbf{Q}_\epsilon}\left\{T \mid h\right\}$, we can just integrate over $\mathcal{X}\left(h\right) \subseteq \mathcal{X}$ to calculate the integral:

$$\int T\left(x\right) d\mathbf{Q}_\epsilon\left(x \mid h\right) = \int_{\mathcal{X}(h) \cap x(\mathcal{E})} T\left(x\right) d\mathbf{Q}_\epsilon\left(x \mid h\right) + \int_{\mathcal{X}(h) \cap (\mathcal{X} \sim x(\mathcal{E}))} T\left(x\right) d\mathbf{Q}_\epsilon\left(x \mid h\right)$$

As previously defined, $x\left(\mathcal{E}_{|h}\right) \equiv \mathcal{X}\left(h\right) \cap x\left(\mathcal{E}\right)$ is the set of equilibrium consistent outcomes with h, and denote $x\left(\sim \mathcal{E}_{|h}\right) := \mathcal{X}\left(h\right) \cap \left(\mathcal{X} \sim x\left(\mathcal{E}\right)\right)$ as the outcomes consistent with h and **not** consistent with any subgame perfect strategy profile. Using these new definitions together with Bayes rule formula for $\mathbf{Q}_\epsilon\left(\cdot \mid h\right)$ we get

$$\int T\left(x\right) d\mathbf{Q}_\epsilon\left(x \mid h\right) = \int_{x\left(\mathcal{E}_{|h}\right)} T\left(x\right) \frac{d\mathbf{Q}_\epsilon\left(x\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)} + \int_{x\left(\sim\mathcal{E}_{|h}\right)} T\left(x\right) \frac{d\mathbf{Q}_\epsilon\left(x\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)} \tag{10.12}$$

We now study the equilibrium conditional measure $Q_n^{\mathcal{E}}\left(\cdot\right)$. Applying Bayes rule and the definition of $Q_n^{\mathcal{E}}$ we get

$$\mathbf{Q}_\epsilon^{\mathcal{E}}\left(B \mid h\right) := \frac{\mathbf{Q}_\epsilon^{\mathcal{E}}\left(B \cap \mathcal{X}\left(h\right)\right)}{\mathbf{Q}_\epsilon^{\mathcal{E}}\left(\mathcal{X}\left(h\right)\right)} \underbrace{=}_{\text{by def.}} \frac{\mathbf{Q}_\epsilon\left(B \cap \mathcal{X}\left(h\right) \cap x\left(\mathcal{E}\right)\right)/\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right) \cap x\left(\mathcal{E}\right)\right)/\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}\right)\right)} =$$

$$= \frac{\mathbf{Q}_\epsilon\left(B \cap x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}\right)\right)}$$

and hence

$$\mathbf{Q}_\epsilon\left(B \cap x\left(\mathcal{E}_{|h}\right) \mid h\right) \underbrace{=}_{\text{by def.}} \frac{\mathbf{Q}_\epsilon\left(B \cap x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)} = \frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)} \mathbf{Q}_\epsilon^{\mathcal{E}}\left(B \mid h\right) \tag{10.13}$$

It will be also useful to define the non-equilibrium conditional measure

$$\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(B\right) \equiv \frac{\mathbf{Q}_\epsilon\left(B \cap \left(X \sim x\left(\mathcal{E}\right)\right)\right)}{\mathbf{Q}_\epsilon\left(X \sim x\left(\mathcal{E}\right)\right)}$$

for which we get, using Bayes rule:

$$\mathbf{Q}_\epsilon\left(B\cap x\left(\sim \mathcal{E}_{|h}\right)\mid h\right)=\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(B\mid h\right) \tag{10.14}$$

Thus we can rewrite the conditional measure $d\mathbf{Q}_\epsilon\left(x\mid h\right)$ as

$$d\mathbf{Q}_\epsilon\left(x\mid h\right)=\begin{cases}\frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}(h)\right)}d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right) & \text{if } x\in x\left(\mathcal{E}_{|h}\right)\\ \frac{\mathbf{Q}_\epsilon\left(x\left(\sim\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}(h)\right)}d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x\mid h\right) & \text{if } x\in x\left(\sim\mathcal{E}_{|h}\right)\\ 0 & \text{elsewhere}\end{cases} \tag{10.15}$$

Using 10.15, we then rewrite 10.12 as

$$\int T\left(x\right)d\mathbf{Q}_\epsilon\left(x\mid h\right)=\frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\int_{x\left(\mathcal{E}_{|h}\right)}T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right)+$$

$$+\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\int_{x\left(\sim\mathcal{E}_{|h}\right)}T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x\mid h\right)$$

so that

$$\int T\left(x\right)d\mathbf{Q}_\epsilon\left(x\mid h\right)-\int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right)=$$

$$=\left[\frac{\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)-\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\right]\int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right)+$$

$$+\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\int T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x\mid h\right)=$$

$$\frac{\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)}\left(\int T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x\mid h\right)-\int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right)\right) \tag{10.16}$$

using in the last equation the fact that $\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right)=\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}_{|h}\right)\right)+\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)$. See that since $x\left(\sim \mathcal{E}_{|h}\right)\subseteq X\sim x\left(\mathcal{E}\right)$, then

$$\mathbf{Q}_\epsilon\left(x\left(\sim \mathcal{E}_{|h}\right)\right)\leq \mathbf{Q}_\epsilon\left(X\sim x\left(\mathcal{E}\right)\right)=1-\mathbf{Q}_\epsilon\left(x\left(\mathcal{E}\right)\right)\leq\epsilon$$

using in the last inequality the fact that m_ϵ is an $\epsilon-$equilibrium model for all $\epsilon\in(0,1)$. Also, because T is bounded, we get that

$$\int T\left(x\right)d\mathbf{Q}_\epsilon^{\sim\mathcal{E}}\left(x\mid h\right)-\int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right)\leq\sup_{x\in X}T\left(x\right)-\inf_{x\in X}T\left(x\right)=\overline{T}-\underline{T}<\infty$$

Taking absolute values on both sides of 10.16, we get

$$\left|\int T\left(x\right)d\mathbf{Q}_\epsilon\left(x\mid h\right)-\int T\left(x\right)d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x\mid h\right)\right|=$$

$$= \frac{\mathbf{Q}_\epsilon \left(x \left(\sim \mathcal{E}_{|h}\right)\right)}{\mathbf{Q}_\epsilon \left(\mathcal{X}\left(h\right)\right)} \left| \int T\left(x\right) d\mathbf{Q}_\epsilon^{\sim \mathcal{E}}\left(x \mid h\right) - \int T\left(x\right) d\mathbf{Q}_\epsilon^{\mathcal{E}}\left(x \mid h\right) \right| \leq$$

$$\leq \epsilon \frac{\overline{T} - \underline{T}}{\underline{p}}$$

using also the assumption that $\mathbf{Q}_\epsilon\left(\mathcal{X}\left(h\right)\right) \geq \underline{p}$ for all $\epsilon \in (0,1)$, proving the desired result. $\square$