

Screening and Adverse Selection in Frictional Markets*

Benjamin Lester
FRB of Philadelphia

Ali Shourideh
University of Pennsylvania – Wharton

Venky Venkateswaran
NYU – Stern

Ariel Zetlin-Jones
Carnegie Mellon University

April 20, 2015

Abstract

We develop a tractable framework for analyzing adverse selection economies with imperfect competition. In our environment, uninformed buyers offer a general menu of screening contracts to privately informed sellers. Some sellers receive offers from multiple buyers while others receive offers from only one buyer, as in Burdett and Judd (1983). This specification allows us to smoothly vary the degree of competition, nesting monopsony and perfect competition à la Rothschild and Stiglitz (1976) as special cases. We show that the unique symmetric mixed-strategy equilibrium exhibits a *strict rank-preserving* property, in that different types of sellers have an identical ranking over the various menus offered in equilibrium. These menus can be all separating, all pooling, or a mixture of both, depending on the distribution of types and the degree of competition in the market. This calls into question the practice of using the incidence of separating contracts as evidence of adverse selection without controlling for market structure. We examine the relationship between *ex-ante* welfare and the degree of competition, and show that in some cases an interior level of frictions maximizes welfare, while in other cases competition is unambiguously bad for welfare. Finally, we study the effects of various policy interventions — such as disclosure and non-discrimination requirements — and show that our model generates new, and perhaps counter-intuitive insights.

Preliminary and Incomplete

*We thank Guido Menzio and Alessandro Lizzeri for useful discussions and comments. The views expressed here are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System.

1 Introduction

Many important markets suffer from *adverse selection*, including markets for insurance, loans, and even certain financial securities. In these markets, a common way to ameliorate the effects of asymmetric information is by designing nonlinear contracts that *screen* (or separate) different types. As a result, a large literature — both theoretical and empirical — has developed in order to study prices, allocations, and welfare in adverse selection markets with nonlinear pricing schedules.

However, perhaps surprisingly, this literature has focused primarily on the two extreme cases of either perfectly competitive or monopolistic market structures.¹ This focus has certainly not been motivated by empirical considerations, as many insurance, credit, and even financial markets are characterized by some degree of *imperfect competition*. Instead, it seems that an important reason for focusing on these two special cases has been a shortage of tractable models that can accommodate the analysis of the more general case.²

As a result, in contrast to the extreme cases of perfect competition and monopoly, much less is known about the continuum of cases in between. Does an equilibrium always exist? Is it unique? What is the nature of the menu of contracts that are offered in equilibrium? Do they pool or screen different types? Do some buyers end up predominantly trading with one type of seller, or do buyers typically trade equally often with different types? Finally, and perhaps most importantly, how does welfare respond to an increase in competition? Should policymakers necessarily be promoting competition and transparency in markets with adverse selection?

In this paper, we construct a simple, parsimonious framework that incorporates adverse selection, screening, and imperfect competition, and then we use this framework to answer all of the questions posed above. The basic building blocks of our model are completely standard: sellers are endowed with a perfectly divisible good, which is either of low or high quality, and this is the seller's private information; buyers offer a seller a menu of contracts, using price-quantity pairs to potentially screen high- and low-quality sellers; and sellers can accept at most

¹We discuss several notable exceptions in the literature review below.

²As [Einav et al. \(2010b\)](#) remark in a recent survey, relative to the progress made along other dimensions in the insurance literature, “there has been much less progress on empirical models of insurance market competition, or empirical models of insurance contracting that incorporate realistic market frictions. One challenge is to develop an appropriate conceptual framework.”

one contract, i.e., contracts are exclusive. Given these assumptions, if all sellers received offers from multiple buyers, our environment is equivalent to the perfectly competitive case studied in, e.g., [Rothschild and Stiglitz \(1976\)](#), [Rosenthal and Weiss \(1984\)](#), and [Dasgupta and Maskin \(1986\)](#). Alternatively, if all sellers receive only a single offer, our environment is equivalent to the monopsonist case studied in, e.g., [Mirrlees \(1971\)](#), [Stiglitz \(1977\)](#), and [Maskin and Riley \(1984\)](#).

However, we depart from the existing literature by assuming that there is uncertainty over the number of offers each seller receives. In particular, in the tradition of [Burdett and Judd \(1983\)](#), each seller receives two offers with probability π and only a single offer with probability $1 - \pi$. Thus, when buyers are designing contracts, they are unsure whether they will be a monopsonist or they will be engaged in Bertrand competition. In this way, we can smoothly vary the degree of competition in the market between the two extremes discussed above by simply varying $\pi \in [0, 1]$.

Now, even though the environment is fairly simple, the characterization of equilibrium is potentially very complicated. This is because, in almost all regions of the parameter space, buyers optimally choose to mix across menus according to a non-generate distribution function.³ Since each menu is comprised of two price-quantity pairs (one for each type), this implies that the key equilibrium object is a probability distribution over four-dimensional offers. A priori, there is no obvious reason to believe that these offers take any particular form; equilibria at the limits of $\pi = 0$ and $\pi = 1$, alone, can be separating or pooling, and can involve trade with only one type or full trade with both types.

Despite these complications, we develop techniques that enable us to provide a complete characterization of the equilibrium set. We first show that any contract can be summarized by the indirect utility it offers the type of seller it is intended for; this reduces the dimensionality of each menu from four to two. Then, we establish a key property of every equilibria: we show that any two menus that are offered in equilibrium are ranked in exactly the same way by both low and high type sellers. In other words, a buyer's offer will be accepted by exactly the same fraction of high- and low-quality sellers. This property of equilibria, which we call "strictly rank preserving," simplifies the characterization even more, as the marginal distribution of offers

³Mixing is to be expected for at least two reasons. First, this is a robust feature of nearly all models in which buyers are both monopsonists and Bertrand competitors with some probability, even without adverse selection or non-linear contracts. Second, even in perfectly competitive markets, it is well known that pure strategy equilibria may not exist in an environment with both adverse selection and non-linear contracts.

for high type sellers can be expressed as a strictly monotonic transformation of the marginal distribution of offers for low type sellers. In section 6, we show that this insight is can be extended to the more general formulation with many types. As we illustrate, the key force behind this result is the complementarity between the surplus provided to different types.

Using these results, we characterize the unique equilibrium for any value of π and any fraction of high- and low-quality sellers. We then exploit this characterization to explore the implications—both positive and normative—of imperfect competition in markets suffering from adverse selection.

We show that, in contrast with many papers in the literature, whether buyers offer sellers separating or pooling menus depends on the underlying distribution of types in the market. However, the nature of equilibrium contracts does not depend on this distribution alone, but also on the market structure. In particular, separating menus are more prevalent when the fraction of high-quality sellers is small or when markets are relatively competitive, while pooling menus emerge when the fraction of high-quality sellers is sufficiently large and market frictions are relatively severe. Interestingly, in some region of the parameter space, for a fixed fraction of high-quality sellers, equilibria can involve all buyers offering pooling contracts, all buyers offering separating contracts, or buyers mixing between the two; it all depends on the degree of competition, captured by π . Hence, using the incidence of separating contracts to identify the severity of adverse selection requires knowledge of the prevailing trading frictions in the market.

Turning to the model's normative implications, we first study the relationship between ex ante welfare and the degree of competition. We show that, when adverse selection is severe — that is, when the fraction of high-quality sellers is sufficiently small — ex ante welfare is inverse U-shaped in π . Therefore, in this region of the parameter space, there is an *interior* level of market frictions that maximizes surplus from trade. When adverse selection is mild, on the other hand, ex ante welfare is monotonic and decreasing in π , so that competition unambiguously hinders the process of realizing gains from trade.

Finally, we demonstrate that, by explicitly modeling adverse selection and imperfect competition in an environment with general contracts, our analysis also yields novel and interesting policy implications. Specifically, we analyze the welfare effects of providing buyers with additional information about sellers' types, or making markets more *transparent*. This is an

important, topical question in a number of insurance and financial markets, given a number of recent policy proposals and/or technological changes. Our main finding is that the desirability of additional transparency also depends on both the distribution of types and the degree of competition. In particular, when adverse selection is relatively mild or the market is relatively competitive, providing buyers with additional information is detrimental to welfare. However, the opposite is true when adverse selection and trading frictions are relatively severe. Thus, again, evaluating the implications of policies aimed at promoting transparency requires knowledge about both the distribution of types in the population and the extent of market frictions.

Literature Review Our paper contributes to a long literature on adverse selection.⁴ Our focus on contracts as screening devices puts us in the tradition of [Rothschild and Stiglitz \(1976\)](#), in contrast to the branch of the literature which restricts attention to single price contracts, following [Akerlof \(1970\)](#). Both approaches have been used extensively in empirical work on markets for financial assets, loans and insurance, ranging from tests of adverse selection to effects of policy interventions.⁵

The main novelty of our analysis — and our primary contribution — is a tractable and flexible specification of imperfect competition without restrictions on contracts. The literature, on the other hand, has generally stayed within the perfectly competitive paradigm.⁶ There are a few notable exceptions. For example, [Guerrieri et al. \(2010\)](#) study an environment with search frictions, where principals post contracts and match bilaterally with agents who direct their search efforts towards specific contracts.⁷ They depart from the perfectly competitive benchmark by specifying a bilateral matching technology that determines the probability that each agent trades (or is rationed) in equilibrium. As is customary in the search literature, this probability depends on the ratio of the measure of principals who offer a specific and the measure of agents that select it. Under a plausible restriction on off-equilibrium beliefs, the

⁴This body of work is too vast to cite every worthy paper. Recent contributions include [Kurlat \(2013\)](#), [Guerrieri and Shimer \(2014b\)](#), [Guerrieri and Shimer \(2014a\)](#), [Guerrieri et al. \(2010\)](#), [Fuchs et al. \(2012\)](#), [Fuchs and Skrzypacz \(2013\)](#), [Azevedo and Gottlieb \(2014\)](#), [Chari et al. \(2014\)](#), [Mahoney and Weyl \(2014\)](#), [Veiga and Weyl \(2012\)](#), [Chang \(2012\)](#), [Benabou and Tirole \(2014\)](#), [Bigio \(2014\)](#) and [Camargo and Lester \(2014\)](#).

⁵For example, [Ivashina \(2009\)](#) and [Chiappori and Salanie \(2000\)](#) test for adverse selection, while [Einav et al. \(2012\)](#) and [Einav et al. \(2010a\)](#) examine effects of policy interventions

⁶This is true, for example, of [Chari et al. \(2014\)](#), [Rosenthal and Weiss \(1984\)](#) and [Azevedo and Gottlieb \(2014\)](#) who allow for general contracts as well as of papers like [Kurlat \(2013\)](#), [Fuchs and Skrzypacz \(2013\)](#) and [Bigio \(2014\)](#) who analyze single-price contracts.

⁷[Chang \(2012\)](#), [Guerrieri and Shimer \(2014b\)](#) and [Guerrieri and Shimer \(2014a\)](#) also study adverse selection under competitive search.

resulting equilibrium is unique.

Our approach to modeling competition is distinct but complementary to the competitive search paradigm. Both approaches present explicit models of trading and allow principals to offer general contracts. There are, however, a few important differences. First, we obtain a unique equilibrium without additional assumptions or refinements. Second, we find that, depending on parameters, equilibrium contracts can be full pooling, separation or a combination of both; in [Guerrieri et al. \(2010\)](#), on the other hand, the equilibrium always features separation. In this sense, our approach has the potential to speak to a richer set of observed outcomes. Finally, our specification allows us to vary the degree of competition in a simple and intuitive way, nesting the well-known limit cases of monopsony and perfect competition as special cases.

The other approach to modeling imperfect competition with selection effects is product differentiation. Two recent examples are [Benabou and Tirole \(2014\)](#) and [Mahoney and Weyl \(2014\)](#). Identical contracts offered by various principals are valued differently by agents due to an orthogonal attribute — “distance” in a Hotelling interpretation or “taste” in a random utility, discrete choice framework. This additional dimension of heterogeneity is the source of market power, and changes in competition are induced by varying the importance of this alternative attribute, i.e. by altering preferences. We take a different approach to modeling (and varying) competition, which holds constant preferences and therefore, the potential social surplus. It is also worth noting that we arrive at very different conclusions about the desirability of competition than the aforementioned papers. In [Benabou and Tirole \(2014\)](#), a trade-off from increased competition arises not due to adverse selection *per se*, but from the need to provide incentives to allocate effort between multiple, imperfectly observable or contractible tasks. Without the issues raised by multi-tasking, competition improves welfare even in the presence of asymmetric information. In [Mahoney and Weyl \(2014\)](#), where attention is restricted to single-price contracts, welfare always increases with competition under adverse selection.

Our formalization of imperfect competition draws from the literature on search frictions and in particular, from the seminal work of [Burdett and Judd \(1983\)](#). In a paper contemporaneous to this one, [Garrett et al. \(2014\)](#) also introduce similar frictions into an environment with private information and screening contracts. Importantly, however, they restrict attention to private values. In other words, the private information of the agents is about their own payoffs and not that of the principals. Under these conditions, screening only serves as a tool for monopsony

rent extraction by the principals. Competition reduces (and ultimately, eliminates) these rents and, along with them, incentives to screen. Thus, when the asymmetric information is about private values, screening disappears under perfect competition. In contrast, with common values, screening plays a central role in mitigating adverse selection problem. As a result, it disappears only when that problem is sufficiently mild - increased competition serves to strengthen incentives to separate.

2 Model

Environment. We consider a market populated by a measure of sellers and a measure of buyers. Each seller is endowed with a single unit of a perfectly divisible good. A fraction $\mu_l \in (0, 1)$ of sellers possess a low (l) quality good, while the remaining fraction $\mu_h = 1 - \mu_l$ possess a high (h) quality good. Buyers and sellers derive utility v_i and c_i , respectively, from consuming each unit of a quality $i \in \{l, h\}$ good. We assume that

$$v_i > c_i \text{ for } i \in \{l, h\}, \tag{1}$$

so that there are gains from trading both high and low quality goods.

There are two types of frictions in the market. First, there is *asymmetric information*: sellers observe the quality of the good they possess while buyers do not, though the probability μ_i that a randomly selected good is quality $i \in \{l, h\}$ is common knowledge. In order to generate the standard “lemons problem,” we focus on the case in which

$$v_l < c_h. \tag{2}$$

The second type of friction is a *search* friction: the buyers in our model post offers, but sellers only sample a finite number of these offers. In particular, we assume that each seller samples one offer with probability $1 - \pi$ and two offers with probability π . Throughout the paper, we refer to sellers with one offer as “captive,” while we refer to those with two offers as “non-captive” sellers.⁸

We allow buyers to post menus that specify different price-quantity pairs. By the revelation

⁸This way of modeling search frictions follows from a long tradition, starting with [Butters \(1977\)](#), [Varian \(1980\)](#), and [Burdett and Judd \(1983\)](#).

principle, we can restrict attention to menus with two pairs, $\{(x_l, t_l), (x_h, t_h)\} \in ([0, 1] \times \mathbb{R}_+)^2$, that specify a quantity $x_i \leq 1$ of the good to be sold in exchange for a transfer t_i from the buyer to the seller, given that the seller reports owning a quality $i \in \{l, h\}$ good. Importantly, we assume that *contracts are exclusive*: if a seller samples two buyers' offers, he can only accept the offer of one buyer. Throughout the paper, we refer to a quantity-transfer pair (x, t) as a "contract," while we refer to a pair of contracts $\{(x_l, t_l), (x_h, t_h)\}$ as a "menu."

Payoffs. A seller who owns a quality i good and accepts a contract $(x_{i'}, t_{i'})$ receives a payoff

$$t_{i'} + (1 - x_{i'})c_i,$$

while a buyer who acquires a quality i good at terms $(x_{i'}, t_{i'})$ receives a payoff

$$-t_{i'} + x_{i'}v_i.$$

Meanwhile, a seller with a quality i good who does not trade receives a payoff c_i , while a buyer who does not trade receives zero payoff.

Strategies and Definition of Equilibrium. Let $\mathbf{z}_i = (x_i, t_i)$ denote the contract that is intended for a seller of type $i \in \{l, h\}$, and let $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$. A buyer's strategy, then, is a distribution across menus, $\Phi \in \Delta([0, 1] \times \mathbb{R}_+)^2$.

A seller's strategy is much simpler: given the available menus, a seller should choose the menu with the contract that maximizes her payoffs, or mix between menus if she is indifferent. Of course, conditional on a menu, the seller chooses the contract which maximizes her payoffs. In what follows, we will take the seller's optimal behavior as given.

A symmetric equilibrium is thus a distribution $\Phi^*(\mathbf{z})$ such that:

1. *Incentive compatibility*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^*(\mathbf{z})$,

$$t_l + c_l(1 - x_l) \geq t_h + c_l(1 - x_h) \tag{3}$$

$$t_h + c_h(1 - x_h) \geq t_l + c_h(1 - x_l). \tag{4}$$

2. *Buyer's optimize*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^*(\mathbf{z})$,

$$\mathbf{z} \in \arg \max_{\mathbf{z}} \sum_{i \in \{l, h\}} \mu_i (v_i x_i - t_i) \left[1 - \pi + \pi \int_{\mathbf{z}'} \chi_i(\mathbf{z}, \mathbf{z}') \Phi^*(d\mathbf{z}') \right], \quad (5)$$

where

$$\chi_i(\mathbf{z}, \mathbf{z}') = \begin{cases} 0 \\ \frac{1}{2} \\ 1 \end{cases} \quad \text{if} \quad t_i + c_i(1 - x_i) \begin{cases} < \\ = \\ > \end{cases} t'_i + c_i(1 - x'_i). \quad (6)$$

The function χ_i represents the seller's optimal choice between two menus. Note that we have assumed that if the seller is indifferent between menus then she chooses among menus with equal probability. Within a given menu, we have assumed that sellers do not randomize; for any incentive compatible contract, sellers choose the contract intended for their type, as in most of the mechanism design literature (see [Myerson \(1979\)](#), [Dasgupta et al. \(1979\)](#), for examples).

It is worth noting that the sellers in our model are heterogenous along two dimensions: the quality of the good that they are selling, and both the number and the type of alternative menus available to them. Moreover, *both* the quality of their good and their available alternatives are privately known. One might argue that buyers should try to screen the second dimension as well as the first, i.e., offer contracts that explicitly depend on whether a seller is captive, along with the details of the alternative menu when the seller is not captive. However, captive and non-captive sellers receive the same payoff from accepting any contract, and hence contracts that explicitly depend on the number or type of outside menus are irrelevant. That is, buyers cannot attain better outcomes if they are permitted to offer multiple menus intended for sellers that have the same quality good but differ with respect to the alternative menus that are available. We have implicitly used this result in our setup above. Note that when contracts are non-exclusive and sellers can trade with multiple buyers, this result no longer holds;

3 Properties of Equilibria

Characterizing the equilibrium described above—a distribution over four-dimensional menus that maximize buyers' payoffs while preserving incentive compatibility—is a daunting task. In this section, we establish a series of results that reduce the dimensionality of the equilibrium

characterization.

First, we show that each menu offered by a buyer can be summarized by the indirect utilities that it delivers to each type of seller, so that equilibrium strategies can in fact be defined by a joint distribution over two-dimensional menus. Then, we establish that the marginal distributions of offers intended for each type of seller are well-behaved, i.e., that they have fully connected support and no mass points.

Finally, we establish that there is a very precise link between the two contracts offered by any buyer, which imposes even more structure on the joint distribution of offers. In particular, we show that any two menus that are offered in equilibrium are ranked in exactly the same way by both low and high type sellers; that is, one menu is strictly preferred by a low type seller if and only if it is also preferred by a high type seller. This property of equilibria, which we call “strictly rank preserving,” simplifies the characterization even more, as the marginal distribution of offers for high type sellers can be expressed as a strictly monotonic transformation of the marginal distribution of offers for low type sellers. This property of equilibria also has important implications regarding the correlation of offers and terms of trade across different types of sellers; we explore these predictions in more detail in Section 5, when we explore the positive implications that emerge from our model in greater detail.

3.1 Utility Representation

As a first step, we establish two results that imply any menu can be summarized by two numbers, (u_l, u_h) , where

$$u_i = t_i + c_i(1 - x_i) \tag{7}$$

denotes the utility received by a type $i \in \{l, h\}$ seller from accepting a contract \mathbf{z}_i .

Lemma 1. *In any equilibrium, for almost all \mathbf{z} in the support of Φ^* , it must be that $x_l = 1$ and $t_l = t_h + c_l(1 - x_h)$.*

In words, Lemma 1 states that all equilibrium menus require that low quality sellers trade their entire endowment, and that the incentive compatibility constraint always binds for low quality sellers. This is reminiscent of the “no-distortion-at-the-top” result in the taxation literature, or that of full-insurance for the high-risk agents in [Rothschild and Stiglitz \(1976\)](#).

Corollary 2. *In equilibrium, any menu of contracts $\{(x_l, t_l), (x_h, t_h)\} \in ([0, 1] \times \mathbb{R}_+)^2$ can be summarized by a pair (u_l, u_h) with $x_l = 1, t_l = u_l$,*

$$x_h = 1 - \frac{u_h - u_l}{c_h - c_l}, \text{ and} \quad (8)$$

$$t_h = \frac{u_l c_h - u_h c_l}{c_h - c_l}. \quad (9)$$

Since $0 \leq x_h \leq 1$, note that the pair (u_l, u_h) must satisfy

$$c_h - c_l \geq u_h - u_l \geq 0 \quad (10)$$

in order to satisfy feasibility. Note that when $u_h = u_l$, Corollary 2 implies that $x_h = 1$ and $t_h = t_l$.

3.2 Recasting the Buyer's Problem and Equilibrium

Buyer's Problem. Given the results above, we can recast the problem of a representative buyer as choosing a menu of indirect utilities, (u_l, u_h) , taking as given the distribution of indirect utilities offered by other buyers. For any menu (u_l, u_h) , buyers must infer the probability that the menu will be accepted by a type $i \in \{l, h\}$ seller. In order to calculate these probabilities, let us define the marginal distributions

$$F_l(u_l) = \int_{z'_l} \mathbf{1} [t'_l + c_l (1 - x'_l) \leq u_l] \Phi (dz'_l)$$

$$F_h(u_h) = \int_{z'_h} \mathbf{1} [t'_h + c_h (1 - x'_h) \leq u] \Phi (dz'_h).$$

$F_l(u_l)$ and $F_h(u_h)$ are the probability distributions of indirect utilities arising from each buyer's mixed strategy. When these distributions are continuous and have no mass points, the probability that a contract intended for a type i seller is accepted is simply $1 - \pi + \pi F_i(u_i)$, i.e., the probability that the seller is captive plus the probability that he is non-captive but receives an offer less than u_i . However, if $F_i(\cdot)$ has a mass point at u_i , then the fraction of non-captive sellers of type i attracted to a contract with value u_i is given by $\tilde{F}_i(u_i) = \frac{1}{2} F_i^-(u_i) + \frac{1}{2} F_i(u_i)$,

where $F_i^-(u_i) = \lim_{u \nearrow u_i} F_i(u)$ is the left limit of F_i at u_i .⁹ Given $\tilde{F}_i(\cdot)$, each buyer solves

$$\max_{u_l \geq c_l, u_h \geq c_h} \mu_l (1 - \pi + \pi \tilde{F}_l(u_l)) \Pi_l(u_h, u_l) + \mu_h (1 - \pi + \pi \tilde{F}_h(u_h)) \Pi_h(u_h, u_l) \quad (11)$$

$$\text{s. t. } c_h - c_l \geq u_h - u_l \geq 0, \quad (12)$$

with

$$\Pi_l(u_h, u_l) \equiv v_l x_l - t_l = v_l - u_l \quad (13)$$

$$\Pi_h(u_h, u_l) \equiv v_h x_h - t_h = v_h - u_h \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_h}{c_h - c_l}. \quad (14)$$

In words, $\Pi_i(u_h, u_l)$ is the buyer's payoff conditional on the offer u_i being accepted by a type i seller. We refer to the objective in (11) as $\Pi(u_h, u_l)$.

Equilibrium. Using the optimization problem described above, we can redefine the equilibrium in terms of the distributions of indirect utilities. In particular, for each u_l , let

$$\begin{aligned} U_h(u_l) &= \arg \max_{u'_h \geq c_h} \Pi(u'_h, u_l) \\ &\text{s. t. } c_h - c_l \geq u'_h - u_l \geq 0. \end{aligned}$$

The equilibrium can then be described by the marginal distributions $\{F_i(u_i)\}_{i \in \{l, h\}}$ together with the requirement that a joint distribution function must exist. In other words, a probability measure μ over the set of feasible (u_l, u_h) 's must exist such that, for each $u_l > u'_l$ and $u_h > u'_h$

$$1 = \mu(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_l \in [c_l, v_h]\})$$

$$F_l^-(u_l) - F_l(u'_l) = \mu(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_l \in (u'_l, u_l)\}), \quad (15)$$

$$F_h^-(u_h) - F_h(u'_h) = \mu(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_h \in (u'_h, u_h)\}). \quad (16)$$

Note that this definition of equilibrium imposes two different requirements. The first is that buyers behave optimally: for each u_l , the joint probability measure puts a positive weight only on $u_h \in U_h(u_l)$. The second is aggregate consistency: the fact that F_l and F_h are marginal distributions associated with a joint measure of menus.

⁹Since F_i is a distribution function it is right continuous and its left limits exists everywhere (it is càdlàg) and it has countable points of discontinuity. We make use of these properties repeatedly throughout the paper.

3.3 Basic Properties of Equilibrium Distributions

In this section, we establish that, in equilibrium, the distributions $F_l(u_l)$ and $F_h(u_h)$ are continuous and have connected support, i.e., there are neither mass points nor gaps in either distribution.

Proposition 3. *The marginal distributions F_l and F_h have connected support. They are also continuous, with the possible exception of a mass point at the lower bound of their support.*

Much like [Burdett and Judd \(1983\)](#), the basic idea behind the proof of Proposition 3 is to come up with deviations that rule out having gaps and mass points in the distribution. The difficulty in our model is that payoffs are interdependent, e.g., a change in the utility offered to low type sellers changes the contract—and hence the profits—that a buyer receives from high type sellers. It is possible, however, to prove these claims sequentially, first for the distribution F_h and then for the distribution F_l . We sketch the proofs here, and present the formal arguments in the Appendix.

To see that F_h has connected support, suppose towards a contradiction that F_h is constant on some interval, and consider an equilibrium menu (\bar{u}_l, \bar{u}_h) with \bar{u}_h equal to the upper bound of this interval. We show that offering a menu (\bar{u}_l, u_h) , with $u_h < \bar{u}_h$ and $F_h(u_h) = F_h(\bar{u}_h)$, is a profitable deviation because the offer u_h attracts the same fraction of high quality sellers but makes more profit per trade. Such a deviation is feasible, however, only if $\bar{u}_h > \bar{u}_l$. If instead $\bar{u}_h = \bar{u}_l$, we show that a deviation to a menu (u_l, \bar{u}_h) , with $u_l < \bar{u}_l$, must be a profitable deviation.

To see that F_h has no mass points, suppose towards a contradiction that it does. If $\Pi_h(u_h, u_l)$ is strictly positive for such a value of u_h , then a small increase in u_h will increase profits by attracting a mass of high quality sellers. If, instead, $\Pi_h(u_h, u_l)$ is not positive, then we can establish several facts. First, since overall profits must be strictly positive for any $\pi \in (0, 1)$, then $\Pi_l(u_l, u_h) > 0$ when $\Pi_h(u_l, u_h) \leq 0$. Second, it must be that $u_h = c_h$, as otherwise it would be profitable to decrease u_h by a small amount and *shed* a positive mass of high type sellers. Finally, if $u_h = c_h$ and $\Pi_h(u_l, u_h) \leq 0$, it must be that $x_h = 0$ —that is, the buyer will not actually trade with high type sellers—in which case it must be that $u_l = c_l$. Therefore, if there is a mass point at $u_h = c_h$, there must also be a mass point at $u_l = c_l$. However, if this is true, then a small increase in u_l is a profitable deviation, which completes the contradiction.

Having shown that F_h is continuous and strictly increasing, we then apply an inductive argument to prove that F_l has connected support and is continuous, with a possible exception at the lower bound of the support. An important step in the induction argument, which we use more generally, is to show that the objective function $\Pi(u_h, u_l)$ is strictly supermodular. We state this here as a lemma.

Lemma 4. *Suppose F_h has connected support and is continuous over its support. Then the profit function is strictly supermodular so that*

$$\Pi(u_{h1}, u_{l1}) + \Pi(u_{h2}, u_{l2}) \geq \Pi(u_{h1}, u_{l2}) + \Pi(u_{h2}, u_{l1}), \forall u_{i1} \geq u_{i2}, i \in \{l, h\}$$

with strict inequality when $u_{i1} > u_{i2}$.

The supermodularity of the buyer's profit function reflects a basic complementarity between the indirect utilities offered to low and high quality sellers. In particular, an increase in the indirect utility offered to low quality sellers relaxes their incentive constraint and allows the buyer to make higher profits from trading high quality sellers. This, in turn, raises the value of attracting more high types by increasing u_h .

An important implication of Lemma 4 is that the correspondence $U_h(u_l)$ is weakly increasing. We use this property to construct deviations similar to those described above in order to rule out gaps and mass points in the distribution F_l at everywhere except possibly at the lower bound of the support. However, in Section 4, we show that mass points at the lower bound only occur in a knife-edge case; generically, the marginal distribution F_l has connected support and no mass points everywhere in its support. We defer the details of the proof to Appendix A.

3.4 Strict Rank Preserving

In this section, we establish that the mapping between a buyer's optimal offer to low and high quality sellers, $U_h(u_l)$, is a well-defined, strictly increasing function. As a result, every equilibrium has the property that menus are *strictly rank preserving*—that is, low and high types share the same ranking over the set of contracts offered in equilibrium—with the possible exception of the knife-edge case discussed above. To formalize this result, the following definition is helpful.

Definition 5. *For any subset \mathcal{U}_l of $\text{Supp}(F_l)$, an equilibrium is **strictly rank-preserving over \mathcal{U}_l** if*

the correspondence $U_h(u_l)$ is a strictly increasing function of u_l for all $u_l \in U_l$. An equilibrium is **strictly rank-preserving** if it is strictly rank-preserving over $\text{Supp}(F_l)$.

Equivalently, an equilibrium is strictly rank-preserving when, for any two points in the equilibrium support (u_l, u_h) and (u'_l, u'_h) , $u_l > u'_l$ if and only if $u_h > u'_h$. Given this terminology, we can now establish one of our key results.

Theorem 6. *Let $\underline{u}_l = \min \text{Supp}(F_l)$. Then all equilibria are strictly rank-preserving over the set $\text{Supp}(F_l) \setminus \{\underline{u}_l\}$.*

The proof of Theorem 6, which can be found in the Appendix, makes use of the key facts established above; namely, that the distributions $F_l(\cdot)$ and $F_h(\cdot)$ are strictly increasing and continuous, and that $\Pi(u_l, u_h)$ is supermodular, which implies that $U_h(u_l)$ is a weakly increasing correspondence. In words, if there exists a $u_l > \underline{u}_l$ and $u'_h > u_h$ such that $u_h, u'_h \in U_h(u_l)$, then the supermodularity of $\Pi(u_l, u_h)$ implies that $[u_h, u'_h] \subset U_h(u_l)$. Since $F_h(\cdot)$ has connected support, this implies that $F_l(\cdot)$ must have a mass point at u_l , which contradicts Proposition 3. A similar argument rules out the possibility that there exist u_h and $u'_l > u_l$ that are offered in equilibrium with $U_h(u'_l) = U_h(u_l) = u_h$. Hence, $U_h(u_l)$ must be a strictly increasing function for all $u_l > \underline{u}_l$.

Moreover, when $F_l(\cdot)$ is continuous everywhere—that is, when there is no mass point at the lower bound of the support—then every menu offered in equilibrium is accepted by exactly the same fraction of low and high quality non-captive sellers. We state this result in the following Corollary to Theorem 6.

Corollary 7. *If F_l and F_h are continuous, then $F_h(U_h(u_l)) = F_l(u_l)$.*

Taken together, theorem 6 and Corollary 7 simplify the search for equilibrium, which we undertake in the next Section. Specifically, if an equilibrium exists in which the marginal distributions F_l and F_h are continuous, then the equilibrium can be described compactly by the marginal F_l along with the policy function $U_h(u_l)$.

4 Construction of Equilibrium

In this section, we use the properties established above to help construct equilibria. Then we show that the equilibrium we construct is unique. In this sense, we characterize the entire set of equilibrium outcomes in our model.

To fix ideas, it is useful to discuss the well-known extreme cases of $\pi = 0$ and $\pi = 1$, i.e., when sellers face a monopsonist and when markets are perfectly competitive, respectively. Using these two extreme cases, we identify a key parameter that governs the structure of the equilibrium set for the general case of $\pi \in (0, 1)$.

4.1 Monopsony and Perfect Competition

We start by describing the equilibrium outcome when sellers face a monopsonist and perfectly competitive markets.

Monopsony. When each seller meets with at most one buyer, i.e., $\pi = 0$, the buyers choose a pair (u_l, u_h) to maximize

$$\mu_l(v_l - u_l) + \mu_h \left[v_h - u_h \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_h}{c_h - c_l} \right],$$

subject to the constraint described in (12). The solution to this problem, summarized in Lemma 8 below, is standard and hence we omit the proof.

Lemma 8. *Let*

$$\phi = 1 - \frac{\mu_h}{\mu_l} \left(\frac{v_h - c_h}{c_h - c_l} \right). \quad (17)$$

When $\pi = 0$, the equilibrium is as follows:

- (i) *if $\phi > 0$, then $u_l = c_l$ with $x_l = 1$ and $u_h = c_h$ with $x_h = 0$;*
- (ii) *if $\phi < 0$, then $u_l = u_h = c_h$ with $x_l = x_h = 1$.*
- (iii) *if $\phi = 0$, then $u_l \in [c_l, c_h]$ with $x_l = 1$ and $u_h = c_h$ with $x_h = 0$.*

Intuitively, the buyer optimally chooses not to trade with the type h seller when $\phi \geq 0$, or

$$\mu_l \geq \mu_h \left(\frac{v_h - c_h}{c_h - c_l} \right). \quad (18)$$

The left hand side of (18) is the marginal benefit of lowering $u_l = t_l$: with probability μ_l , the buyer pays a marginal unit less in exchange for the low quality good. The right hand side of (18) is the cost of a small decrease in u_l : since reducing u_l tightens the incentive compatibility constraint, the buyer must trade a smaller amount with high quality sellers. Since the objective is linear, $x_h = 0$ when (18) is satisfied. On the other hand, when (18) is violated, the buyer pools high and low quality sellers, offering c_h in exchange for one unit of either good.

Given this interpretation, the parameter ϕ can be thought of as capturing the buyer's net loss from a marginal increase in the utility offered to the low quality seller, taking into account the fact that such an increase relaxes the incentive constraint and leads to higher profits from trading with the high quality seller. This loss is positive when the fraction of high quality sellers, μ_h , is small and/or the gains from trading with high quality sellers, $v_h - c_h$, is relatively small.

Note that an important feature of the monopsony allocations is that the buyer makes non-negative profits on each type when $\phi > 0$. However, when $\phi < 0$, the monopolist subsidizes the low quality sellers in order to be able to trade with high quality sellers, i.e., allocations involve *cross-subsidization*. As we will show, this pattern extends to the set of equilibria in our general model with $\pi \in (0, 1)$.

Bertrand Competition. When markets are perfectly competitive, i.e., when $\pi = 1$, our setup becomes the same as that in [Riley \(1975\)](#), [Riley \(1979\)](#) and [Rosenthal and Weiss \(1984\)](#), and similar to that of [Rothschild and Stiglitz \(1976\)](#). As is typical in adverse selection economies, a pure strategy equilibrium in our model must satisfy separation ($u_h > u_l$) and buyers must earn zero profits on each type of seller ($u_l = v_l$ and $\Pi_h(u_h, v_l) = 0$). Indeed, when μ_h is sufficiently small, so that $\phi \geq 0$, the unique equilibrium is the least-cost separating contract: low type sellers earn $u_l = v_l$ and high type sellers receive v_h per unit sold on a fraction of their endowment, which is sufficiently small to ensure that incentive compatibility holds.

However, when μ_h is larger, so that $\phi < 0$, one can construct a pooling menu which yields buyers strictly positive payoffs, and hence dominates the pure strategy equilibrium described above. In this case, no pure strategy equilibrium exists, and thus we turn our attention to equilibria in mixed strategies, as in [Dasgupta and Maskin \(1986\)](#) and [Rosenthal and Weiss \(1984\)](#).

In the equilibrium that emerges, buyers mix across menus (u_l, u_h) in which the payoff from trading with low types is negative, the payoff from trading with high types is positive, but the two payoffs offset each other exactly, so that buyers ultimately break even. This is true across each contract, independent of the marginal distributions. In order to ensure that this is an equilibrium, however, the marginal distribution $F_l(\cdot)$ must be constructed to ensure that pure strategy deviations (either pooling or cream-skimming) attract disproportionately many

low quality sellers and therefore are not profitable. The following lemma summarizes this equilibrium in the Bertrand game; the proof is in the Appendix.

Lemma 9. *When $\pi = 1$, the unique equilibrium of the game is as follows:*

(i) *if $\phi \geq 0$, then $u_l = v_l$ with $x_l = 1$ and $u_h = \frac{v_h(c_h - c_l) + v_l(v_h - c_h)}{v_h - c_l}$ with $x_h = \frac{v_l - c_l}{v_h - c_l}$;*

(ii) *if $\phi < 0$, then symmetric equilibrium is described by the distribution*

$$F_l(u_l) = \left(\frac{u_l - v_l}{\mu_h(v_h - v_l)} \right)^{-\phi} \quad (19)$$

for value of the low type, with $\text{Supp}(F_l) = [v_l, \bar{v}]$ where $\bar{v} = \mu_h v_h + \mu_l v_l$. Furthermore, $U_h(u_l)$ satisfies

$$\mu_h \Pi_h(U_h(u_l), u_l) + \mu_l \Pi_l(U_h(u_l), u_l) = 0. \quad (20)$$

Note, again, that the equilibrium features cross-subsidization when $\phi < 0$, but not when $\phi \geq 0$.

4.2 Imperfect Competition

We are now ready to analyze the general model with imperfect competition, i.e., when $\pi \in (0, 1)$. As in the two cases studied above, the sign of the parameter ϕ is a key determinant of the structure of the equilibrium outcome. Hence, we consider the three relevant cases— $\phi > 0$, $\phi < 0$, and $\phi = 0$ —sequentially.

4.2.1 No Cross-subsidizing Equilibria ($\phi > 0$)

We proceed by first constructing an equilibrium when $\phi > 0$, and then establishing in Section 4.3 that this equilibrium is unique. We construct the equilibrium distribution F_l as the solution to the following differential equation

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} (v_l - u_l) = \phi \quad (21)$$

along with the boundary condition $\min \text{Supp}(F_l) = c_l$. The policy function, $U_h(u_l)$, satisfies

$$(1 - \pi + \pi F_l(u_l)) [\mu_h \Pi_h(U_h(u_l), u_l) + \mu_l (v_l - u_l)] = \mu_l (1 - \pi) (v_l - c_l). \quad (22)$$

To understand this equilibrium, note that the differential equation (21) describes the first order condition for a buyer with respect to u_l when facing the distribution $F_l(u_l)$. In this sense,

(21) describes necessary conditions for an equilibrium because it ensures local deviations away from equilibrium menus are not profitable. Implicitly, this first order condition requires three assumptions. First, that there is not a mass point at the lower bound of the support of $F_l(u_l)$. Second, that for each menu (u_l, u_h) , the utility for the high quality seller is strictly greater than that for the low quality seller; we call such an equilibrium a *separating equilibrium*. Lastly, that the implied quantity traded by the high quality seller is interior in all trades, i.e., $0 < x_h = (u_h - u_l) / (c_h - c_l) < 1$, except possibly at the boundary of the support of F_l . We call an equilibrium that satisfies this latter property an *interior equilibrium*.

Equation (21) is familiar from basic production theory and resembles optimality condition of a monopolist facing a demand function. For example, in equation (21), the first term on the left-hand side can be interpreted as the elasticity of demand—it represents the percentage change in the fraction of low quality sellers attracted to the contract from a one percent change in the utility offered to the low type seller. Since the second term is profits per trade with low quality sellers, the left-hand side captures the marginal benefit to the buyer of increasing u_l . On the right-hand side of (21), ϕ represents the marginal cost of increasing the utility of the low quality seller, taking into account the fact that increasing u_l relaxes the incentive constraint. The fact that ϕ incorporates this additional benefit of increasing u_l implies that $\phi < 1$, i.e., an increase in the utility of the low quality seller decreases profits by less than 1.

The boundary condition requires that the lowest utility offered to the low quality seller is equal to c_l . From (22), and using the fact that $F_l(c_l) = 0$, we find $U_h(c_l) = c_h$. That is, the worst menu offered in such an equilibrium coincides with the monopsony outcome. Loosely speaking, if the worst equilibrium menu offers more utility to high quality sellers than c_h , then a buyer offering this worst menu could increase profit by decreasing u_h —his payoff from trading with high type sellers would increase without changing the payoffs from trading with low types. On the other hand, if the worst menu offers more utility to low quality sellers than c_l , the buyer could profit by decreasing u_l and u_h —the gains associated with trading from low types would exceed the losses associated from trading with high types precisely because $\phi > 0$.

The final step in our construction of the equilibrium is to determine the policy function, $U_h(u_l)$. Since the worst menu offered in equilibrium is the monopsony contract and all menus offered must earn equal profits, $U_h(u_l)$ is determined by the equal profit condition, equation (22). The right-hand side of (22) define monopsony profits. The left-hand side of (22) defines

profits earned from the menu $(u_l, U_h(u_l))$ given the distribution F_l . The following proposition asserts that our construction of F_l and $U_h(u_l)$ from (21) and (22) constitutes an equilibrium.

Proposition 10. *If $\phi > 0$, there exists an interior, separating equilibrium with continuous distributions (F_l, F_h) . The equilibrium distribution F_l is characterized by the differential equation (21); the distribution F_h satisfies $F_h(U_h(u_l)) = F_l(u_l)$ with $U_h(u_l)$ determined by (22).*

As discussed above, the differential equation (21) ensures that all local deviations by a buyer from an equilibrium menu are unprofitable. To complete the proof that F_l and $U_h(u_l)$ are an equilibrium, we need only ensure that no global deviations are profitable; we establish that this is true in Appendix A. Notice also from (21) that since $\phi > 0$, our equilibrium satisfies $v_l > u_l$ for all menus in equilibrium. That is, buyers earn strictly positive profits from trading with low quality sellers. It is straightforward to show that buyers also earn strictly positive profits from trading with high quality sellers. Therefore, the equilibrium we have constructed features no cross-subsidization.

4.2.2 Cross-subsidizing Equilibria ($\phi < 0$)

We now turn our attention to the case of $\phi < 0$. As with monopsony and perfect competition, cross-subsidization occurs in the region with $\phi < 0$. Notice that under Bertrand competition cross-subsidization is achieved with separating menus. In contrast, under monopsony cross-subsidization is achieved through a pooling menu. It is then natural to expect equilibrium menus to feature pooling when $\phi < 0$ and markets are imperfectly competitive. We show that in fact this is the case when the number of high quality sellers is high enough. We proceed by constructing an equilibrium when $\phi < 0$, and then establishing in Section 4.3 that this equilibrium is unique.

Our equilibrium construction relies on finding a threshold, \hat{u}_l , in the support of the distribution F_l . For utility levels in the support of F_l , equilibrium menus feature pooling with $U_h(u_l) = u_l$; above the threshold \hat{u}_l , equilibrium menus feature separation with $U_h(u_l) > u_l$. We find the threshold \hat{u}_l as follows. Let $\bar{\Pi}(\hat{u}_l)$ be defined as

$$\bar{\Pi}(\hat{u}_l) = (1 - \pi)(\mu_h(v_h - c_h) + \mu_l(v_l - \min\{c_h, \hat{u}_l\})). \quad (23)$$

and let $H(\hat{u}_l)$ satisfy

$$(1 - \pi + \pi H(\hat{u}_l)) (\mu_h v_h + \mu_l v_l - u_l) = \bar{\Pi}(\hat{u}_l). \quad (24)$$

Finally, define $\bar{u}_l(\hat{u}_l)$ as

$$\bar{u}_l(\hat{u}_l) = v_l + (\hat{u}_l - v_l) [1 - \pi + \pi H(\hat{u}_l)]^{\frac{1}{\phi}}. \quad (25)$$

Then, the threshold, \hat{u}_l is the value satisfying

$$\mu_h v_h + \mu_l v_l - \bar{u}_l(\hat{u}_l) = \bar{\Pi}(\hat{u}_l). \quad (26)$$

If the solution satisfies $\hat{u}_l \in (c_h, \bar{u}_l(\hat{u}_l))$, then, we let the equilibrium distribution F_l be given by

$$(1 - \pi + \pi F_l(u_l)) (\mu_h v_h + \mu_l v_l - u_l) = \bar{\Pi}(\hat{u}_l) \quad (27)$$

for $u_l \leq \hat{u}_l$. For such values of u_l , we set $U_h(u_l) = u_l$ so that menus in this region feature pooling. For values of $u_l \geq \hat{u}_l$, F_l satisfies (21) together with the boundary condition

$$1 - \pi + \pi F_l(\hat{u}_l) = \frac{\bar{\Pi}(\hat{u}_l)}{(\mu_h v_h + \mu_l v_l - \hat{u}_l)}$$

and with $U_h(u_l)$ determined by the equal profit condition

$$(1 - \pi + \pi F_l(u_l)) [\mu_h \Pi_h(U_h(u_l), u_l) + \mu_l (v_l - u_l)] = \bar{\Pi}(\hat{u}_l). \quad (28)$$

In this region of the support of F_l , $U_h(u_l) > u_l$ so the menus are separating. The support of the distribution F_l is given by $[c_h, \bar{u}_l(\hat{u}_l)]$.

If the solution to (26) satisfies $\hat{u}_l \leq c_h$, then F_l satisfies (21) with the boundary condition $\min \text{Supp}(F_l) = \hat{u}_l$ and $U_h(u_l)$ is determined by (28). If the solution to (26) satisfies $\hat{u}_l \geq \bar{u}_l(\hat{u}_l)$, then F_l satisfies (27), $\text{Supp}(F_l) = [c_h, \hat{u}_l]$, and $U_h(u_l) = u_l$ for all $u_l \in \text{Supp}(F_l)$.

Relative to our construction when $\phi > 0$, it is immediate that our construction when $\phi < 0$ is more complicated. Ultimately, this computation relies on finding a solution to the non-linear equation (26). Note that it is possible for the threshold that solves (26) may be interior to the support of F_l or at either end of the support. Thus, the constructed equilibrium may be separating for almost all menus, pooling for all menus, or pooling for some menus and

separating for others. We discuss below parameter values such that each of these cases are possible. Importantly, our equilibrium provides a straightforward partition of the support of F_l into pooling and separating region which simplifies the characterization greatly. Note also that the equation (26) may have multiple solutions; in this case, we characterize the equilibrium using the lowest solution \hat{u}_l .

To understand the equilibria we have constructed, assume that the threshold \hat{u}_l satisfies $\hat{u}_l \in (c_h, \bar{u}_l(\hat{u}_l))$, and consider first the pooling region $u_l \in [\underline{u}_l, \hat{u}_l]$. Since buyers must be indifferent between offering any menu in the pooling region, each of these menus must earn the same profits in spite of the fact that better utility levels attract a greater number of sellers. Hence, equation (27) determines the distribution $F_l(u_l)$ in this interval. In addition, profits must satisfy (23) since the worst contract is a pooling menu with a utility offer of $u_l = c_h$. The object, $H(\hat{u}_l)$, determined in (24) specifies the distribution F_l at the upper bound of the pooling region.

Next consider the separating region. As in Section 4.2.1, local deviations must be unprofitable which requires F_l to satisfy (21) and that buyers earn equal profits at all equilibrium menus determines $U_h(u_l)$ in this region according to (28) as a function of $F_l(u_l)$. To solve the differential equation (21) we must specify a boundary condition. Continuity of the distribution F_l at the threshold, \hat{u}_l , yields the necessary condition given by equation (24). The solution to the differential equation is given by

$$1 - \pi + \pi F_l(u_l) = (1 - \pi + \pi F_l(\hat{u}_l)) \left(\frac{\hat{u}_l - v_l}{u_l - v_l} \right)^\phi. \quad (29)$$

Having solved the differential equation exactly, we then determine the upper bound of the support of F_l by solving for $F_l(\bar{u}_l) = 1$ using equation (29). In a slight abuse of notation, we denote this upper bound $\bar{u}_l(\hat{u}_l)$, which is given by equation (25).

The last step in the construction is to ensure that at the upper bound of the support of F_l , the pooling menu $(\bar{u}_l(\hat{u}_l), \bar{u}_l(\hat{u}_l))$ yields profits exactly equal to the profits at the lower bound $\bar{\Pi}(\hat{u}_l)$ according to (28). By construction, then, we have imposed that the best menu offered in equilibrium is a pooling menu. When $\phi < 0$, it is straightforward to show that any equilibrium features this property. If the best menu offered in equilibrium were a separating menu, then a buyer offering this menu purchases strictly fewer than 1 unit of goods from each high quality seller; an increase in the utility offered to low quality sellers necessarily improves the buyer's

profits by increasing the amount purchased from high quality sellers with no impact on the number of sellers the buyer attracts. These results lead us to the non-linear equation (26) which we solve to determine the threshold, \hat{u}_l .

To prove that our constructed distribution F_l along with the policy function $U_h(u_l)$ constitute an equilibrium we must prove that there are no profitable local deviations in the pooling region and that there are no global deviations. The following proposition asserts that our construction of F_l and $U_h(u_l)$ using the threshold \hat{u}_l yield an equilibrium and provides a full characterization of the threshold \hat{u}_l with respect to the support of F_l .

Proposition 11. *If $\phi < 0$, there exists a threshold equilibrium with continuous distributions (F_l, F_h) with F_l defined by (26) for $u_l < \hat{u}_l$ and (28) for $u_l \geq \hat{u}_l$, $U_h(u_l) = u_l$ for $u_l < \hat{u}_l$, $U_h(u_l)$ given by (28) for $u_l \geq \hat{u}_l$ and $F_h(U_h(u_l)) = F_l(u_l)$. There exists two cutoffs $\phi_2 < \phi_1 < 0$ such that:*

1. *When $\phi_2 < \phi < \phi_1$, the threshold \hat{u}_l is interior to the support of F_l ,*
2. *When $\phi_1 \leq \phi$, the threshold \hat{u}_l satisfies $\hat{u}_l = \min \text{Supp}(F_l)$,*
3. *When $\phi \leq \phi_2$, the threshold \hat{u}_l satisfies $\hat{u}_l = \max \text{Supp}(F_l)$.*

Proposition 11 describes three types of equilibria: semi-separating equilibria when $\phi_2 < \phi < \phi_1$, separating when $\phi_1 \leq \phi$, and pooling when $\phi \leq \phi_2$. The cutoffs, ϕ_1 and ϕ_2 represent necessary conditions for existence of an equilibrium with only separating menus ($\phi \geq \phi_1$) or with only pooling menus ($\phi \leq \phi_2$).

Consider first the cutoff ϕ_1 . An equilibrium with only separating menus has $U_h(u_l) > u_l$ for almost all u_l in the support of F_l . In particular, at the lower bound of the support of F_l , in this case given by the threshold \hat{u}_l , the equilibrium must satisfy $\hat{u}_l \leq c_h$. In the Appendix, we show that this condition is satisfied for the constructed \hat{u}_l if and only if

$$c_h \geq v_l + \frac{\pi(1 - \mu_l)(v_h - v_l)}{(1 - \pi) \left[(1 - \pi)^{\frac{1-\phi}{\phi}} - 1 \right]}. \quad (30)$$

The condition (30) yields an lower bound on ϕ , which we refer to as ϕ_1 , such that a separating equilibrium exists. Note also that a lower bound on ϕ is equivalent to an upper bound on μ_h .

Next consider the cutoff ϕ_2 . An equilibrium with only pooling menus has $\bar{\Pi}(\hat{u}_l) = (1 - \pi)(\mu_h v_h + \mu_l v_l - c_h)$ and F_l determined by (27). To verify this distribution F_l yields an equilibrium, we must ensure that the standard cream-skimming deviation is not profitable relative

to any pooling menu (u_l, u_l) with u_l in the support of F_l . Such a cream-skimming deviation, locally, is of the form $(u_l - \varepsilon, u_l)$. The decrease in u_l can be attractive for two reasons: first, it decreases the loss in a trade with a low quality seller; and second, it decreases the number of non-captive low quality sellers that the buyer attracts, inducing a more favorable mix of non-captive high and low quality sellers. Since the best pooling menu offers the largest subsidy to low quality sellers, one can show that a buyer's incentive to cream-skim are strongest at the most generous menu offered in equilibrium, which occurs at the threshold \hat{u}_l . Thus, if cream-skimming is not profitable at the pooling menu offering utility \hat{u}_l , then it is not profitable at any pooling menu offering utility less than \hat{u}_l . As we show in the Appendix, cream-skimming is not profitable at the pooling menu offering utility \hat{u}_l if, and only if,

$$1 - \pi \geq \frac{\mu_h v_h + \mu_l v_l - v_l}{(1 - \phi)(\mu_h v_h + \mu_l v_l - c_h)}. \quad (31)$$

The condition (31) yields an upper bound on ϕ , which we refer to as ϕ_2 , such that a pooling equilibrium exists. One can prove that $\phi_2 < \phi_1$ so that if a pooling equilibrium exists, then a separating equilibrium does not and vice versa.

When ϕ lies between these cutoffs, $\phi_2 < \phi < \phi_1$, one can prove that the threshold \hat{u}_l is interior to the support of the constructed F_l . To confirm that the candidate threshold equilibrium is indeed an equilibrium, we must prove that buyers have no profitable deviations. By construction, buyers have no incentive to deviate locally to alternative menus in the separating region. Hence, we need only check deviations in the pooling interval. As in the pooling equilibrium, it suffices to check that cream-skimming is not profitable at the pooling menu offering utility \hat{u}_l . In the Appendix, we show that when $\phi_2 < \phi < \phi_1$, such cream-skimming deviations are not profitable at \hat{u}_l .

4.2.3 Equilibria with Mass Points ($\phi = 0$)

When $\phi \neq 0$, we have described the construction of equilibria assuming that the distributions F_l and F_h do not feature mass points. Here, we construct an equilibrium $\phi = 0$ that does feature mass points. In particular, we construct an equilibrium in which the distribution F_l assigns full mass to $u_l = v_l$. The distribution F_h is determined by the equation

$$(1 - \pi + \pi F_h(u_h)) \mu_h \Pi_h(u_h, v_l) = (1 - \pi) \mu_l (v_l - c_l) \quad (32)$$

with $\text{Supp}(F_h) = \left[c_h, c_h + \pi \frac{(v_l - c_l)(v_h - c_h)}{v_h - c_l} \right]$.

To understand this equilibrium, note that (32) is determined to ensure that buyers have no incentive to deviate by changing only u_h . Next, consider the incentives of a buyer to deviate by perturbing u_l away from v_l . A marginal increase in u_l leads to a marginal increase in profits earned per high quality seller attracted. However, the number of non-captive low quality sellers increases discretely. A marginal decrease in u_l is also unprofitable. Such a deviation increases profits earned per captive low-quality seller but decrease profits earned per high-quality seller. It is straightforward to show that when $\phi = 0$, the losses from this deviation are larger than the gains. The next proposition asserts that the equilibrium we have constructed is indeed an equilibrium.

Proposition 12. *When $\phi = 0$, the pair of distribution functions - F_l degenerate at v_l and F_h satisfying (32) - constitute an equilibrium.*

4.3 Uniqueness

In this section, we establish uniqueness of the equilibrium we have constructed. We proceed by discussing the key results needed to ensure uniqueness for each ϕ .

Consider first the separating equilibrium we construct when $\phi > 0$. We demonstrate uniqueness of the separating equilibrium by ruling out the two relevant alternatives. To be more precise, we first show that there are no equilibria with mass points when $\phi > 0$. Then we establish that there are no equilibria that feature pooling, i.e., there are no equilibria in which $U_h(u_l) = u_l$ for some $u_l \in \text{Supp}(F_l)$.

Next, consider the threshold equilibrium we construct when $\phi < 0$. We demonstrate uniqueness of the threshold equilibrium as follows: first, we show that any equilibrium features pooling at the upper bound of the support of F_l ; second, we prove that any equilibrium features at most one interval of pooling followed by at most one interval of separation; third, we prove that the equilibria characterized in Proposition 11 are mutually exclusive so that equilibria without mass points are unique; fourth, we prove no equilibrium features mass points demonstrating uniqueness of the equilibrium characterized in Proposition 11.

Finally, when $\phi = 0$, it is straightforward to prove that in any equilibrium, F_l must be degenerate at $u_l = v_l$. We summarize these results in the following Theorem.

Theorem 13. For any (π, ϕ) , the applicable equilibrium constructed in Section 4.2.1, 4.2.2, or 4.2.3 is unique.

5 Discussion

In this section, we explore the implications of the equilibrium characterization provided above. We begin with positive implications, paying particular attention to how the nature of the contracts that are offered in equilibrium are affected by both the severity of the adverse selection problem and the degree of competition. We show that, in contrast with many papers in the literature, whether buyers offer sellers separating or pooling menus depends on the underlying distribution of types in the market. Moreover, this decision depends systematically on the underlying market structure: separating menus are more prevalent when markets are more competitive, while pooling menus emerge when markets are more frictional. Hence, observing separating contracts in a market is not necessarily a sign of severe adverse selection; identifying the severity of information frictions requires knowledge of the prevailing trading frictions.

We then explore some of the normative implications, fleshing out the affect of competition on ex ante welfare in different regions of the parameter space. We show that, when adverse selection is severe, ex ante welfare is inverse U-shaped in π , so that there is an *interior* level of frictions that maximizes surplus from trade. When adverse selection is mild, on the other hand, ex ante welfare is monotonic and decreasing in π , so that competition unambiguously hinders the process of realizing gains from trade.

Finally, we demonstrate that explicitly modeling competition and allowing for general contracts yields novel and interesting policy implications. Specifically, we analyze the effect of making additional information about sellers' types available to buyers on *ex-ante* welfare. This is a very topical question in the context of recent developments, both due to policy and/or technological changes, in a number of insurance and financial markets. Our main finding is that the desirability of additional information depends both on the distribution of types and the degree of competition. In particular, when adverse selection is relatively mild to begin with or when the market is very competitive, additional information is detrimental to welfare. The opposite is true when adverse selection and trading frictions are relatively severe. Thus, evaluating the implications of these policies requires knowledge of the distribution of types in the

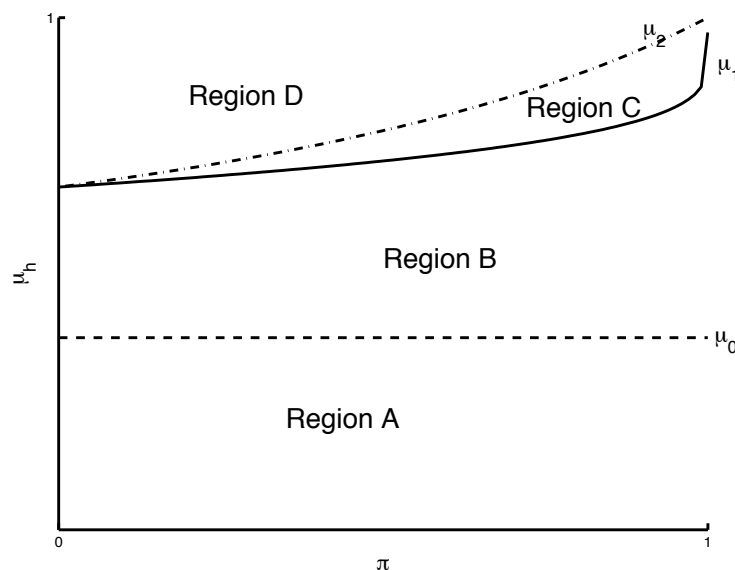


Figure 1: Equilibrium thresholds. Region A: no cross-subsidization, separating. Region B: cross-subsidization, separating. Region C: cross-subsidization, pooling at low μ_l , separating at high μ_l . Region D: cross-subsidization, pooling.

population as well as the extent of frictions.

5.1 Equilibrium Contracts, Adverse Selection, and Competition

Figure 1 depicts the various types of equilibria that arise for different values of μ_h and π , which represent the fraction of high quality sellers and the degree of competition in the market, respectively. The figure shows that for each value of $\pi \in (0, 1)$, there are three critical cutoffs: μ_0 , μ_1 , and μ_2 . When the adverse selection problem is relatively severe — that is, when $\mu_h < \mu_0$ — buyers optimally choose to offer separating contracts (region A). Moreover, these contracts do not feature cross subsidization; all contracts offered in equilibrium yield a positive expected payoff from both low and high type sellers. As μ_h increases, however, the incentive to trade larger quantities with high type sellers increases, as such sellers are more abundant. Indeed, for all $\mu_h > \mu_0$ (regions B,C and D), all equilibrium contracts feature cross-subsidization, as low quality sellers earn information rents that relax incentive constraints and enable buyers to trade larger quantities with high quality sellers.

In region B, i.e. when $\mu_h \in (\mu_0, \mu_1)$, all equilibrium menus are composed of separating contracts. In this region, even though a monopsonist would like to pool low and high quality sellers, such an offer cannot be sustained. The reason, as we discussed above, is that *cream-*

skimming is still optimal in this region: a buyer could profitably deviate by offering less utility to low types, while offering the same utility to high types by trading a smaller quantity at a higher price. Since μ_h is still relatively small in this region, the benefits of trading with fewer low types (and hence suffering fewer losses) outweigh the cost of extracting less rents from high types.

As μ_h continues to increase, however, the benefits of cream-skimming deviations subside while the costs grow. As a result, in region C, when $\mu_h > \mu_1$, at least some pooling menus can be sustained in equilibrium. In particular, whenever some pooling menus are offered, they will be the menus that deliver low levels of utility to sellers. Intuitively, those buyers whose offers fall in the lower portion of the distribution of indirect utilities earn most of their profits from captive sellers; the benefits of altering the mix of low and high quality non-captive sellers they attract by deviating to a cream-skimming menu are small relative to the costs of extracting fewer rents from the mass of captive, high quality sellers. Conversely, those buyers offering indirect utilities at the top of the distribution trade with a large fraction of non-captive sellers, and hence the cream skimming deviation may still be profitable. Thus, when $\mu_h \in (\mu_1, \mu_2)$, menus that offer high indirect utilities remain separating menus. However, when $\mu_h > \mu_2$ (region D), cream-skimming is not profitable anywhere in the distribution of equilibrium offers, and all buyers offer pooling menus. Overall, comparative statics with respect to μ_h reveal that the underlying distribution of asset quality affects whether menus feature cross-subsidization *and* whether they feature pooling or separating contracts. Interestingly, this is not the case in related models like [Guerrieri et al. \(2010\)](#).

Figure 1 also shows that pooling is more prevalent when markets are less competitive. In particular, when μ_h is sufficiently high, the equilibrium features at least some (if not all) pooling menus. However, for similar reasons to those discussed above, the incentive to deviate to a cream-skimming menu grows as π increases and a larger fraction of profits are generated from trades with non-captive sellers. When π gets sufficiently close to 1, in fact, no pooling menus can be sustained. Note that these last results imply that the relationship between the contracts that are traded in a market and the severity of the adverse selection problem can ultimately depend heavily on the degree of competition in that market. In particular, the observation that separating contracts are observed in market A and not in market B does *not* necessarily imply that adverse selection is more severe in market A than B.

5.2 Ex-ante Welfare and Competition with Adverse Selection

We now turn our attention to the relationship between ex ante welfare and competition, as captured by π , for varying degrees of adverse selection, as captured by μ_h . To start, note that ex ante welfare is given by

$$\begin{aligned} \mu_l v_l &+ (1 - \pi) \int \mu_h (v_h x_h(u_l) + c_h (1 - x_h(u_l))) dF_l(u_l) \\ &+ \pi \int \mu_h (v_h x_h(u_l) + c_h (1 - x_h(u_l))) d(F_l(u_l)^2) \end{aligned} \quad (33)$$

where

$$x_h(u_l) = 1 - \frac{u_h(u_l) - u_l}{c_h - c_l}. \quad (34)$$

The first term represents the total surplus from trade between low quality sellers and buyers; since $x_l = 1$ with probability 1, all low quality trades generate a surplus of v_l , the buyers' valuation. The second term in (33) captures the total surplus created from trades between buyers and captive high-quality sellers; a captive high-quality seller trades a quantity $x_h(u_l)$, where u_l is drawn from $F_l(u_l)$. Likewise, the last term in (33) captures the total surplus created from trades between buyers and non-captive high-quality sellers; since non-captive sellers choose the maximum indirect utility among the two offers they receive, they trade $x_h(u_l)$ with u_l drawn from $F_l(u_l)^2$.

Using this definition of ex ante welfare, we show that ex ante welfare is inverse U-shaped in π when adverse selection is severe; that is, there is an *interior* level of frictions that maximize the surplus generated from trade. On the other hand, when adverse selection is mild, ex ante welfare is strictly decreasing in π ; that is, more competition is unambiguously bad for welfare. The following proposition summarizes.

Proposition 14. *When $\mu_h < \mu_0$, there exists a $\pi \in (0, 1)$ that maximizes ex ante welfare. On the other hand, when $\mu_h > \mu_0$, ex ante welfare is maximized at $\pi = 0$.*

Two key features of our equilibrium can account for the first result. First, when π is sufficiently large, $x_h(u_l)$ is inverse U-shaped in u_l . Second, as π increases, $F_l(u_l)$ increases in the sense of first order stochastic dominance; that is, F_l shifts to the right, putting more weight on offers with lower values of x_h . This drives down the total quantity traded with high quality sellers, and thus ex ante welfare.

But why is $x_h(u_l)$ decreasing for high values of u_l when π is sufficiently close to 1? Intuitively, when $\phi > 0$, buyers compete for low quality sellers more intensely than they compete for high quality sellers, since the former are relatively more abundant. Thus, offers to low quality sellers are clustered very tightly at the top of F_l ; demand in this region is very elastic and, in order to satisfy the equal profits condition, prices offered to low quality sellers have to rise relatively slowly.¹⁰ Competition for high quality sellers, on the other hand, is less fierce. As a result, offers to high quality sellers are more dispersed at the top of F_h ; demand in this region is more inelastic, and hence prices offered to high quality sellers have to rise very quickly. In order to achieve this increase, the quantity traded with high quality sellers has to fall.

To see the mathematics behind this result, note that (8) implies that $x_h(u_l) = x_h(U_h(u_l))$ is decreasing if and only if $U'_h(u_l) > 1$. Moreover, from (22), one can derive

$$U'_h(u_l) = \left[\frac{\phi(c_h - c_l)}{v_h - c_l} \right] \times \frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l}. \quad (35)$$

Therefore, as $\pi \rightarrow 1$, $U'_h(u_l) > 1$ if u_l converges to v_l faster than $\Pi_h(u_l, U_h(u_l))$ converges to zero. This is exactly the case when $\phi > 0$, and the incentive to outbid other buyers to trade with non-captive, low quality sellers is strong relative to the incentive to outbid other buyers for non-captive, high quality sellers.

The second result — that ex ante welfare is maximized at $\pi = 0$ when $\phi < 0$ — is more straightforward. Since a monopsonist offers a pooling contract in this region of the parameter space, all gains from trade are exhausted. By introducing competition, one simply increases a buyer's incentive to deviate to a cream-skimming menu. In order to make such a deviation unprofitable, equilibrium contracts offer high quality sellers a higher price but a lower quantity to trade, and this causes a decline in ex ante welfare.

5.3 Information, Disclosure, Discrimination, and Welfare

In this section, we show that explicitly modeling imperfect competition in an environment with both adverse selection and general contracts reveals new and interesting implications regarding the impact of certain policy choices and technological innovations. In other words, several normative insights that have been derived under a particular market structure (e.g., perfect competition) or with *ad-hoc* restrictions on contracts may not be particularly robust.

¹⁰Indeed, from (21), one can see that as π tends to 1, $\lim_{u_l \rightarrow v_l} f_l(u_l) = \infty$.

We demonstrate this insight by analyzing what happens when buyers have better information about sellers. Consider an environment in which the probability that a particular seller is high quality, μ'_h , is a random variable drawn from $G(\mu'_h)$. In the baseline case, i.e. without additional information, buyer behavior and therefore, the equilibrium outcome is determined by the unconditional average, $\mu_h = \int \mu'_h dG(\mu'_h)$. and π . If buyers have additional information about the seller's type, denoted \mathcal{J} , it induces a set of conditional distributions, $\hat{G}(\mu'_h, \mathcal{J})$. For each such \mathcal{J} , the equilibrium is determined by the conditional mean $\mu_h(\mathcal{J}) = \int \mu'_h d\hat{G}(\mu'_h, \mathcal{J})$. Obviously, $\mu_h = \mathbb{E}[\mu_h(\mathcal{J})]$, where the expectation is taken over the realizations of \mathcal{J} . In this sense, more information induces a mean-preserving spread of the distribution of the average asset quality, μ_h . Below, we discuss how this is a natural way to model additional information - whether due to policies or other changes - in insurance and financial markets and examine its effects on welfare.

1. **Non-discrimination in insurance markets.** Under the Affordable Care Act, insurance providers are not allowed to discriminate based on certain observable characteristics (e.g. health factors). In other words, participants with known differences in risk must be offered the same menus (of benefits, premiums etc.). In the context of our model, we can interpret these differences in risk as differences in μ_h across subgroups of the population. The distribution of group-specific μ_h is a mean-preserving spread of the unconditional distribution. Then, under the non-discrimination policy, insurance companies would offer contracts based on the pooled (or population) μ_h . In the absence of such a policy, i.e. if companies were free to offer different terms to different subgroups, the set of contracts offered in equilibrium to each subgroup would be based on their μ_h .
2. **Credit scores in consumer loan markets.** The development of standardized scoring systems to assess borrower creditworthiness is widely believed to have played an important role in the expansion of US consumer credit markets over the last three decades¹¹. Again, interpreting these credit scores as signals about borrower types, it is easy to see that they induce a mean-preserving spread of the distribution of the average asset quality.
3. **Transparency in financial markets.** An important debate in financial regulation relates to disclosure of trades conducted in decentralized or over-the-counter settings. Many

¹¹See [Chatterjee et al. \(2011\)](#) and [Einav et al. \(2013\)](#)

securities in the US are now subject to rules requiring public dissemination of trade information. For example, all transactions in corporate bonds and asset-backed securities are now required to be reported within 15 minutes through a centralized reporting system called Trade Reporting and Compliance Engine (TRACE). This is intended to induce "price discovery and improved bargaining power of previously uninformed participants", and in turn " a decrease in bond price dispersion and an increase in trading activity" (Levitt (1999)). To map this into our mean-preserving spread experiment, suppose that the fraction of high quality assets is state-dependent. Disclosure of past trades essentially generates a signal about the state of the world (i.e. about μ_h) and thus, induces a mean-preserving spread of the unconditional distribution.

To start, let us define welfare explicitly in terms of μ_h , so that

$$\begin{aligned}
 W(\mu_h) &= (1 - \mu_h) v_l + \mu_h v_h X_h(\mu_h) + \mu_h c_h (1 - X_h(\mu_h)) \\
 \text{where } X_h(\mu_h) &\equiv \int x_h(u) dR(u, \mu_h) \\
 \text{and } R(u, \mu_h) &= (1 - \pi) F_l(u, \mu_h) + \pi F_l^2(u, \mu_h).
 \end{aligned}$$

Note that, for expositional simplicity, we have retained the dependence of allocations and the distribution on μ_h , while suppressing the dependence on π . The effect of a mean-preserving thus depends on the curvature of $W(\mu_h)$. If W is convex (concave) in μ_h over the relevant region, a mean-preserving spread increases (reduces) welfare.

It is important to mention that we are interested in local mean-preserving spreads. Additional information of sufficiently high quality always improves welfare. To see this, consider the extreme case of a perfectly informative signal. This eliminates the adverse selection problem and achieves first-best welfare, irrespective of the degree of competition - clearly, not a very interesting (or realistic) experiment. Therefore, in what follows, we restrict attention to 'small' spreads.

Before considering the general case, it is worthwhile to examine the limiting cases of monopsony and Bertrand competition. Recall from Lemma 8, that under monopsony, $x_h = 0$ when $\mu_h < \bar{\mu}_0$, so $W(\mu_h) = (1 - \mu_h) v_l + \mu_h c_h = v_l + \mu_h (c_h - v_l)$. When $\mu_h \geq \bar{\mu}_0$, the monopsonist pools so $x_h = 1$, so we have $W(\mu_h) = v_l + \mu_h (v_h - v_l)$. In other words, under monopsony, W consists of two piece-wise linear segments, with a discontinuity at $\mu_h = \bar{\mu}_0$. Directly, a

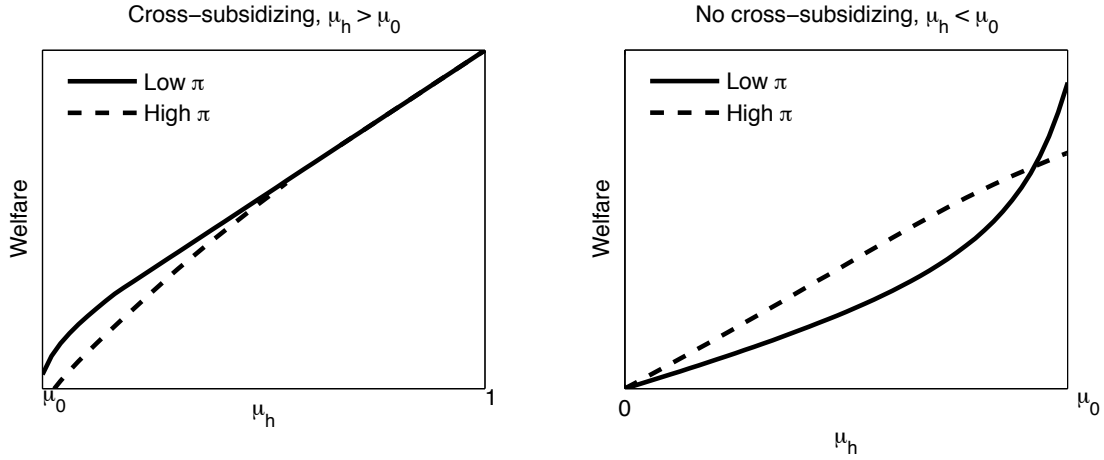


Figure 2: Welfare as a function of μ_h

mean-preserving spread is welfare-neutral unless such spreading spans the discontinuity. At the other extreme, when $\pi = 1$, W is also linear in the region where a pure strategy equilibrium exists (i.e. which $\mu_h < \bar{\mu}_0$). This is because x_h is pinned down entirely by the incentive constraint and independent of μ_h . Again, within this region, the policy experiments have no effect on welfare. The analysis of the case with mixed strategies is more involved and closely related to the general case of $\pi \in (0, 1)$ to which we turn next.

Our results for $\pi \in (0, 1)$ are depicted in Figure 2, which shows the effect of mean preserving spreads for the two cases of cross-subsidizing and no-cross subsidizing equilibria. Each panel plots the W over an interval of μ_h , for two different values of π . The first panel considers an interval entirely in the cross-subsidizing region, i.e. when $\mu_h > \bar{\mu}_0$. In this region, W is concave, both for high and low π , implying that mean-preserving spreads are detrimental. To see why, first note that higher μ_h shifts the distributions of u_l and u_h to the right. This is intuitive - as the probability of high quality assets increases, the adverse selection becomes less severe, so both seller types receive higher utilities. This also results in higher x_h , increasing overall welfare. However, this change in the distribution becomes more gradual as μ_h rises closer to the pooling region. In other words, a reduction in μ_h has a bigger effect on the distribution - and consequently on expected trade X_h and welfare - than an increase of similar magnitude.

The opposite happens in the no-cross subsidization region, provided π is sufficiently low. In this region, W is convex, so mean-preserving spreads *increase* welfare. With π is high, however, there is an additional effect. Recall from Section 5.2 that x_h is decreasing in u_l when π is high.

This moderates the direct gains from a rightward shift in the distribution of utilities induced by higher μ_h . The combined effect of these two forces results in a region where W is concave, making mean-preserving spreads undesirable. Thus, in the no cross subsidization region, the curvature of W varies with π so mean-preserving spreads can be good or bad for welfare¹².

Thus, the concavity of W depends in a rich way both on the type of equilibrium (cross-subsidizing vs no cross subsidization) as well as on π . Broadly speaking, our results suggest that providing additional information to buyers is less likely to be desirable when the adverse selection problem is relatively mild to begin with (in the sense that μ_h is high) or when the market is very competitive (i.e. when π is high). Therefore, in order to assess even the qualitative effect of policies like the ones described at the beginning of the section, one needs to first accurately measure both the degree of competition and the severity of adverse selection.

6 General Model with Many Types

In this section, we modify our environment to allow for more than two types and show that the key insight about equilibria in the two type case - strict rank-preserving menus - extends to this more general case.

Formally, there is one seller endowed with a single unit of a perfectly divisible good of type $i \in \{1, 2, \dots, N\}$. The probability of type i is given by μ_i . As before, buyers and sellers derive utility v_i and c_i with $v_i > c_i \forall i$. Without loss of generality, we order the types so that $c_1 < c_2 < \dots < c_N$, i.e. type-1 is the lowest seller type. We maintain the 'lemons assumption', which in this setting amounts to $c_{i+1} > v_i \ i = 1, 2, \dots, N-1$.

Strategies and definition of equilibrium: Let $z_i = (x_i, t_i)$ denote the contract intended for type i . It specifies the quantity that changes hands and the transfer. Correspondingly, we let $\mathbf{z} = (z_1, z_2, \dots, z_N)$ denote a menu of contracts offered by each buyer. A buyer's strategy is a distribution Φ over such menus, while the seller simply chooses the contract with the highest payoff from the menus available to her.

A symmetric equilibrium is given by a distribution Φ^* such that, for almost all \mathbf{z} in the support of Φ^* the following properties are satisfied

¹²Numerical calculations, however, suggest that the concavity of W is relatively mild and occurs only over a narrow region of the parameter space - in other words, W appears to be mostly convex in the no cross subsidization region.

1. *Incentive compatibility*

$$t_i + c_i (1 - x_i) \geq t_j + c_i (1 - x_j)$$

2. *Buyers optimize*

$$\mathbf{z} \in \arg \max_{\mathbf{z}} \sum_{i \in \{l, h\}} \mu_i (v_i x_i - t_i) \left[1 - \pi + \pi \int_{\mathbf{z}'} \chi_i(\mathbf{z}, \mathbf{z}') \Phi^*(d\mathbf{z}') \right]$$

where

$$\chi_i(\mathbf{z}, \mathbf{z}') = \begin{cases} 0 \\ \frac{1}{2} \\ 1 \end{cases} \quad \text{if} \quad t_i + c_i(1 - x_i) \begin{cases} < \\ = \\ > \end{cases} t'_i + c_i(1 - x'_i).$$

As in the two type, we first show that a menu can be summarized by a vector of indirect utilities it offers to each type: $\mathbf{u} = (u_1, \dots, u_N)$, where $u_i = t_i + c_i (1 - x_i)$. The next result follows from the same logic as that underlying Lemma 1.

Lemma 15. *In almost all equilibrium menus, $x_1 = 1$ and the local downward IC's are binding, i.e.*

$$t_i + c_i (1 - x_i) = t_{i+1} + c_i (1 - x_{i+1}) \quad i = 1, 2, \dots, N - 1$$

This result allows us to recast the buyer's problem in terms of the utility vector \mathbf{u} . Formally, define a family of distribution functions

$$F_i(u_i) = \int_{\mathbf{z}'_i} \mathbf{1} [t'_i + c_i (1 - x'_i) \leq u_i] \Phi(d\mathbf{z}'_i) \quad i = 1, 2, \dots, N$$

Given $\{F_i\}$, the buyer solves

$$\begin{aligned} & \max_{u_i \geq c_i} \sum_{i=1}^N \mu_i (1 - \pi + \pi F_i(u_i)) \Pi_i(u_1, u_2, \dots, u_N) \\ \text{subject to} & \quad c_{i+1} - c_i \geq u_{i+1} - u_i \geq 0 \\ \text{with} & \quad \Pi_i(u_1, u_2, \dots, u_N) = v_i - u_i \left(\frac{v_i - c_{i-1}}{c_i - c_{i-1}} \right) + u_{i-1} \left(\frac{v_i - c_i}{c_i - c_{i-1}} \right) \end{aligned}$$

Note that the profit function Π_i is increasing the u_{i-1} , reflecting the value of relaxing the incentive constraint of type $i - 1$. Let $\Pi(u_1, u_2, \dots, u_N)$ denote the objective in the above problem.

Define

$$\mathbb{U}(u_1) = \left\{ (u_2, \dots, u_N) : (u_2, \dots, u_N) \in \arg \max_{\{\tilde{u}_j\}_{j=2}^N} \Pi(u_1, \tilde{u}_2, \dots, \tilde{u}_N) \right\}$$

Definition 16. An equilibrium is *strictly rank-preserving* if, for all $u_1 \in \text{Supp}(F_1)$, $\mathbb{U}(u_1)$ is single-valued and strictly increasing, element by element, i.e. if $u_1 > u'_1$, then every element of $\mathbb{U}(u_1)$ is strictly larger than its counterpart in $\mathbb{U}(u'_1)$.

We are now ready to state the main result of this section. It shows that, except in a knife-edge case where the cost of providing utility to the lowest type is exactly zero, the equilibrium is strictly rank-preserving.

Theorem 17. If $\phi_1 = 1 - \frac{\mu_2}{\mu_1} \left(\frac{v_2 - c_2}{c_2 - c_1} \right) \neq 0$, the equilibrium is strictly rank-preserving.

The strategy for the proof, laid out in the Appendix, closely follows the one we used for Theorem 6 in the two-type case. The key step is to show that the distribution functions, $\{F_i\}_{i=1}^N$ are continuous and have connected support. We proceed by induction - we start by proving this for F_N , the highest quality type. We then use this result and the supermodularity of the profit function to inductively rule out flat spots in the distribution functions F_i .

The next step in the proof is show that if the equilibrium exhibits masspoints in any F_i , they must be comprised of menus which make zero profits from the lowest type¹³, i.e. where $u_1 = v_1$. For such a masspoint to be sustainable when $\pi \in (0, 1)$, the net marginal cost of providing utility to the lowest type, the coefficient ϕ_1 in the statement of the theorem, must be exactly zero. If this condition is violated, i.e. if $\phi_1 \neq 0$, deviations from $u_1 = v_1$ yield strictly higher profits. In other words, if $\phi_1 \neq 0$, masspoints cannot occur in *any* equilibrium F_i . Finally, the fact that all the F_i 's are continuous and strictly increasing can be used to rule out regions where $\mathbb{U}(u_1)$ is weakly-increasing or is a correspondence, completing the proof.

¹³This is analogous to our finding in the two type case that masspoints in F_1 can only occur at $u_1 = v_1$.

References

- AKERLOF, G. A. (1970): "The market for" lemons": Quality uncertainty and the market mechanism," *The quarterly journal of economics*, 488–500. 5
- AZEVEDO, E. M. AND D. GOTTLIEB (2014): "Perfect Competition in Markets with Adverse Selection," *The Wharton School Research Paper*. 5
- BENABOU, R. AND J. TIROLE (2014): "Bonus Culture: Competitive Pay, Screening, and Multitasking," *Journal of Political Economy*, forthcoming. 5, 6
- BIGIO, S. (2014): "Endogenous Liquidity and the Business Cycle," Tech. rep. 5
- BURDETT, K. AND K. L. JUDD (1983): "Equilibrium Price Dispersion," *Econometrica*, 51, 955–69. 3, 6, 7, 13
- BUTTERS, G. R. (1977): "Equilibrium Distributions of Sales and Advertising Prices." *Review of Economic Studies*, 44. 7
- CAMARGO, B. AND B. LESTER (2014): "Trading dynamics in decentralized markets with adverse selection," *Journal of Economic Theory*, 153, 534–568. 5
- CHANG, B. (2012): "Adverse selection and liquidity distortion," *Available at SSRN 1701997*. 5
- CHARI, V., A. SHOURIDEH, AND A. ZETLIN-JONES (2014): "Reputation and persistence of adverse selection in secondary loan markets," *American Economic Review*, 104, 4027–70. 5
- CHATTERJEE, S., D. CORBAE, AND J.-V. RIOS-RULL (2011): "A theory of credit scoring and competitive pricing of default risk," *Unpublished paper, University of Minnesota*. [672]. 31
- CHIAPPORI, P.-A. AND B. SALANIE (2000): "Testing for asymmetric information in insurance markets," *Journal of Political Economy*, 108, 56–78. 5
- DASGUPTA, P., P. HAMMOND, AND E. MASKIN (1979): "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility," *The Review of Economic Studies*, 185–216. 9
- DASGUPTA, P. AND E. MASKIN (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *The Review of Economic Studies*, 53, 1–26. 3, 17, 40

- EINAV, L., A. FINKELSTEIN, AND M. R. CULLEN (2010a): “Estimating welfare in insurance markets using variation in prices,” *Quarterly Journal of Economics*, 125. 5
- EINAV, L., A. FINKELSTEIN, AND J. LEVIN (2010b): “Beyond Testing: Empirical Models of Insurance Markets,” *Annu. Rev. Econ.*, 2, 311–36. 2
- EINAV, L., M. JENKINS, AND J. LEVIN (2012): “Contract pricing in consumer credit markets,” *Econometrica*, 80, 1387–1432. 5
- (2013): “The impact of credit scoring on consumer lending,” *The RAND Journal of Economics*, 44, 249–274. 31
- FUCHS, W., A. ORY, AND A. SKRZYPACZ (2012): “Transparency and distressed sales under asymmetric information,” *Unpublished manuscript*. 5
- FUCHS, W. AND A. SKRZYPACZ (2013): “Costs and benefits of dynamic trading in a lemons market,” . 5
- GARRETT, D. F., R. D. GOMES, AND L. MAESTRI (2014): “Competitive Screening Under Heterogeneous Information,” . 6
- GUERRIERI, V. AND R. SHIMER (2014a): “Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality,” *The American Economic Review*, 104. 5
- (2014b): “Markets with multidimensional private information,” Tech. rep., National Bureau of Economic Research. 5
- GUERRIERI, V., R. SHIMER, AND R. WRIGHT (2010): “Adverse selection in competitive search equilibrium,” *Econometrica*, 78, 1823–1862. 5, 6, 28
- IVASHINA, V. (2009): “Asymmetric information effects on loan spreads,” *Journal of Financial Economics*, 92, 300–319. 5
- KURLAT, P. (2013): “Lemons markets and the transmission of aggregate shocks,” *The American Economic Review*, 103, 1463–1489. 5
- LEVITT, A. (1999): “Testimony to the U.S. Congress House of Representatives House Subcommittee on Finance and Hazardous Materials,” Available at: <http://www.sec.gov/news/testimony/testarchive/1999/tsty0499.htm>. 32

- MAHONEY, N. AND E. G. WEYL (2014): "Imperfect Competition in Selection Markets," Tech. rep., National Bureau of Economic Research. 5, 6
- MASKIN, E. AND J. RILEY (1984): "Monopoly with incomplete information," *The RAND Journal of Economics*, 15, 171–196. 3
- MIRRELES, J. (1971): "An Exploration in the Theory of Optimum Income Taxation," *The Review of Economic Studies*, 38, 175–208. 3
- MYERSON, R. B. (1979): "Incentive Compatibility and the Bargaining Problem," *Econometrica*, 47, 61–74. 9
- RILEY, J. G. (1975): "Competitive Signalling," *Journal of Economic Theory*, 10, 174–186. 17
- (1979): "Informational Equilibrium," *Econometrica*, 47, 331–59. 17
- ROSENTHAL, R. W. AND A. M. WEISS (1984): "Mixed Strategy Equilibrium in a Market with Asymmetric Information," *Review of Economic Studies*, 51, 333–342. 3, 5, 17
- ROTHSCHILD, M. AND J. STIGLITZ (1976): "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, 90, 629–649. 3, 5, 10, 17
- STIGLITZ, J. E. (1977): "Monopoly, non-linear pricing and imperfect information: the insurance market," *The Review of Economic Studies*, 407–430. 3
- VARIAN, H. R. (1980): "A Model of Sales," *American Economic Review*, 70, 651–59. 7
- VEIGA, A. AND E. G. WEYL (2012): "Multidimensional product design," *University of Chicago mimeo*. 5

Appendix

A Proofs

A.1 Proof of Lemma 1

Both results are similar to existing results (see, for example, [Dasgupta and Maskin \(1986\)](#)), and thus we keep the exposition brief. To establish the first result, consider a contract $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ with $x_l < 1$ and $t_l \in \mathbb{R}_+$ yielding a low quality seller utility u_l . Now, consider a deviation $\mathbf{z}' = (\mathbf{z}'_l, \mathbf{z}_h)$ with $x'_l = x_l + \epsilon$ for $\epsilon \in (0, 1 - x_l]$ and $t'_l = t_l + \epsilon c_l$. Note that $u'_l = u_l$, so that \mathbf{z}_l and \mathbf{z}'_l are accepted with the same probability, but

$$x_l v_l - t_l < x_l v_l - t_l + \epsilon(v_l - c_l) = x'_l v_l - t'_l,$$

so that \mathbf{z}'_l earns the buyer a higher payoff when it's accepted.

Second, to establish that the incentive compatibility constraint is always binding for low quality sellers, consider a contract $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ with $t_l > t_h + c_l(1 - x_h)$. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}_l, \mathbf{z}'_h)$ with $x'_h = x_h + \epsilon$ and $t'_h = t_h + \epsilon c_h$ for $\epsilon \in \left(0, \frac{t_l - t_h - c_l(1 - x_h)}{c_h - c_l}\right]$, which is a non-empty interval by assumption. The upper bound on ϵ ensures that the incentive compatibility constraint on type l sellers is not violated. In addition, note that $u'_h = u_h$, so that \mathbf{z}_h and \mathbf{z}'_h are accepted with the same probability, but

$$x_h v_h - t_h < x_h v_h - t_h + \epsilon(v_h - c_h) = x'_h v_h - t'_h,$$

so that \mathbf{z}'_h earns the buyer a higher payoff when it's accepted. ■

A.2 Proof of Proposition 3

We prove this result in a through a sequence of three lemmas.

Lemma 18. *The marginal distribution F_h is continuous and strictly increasing.*

Proof. Suppose not. Then there exists a contract (u_l, u_h) in the support of the equilibrium distribution such that either:

1. F_h has a *flat*, i.e., $\exists \delta > 0$ such that $F_h(u_h) - F_h(u_h - \delta) = 0$.
2. F_h has a *mass point*, i.e., $\exists \delta > 0$ such that $F_h(u_h) - F_h(u'_h) > \delta$ for all $u'_h < u_h$.

We start by showing that F_h cannot have a flat region.

There are two relevant cases.

1. **Case 1:** $u_h > u_l$.

Consider the menu $(u_h - \epsilon, u_l)$ for some $0 < \epsilon < \delta$, which implies $F_h(u_h - \epsilon) = F_h(u_h)$. This menu generates strictly higher profits than (u_h, u_l) since,

$$\Pi(u_h - \epsilon, u_l) - \Pi(u_h, u_l) = \mu_h (1 - \pi + \pi F_h(u_h)) \frac{v_h - c_l}{c_h - c_l} \epsilon > 0.$$

which is a contradiction.

2. **Case 2:** $u_h = u_l = u$.

Suppose that F_h has a flat. We prove a contradiction by working with three different deviations. Note first that if F_l has a flat in an interval of the form $(u - \epsilon', u)$, then a small equal reduction in both values of u_l and u_h increases profits. Therefore, it must be that F_l does not have a flat in a neighborhood below u . Furthermore, since (u, u) is offered in equilibrium a small reduction in both u_h and u_l should not raise profits. In other words, we must have

$$\mu_h(1 - \pi + \pi F_h(u)) - \underline{D}^- \hat{\Pi}_l(u) \mu_l \leq 0 \quad (36)$$

where $\hat{\Pi}_l(u_l) = (1 - \pi + \pi F(u_l))(q_l - u_l)$ and \underline{D}^- is a notion of derivative defined as

$$\underline{D}^- \hat{\Pi}_l(u) = \liminf_{x \nearrow u} \frac{\hat{\Pi}_l(x) - \hat{\Pi}_l(u)}{x - u}$$

Similarly, a small reduction in u_l should not raise the profits either and therefore, we must have

$$- \mu_h(1 - \pi + \pi F_h(u)) \frac{v_h - c_h}{c_h - c_l} - \underline{D}^- \hat{\Pi}_l(u) \mu_l \leq 0 \quad (37)$$

Since the left hand side of (37) is strictly lower than that of (36), we must have that

$$- \mu_h(1 - \pi + \pi F_h(u)) \frac{v_h - c_h}{c_h - c_l} - \underline{D}^- \hat{\Pi}_l(u) \mu_l < 0 \quad (38)$$

or

$$0 < \mu_h(1 - \pi + \pi F_h(u)) \frac{v_h - c_h}{c_h - c_l} + \underline{D}^- \hat{\Pi}_l(u) \mu_l \quad (39)$$

Note that since F_l does not have a flat in a neighborhood below u , there must exist a sequence $\{u_l^k\}_{k=0}^\infty$ where $u_l^k < u$ and

$$u_l^k \in \text{Supp}(F_l), \lim_{k \rightarrow \infty} u_l^k = u$$

Since u_l^k is in the support of F_l , there must exist u_h^k 's such that profits evaluated at (u_l^k, u_h^k) is equal to the equilibrium profits, $\bar{\Pi}$. Note that we must have $u_l^k \leq u_h^k$. Furthermore, since F_h is flat in the interval $[u - \delta, u]$, it must be that for k large enough $u_h^k \geq u$. Therefore, for k large enough, $u_l^k < u \leq u_h^k$. We can use the definition of the derivative \underline{D}^- , (39) implies that

$$0 < \mu_h(1 - \pi + \pi F_h(u)) \frac{v_h - c_h}{c_h - c_l} + \frac{\hat{\Pi}_l(u) - \hat{\Pi}_l(u_l^k)}{u - u_l^k} \mu_l$$

which implies that

$$\begin{aligned} 0 &< \mu_h(1 - \pi + \pi F_h(u)) \frac{v_h - c_h}{c_h - c_l} (u - u_l^k) + (\hat{\Pi}_l(u) - \hat{\Pi}_l(u_l^k)) \mu_l \\ &\leq \mu_h(1 - \pi + \pi F_h(u_h^k)) \frac{v_h - c_h}{c_h - c_l} (u - u_l^k) + (\hat{\Pi}_l(u) - \hat{\Pi}_l(u_l^k)) \mu_l \end{aligned}$$

Using the fact that $\mu_l \hat{\Pi}_l(u_l^k) + \mu_h(1 - \pi + \pi F_h(u_h^k)) \Pi_h(u_l^k, u_h^k) = \bar{\Pi}$, and adding $\bar{\Pi}$ to the above inequality, we have

$$\begin{aligned} \bar{\Pi} &< \bar{\Pi} + \mu_h(1 - \pi + \pi F_h(u_h^k)) \frac{v_h - c_h}{c_h - c_l} (u - u_l^k) + (\hat{\Pi}_l(u) - \hat{\Pi}_l(u_l^k)) \mu_l \\ &= \mu_h(1 - \pi + \pi F_h(u_h^k)) \left[\frac{v_h - c_h}{c_h - c_l} (u - u_l^k) + \Pi_h(u_l^k, u_h^k) \right] + \hat{\Pi}_l(u) \mu_l \\ &= \mu_h(1 - \pi + \pi F_h(u_h^k)) \Pi_h(u, u_h^k) + \hat{\Pi}_l(u) \mu_l \end{aligned}$$

This implies that the menu represented by (u, u_h^k) is a profitable deviation which is a contradiction. Therefore, in this case, F_l cannot have a flat.

Next, we show that F_h cannot have a mass point. Suppose that u_h exists such that $F_h(u_h) > \lim_{u \nearrow u_h} F_h(u)$. Note that $u_l \leq u_h$ must exist so that profits from the menu (u_l, u_h) is equal to equilibrium profits $\bar{\Pi}$. There are two possibilities:

1. $\Pi_h(u_l, u_h) > 0$: In this case a small increase in u_h leads to a discontinuous increase in the probability of trading with high quality sellers while the profits per trade with high quality sellers change continuously. Thus a small enough increase in u_h is a profitable deviation.
2. $\Pi_h(u_l, u_h) \leq 0$: In this case, it must be that profits from the low quality sellers are non-negative (since equilibrium profits must be non-negative) and therefore, $u_l \leq v_l < c_h \leq u_h$. Therefore, if $u_h > c_h$, a downward deviation in u_h is profitable since it is feasible and it leads to higher profits from trading with high quality sellers. Therefore, we must have $u_h = c_h$. In this case, non-positivity of the profits from high quality sellers imply that

$$\Pi(u_l, c_h) = v_h - c_h \frac{v_h - c_l}{c_h - c_l} + \frac{v_h - c_h}{c_h - c_l} u_l = \frac{v_h - c_h}{c_h - c_l} (u_l - c_l) \leq 0$$

Thus, we must have that $u_l = c_l$. This implies that $c_h \in U_h(c_l)$. Using the consistency requirement for the equilibrium, we must have that

$$\begin{aligned} F_h(c_h) - F_h^-(c_h) &= \mu(\{(c_l, c_h)\}) > 0 \\ F_l(c_l) - F_l^-(c_l) &= \mu(\{(c_l, u_h), u_h \in U_h(c_l)\}) \geq \mu(\{(c_l, c_h)\}) > 0 \end{aligned}$$

Thus F_l also has a mass point at c_l . Now, consider the equilibrium menu (c_l, c_h) and consider the deviation to $(c_l + \varepsilon, c_h)$ for small and positive ε . This deviation leads to a discontinuous increase in the probability of trade with low quality sellers and thus increases the profits for low quality sellers. Furthermore, since the function Π_h is increasing in u_l , this deviation also raises profits from the high quality sellers and thus is a profitable deviation. This establishes the contradiction.

Therefore, F_h cannot have a mass point. ■

Lemma 19. *The profit function $\Pi(u_l, u_h)$ is strictly super-modular.*

Proof. Suppose $u_{l2} > u_{l1}$ and $u_{h2} > u_{h1}$. Then, letting $\xi_1 \equiv \frac{v_h - c_h}{c_h - c_l} > 0$ and $\xi_2 \equiv \frac{v_h - c_l}{c_h - c_l} > 0$,

$$\begin{aligned}
& \Pi(u_{l1}, u_{h2}) - \Pi(u_{l1}, u_{h1}) \\
&= \mu_h \{ [1 - \pi + \pi F_h(u_{h2})] \Pi_h(u_{l1}, u_{h2}) - [1 - \pi + \pi F_h(u_{h1})] \Pi_h(u_{l1}, u_{h1}) \} \\
&= \mu_h \{ [1 - \pi + \pi F_h(u_{h2})] [v_h + \xi_1 u_{l1} - \xi_2 u_{h2}] - [1 - \pi + \pi F_h(u_{h1})] [v_h + \xi_1 u_{l1} - \xi_2 u_{h1}] \} \\
&< \mu_h \{ [1 - \pi + \pi F_h(u_{h2})] [v_h + \xi_1 u_{l2} - \xi_2 u_{h2}] - [1 - \pi + \pi F_h(u_{h1})] [v_h + \xi_1 u_{l2} - \xi_2 u_{h1}] \} \\
&= \Pi(u_{l2}, u_{h2}) - \Pi(u_{l2}, u_{h1}),
\end{aligned}$$

where the inequality follows from the fact that F_h is strictly increasing, and hence

$$\pi \xi_1 (u_{l2} - u_{l1}) [F_h(u_{h2}) - F_h(u_{h1})] > 0.$$
■

Lemma 20. *The correspondence $U_h(u_l)$ is weakly increasing. That is, if $u_{l2} > u_{l1}$ then $u_{h2} \geq u_{h1}$ for all $u_{h1} \in U_h(u_{l1})$ and $u_{h2} \in U_h(u_{l2})$.*

Proof. Suppose by way of contradiction that $u_{l2} > u_{l1}$ but $u_{h2} < u_{h1}$ for some $u_{h1} \in U_h(u_{l1})$ and $u_{h2} \in U_h(u_{l2})$. Since $u_{l1} < u_{l2} \leq u_{h2} < u_{h1} \leq u_{l1} + v_h - v_l < u_{l2} + v_h - v_l$, all pairs $\{(u_{li}, u_{hj})\}_{i,j \in \{1,2\}}$ satisfy the constraint set. Then, since $u_{hi} \in U_h(u_{li})$ for $i \in \{1,2\}$, it must be that $\Pi(u_{l1}, u_{h1}) \geq \Pi(u_{l1}, u_{h2})$ and $\Pi(u_{l2}, u_{h2}) \geq \Pi(u_{l2}, u_{h1})$, so that

$$\Pi(u_{l1}, u_{h1}) + \Pi(u_{l2}, u_{h2}) \geq \Pi(u_{l1}, u_{h2}) + \Pi(u_{l2}, u_{h1}). \quad (40)$$

However, since $u_{l1} < u_{l2}$ and $u_{h2} < u_{h1}$, the strict supermodularity of the profit function implies

$$\Pi(u_{l1}, u_{h1}) + \Pi(u_{l2}, u_{h2}) < \Pi(u_{l1}, u_{h2}) + \Pi(u_{l2}, u_{h1}),$$

which contradicts (40). ■

Lemma 21. *The marginal distribution F_l is strictly increasing and continuous (except possibly at its lower bound.)*

Proof. To show that F_l is strictly increasing, we suppose that it has a flat region, i.e. there exists $u_{l1} < u_{l2}$ such that $F_l(u_{l1}) = F_l(u_{l2}) = F$ for some F and all $u_l \in (u_{l1}, u_{l2})$, $F_l(u_l) = F$. Let $u_{h1} = \sup U_h(u_{l1})$ and $u_{h2} = \inf U_h(u_{l2})$. Since $u_{l2} \geq u_{l1}$, Lemma 20 implies $u_{h2} \geq u_{h1}$.

Next, we argue that $u_{h2} = u_{h1}$. To see this, suppose instead that $u_{h2} > u_{h1}$. By Lemma 18, since F_h has connected support, for any sub-interval $[u, u'] \subset [u_{h1}, u_{h2}]$, $F_h(u') - F_h(u) > 0$. This means that there exists a positive measure of $u_h \in (u_{h1}, u_{h2})$, such that for each u_h there exists u_l with $u_h \in U_h(u_l)$. By our initial contradiction assumption, we must have $u_l < u_{l1}$ or $u_l > u_{l2}$. If $u_l < u_{l1}$ then the fact that $u_h > u_{h1}$ contradicts the weak rank preserving property. Similarly, if $u_l > u_{l2}$, then the fact that $u_h < u_{h2}$ contradicts Lemma 20. Hence, it must be that $u_{h2} = u_{h1} = \hat{u}_h$.

Since F_h has connected support and no mass, $\Pi(u_l, u_h)$ is continuous in u_h and therefore, $\hat{u}_h = \max U_h(u_{l1})$ and $\hat{u}_h = \min U_h(u_{l2})$. That is $\Pi(u_{l1}, \hat{u}_h) = \Pi(u_{l2}, \hat{u}_h) = \bar{\Pi}$ where $\bar{\Pi}$ is equilibrium profits.

Note that

$$\forall u_l \in [u_{l1}, u_{l2}], \tilde{\Pi}(u_l, u_h) = (1 - \pi + \pi F) \mu_l (q_l - u_l) + \mu_h (1 - \pi + \pi F_h(u_h)) \Pi_h(u_l, u_h) \quad (41)$$

which is a linear function in u_l . Hence, if $\frac{\partial}{\partial u_l} \tilde{\Pi}(u_l, \hat{u}_h) \neq 0$, we would be able to construct profitable deviations (either $\tilde{\Pi}(u_{l1} + \varepsilon, \hat{u}_h) > \bar{\Pi}$ or $\tilde{\Pi}(u_{l2} - \varepsilon, \hat{u}_h) > \bar{\Pi}$). This means for all $u_l \in [u_{l1}, u_{l2}]$, $\frac{\partial}{\partial u_l} \tilde{\Pi}(u_l, \hat{u}_h) = 0$; that is, profits must be constant in this interval so that $\tilde{\Pi}(u_l, \hat{u}_h) = \bar{\Pi} \forall u_l \in [u_{l1}, u_{l2}]$.

Furthermore, note that the profit function in (41) is differentiable with respect to u_h . Since this derivative is linear with respect to u_l , there must exist a value of $u_l \in (u_{l1}, u_{l2})$ so that $\frac{\partial}{\partial u_h} \tilde{\Pi}(u_l, \hat{u}_h) \neq 0$. Therefore when $\frac{\partial}{\partial u_h} \tilde{\Pi}(u_l, \hat{u}_h) > 0$, a deviation of the form $(u_l, \hat{u}_h + \varepsilon)$ with $\varepsilon > 0$ small is profitable and vice versa. This establishes the contradiction and hence F_l must not have a flat.

Next, we show that F_l is continuous except possibly at its lower bound. That is, suppose that $u_l \in \text{Supp}(F_l)$ exists such that $u_l > \min \text{Supp}(F_l)$ and $F_l(u_l) > F_l^-(u_l)$. There are two possible cases:

1. $u_l \neq v_l$: Since F_h has no mass point, it must be that $U_h(u_l)$ is an interval when F_l puts a mass point on u_l . This implies that there are u_h 's in $U_h(u_l)$ that are interior in the sense that $u_h > u_l$. Now consider a deviation to $u_l + \varepsilon$ for a small and positive ε . This increases the number of low types by a large amount. This more than offsets the decline in profits per low type (which vary in a continuous fashion) for small enough ε and is therefore a profitable deviation. Note that this is a feasible deviation since u_h is interior. A similar deviation in the opposite direction rules out $u_l > v_l$.
2. $u_l = v_l$: In this case, we show that $F_l^-(v_l) = 0$ which implies that the masspoint is at the lower bound of the support.

Suppose that $F_l^-(v_l) > 0$ and that $\phi > 0$. First, since F_h is continuous and strictly increasing, if F_l has a masspoint at v_l , it must be that $U_h(v_l)$ is a non-degenerate interval. Let $\hat{u}_h = \min U_h(v_l)$. Note that F_l is continuous and strictly increasing for all $u_l \neq v_l$, which implies that $U_h(u_l)$ is a strictly increasing function in that region. This implies that $F_h(u_h) \geq F_l(v_l)$, $\forall u_h \in U_h(u_l)$ with equality only for $u_h = \hat{u}_h$. Furthermore, for all $u_l < v_l$, $F_l(u_l) = F_h(U_h(u_l))$. Now, consider the change in profits from offering $(u_l + \varepsilon, U_h(u_l))$ for a value of u_l close to v_l and a small but positive value of ε . Since $c_h > v_l$, for values of u_l close enough to v_l , the allocations must be separating and therefore a small increase in u_l is feasible. Furthermore, since F_l cannot have a mass point and flat below v_l , the profit function must be differentiable and its derivative with respect to ε should be given by

$$\mu_h(1 - \pi + \pi F_l(u_l)) \frac{v_h - c_h}{c_h - c_l} - \mu_l(1 - \pi + \pi F_l(u_l)) + \mu_l \pi f_l(u_l) (v_l - u_l)$$

Optimality of $U_h(u_l)$ implies that the above must be zero. This implies that in a neighborhood of v_l , F_l must satisfy

$$1 - \pi + \pi F_l(u_l) = \kappa (v_l - u_l)^{-\phi}$$

for some positive value of κ which implies that $F_l(v_l) = \infty$ which is a contradiction.

Note that when $\phi \leq 0$, $F_l^-(v_l) > 0$ gurantees a profitable deviation since a small increase in u_l not only (weakly) increases profits per trade but also attracts a higher number of low quality sellers from which profits are positive. This establishes the claim. ■

A.3 Proof of Proposition 10

Separating Allocation. We first show that the equilibrium allocations constructed in (21) and (22) are indeed separating. Note that solution to the differential equation in (22) together with boundary condition $F_l(c_l) = 0$, must satisfy

$$1 - \pi + \pi F_l(u_l) = (1 - \pi)(v_l - c_l)^\phi (v_l - u_l)^{-\phi} \quad (42)$$

Therefore, from (22), $U_h(u_l)$ must satisfy

$$\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l (v_l - u_l) = \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi$$

or

$$U_h(u_l) = \frac{\mu_h v_h + \mu_l v_l - \mu_l \phi u_l - \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi}{\mu_h \frac{v_h - c_l}{c_h - c_l}}$$

For this to be a separating allocation, we must show that

$$u_l + c_h - c_l \geq U_h(u_l) > u_l, \forall u_l \in \text{Supp}(F_l) \quad (43)$$

where

$$\text{Supp}(F_l) = \left[c_l, v_l - (1 - \pi)^{\frac{1}{\phi}} (v_l - c_l) \right]$$

The right hand side inequality is satisfied if and only if

$$\mu_h v_h + \mu_l v_l > \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi + u_l \quad (44)$$

Let the function $H(u_l)$ = be defined as the right hand side of the above inequality. Then $H(u_l)$ is a concave function:

$$\begin{aligned} H'(u_l) &= -\phi \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{\phi-1} + 1 \\ H''(u_l) &= \phi(\phi - 1) \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{\phi-2} < 0 \end{aligned}$$

where the last inequality follows from $\phi < 1$. Note further that, since $\phi < 1$, $H'(v_l) = -\infty$ and $H'(c_l) = 1 - \phi \mu_l > 0$. Therefore, to show (44), it is sufficient to show that it is satisfied where $H(u_l)$ is maximized in the interval $[c_l, v_l]$. The function $H(u_l)$ is maximized at u_l^* given by

$$\phi \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l^*)^{\phi-1} = 1 \rightarrow u_l^* = v_l - (\phi \mu_l)^{\frac{1}{1-\phi}} (v_l - c_l)$$

Therefore

$$\begin{aligned}
H(u_l^*) &= \frac{1}{\phi}(v_l - u_l^*) + u_l^* \\
&= \frac{(\phi\mu_l)^{\frac{1}{1-\phi}}(v_l - c_l)}{\phi} + v_l - (\phi\mu_l)^{\frac{1}{1-\phi}}(v_l - c_l) \\
&= v_l + (v_l - c_l) \mu_l^{\frac{1}{1-\phi}} \phi^{\frac{\phi}{1-\phi}} [1 - \phi]
\end{aligned}$$

Note that since $c_h \geq v_l$, we must have the following

$$(\phi\mu_l)^{\frac{\phi}{1-\phi}} < 1 \leq \frac{(c_h - c_l)(v_h - v_l)}{(v_l - c_l)(v_h - c_h)}$$

The above inequality implies that

$$(v_l - c_l) (\phi\mu_l)^{\frac{\phi}{1-\phi}} \frac{v_h - c_h}{c_h - c_l} < (v_h - v_l) \rightarrow (v_l - c_l) \mu_l (\phi\mu_l)^{\frac{\phi}{1-\phi}} \frac{\mu_h v_h - c_h}{\mu_l c_h - c_l} < \mu_h (v_h - v_l)$$

Hence,

$$(v_l - c_l) \mu_l (\phi\mu_l)^{\frac{\phi}{1-\phi}} (1 - \phi) < \mu_h (v_h - v_l)$$

and

$$v_l + (v_l - c_l) \mu_l (\phi\mu_l)^{\frac{\phi}{1-\phi}} (1 - \phi) < \mu_h (v_h - v_l) + v_l$$

or

$$\max_{u_l \in [c_l, v_l]} H(u_l) < \mu_h v_h + \mu_l v_l$$

which in turn implies that $U_h(u_l) > u_l$ for all $u_l \in \text{Supp}(F_l)$.

Next, we show the left inequality in (43). For this to hold, we must have

$$\frac{\mu_h v_h + \mu_l v_l - \mu_l \phi u_l - \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi}{\mu_h \frac{v_h - c_l}{c_h - c_l}} \leq u_l + c_h - c_l$$

or equivalently

$$\mu_h c_l + \mu_l v_l \leq u_l + \mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi, \forall u_l \in \text{Supp}(F_l) \subset [c_l, v_l] \quad (45)$$

Since, the right hand side of (45) is a concave function, it takes its minimum values at the extremes of the interval $[v_l, c_l]$. These values are given by v_l and $\mu_l v_l + \mu_h c_l$ both of which are at least as large the left side of (45). Hence, (45) must be satisfied for all $u_l \in [v_l, c_l]$.

Global Deviations. Note that our conditions (21) and (22) imply that local deviations with respect to u_h and u_l are not profitable. It, thus, remains to show that for all (u'_h, u'_l)

$$\Pi(u'_h, u'_l) = \mu_l (1 - \pi + \pi F_l(u'_l)) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_h(u'_h)) \Pi_h(u'_h, u'_l) \leq \mu_l (1 - \pi) (v_l - c_l)$$

We consider two cases:

1. $u'_h > \max \text{Supp} (F_h) = \bar{u}_h$: In this case, $1 - \pi + \pi F_h (u'_h) = 1$. If $u'_l > \max \text{Supp} (F_l) = \bar{u}_l$, then the profits from this menu are given by

$$\mu_l (v_l - u'_l) + \mu_h \Pi_h (u'_h, u'_l)$$

Since $\phi > 0$, the above function is decreasing in (u'_h, u'_l) and therefore

$$\mu_l (v_l - u'_l) + \mu_h \Pi_h (u'_h, u'_l) < \mu_l (v_l - \bar{u}_l) + \mu_h \Pi_h (\bar{u}_h, \bar{u}_l) = \mu_l (1 - \pi) (v_l - c_l)$$

When $u'_l \leq \max \text{Supp} (F_l)$, the partial derivative of the above profit function with respect to u'_l is given by

$$\begin{aligned} & -\mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l) (v_l - u'_l) + \mu_h \frac{v_h - c_h}{c_h - c_l} \geq \\ & -\mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (u'_l)) \frac{v_h - c_h}{c_h - c_l} = 0 \end{aligned}$$

Thus for a given value of u'_h , we must have

$$\Pi (u'_h, u'_l) \leq \Pi (u'_h, \bar{u}_l) < \Pi (\bar{u}_h, \bar{u}_l) = \mu_l (1 - \pi) (v_l - c_l)$$

This proves the claim.

2. $u'_h \in [c_h, \bar{u}_h]$. In this case, we must have there exists \tilde{u}_l such that $u'_h = U_h (\tilde{u}_l)$ and thus $F_h (u'_h) = F_l (\tilde{u}_l)$. We can thus write the profit function as

$$\mu_l (1 - \pi + \pi F_l (u'_l)) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (\tilde{u}_l)) \Pi_h (u'_h, u'_l) \quad (46)$$

We show that the above function is maximized at $u'_l = \tilde{u}_l$. This proves our claim.

To show this, we show that the function is strictly concave for values of $u'_l \in \text{Supp} (F_l)$ and decreasing for values of $u'_l > \bar{u}_l$. When $u'_l \in \text{Supp} (F_l)$, since Π_h is linear in u'_l , the second derivative of (46) with respect to u'_l is given by

$$\frac{\partial^2}{\partial (u'_l)^2} \mu_l (1 - \pi + \pi F_l (u'_l)) (v_l - u'_l)$$

Using (42), we can rewrite the above as

$$\begin{aligned} \frac{\partial^2}{\partial (u'_l)^2} \mu_l (1 - \pi + \pi F_l (u'_l)) (v_l - u'_l) &= \frac{\partial^2}{\partial (u'_l)^2} \mu_l (1 - \pi) (v_l - c_l)^\phi (v_l - u'_l)^{1-\phi} \\ &= (\phi - 1) \phi \mu_l (1 - \pi) (v_l - c_l)^\phi (v_l - u'_l)^{-1-\phi} < 0 \end{aligned}$$

This implies that (46) is strictly concave in u'_l for values of $u'_l \in \text{Supp} (F_l)$. When $u'_l > \bar{u}_l$, $1 - \pi + \pi F_l (u'_l) = 1$ and thus the value in (46) is given by

$$\mu_l (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (\tilde{u}_l)) \Pi_h (u'_h, u'_l)$$

The derivative of this function with respect to u'_l is given by

$$-\mu_l + \mu_h (1 - \pi + \pi F_l(\tilde{u}_l)) \frac{v_h - c_h}{c_h - c_l} < -\mu_l + \mu_h \frac{v_h - c_h}{c_h - c_l} = -\mu_l \phi < 0$$

Therefore, (46) is maximized at a value of u'_l which equates the partial derivative of (46) with zero. This value must satisfy

$$-\mu_l (1 - \pi + \pi F_l(u'_l)) + \mu_l \pi f_l(u'_l) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l(\tilde{u}_l)) \frac{v_h - c_h}{c_h - c_l} = 0$$

Note that since (46) is strictly concave, at most one u'_l exists that satisfies the above. The differential equation (21) implies that $u'_l = \tilde{u}_l$ is a solution to the above equation. This implies that (46) must be maximized at $u'_l = \tilde{u}_l$. This concludes the proof. ■

A.4 Proof of Proposition 11

We first show that the thresholds, ϕ_1 and ϕ_2 defined as the values of ϕ that satisfy (30) and (31) represent lower and upper bounds on ϕ respectively and that $\phi_2 < \phi_1 < 0$. We state these results as Lemma 22. Next, we show for each case described in the proposition, that there exist no profitable local or global deviations.

Lemma 22. *Let ϕ_1 and ϕ_2 satisfy (30) and (31) respectively. Then (30) is satisfied for all $\phi_1 \leq \phi \leq 0$ and (31) is satisfied for all $\phi \leq \phi_2$. Moreover, $\phi_2 < \phi_1 < 0$.*

Proof. First, note that equation (30) which determines the threshold ϕ_1 can be re-written as

$$(1 - \pi)^{\frac{1-\phi}{\phi}} \geq \frac{\pi}{1 - \pi} \frac{v_h - v_l}{c_h - v_l} \mu_h + 1, \quad (47)$$

or, after taking logs and substituting for ϕ ,

$$\frac{\mu_h (v_h - c_h)}{c_h - c_l - \mu_h (v_h - c_l)} \log(1 - \pi) - \log(\mu_h \pi (v_h - v_l) + (1 - \pi)(c_h - v_l)) - \log[(1 - \pi)(c_h - v_l)] \geq 0. \quad (48)$$

We show that the left-hand side of (48) is a decreasing function of μ_h which is strictly satisfied when μ_h is such that $\phi = 0$ and is weakly violated when $\mu_h = 1$. Hence, there is a unique threshold μ_1 (and implied threshold ϕ_1) such that for all $\mu_h \leq \mu_1$ and $\phi < 0$, the separating condition (30) is satisfied. Differentiating the left-hand side of (48) with respect to μ_h we obtain

$$\log(1 - \pi) \frac{(v_h - c_h)(c_h - c_l)}{[c_h - c_l - \mu_h (v_h - c_l)]^2} - \frac{\pi (v_h - v_l)}{\mu_h \pi (v_h - v_l) + (1 - \pi)(c_h - v_l)}$$

which is negative for all $\pi \leq 1$. Next, as $\phi \rightarrow 0$ from below, it is immediate that (47) is satisfied since the left-hand side tends to infinity. As $\mu_h \rightarrow 1$, the term $(1 - \phi)/\phi \rightarrow -1$ and so (47) tends to the requirement that

$$1 \geq \pi \frac{v_h - v_l}{c_h - v_l} + (1 - \pi)$$

which is violated since $c_h < v_h$.

Next, consider equation (31) which determines the threshold ϕ_2 . Substituting for ϕ and straightforward algebra shows this equation is equivalent to

$$\mu_h (v_h - v_l) \left[1 + (1 - \pi) \frac{v_h - c_h}{c_h - c_l} \right] \geq v_h - v_l + (c_h - v_l) (1 - \pi) \frac{v_h - c_h}{c_h - c_l}. \quad (49)$$

Clearly, (49) represents a lower bound on μ_h , or, equivalently, an upper bound on ϕ . Note that this equation is necessarily satisfied at $\mu_h = 1$. It is immediate that when μ_h is such that $\phi = 0$, equation (31) is violated since $c_h > v_l$.

We now prove that if $\phi \leq \phi_2$, then $\phi < \phi_1$ to establish that $\phi_2 < \phi_1$. Suppose $\phi \leq \phi_2$ and let $\bar{v} = \mu_h v_h + \mu_l v_l$ so that

$$1 - \pi \geq \frac{\bar{v} - v_l}{(1 - \phi) (\bar{v} - c_h)}. \quad (50)$$

Note that (50) implies

$$1 - \phi \geq \frac{\bar{v} - v_l}{(\bar{v} - c_h) (1 - \pi)} > \frac{\bar{v} - v_l}{\bar{v} - c_h}$$

and so

$$-\phi > \frac{c_h - v_l}{\bar{v} - c_h}.$$

We will show that (30) is violated. First, we re-arrange (30) as

$$(1 - \pi) \left[(1 - \pi)^{\frac{1-\phi}{\phi}} - 1 \right] (c_h - v_l) - \pi \mu_h (v_h - v_l) \geq 0$$

or, using straightforward algebra, as

$$(1 - \pi) (\bar{v} - c_h) + (1 - \pi)^{\frac{1}{\phi}} (c_h - v_l) \geq \bar{v} - v_l.$$

We show that if (50) holds, then

$$(1 - \pi) (\bar{v} - c_h) + (1 - \pi)^{\frac{1}{\phi}} (c_h - v_l) < \bar{v} - v_l.$$

Towards this end, define a function

$$H(\pi) = (1 - \pi) (\bar{v} - c_h) + (1 - \pi)^{\frac{1}{\phi}} (c_h - v_l).$$

We argue that

$$H\left(1 - \frac{\bar{v} - v_l}{(1 - \phi) (\bar{v} - c_h)}\right) < \bar{v} - v_l \quad (51)$$

and that this result implies $H(\pi) < \bar{v} - v_l$. Note that using the expression for $H(\pi)$, we have

$$H\left(1 - \frac{\bar{v} - v_l}{(1 - \phi) (\bar{v} - c_h)}\right) = \frac{\bar{v} - v_l}{(1 - \phi) (\bar{v} - c_h)} (\bar{v} - c_h) + \left(\frac{\bar{v} - v_l}{(1 - \phi) (\bar{v} - c_h)}\right)^{\frac{1}{\phi}} (c_h - v_l). \quad (52)$$

Straightforward algebra can be applied to (52) to show that (51) holds if and only if

$$\left(\frac{\bar{v} - v_l}{(1 - \phi)(\bar{v} - c_h)} \right)^{\frac{1-\phi}{\phi}} < -\phi \frac{\bar{v} - c_h}{c_h - v_l}.$$

Raising both sides of the last inequality by ϕ , we obtain (51) holds if and only if

$$\left(\frac{\bar{v} - v_l}{(1 - \phi)(\bar{v} - c_h)} \right)^{1-\phi} > \left(-\phi \frac{\bar{v} - c_h}{c_h - v_l} \right)^\phi$$

or

$$\left(\frac{c_h - v_l}{\bar{v} - c_h} \right)^\phi \left(\frac{\bar{v} - v_l}{\bar{v} - c_h} \right)^{1-\phi} > (-\phi)^\phi (1 - \phi)^{1-\phi}. \quad (53)$$

Let $B(x) = x^\phi(1+x)^{1-\phi}$ and since $(\bar{v} - v_l) / (\bar{v} - c_h) = 1 + (c_h - v_l) / (\bar{v} - c_h)$, (53) can be written as

$$B\left(\frac{c_h - v_l}{\bar{v} - c_h}\right) > B(-\phi).$$

Observe that

$$B'(x) = B(x) \left(\frac{\phi}{x} + \frac{1-\phi}{1+x} \right) = B(x) \frac{\phi + x}{x(1+x)}$$

so that for $0 < x < -\phi$, $B'(x) < 0$. Since $-\phi > (c_h - v_l) / (\bar{v} - c_h)$ and $B(x)$ is strictly decreasing, we have proved (53) and, therefore, (51) are satisfied.

Finally, note that the function $H(x)$ is convex since

$$\begin{aligned} H'(\pi) &= -(\bar{v} - c_h) - \frac{1}{\phi} (1 - \pi)^{\frac{1}{\phi}-1} (c_h - v_l) \\ H''(\pi) &= \frac{1}{\phi} \left(\frac{1}{\phi} - 1 \right) (1 - \pi)^{\frac{1}{\phi}-2} (c_h - v_l) > 0. \end{aligned}$$

Additionally, $H(0) = \bar{v} - v_l$, $H'(0) \leq 0$ when $-\phi \geq (c_h - v_l) / (\bar{v} - c_h)$ and $\lim_{\pi \rightarrow 1} H(\pi) = \infty$. Thus, there is a unique value $\pi^s > 0$ such that for all $\pi < \pi^s$, $H(\pi) \leq \bar{v} - v_l$. Since (51) holds, and since $\pi \leq 1 - (\bar{v} - v_l) / ((1 - \phi)(\bar{v} - c_h))$, we must have $H(\pi) < \bar{v} - v_l$. This proves $\phi < \phi_1$ as desired. ■

We now prove that in each region ($\phi \leq \phi_2$, $\phi_2 < \phi < \phi_1$, $\phi_1 \leq \phi < 0$), there are no local or global deviations for buyers which earn strictly positive profits.

Case 1: Only Separating Menus. Here, it suffices to rule out global deviations since the distribution $F_l(u_l)$ is chosen to ensure no local deviations are profitable. To rule out global deviations a proof similar to that of Proposition 10 can be used. Here, we show that for a given value of u'_h , the profit function is strictly concave in u'_l and therefore it must be maximized at $u'_l = U_h^{-1}(u'_h)$ – since at this value its derivative is set to zero.

The profits are given by

$$\mu_l (1 - \pi + \pi F_l(u'_l)) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_h(u'_h)) \Pi_h(u'_h, u'_l)$$

Since Π_h is linear in u'_l , the second derivative of the above function is equal to the second derivative of the profits for the low quality sellers. Using differential equation (21), we know that $(1 - \pi + \pi F_l(u'_l)) = \kappa (u'_l - v_l)^{-\phi}$ for some constant κ . Therefore, we have

$$\begin{aligned} \frac{\partial^2}{\partial (u'_l)^2} \mu_l (1 - \pi + \pi F_l(u'_l)) (v_l - u'_l) &= -\mu_l \kappa \frac{\partial^2}{\partial (u'_l)^2} (u'_l - v_l)^{1-\phi} \\ &= -\mu_l \kappa (1 - \phi) (-\phi) (u'_l - v_l)^{-1-\phi} < 0 \end{aligned}$$

This completes the proof.

Case 2: Only Pooling Menus. We first prove that no local deviations in the pooling equilibrium yield strictly positive profits. Below we demonstrate global deviations are also unprofitable. Recall that in a pooling equilibrium, the distributions $F_l(u_l)$ satisfies

$$(1 - \pi + \pi F_l(u_l)) [\bar{v} - u_l] = (1 - \pi) (\bar{v} - c_h) \quad (54)$$

where $\bar{v} = \mu_h v_h + \mu_l v_l$, $U_h(u_l) = u_l$, $F_h(u_l) = F_l(u_l)$, and $\text{Supp}(F_l) = [c_h, \bar{v} - (1 - \pi) (\bar{v} - c_h)]$. Fix any utility, u_l , interior to the support of F_l and consider a perturbation that increases u_h by ε ($\hat{u}_h = u_l + \varepsilon$) holding u_l fixed. Profits from such a deviation satisfy

$$\begin{aligned} &\mu_l (1 - \pi + \pi F_l(u_l)) (v_l - u_l) + \mu_h (1 - \pi + \pi F_l(u_l + \varepsilon)) \Pi_h(u_l + \varepsilon, u_l) \\ &= \mu_l (1 - \pi + \pi F_l(u_l)) (v_l - u_l) + \mu_h (1 - \pi + \pi F_l(u_l + \varepsilon)) \left[v_h - u_l - \varepsilon \frac{v_h - c_l}{c_h - c_l} \right]. \end{aligned}$$

If local deviations are unprofitable, this function must be maximized at $\varepsilon = 0$, so that F_l must satisfy

$$\mu_h \pi f_l(u_l) [v_h - u_l] - \mu_h (1 - \pi + \pi F_l(u_l)) \frac{v_h - c_l}{c_h - c_l} \leq 0.$$

Totally differentiating (54) yields the following relationship between F_l and f_l ,

$$\pi f_l(u_l) (\bar{v} - u_l) = (1 - \pi + \pi F_l(u_l)) \quad (55)$$

so that for local deviations to be unprofitable, we require

$$\mu_h \pi f_l(u_l) [v_h - u_l] - \mu_h \pi f_l(u_l) (\bar{v} - u_l) \frac{v_h - c_l}{c_h - c_l} \leq 0.$$

Since F_l is continuous in our constructed equilibrium, we may simplify this condition using straightforward algebra as

$$u_l (v_h - c_h) \leq \bar{v} (v_h - c_l) - v_h (c_h - c_l).$$

Consequently, we see that it suffices to check that this deviation is unprofitable at $\max \text{Supp}(F_l)$. Using $u_l = \bar{v} - (1 - \pi) (\bar{v} - c_h)$, simple algebraic manipulations show that this local deviation

is unprofitable as long as

$$\frac{\bar{v} - v_l}{(1 - \phi)(\bar{v} - c_h)} \leq 1 - \pi. \quad (56)$$

In order to rule out global deviations, we show that for any value of $u'_h \in \text{Supp}(F_l)$, when $u'_l \leq u'_h$, the profit function is increasing in u'_l . This implies that profits are maximized at $u'_l = u'_h$ and therefore there are no profitable deviation. To see this, note that profits are given by

$$\mu_l (1 - \pi + \pi F_l(u'_l)) (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l(u'_h)) \Pi_h(u'_h, u'_l)$$

The derivative of this function with respect to u'_l is given by

$$\begin{aligned} & \mu_l \pi f_l(u'_l) (v_l - u'_l) - \mu_l (1 - \pi + \pi F_l(u'_l)) + \mu_h (1 - \pi + \pi F_l(u'_h)) \frac{v_h - c_h}{c_h - c_l} \geq \\ & \mu_l \pi f_l(u'_l) (v_l - u'_l) - \mu_l (1 - \pi + \pi F_l(u'_l)) + \mu_h (1 - \pi + \pi F_l(u'_l)) \frac{v_h - c_h}{c_h - c_l} = \\ & \mu_l \pi f_l(u'_l) (v_l - u'_l) - \mu_l \phi (1 - \pi + \pi F_l(u'_l)) \end{aligned} \quad (57)$$

Using (55), we can write the above as

$$\begin{aligned} & \mu_l \frac{v_l - u'_l}{\bar{v} - u'_l} (1 - \pi + \pi F_l(u'_l)) - \mu_l \phi (1 - \pi + \pi F_l(u'_l)) = \\ & \mu_l (1 - \pi + \pi F_l(u'_l)) \left[\frac{v_l - u'_l}{\bar{v} - u'_l} - \phi \right] = \mu_l (1 - \pi + \pi F_l(u'_l)) \left[1 + \frac{v_l - \bar{v}}{\bar{v} - u'_l} - \phi \right] \end{aligned}$$

Since $u'_l \leq u'_h \leq \max \text{Supp}(F_l)$, the expression in the bracket takes its minimum value at $u'_l = \max \text{Supp}(F_l)$ and we have

$$1 + \frac{v_l - \bar{v}}{\bar{v} - u'_l} - \phi \geq 1 + \frac{v_l - \bar{v}}{(1 - \pi)(\bar{v} - c_h)} - \phi \geq 0$$

where the second inequality follows from (56). This implies that the expression in (57) is positive and concludes the proof.

Case 3: Equilibrium Features Pooling and Separating Menus. As in Case 1 with only separation, the distribution F_l for $u_l \in [\hat{u}_l, \max \text{Supp}(F_l)]$ is chosen to ensure local deviations are not profitable. Thus, we need only prove that local deviations are not profitable in the pooling region and that no global deviations are profitable. As in Case 2 with only pooling menus, it suffices to ensure that at the upper bound of the pooling region, \hat{u}_l , no local deviations are profitable, or

$$\hat{u}_l (v_h - c_h) \leq \bar{v} (v_h - c_l) - v_h (c_h - c_l). \quad (58)$$

To prove that (58) holds, first note that since $\phi_2 < \phi < \phi_1$, we have $c_h < \hat{u}_l < \bar{u}(\hat{u}_l)$. To see this, guess that $\hat{u}_l > c_h$ (we will verify it later), in which case \hat{u}_l satisfies

$$\bar{v} - \left\{ v_l + (\hat{u}_l - v_l) \left[(1 - \pi) \frac{\bar{v} - c_h}{\bar{v} - \hat{u}_l} \right]^{\frac{1}{\phi}} \right\} - (1 - \pi)(\bar{v} - c_h) = 0. \quad (59)$$

Let $H(u)$ denote the left-hand side of (59). Observe that one solution to $H(u) = 0$ is given by

$$\bar{u} = \bar{v} - (1 - \pi)(\bar{v} - c_h).$$

This solution occurs when $\bar{u}(\hat{u}_l) = \hat{u}_l$. For $\hat{u}_l < \bar{u}$, we have $\bar{u}_l(\hat{u}_l) > \bar{u}_l$. We now show that when $\phi_2 < \phi < \phi_1$, then there is another solution with $\hat{u}_l \in (c_h, \bar{u})$. We show this by proving that $H'(\bar{u}) > 0$, $H(\bar{u}) = 0$, $H(c_h) > 0$ and $H''(u) \geq 0$. Then, there exists a unique threshold, \hat{u}_l such that $H(u) = 0$. Note that

$$H'(u) = - \left[(1 - \pi) \frac{\bar{v} - c_h}{\bar{v} - u} \right]^{\frac{1}{\phi}} - (u - v_l) \frac{1}{\phi} \left[(1 - \pi) \frac{\bar{v} - c_h}{\bar{v} - u} \right]^{\frac{1}{\phi} - 1} (1 - \pi)(\bar{v} - c_h)(\bar{v} - u)^{-2}.$$

Algebraic computations, available on request, then demonstrate that $H''(u) \geq 0$. Next, observe that by construction, $H(\bar{u}) = 0$ and

$$H'(\bar{u}) = -1 - \frac{1}{\phi} \frac{\bar{u} - v_l}{\bar{v} - \bar{u}}.$$

Substituting for \bar{u} and re-arranging terms, we find that

$$H'(\bar{u}) = \frac{1 - \phi}{\phi} \left[1 - \frac{\bar{v} - v_l}{(1 - \pi)(1 - \phi)(\bar{v} - c_h)} \right].$$

When $\phi > \phi_2$, the term in brackets is negative so that $H'(\bar{u}) > 0$.

Finally, note that $H(c_h)$ satisfies

$$\begin{aligned} H(c_h) &= \bar{v} - v_l - (c_h - v_l)(1 - \pi)^{\frac{1}{\phi}} - (1 - \pi)(\bar{v} - c_h) \\ &= \pi(\bar{v} - v_l) - (c_h - v_l)(1 - \pi)^{\frac{1}{\phi}} + (1 - \pi)(c_h - v_l) \\ &= \pi(\bar{v} - v_l) - (c_h - v_l) \left[(1 - \pi)^{\frac{1}{\phi}} - (1 - \pi) \right] \\ &= \frac{1}{(1 - \pi)^{\frac{1}{\phi}} - (1 - \pi)} \left[v_l + \frac{\pi(\bar{v} - v_l)}{(1 - \pi)^{\frac{1}{\phi}} - (1 - \pi)} - c_h \right]. \end{aligned}$$

The fact that $\phi < \phi_1 < 0$ implies that the term in brackets is strictly positive and the leading fraction is also positive so that $H(c_h) > 0$.

We now prove that (58) is satisfied at \hat{u}_l . Algebra, available upon request shows that (58) may be written as

$$\hat{u}_l \leq \frac{-\phi}{1 - \phi} \bar{v} + \frac{1}{1 - \phi} v_l.$$

We demonstrate that this condition is verified by proving that $H\left(\frac{-\phi}{1 - \phi} \bar{v} + \frac{1}{1 - \phi} v_l\right) \leq 0$. To see

this, note that

$$\begin{aligned}
& H\left(\frac{-\phi}{1-\phi}\bar{v} + \frac{1}{1-\phi}v_l\right) \\
&= \bar{v} - v_l - \left(\frac{-\phi}{1-\phi}\bar{v} + \frac{1}{1-\phi}v_l - v_l\right) \left[(1-\pi) \frac{\bar{v} - c_h}{\bar{v} - \frac{-\phi}{1-\phi}\bar{v} - \frac{1}{1-\phi}v_l} \right]^{\frac{1}{\phi}} \\
&\quad - (1-\pi)(\bar{v} - c_h) \\
&= \bar{v} - v_l + \frac{\phi}{1-\phi}(\bar{v} - v_l) \left[\frac{(1-\pi)(1-\phi)(\bar{v} - c_h)}{\bar{v} - v_l} \right]^{\frac{1}{\phi}} - (1-\pi)(\bar{v} - c_h) \\
&= (\bar{v} - v_l) \left[\frac{\bar{v} - v_l - (1-\pi)(\bar{v} - c_h)}{\bar{v} - v_l} + \phi \frac{(1-\phi)^{\frac{1}{\phi}-1} (1-\pi)^{\frac{1}{\phi}} (\bar{v} - c_h)^{\frac{1}{\phi}}}{(\bar{v} - v_l)^{\frac{1}{\phi}}} \right]. \tag{60}
\end{aligned}$$

We now show that the term in brackets on the right side of (60) is negative. To simplify notation, define $\xi = (1-\pi)(\bar{v} - c_h) / (\bar{v} - v_l)$. Since $\phi > \phi_2$, we have that $\xi < 1/(1-\phi)$. As well, the term in brackets can be written compactly as

$$1 - \xi + \phi(1-\phi)^{\frac{1}{\phi}-1} \xi^{\frac{1}{\phi}}.$$

Let $G(\xi) = 1 - \xi + \phi(1-\phi)^{\frac{1}{\phi}-1} \xi^{\frac{1}{\phi}}$ and observe that for $\xi \leq 1/(1-\phi)$, we have

$$G'(\xi) = -1 + [(1-\phi)\xi]^{\frac{1}{\phi}-1} \geq 0$$

so that for low values of ξ , $G(\xi)$ is an increasing function. Finally, we see that $G(1/(1-\phi)) = 0$ so that $G(\xi) \leq G(1/(1-\phi)) \leq 0$ which ensures the term in brackets in (60) is indeed negative as desired.

To rule out global deviations, one can use the arguments provided in the two cases above (when equilibrium is fully separating or pooling) in each region of the Supp (F_h). The argument is the exact replication of the above arguments and is thus omitted. ■

A.5 Proof of Proposition 12

We show that there are no profitable deviations. In other words,

$$\forall (u'_h, u'_l) : \mu_h(1-\pi + \pi F_l(u'_l)) \Pi_h(u'_h, u'_l) + \mu_l(1-\pi + \pi F_l(u'_l))(v_l - u'_l) \leq (1-\pi)\mu_l(v_l - c_l)$$

We consider two cases:

1. $u'_h > \max \text{Supp}(F_h) = \bar{u}_h$: In this case, when $u'_l > v_l$, the profit function is given by

$$\mu_h \Pi_h(u'_h, u'_l) + \mu_l(v_l - u'_l)$$

Since $\phi = 0$, the above function is invariant to changes in u'_l and is strictly decreasing in u'_h . Therefore, its value must be less than its value evaluated at (\bar{u}_h, v_l) which gives the equilibrium profits. When, $u'_l \leq v_l$, the profits are given by $\mu_h \Pi_h(u'_h, u'_l)$ which is

decreasing in u'_h and therefore

$$\mu_h \Pi_h (u'_h, u'_l) + \mu_l (1 - \pi) (v_l - u'_l) < \mu_h \Pi_h (\bar{u}_h, u'_l) + \mu_l (1 - \pi) (v_l - u'_l)$$

Note that the right hand side of the above inequality is a linear function of u'_l whose derivative is given by

$$\begin{aligned} \mu_h \frac{v_h - c_h}{v_l - c_l} - \mu_l (1 - \pi) &= \mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l + \mu_l \pi \\ &= -\mu_l \phi + \mu_l \pi = \mu_l \pi > 0. \end{aligned}$$

Therefore, we must have that

$$\mu_h \Pi_h (\bar{u}_h, u'_l) + \mu_l (1 - \pi) (v_l - u'_l) \leq \mu_h \Pi_h (\bar{u}_h, v_l) = (1 - \pi) \mu_l (v_l - c_l)$$

where the last equality follows from (32).

2. $u'_h \in [c_h, \bar{u}_h]$. In this case, when $u'_l > v_l$, profits are given by

$$\begin{aligned} \mu_h (1 - \pi + \pi F_l (u'_h)) \Pi_h (u'_h, u'_l) + \mu_l (v_l - u'_l) &\leq \mu_h (1 - \pi + \pi F_l (u'_h)) \Pi_h (u'_h, v_l) \\ &= (1 - \pi) \mu_l (v_l - c_l) \end{aligned}$$

where the inequality is satisfied since $u'_l > v_l$ and the last equality follows from (32).

When $u'_l \leq v_l$, profits are given by

$$\mu_h (1 - \pi + \pi F_l (u'_h)) \Pi_h (u'_h, u'_l) + \mu_l (1 - \pi) (v_l - u'_l)$$

The above function is linear in u'_l and its derivative is given by

$$\begin{aligned} \mu_h (1 - \pi + \pi F_l (u'_h)) \frac{v_h - c_h}{c_h - c_l} - \mu_l (1 - \pi) &= (1 - \pi) \left(\mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l \right) + \pi F_l (u'_h) \frac{v_h - c_h}{c_h - c_l} \\ &= \pi F_l (u'_h) \frac{v_h - c_h}{c_h - c_l} \geq 0 \end{aligned}$$

Therefore it is maximized at $u'_l = v_l$. This completes the proof. ■

A.6 Proof of Theorem 13

We prove uniqueness of the equilibrium first for $\phi > 0$ and then for $\phi < 0$. Finally, we show uniqueness for $\phi = 0$.

Case 1: $\phi > 0$. As discussed in the text, we need to prove that no equilibrium features a mass point in F_l and that no equilibrium features pooling ($U_h(u_l) = u_l$) on any interval in the support of F_l . We state these results and proofs as independent lemmas below. These results imply that any equilibrium is separating and has continuous distribution F_l ; since there is a unique

solution to the differential equation (21) with the boundary condition $\min \text{Supp}(F_l) = c_l$, we have then shown that the equilibrium described in Proposition 10 is unique.

Lemma 23. *If $\phi > 0$, then F_l has no mass points.*

Proof. We have already shown that if F_l has a mass point, then it must occur at the lower bound of the support of F_l which satisfies $\min \text{Supp}(F_l) = c_l$. Suppose by way of contradiction that F_l has a mass point at c_l . Since F_h has no mass point, $U_h(u_l)$ must be a correspondence at $u_l = c_l$. Hence, there exists a $u_h \in U_h(u_l)$ such that $u_h > u_l = c_l$. Consider a deviation by a buyer to the menu $(c_l + \varepsilon, u_h)$. Such a deviation induces a jump in the number of non-captive low quality sellers attracted while decreasing the profits earned per low quality seller marginally. Hence, this deviation yields strictly positive profits, implying a contradiction. ■

Lemma 24. *If $\phi > 0$, in any equilibrium, there is no interval $[u_1, u_2] \in \text{Supp } F_l$ such that $U_h(u_l) = u_l$ for $u_l \in [u_1, u_2]$. That is, there is no interval which features pooling menus.*

Proof. Suppose by way of contradiction that a pooling interval exists. We will show that this necessarily implies that the separating equilibrium we construct in Proposition 10 must not exist.

Towards this end, we first prove that there can be at most one interval of pooling and that the pooling thresholds u_1 and u_2 are uniquely determined. First, note that since the worst equilibrium menu is the menu (c_l, c_h) which satisfies $U_h(c_l) > c_l$, the equilibrium near the lower bound of the support must be separating. Let $[u_1, u_2]$ denote the lowest interval of pooling and assume $u_2 < \max \text{Supp}(F_l)$. We show that for all $u_l > u_2$, $U_h(u_l) > u_l$. Since the equilibrium is locally pooling below u_2 , $\lim_{u_l \nearrow u_2} U_h'(u_l) = 1$. If $u_2 < \max \text{Supp}(F_l)$, the equilibrium must be locally separating above u_2 so that $U_h(u_l)$ satisfies

$$(1 - \pi + \pi F_l(u_l)) [\mu_l (v_l - u_l) + \mu_h \Pi_h(u_l, U_h(u_l))] = (1 - \pi) (v_l - c_l)$$

with $F_l(u_l)$ satisfying the differential equation

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} = \frac{\phi}{v_l - u_l}. \quad (61)$$

Twice differentiating the equal profit condition with respect to u_l , we obtain

$$U_h''(u_l) = \phi(1 - \phi) \frac{(1 - \pi) (v_l - c_l) (v_h - v_l)}{(1 - \pi + \pi F_l(u_l)) (v_l - u_l)^2 \mu_h (v_h - c_l)} > 0$$

where the inequality holds when $\phi > 0$ (recall $\phi \leq 1$) so that U_h is convex. This means that for all $u_l > u_2$, $U_h(u_l) > u_l$. We conclude that there can be at most one region of pooling menus.

For the pooling region to exist, cream-skimming deviations must be unprofitable at the upper threshold of the pooling interval, u_2 so that

$$u_2 \leq \bar{v} + (\bar{v} - v_h) \frac{c_h - c_l}{v_h - c_h} \equiv v_{\text{pool}}.$$

The lower threshold is determined by imposing the boundary condition $U_h(u_1) = u_1$ on the differential equation (61). Solving the differential equation, we find u_1 satisfies

$$\bar{v} = u_1 + \mu_l (v_l - c_l)^{1-\phi} (v_l - u_1)^\phi. \quad (62)$$

We now analyze solutions to the non-linear equation (62). Let

$$H(u) = u + \mu_l (v_l - c_l)^{1-\phi} (v_l - u)^\phi$$

so that

$$H'(u) = 1 - \phi \mu_l (v_l - c_l)^{1-\phi} (v_l - u)^{\phi-1}$$

Straightforward calculations show that $H(u)$ is a concave function when $\phi \geq 0$ and $u \leq v_l$. Since $H'(v_l) < 0$, we have that $\arg \max_u H(u) < v_l$. Since the pooling region exists, by assumption, we must have $\max_u H(u) \geq \bar{v}$. Since $U_h(u_1) \rightarrow u_1$ as $u_1 \rightarrow u_1$, we must have $U'_h(u_1) \leq 1$ as $u_1 \rightarrow u_1$. Or, using the equal profit condition again

$$-\mu_l \phi - \mu_h \frac{v_h - c_l}{c_h - c_l} U'_h(u_1) = H'(u) - 1. \quad (63)$$

Using (63), one can show that requiring $U'_h(u_1) \leq 1$ is equivalent to requiring $H'(u) \geq 0$. Hence, u_1 is uniquely determined as the lowest solution to $H(u) = \bar{v}$. Extensive algebraic manipulations, available upon request, can be used to show that $H'(v_{\text{pool}}) < 0$ so that by concavity of $H(u)$, we know that $u_1 \leq v_{\text{pool}}$. A similar argument, applied to the upper bound implies that the upper threshold is also uniquely determined if $u_2 < \text{Supp}(F_l)$.

Now, note that the equilibrium we describe in Proposition 10 satisfies $\max \text{Supp}(F_l) = v_l - (1 - \pi)^{1/\phi} (v_l - c_l)$. If the pooling interval exists, it is immediate that $u_1 \leq v_l - (1 - \pi)^{1/\phi} (v_l - c_l)$. (If $u_1 > v_l - (1 - \pi)^{1/\phi} (v_l - c_l)$, then $F(u_1) > 1$). If the threshold u_1 satisfies

$$u_1 < v_l - (1 - \pi)^{1/\phi} (v_l - c_l), \quad (64)$$

then the equilibrium with only separating menus described in Proposition 10 does not exist, which is a contradiction. If such an equilibrium exists when u_1 satisfies the inequality (64), then for some values $u_1 \in [u_1, v_l - (1 - \pi)^{1/\phi} (v_l - c_l)]$, we have $U_h(u_1) = u_1$ violating the separating feature of the equilibrium. ■

Case 2: $\phi < 0$. To prove the equilibrium characterized in Proposition 11 is unique, we first prove that in any equilibrium with $\phi < 0$, if $\bar{u} = \max \text{Supp}(F_l)$, then $U_h(\bar{u}) = \bar{u}$ so that the best menu in equilibrium is a pooling menu. We then demonstrate that F_l has no mass points when $\phi < 0$. Next, We then prove that if the equilibrium has a pooling region, the region begins at the lower bound of the support of F_l or ends at the upper bound of F_l . Additionally, if the equilibrium features a separating region, this region must end at the upper bound of the support of F_l . These results imply that any equilibrium must take one of the three forms described in Proposition 11: only separating, only pooling, or semi-separating. Finally, we show that the necessary conditions for each type of equilibrium to exist are mutually exclusive so that at most one type of equilibrium exists for each region of the parameter space, ensuring our equilibrium is unique for all $\phi < 0$. We prove these results in the sequence of following lemmas.

Lemma 25. *If $\phi < 0$, then F_l has no mass points.*

Proof. Suppose by way of contradiction that F_l has a mass point at the lower bound of the support of F_l . Then it must be that $\min \text{Supp}(F_l) = v_l$. Exactly as in Lemma 23, if F_l has a

mass point at some $u_l < v_l$, then a deviation to a menu $(u_l \pm \varepsilon, u_h)$ must be feasible for some $u_h \in U_h(u_l)$ and earn strictly positive profits.

Next note that if F_l has a mass point at v_l , then we must have $F_l^+(v_l) = 1$. If instead $F_l^+(v_l) = m < 1$, then, since $U_h(v_l)$ is an interval, there must be an interval of u_l such that for $u_l \in (v_l, v_l + \varepsilon)$ for $\varepsilon > 0$ the equilibrium menus are separating, or $U_h(u_l) > u_l$. In this interval of $\text{Supp}(F_l)$, since local deviations must be unprofitable, F_l must satisfy

$$1 - \pi + \pi F_l(u_l) = C (u_l - v_l)^{-\phi}$$

for some positive constant C . But then,

$$\lim_{u_l \searrow v_l} 1 - \pi + \pi F_l(u_l) = \lim_{u_l \searrow v_l} C (u_l - v_l)^{-\phi} = 0$$

which contradicts the fact that

$$\lim_{u_l \searrow v_l} 1 - \pi + \pi F_l(u_l) = 1 - \pi + \pi F_l(v_l) = 1 - \pi(1 - m) > 0.$$

Hence, $F_l^+(v_l) = 1$.

As a consequence, we have that $\text{Supp}(F_l) = \{v_l\}$ while $\max \text{Supp}(F_h) > \underline{u}_h \geq v_l = \max \text{Supp}(F_l)$ (this last fact follows since F_h is continuous). Let $\bar{u}_h = \max \text{Supp}(F_h)$ and consider the profits a buyer would obtain by following a deviation to a menu $(v_l + \varepsilon, \bar{u}_h)$. Profits at the original contract satisfy

$$\mu_h \left(v_h - \bar{u}_h \frac{v_h - c_l}{c_h - c_l} + v_l \frac{v_h - c_h}{c_h - c_l} \right)$$

while the deviation contract yields profits equal to

$$\begin{aligned} & \mu_h \left(v_h - \bar{u}_h \frac{v_h - c_l}{c_h - c_l} + (v_l + \varepsilon) \frac{v_h - c_h}{c_h - c_l} \right) + \mu_l (v_l - (v_l + \varepsilon)) \\ &= \mu_h \left(v_h - \bar{u}_h \frac{v_h - c_l}{c_h - c_l} + v_l \frac{v_h - c_h}{c_h - c_l} \right) - \mu_l \varepsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} \varepsilon. \end{aligned}$$

Since the increment to profits earned from this deviation is given by

$$-\mu_l \varepsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} \varepsilon = -\mu_l \phi \varepsilon$$

and since $\phi < 0$, this deviation earns strictly greater profits than the equilibrium menu (v_l, \bar{u}_h) , which yields the necessary contradiction. \blacksquare

Lemma 26. *If $\phi < 0$, then the best equilibrium menu is a pooling menu.*

Proof. Let $\bar{u} = \max \text{Supp}(F_l)$ and suppose for contradiction that $U_h(\bar{u}) > \bar{u}$. Consider a deviation menu with $(u_l, u_h) = (\bar{u} + \varepsilon, U_h(\bar{u}))$. Since $U_h(\bar{u}) > \bar{u}$, this menu is incentive compatible and has $F_l(u_l) = F_l(u_h) = 1$. This menu increases the buyer's profits relative to the menu $(\bar{u}, U_h(\bar{u}))$ by the amount

$$-\mu_l \varepsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} \varepsilon = -\mu_l \phi \varepsilon > 0$$

where the inequality follows from $\phi < 0$. This profitable deviation yields the necessary contradiction so that we must have $U_h(\bar{u}) = \bar{u}$. ■

Lemma 27. *If $\phi < 0$ and an equilibrium features $[u_1, u_2] \subseteq \text{Supp}(F_l)$ such that $U_h(u_1) = u_1$ for $u_1 \in [u_1, u_2]$, then either $u_1 = \min \text{Supp}(F_l)$ or $u_2 = \max \text{Supp}(F_l)$.*

Proof. Suppose for contradiction that an interval of pooling exists and $u_1 > \min \text{Supp}(F_l)$ and $u_2 < \max \text{Supp}(F_l)$. Then there must exist intervals sufficiently close to and below u_1 and above u_2 respectively in which the equilibrium menu feature separation. Consequently, we must have $\lim_{u_1 \nearrow u_1} U'_h(u_1) \leq 1$ and $\lim_{u_1 \searrow u_2} U'_h(u_1) \geq 1$. In any region with $U_h(u_1) > u_1$, the distribution F_l must also satisfy

$$\frac{\pi f_l(u_1)}{1 - \pi + \pi F_l(u_1)} = \frac{-\phi}{u_1 - v_l}$$

to rule out profitable local deviations and U_h must satisfy

$$\bar{v} - \mu_l \phi u_1 - \mu_h \frac{v_h - c_l}{c_h - c_l} U_h(u_1) = \bar{\Pi} (1 - \pi + \pi F_l(u_1))^{-1}$$

to ensure all menus earn the equilibrium level of profits which we denote with $\bar{\Pi}$.

Using these features of the conjectured equilibrium, we have that in the separating regions, $U'_h(u_1)$ satisfies

$$-\mu_l \phi - (1 - \mu_l \phi) U'_h(u_1) = \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_1)} \frac{\phi}{u_1 - v_l}.$$

Moreover, the second derivative of U_h satisfies

$$-(1 - \mu_l \phi) U''_h(u_1) = \frac{\bar{\Pi} \pi f_l(u_1)}{[1 - \pi + \pi F_l(u_1)]^2} \frac{\phi}{u_1 - v_l} + \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_1)} \frac{-\phi}{[u_1 - v_l]^2}$$

which implies $U''_h(u_1) \leq 0$ since $\phi < 0$. However, the pooling region requires $U_h^+(u_2) \geq 1 \geq U_h^-(u_1)$ which contradicts the concavity of U_h given that $u_1 < u_2$. Hence, either $u_1 = \min \text{Supp}(F_l)$ or $u_2 = \max \text{Supp}(F_l)$. ■

Lemma 28. *If $\phi < 0$ and an equilibrium features $[u_1, u_2] \subseteq \text{Supp}(F_l)$ such that $U_h(u_1) > u_1$ for $u_1 \in (u_1, u_2)$, then $u_2 = \max \text{Supp}(F_l)$.*

Proof. Suppose such a separating interval exists with $\min \text{Supp}(F_l) < u_2 < \max \text{Supp}(F_l)$. Then there exists a pooling interval $[u_2, \bar{u}]$ for some \bar{u} . Since $u_2 > \min \text{Supp}(F_l)$, Lemma 27 implies that $\bar{u} = \max \text{Supp}(F_l)$. Since the conjectured equilibrium features separation in $[u_1, u_2]$ with $U_h(u_1) \rightarrow u_1$ as $u_1 \rightarrow u_2$, we must have $U_h^-(u_2) \leq 1$ or

$$\frac{1}{1 - \mu_l \phi} \left[-\mu_l \phi + \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_2)} \frac{-\phi}{u_2 - v_l} \right] \leq 1$$

which can be re-written as

$$-\phi \bar{\Pi} \leq [1 - \pi + \pi F_l(u_2)] (u_2 - v_l).$$

Since $u_2 < \bar{u}$, $F(u_2) < 1$ so that

$$-\phi\bar{\Pi} < u_2 - v_l. \quad (65)$$

Moreover, since the best menu is pooling with utility \bar{u} , equilibrium profits satisfy

$$\bar{\Pi} = \bar{v} - \bar{u}$$

equation (65) implies

$$v_l - \phi(\bar{v} - \bar{u}) < u_2 < \bar{u}$$

or

$$v_l - \phi\bar{v} < (1 - \phi)\bar{u} \quad (66)$$

Since the equilibrium features pooling in the interval $[u_2, \bar{u}]$, we must have cream-skimming is not a profitable deviation, in particular from the pooling menu (\bar{u}, \bar{u}) , or

$$\bar{u} < \bar{v} \frac{v_h - c_l}{v_h - c_h} - v_h \frac{c_h - c_l}{v_h - c_h}. \quad (67)$$

Conditions (66) and (67) then require

$$\frac{v_l - \phi\bar{v}}{1 - \phi} < \bar{v} \frac{v_h - c_l}{v_h - c_h} - v_h \frac{c_h - c_l}{v_h - c_h}.$$

By substituting for ϕ and using straightforward algebraic manipulations we show that

$$\frac{v_l - \phi\bar{v}}{1 - \phi} = \bar{v} \frac{v_h - c_l}{v_h - c_h} - v_h \frac{c_h - c_l}{v_h - c_h}$$

which yields the needed contradiction. ■

Since the only possible equilibria when $\phi < 0$, then, are fully separating (except at the upper bound of the support of F_l), fully pooling, or semi-separating, we need only prove that only one of these equilibria may exist for any value of ϕ . We have already shown in the proof of Proposition 11 that $\phi_2 < \phi_1 < 0$. Recall that a necessary condition for a fully pooling equilibrium is that $\phi \leq \phi_2$. Hence, there is no fully pooling equilibrium when $\phi > \phi_2$. Similarly, a necessary condition for a fully separating equilibrium is that $\phi \geq \phi_1$ so that when $\phi < \phi_1$, no fully separating equilibrium exists. This means that in the interval $\phi_2 < \phi < \phi_1$, the only possible equilibrium is a semi-separating equilibrium. Moreover, the threshold in the semi-separating equilibrium is interior to the support of F_l only if ϕ lies between ϕ_2 and ϕ_1 . Hence, at most one of these types of equilibria may exist for any value of $\phi < 0$, proving that the equilibrium described in Proposition 13 is unique.

A.7 Proof of Proposition 14

When $\phi < 0$, it is immediate that welfare is (weakly) maximized when $\pi = 0$. To prove that welfare is maximized for $\pi \in (0, 1)$ when $\phi > 0$ we prove that our measure of welfare is decreasing in π at $\pi = 1$. Since welfare associated with the endpoint $\pi = 1$ is strictly larger than at the other extreme, when $\pi = 0$, this finding necessarily implies that welfare is maximized at an interior π .

To show that welfare is decreasing at $\pi = 1$, we first demonstrate that the volume of loans sold by high-quality sellers is decreasing in the level of indirect utility offered to low-quality sellers when this level is sufficiently close to v_l . We state this result as the following Lemma. Below, we use this finding to prove that welfare is decreasing in π at $\pi = 1$.

Lemma 29. *As u_l tends to v_l , the volume of loans sold by high-quality sellers at the associated u_h is decreasing.*

Proof. First, note that for any equilibrium menu, $(u_l, U_h(u_l))$, the associated volume of loans sold by high quality sellers satisfies

$$x(u_l) = 1 - \frac{U_h(u_l) - u_l}{c_h - c_l}.$$

Thus, $x(u_l)$ is decreasing if and only if $U'_h(u_l) > 1$. We determine $U'_h(u_l)$ using properties of the equilibrium, namely equations (21) and (22). Specifically, if we differentiate (22) with respect to u_l , we find

$$\mu_h \frac{v_h - c_l}{c_h - c_l} U'_h(u_l) = -\mu_l \phi + \frac{\mu_l (1 - \pi) (v_l - c_l)}{1 - \pi + \pi F_l(u_l)} \frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)}.$$

Using (21), we can simplify this expression as

$$\mu_h \frac{v_h - c_l}{c_h - c_l} U'_h(u_l) = -\mu_l \phi + \frac{\mu_l (1 - \pi) (v_l - c_l)}{1 - \pi + \pi F_l(u_l)} \frac{\phi}{v_l - u_l}.$$

Using (22) again to replace $\mu_l (1 - \pi) (v_l - c_l) / [1 - \pi + \pi F_l(u_l)]$ yields equation (35) in the text, which we replicate here.

$$\begin{aligned} \mu_h \frac{v_h - c_l}{c_h - c_l} U'_h(u_l) &= -\mu_l \phi + \frac{\phi [\mu_h \Pi_h(U_h(u_l), u_l) + \mu_l (v_l - u_l)]}{v_l - u_l} \\ \Rightarrow U'_h(u_l) &= \left[\frac{\phi (c_h - c_l)}{v_h - c_l} \right] \times \frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l}. \end{aligned}$$

We show that as $u_l \rightarrow v_l$, $U'_h(u_l) \rightarrow \infty$. To see this, note that

$$\frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l} = \frac{\mu_l}{\mu_h} \left[\frac{(1 - \pi) (v_l - c_l)}{v_l - u_l} \frac{1}{1 - \pi + \pi F_l(u_l)} - 1 \right].$$

When $\phi > 0$, the equilibrium distribution (that is, the solution to the differential equation) satisfies

$$1 - \pi + \pi F_l(u_l) = (1 - \pi) \left(\frac{v_l - c_l}{v_l - u_l} \right)^\phi$$

and hence

$$\frac{\Pi_h(U_h(u_l), u_l)}{v_l - u_l} = \frac{\mu_l}{\mu_h} \left[\left(\frac{v_l - c_l}{v_l - u_l} \right)^{1-\phi} - 1 \right].$$

Since $\phi > 0$, as $u_l \rightarrow v_l$, we see that this term tends to infinity. ■

Next, we show that welfare is a decreasing function of π at $\pi = 1$.

Note ex-ante welfare is given by $\mu_l (v_l - c_l) + (v_h - c_h)$ times the expected trade by the high quality sellers which is given by

$$\begin{aligned}
& (1 - \pi) \mu_h \int_{c_l}^{\bar{u}_l} x_h(u_l) f(u_l) du_l + \pi \mu_h \int_{c_l}^{\bar{u}_l} 2x_h(u_l) f(u_l) F(u_l) du_l \\
&= \mu_h \int_{c_l}^{\bar{u}_l} x_h(u_l) f(u_l) [1 - \pi + 2\pi F(u_l)] du_l \\
&= \mu_h \int_{c_l}^{\bar{u}_l} \left(\frac{\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi + u_l - \mu_h c_l - \mu_l v_l}{\mu_h (v_h - c_l)} \right) \left(\frac{1 - \pi}{\pi} \phi (v_l - c_l)^\phi (v_l - u_l)^{-1-\phi} \right) \\
&\quad \left[(1 - \pi) + 2(1 - \pi) \left(\frac{v_l - c_l}{v_l - u_l} \right)^\phi - 2(1 - \pi) \right] du_l \\
&= \phi \mu_h \frac{(1 - \pi)^2}{\pi} \int_{c_l}^{\bar{u}_l} \frac{\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi + u_l - \mu_h c_l - \mu_l v_l}{\mu_h (v_h - c_l)} (v_l - c_l)^\phi (v_l - u_l)^{-1-\phi} \\
&\quad \left[2 \left(\frac{v_l - c_l}{v_l - u_l} \right)^\phi - 1 \right] du_l \\
&= \phi \frac{(v_l - c_l)^\phi (1 - \pi)^2}{v_h - c_l \pi} \int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^\phi + u_l - \mu_h c_l - \mu_l v_l \right) (v_l - u_l)^{-1-\phi} \\
&\quad \left[2 \left(\frac{v_l - c_l}{v_l - u_l} \right)^\phi - 1 \right] du_l
\end{aligned}$$

Expected trade by the high type is $\phi \frac{(v_l - c_l)^\phi}{v_h - c_l}$ times

$$\begin{aligned}
& \frac{(1 - \pi)^2}{\pi} \int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - \mu_h c_l - \mu_l v_l) (v_l - u_l)^{-1-\phi} \right) \\
&\quad \left[2 \left(\frac{v_l - c_l}{v_l - u_l} \right)^\phi - 1 \right] du_l \\
&= \frac{(1 - \pi)^2}{\pi} 2 (v_l - c_l) \left(\int_{c_l}^{\bar{u}_l} \mu_l (v_l - u_l)^{-1-\phi} + (u_l - \mu_h c_l - \mu_l v_l) (v_l - u_l)^{-1-2\phi} (v_l - c_l)^{\phi-1} du_l \right) \\
&\quad - \frac{(1 - \pi)^2}{\pi} \left(\int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - \mu_h c_l - \mu_l v_l) (v_l - u_l)^{-1-\phi} \right) du_l \right) \\
&= \frac{(1 - \pi)^2}{\pi} 2 (v_l - c_l) \left(\int_{c_l}^{\bar{u}_l} \mu_l (v_l - u_l)^{-1-\phi} + (u_l - v_l + \mu_h (v_l - c_l)) (v_l - u_l)^{-1-2\phi} (v_l - c_l)^{\phi-1} du_l \right) \\
&\quad - \frac{(1 - \pi)^2}{\pi} \left(\int_{c_l}^{\bar{u}_l} \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - v_l + \mu_h (v_l - c_l)) (v_l - u_l)^{-1-\phi} \right) du_l \right)
\end{aligned}$$

The indefinite integral in the first term:

$$\begin{aligned}
& \int \left[\mu_l (v_l - u_l)^{-1-\phi} + (u_l - v_l + \mu_h (v_l - c_l)) (v_l - u_l)^{-1-2\phi} (v_l - c_l)^{\phi-1} \right] du_l \\
&= - \int \left[\mu_l z^{-1-\phi} + (-z + \mu_h (v_l - c_l)) z^{-1-2\phi} (v_l - c_l)^{\phi-1} \right] dz \\
&= - \left[-\mu_l \frac{z^{-\phi}}{\phi} - (v_l - c_l)^{\phi-1} \frac{z^{1-2\phi}}{1-2\phi} - \mu_h (v_l - c_l) (v_l - c_l)^{\phi-1} \frac{z^{-2\phi}}{2\phi} \right] \\
&= \mu_l \frac{z^{-\phi}}{\phi} + (v_l - c_l)^{\phi-1} \frac{z^{1-2\phi}}{1-2\phi} + \mu_h (v_l - c_l)^\phi \frac{z^{-2\phi}}{2\phi} \\
&= \mu_l \frac{(v_l - u_l)^{-\phi}}{\phi} + (v_l - c_l)^{\phi-1} \frac{(v_l - u_l)^{1-2\phi}}{1-2\phi} + \mu_h (v_l - c_l)^\phi \frac{(v_l - u_l)^{-2\phi}}{2\phi}
\end{aligned}$$

and in the second term

$$\begin{aligned}
&= \int \left(\mu_l (v_l - c_l)^{1-\phi} (v_l - u_l)^{-1} + (u_l - v_l + \mu_h (v_l - c_l)) (v_l - u_l)^{-1-\phi} \right) du_l \\
&= - \int \left(\mu_l (v_l - c_l)^{1-\phi} z^{-1} + (-z + \mu_h (v_l - c_l)) z^{-1-\phi} \right) dz \\
&= - \left[\mu_l (v_l - c_l)^{1-\phi} \ln z - \frac{z^{1-\phi}}{1-\phi} - \mu_h (v_l - c_l) \frac{z^{-\phi}}{\phi} \right] \\
&= -\mu_l (v_l - c_l)^{1-\phi} \ln z + \frac{z^{1-\phi}}{1-\phi} + \mu_h (v_l - c_l) \frac{z^{-\phi}}{\phi} \\
&= -\mu_l (v_l - c_l)^{1-\phi} \ln (v_l - u_l) + \frac{(v_l - u_l)^{1-\phi}}{1-\phi} + \mu_h (v_l - c_l) \frac{(v_l - u_l)^{-\phi}}{\phi}
\end{aligned}$$

Using the above two integrals and extensive algebra implies that expected value of trade is given by

$$\begin{aligned}
&= \frac{1}{\pi} (v_l - c_l)^{1-\phi} 2 \left[\mu_l \frac{(1-\pi)\pi}{\phi} + \frac{(1-\pi)^{\frac{1}{\phi}} - (1-\pi)^2}{1-2\phi} + \mu_h \frac{1 - (1-\pi)^2}{2\phi} \right] \\
&\quad - \frac{1}{\pi} (v_l - c_l)^{1-\phi} \left[-\mu_l \frac{(1-\pi)^2}{\phi} \ln(1-\pi) + \frac{(1-\pi)^{\frac{1+\phi}{\phi}} - (1-\pi)^2}{1-\phi} + \mu_h \frac{(1-\pi)\pi}{\phi} \right]
\end{aligned}$$

Using the above formula, the derivative of expected value of trade by the high quality seller

w.r.t π is given by

$$\begin{aligned}
& -\frac{1}{\pi^2} (v_l - c_l)^{1-\phi} 2 \left[\mu_l \frac{(1-\pi)\pi}{\phi} + \frac{(1-\pi)^{\frac{1}{\phi}} - (1-\pi)^2}{1-2\phi} + \mu_h \frac{1 - (1-\pi)^2}{2\phi} \right] \\
& + \frac{1}{\pi} (v_l - c_l)^{1-\phi} 2 \left[\mu_l \frac{1-2\pi}{\phi} + \frac{-\frac{1}{\phi}(1-\pi)^{\frac{1}{\phi}-1} + 2(1-\pi)}{1-2\phi} + \mu_h \frac{2(1-\pi)}{2\phi} \right] \\
& + \frac{1}{\pi^2} (v_l - c_l)^{1-\phi} \left[-\mu_l \frac{(1-\pi)^2}{\phi} \ln(1-\pi) + \frac{(1-\pi)^{\frac{1+\phi}{\phi}} - (1-\pi)^2}{1-\phi} + \mu_h \frac{(1-\pi)\pi}{\phi} \right] \\
& - \frac{1}{\pi} (v_l - c_l)^{1-\phi} \left[\mu_l \frac{2(1-\pi)}{\phi} \ln(1-\pi) + \mu_l \frac{(1-\pi)}{\phi} - \frac{\frac{1+\phi}{\phi}(1-\pi)^{\frac{1}{\phi}} - 2(1-\pi)}{1-\phi} + \mu_h \frac{1-2\pi}{\phi} \right]
\end{aligned}$$

Evaluated at $\pi = 1$, this becomes

$$\begin{aligned}
& - (v_l - c_l)^{1-\phi} \frac{\mu_h}{\phi} + (v_l - c_l)^{1-\phi} \left[\frac{-2\mu_l}{\phi} \right] - (v_l - c_l)^{1-\phi} \left[\frac{-\mu_h}{\phi} \right] \\
& = \frac{(v_l - c_l)^{1-\phi}}{\phi} [-\mu_h - 2\mu_l + \mu_h] = -2\mu_l \frac{(v_l - c_l)^{1-\phi}}{\phi} \\
& < 0
\end{aligned}$$

A.8 Proof of Theorem 17

Lemma 30. *In almost all equilibrium menus, $x_1 = 1$ and the local downward IC's are binding.*

Proof. Suppose not. Then adding $C_{i,i+1} + C_{i+1,i} > 0$ and hence $(c_{i+1} - c_i)(x_i - x_{i+1}) > 0$ or $x_{i+1} < x_i$. Consider the deviation $(x_1, \dots, \tilde{x}_{i+1}, x_{i+2}, \dots, x_N)$ and $(t_1, \dots, t_i, \tilde{t}_{i+1}, t_{i+2}, \dots, t_N)$ where $\tilde{t}_{i+1} = t_{i+1} + v_{i+1}\varepsilon$ and $\tilde{x}_{i+1} = x_{i+1} + \varepsilon$ for small $\varepsilon > 0$. Since the utility of none of the types change, this deviation can only affect $C_{i,i+1}$ and $C_{i+2,i+1}$. These are given by

$$\begin{aligned}
\tilde{C}_{i,i+1} &= u_i - u_{i+1} - (c_i - c_{i+1})(1 - \tilde{x}_{i+1}) = u_i - u_{i+1} - (c_i - c_{i+1})(1 - x_{i+1}) - \varepsilon(c_{i+1} - c_i) \\
\tilde{C}_{i+2,i+1} &= u_{i+2} - u_{i+1} - (c_{i+2} - c_{i+1})(1 - \tilde{x}_{i+1}) = u_{i+2} - u_{i+1} - (c_{i+2} - c_{i+1})(1 - x_{i+1}) + \varepsilon(c_{i+2} - c_{i+1})
\end{aligned}$$

Since the original allocation satisfies $C_{i,i+1} > 0$. If ε is small enough, the above value of $\tilde{C}_{i,i+1} \geq 0$ is satisfied. Furthermore, for $\varepsilon > 0$, $\tilde{C}_{i+2,i+1} > C_{i+2,i+1} \geq 0$. The difference between the profits from these two allocations is given by $(v_{i+1} - c_{i+1})\varepsilon > 0$. Hence there exists a profitable deviation.

Q.E.D.

Claim 31. $\forall \mathbf{u} \in \mathcal{U}, \Pi_N(\mathbf{u}_N, \mathbf{u}_{N-1}) \geq 0$.

1. **Proof:** Suppose not. Then there is a profitable deviation which involves lowering u_N . To be more precise, there are two cases:

(a) $u_N > c_N$: In this case, take the value of i such that $u_N = \dots = u_i > u_{i-1}$; when the allocation has pooling for all types set $i = 1$. Then since $\Pi_N(\mathbf{u}_N, \mathbf{u}_{N-1}) < 0$, then for

all $j > i + 1$, $\Pi_j(u_j, u_{j-1}) < 0$ and decreasing the value of u_j for $j = i, i + 1, \dots, N$ by a small amount increases profits – either some customers are lost or profits are increases which makes the deviation profitable.

(b) $u_N = c_N$: If $u_N = c_N$, then since $u_{N-1} \geq c_{N-1}$, we have

$$\begin{aligned} \Pi_N(u_N, u_{N-1}) &= v_N - \frac{v_N - c_N}{c_N - c_{N-1}}c_N + \frac{v_N - c_{N-1}}{c_N - c_{N-1}}u_{N-1} \\ &\geq v_N - \frac{v_N - c_{N-1}}{c_N - c_{N-1}}c_N + \frac{v_N - c_N}{c_N - c_{N-1}}c_{N-1} \\ &= v_N - \frac{v_N(c_N - c_{N-1})}{c_N - c_{N-1}} = 0 \end{aligned}$$

which confirms the claim.

(c) **Discussion:** Can we rule out $\Pi_N = 0$. Note that if this is the case, it must be that $u_N = c_N$ and $u_{N-1} = c_{N-1}$. Why is this? because if $u_N > c_N$ then a decline in u_N increases profits. But then if $u_N = c_N$ and $\Pi_N = 0$ it must be that $u_{N-1} = c_{N-1}$.

Claim 32. If F_N has a mass point at u_N , then it must be that for all $\mathbf{u} \in \mathcal{U}$ with $u_N = u_N$ must satisfy $\Pi_N(u_N, u_{N-1}) = 0$.

1. **Proof:** Suppose that $\Pi_N(u_N, u_{N-1}) > 0$ (From Claim 1 we cannot have $\Pi_N < 0$) and that $F_N^+(u_N) > F_N^-(u_N)$. Then there must be a positive measure of allocations where type- N customers are shared equally. Hence an infinitesimal increase in u_N attracts all of these type- N sellers and since $\Pi_N > 0$ the decline in the profits from increasing u_N is infinitesimal compared to the increase in profits from attracting a mass of type- N 's.

(a) **Corollary.** If F_N has a mass point at u_N , then it must be that $u_N = c_N$ and that for all $\mathbf{u} \in \mathcal{U}$ with $u_N = c_N$, we must have $u_{N-1} = c_{N-1}$.

Claim 33. F_N does not have a mass point.

1. **Proof.** The line of argument following the above Corollary can be extended inductively. Following the corollary, the consistency condition (??) implies that F_{N-1} must also put mass at c_{N-1} . The situation is then similar for type $N - 1$. In particular, we can show that at any \mathbf{u} with $u_{N-1} = c_{N-1}$, we must have that $\Pi_{N-1} = 0$ and $u_{N-2} = c_{N-2}$. Continuing in this way, we must have that the underlying measure, μ , must put a mass on $\mathbf{u} = (c_1, \dots, c_N)$. Then an upward deviation in u_1 yields strictly positive profits, yielding the contradiction.

Claim 34. F_N has a connected support.

1. **Proof:** Suppose that it does not. That is there exists \hat{u}_N and \tilde{u}_N such that $F_N(\hat{u}_N) = F_N(\tilde{u}_N) = F < 1$ with $\hat{u}_N < \tilde{u}_N \in \mathcal{U}^{N14}$. Then we show that it is possible to construct a profitable deviation. Note that when F_N is flat in the interval (\hat{u}_N, \tilde{u}_N) , so is \tilde{F}_N . Note, futher, that since $\hat{u}_N \geq c_N$ it must be that $\tilde{u}_N > c_N$. To show this, we consider a few options for deviations:

¹⁴I think this is an important assumption but we can probably show that it is without loss of generality. At this point I am fucking tired of thinking about it!

- (a) Suppose that $\mathbf{u} \in \mathcal{U}$ exists such that $\mathbf{u}_N = \tilde{u}_N$ with $\mathbf{u}_{N-1} < \tilde{u}_N$. Then for a small ε , the vector $\mathbf{u} - (\underbrace{0, \dots, 0}_{N-1}, \varepsilon)$ is incentive compatible and individually rational. Moreover, since this deviation does not change profits from types 1 to $N-1$ and it increases profits from type N seller without changing the measure of non-captive N -types that are attracted (recall that $\tilde{F}_N(\tilde{u}_N - \varepsilon) = \tilde{F}_N(\tilde{u}_N) = F$ for $\varepsilon > 0$ small enough).
- (b) Suppose that for all $\mathbf{u} \in \mathcal{U}$ with $\mathbf{u}_N = \tilde{u}_N$ satisfy $\mathbf{u}_{N-1} = \tilde{u}_N$. More specifically, consider the highest value of i for which $\mathbf{u} \in \mathcal{U}$ exists such that $\mathbf{u}_N = \tilde{u}_N$ and $\mathbf{u}_N = \dots = \mathbf{u}_i > \mathbf{u}_{i-1}$. There are two possibilities:
- i. There exists a $\varepsilon > 0$ small enough such that

$$[\tilde{u}_N - \varepsilon, \tilde{u}_N]^{N-i} \cap \mathcal{U}(i, \dots, N-1) = \emptyset.$$

In this case, consider the vector $\mathbf{u} - (\underbrace{0, \dots, 0}_{i-1\text{-times}}, \underbrace{\varepsilon/2, \dots, \varepsilon/2}_{N-i})$ which is incentive compatible and individually rational. As before the profits from types 1 to $i-1$ do not change since $[\tilde{u}_N - \varepsilon, \tilde{u}_N]^{N-i+1} \cap \mathcal{U}(i, \dots, N) = \emptyset$, $\tilde{F}_j(\mathbf{u})$ is flat, $\forall j = i, \dots, N$ in the interval $[\tilde{u}_N - \varepsilon, \tilde{u}_N]$ and hence this deviation increases profits.

- ii. There exists a sequence $\{\mathbf{u}^k\}_{k=1}^{\infty} \in \mathcal{U}$ whose projection onto the elements $i, \dots, N-1$ belongs to the set $(-\infty, \tilde{u}_N]^{N-i}$ and converges to \mathbf{u} . Since \tilde{F}_N is flat close to \tilde{u}_N for k high enough, it must be that $\mathbf{u}_N^k \geq \tilde{u}_N$. Optimality of \mathbf{u} implies that a feasible downward deviation of the form $\mathbf{u} - (\varepsilon_1, \dots, \varepsilon_N)$ with $\varepsilon_{N-1} = \varepsilon_N$ should not increase profits. This implies that we must have the following condition:

$$\mu_N (1 - \pi + \pi \tilde{F}_N(\tilde{u}_N)) - \underline{D}^- \tilde{\Pi}_{1, \dots, N-1}(\mathbf{u}) \leq 0 \quad (68)$$

where $\tilde{\Pi}_{1, \dots, N-1}(\cdot)$ is the total profit from types 1 to $N-1$ and \underline{D}^- is the derivative as defined below

$$\underline{D}^- \tilde{\Pi}_{1, \dots, N-1}(\mathbf{u}) = \liminf_{t \searrow 0} \frac{\sup_{\hat{\mathbf{u}} \in \mathcal{V}(t)} \tilde{\Pi}_{1, \dots, N-1}(\mathbf{u}) - \tilde{\Pi}_{1, \dots, N-1}(\hat{\mathbf{u}})}{t}$$

with

$$\mathcal{V}(t) = \{\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_{N-1}) : \hat{\mathbf{u}} \text{ satisfies } (??), (??), \hat{u}_{N-1} < \tilde{u}_N; |\tilde{u}_N - \hat{u}_{N-1}| = t\}$$

Note that deviation of the form $\mathbf{u} - (\underbrace{0, \dots, 0}_{i-1\text{-times}}, \varepsilon_i, \dots, \varepsilon_{N-1}, 0)$ for ε_j 's small enough leads to the following inequality:

$$-\mu_N (1 - \pi + \pi \tilde{F}_N(\tilde{u}_N)) \frac{v_N - c_N}{c_N - c_{N-1}} - \underline{D}^- \tilde{\Pi}_{1, \dots, N-1}(\mathbf{u}) \leq 0 \quad (69)$$

The left side of (69) is less than that of (68) and therefore it must be negative.

Since \mathbf{u}^k converges to \mathbf{u} , it must be that for some value of k high enough,

$$-\mu_N (1 - \pi + \pi \tilde{F}_N(\tilde{u}_N)) \frac{v_N - c_N}{c_N - c_{N-1}} - \frac{\tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}) - \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}^k)}{\tilde{u}_N - \mathbf{u}_{N-1}^k} < 0$$

Hence, we have

$$0 < \mu_N (1 - \pi + \pi \tilde{F}_N(\tilde{u}_N)) \frac{v_N - c_N}{c_N - c_{N-1}} (\tilde{u}_N - \mathbf{u}_{N-1}^k) + \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}) - \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}^k)$$

Since $\mathbf{u}_N^k \geq \tilde{u}_N$, we must have that

$$0 < \mu_N (1 - \pi + \pi \tilde{F}_N(\mathbf{u}_N^k)) \frac{v_N - c_N}{c_N - c_{N-1}} (\tilde{u}_N - \mathbf{u}_{N-1}^k) + \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}) - \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}^k)$$

Adding $\mu_N (1 - \pi + \pi \tilde{F}_N(\mathbf{u}_N^k)) \Pi_N(\mathbf{u}_N^k, \mathbf{u}_{N-1}^k)$ to both sides of the above inequality, we can write

$$\begin{aligned} & \mu_N (1 - \pi + \pi \tilde{F}_N(\mathbf{u}_N^k)) \Pi_N(\mathbf{u}_N^k, \mathbf{u}_{N-1}^k) + \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}^k) \\ & < \mu_N (1 - \pi + \pi \tilde{F}_N(\mathbf{u}_N^k)) \left[\frac{v_N - c_N}{c_N - c_{N-1}} (\tilde{u}_N - \mathbf{u}_{N-1}^k) + \Pi_N(\mathbf{u}_N^k, \mathbf{u}_{N-1}^k) \right] + \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u}) \end{aligned}$$

or

$$\bar{\Pi} < \mu_N (1 - \pi + \pi \tilde{F}_N(\mathbf{u}_N^k)) + \tilde{\Pi}_{1,\dots,N-1}(\mathbf{u})$$

Hence the vector $(\mathbf{u}_1, \dots, \mathbf{u}_{N-1}, \mathbf{u}_N^k)$ is a profitable deviation. This proves the claim.

Lemma 35. *Lemma: If $\{F_s\}_{s=i+1}^N$ has connected support and no mass points, then F_i has connected support.*

Proof: TO BE COMPLETED.

Lemma 36. *: If $\{F_s\}_{s=i+1}^N$ has connected support and contain no mass points, then F_i has no masspoints unless*

$$\phi_j = \mu_j \left(\frac{v_j - c_{j-1}}{c_j - c_{j-1}} \right) - \mu_{j+1} \left(\frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j} \right) = 0 \quad \forall \quad 1 \leq j \leq i$$

Strategy for proof: We first show that if F_i has a masspoint, it must be the case the equilibrium distribution puts full mass on menus with utilities u_1, u_2, \dots, u_i equal to the least cost separating (LCS) utilities, i.e.

$$\begin{aligned} u_1^{\text{LCS}} &= v_1 \\ u_j^{\text{LCS}} &= u_{j-1}^{\text{LCS}} + (1 - x_j^{\text{LCS}}) (c_j - c_{j-1}) \quad \text{where} \quad x_j^{\text{LCS}} = \frac{u_{j-1}^{\text{LCS}} - c_{j-1}}{v_j - c_{j-1}} \end{aligned}$$

We then show that, if $\phi_j \neq 0$ for some $j \leq i$, we can construct a profitable deviation.

We proceed in steps. The following two claims show that if there is a mass at u_i , then $\Pi_1 = \Pi_2 = \dots = \Pi_i = 0$ for every menu that offers u_i to type- i . This pins down the unique set of indirect utilities to the first i types that can be part of a masspoint in F_i . Then, we use supermodularity and the resulting rank preserving (weak) property to show that, if $\phi_j > 0$ for some $j \leq i$, increasing u_{j-1} raises profits. If $\phi_j < 0$, we decrease u_{j-1} .

Claim 37. *If F_i has a mass point at u_i , then for all $\mathbf{u} \in \mathcal{U}$ with $\mathbf{u}_i = u_i$ must satisfy $\Pi_i(\mathbf{u}_{i-1}, \mathbf{u}_i) = 0$.*

Proof. Suppose otherwise. If $\Pi_i(\mathbf{u}_{i-1}, \mathbf{u}_i) > 0$, then increasing \mathbf{u}_i leads to a discrete jump in the probability of winning, but the reduction in profits conditional on trade can be made arbitrarily small. Thus, $\Pi_i(\mathbf{u}_{i-1}, \mathbf{u}_i) \leq 0$. Now, to rule out $\Pi_i(\mathbf{u}_{i-1}, \mathbf{u}_i) < 0$, find the lowest $k \geq 0$ such that $\mathbf{u}_{i-k-1} < \mathbf{u}_i$. Applying the above argument, we can show that $\Pi_{i-s}(\mathbf{u}_{i-s-1}, \mathbf{u}_{i-s}) \leq 0$ for $s \leq k$. Now, decrease all the utilities i.e. $(\mathbf{u}_{i-k}, \mathbf{u}_{i-k+1}, \dots, \mathbf{u}_i)$ by $\varepsilon > 0$. As $\varepsilon \rightarrow 0$, the resulting change in profits converges to

$$\pi \sum_{s=0}^k \mu_{i-s} [F_{i-s}^-(\mathbf{u}_{i-s}) - F_{i-s}(\mathbf{u}_{i-s})] \Pi_{i-s}(\mathbf{u}_{i-s-1}, \mathbf{u}_{i-s}) > 0$$

For example, if $\mathbf{u}_{i-1} < \mathbf{u}_i$, consider a deviation which reduces \mathbf{u}_i by ε . As $\varepsilon \rightarrow 0$, the resulting change in profits converges to

$$\pi \mu_i [F_i^-(\mathbf{u}_i) - F_i(\mathbf{u}_i)] \Pi_i(\mathbf{u}_{i-1}, \mathbf{u}_i) > 0$$

QED.

Claim 38. *If F_i has a masspoint at u_i , then it must be that all $\mathbf{u} \in \mathcal{U}$ with $\mathbf{u}_i = u_i$ must satisfy $\mathbf{u}_{i-k} = u_{i-k}^{\text{LCS}}$, $k = 0, 1, 2, \dots, i-1$ where u_j^{LCS} are defined by¹⁵*

$$\begin{aligned} u_1^{\text{LCS}} &= v_1 \\ u_j^{\text{LCS}} &= u_{j-1}^{\text{LCS}} + (1 - x_j^{\text{LCS}}) (c_j - c_{j-1}) \quad \text{where} \quad x_j^{\text{LCS}} = \frac{u_{j-1}^{\text{LCS}} - c_{j-1}}{v_j - c_{j-1}} \end{aligned}$$

Proof: By the previous claim, all $\mathbf{u} \in \mathcal{U}$ with $\mathbf{u}_i = u_i$ satisfy $\Pi_i(\mathbf{u}_{i-1}, \mathbf{u}_i) = 0$, which means there must be a masspoint at in F_{i-1} at u_{i-1} , with $\Pi_i(u_{i-1}, u_i) = 0$. Then, all $\mathbf{u} \in \mathcal{U}$ with $\mathbf{u}_i = u_i$ and $\mathbf{u}_{i-1} = u_{i-1}$, we must have $\Pi_{i-1}(\mathbf{u}_{i-2}, \mathbf{u}_{i-1}) = 0$. To see why, suppose otherwise. If $\Pi_{i-1}(\mathbf{u}_{i-2}, \mathbf{u}_{i-1}) > 0$, then an arbitrarily small increase in \mathbf{u}_{i-1} yields strictly higher profits. If $\Pi_{i-1}(\mathbf{u}_{i-2}, \mathbf{u}_{i-1}) < 0$, we decrease \mathbf{u}_{i-1} by $\varepsilon > 0$. This deviation reduces (in a discrete fashion) the probability of (loss-making) trades with type- $i-1$, but leads to continuous losses from type- i . Therefore, as $\varepsilon \rightarrow 0$, it delivers strictly higher profits¹⁶. By induction, this argument leads to mass at some \mathbf{u}_1 with $\Pi_1(\mathbf{u}_1) = 0$, implying $\mathbf{u}_1 = v_1 = u_1^{\text{LCS}}$. But, since $\Pi_2(\mathbf{u}_1, \mathbf{u}_2) = 0$, $\mathbf{u}_2 = u_2^{\text{LCS}}$ and so on.

QED.

¹⁵These are the utilities associated with the least-cost separating outcome.

¹⁶Note: If $\mathbf{u}_{i-1} = \mathbf{u}_{i-2} = \dots = \mathbf{u}_{i-1-k}$, then we reduce all these utilities in a symmetric fashion.

Claim 39. *If F_i has a masspoint at u_i , then $\phi_j = 0$ for all $j \leq i$.*

Proof. Suppose otherwise. First, suppose $\phi_i < 0$. Note that since F_{i+1} has no masspoints, $U_{i+1}(u_i)$ must be an interval. Now, we increase u_i by $\varepsilon > 0$ for some menu with $u_{i+1} = u_{i+1} > \min U_{i+1}(u_i)$. (CANT WE JUST USE THE MINIMUM OF U_i HERE ?). As $\varepsilon \rightarrow 0$, the change in profits from this deviation (per ε) goes to

$$-(1 - \pi + \pi F_i(u_i)) \left(\frac{v_i - c_{i-1}}{c_i - c_{i-1}} \right) \mu_i + (1 - \pi + \pi F_{i+1}(u_{i+1})) \left(\frac{v_{i+1} - c_{i+1}}{c_{i+1} - c_i} \right) \mu_{i+1}$$

where we use the fact that F_i is right-continuous. The strict supermodularity of the profit function implies

$$F_i(u_i) \leq F_{i+1}(u_{i+1})$$

Also, since $\phi_i < 0$ by assumption, we have

$$\left(\frac{v_i - c_{i-1}}{c_i - c_{i-1}} \right) \mu_i < \left(\frac{v_{i+1} - c_{i+1}}{c_{i+1} - c_i} \right) \mu_{i+1}$$

Together, these two inequalities imply that the change in profits is strictly positive. TO BE COMPLETED.

Claim 40. *If all the F_i are continuous and strictly increasing, then $\mathbb{U}(u_1)$ is a strictly increasing function.*

Proof: Suppose otherwise. If $\mathbb{U}(u_1)$ is only weakly increasing in some element u_j over some region, that implies a masspoint at u_j , contradicting the previous claims. If it is a correspondence for some u_1 , it implies a masspoint in F_1 , again contradicting the previous results. TO BE COMPLETED.