A Tractable Model of Monetary Exchange with Ex-Post Heterogeneity*

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Abstract

We construct a continuous-time, pure currency economy with the following three key features. First, our modelled economy incorporates idiosyncratic uncertainty— households receive infrequent and random opportunities of lumpy consumption—and displays an endogenous, non-degenerate distribution of money holdings. Second, our model is tractable: properties of equilibria can be obtained analytically, and equilibria can be solved in closed form in a variety of cases. Third, our model admits as a special, limiting case the quasi-linear economy of Lagos and Wright (2005) and Rocheteau and Wright (2005). We use our modeled economy to obtain new insights on the effects of anticipated inflation on individual spending behavior, the social benefits and output effects of inflationary transfer schemes, and transitional dynamics following unanticipated monetary shocks.

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1 Introduction

We construct a continuous-time, pure currency economy with the following three key features. First, our economy incorporates idiosyncratic uncertainty and displays an endogenous, non-degenerate distribution of money holdings, as in Bewley (1980, 1983). The nature of the idiosyncratic risk is analogous to the one in random-matching monetary models (Shi, 1995; Trejos and Wright, 1995).[1] Second, our model is tractable, by which we mean that properties of equilibria can be obtained analytically, and equilibria (including value functions and distributions) can be solved in closed form in a variety of cases. Third, our model admits as a special, limiting case the quasi-linear economy of Lagos and Wright (2005) and Rocheteau and Wright (2005) that has become the workhorse paradigm for monetary theory. We use our model to revisit classical, yet topical, questions in monetary economics: the effects of anticipated inflation on individual spending behavior, the social benefits and output effects of inflationary transfer schemes, and transitional dynamics following unanticipated monetary shocks.

Pure currency economies are environments in which contracts involving intertemporal obligations are unfeasible, due to lack of monitoring and enforcement, and in which currency is the only durable object that can serve as means of payment. Despite being far remote from actual economies, they are critical constructions for our understanding of monetary exchange, and by extension of monetary policy. Following Bewley (1980, 1983), pure currency economies typically feature idiosyncratic uncertainty (in the recent literature, random matching shocks), which generates a precautionary demand for liquidity. Despite the presence of uninsurable idiosyncratic risk, the most recent and widely used models (e.g., Shi, 1997; Lagos and Wright, 2005) deliver equilibria with degenerate distributions of money holdings.[2] These models are solvable in closed form and can easily be integrated with the standard representative-agent model used in macroeconomics, which led to a variety of insights on the role of money and monetary policy.[3] Yet, this gain in tractability comes at a cost: models with degenerate distributions miss a fundamental trade-off for

[1] Precisely, households receive infrequent and random opportunities of lumpy consumption. In the Shi (1995) and Trejos and Wright (1995) models random opportunities for lumpy consumption take place in pairwise meetings and the terms of trade are determined via ex-post bargaining. Relative to Shi (1995) and Trejos and Wright (1995) the distribution of money holdings is unharnessed by assuming that money is perfectly divisible and by removing the unit upper bound on money holdings, and all markets are competitive, as in Bewley (1980, 1983) or Rocheteau and Wright (2005).

[2] In Shi's (1997) model households are composed of a large number of members who pool their money holdings in order to insure themselves against the idiosyncratic risks associated with decentralized market activities. In the Lagos and Wright (2005) model the pooling of money holdings is achieved through a competitive market that opens periodically and quasi-linear preferences that eliminate wealth effects.

[3] Some of these insights are surveyed in Williamson and Wright (2010a,b) and Nosal and Rocheteau (2011).

policy between promoting self-insurance by enhancing the rate of return of currency and providing risk sharing through transfers of money (Wallace, 2014). In the absence of ex-post heterogeneity, monetary policy is exclusively about enhancing the rate of return of currency, thereby making the Friedman rule omnipotent. Some versions of the search model that incorporate this trade-off have been studied numerically (see the literature review), but these versions are much harder to grasp due to the complex interactions between ex-post bargaining and the endogenous distribution of asset holdings.

In our model, ex-ante identical households, who enjoy consumption and leisure flows, have the possibility to trade continuously in competitive spot markets. At some random times they receive idiosyncratic preference shocks that generate utility for lumps of consumption. These spending opportunities represent large shocks that cannot be paid for by a contemporaneous income flow (e.g., health shocks, large housing repairs, and so on). Following Kocherlakota (1998) lack of enforcement and anonymity prevent households from borrowing to finance these shocks, thereby creating a role for liquidity. Because the sequences of shocks are independent across households, the model generates heterogeneous individual histories and hence, possibly, heterogeneous holdings of money.

We provide a detailed characterization of the household's consumption and saving problems under a minimal set of assumptions on preferences. We show that in equilibrium agents have a target for their real balances, which depends on their rate of time preference, the inflation rate, and the frequency of consumption opportunities. They approach this target gradually over time by saving a fraction of their labor income flow. When they are hit by a preference shock for lumpy consumption, agents deplete their money holdings in full, if their wealth is below a threshold, or partially otherwise. Given the household's optimal consumption-saving behavior, we can characterize the stationary distribution of real money holdings in the population, and we solve for the value of money, thereby establishing the existence of an equilibrium. Moreover, under zero money growth ("laissez-faire"), the steady-state monetary equilibrium is unique, and it is near-efficient when households are patient, i.e., households are better off in a monetary equilibrium than under the full-insurance allocation with slightly scaled-down labor endowments (for a definition, see Green and Zhou, 2005).

If the money growth rate is large enough, then households exhaust their money holdings periodically, as in Shi (1997) or Lagos and Wright (2005), which keeps the model tractable since real balances only depend on the timing of the most recent shock. In contrast to Shi (1997) and Lagos and Wright (2005), however, households who accumulate their money holdings slowly through time. This implies in turn that they hold different amounts of money at the time when they trade, which makes distributional considerations relevant.

We study in details the special case where households have linear preferences over consumption and labor flows. This version of the model is worth investigating for at least two reasons. First, it admits as a limit the New-Monetarist model of Lagos and Wright (2005) or Rocheteau and Wright (2005), LRW thereafter, thereby allowing for a clear comparison with the literature. Second, the LRW version allows us to isolate a single parameter, the size of households' labor endowments, $\bar{h}$, that determines the speed at which households insure themselves against preference shocks, and that parametrizes the trade-off for policy between self-insurance and risk sharing.

If labor endowments are large, there is limited ex-post heterogeneity and hence the risk-sharing benefits of lump-sum transfers of money are small relative to the distortions induced by the inflation tax on the desired level of self-insurance. In the limit, when labor endowment are infinite, the equilibrium approaches the one in LRW, with a degenerate distribution and linear value functions, where welfare and output increase with the rate of return of currency. As a result inflation is detrimental to both output and welfare.

In contrast, when labor endowments are small, the first-best level of aggregate output is implemented for an interval of positive inflation rates. Output is higher than the laissez-faire level because inflation induces the richest households in the economy to keep working in order to mitigate the erosion of their real balances. For inflation rates that are neither too low nor too high, aggregate real balances are equal to the first-best consumption level so that the only inefficiency afflicting the economy is due to imperfect risk sharing. In that case moderate inflation raises welfare by reducing the dispersion of real balances and by reallocating consumption from households with low marginal utilities to the ones with higher marginal utilities. If preferences for lumpy consumption are linear with a satiation point, as in Green and Zhou (2005), such positive inflation rates implement a first-best allocation that could not be obtained under laissez-faire.

We calibrate our model using targets from the distribution of balances of transaction accounts in the 2013 Survey of Consumer Finance (SCF) and we provide comparative statics and measures of the welfare cost of inflation. We use this calibrated example to illustrate the negative relationship between the optimal inflation rate and the size of the labor endowment, $\bar{h}$. For small values of $\bar{h}$

the optimal inflation rate is positive while for sufficiently large values of $\bar{h}$ the optimal inflation is 0—negative inflation rates are not feasible in a pure currency economy with no enforcement.

As conjectured by Wallace (2014) lump-sum transfer schemes can be too restrictive to exploit effectively the trade-off between risk sharing and self-insurance. We illustrate this point by constructing an incentive-compatible, inflationary transfer scheme that improves welfare in economies with large labor endowments. This scheme assigns a lump-sum amount of money to the poorest households, thereby improving risk sharing, and a quantity of money that increases linearly with real balances to the richest households, thereby promoting self insurance. This transfer scheme is designed to keep households' incentives to self-insure unchanged despite the positive inflation rate. As a result, it raises aggregate real balances, and it increases social welfare relative to the laissez-faire equilibrium. By providing conditions on $\bar{h}$ for when lump-sum transfer schemes or regressive schemes raise welfare relative to laissez-faire, our model provides a first step toward a characterization of optimal interventions in pure currency economies.

To conclude, we study two specifications of the model that are solvable in closed form and provide additional insights on the effects of money growth in the presence of ex-post heterogeneity. First, we assume quadratic preferences and show that policy functions are linear in real balances, which allows us to solve both steady states and transitional dynamics in closed form. A one-time increase in the money supply leads to a one-time increase in the price level and no effect on aggregate real quantities despite a redistribution of wealth across households that affects individual consumption and labor supply decisions. The mean-preserving decrease in the distribution of real balances raises society's welfare.

Second, we assume that the utility over lumpy consumption is linear and the marginal utility from lumpy consumption is stochastic. In this case agents adopt an optimal stopping rule to spend their real balances. As inflation increases, households spend their real balances more often on goods that are less valuable to them. It is a manifestation of the so-called "hot potato" effect of inflation that has proven hard to capture in models with degenerate distributions (e.g., Lagos and Rocheteau, 2005).[4]

[4] The "hot potato" effect of inflation has also been studied by Ennis (2009), Nosal (2011), and Liu, Wang, and Wright (2011).

Literature

Our model is related to the literature on incomplete markets as surveyed in Ljungqvist and Sargent (2004, chapters 16-17) and Heathcote, Storesletten, and Violante (2009). Households who are subject to idiosyncratic shocks on their endowments or preferences, but do not have access to complete markets, accumulate assets in an attempt to smooth their consumption across time and states. This asset takes the form of fiat money in Bewley (1980, 1983), physical capital in Aiyagari (1994), and private IOUs in Huggett (1993). In contrast to the incomplete-market literature, but in the tradition of monetary theory, market incompleteness is not exogenous in our model: contracts involving intertemporal obligations are not incentive-feasible due to the absence of enforcement and monitoring technologies. One implication from making the frictions that render money essential explicit is that contraction of the money supply through taxation is inconsistent with the lack of enforcement technology (Wallace, 2014). Hence, we will consider positive money growth rates throughout the paper.

Bewley economies related to ours include Scheinkman and Weiss (1986), Imrohoroglu (1992), Green and Zhou (2005), Algan, Challe, and Ragot (2011), and Dressler (2011), Lippi, Ragni, and Trachter (2014), among others. Scheinkman and Weiss (1986) describe a continuous-time environment with quasi-linear preferences, a specification close to the LRW case. They assume ex-ante heterogeneity across agents and they consider aggregate shocks on endowments, while we assume idiosyncratic preference shocks for lumpy consumption in the tradition of the search-based monetary literature. Algan, Challe, and Ragot (2011) also have a Bewley model with quasi-linear preferences but they assume discrete time and they focus on equilibria with limited ex-post heterogeneity.[5] Both Imrohoroglu (1992) and Dressler (2011) study the welfare cost of inflation. We also provide a calibration of our model and an estimate for the cost of inflation even though our contribution is mainly methodological and qualitative. Green and Zhou (2005) adopt mechanism design to investigate the efficiency property of a Bewley monetary economy with spot trade allowing for both endowment and preference shocks. They show that monetary spot trading is nearly efficient ex ante if agents are very patient, a property that also holds in our model (see our Proposition 6). Moreover, we adapt the example from Green and Zhou (2005, Section 6) to our environment in

[5] The paper by Algan, Challe, and Ragot (2011) is more closely related to our companion paper, Rocheteau, Weill, and Wong (2015), where we consider a discrete-time version of the model with quasi-linear preferences in order to study analytically the transitional dynamics following one-time money injections. A key difference is that we assume random matching and bargaining.

Section 4.2.

Our description of idiosyncratic preference shocks for lumpy consumption in a continuous-time environment is similar to the formalization in the search-theoretic models of Shi (1995) and Trejos and Wright (1995), where prices are determined through bargaining. Discrete-time search-theoretic models with bargaining and nondegenerate distribution of money holdings include Camera and Corbae (1999), Zhu (2005), Molico (2006), Chiu and Molico (2010, 2011). Relative to the literature, we describe an environment in continuous time where there is no search and bargaining. Moreover, in contrast to Chiu and Molico (2010, 2011), we do not assume the coexistence of different (centralized and decentralized) markets that open periodically.[6] Diamond and Yellin (1990), Green and Zhou (1998, 2002), Zhou (1999), and Menzio, Shi, and Sun (2013) assume price posting. In Green and Zhou (1998, 2002) and Zhou (1999) search is undirected and goods are indivisible, which leads to a continuum of steady states. In contrast, our competitive pricing is non-strategic and the laissez-faire monetary equilibrium is unique. Finally, our model is also tractable to handle money growth and inflationary transfer schemes.

Alvarez and Lippi (2013) introduce similar preference shocks for lumpy consumption as ours in a Baumol-Tobin model. Relative to Baumol-Tobin, our model has a single asset, fiat money, and households are not subject to a cash-in-advance constraint—in the absence of shocks they would not accumulate money and they would finance their flow consumption with their labor only. Moreover, we do not take the consumption path (both in terms of flows and jump sizes) as exogenous neither do we assume that labor income is exogenous.

2 The environment

Time, $t \in \mathbb{R}_+$, is continuous and goes on forever. The economy is populated with a unit measure of infinitely-lived households who discount the future at rate r. There is a single perishable consumption good produced according to a linear technology that transforms h units of labor into h units of output. Households have a finite endowment of labor per unit of time, $\bar{h} < \infty$.

Households value consumption, c, and leisure flows, ℓ, according to an increasing and concave instantaneous utility function, $u(c, \ell)$. We assume that both consumption and leisure are normal goods, that $u(c, \ell)$ is bounded above, i.e. $\sup_{c \geq 0} u(c, \bar{h}) \equiv \|u\| < \infty$, and that it is bounded

[6] Rocheteau, Weill, and Wong (2015) study a discrete-time version of the model with search and bargaining and alternating market structures. The model remains tractable and can be used to study transitional dynamics following one-time money injections.

below so that we can normalize $u(0,0) = 0$. In addition to consuming and producing in flows, households receive preference shocks that generate lumps of utility for the consumption of discrete quantities of the good. Lumpy consumption opportunities represent large shocks (e.g., replacement of durables, health events) that require immediate spending.[7] These shocks occur at Poisson arrival times, $\{T_n\}_{n=1}^{\infty}$, with intensity α. The utility of consuming y units of goods at time T_n is given by the increasing, concave, and bounded utility function, $U(y)$, and we normalize $U(0) = 0$. Taken together, the lifetime expected utility of a household can be written as:

$$\mathbb{E}\left[\int_0^{+\infty} e^{-rt} u\left(c_t, \bar{h} - h_t\right) dt + \sum_{n=1}^{\infty} e^{-rT_n} U\left(y_{T_n}\right)\right], \tag{1}$$

given some adapted and left-continuous processes for c_t, h_t, and y_t. We impose the following additional regularity conditions on households' utility functions. First, $U(y)$ is strictly increasing, strictly concave, twice continuously differentiable, and satisfies the Inada condition $U'(0) = +\infty$. Second, $u(c, \ell)$ can have either one of the following two specifications:

1. **Smooth-Inada (SI) preferences:** $u(c, \ell)$ is strictly concave, twice continuously differentiable, and satisfies Inada conditions with respect to both arguments, i.e., $u_c(0, \ell) = \infty$ and $u_c(\infty, \ell) = 0$ for all $\ell > 0$, $u_\ell(c, 0) = \infty$ for all $c > 0$;

2. **Linear preferences:** $u(c, \ell) = \min\{c, \bar{c}\} + \ell$, for some $\bar{c} \geq 0$.

The first specification facilitates the analysis because it implies smooth policy functions for households and strictly positive consumption and labor flows. The second specification corresponds to the quasi-linear preferences commonly used in monetary theory since Lagos and Wright (2005) to eliminate wealth effects and to obtain equilibria with degenerate distributions of money balances.[8] In our model, distributions will be *non*-degenerate even under quasi-linear preferences, because the feasibility constraint on labor, $h \leq \bar{h}$, will be binding for some agents in equilibrium. However, these preferences will facilitate the comparison of our model to the literature—we will obtain the model with competitive pricing from Rocheteau and Wright (2005) as a limiting case—and they will

[7] Following Shi (1995) and Trejos and Wright (1995) one could also interpret the preference shocks as random consumption opportunities in a decentralized goods market with search-and-matching frictions. For such an interpretation, see Rocheteau, Weill, and Wong (2015), or Appendix C. Finally, one could also consider lumpy production shocks that provide opportunities to produce lumpy quantities of the good.

[8] Lagos and Wright (2005) assume quasi-linear preferences of the form $u(c) + \ell$. See also Scheinkman and Weiss (1986) for similar preferences. The fully linear specification comes from Lagos and Rocheteau (2005). Wong (2014) shows in the context of a discrete-time model that the same results are maintained under a more general class of preferences.

also simplify greatly policy functions allowing us to obtain closed-form solutions for all equilibrium objects, including value functions and distributions.

In order to make money essential we assume that households cannot commit and there is no monitoring technology (Kocherlakota, 1998). As a result households cannot borrow to finance lumpy consumption since otherwise they would default on their debt. The only asset in the economy is fiat money: a perfectly recognizable, durable and intrinsically worthless object. The supply of money, denoted M_t, grows at a constant rate, $\pi \geq 0$, through lump-sum transfers to households. Trades of money and goods take place in spot competitive markets. The price of money in terms of goods is denoted ϕ_t.

Full-insurance allocations

Suppose that households pool together their labor endowments to insure themselves against the idiosyncratic preference shocks for lumpy consumption. Their maximization problem is:

$$\max_{c_t,h_t,y_t} \int_0^{+\infty} e^{-rt} \left[u\left(c_t, \bar{h} - h_t\right) + \alpha U\left(y_t\right)\right] dt \tag{2}$$

$$\text{s.t.} \quad c_t + \alpha y_t = h_t. \tag{3}$$

The paths for consumption and hours are chosen so as to maximize the household's ex-ante expected utility, (2), subject to the feasibility constraint, (3), that specifies that the aggregate consumption flows across all households, c_t, plus the lumpy consumption of the α households with a preference shock, αy_t, must equal aggregate output, h_t. Under SI preferences the solution to (2)-(3) is $(c_t, h_t, y_t) = (c^{FI}, h^{FI}, y^{FI})$ for all t, where the full-insurance allocation solves

$$u_c\left(c^{FI}, \bar{h} - h^{FI}\right) = u_\ell\left(c^{FI}, \bar{h} - h^{FI}\right) = U'\left(y^{FI}\right). \tag{4}$$

Households equalize the marginal utility of flow consumption, the marginal utility of leisure, and the marginal utility of lumpy consumption, and they pay a constant insurance premium, αy^{FI}. Under linear preferences,

$$h^{FI} = \alpha y^{FI} = \min\{\alpha y^\star, \bar{h}\}, \tag{5}$$

and $c^{FI} = 0$, where $y^\star$ is the quantity that equalizes the marginal utility of lumpy consumption and the unit marginal disutility of work, $U'\left(y^\star\right) = 1$. If labor endowments are sufficiently large, then the first-best allocation is such that households consume $y^\star$ whenever they receive a preference shock and they supply $\alpha y^\star$ of their labor endowment. If labor endowments are small, $\bar{h} < \alpha y^\star$, then $y^\star$ is not feasible, so households supply their whole labor endowment, $\bar{h}$, and they share the output equally among the α households with a desire to consume.

3 Stationary Monetary Equilibrium

In this section we study stationary, monetary equilibria featuring a constant rate of return of money, $\dot{\phi}_t/\phi_t = -\pi$, and a time-invariant distribution of real balances. We start by characterizing the consumption-saving problem of a representative household. Next, we study the stationary distribution of real balances induced by the household's optimal behavior. We then establish the existence of a stationary equilibrium. The last subsection describes a particular class of monetary equilibria for which households' policy functions and stationary distribution are easily characterized.

3.1 The household's problem

We first analyze the household's problem in a stationary equilibrium in which the inflation rate is equal to the growth rate of the money supply, $\pi \geq 0$. Let $W(z)$ denote the maximum attainable lifetime utility of a household holding z units of real balances (money balances expressed in terms of the consumption good) at the beginning of time. In a supplementary appendix we provide a formal optimality-verification argument to show that it solves the following Bellman equation:

$$W(z) = \sup \int_0^\infty e^{-(r+\alpha)t} \Big(u\left(c_t, \bar{h} - h_t\right) + \alpha \big[U\left(y_t\right) + W\left(z_t - y_t\right) \big] \Big)\, dt, \tag{6}$$

with respect to a measurable plan $\{c_t, h_t, y_t, z_t : t \geq 0\}$ and subject to:

$$z_0 = z \tag{7}$$

$$0 \leq y_t \leq z_t \tag{8}$$

$$\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon. \tag{9}$$

The effective discount factor, $e^{-(r+\alpha)t}$, in the household's objective, (6), is equal to the time discount factor, e^{-rt}, multiplied by the probability that no preference shock occurs during the time interval $[0, t)$, i.e., $\Pr\left(T_1 \geq t\right) = e^{-\alpha t}$. It multiplies the household's expected utility at time t, conditional on $T_1 \geq t$. The first term of the period utility is the utility flow of consumption and leisure, $u\left(c_t, \bar{h} - h_t\right)$. The second term is the expected utility associated with a preference shock at time t, an event occurring with Poisson intensity α, which is the sum of $U\left(y_t\right)$ from consuming a lump of y_t units of consumption good and the continuation utility $W\left(z_t - y_t\right)$ from keeping $z_t - y_t$ real balances.

Equation (7) is the initial condition for real balances, and (8) is a feasibility constraint stating that real balances must remain positive before and after a preference shock. The fact that $y_t \leq z_t$ follows from the absence of enforcement and monitoring technologies that prevent households from issuing debt. Finally, (9) is the law of motion for real balances. The rate of change in real balances is equal to the household's output flow net of consumption, $h_t - c_t$, plus the negative flow return on currency, $-\pi z$, and a flow lump-sum transfer of real balances, $\Upsilon = \pi \phi M \geq 0$. If $\pi > 0$, the real value of the lump-sum transfer depends on the value of fiat money, ϕ, which is endogenous (see Section 3.3).

Theorem 1 *There is a unique bounded solution, $W(z)$, to (6). It is strictly increasing, strictly concave, continuously differentiable over $[0, \infty)$. It is twice continuously differentiable over $(0, \infty)$, except perhaps under linear preference, when this property may fail for at most two points. Moreover,*

$$W'(0) < \frac{r+\alpha}{\bar{h}} \left(\frac{\|u\|}{r} + \alpha \frac{\|U\|}{r} \right), \quad \lim_{z \to 0} W''(z) = -\infty, \text{ and } \lim_{z \to \infty} W'(z) = 0.$$

Finally, W solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$rW(z) = \max \left\{ u(c, \bar{h} - h) + \alpha \left[U(y) + W(z-y) - W(z) \right] + W'(z) \left(h - c - \pi z + \Upsilon \right) \right\}, \tag{10}$$

with respect to $(c, h, \bar{h})$ and subject to $c \geq 0$, $0 \leq h \leq \bar{h}$ and $0 \leq y \leq z$.

The first part of Theorem 1 follows from standard dynamic programming arguments according to which the optimization problem, (6), defines a contraction mapping from the set of continuous, increasing, concave, and bounded functions into itself. The fact that $W'(\infty) = 0$ follows from concavity and boundedness. A perhaps surprising result is that $W'(0) < \infty$ even though $U'(0) = \infty$. Intuitively, a household with depleted money balances, $z = 0$, has a finite marginal utility for real balances because it has some positive time to accumulate real balances before his next opportunity for lumpy consumption, $\mathbb{E}[T_1] = 1/\alpha > 0$.

The main technical challenge in Theorem 1 is to establish that W is sufficiently smooth, i.e., it admits continuous derivatives of sufficiently high order. These properties are important to ensure that the policy functions are well behaved. For instance, under SI preferences, twice continuous differentiability of W ensures the continuous differentiability of the saving rate, and implies that the ODE (9) has a unique solution. Having well-behaved policy functions will also allow us to apply standard theorems in order to establish the existence of a unique stationary distribution of real balances, and to show that the mean of the distribution, ϕM, is continuous in $\Upsilon = \pi\phi M$, which facilitates the proof of existence of an equilibrium.[9]

The HJB equation, (10), has a standard interpretation as an asset-pricing condition. If we think of $W(z)$ as the price of an asset, the opportunity cost of holding that asset is $rW(z)$. The asset yields a utility flow, $u(c,\ell)$, and a capital gain, $U(y)+W(z-y)-W(z)$, in the event of a preference shock with Poisson arrival rate α. Finally, the value of the asset changes over time due to the accumulation of real balances, which gives the last term on the right side of (10), $W'(z_t)\dot{z}_t$.

Optimal lumpy consumption. From (10) a household chooses its optimal lumpy consumption in order to solve:

$$V(z) = \max_{0\leq y\leq z} \left\{U(y) + W(z-y)\right\}. \tag{11}$$

In words, a household chooses its level of consumption in order to maximize the sum of its current utility, $U(y)$, and its continuation utility with $z-y$ real balances, $W(z-y)$. Because $U'(0)=\infty$ but $W'(0)<\infty$, a household always finds it optimal to choose strictly positive lumpy consumption, $y(z)>0$. Hence, the first-order condition of (11) is

$$U'(y) \geq W'(z-y), \tag{12}$$

with an equality if $y=z$. The following proposition provides a detailed characterization of the solution to (12).

Proposition 1 ***(Optimal Lumpy Consumption)*** *The unique solution to (12), $y(z)$, admits the following properties:*

1. *$y(z)$ is continuous and strictly positive for any $z>0$.*

2. *Both $y(z)$ and $z-y(z)$ are increasing and satisfy $\lim_{z\to\infty} y(z) = \lim_{z\to\infty} z-y(z) = \infty$.*

3. *$y(z)=z$ if and only if $z\leq \bar{z}_1$, where $\bar{z}_1>0$ solves $U'(\bar{z}_1)=W'(0)$.*

Finally, $V(z)$, is strictly increasing, strictly concave, continuously differentiable with $V'(z) = U'[y(z)]$.

[9] We establish these smoothness properties by adapting arguments from the theory of viscosity solutions (see, e.g., Bardi and Capuzzo-Dolcetta, 1997) in order to obtain a version of HJB that does not require continuous differentiability. Based on this weaker HJB equation, we can show that W must, in fact, be continuously differentiable. We then go on to prove that, under SI preferences, W is twice continuously differentiable over $z\in(0,z)$. While this property may not hold for at most two points with linear preferences, we show later that it does hold in equilibrium over the support of the distribution of real balances.

The properties of the solution, $y(z)$, follow directly from (12). The left side of (12) is decreasing in y from $U'(0)=\infty$ to $U'(\infty)=0$ while the right side of (12) is increasing in y. Hence, there is a unique solution, $y(z)$, to (12). An increase in z reduces the marginal utility of real balances, $W'(z-y)$, leading the household to increase both its lumpy consumption, $y(z)$, and its cash-on-hand after lumpy consumption, $z-y(z)$. When real balances go to infinity, $z\to\infty$, $y(z)$ must go to infinity since otherwise $W'(z-y)$ would go to zero and $U'(y)$ would remain bounded away from zero, thereby violating (12). A similar argument applies to the cash-on-hand after lumpy consumption, $z-y(z)$. Finally, Proposition 1 shows that, as long as real balances are below some threshold $\bar{z}_1$, the household finds it optimal to deplete his real balance in full upon receiving a preference shock. This follows because the utility derived from spending a small amount of real balances, $U'(0)=\infty$, is larger than the benefit from holding onto it, $W'(0)<\infty$. By induction we can construct a sequence of threshold real-balances, $\{\bar{z}_n\}_{n=1}^{+\infty}$, such that:

$$\begin{aligned} z &\in [0,\bar{z}_1) \Longrightarrow z-y(z)=0 \\ z &\in [\bar{z}_n,\bar{z}_{n+1}) \Longrightarrow z-y(z)\in[\bar{z}_{n-1},\bar{z}_n)\,, \;\; \forall n\geq 1. \end{aligned}$$

In words, if a household's real balances belong to the interval $[\bar{z}_n,\bar{z}_{n+1})$, the post-trade real balances of the household following a preference shock, $z-y(z)$, belong to the adjacent interval, $[\bar{z}_{n-1},\bar{z}_n)$. Hence, the household is insured against n consecutive preference shocks, i.e., it would take n shocks to deplete the real balances of the household. The properties of lumpy consumption, $y(z)$, and post-trade real balances, $z-y(z)$, are illustrated in Figure 1. Finally, the properties of $V(z)$ follow directly from the concavity of the problem and an application of the envelope theorem.

Optimal saving. Next, we characterize a household's optimal saving behavior. We first define the saving rate correspondence:

$$s(z) \;\;\equiv\;\; \big\{h-c-\pi z+\Upsilon \,:\, (h,c) \text{ solves } (10)\big\}. \tag{13}$$

Proposition 2 ***(Optimal Saving Rate)*** *The saving-rate correspondence, $s(z)$, is upper hemi-continuous, convex, decreasing, strictly positive near $z=0$, and admits a unique $z^\star\in(0,\infty)$ such that $0\in s(z^\star)$.*

1. ***SI preferences.*** *The saving-rate correspondence is singled-valued, strictly decreasing, and continuously differentiable over $(0,\infty)$.*

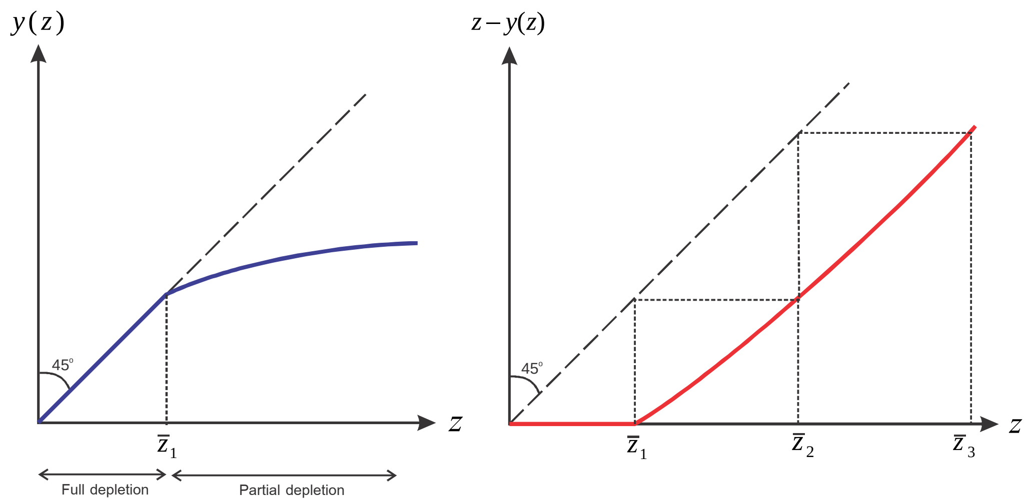

Figure 1: Left panel: Lumpy consumption. Right panel: Post-trade real balances.

2. ***Linear preferences.*** *The saving rate is equal to:*

$$s(z) = \left\{ \begin{array}{c} \bar{h} - \pi z + \Upsilon \\ [-\bar{c} - \pi z + \Upsilon, \bar{h} - \pi z + \Upsilon] \\ -\bar{c} - \pi z + \Upsilon \end{array} \right. \quad \textit{if} \;\; W'(z) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right. 1. \tag{14}$$

The first part of Proposition 2 highlights three general properties of households' saving behavior. The first property states that households save less when they hold larger real balances. This follows because flow consumption and leisure are normal goods, hence $h - c$ decreases with z, and because the inflation tax, πz, increases with z. The second property of $s(z)$ is that it is strictly positive near zero. This result follows from Theorem 1 according to which $W'(0) < \infty$. The only way the marginal utility of wealth can remain bounded near $z = 0$ is if a household with depleted money balances saves enough to keep its real balances bounded away from zero at its next preference shock. The third property is that households have a target, $z^\star$, for their real balances. For all $z < z^\star$, the saving rate is strictly positive and finite whereas at $z = z^\star$ the saving rate is zero. We prove that $z^\star < \infty$ by showing that $s(z)$ is negative for z large enough.[10]

The second part of Proposition 2 provides a tighter characterization of $s(z)$ under our two preference specifications. Under SI preferences, the HJB equation, (10), defines a strictly concave optimization problem leading to a smooth and strictly decreasing saving rate. Indeed, the first-order

[10] When inflation is strictly positive, $\pi > 0$, $s(z) < 0$ follows from the observation that $\bar{h} < \infty$ cannot offset the inflation tax, $-\pi z$, when z is large. When $\pi = \Upsilon = 0$, the result comes from Theorem 1 according to which $W'(z)$ goes to zero as z becomes large. Indeed from (10) the household's labor supply becomes zero, implying a negative saving rate.

conditions for consumption and leisure are

$$u_c(c,\ell) = u_\ell(c,\ell) = W'(z). \tag{15}$$

Given that flow consumption and leisure are assumed to be normal goods, it follows that c and ℓ increase with z and so decrease with the marginal value of real balances. Under linear preferences (10) defines a linear optimization problem delivering a bang-bang solution for the saving rate. Households work maximally and consume nothing when real balances are low enough and $W'(z) > 1$. They stop working while consuming maximally when real balance are large enough and $W'(z) < 1$.

Next, we study the time path of a household's real balances, namely, the solution to the initial value problem

$$\dot{z}_t = s\left(z_t\right) \text{ with } z_0 = 0. \tag{16}$$

Under linear preferences this problem is well defined ($s(z)$ is single-valued) for all z except when $W'(z) = 1$ in which case there are multiple optimal saving rates. For such real balances we pick the saving rate that is closest to zero. As a result if $z^\star$ is such that $W'(z^\star) = 1$, this ensures that real balance remain constant and equal to their stationary point.[11] Given the unique solution to (16), we can define the time to reach z from $z_0 = 0$:

$$\mathcal{T}(z) \equiv \inf\{t \geq 0 \,:\, z_t \geq z |\, z_0 = 0\}. \tag{17}$$

All in all, we find:

Proposition 3 ***(Optimal Path of Real Balances)*** *The initial value problem (16), has a unique solution. This solution is strictly increasing for all $t \in [0, \mathcal{T}(z^\star))$, where $z_{\mathcal{T}(z^\star)} = z^\star$, and it is constant and equal to $z^\star$ for all $t \geq \mathcal{T}(z^\star)$. Under SI preferences, $\mathcal{T}(z^\star) = \infty$. Under linear preferences, $\mathcal{T}(z^\star) < \infty$ if and only if $0 \in (-\bar{c} - \pi z^\star + \Upsilon, \bar{h} - \pi z^\star + \Upsilon)$.*

Proposition 3 shows that a household accumulates real money balances until it reaches its target $z^\star$. Under SI preferences $s(z)$ is continuously differentiable to the left of $z^\star$, which implies by a standard approximation argument that real balances only reach their target asymptotically at an exponential speed dictated by $|s'(z^\star)|$. Under linear preferences the saving rate may fail to be

[11] Under SI preferences a technical difficulty arises because $s(z)$ is not continuously differentiable at $z = 0$, and hence the standard existence and uniqueness theorems for ODEs do not apply. One can nevertheless construct the unique solution of (16) by starting the ODE at some $z > 0$ and "run it backward" until it reaches zero.

continuously differentiable at $z^\star$ and, as a result, the target may be reached in finite time. For instance, in the laissez-faire economy where $\pi = \Upsilon = 0$, the saving rate jumps downward at the target $z^\star$, i.e., $s(z) = \bar{h} > 0$ for all $z < z^\star$, while $s(z^\star) = 0$. Clearly, this implies that the target is reached in finite time $\mathcal{T}(z^\star) = z^\star/\bar{h} < +\infty$. In Figure 2 we illustrate the path for real balances and the spending behavior of a household subject to random preference shocks.

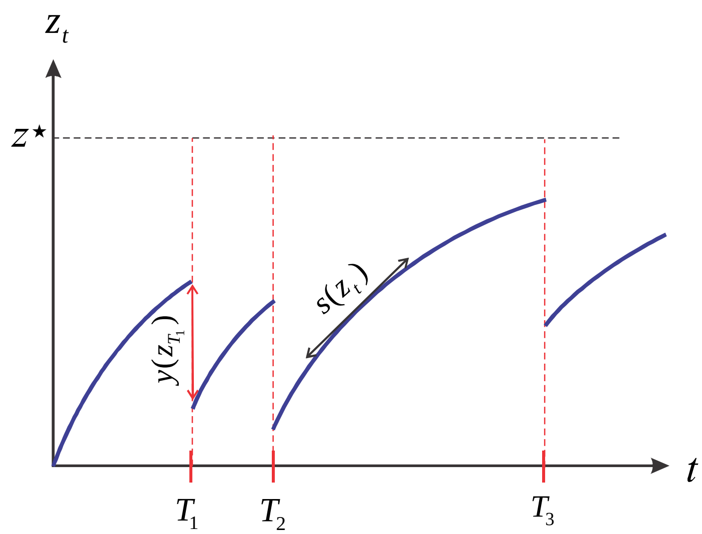

Figure 2: Optimal path of real balances

3.2 The stationary distribution of real balances

We now show that the household's policy functions, $y(z)$ and $s(z)$, induce a unique stationary distribution of real balances over the support $[0, z^\star]$. To this end, we define the minimal time that it takes for a household with z real balances at the time of a preference shock to accumulate strictly more than z' real balances following that shock:

$$\Delta(z, z') \equiv \max\left\{\mathcal{T}(z'_+) - \mathcal{T}\left[z - y(z)\right], 0\right\}, \tag{18}$$

for $z, z' \in [0, z^\star]$. Notice that $\Delta(z, z^\star) = \infty$ since the household never accumulates more than the target. Let $F(z)$ denote the cumulative distribution function of a candidate stationary equilibrium. It must solve the fixed-point equation:

$$1 - F(z') = \int_0^\infty \alpha e^{-\alpha u} \int_0^\infty \mathbb{I}_{\{u \geq \Delta(z,z')\}} dF(z)\, du = \int_0^\infty e^{-\alpha\Delta(z,z')}\, dF(z), \tag{19}$$

where the second equality is obtained by changing the order of integration. The right side of (19) calculates the measure of households with real balances strictly greater than z'. First, it partitions the population into cohorts indexed by the date of their last preference shock. There is a density measure, $\alpha e^{-\alpha u}$, of households who had their last preference shocks u periods ago. Second, in each cohort there is a fraction $dF(z)$ of households who held z real balances immediately before the shock. Those households consumed $y(z)$, which left them with $z - y(z)$ real balances. If $u \geq \Delta(z, z')$, then sufficient time has elapsed since the preference shock for their current holdings to be strictly greater than z'.

The fixed-point problem in (19) can be reduced to finding a stationary distribution of the discrete-time Markov process with transition probability function:

$$Q(z, [0, z']) \equiv 1 - e^{-\alpha \Delta(z, z')} \tag{20}$$

The function Q is the transition probability of the discrete-time Markov process that samples the real balances of a given household at the times $\{T_n\}_{n=1}^{\infty}$ of its preference shocks. It is monotone in the sense of first-order stochastic dominance, i.e., a household who had higher real balances at its last preference shock tends to have higher current real balances. One can also show that Q satisfies the Feller property, as well as an appropriate mixing condition allowing us to apply Theorem 12.12 and 12.13 in Stokey, Lucas, and Prescott (1989). We obtain:

Proposition 4 ***(Stationary Distribution of Real Balances)*** *The fixed point problem, (19), admits a unique solution, $F(z)$. This solution is continuous in the lump-sum transfer parameter, Υ, in the sense of weak-convergence.*

In addition to obtaining existence and uniqueness of a stationary distribution, Proposition 4 shows that F is continuous in Υ because all policy functions are appropriately continuous in that parameter. This continuity property is helpful to establish equilibrium existence, as it ensures that the market-clearing condition is continuous in the price of money, ϕ. Moreover, F has no mass point, except maybe at the target, $z^\star$. One might find it counterintuitive that there is no mass point at $z = 0$, given that we showed in Proposition 1 that a large flow of household, $\alpha F(\bar{z}_1)$, deplete their money holdings at each point in time. However, we also showed in Proposition 2 that this flow of households immediately accumulate real balances, $s(0) > 0$, in order to have strictly positive real balance at their next preference shock.

3.3 The real value of money

We look for a stationary equilibrium where aggregate real balances, $\phi_t M_t$, are constant over time, i.e., $\dot{\phi}_t / \phi_t = -\pi$. The household's path for real balances, z_t, depends on aggregate real balances because the real value of the lump-sum transfer received by each household is proportional to ϕM, i.e., $\Upsilon = \pi\phi M$. By definition aggregate real balances solve:

$$\phi M = \int_0^\infty z dF(z \mid \pi\phi M), \tag{21}$$

where the right side makes it explicit that the stationary distribution depends on the lump-sum transfer, $\Upsilon = \phi\pi M$. From (21) money is neutral as aggregate real balances are determined independently of M. As it is standard, however, a change in the money growth rate will have real effects by affecting the rate of return on household's savings. We are now in position to define an equilibrium.

Definition 1 *A stationary monetary equilibrium is composed of a value function, $W(z)$, a distribution of real balances, $F(z)$, and a price, $\phi > 0$, solving (6), (19), and (21).*

In order to establish the existence of an equilibrium we study (21) at its boundaries. As ϕM approaches zero the left side of (21) goes to zero, but the right side remains strictly positive. Indeed, from Proposition 2 households accumulate strictly positive real balances even when they receive no lump-sum transfer, $\Upsilon = 0$. As ϕM tends to infinity, the left side of (21) goes to infinity while the right side of (21) remains bounded because when Υ goes to infinity households supply no labor and enjoy a strictly positive consumption flow, $h = 0$ and $c > 0$, which contradicts market clearing on the goods market. Finally, Proposition 4 established that the stationary distribution, F, is continuous in $\Upsilon = \phi\pi M$. Hence, we can apply the intermediate value theorem and we obtain:

Proposition 5 ***(Existence and Uniqueness.)*** *For all $\pi \geq 0$ there exists a stationary monetary equilibrium. Moreover, the laissez-faire equilibrium, $\pi = \Upsilon = 0$, is unique.*

From Proposition 5 a monetary equilibrium exists for all inflation rates. Indeed, we showed in Proposition 2 that, as a result of the Inada condition on $U(y)$, the saving rate, $s(z)$, is always strictly positive near $z = 0$. In the laissez-faire where $\pi = \Upsilon = 0$ the equilibrium has a simple recursive structure allowing to prove uniqueness. From Theorem 1 the value and policy functions are uniquely determined independently of F. From Proposition 4, F is uniquely determined given the policy functions.

We now establish that the laissez-faire monetary equilibrium, $\pi = 0$, is nearly efficient when households are patient. We adapt Green and Zhou's (2005) definition of near efficiency as follows. An equilibrium allocation is said to be δ-efficient, for $\delta \in (0, 1]$, if it is weakly preferred ex-ante by households to the full risk-sharing allocation of the environment in which the labor endowment, $\bar{h}$, is shrunken by a factor δ.

Proposition 6 ***(Near-Efficiency of the Laissez-Faire Monetary Equilibrium)*** *For all $\delta \in (0,1)$ there is a $\bar{r}_\delta > 0$ such that for all $r < \bar{r}_\delta$ the laissez-faire monetary equilibrium is δ-efficient.*

Proposition 6 is analogous to Proposition 2 in Green and Zhou (2005) and Theorem A in Levine and Zame (2002). In the spirit of Folk Theorems for repeated games it states that patient households are as well off in the laissez-faire monetary equilibrium than under the full-insurance allocation with a slightly scaled-down labor endowment. The logic of the proof goes as follows. Suppose that households adopt the following strategy. They consume a flow consumption c^{FI}, they supply a flow labor $h < h^{FI}$, and they consume $y(z) = y^{FI}$ whenever $z \geq y^{FI}$ and $y(z) = 0$ otherwise. Households mimic the full-insurance consumption behavior whenever it is possible given their real balances, and they supply a flow of labor no greater than the first-best level. Interestingly, this strategy is almost identical to the inventory accumulation strategy in the environment described in Diamond and Yellin (1985). One can show that on average households spend a fraction $(h - c^{FI})/\alpha y^{FI}$ of their time with $z \geq y^{FI}$. As h approaches h^{FI}, households' real balances are almost always larger than y^{FI}. Hence, provided that households are very patient, the average household's utility approaches the full-insurance one.

3.4 Equilibria with full depletion of real balances

In this section we study the class of equilibria with *full depletion*, in which households find it optimal to spend all their money holdings whenever a preference shock occurs, i.e., $y(z) = z$ for all $z \in [0, z^\star]$. In this case our model becomes very tractable and we obtain a tight characterization of decision rules and distributions. We also show that full depletion occurs under appropriate parameter restrictions.

The optimal path for real balances under full depletion. The ODE for the optimal path of real balances, (9), can be rewritten as:

$$\dot{z}_t = h(\lambda_t) - c(\lambda_t) - \pi z_t + \Upsilon, \tag{22}$$

where $\lambda_t \equiv W'(z_t)$ is the marginal value of real balance, while $h(\lambda_t)$ and $c(\lambda_t)$ are the solutions to

$$\max_{c \geq 0, h \leq \bar{h}} \left\{ u(c, \bar{h} - h) + \lambda \left(h - c - \pi z + \Upsilon \right) \right\}. \tag{23}$$

To solve for λ_t we apply the envelope condition to differentiate the HJB (10) with respect to z along the optimal path of money holdings. This leads to the ODE:

$$r\lambda_t = \alpha \left[U'(z_t) - \lambda_t\right] - \pi\lambda_t + \dot{\lambda}_t, \tag{24}$$

where we used that $V'(z_t) = U'[y(z_t)] = U'(z_t)$ from Proposition 1. According to the first term on the right side of (24) a household enjoys a surplus $U'(z_t) - \lambda_t$ from spending his marginal real balances in the event of a preference shock with arrival rate α. The second term corresponds to the inflation tax that erodes the value of money at rate π, and the third term is the change in the marginal value of real balances as the household accumulates more money over time.

The pair, (z_t, λ_t), solves a system of two ODEs, (22) and (24).[12] We represent the phase diagram associated with this system in Figure 3. One can show that the stationary point of this system is a saddle point and the optimal solution to the household's problem is the associated saddle path. In the laissez-faire economy with $\pi = \Upsilon = 0$ the z-isocline is horizontal and the dynamic system is independent of the distribution of real balances.

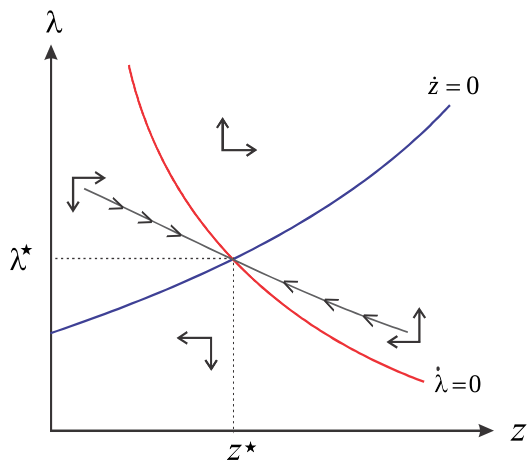

Figure 3: Phase diagram of an equilibrium with full depletion of real balances

The stationary distribution of real balances under full depletion. Under full depletion, $y(z) = z$, it follows that $\Delta(z, z') = \mathcal{T}(z'_+)$. Hence, from (20), the transition probability function,

$$Q(z, [0, z']) = 1 - e^{-\alpha \mathcal{T}(z'_+)}, \tag{25}$$

[12] A similar system of ODE would also hold under partial depletion, but $U'(z_t)$ would be replaced by $U'[y(z_t)]$. Hence, in order to solve for this system of ODE, one also needs to solve for the unknown function $y(z)$. In Appendix B, we provide a numerical solution to this problem.

does not depend on z. In words, the probability that a household holds less than z' is independent on his real balances just before his last lumpy consumption opportunity, z. This result is intuitive since households "re-start from zero" after a lumpy consumption opportunity. It then follows that the stationary probabilities must coincide with the transition probabilities, i.e.

$$F(z') = Q\left(z, [0, z']\right). \tag{26}$$

Finally, the equilibrium equation for the price level, (21), simplifies as well:

$$\phi M = \int_0^\infty z dF(z \,|\, \phi\pi M) = \int_0^\infty \left[1 - F(z \,|\, \phi\pi M)\right] dz = \int_0^{z^\star} e^{-\alpha \mathcal{T}(z \,|\, \phi\pi M)} dz, \tag{27}$$

where our notation highlights that the time to accumulate real balances, $\mathcal{T}$, is a function of the real money transfer, $\Upsilon = \phi\pi M$.

Verifying full depletion. The above two paragraphs showed how to solve for households' policy functions and for the stationary distribution of real balances, assuming that households find it optimal to deplete their money holdings in full when a lumpy consumption opportunity occurs. From the first-order condition, (12), $y(z) = z$ is optimal if and only if

$$U'(z^\star) \geq W'(0) = \lambda_0. \tag{28}$$

According to (28) the marginal utility of consumption when $y(z^\star) = z^\star$ must be greater than the marginal value of money at $z = 0$. In order to verify this condition one must solve for the equilibrium price, ϕ, and the associated real transfer, $\Upsilon = \phi\pi M$. We turn to this task in the following proposition.

Proposition 7 (Sufficient Conditions for Full Depletion) *Under either SI or linear preferences, there exists a threshold for the inflation rate, π_F, such that, for all $\pi \geq \pi_F$, all stationary monetary equilibria feature full depletion.*

Under Linear preference, there exists a threshold for the labor endowment, $\bar{h}_F$, such that, for all $\bar{h} \geq \bar{h}_F$, there exists a unique stationary monetary equilibrium, and this equilibrium must feature full depletion.

Proposition 7 identifies two conditions on exogenous parameters ensuring full depletion. If inflation is large enough, then money holdings become "hot potatoes": they depreciate quickly so that households always find it optimal to spend all their money when given the opportunity. Under linear preferences, if the labor endowment is large enough, then households spend all of their money holdings when a preference shock hits because they anticipate that they can rebuild their money inventories quickly.

4 The quasi-linear economy

In this section we provide a detailed characterization of the model under linear preferences. We describe first the laissez-faire equilibrium under a constant money supply, $\pi = 0$. We show that the model can be solved in closed form for a broad set of parameter values, and it admits at the limit when labor endowment grow very large, $\bar{h} \to \infty$, the LRW equilibrium with linear value functions and a degenerate distribution of money holdings. Next, we turn to equilibria with money growth to show analytically how the effects of inflation on output and welfare depend on $\bar{h}$. We pursue our investigation with a calibrated example where we target the distribution of the balances of transaction accounts in the 2013 Survey of Consumer Finance (SCF).

4.1 Laissez-faire

We focus on equilibria with full depletion of real balances. From (23) households choose $\dot{z} = h \leq \bar{h}$ to maximize $\dot{z}(\lambda - 1)$ where λ solves the envelope condition (24). The solution is such that $z_t = \bar{h}t$ for all $t \leq \mathcal{T}(z^\star) = z^\star/\bar{h}$, where t is the length of time since the last preference shock, and $z^\star$ is the stationary solution to (24). The marginal value of money at the target is $\lambda = 1$ because a household who keeps his real balances constant must be indifferent between working at a disutility cost of one in order to accumulate one unit of real balances worth λ and not working. From (24):

$$U'(z^\star) = 1 + \frac{r}{\alpha}. \tag{29}$$

The marginal utility of lumpy consumption is equal to the marginal disutility of labor augmented by a wedge, r/α, due to discounting. If households are more impatient, or if preference shocks are less frequent, households reduce their targeted real balances.

From (25)-(26) the steady-state distribution of real balances is a truncated exponential distribution,

$$F(z) = 1 - e^{-\frac{\alpha z}{\bar{h}}} \, \mathbb{I}_{\{z<z^\star\}} \text{ for all } z \in \mathbb{R}_+. \tag{30}$$

Note that it has a mass point at the targeted real balances, $1 - F(z^\star) = e^{-\alpha z^\star/\bar{h}}$, which is increasing with $\bar{h}$. From market clearing, (27), aggregate real balances are:

$$\phi M = \frac{\bar{h}}{\alpha}\left(1 - e^{-\frac{\alpha z^\star}{\bar{h}}}\right). \tag{31}$$

Aggregate real balances are smaller than the target, $\phi M < z^\star$, and they are increasing with the household's labor endowment. They do not depend on the nominal stock of money—money is neutral.

We now check the condition for full depletion of money balances, (28). Integrating (24) over $[t, \mathcal{T}(z^\star)]$ and using the change of variable $z = \bar{h}t$ we obtain a closed-form expression for λ as a function of z,

$$\lambda(z) = 1 + \alpha \int_z^{z^\star} e^{-\left(\frac{r+\alpha}{\bar{h}}\right)(x-z)} \left[\frac{U'(x) - U'(z^\star)}{\bar{h}}\right] dx. \tag{32}$$

The marginal value of real balances is equal to the marginal disutility of labor, one, plus a discounted sum of the differences between the marginal utility of lumpy consumption on the equilibrium path, $U'(z_t)$, and at the target, $U'(z^\star)$. It is easy to check that $\lambda'(z) < 0$, i.e., the value function is strictly concave, and as z approaches $z^\star$ the marginal value of real balances approaches one. From (32) the condition for full depletion, (28), can be expressed as

$$\frac{r}{\alpha} \geq \alpha \int_0^{\mathcal{T}(z^\star)} e^{-(r+\alpha)\tau} \left\{U'\left[z(\tau)\right] - U'(z^\star)\right\} d\tau. \tag{33}$$

The right side of (33) is monotone decreasing in $\bar{h}$ (since $\mathcal{T}(z^\star) = z^\star/\bar{h}$) and it approaches 0 as $\bar{h}$ tends to $+\infty$. So, we extend Proposition 7 by showing that the equilibrium features full depletion if and only if $\bar{h}$ is above some threshold. Alternatively, (33) holds if households are sufficiently impatient because the cost of holding money outweighs the insurance benefits from hoarding real balances. Finally, from (32) we are able to compute the value function in closed form:

$$W(z) = z + W(z^\star) - z^\star - \frac{\alpha}{r+\alpha} \int_z^{z^\star} \left[1 - e^{-\frac{(r+\alpha)(u-z)}{\bar{h}}}\right] \left[U'(u) - U'(z^\star)\right] du \quad \forall z < z^\star \tag{34}$$

$$W(z^\star) = \frac{\alpha}{r} \left\{U(z^\star) - z^\star - \frac{\alpha}{r+\alpha} \int_0^{z^\star} \left[1 - e^{-\frac{(r+\alpha)u}{\bar{h}}}\right] \left[U'(u) - U'(z^\star)\right] du\right\} \tag{35}$$

The first term on the right side of (34) is linear in wealth, which is reminiscent of the linear value function in LRW. However, the last term is strictly concave: it measures the distance between the marginal utility of consumption, $U'(y)$, and its target, $U'(z^\star) = 1$, as households accumulate real balances slowly through time.

The following proposition establishes that as households' labor endowment tends to infinity the equilibrium approaches an equilibrium with degenerate distribution and linear value function analogous to the one in LRW.[13]

[13] The proof follows directly from (30), (31), and (34), and it is therefore omitted.

Proposition 8 ***(Convergence to LRW)*** *As $\bar{h} \to \infty$ the measure of households holding $z^\star$ tends to one, the value of money approaches $z^\star/M$, and $W(z)$ converges uniformly to $z - z^\star + \alpha \left[U(z^\star) - z^\star\right]/r$.*

4.2 Output and welfare effects of inflation

We now show that our model has a rich set of predictions for the effects of inflation on output and welfare. We focus first on equilibria with full depletion, $y(z^\star) = z^\star$. (We explore equilibria with partial depletion numerically with a calibrated version of our model in the next subsection). In the presence of money growth, $\pi > 0$, the target for real balances can take two expressions depending on whether the feasibility constraint, $h(z^\star) \leq \bar{h}$, is slack or binding:

$$z^\star \equiv \min\left\{z_s, z_b\right\}, \tag{36}$$

where

$$z_s \equiv \left(U'\right)^{-1}\left(1 + \frac{r+\pi}{\alpha}\right), \quad \text{and} \quad z_b \equiv \frac{\bar{h}}{\pi} + \phi M. \tag{37}$$

The quantity z_s is the target level of real balance in a slack-labor regime that solves (24) when $\lambda = 1$ and $\dot{\lambda} = 0$. It equalizes the marginal utility of lumpy consumption, $U'(z)$, and the cost of holding real balances, $1 + (r+\pi)/\alpha$, thereby generalizing (29) by replacing r with $r + \pi$. It can be interpreted as the *ideal target* that households aim for. It is feasible to reach it only if $\bar{h} + \Upsilon \geq \pi z_s$. The quantity z_b defined by (37) is the target level of real balances in a binding-labor regime. It is the highest level of real balances that is feasible to accumulate given households' finite labor endowment, $\bar{h}$, the inflation tax on real balances, πz, and the lump-sum transfer, $\Upsilon = \pi\phi M$, i.e., it is the stationary solution to (22) with $h(\lambda) - c(\lambda) = \bar{h}$. So z_b is a *constrained target*. From (36) the *effective target*, $z^\star$, is the minimum between these two quantities. Given Υ, a higher inflation rate reduces $z^\star$. Moreover, as π goes to infinity, $z^\star$ goes to 0.

From (22) the trajectory for individual real balances is $z_t = z_b(1 - e^{-\pi t})$, and the time to reach real balance z is $\mathcal{T}(z \,|\, \pi\phi M) = -\frac{1}{\pi}\log\left(1 - \frac{z}{z_b}\right)$. Hence, if $z_b \leq z_s$ then households reach $z^\star = \min\{z_b, z_s\}$ only asymptotically, and the distribution of real balances has no mass point. In contrast, if $z_b > z_s$ then households reach $z^\star$ in finite time and the distribution has a mass point at $z = z^\star$.

Substituting the closed-form expressions for $\mathcal{T}(z \,|\, \pi\phi M)$ and z_b into the market-clearing condition, (27) we obtain after a few lines of algebra that aggregate real balances solve:

$$\frac{\phi M}{\bar{h}/\pi + \phi M} = \frac{\pi}{\alpha + \pi}\left\{1 - \left(1 - \min\left\{1, \frac{z_s}{\bar{h}/\pi + \phi M}\right\}\right)^{\frac{\alpha+\pi}{\pi}}\right\}. \tag{38}$$

Clearly, the left-hand side is strictly increasing in ϕ and the right-hand side is decreasing in ϕ. Hence, this equation has a unique solution and there is a unique candidate equilibrium with full depletion. Suppose first that $z^\star = z_b \leq z_s$. In this regime all households supply $\bar{h}$. The solution to (38) is $\phi M = \bar{h}/\alpha$. So both aggregate output, $H = \bar{h}$, and aggregate real balances are independent of the inflation rate. Substituting $\phi M = \bar{h}/\alpha$ into (37) the condition $z_b \leq z_s$ can be expressed as $\bar{h}/\pi + \bar{h}/\alpha \leq z_s$. The second regime is such that $z^\star = z_s \leq z_b$, in which case (38) has a unique solution, $\phi M \in (0, \bar{h}/\alpha]$. Finally, the condition for full depletion of money balances is given by (33) where r is replaced with $r + \pi$ and $\mathcal{T}(z \,|\, \pi\phi M) = -\frac{1}{\pi}\log\left(1 - \frac{z}{z_b}\right)$.

The following proposition shows that the effects of money growth on aggregate output and the household ex-ante welfare are qualitatively different depending on the size of $\bar{h}$, where output and welfare are defined by

$$\begin{aligned}
\mathcal{H}(\pi, \bar{h}) &\equiv \int h(z; \pi, \bar{h}) dF(z; \pi, \bar{h}) \\
\mathcal{W}(\pi, \bar{h}) &\equiv \int \left[-h(z; \pi, \bar{h}) + \alpha U(z)\right] dF(z; \pi, \bar{h}).
\end{aligned}$$

The pointwise limits for output and welfare when individual labor endowments go to infinity are denoted by $\mathcal{H}^\infty(\pi) \equiv \lim_{\bar{h}\to\infty} \mathcal{H}(\pi, \bar{h})$ and $\mathcal{W}^\infty(\pi) \equiv \lim_{\bar{h}\to\infty} \mathcal{W}(\pi, \bar{h})$.

Proposition 9 ***(Output and welfare effects of inflation.)*** *In the quasi-linear economy:*

(i) Large labor endowment. *Both $\mathcal{H}^\infty(\pi)$ and $\mathcal{W}^\infty(\pi)$ are decreasing with π.*

(ii) Small labor endowment. *If $U(z)/\left[zU'(z)\right]$ is bounded above near zero, then there exists some minimum inflation rate, $\underline{\pi}$, and a continuous function $\bar{H} : [\underline{\pi}, \infty) \to \mathbb{R}_+$ with limits $\lim_{\pi\to 0} \bar{H}(\pi) = \lim_{\pi\to\infty} \bar{H}(\pi) = 0$, such that, for all $\pi \geq \underline{\pi}$ and $h \in \left[0, \bar{H}(\pi)\right]$, there exists an equilibrium with binding labor and full depletion. In this equilibrium $\mathcal{H}(\pi, \bar{h})$ attains its first-best level, $\bar{h}$, and $\mathcal{W}(\pi, \bar{h})$ increases with π.*

(iii) Large inflation. *As $\pi \to \infty$, $\mathcal{H}(\pi, \bar{h}) \to 0$ and $\mathcal{W}(\pi, \bar{h}) \to 0$.*

The size of the labor endowment, $\bar{h}$, determines the speed at which households can reach their desired level of insurance against preference shocks by accumulating real balances. As a result, $\bar{h}$ is a key parameter to determine the relative effects of inflation on risk-sharing and self-insurance. With large labor endowments, $\bar{h} \to \infty$, there is no role for risk-sharing role as all households reach their target almost instantly. The only consideration that is relevant for policy is the negative effect of the inflation tax on the incentives to self-insure as measured by $z^\star$. Hence, aggregate output, which is approximately $\alpha z^\star$, and social welfare, approximately, $\alpha\left[U(z^\star) - z^\star\right]$, are decreasing with the inflation rate. These are the standard comparative statics in models with degenerate distributions (e.g., Lagos and Wright, 2005).

With small labor endowments, risk-sharing considerations dominate because even though π reduces $z^\star$ it takes a long time for households to reach their desired level of insurance. Indeed, in the laissez-faire the time that it takes, in the absence of any shock, to reach the target, $\mathcal{T}(z^\star) = z^\star/\bar{h}$, can be arbitrarily large when $\bar{h}$ is small. Consider the regime where the equilibrium features both full depletion, $y(z^\star) = z^\star$, and binding labor, $h(z^\star) = \bar{h}$. This regime occurs when the inflation rate is neither too low nor too high. Because households cannot reach their ideal target, z_s, they all supply $\bar{h}$ irrespective of their wealth, and aggregate output is constant and equal to $\bar{h}$. This output level is also the full-insurance one, $h^{FI} = \bar{h}$. Indeed, the condition for the binbing labor constraint is $\bar{h}/\pi + \bar{h}/\alpha \leq z_s < y^\star$, which implies $\bar{h} < \alpha y^\star$, and from (5) $h^{FI} = \bar{h}$. In addition, aggregate real balances are equal to the first-best level of consumption, $\phi M = \bar{h}/\alpha$. So, risk-sharing is the only consideration for policy as the only source of inefficiency arises from the non-degenerate distribution of real balances. Wealthy households who hold more real balances than the socially desirable level of consumption, $z > \bar{h}/\alpha$, pay a tax equal to $\pi(z - \bar{h}/\alpha)$ while the poor households who hold less real balances than the socially-desirable level of consumption, $z < \bar{h}/\alpha$, receive a subsidy equal to $\pi(\bar{h}/\alpha - z)$. Hence, moderate inflation moves individual consumption levels toward the first best, thereby smoothing consumption across households and raising household's ex-ante welfare.

Even though an equilibrium can attain the first-best level of output for intermediate inflation rates when $\bar{h}$ is sufficiently small, it fails to implement the first-best allocation because there is ex-post heterogeneity across risk-averse households and ex-post dispersion in the marginal utility over lumpy consumptions. Suppose, instead, that the utility for lumpy consumption is linear with a satiation point, $U(y) = A\min\{y, \bar{y}\}$. In this case, we can construct an equilibrium in which positive inflation leads households to work full time, $h = \bar{h}$, and in which households' lumpy consumption remain smaller than $\bar{y}$. Hence, there is no dispersion in their marginal utilities for lumpy consumptions. Just as in Green and Zhou (2005, Section 6), this economy implements the first best for positive inflation rates. To see this, we assume that $\bar{h} < \alpha\bar{y}$, which implies that first-best allocations are such that all households supply $h = \bar{h}$. Assuming that $z^\star \leq \bar{y}$, the ODE for the marginal value of real balances, (24), becomes

$$(r + \pi)\lambda_t = \alpha(A - \lambda_t) + \dot{\lambda}_t, \quad \forall t \in [0, \mathcal{T}(z^\star)], \tag{39}$$

with $\dot{\lambda}_{\mathcal{T}(z^\star)} = 0$. With Poisson arrival rate, α, the household spends all his real balances, which generates a marginal surplus equal to $A - \lambda$. The solution is $\lambda_t = \mathbb{E}\left[e^{-(r+\pi)T_1} A\right] = \alpha A/(r + \pi + \alpha)$, for all $t \in \mathbb{R}_+$. Moreover, $A > \lambda_0 = \alpha A/(r + \pi + \alpha)$ holds so that full depletion is optimal. The equilibrium features full employment, $h = \bar{h}$, if the target for real balances is $z^\star = z_b = \bar{h}(1/\pi + 1/\alpha) \leq z_s$, where $z_s \geq \bar{y}$ if $A \geq 1 + (r + \pi)/\alpha$. This leads to the following proposition.

Proposition 10 ***(Implementation of the first best.)*** *Assume $U(y) = A \min\{y, \bar{y}\}$. Sufficient conditions for a monetary equilibrium to implement a first best are:*

$$\frac{\alpha \bar{h}}{\alpha \bar{y} - \bar{h}} \leq \pi \leq \alpha\,(A - 1) - r. \tag{40}$$

Provided that the rate of time preference is not too large and labor endowment is sufficiently low, there is a range of inflation rates that implement a first-best allocation. The inflation rate cannot be too low since otherwise some households find it optimal not to work. For instance, if $\pi = 0$ households reach their target in a finite amount of time, and hence a fraction of them do not supply any labor. The inflation rate cannot too high or households will not find it optimal to self insure by accumulating real balances.

4.3 Calibrated example

In the following we pursue our investigation with a calibrated example. We no longer need to restrict our attention to equilibria featuring full depletion of real balances.[14] We normalize a unit of time to a year and we set $r = 4\%$. The average inflation rate is $\pi = 2\%$. We adopt a CRRA specification for the utility of lumpy consumption: $U(y) = y^{1-a}/(1 - a)$ over the relevant range $y \in [0, 1]$. Provided that $\bar{h} \geq \alpha$ the first-best level of lumpy consumption is 1. The remaining parameters, a, α, and $\bar{h}$, are calibrated to the distribution of the balances of transaction accounts in the 2013 SCF.[15]

[14] While equilibria with partial depletion of real balances, $y(z^\star) < z^\star$, cannot be solved analytically, they can be easily computed numerically from a system of delayed differential equations. See Appendix B for details.

[15] Transaction accounts in SCF include checking, savings, money market, and call accounts, but they do not include currency. Hence, we interpret money holdings in the model as pre-loaded (no credit involved) payment accounts mainly used for transactional purposes.

We adopt the following three targets: the ratio of the balances of the 80th-percentile household to the average balances, $F^{-1}(.8)/\phi M$,[16] the ratio of the average balances to the average income, $\phi M/H$, and the semi-elasticity of money demand, $\eta \equiv \partial \log \phi M/(100 \times \partial \pi)$. In the SCF of 2013 $F^{-1}(.8)/\phi M = 1.23$ and $\phi M/H = .39$.[17] Aruoba, Waller and Wright (2011) estimate that $\eta = -.06$. These calibration targets are matched with $a = .31$, $\alpha = 3.21$ and $\overline{h} = 6.26$. So lumpy consumption shocks occur every 3.6 months on average and the annual labor endowment is more than 6 times the first-best level of consumption in the event of a shock. In the calibrated economy, the welfare cost relative to the first best is 1.29% of total consumption.[18]

The middle and right panels in the bottom row of Figure 4 plot the distribution of real balances and the household's lumpy-consumption rule for this parametrization. The threshold below which there is full depletion, $\bar{z}_1$, is about 0.54 while the target for real balances is about $z^\star = 1.6$, i.e., the equilibrium features partial depletion. We increase the inflation rate from $\pi = 2\%$ to $\pi = 10\%$. From the bottom left panel $\bar{z}_1$ increases to 0.56 while $z^\star$ decreases to 1.2. The bottom left panel shows that $y(z)$ increases for all z. So households target lower real balances and they spend more given a level of wealth when a preference shock happens, in accordance with a 'hot potato' effect. From the top panels aggregate output and aggregate real balances decrease with inflation. The welfare cost of 10 percent inflation is about 1.27 % of total consumption.[19]

In order to illustrate the non-monotone effects of inflation on output and welfare we keep the same parametrization but we allow $\bar{h}$ to vary. In Figure 5 we distinguish four regimes: full versus partial depletion of real balances ($y(z^\star) = z^\star$ versus $y(z^\star) < z^\star$) and slack versus binding labor constraint ($h(z^\star) < \bar{h}$ versus $h(z^\star) = \bar{h}$). For sufficiently high π and sufficiently low $\bar{h}$, the equilibrium features full depletion and binding labor: this the area marked III in the figure. The lower

[16] We pick the 80th percentile as the calibration target for two reasons. First, in the data, the median of balances is much lower than the mean: the median to mean ratio is 0.11. Second, the model with linear utility tends to overstate the median since all income is saved.

[17] In SCF 2013 93.2% of all households have transaction accounts. Conditional on those having transaction accounts, the average balances are \$36.3k (in 2013 dollars). The average income is \$87k. Thus, the average balances to the average income ratio is $0.932 \times 36.3/87 = 0.3889$.

[18] The welfare loss relative to the first best with full insurance is measured by the percentage of households' consumption that a social planner would be willing to give up to have constant lumpy consumption (subject to the same output level so that it is feasible).

[19] As it is standard in the literature our measure of the welfare cost of inflation is equal to the percentage of households' consumption that a social planner would be willing to give up to have zero inflation instead of π. This measure does not take into account transitional dynamics. Our estimate of the welfare cost of inflation is consistent with the ones in discrete-time models with alternating competitive markets and ex-ante heterogeneous buyers and sellers. In the absence of distributional considerations Rocheteau and Wright (2009) found a welfare cost of 10 percent inflation equal to 1.54% of GDP. Dressler (2011) departs from quasi-linear preferences in order to obtain a non-degenerate distribution and finds a cost of inflation equal to 1.23%. Imrohoroglu (1992) in a Bewley model with income shocks found a welfare cost of inflation equal to 1.07%.

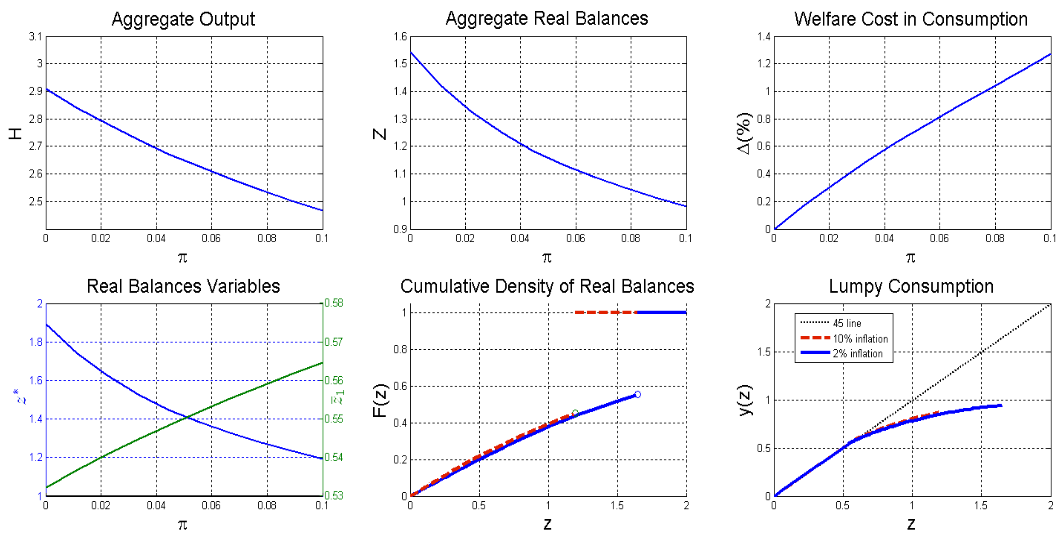

Figure 4: Calibrated example

bond for inflation and the upper boundary of this area correspond respectively to $\underline{\pi}$ and $\log\left[\bar{H}(\pi)\right]$ in Proposition 9. As π is reduced below $\underline{\pi}\approx$ 240% the equilibrium features partial depletion of real balances (areas I and II). Finally, provided that $\bar{h}$ is sufficiently large, the equilibrium features both full depletion and slack labor (area IV). The equilibria we have characterized in closed form correspond to the areas III and IV. Note that for all $\bar{h} \leq \alpha = 3.21$ (i.e., $\log \bar{h} \leq 1.14$) the first-best level of output is $h^{FI} = \bar{h}$. It is achieved in areas II and III.

Propositions 9 and 10 established that for high $\bar{h}$ inflation is detrimental to society's welfare whereas for low $\bar{h}$ positive inflation implemented with lump-sum transfers raises welfare relative to the laissez-faire. In Figure 5 we illustrate these results by plotting with a black, thick curve the welfare-maximizing inflation rate as a function of the labor endowment. As $\bar{h}$ increases the optimal inflation rate decreases, and for sufficiently high value of $\bar{h}$ the laissez-faire equilibrium dominates any equilibrium with positive inflation. Moreover, when equilibria with full depletion and binding labor exist (region III) then the optimal inflation rate is the highest one that is consistent with such an equilibrium. A higher inflation rate would relax the labor constraint (region IV) and would reduce output below its efficient level, $\bar{h}$. For values of $\bar{h}$ that are large enough such that region III does not exist, then the optimal inflation rate corresponds to an equilibrium with slack labor and partial depletion (region I). In region IV with slack labor and full depletion a reduction of the inflation rate is always welfare improving.

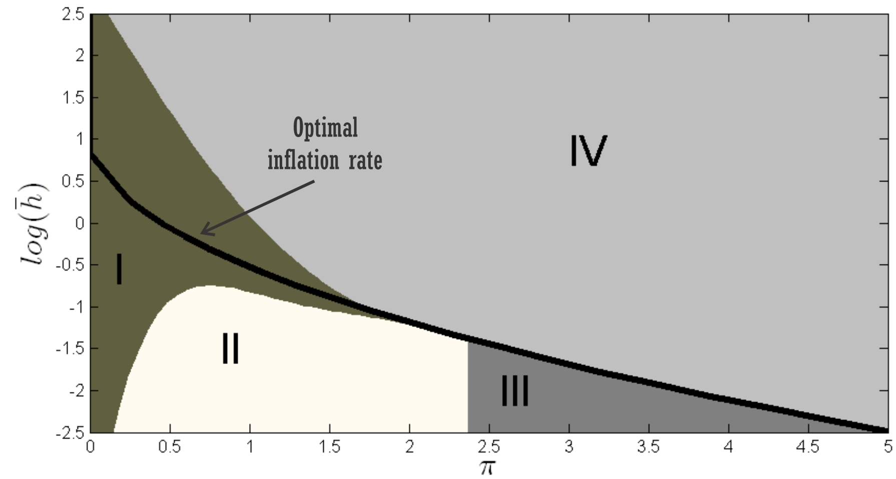

Figure 5: Region I: Slack labor & Partial depletion; Region II: Binding labor & Partial depletion; Region III: Binding labor & Full depletion; Region IV: Slack labor & Full depletion.

In the top panels of Figure 6 we plot aggregate output, real balances, and the welfare cost of inflation in the case where $\bar{h} = 0.15$ (i.e., $\log(\bar{h}) = -1.90$). As shown in Figure 5 as inflation increases the economy transitions between different regimes with different comparative statics. In the first regime with slack labor, $h(z^\star) < \bar{h}$, and partial depletion, $y(z^\star) < z^\star$, an increase in inflation leads to higher employment and output, lower aggregate real balances, and higher welfare (a negative welfare cost). If inflation increases further, then the labor constraint, $h(z^\star) \leq \bar{h}$, binds and aggregate output is independent of the inflation rate. In the third regime both aggregate output and real balances are independent of the inflation rate, but welfare is still increasing with inflation. For large inflation rates aggregate output, real balances, and welfare fall. In the bottom panels of Figure 6 we plot the same variables for a larger labor endowment, $\bar{h} = 1$. For low inflation rates the economy is in a regime with slack labor and partial depletion, and it transitions to a regime with full depletion for larger inflation rates. Except for very low inflation rates, inflation affects negatively both output and aggregate real balances. The optimal inflation rate is strictly positive, corresponding to an equilibrium featuring partial depletion of real balances. These examples illustrate the key role played by $\bar{h}$ for the output and welfare effects of inflation.

4.4 Beyond lump-sum transfers

We showed in Proposition 9 that when $\bar{h}$ is sufficiently large then inflation implemented through lump-sum transfers is welfare-worsening as the cost of lowering $z^\star$ outweighs the risk-sharing benefits

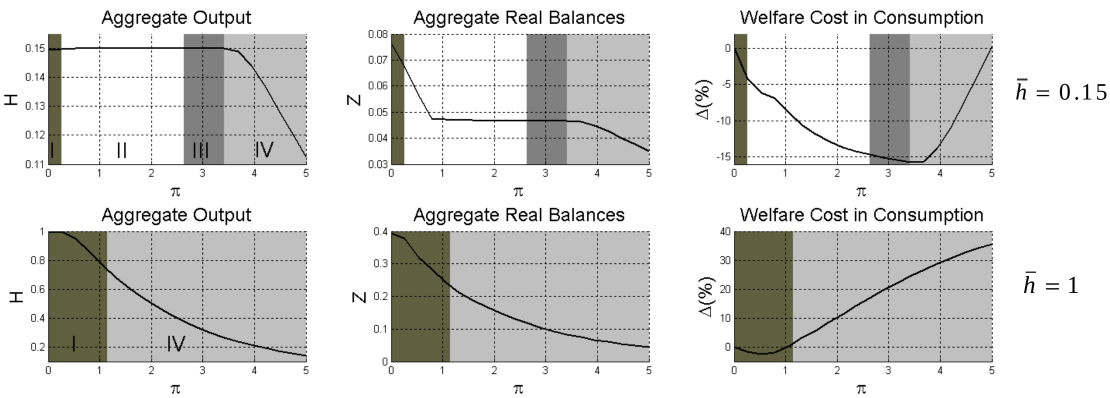

Figure 6: Non-monotonic effects of inflation on output and welfare. Top panels: $\bar{h} = 0.15$. Bottom panels: $\bar{h} = 1$.

associated with lump-sum transfers. In contrast, Wallace (2014) conjectures that money creation is almost always optimal in pure-currency economies, but it might not necessarily be produced by way of lump-sum transfers. In accordance with this conjecture, we establish in the following that inflation is optimal once one allows for more general, incentive-compatible, transfer schemes.[20] Suppose new money, $\dot{M} = \pi M$, is injected through the following transfer scheme:

$$\tau(z) = \begin{cases} \tau_0 & z \leq z_\pi^\star \\ \tau_z z - \tau_1 \quad \text{if} & z \in (z_\pi^\star, z_0^\star] \\ \pi z & z > z_0^\star \end{cases}, \tag{41}$$

where $z_\pi^\star$ solves $U'(z_\pi^\star) = 1 + (r+\pi)/\alpha$. The real transfer, $\tau(z)$, is non-negative (which is consistent with an economy with no enforcement), and it is non-decreasing so that households have no incentive to hide some of their money balances. Hence, $\tau_0 \geq 0$ and $\tau_z \geq 0$. Moreover, we assume that $\tau(z)$ is continuous, $\tau_z = (\pi z_0^\star - \tau_0)/(z_0^\star - z_\pi^\star)$ and $\tau_1 = (\pi z_\pi^\star - \tau_0)\, z_0^\star/(z_0^\star - z_\pi^\star)$. From the government budget constraint, the sum of the transfers to households net of the inflation tax must be zero, $\int [\tau(z) - \pi z]\, dF_\tau(z) = 0$, where the distribution F_τ is now indexed by the transfer scheme. Hence, $\tau_z \geq \pi$ and $\tau_1 \leq 0$. So the first tier is a lump-sum transfer, the second tier is a linear regressive transfer, and the third tier is neutral.

Our proposed scheme, illustrated in Figure 7, takes into account the trade-off between self-insurance and risk sharing in economies with non-degenerate distributions. It has a regressive component that guarantees a positive rate of return on real balances above a threshold. As a result

[20]This result is consistent with Wallace (2014) conjecture according to which in pure-currency economies with nondegenerate distribution of money, there are transfer schemes financed by money creation that improve ex ante representative-agent welfare relative to what can be achieved holding the stock of money fixed. Andolfatto (2010) shows that a regressive transfer scheme is optimal in the context of the Rocheteau and Wright (2005) model.

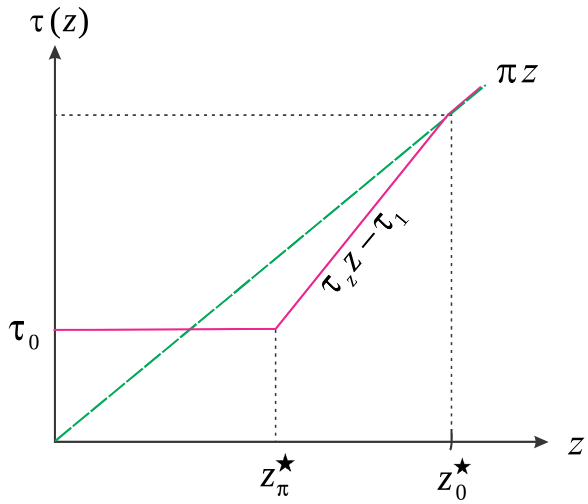

Figure 7: A socially beneficial inflationary scheme

of this component households accumulate the same amount they would accumulate in the laissez-faire equilibrium, $z_0^\star$. It has a lump-sum component that improves risk sharing by transferring wealth from the richest households to the poorest ones.

Proposition 11 ***(Socially beneficial inflation)*** *Suppose that $\bar{h}$ is sufficiently large so that (33) holds. There is a transfer scheme, $\tau(z)$ given by (41), with $\pi > 0$ that raises society's welfare relative to the laissez-faire.*

In order to prove that the transfer scheme is socially beneficial we show that it not only redistributes wealth but it also raises aggregate balances. In order to make the second claim we establish that it takes longer to accumulate $z_0^\star$ under the transfer scheme, τ, than under laissez faire. Relative to laissez faire, households accumulate real balances at a faster pace when they are poor, because $\tau(z) - \pi z > 0$, and at a slower pace when they are rich, because $\tau(z) - \pi z < 0$. Even though the sum of the net transfers across households is zero, only a fraction of the households become sufficiently rich to be net contributors to the scheme before they are hit by a new preference shock. As a result, the burden on the rich households outweighs the subsidies they received while being poor, and hence they reach their desired real balances later relative to the laissez faire. It follows that there is a larger fraction of households who are producing making aggregate real balances larger under the inflationary scheme. In summary, the transfer scheme, τ, raises society's welfare by redistributing a higher stock of real balances from rich to poor households without giving incentives to households to lower their targeted real balances.

5 Other applications

In the following we depart from the linear specification for $u(c, \bar{h} - h)$ adopted in Section 4 in order to illustrate additional insights and other tractable cases of our model. We first provide an example with quadratic preferences allowing us to characterize in closed form the transitional dynamics following a one-time money injection. Second, we assume general preferences over c and h but linear and stochastic preferences over lumpy consumption in order to discuss the effects of inflation on households' spending behavior.

5.1 Money in the short run

Suppose now that preferences are quadratic: $U(y) = Ay - y^2/2$ and $u(c, \bar{h} - h) = \varepsilon c - c^2/2 - h^2/2$.[21] From (15), and assuming interiority, the optimal choices of consumption and labor are $c_t = \varepsilon - \lambda_t$ and $h_t = \lambda_t$. Under full depletion of real balances the stationary solution to the system of ODEs, (22)-(24), is $\lambda^\star = \varepsilon/2$ and $z^\star = A - (1 + r/\alpha)\, \varepsilon/2$. We assume that $A > (1 + r/\alpha)\, \varepsilon/2$ to guarantee $z^\star > 0$. Along the saddle path trajectory

$$\lambda(z) = \frac{\nu}{2} (z - z^\star) + \lambda^\star, \tag{42}$$

where $\nu = \left(r + \alpha - \sqrt{(r+\alpha)^2 + 8\alpha} \right) /2 < 0$. It follows that the household's policy functions are:

$$c(z) = \frac{\varepsilon - \nu (z - z^\star)}{2} \tag{43}$$

$$h(z) = \frac{\varepsilon + \nu (z - z^\star)}{2}. \tag{44}$$

As households get richer their marginal value of wealth decreases, their consumption flow increases, and their supply of labor decreases. The condition for full depletion is $A - z^\star > -\nu z^\star/2 + \lambda^\star$ and $c(z)$ is interior for all z if $c(0) \geq 0$. It can be checked that the set of parameter values for which these restrictions hold is non-empty.

The saddle path of (22)-(24) is such that $z_t = z^\star \left(1 - e^{\nu t}\right)$ where t is the length of the time interval since the last preference shock. Using that t is exponentially distributed the distribution of real balances is:

$$F(z) = 1 - \left(\frac{z^\star - z}{z^\star} \right)^{-\frac{\alpha}{\nu}} \quad \text{for all } z \leq z^\star. \tag{45}$$

[21] Notice that these preferences do not satisfy the Inada conditions imposed earlier. But previous results are not needed as we are able to solve the equilibrium in closed form.

In contrast to the model of Section 4 the distribution of real balances has no mass point at $z^{\star}$ as households reach their target asymptotically. Market clearing gives

$$\phi M = \int_0^{z^{\star}} [1 - F(z)]\, dz = \frac{\nu}{\nu - \alpha} z^{\star}. \tag{46}$$

As before aggregate real balances depend on all preference parameters (r, ε, A) but not on M: money is neutral in the long run.

We now turn to the transitional dynamics following a one-time increase in the money supply, from M to γM, where $\gamma > 1$. We conjecture the existence of an equilibrium where the value of money adjusts instantly to its new steady-state value, ϕ/γ. Along the equilibrium path aggregate real balances, $Z = \phi M$, are constant. To check that our proposed equilibrium is indeed an equilibrium we show that the goods market clear at any point in time. From (43) and (44) it is easy to check that aggregate consumption is $C \equiv \int c(z) dF_t(z) + \alpha \int z dF_t(z) = [\varepsilon - \nu (Z - z^{\star})]/2 + \alpha Z$ while aggregate output is $H = \int h(z) dF_t(z) = [\varepsilon + \nu (Z - z^{\star})]/2$. From (46) it follows that $C + \alpha Z = H$, i.e., the goods market clear. The predictions of the model for aggregate quantities are consistent with the quantity theory: the price level moves in proportion to the money supply and real quantities are unaffected. So, from an aggregate viewpoint, money is neutral in the short run.[22]

However, money affects the distribution of real balances and consumption levels across households, which is relevant for welfare under strictly concave preferences. We compute society's welfare at the time of the money injection as $\int W(z) dF_0(z)$ where

$$F_0(z) = F[\gamma z - (\gamma - 1) Z]. \tag{47}$$

According to (47) the measure of households who hold less than z real balances immediately after the money injection is equal to the measure of households who were holding less than $\gamma z - (\gamma - 1)Z$ just before the shock: they received a lump-sum transfer of size $(\gamma - 1)Z$ and their real wealth is scaled down by a factor γ^{-1} due to the increase in the price level. The value function, $W(z)$, being strictly concave ($\lambda(z)$ is a decreasing function of z), the reduction in the spread of the distribution leads to an increase in welfare.

5.2 Inflation and velocity

Suppose now that $U(y) = Ay$ where A is an i.i.d. draw from some distribution $\Psi(A)$. We will use this version of the model to capture the common wisdom according to which households spend

[22] In Rocheteau, Weill, and Wong (2015) we study transitional dynamic following one-time money injections in a discrete-time version of our model with search and bargaining and linear preferences. We show that the money injection affects the rate of return of money, aggregate real balances, and output levels.

their real balances faster on less valuable commodities as inflation increases, thereby generating a misallocation of resources.[23]

We conjecture that $W(z)$ is linear with slope λ. The HJB equation, (10), becomes:

$$rW(z) = \max_{c,h} \left\{ u(c, \bar{h} - h) + \alpha \int V(z) + \lambda(h - c - \pi z + \Upsilon) \right\} \tag{48}$$

where $V(z) \equiv \int V(z, A) d\Psi(A)$ with

$$V(z, A) \equiv \max_{0 \leq y \leq z} \{Ay + W(z - y)\} = \max_{0 \leq y \leq z} (A - \lambda)\, y + W(z). \tag{49}$$

From (49) the household spends all his real balances whenever $A > \lambda$. Differentiating (48) and using that $V'(z) - \lambda = \int_{\lambda}^{\bar{A}} (A - \lambda) d\Psi(A)$, λ solves:

$$(r + \pi)\, \lambda = \alpha \left[\int_{\lambda}^{\bar{A}} (A - \lambda)\, d\Psi(A) \right] = \alpha \int_{\lambda}^{\bar{A}} [1 - \Psi(A)]\, dA. \tag{50}$$

Equation (50) has the interpretation of an optimal stopping rule. According to the left side of (50) by spending his real balances the household saves the opportunity cost of holding money as measured by $r + \pi$. According to the middle term in (50) if the household does not spend his real balances, then he has to wait for the next preference shock with $A \geq \lambda$. Such a shock occurs with Poisson arrival rate $\alpha\, [1 - \Psi(\lambda)]$, in which case the expected surplus from spending one unit of real balances is $\mathbb{E}\, [A - \lambda |\, A \geq \lambda] = \int_{\lambda}^{\bar{A}} (A - \lambda)\, d\Psi(A) / [1 - \Psi(\lambda)]$. Finally, the right side of (50) is obtained by integration by parts. It is easy to check that there is a unique, $\lambda^{\star}$, solution to (50), and this solution is independent of the household's real balances as initially guessed. As inflation increases $\lambda^{\star}$ decreases and, in accordance with a "hot potato" effect, households spend their money holdings on goods for which they have a lower marginal utility of consumption. Given $\lambda^{\star}$ the flow of consumption, $c^{\star}$, and hours, $h^{\star}$, are given by (15).

The real balances of a household who depleted his money holdings t periods ago are $z_t = (h^{\star} - c^{\star} + \pi\phi M)\, (1 - e^{-\pi t})/\pi$. The probability that a household does not receive a preference shock with $A \geq \lambda^{\star}$ over a time interval of length t is $e^{-\alpha[1 - \Psi(\lambda^{\star})]t}$. Consequently,

$$F(z) = 1 - \left[\frac{h^{\star} - c^{\star} + \pi(\phi M - z)}{h^{\star} - c^{\star} + \pi\phi M} \right]^{\frac{\alpha[1-\Psi(\lambda)]}{\pi}} \quad \text{for all } z \leq \frac{h^{\star} - c^{\star} + \pi\phi M}{\pi}. \tag{51}$$

By market clearing, (21),

$$\phi M = \frac{h^{\star} - c^{\star}}{\alpha\, [1 - \Psi(\lambda^{\star})]}. \tag{52}$$

[23] This wisdom has proved difficult to formalize in models with degenerate distributions. See Lagos and Rocheteau (2005), Ennis (2009), Liu, Wang, and Wright (2011), and Nosal (2011) for several attempts to generate a 'hot potato' effect in this class of models.

Aggregate real balances fall with inflation because households save less, $h^\star - c^\star$ is lower, and because they spend their real balances more rapidly, $\alpha\left[1-\Psi(\lambda^\star)\right]$ increases. The velocity of money, denoted $\mathcal{V}$, is defined as nominal aggregate output divided by the stock of money. From (52),

$$\mathcal{V} \equiv \frac{h^\star}{\phi M} = \frac{\alpha\left[1-\Psi(\lambda^\star)\right]}{1-\frac{c^\star}{h^\star}}. \tag{53}$$

The velocity of money increases with inflation for two reasons: households spend their real balances more often following preference shocks, $1-\Psi(\lambda^\star)$ increases, and the saving rate, $(h^\star - c^\star)/h^\star$, decreases. A monetary equilibrium exists if $h^\star - c^\star > 0$, which holds if the inflation rate is not too large and the preference shocks are sufficiently frequent.

Finally, if preferences over flow consumption and leisure are also linear, then all households supply $\bar{h}$ provided that $\pi < \alpha \int_1^{\bar{A}} \left[1-\Psi(A)\right] dA - r$. So inflation has not effect on aggregate output. Welfare at a steady-state monetary equilibrium is

$$\mathcal{W} = \alpha \int\int AzdF(z)d\Psi(A) - \bar{h} = \bar{h}\left[\frac{\int_{\lambda^\star}^{\bar{A}} Ad\Psi(A)}{1-\Psi(\lambda^\star)} - 1\right].$$

It is increasing with $\lambda^\star$ and hence decreasing with π. As inflation increases output is consumed by households with lower marginal utilities, which reduces social welfare.

6 Conclusion

We constructed a continuous-time, pure currency economy in which households are subject to idiosyncratic preference shocks for lumpy consumption. We offered a complete characterization of steady-state equilibria for general preferences and we proved existence of equilibrium. We provided closed-form solutions for a class of equilibria where households deplete their money holdings in full periodically and for special classes of preferences. We studied both analytically and numerically a version of our economy with quasi-linear preferences resembling the New-Monetarist framework of Lagos and Wright (2005) and Rocheteau and Wright (2005). In the presence of finite labor endowments the equilibrium of this economy features a non-degenerate distribution of real balances and a trade-off for policy between self-insurance and risk sharing parameterized by the size of labor endowments. The model has a variety of new insights, including non-monotonic effects of inflation on output and welfare and the optimality of inflationary transfer schemes.

Due to its tractability our model is amenable to various extensions. One can introduce search and bargaining and study transitional dynamics following monetary shocks (Rocheteau, Weill, and

Wong, 2015). One can study a version of the model with liquid and illiquid assets (i.e., assets that cannot be used to finance lumpy consumption) and their implications for asset prices. Finally, while our focus has been on pure currency economy one could replace currency with private IOUs under limited commitment (e.g., as in Huggett, 1993) to study the dynamics of debt accumulation and inside money when IOUs can also be used as means of payment.

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Appendix A: Proofs of Propositions

PROOF OF THEOREM 1. The Theorem summarizes a series of results from the Supplementary Appendix. Lemma II.3 shows that the Bellman equation has a unique bounded solution, and that this solution is concave, continuous, and increasing. Lemma II.5 shows that the value function is strictly increasing. Lemma II.8 shows that the value function is a viscosity solution of the HJB equation. Proposition II.11 uses this result to show that the value function is continuously differentiable, and that its derivative is strictly decreasing. This implies that the value function is strictly concave, that it is twice continuously differentiable almost everywhere, and that it is a classical solution of the HJB equation, i.e., that it satisfies (10).

Proposition II.14 shows that the value function is twice continuously differentiable in a neighborhood of any $z > 0$, except perhaps if saving rate is zero, and under linear preferences if $W'(z) = 1$. Since there is a unique level of real balance such that the saving rate is zero (see Proposition 2), and since $W'(z)$ is strictly decreasing, this means that the value function is twice continuously differentiable except for two levels of real balance. Under SI preferences, Proposition IV.4 shows that the value function is twice continuously differentiability obtains even when the saving rate is zero. Under linear preferences, Lemma VI.6 shows that, in equilibrium, the value function is twice continuously differentiable over the support of real balance.

To derive the bound on the derivative at $z = 0$, consider some small $\varepsilon > 0$. By working full time, $h_t = \bar{h}$ and consuming nothing, the household can reach ε at time T_ε solving $z_{T_\varepsilon} = \varepsilon$, where $\dot{z}_t = \bar{h} + \Upsilon - \pi z_t$. Solving this ODE explicitly shows that

$$T_\varepsilon = -\frac{1}{\pi} \log\left(1 - \frac{\pi}{\bar{h} + \Upsilon}\varepsilon\right) = \frac{\varepsilon}{\bar{h} + \Upsilon} + o(\varepsilon). \tag{54}$$

Clearly, since utility flows are bounded below by zero, we must have that $W(0) \geq e^{-(r+\alpha)T_\varepsilon} W(\varepsilon)$, which implies in turn that:

$$0 \leq W(\varepsilon) - W(0) \leq \left(1 - e^{-(r+\alpha)T_\varepsilon}\right) W(\varepsilon)$$

Dividing both side by ε and taking the limit $\varepsilon \to 0$, we obtain that:

$$W'(0) \leq \frac{r + \alpha}{\bar{h} + \Upsilon} W(0).$$

By taking the sup norm on both side of the Bellman equation we obtain that $r\|W\| \leq \|u\| + \alpha\|U\|$, and the result follow. Finally, the result that $\lim_{z\to\infty} W'(z) = 0$ follow from the fact that $W(z)$

is concave and bounded (see Corollary II.4 in the Supplementary Appendix), and the result that $\lim_{z\to 0} W''(z) = \infty$ is shown in Corollary II.17 of the Supplementary Appendix. ■

PROOF OF PROPOSITION 1. The results follow directly because $U(y)$ and $W(z)$ are both strictly concave and continuously differentiable, because $U'(0) = \infty$ while $W'(0) < \infty$, and because $U'(\infty) = W'(\infty) = 0$. ■

PROOF OF PROPOSITION 2. Note that the saving rate correspondence can be written as $s(z) = h - c - \pi z + \Upsilon$, where $(h, c) \in X(\lambda)$ and $X(\lambda) = \arg\max \left\{u(c, \bar{h} - h) + \lambda\,(h - c - \pi z + \Upsilon)\right\}$, with respect to $c \geq 0$, $0 \leq h \leq \bar{h}$. With linear preferences, $X(\lambda) = (\bar{c}, 0)$ if $\lambda < 1$, $X(\lambda) = [0, \bar{c}] \times [0, \bar{h}]$ if $\lambda = 1$ and $X(\lambda) = (0, \bar{h})$ if $\lambda > 1$. With SI preferences, one easily verify that $X(\lambda)$ is singled-valued and continuous, that the optimal consumption choice, $c(\lambda)$, is strictly decreasing, and that the optimal labor choice, $h(\lambda)$, is increasing (see Lemma I.2 in the Supplementary Appendix for details). Combined with the fact, established in Theorem 1, that $W'(z)$ is strictly decreasing and continuous, all the statements of the Lemma follow except for $s(z) > 0$.

To establish that $s(z) > 0$ near zero, recall that the value function is twice differentiable almost everywhere. Consider any $z > 0$ such that $W''(z)$ exists. Then, we can apply the envelope condition to the right-side of the HJB equation (see Theorem 1 in Milgrom and Segal, 2002). We obtain that:

$$(r + \alpha + \pi)W'(z) = \alpha V'(z) + W''(z)s(z).$$

From Proposition 1 we have that $V'(z) = U'\left[y(z)\right]$. Since $y(z) \leq z$ and $\lim_{z\to 0} U'(z) = \infty$, it follows that $\lim_{z\to 0} V'(z) = \infty$ as well. Since $W'(0) < \infty$ this implies, together with the above equation, that $\lim_{z\to 0} W''(z)s(z) = -\infty$. Since $W''(z) \leq 0$, it then follows that $s(z) > 0$ for some z close enough to zero. Since $s(z)$ is decreasing, it follows that $s(z) > 0$ for all z close enough to zero. ■

PROOF OF PROPOSITION 3. The proof from results in two sections of the Supplementary Appendix: Section IV.3, which studies the initial value problem in the case of SI preferences, and Section V.2, which explicitly solve for the solution of this problem in the case of linear preferences. ■

PROOF OF PROPOSITION 4. See Section VI.1 of the Supplementary Appendix for the detailed application of Theorem 12.12 and 12.13 in Stokey, Lucas, and Prescott (1989). ■

Lemma 1 *At the target level of real balance, $z^\star$:*

$$W'(z^\star) = \frac{\alpha}{r + \alpha + \pi} V'(z^\star).$$

PROOF OF LEMMA 1. Note that, at $z^\star$, there exists some optimal consumption and labor choices, $(c^\star, h^\star)$ such that $h^\star - c^\star - \pi z^\star + \Upsilon$. Hence, from the HJB:

$$\begin{aligned}
(r+\alpha)W(z) \quad &\geq u(c^\star, \bar{h} - h^\star) + \alpha V(z) + W'(z)\left(h^\star - c^\star - \pi z + \Upsilon\right) \\
(r+\alpha)W(z^\star) \quad &= u(c^\star, \bar{h} - h^\star) + \alpha V(z^\star),
\end{aligned}$$

where the inequality in the first equation follows because we evaluate the right-side of the HJB at a point that may not be achieve the maximum. Taking the difference between these two equations, and recalling that $h(z^\star) - c(z^\star) - \pi z^\star + \Upsilon = 0$, we obtain that:

$$(r+\alpha)\left[W(z) - W(z^\star)\right] \geq \alpha\left[V(z) - V(z^\star)\right] - \pi W'(z)\left(z - z^\star\right)$$

The result follows by dividing both sides by $z - z^\star$, for $z > z^\star$ and then for $z < z^\star$, and taking the limit as $z \to z^\star$, keeping in mind that $V(z)$ is differentiable at $z^\star$ and that $W(z)$ is continuously differentiable. ■

Lemma 2 *Under either SI or linear preferences, for* $z \in [0, z^\star]$*:*

$$W'(z) = \frac{\alpha}{r+\alpha+\pi}\int_z^{z^\star} V'(x)dG(x\,|\,z), \ \ \textit{where } G(x\,|\,z) \equiv 1 - e^{-(r+\alpha+\pi)[\mathcal{T}(x_+) - \mathcal{T}(z)]}$$

PROOF OF LEMMA 2. First, recall from Theorem 1 that the value function is twice continuously differentiable over $(0,\infty)$, except perhaps under linear preferences, when this property may not hold for at most two points. Hence, we can take derivatives on the right-side of the HJB equation along the path of real balance z_t, except perhaps at two points. Applying the envelope condition, we obtain that:

$$(r+\alpha+\pi)W'(z_t) \quad = \alpha V'(z_t) + W''(z_t)\,\dot{z}_t$$

if $z_t < z^\star$, except perhaps at two points. At $z = z^\star$, Lemma 1 shows that

$$(r+\alpha+\pi)W'(z^\star) = \alpha V'(z^\star)$$

In all cases we can integrate this formula forward over the time interval $[t, \mathcal{T}(z^\star)]$ and we obtain that:

$$W'(z_t) = \int_t^{\mathcal{T}(z^\star)} \alpha V'(z_s))\, e^{-(r+\alpha+\pi)(s-t)}\, ds + e^{-(r+\alpha+\pi)[\mathcal{T}(z^\star) - t]}\frac{\alpha V'(z^\star)}{r+\alpha+\pi} \tag{55}$$

Consider the first integral in equation (55). The inverse of z_t when restricted to the time interval $[0, \mathcal{T}(z^\star)]$ is the strictly increasing function $\mathcal{T}(x)$, the time to reach the real balance x starting from time zero. Let $M(x) \equiv 1 - e^{-(r+\alpha+\pi)[\mathcal{T}(x)-t]}$ and note that $M \circ z(s) = 1 - e^{-(r+\alpha+\pi)(s-t)}$. With these notations, the first integral can be written:

$$\int_t^{\mathcal{T}(z^\star)} \alpha V'\left[z_s\right] e^{-(r+\alpha+\pi)(s-t)}\, ds = \tfrac{\alpha}{r+\alpha+\pi} \int_t^{\mathcal{T}(z^\star)} V' \circ z_s \, d\left[M \circ z_s\right]$$
$$= \tfrac{\alpha}{r+\alpha+\pi} \int_{x\in[z,z^\star]} V'(x)\, dM(x) = \tfrac{\alpha}{r+\alpha+\pi} \int_{x\in[z,z^\star)} V'(x)\, dG(x \,|\, z).$$

where the second equality follows by an application of the change of variable formula for Lebesgue-Stieltjes integral (see Carter and van Brunt, 2000, Theorem 6.2.1), and the second line follows because $G(x \,|\, z) = M(x)$ for all $x \in [z, z^\star)$. The result follows by noting that the second integral can be written: $\frac{\alpha V'(z^\star)}{r+\alpha+\pi} \times \left[G(z^\star \,|\, z) - G(z^\star_- \,|\, z)\right]$. ■

PROOF OF PROPOSITION 7. To establish the first point of the Proposition, we note that, at the target $z^\star$,

$$0 = h^\star - c^\star - \pi z^\star + \Upsilon.$$

where $(c^\star, h^\star)$ are optimal consumption and labor choices when $z = z^\star$. Since, in equilibrium, $\Upsilon = \pi \int_0^{z^\star} z dF(z) < z^\star$, we obtain that $h^\star - c^\star > 0$. This implies that the marginal value of real balance satisfies $W'(z^\star) \geq \underline{\lambda} > 0$, where the constant $\underline{\lambda}$ is independent of the rate of inflation, π. Namely, with linear preferences, $\underline{\lambda} = 1$. With SI preferences, $\underline{\lambda}$ solves $h(\underline{\lambda}) - c(\underline{\lambda}) = 0$. Next, we use Lemma 1:

$$(r + \alpha + \pi) W'(z^\star) = U'\left[y(z^\star)\right].$$

Since $W'(z^\star) \geq \underline{\lambda}$, this implies that $\lim_{\pi\to\infty} y(z^\star) = 0$. Finally, since we have established in Theorem 1 that $W'(0) \leq (r+\alpha)/\overline{h} \times (\|u\| + \alpha \|U\|)/r$, we obtain that $W'(0) < U'\left[y(z^\star)\right]$ for π large enough. Therefore, the solution of the optimal lumpy consumption problem is $y(z^\star) = z^\star$, i.e., there is full depletion. We conclude that $\lim_{\pi\to\infty} z^\star = \lim_{\pi\to\infty} y(z^\star) = 0$. Finally aggregate output is, by market clearing, equal to $\alpha \int_0^{z^\star} y(z) dF(z)$, which is less than $z^\star$. Hence, aggregate output also goes to zero as $\pi \to \infty$.

The second part of the Proposition, which deals with linear preference, requires some notations and results from Section 4. The proof can be found at the beginning of the proof of Proposition 9, in the paragraph “(i) Large labor endowments". ■

PROOF OF PROPOSITION 6.

Recall from (2)-(3) that the full-insurance allocation is determined by:

$$(c^{FI}, y^{FI}, h^{FI}) \in \arg\max \left\{u(c, \bar{h} - h) + \alpha U(y)\right\}$$
$$\text{s.t.} \quad \alpha y + c = h \quad \text{and} \quad h \leq \bar{h}.$$

We propose arbitrary strategies for households and evaluate these strategies in order to obtain a lower bound for their lifetime expected utility in a laissez-faire monetary equilibrium, $W(z)$. The strategy we are considering for the accumulation of real balances replicates the inventory-accumulation strategy in Diamond and Yellin (1985). Households consume a constant y units of lumpy consumption whenever their real balances allow them to do so, $z \geq y$, and nothing otherwise. Moreover, their flow consumption is constant and equal to c, and their flow labor supply is constant and equal to h. Their flow saving rate is $\dot{z} = s = h - c > 0$. Denote $u = u(c, \bar{h} - h)$ and $U = U(y)$. Notice also that this strategy coincides with the one sustaining the full-insurance allocation, provided that (c, h, y) is appropriately chosen, except when $z < y$.

The value of a household following that strategy starting with z real balances is:

$$\tilde{W}(z) = \int_0^{+\infty} e^{-rt} \left(u + \alpha U \int_y^{+\infty} p(z', t; z) dz'\right) dt, \tag{56}$$

where $p(z', t; z)$ is the probability that the households has z' real balances at time t starting from z at $t = 0$. From (56) the household enjoys a constant utility flow, u, and the utility of lumpy consumption, U, with probability αdt, if his real balances at time t are greater than y, with probability $\int_y^{+\infty} p(z', t; z) dz'$. The household value function can be rewritten as:

$$\tilde{W}(z) = \frac{u}{r} + \alpha U \int_0^{+\infty} e^{-rt} \int_y^{+\infty} p(z', t; z) dz' dt. \tag{57}$$

Except for the first term, this corresponds to (4.2) in Diamond and Yellin (1985). It is clear that $\tilde{W}(z)$ is non-decreasing in z. So in the following we will focus on $\tilde{W}(0)$ which will give us a lower bound for $W(z)$. For all $z < y$ the value function satisfies the following HJB equation:

$$r\tilde{W}(z) = u + \tilde{W}'(z)s. \tag{58}$$

The household enjoys the utility flow u but do not consume when preference shocks for lumpy consumption occur (since the strategy specifies $y(z) = 0$ for all $z < y$). Moreover, he accumulates real balances at rate s, which leads to an increase in his value function equal to $\tilde{W}'(z)s$. Using the integrating factor method the value of the household with depleted real balances is:

$$\tilde{W}(0) = e^{-\frac{r}{s}y}\tilde{W}(y) + \frac{u}{r}\left(1 - e^{-\frac{r}{s}y}\right). \tag{59}$$

The value of a household with depleted real balances is equal to his value with y real balances discounted at rate r over a time interval of length y/s, the time that it takes to reach y; it is augmented with the discounted sum of the utility flow, u, over that period. Define:

$$H(x) = \int_0^x \int_0^\infty re^{-rt} p(z, t; y) dtdz.$$

The quantity $G(x)$ is the average probability that the household will hold less than x real balances over his lifetime, if he starts with y, with weights decreasing at rate r over time. From (57) the value of a household with y real balances is:

$$\tilde{W}(y) = \frac{u + \alpha \left[1 - H(y)\right] U}{r}. \tag{60}$$

The first term on the right side of (60) is simply the discounted sum of utility flows, u/r. The second term on the right side is the discounted sum of the utilities from lumpy consumption, taking into account that such opportunities occur on average at rate $\alpha[1 - G(y)]$. Substituting $\tilde{W}(y)$ by its expression into (59):

$$\tilde{W}(0) = e^{-\frac{r}{s}y} \frac{\alpha U}{r} \left[1 - H(y)\right] + \frac{u}{r}. \tag{61}$$

We obtain the term, $1 - H(y)$, directly from Diamond and Yellin (1985, eq. (4.12)),

$$1 - H(y) = \frac{\frac{r}{\alpha}\left(1 + \frac{r}{s\theta}\right)}{1 - e^{-\frac{r}{s}y}}, \tag{62}$$

where θ is the unique positive root of:

$$s\theta = \alpha \left(e^{\theta y} - 1\right) - r. \tag{63}$$

Note that when $r = 0$ (63) is the characteristic equation for the stationary distribution of real balances. Moreover, from (62),

$$\lim_{r \to 0} \left[1 - H(y)\right] = \frac{s}{\alpha y},$$

which is the fraction of households with real balances larger than y in a steady state where all households play the inventory-accumulation strategy of Diamond and Yellin (1985, eq. (3.9)). Substituting $1 - H(y)$ from (62) into (61):

$$\tilde{W}(0) = e^{-\frac{r}{s}y} \frac{\alpha U}{r} \frac{\frac{r}{\alpha}\left(1 + \frac{r}{s\theta}\right)}{1 - e^{-\frac{r}{s}y}} + \frac{u}{r}. \tag{64}$$

Now, we set (y, c, h) equal to their levels at the full-insurance allocation, (y^{FI}, c^{FI}, h^{FI}). So, households follow a strategy that implements the full insurance allocation except then their real balances are lower than y^{FI}, in which case they do not consume if a preference shock occurs. The saving rate is then $s = \alpha y^{FI}$ and the value of the household with depleted real balances is:

$$\tilde{W}^{FI}(0) = e^{-\frac{r}{\alpha}} \frac{\alpha U}{r} \frac{\frac{r}{\alpha}\left(1 + \frac{r}{s\theta}\right)}{1 - e^{-\frac{r}{\alpha}}} + \frac{u}{r}. \tag{65}$$

Multiplying by r both sides and taking the limit as r goes to 0:

$$\lim_{r \to 0} r\tilde{W}^{FI}(0) \to \alpha U(y^{FI}) + u(c^{FI}, \bar{h} - h^{FI}). \tag{66}$$

The right side of (66) is the flow lifetime expected utility of a household at the full-insurance allocation.

The ex-ante expected utility of the household in the laissez-faire monetary equilibrium is measured by $\int rW(z)dF(z)$. Because households follow their optimal strategy, $W(z) \geq \tilde{W}(z) \geq \tilde{W}(0)$ for all z. Moreover, the ex-ante expected utility of a household in the laissez-faire monetary equilibrium is bounded above by the utility at the full-insurance allocation, $\int W(z)dF(z) \leq \left[u(c^{FI}, \bar{h} - h^{FI}) + \alpha U(y^{FI})\right]/r$. Hence:

$$r\tilde{W}^{FI}(0) \leq \int rW(z)dF(z) \leq u(c^{FI}, \bar{h} - h^{FI}) + \alpha U(y^{FI}). \tag{67}$$

From (66) it follows that:

$$\lim_{r \to 0} \int rW(z)dF(z) = u(c^{FI}, \bar{h} - h^{FI}) + \alpha U(y^{FI}). \tag{68}$$

Let $(h_\delta^{FI}, c_\delta^{FI}, y_\delta^{FI})$ denote the full-insurance allocation corresponding to an environment where labor endowments have been scaled down by a factor $\delta < 1$. Since $u(c_\delta^{FI}, \delta\bar{h} - h_\delta^{FI}) + \alpha U(y_\delta^{FI}) < u(c^{FI}, \bar{h} - h^{FI}) + \alpha U(y^{FI})$ for all $\delta < 1$, and from (68), it follows that for all $\delta \in (0, 1)$ there is a $\bar{r}_\delta > 0$ such for all $r < \bar{r}_\delta$,

$$\int rW(z)dF(z) > u(c_\delta^{FI}, \delta\bar{h} - h_\delta^{FI}) + \alpha U(y_\delta^{FI}).$$

The ex-ante expected utility of the household is larger the laissez-faire monetary equilibrium than the full-insurance allocation when labor endowments have been scaled down by a factor δ. ■

PROOF OF PROPOSITION 9. Part (i): Large labor endowment. Fix some $\pi \geq 0$. We first note that $y(z^\star) \leq z^\star \leq z_s$, hence equilibrium aggregate demand is bounded by αz_s independently of $\bar{h}$. Equilibrium aggregate supply can be written:

$$F(z_-^\star)\bar{h} + \left[1 - F(z_-^\star)\right] h^\star.$$

To remain bounded as $\bar{h} \to \infty$, it must be the case that $\lim_{\bar{h}\to\infty} F(z^\star_-) = 0$. This also implies that, for $\bar{h}$ large enough, there is an atom at $z^\star$, so that $W'(z^\star) = 1$ and $z^\star = z_s$. Because F converge to a Dirac distribution concentrated at $z^\star = z_s$, we have that $\lim_{\bar{h}\to\infty} \phi M = z_s$.

Next we argue that, as $\bar{h}$ is large enough, $y(z^\star) = y(z_s) = z_s$, i.e., all equilibria must feature full depletion. For this we use the expression for $W'(z)$ derived in Lemma 2:

$$\begin{aligned} W'(0) &= \frac{\alpha}{r+\alpha+\pi}\int_0^{z^\star} U'\left[y(z)\right] dG(z\,|\,0) \leq \frac{\alpha}{r+\alpha+\pi}\int_0^{z^\star} \max\{U'\left(z\right), W'(0)\}\, dG(z\,|\,0) \\ &\leq \frac{\alpha}{r+\alpha+\pi}\left[G(z_s^-\,|\,0)\int_{z\in[0,z_s)} \max\left\{U'(z), W'(0)\right\}\frac{dG(z\,|\,0)}{G(z_s^-\,|\,0)} + \left[1-G(z_s^-\,|\,0)\right]\max\{U'(z_s), W'(0)\}\right], \end{aligned}$$

as long as $\bar{h}$ is large enough. To obtain the inequality of the first line, we have used that $U'\left[y(z)\right] = U'(z)$ if there is full depletion, while $U'\left[y(z)\right] = W'\left[z - y(z)\right] \leq W'(0)$ if there is partial depletion. To obtain the second line, we have used that $z^\star = z_s$ as long as $\bar{h}$ is large enough. Substituting the expression for $\mathcal{T}(z\,|\,\pi\phi M)$ into the definition of $G(z\,|\,0)$, we obtain that:

$$G(z\,|\,0) = \begin{cases} 1 - \left(1 - \frac{z}{z_b}\right)^{1+\frac{r+\alpha}{\pi}} & \text{if} \quad z < z_s \\ 1 & \text{if} \quad z = z_s. \end{cases}$$

Given that z_b goes to infinity as $\bar{h}$ goes to infinity, one sees that $G(z\,|\,0)$ converges weakly to a Dirac distribution concentrated at z_s. We also have:

$$\frac{G'(z\,|\,0)}{G(z_s^-\,|\,0)} = \left(1 + \frac{r+\alpha}{\pi}\right)\frac{\frac{1}{z_b}\left(1-\frac{z}{z_b}\right)^{\frac{1+\alpha}{\pi}}}{1-\left(1-\frac{z_s}{z_b}\right)^{1+\frac{1+\alpha}{\pi}}} \leq \frac{\left(1+\frac{r+\alpha}{\pi}\right)\frac{1}{z_b}}{1-\left(1-\frac{z_s}{z_b}\right)^{1+\frac{1+\alpha}{\pi}}} \to \frac{1}{z_s},$$

as $\bar{h}$ and thus z_b go to infinity. Thus the conditional probability distribution $G(z\,|\,0)/G(z_s^-\,|\,0)$ has a density that can be bounded uniformly in $\bar{h}$. Finally, our bound for $W'(0)$ in Theorem 1 can be written, in the case of linear preferences

$$W'(0) \leq \frac{r+\alpha}{\bar{h}}\left(\frac{\bar{h}+\bar{c}}{r} + \alpha\frac{\|U\|}{r}\right) \to 1 + \frac{\alpha}{r},$$

as $\bar{h} \to \infty$. Taken together, these observations imply that:

$$\int_{z\in[0,z_s)} \max\left\{U'(z), W'(0)\right\}\frac{dG(z\,|\,0)}{G(z_s^-\,|\,0)} \leq \frac{1}{z_s}\int_0^{z_s} \max\left\{U'(z), 1+\frac{\alpha}{r}\right\} dz + \varepsilon$$

for some $\varepsilon > 0$ as long as $\bar{h}$ is large enough (note that the integral on the right-hand side is well defined since $U(z) = \int_0^z U'(z)\,dx$). Together with the fact that $G(z_s^-\,|\,0) \to 0$ as $\bar{h} \to \infty$, we obtain that:

$$G(z_s^-\,|\,0)\int_{z\in[0,z_s)} \max\left\{U'(z), W'(0)\right\}\frac{dG(z\,|\,0)}{G(z_s^-\,|\,0)} \to 0 \text{ and } 1 - G(z_s^-\,|\,0) \to 1$$

as $\bar{h} \to \infty$. Hence, for any $\varepsilon > 0$, $W'(0) \leq \frac{\alpha}{r+\alpha+\pi} \max\{U'(z_s), W'(0)\} + \varepsilon$ as long as $\bar{h}$ is large enough. Picking $\varepsilon < \frac{r+\alpha}{r+\alpha+\pi} U'(z_s)$, we obtain that $W'(0) < \max\{W'(0), U'(z_s)\}$, which implies that $W'(0) < U'(z_s)$, for $\bar{h}$ large enough, i.e., there is full depletion.

Because $H = \alpha\phi M$ under full depletion, and because the distribution of real balance converges towards a Dirac concentrated at z_s, we obtain that $\lim_{\bar{h}\to\infty} H = H^{\infty}(\pi) = \alpha z_s$, which is decreasing in π. Aggregate welfare can be written as $\alpha \int U(z)\, dF(z) - H$, the average utility enjoyed from lumpy consumption net of the average disutility of supplying labor. As $\bar{h} \to \infty$, F converges weakly to a Dirac concentrated at z_s, and H converges to αz_s. It follows that welfare converges to $\mathcal{W}^{\infty} = \alpha\,[U(z_s) - z_s]$, which is decreasing with π.

Part (ii): Small labor endowment. We have shown that there exists a unique candidate equilibrium with full depletion. In this candidate equilibrium, the condition for binding labor is that $z_s \geq z_b$ or, using the definition of z_s:

$$U'(z_b) \leq 1 + \frac{r+\pi}{\alpha}.$$

Recall that $z_b = \frac{\bar{h}}{\pi} + \frac{\bar{h}}{\alpha}$ is an increasing function of $\bar{h}$. Since marginal utility is decreasing, the condition for binding labor can be written:

$$\bar{h} \in \left[0, \bar{H}(\pi)\right] \text{ where } \bar{H}(\pi) = \frac{\alpha\pi}{\alpha+\pi} \left(U'\right)^{-1} \left(1 + \frac{r+\pi}{\alpha}\right).$$

One immediately sees that $\lim_{\pi\to 0} \bar{H}(\pi) = \lim_{\pi\to\infty} \bar{H}(\pi) = 0$.

Next, we turn to the sufficient condition for full depletion. Using Lemma 2 we have, in the candidate equilibrium with full depletion:

$$W'(0) \quad = \quad \frac{\alpha}{r+\alpha+\pi} \int_0^{z_b} U'(z) dG(z) \text{ where } G(z) = 1 - \left(1 - \frac{z}{z_b}\right)^{1+\frac{r+\alpha}{\pi}}$$

Substituting in the expression for $G(z)$ in the integral, we obtain:

$$W'(0) \quad = \quad \frac{\alpha}{\pi} \frac{1}{z_b} \int_0^{z_b} U'(z) \left(1 - \frac{z}{z_b}\right)^{\frac{r+\alpha}{\pi}} dz \leq \frac{\alpha}{\pi} \frac{U(z_b)}{z_b},$$

where the inequality follows by using $(1 - z/z_b)^{\frac{r+\alpha}{\pi}} \leq 1$, integrating, and keeping in mind that $U(0) = 0$. Full depletion obtains if $W'(0) \leq U'(z_b)$. Using the above upper bound for $W'(0)$, we obtain that a sufficient condition for full depletion is:

$$\frac{\pi}{\alpha} \geq \frac{U(z_b)}{z_b U'(z_b)}.$$

Note that $z_b \leq (U')^{-1}(1)$, that the function $z \mapsto [U(z) - U(0)] / [zU'(z)]$ is continuous over $(0, (U')^{-1}(1)]$ and, by our maintained assumption in the Lemma, bounded near zero. Hence it is bounded over the closed interval $[0, (U')^{-1}(1)]$. Therefore the condition for full depletion is satisfied if:

$$\pi \geq \underline{\pi} \equiv \alpha \times \sup_{z \in [0,(U')^{-1}(1)]} \frac{U(z)}{zU'(z)}.$$

Output effect of inflation. In the regime with binding labor, $h(z) = \bar{h}$ for all $z \in$supp(F). Hence, for all $\bar{h} \in \left[0, \hat{h}\right]$ and for all $\pi \in [\underline{\pi}, \bar{\pi}]$, $H = \bar{h}$.

Welfare effect of inflation. From (38) in the regime with binding labor, $\phi M = \bar{h}/\alpha$. Hence, an increase in the money growth rate through lump-sum transfers is a mean-preserving reduction in the distribution of real balances. In this regime social welfare is measured by

$$\mathcal{W} = \int \left[-h(z) + \alpha U(z)\right] dF(z) = -\bar{h} + \alpha \int U(z) dF(z).$$

Given the strict concavity of $U(y)$ money growth leads to an increase in welfare.

Part (iii): Large inflation. From (36), as $\pi \to \infty$, $z^\star \to 0$, $\phi M \to 0$, $H \to 0$, and $\mathcal{W} \to 0$. ∎

PROOF OF PROPOSITION 11.

The proof is structured as follows. Given a policy, (π, τ), we conjecture that households behave as follows: $y(z) = z$ for all $z \in [0, z_0^\star]$; $h(z) = \bar{h}$ for all $z < z_0^\star$, and $h(z_0^\star) = 0$. We also assume that parameters are such that $\bar{h} + \tau(z) - \pi z > 0$ for all $z \in [0, z_0^\star)$. Given this conjecture we will show that: (i) Aggregate real balances under τ are larger than under laissez faire ($\tau_0 = \tau_1 = \tau_z = 0$). (ii) Welfare under τ is larger than under laissez-faire. The second part of the proof will consist in checking that: (iii) For π small enough, there is a transfer scheme, τ, of the form described in (41), that balances the government budget; (iv) Households' conjectured behavior is optimal.

Guessing that the equilibrium features full depletion, and keeping in mind that $\tau(z_0^\star) = \pi z_0^\star$ by construction, the government budget constraint under the transfer scheme, τ, is:

$$\int \left[\tau(z) - \pi z\right] dF_\tau(z) = \int_0^{\mathcal{T}(z_0^\star;\tau)} \left\{\tau\left[z(t)\right] - \pi z(t)\right\} \alpha e^{-\alpha t} dt = 0, \tag{69}$$

where $\mathcal{T}(z_0^\star; \tau)$ is the time to accumulate $z_0^\star$ under the transfer scheme τ and $z(t)$ is the solution to

$$\begin{aligned} \dot{z} &= \bar{h} + \tau(z) - \pi z \;\; \text{for all } z < z_0^\star. \\ &= 0 \text{ if } z = z_0^\star. \end{aligned} \tag{70}$$

We denote $Z_\tau \equiv \int [1 - F_\tau(z)]\, dz$ the aggregate real balances under the transfer scheme, τ, and $Z_0 \equiv \int [1 - F_0(z)]\, dz$ the aggregate real balances under laissez faire. Moreover, denote $\mathcal{T}_\tau \equiv \mathcal{T}(z_0^\star;\tau)$ and $\mathcal{T}_0 = \mathcal{T}(z_0^\star, 0)$ under laissez-faire.

RESULT #1: $\mathcal{T}_\tau > \mathcal{T}_0$ and $Z_\tau > Z_0$.

PROOF: By construction the transfer scheme in (41) is such that there is a level of real balances, $z_{\hat{t}}$, with $\hat{t} \in (0, \mathcal{T}_\tau)$, below which the net transfer to the household is positive, since $\tau_0 > 0$, and above which the net transfer is negative, since from (69) the sum of those transfers must be 0:

$$\begin{aligned} \tau(z_t) - \pi z_t &> 0 \text{ for all } t \in \left(0, \hat{t}\right) \\ \tau(z_t) - \pi z_t &< 0 \text{ for all } t \in \left(\hat{t}, \mathcal{T}_\tau\right). \end{aligned}$$

Dividing the government budget constraint by $\alpha e^{-\alpha \hat{t}}$, (69) becomes:

$$\int_0^{\hat{t}} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt + \int_{\hat{t}}^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt = 0. \tag{71}$$

Given $\hat{t}$, $\alpha e^{-\alpha t}/\alpha e^{-\alpha \hat{t}}$ is decreasing in t, $\alpha e^{-\alpha t}/\alpha e^{-\alpha \hat{t}} > 1$ for all $t < \hat{t}$ and $\alpha e^{-\alpha t}/\alpha e^{-\alpha \hat{t}} < 1$ for all $t > \hat{t}$. It follows that

$$\int_0^{\hat{t}} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt > \int_0^{\hat{t}} [\tau(z_t) - \pi z_t]\, dt \tag{72}$$

$$\int_{\hat{t}}^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt < \int_{\hat{t}}^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t]\, dt. \tag{73}$$

From (71) and the two inequalities, (72)-(73),

$$\int_0^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt = 0 > \int_0^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t]\, dt. \tag{74}$$

From (70) and (74),

$$\int_0^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t]\, dt = \int_0^{\mathcal{T}_\tau} \left[\dot{z}_t - \bar{h}\right] dt = z_0^\star - \bar{h}\mathcal{T}_\tau < 0,$$

where we used that $z_0 = 0$ and $z_{\mathcal{T}_\tau} = z_0^\star$. So $\mathcal{T}_\tau > \mathcal{T}_0 = z_0^\star/\bar{h}$. As a result the measure of households holding their targeted real balances is

$$1 - F_\tau(z_0^\star) = e^{-\alpha \mathcal{T}_\tau} < e^{-\alpha z_0^\star/\bar{h}} = 1 - F_0(z_0^\star).$$

The law of motion for aggregate real balances is $\dot{Z}_\tau = F_\tau(z_0^\star)\bar{h} - \alpha Z_\tau$. At a steady state, $Z_\tau = F_\tau(z_0^\star)\bar{h}/\alpha$, which is larger than $Z_0 = F_0(z_0^\star)\bar{h}/\alpha$ under laissez faire.$\blacksquare$

Social welfare is measured by the sum of utilities across households:

$$\mathcal{W}_\tau = \int \left[-h(z) + \alpha U(z)\right] dF_\tau(z) = \alpha \int \left[-z + U(z)\right] dF_\tau(z), \tag{75}$$

where the second equality is obtained by market clearing, $\int h(z) dF_\tau(z) = \alpha \int y(z) dF_\tau(z) = \alpha \int z dF_\tau(z)$. Using that $U(z) - z = \int_0^z \left[U'(x) - 1\right] dx + U(0)$ (75) can be rewritten as

$$\mathcal{W}_\tau = \alpha \int \int \left[U'(x) - 1\right] \mathbb{I}_{\{0 \le x \le z\}} dx dF_\tau(z) + \alpha U(0). \tag{76}$$

Changing the order of integration,

$$\begin{aligned} \int \int \left[U'(x) - 1\right] \mathbb{I}_{\{0 \le x \le z\}} dx dF_\tau(z) &= \int \int \mathbb{I}_{\{0 \le x \le z\}} dF_\tau(z) \left[U'(x) - 1\right] dx \\ &= \int \left[1 - F_\tau(z)\right] \left[U'(x) - 1\right] dx. \end{aligned} \tag{77}$$

Plugging (77) into (76):

$$\mathcal{W}_\tau = \alpha \int \left[1 - F_\tau(x)\right] \left[U'(x) - 1\right] dx + \alpha U(0). \tag{78}$$

RESULT #2: *Social welfare under τ is higher than welfare at the laissez-faire.*

PROOF: The welfare gain under τ relative to laissez faire is:

$$\begin{aligned} \mathcal{W}_\tau - \mathcal{W}_0 &= \alpha \int \left[1 - F_\tau(x)\right] \left[U'(x) - 1\right] dx - \alpha \int \left[1 - F_0(x)\right] \left[U'(x) - 1\right] dx \\ &= \alpha \int \left[F_0(x) - F_\tau(x)\right] \left[U'(x) - 1\right] dx. \end{aligned} \tag{79}$$

Given our conjecture that the equilibrium features full depletion we have:

$$F_\tau\left(z_{\tau,t}\right) = F_0\left(z_{0,t}\right) = 1 - e^{-\alpha t},$$

where $z_{\tau,t}$ and $z_{0,t}$ denote the real balances of a household who received his last preference shock t periods ago under the transfer scheme τ and under laissez-faire, respectively. Integrating the law of motion of real balances, (70):

$$\begin{aligned} z_{0,t} &= \bar{h} t \quad \text{for all } t < z_0^\star / \bar{h} \\ z_{\tau,t} &= \bar{h} t + \int_0^t \left[\tau\left(z_{\tau,x}\right) - \pi z_{\tau,x}\right] dx \quad \text{for all } t \le \mathcal{T}_\tau. \end{aligned}$$

By definition of the transfer scheme,

$$\begin{aligned} \tau\left(z_t\right) - \pi z_t &> 0 \text{ for all } t \in \left(0, \hat{t}\right) \\ \tau\left(z_t\right) - \pi z_t &< 0 \text{ for all } t \in \left(\hat{t}, \mathcal{T}_\tau\right), \end{aligned}$$

and, from (69), $\int_0^{\mathcal{T}_\tau} [\tau(z_{\tau,x}) - \pi z_{\tau,x}]\, dx < 0$. It follows that there is a $\tilde{t} \in (\hat{t}, \mathcal{T}_0)$ such that $z_{0,\tilde{t}} = z_{\tau,\tilde{t}} = z_s$. For all $t \in (0, \tilde{t})$, $z_{0,t} < z_{\tau,t}$. For all $t \in (\tilde{t}, \mathcal{T}_\tau)$ and $z_{0,t} > z_{\tau,t}$. Equivalently, $F_\tau(z) < F_0(z)$ for all $z < z_s$ and $F_\tau(z) > F_0(z)$ for all $z > z_s$. From (79):

$$\mathcal{W}_\tau - \mathcal{W}_0 = \alpha \left\{ \int_0^{z_s} [F_0(x) - F_\tau(x)] \left[U'(x) - 1\right] dx + \int_{\tilde{z}}^{z_0^\star} [F_0(x) - F_\tau(x)] \left[U'(x) - 1\right] dx \right\}. \quad (80)$$

By the definition of z_s and the fact that $U'(z)$ is decreasing:

$$\begin{aligned} [F_0(x) - F_\tau(x)] \left[U'(x) - 1\right] &> [F_0(x) - F_\tau(x)] \left[U'(\tilde{x}) - 1\right] \quad \text{for all } x \in (0, z_s) \\ [F_0(x) - F_\tau(x)] \left[U'(x) - 1\right] &> [F_0(x) - F_\tau(x)] \left[U'(\tilde{x}) - 1\right] \quad \text{for all } x \in (z_s, z_0^\star). \end{aligned}$$

Plugging these two inequalities into (80):

$$\mathcal{W}_\tau - \mathcal{W}_0 \geq \alpha \left[U'(z_s) - 1\right] \int_0^{z_0^\star} [F_0(x) - F_\tau(x)]\, dx. \quad (81)$$

We proved that

$$\int z dF_\tau(z) = \int [1 - F_\tau(z)]\, dz \geq \int z dF_0(z) = \int [1 - F_0(z)]\, dz.$$

Hence, $\int_0^{z_0^\star} [F_0(x) - F_\tau(x)]\, dx > 0$. Moreover, $U'(z_s) \geq U'(z_0^\star) = 1 + r/\alpha$. Hence, $U'(\hat{z}) - 1 > 0$. It follows from (81) that $\mathcal{W}_\tau > \mathcal{W}_0$.■

The transfer scheme, τ, is fully characterized by π and τ_0 since $\tau_z = (\pi z_0^\star - \tau_0) / (z_0^\star - z_\pi^\star)$ and $\tau_1 = (\pi z_\pi^\star - \tau_0) z_0^\star / (z_0^\star - z_\pi^\star)$. We now establish that for a given inflation rate, π, there exists a lump-sum component, τ_0, that balances the government budget.

RESULT #3: *For π sufficiently small, there is a $\tau_0 \in (0, \pi z_\pi^\star)$ such that $\int [\tau(z) - \pi z]\, dF_\tau(z) = 0$ holds and $\dot{z}_t > 0$ for all $t \in [0, \mathcal{T}(z_0^\star))$.*

PROOF: The government budget constraint, (69), can be re-expressed as

$$\Gamma(\tau_0) \equiv \int_0^{\mathcal{T}(z_0^\star, \tau_0)} \{\tau[z(t)] - \pi z(t)\}\, \alpha e^{-t} dt = 0,$$

By direct integration of the ODE for real balance, (70), one obtains that both $z(t)$ and $\mathcal{T}(z_0^\star, \tau_0)$ are continuous functions of τ_0. Since $\tau(z)$ is, by construction, continuous in z, we obtain that $\Gamma(\tau_0)$ is also continuous in τ_0. If $\tau_0 = 0$, then from (41):

$$\tau(z) - \pi z = \left\{ \begin{array}{c} -\pi z \\ \left(\frac{\pi z_\pi^\star}{z_0^\star - z_\pi^\star}\right)(z - z_0^\star) \end{array} \right. \text{if} \begin{array}{c} z \leq z_\pi^\star \\ z \in (z_\pi^\star, z_0^\star] \end{array}.$$

Hence, $\tau(z_t) - \pi z_t < 0$ for all $t \in (0, \mathcal{T}(z_0^\star))$ and $\Gamma(0) < 0$. If $\tau_0 = \pi z_\pi^\star$, then from (41):

$$\tau(z) = \left\{ \begin{array}{c} \pi z_\pi^\star \\ \pi z \end{array} \right. \text{if} \begin{array}{c} z \leq z_\pi^\star \\ z \in (z_\pi^\star, z_0^\star] \end{array}$$

Consequently, $\tau[z(t)] - \pi z(t) > 0$ for all $t < \mathcal{T}(z_\pi^\star)$ and $\tau(z_t) - \pi z_t = 0$ for all $t \geq T_\pi^\star$. Hence, $\Gamma(\pi z_\pi^\star) > 0$. By the Intermediate Value Theorem there is $\tau_0 \in (0, \pi z_\pi^\star)$ such that $\Gamma(\tau_0) = 0$. Finally, for the transfer scheme to be feasible, it must be that $\dot{z} > 0$ for all $z < z_0^\star$. This requires $\bar{h} + \tau_0 - \pi z_\pi^\star > 0$, since net transfers achieve their minimum at $z = z_\pi^\star$. This condition will be satisfied for π sufficiently small. ■

Finally, we need to check that household's conjectured behavior is optimal: households find it optimal to supply $\bar{h}$ units of labor until they reach $z_0^\star$ and to deplete their money holdings in full when a preference shock occurs. The ODE for the marginal value of money is:

$$(r+\pi)\lambda_t = \alpha\left[U'(z_t) - \lambda_t\right] + \lambda_t \tau'(z_t) + \dot{\lambda}_t. \tag{82}$$

RESULT #4: *For* π and τ_0 *sufficiently small the solution to (82) is such that:* $\lambda_t > 1$ *for all* $t < T_\tau$, $\lambda_{T_\tau} = 1$, *and* $\lambda_t \leq U'(z_0^\star)$ for all $t \in [0, T_\tau]$.

PROOF: Integrating (82), we obtain that the marginal value of money solves:

$$\lambda_t = 1 + \int_t^{\mathcal{T}(z_\pi^\star)} e^{-(r+\pi+\alpha)(s-t)}\alpha\left\{U'(z_s) - U'(z_\pi^\star)\right\} ds + e^{-(r+\pi+\alpha)[\mathcal{T}(z_\pi^\star)-t]}\left[\lambda_{\mathcal{T}(z_\pi^\star)} - 1\right], \tag{83}$$

for all $t \leq \mathcal{T}_\pi^\star$, and

$$\lambda_t = 1 + \int_t^{\mathcal{T}_\tau} \alpha e^{-(r+\pi+\alpha-\tau_z)(s-t)}\left[U'(z_s) - U'(z_0^\star) + \frac{\tau_z - \pi}{\alpha}\right] ds, \tag{84}$$

for all $t \geq \mathcal{T}(z_\pi^\star)$, where we used that $\lambda_{\mathcal{T}_\tau} = 1$ and $\mathcal{T}(z_\pi^\star) = -\frac{1}{\pi}\ln\left[1 - \pi z_\pi^\star/(\bar{h} + \tau_0)\right]$. For all $t \in (\mathcal{T}_\pi^\star, \mathcal{T}_\tau)$, $z_t < z_0^\star$ and hence $U'(z_t) > U'(z_0^\star)$. Given that $\tau_z > \pi$ it follows from (84) that $\lambda_t > 1$ for all $t \in (\mathcal{T}_\pi^\star, \mathcal{T}_\tau)$. Similarly, for all $t < \mathcal{T}_\pi^\star$, $z_t < z_\pi^\star$ and hence $U'(z_t) > U'(z_\pi^\star)$. Given that $\lambda_{\mathcal{T}(z_\pi^\star)} > 1$, it follows from (83) that $\lambda_t > 1$ for all $t \leq \mathcal{T}(z_\pi^\star)$.

For the second part of the Lemma we note that, when $\pi = \tau_0 = 0$, we have that $\lambda_t < U'(z_0^\star)$ for all $t \in [0, \mathcal{T}(z_0^\star)]$. Since λ is continuous with respect to (t, π, τ_0) and since $\mathcal{T}(z_0^\star)$ is finite at $\pi = \tau = 0$, we obtain by uniform continuity that $\lambda_t < U'(z_0^\star)$ for (π, τ_0) sufficiently small. ■

Note that, by Result #3, the τ_0 balancing the government budget constraint is less than $\pi z_\pi^\star$, hence it goes to zero as π goes to zero. Hence, when π is sufficiently small, the solution to the ODE (82) satisfies all the properties of Result #4. This allows us to construct a candidate value function for all $z \in [0, z_0^\star]$. Namely, we let $\lambda(z) \equiv \lambda_{\mathcal{T}(z)}$, where λ_t is the solution to the ODE (82):

$$W(z) \equiv W(0) + \int_0^z \lambda(x)\, dx \text{ where } rW(0) \equiv \lambda(0)\left(\bar{h} + \tau_0\right).$$

Next, we construct a candidate value function for $z \geq z_0^\star$:

RESULT #5: *For* π and τ_0 *sufficiently small, and for, there exists a continuously differentiable and bounded function* $W(z)$*, and two absolutely continuous functions* $V(z)$ and $\lambda(z)$ *such that*:

- *For* $z \le z_0^\star$, $W(z)$, $V(z)$ *and* $\lambda(z)$ *are the function constructed following Result #4.*
- For $z \ge z_0^\star$:

$$\begin{aligned} W(z) &= W(z_0) + \int_{z_0}^{z} \lambda(x)\, dx && (85)\\ V(z) &= \max_{y\in[0,z]} U(y) + W(z-y) && (86)\\ (r+\alpha)\lambda(z) &= V'(z) - \bar{c}\lambda'(z) \text{ almost everywhere} && (87)\\ \lambda(z) &\in [0,1]. && (88) \end{aligned}$$

Proof. We construct a solution to the problem (85)-(88) as follows. Suppose that we have constructed a solution over some interval $[z_0^\star, Z]$, where $Z \ge z_0^\star$. We first observe that:

$$U'(z_0^\star) = 1 + \frac{r}{\alpha} \ge \sup_{x\in[0,z_0^\star]} \lambda(x) = \sup_{x\in[0,Z]} \lambda(x) \tag{89}$$

where the first equality and the first inequality follow from our construction of $W(z)$ and $\lambda(z)$ over $[0, z_0^\star]$, and the last equality follows because $\lambda(z) \le 1$ for $z \in [z_0^\star, Z]$. We now show how to extend this solution over the interval $[Z, Z + z_0^\star]$ as follows. First, we let:

$$\tilde{V}(z) \equiv \max_{y\in[z-Z,z]} U(y) + W(z-y), \tag{90}$$

which is well defined for all $z \in [Z, Z + z_0^\star]$, given that have constructed $W(z)$ for all $z \le Z$ and since $z - y \le Z$ by the choice of our constraint set. Note that, in principle, the function $\tilde{V}(z)$ differs from $V(z)$ because it imposes the constraint that $y \ge z - Z$. Our goal is to show that, nevertheless, $\tilde{V}(z) = V(z)$. Precisely, if one extends $\lambda(z)$ over $[Z, Z + z_0^\star]$ using (87), and define $W(z)$ using (85), then the household never finds it optimal to choose $y < z_0^\star$, implying that the additional constraint we imposed to define $\tilde{V}(z)$ is not binding.

We first establish that $\underline{\tilde{V}(z) \text{ is absolutely continuous and } \tilde{V}'(z) \le U'(z_0^\star)}$. Consider first $z \in [Z, Z + z_0^\star/2]$. Given (89), it follows that the solution to (90) must be greater than $z_0^\star$. By implication since $z - Z \le z_0^\star/2$, the solution y to (90) must be greater than $z - Z + z_0^\star/2$. Given this observation and after making the change of variable $x = z - y$, we obtain that

$$\tilde{V}(z) \equiv \max_{x\in[0,Z-z_0^\star/2]} U(z-x) + W(x).$$

The objective is continuously differentiable with respect to z, and its partial derivative is $U'(z-x) \leq U'(z_0^\star/2)$ given that $z \leq Z+z_0^\star/2$ and $x \leq Z-z_0^\star/2$. Proceeding to the interval $z \in [Z+z_0^\star/2, Z+z_0^\star]$, we make the change of variable $x = z-y$ in (90) and obtain that $\tilde{V}(z) = \max_{x\in[0Z]} U(z-x)+W(x)$. Again, the objective is continuously differentiable with a partial derivative with respect to z equal to $U'(z-x) \leq U'(z_0^\star/2)$, since $z \geq Z+z_0^\star/2$ and $x \leq Z$. Hence, in both cases, given that the objective has a bounded partial derivative with respect to z, we can apply Theorem 2 in Milgrom and Segal (2002): $\tilde{V}(z)$ is absolutely continuous and the envelope condition holds, i.e., $\tilde{V}'(z) = U'[y(z)]$ whenever this derivative exists. By condition (89), it follows that $y(z) \geq z_0^\star$, hence $\tilde{V}'(z) \leq U'(z_0^\star)$, as claimed.

Next, we construct a solution over $[Z, Z+z_0^\star]$. Given that the function $\tilde{V}(z)$ constructed above is absolutely continuous, we can integrate the ODE (87) with $\tilde{V}'(z)$ and we obtain a candidate solution:

$$\tilde{\lambda}(z) = \lambda(Z)e^{-\frac{r+\alpha}{\bar{c}}(z-Z)} + \frac{\alpha}{\bar{c}}\int_Z^z \tilde{V}'(x)e^{-\frac{r+\alpha}{\bar{c}}(z-x)}\,dx$$

Given that $\lambda(Z) \leq 1$ and $\tilde{V}'(x) \leq U'(z_0^\star) = 1 + r/\alpha$, one sees after direct integration that $\tilde{\lambda}(z) \leq 1 \leq U'(z_0^\star)$ for all $z \in [Z, Z+z_0^\star]$. Now let

$$\tilde{W}(z) = W(Z) + \int_Z^z \tilde{\lambda}(x)\,dx.$$

We now show that, if we extend $W(z)$ by $\tilde{W}(z)$, $\lambda(z)$ by $\tilde{\lambda}(z)$, and $V(z)$ by $\tilde{V}(z)$ over the interval $[Z, Z+z_0^\star]$, we obtain a solution of the problem (85)-(87) over $[Z, Z+z_0^\star]$: indeed, we have just shown that $\tilde{\lambda}(z) = \tilde{W}'(z) \leq U'(z_0^\star)$ for all $z \in [Z, Z+z_0^\star]$, implying that the constraint $y \geq z - Z$ we imposed in the definition of $\tilde{V}(z)$ is not binding. That is:

$$V(z) = \max_{y\in[0,z]} U(y) + W(z-y) = \max_{y\in[z-Z,Z]} U(y) + W(z-y) = \tilde{V}(z).$$

Hence, we have extended the solution from $[z_0^\star, Z]$ to $[Z, Z+z_0^\star]$. Notice that the argument does not depend on Z: hence, we can start with $Z = z_0^\star$, and repeat this extension until we obtain a solution defined over $[z_0^\star, \infty)$.

Finally, we show that $W(z)$ is bounded. By construction we have:

$$\begin{aligned}
\lambda(z) &= \lambda(z_0^\star)e^{-\frac{r+\alpha}{\bar{c}}\left(z-z_0^\star\right)} + \frac{\alpha}{\bar{c}}\int_{z_0^\star}^z V'(x)e^{-\frac{r+\alpha}{\bar{c}}\left(x-z_0^\star\right)}\,dx \\
W(z) &= W(z_0^\star) + \int_{z_0^\star}^z \lambda(y)\,dy.
\end{aligned}$$

Plugging the first equation into the second, keeping in mind that $\lambda(z_0^\star) = 1$, and changing the order of integration we obtain:

$$\begin{aligned} W(z) &= W(z_0^\star) + \frac{\bar{c}}{r+\alpha}\left(1 - e^{-\frac{r+\alpha}{\bar{c}}(z - z_0^\star)}\right) + \frac{\alpha}{r+\alpha}\int_{z_0^\star}^{z} V'(x)\left[1 - e^{-\frac{r+\alpha}{\bar{c}}(z-x))}\right] dy \\ &\leq W(z_0^\star) + \frac{\bar{c}}{r+\alpha} + \frac{\alpha}{r+\alpha}\left[V(z) - V(z_0^\star)\right] \\ &\leq W(z_0^\star) + \frac{\bar{c}}{r+\alpha} + \frac{\alpha}{r+\alpha}\left[W(z) + \|U\| - W(z_0^\star)\right], \end{aligned}$$

where the first inequality follows because $1 - e^{-\frac{r+\alpha}{\bar{c}}(z-x)} \leq 1$ for all $x \in [z_0^\star, z]$, and the second inequality because $W(z) \leq V(z) \leq W(z) + \|U\|$. Rearranging and simplifying we obtain that

$$W(z) \leq W(z_0) + \frac{\bar{c} + \alpha\|U\|}{r},$$

establishing the claim. ■

RESULT #6: *For π sufficiently small and τ_0 chosen, as in RESULT #3, to balance the government budget constraint, the households conjectured behavior is optimal.*

Proof. Consider the candidate value function constructed in Result #4 and #5. By construction, $W(z)$ is continuously differentiable and it solves the HJB equation:

$$(r+\alpha)W(z) = \max_{c\geq 0, 0\leq h\leq \bar{h}, 0\leq y\leq z}\left\{\min\{c, \bar{c}\} + \bar{h} - h + \alpha\left[U(y) + W(z-y)\right] + W'(z)\left[h - c + \tau(z) - \pi z\right]\right\}.$$

Then, the optimality verification argument of Section VII in the supplementary appendix establishes that $W(z)$ is equal to the maximum attainable utility of a households, and that the associated decision rules are optimal. ■

Appendix B: Numerical methods

B.1. Construction of a Stationary Equilibrium

In this section we outline an efficient numerical algorithm to construct a stationary equilibrium in a general setting, no matter the equilibrium features full or partial depletion of money after a liquidity shock. We guess and verify a stationary equilibrium with a numerical method, where the target real balances is finite and the value function is strictly concave, increasing and twice differentiable.

As shown in the mathematical appendix, the value function $W(z)$ solves the following HJB equation:

$$(r+\alpha) W(z) = \max_{c \geq 0, h \in [0, \bar{h}]} \left\{ u\left(c, \bar{h} - h\right) + \alpha V(z) + W'(z) \left[h - c - \pi(z - \Upsilon)\right] \right\} \tag{91}$$

where $V(z) \equiv \max_{y \in [0,z]} \{U(y) + W(z-y)\}$. Denote $\lambda = W'(z)$, and $c(\lambda)$ and $h(\lambda)$ are the maximizers to the above. Let $y(z)$ denote the solution of y to V. Then $V'(z)$ and $y'(z)$ are given by

$$V'(z) = U'[y(z)] \tag{92}$$

$$y'(z) = \begin{cases} 1 \text{ if } U'(z) \geq W'(0) \\ \frac{W''[z-y(z)]}{W''[z-y(z)] + U''[y(z)]} \text{ if } U'[y(z)] = W'[z - y(z)] \end{cases} \tag{93}$$

Under the premise that W is twice differentiable, the equilibrium dynamics of household's state and co-state is given by the following system of differential equations:

$$\dot{z} = h(\lambda) - c(\lambda) - \pi(z - \Upsilon), \tag{94}$$

$$\dot{\lambda} = (r + \alpha + \pi) \lambda - \alpha V'(z). \tag{95}$$

A challenge to computation is that when $y(z) \neq z$ (periodic partial depletion of money), then $V'(z)$ depends on the shape of the value function $W(z)$. Also, the stationary point $(z^\star, \lambda^\star)$, which is given by

$$\begin{aligned} h(\lambda^\star) &= c(\lambda^\star) + \pi(z^\star - \Upsilon), \\ \frac{r + \alpha + \pi}{\alpha} \lambda^\star &= V'(z^\star), \end{aligned}$$

also depends on the shape of the value function $W(z)$ through $V'(z)$. In other words, unlike the standard growth model, the system of ODE (94) and (95) is not sufficient to compute the equilibrium dynamics: we have to solve $W(z)$ in order to compute $V'(z)$ and the stationary point

$(z^\star, \lambda^\star)$. But the HJB (91) implies that $W(z)$ also depends on $V(z)$ - in sum, the problem is lack of recursive structure.

Our novel, recursive method to solve this problem involves two key elements: first rewrite the equilibrium as a system of delay differential equations (DDE), and second modify the time-elimination method (Mulligan and Sala-i-Martin 1993) to solve this system of DDE. The time-elimination method allows us to change the state variable from t to $\zeta \equiv -\lambda$, and then the equilibrium dynamics is fully characterized by the "stable arm" function $z(\zeta)$ with initial condition $z(\zeta_0) = 0$ and boundary condition $z(\zeta^\star) = z^\star$. The stable arm $z(\zeta)$ is well-defined and unique under the premise that the value function is strictly concave, increasing and twice differentiable (see mathematical appendix for details).

B.1.1. Computing the Stable Arm: DDE

To compute the system with time-elimination method, we first formulate $z(\zeta; \lambda_0, \Upsilon)$ given $\lambda(0) = \lambda_0$ and Υ. By eliminating the time in (94) and (95), the slope of the stable arm is

$$z'(\zeta) = \frac{\dot{z}}{-\dot{\lambda}} = \frac{h(-\zeta) - c(-\zeta) - \pi(z - \Upsilon)}{(r + \alpha + \pi)\zeta + \alpha\Omega(\zeta)}. \tag{96}$$

where $\Omega(\zeta) \equiv V'[z(\zeta)]$ is differentiable. We depress the dependence on λ_0, Υ in z and Ω unless confusion could arise. For computation purpose, we want to formulate $\Omega(\zeta)$ in a recursive way. A trick is to decompose z into y and $z - y$, which is then given by

$$z = \underbrace{(U')^{-1}(\Omega)}_{y} + \underbrace{\lambda^{-1}[\min\{\Omega, \lambda_0\}]}_{z-y}. \tag{97}$$

Notice that

$$\frac{d\lambda^{-1}(\Omega)}{d\Omega} = z'(-\Omega).$$

Differentiate (97) with respect to ζ, then we have $\Omega(\zeta)$ as the solution to the following delay differential equation (DDE):

$$\Omega'(\zeta) = z'(\zeta)\left[U''\left[(U')^{-1}[\Omega(\zeta)]\right]^{-1} + \mathbb{I}[\Omega(\zeta) < \lambda_0]\, z'(-\Omega)\right]^{-1}, \tag{98}$$

where $\mathbb{I}(\Omega < \lambda_0)$ is the indicator function that $\mathbb{I}(\Omega < \lambda_0) = 1$ if $\Omega < \lambda_0$ (partial depletion), otherwise $\mathbb{I}(\Omega < \lambda_0) = 0$ (full depletion). Equation (98) is a DDE since it depends on the current state variables z (through $z'(\zeta)$) and Ω, as well as the lagged state variable $-\Omega$, which is lagged (interpreting the state ζ as the new "time") since $-\Omega \leq \zeta$. That is why we formulate the state variable as $\zeta \equiv -\lambda$, because we can formulate $\Omega(\zeta)$ as the solution to a self-containing DDE not involving $W(z)$ any more. Now the equilibriums is sufficiently characterized by the system of DDE (96) and (98).

To solve the system of $z(\zeta)$ and $\Omega(\zeta)$, given Υ and $\zeta_0 = \lambda_0$ we start integrating (96) and (98) from the boundary condition $z(-\lambda_0) = 0$ and $\Omega(-\lambda_0) = U'(0)$, which is well-defined by assuming that either $U'(0)$ is bounded or we start with some arbitrarily large values. The integration will result in two functions $z(\zeta)$ and $\Omega(\zeta)$ given λ_0 and Υ. For later use, define $\zeta^\star(\lambda_0, \Upsilon)$ and $z^\star(\lambda_0, \Upsilon)$ as the solution to $h(-\zeta^\star) - c(-\zeta^\star) - \pi(z^\star - \Upsilon) = 0$ and $z^\star = z(\zeta^\star)$.

B.1.2. Computing the Distribution

Having computed $z(\zeta)$ and $\Omega(\zeta)$, we invert the system back to $\lambda(z)$ and $\Omega(z)$ using the definition $\zeta = -\lambda(z)$. It is invertible since $W'(z)$ is monotone. Notice that by definition we have $y(z) = (U')^{-1}[\Omega(z)]$. Define $\varphi(z)$ as the solution to

$$\varphi - y(\varphi) = z. \tag{99}$$

In other words, $\varphi(z)$ is the level of real balances before the liquidity shock such that the household will deplete the real balances up to z after the shock. Notice that since $y(z) > 0$ for all $z > 0$, we have $\varphi(z) > z$. Differentiating (99) we have for all $z > 0$

$$\varphi'(z) = \frac{W''[z - y(z)] + U''[\varphi(z)]}{U''[y(z)]} \geq 0.$$

so $\varphi(z)$ is strictly increasing for all $z > 0$. Define $z_d \equiv z^\star - y(z^\star)$, then we have $\varphi(z) \leq z^\star$ if and only if $z \leq z_d$. Define $s(z) \equiv h[\lambda(z)] - c[\lambda(z)] - \pi(z - \Upsilon)$. which is differentiability continuous, bounded and positive for all $z \in [0, z^\star)$. The equilibrium density of real balance $f(z)$ solves the Kolmogorov forward equation (hereafter KFE, which is derived later):

$$\partial_z[s(z) f(z)] = \begin{cases} -\alpha f(z), \text{ if } z > z_d, \\ -\alpha f(z) + \alpha \frac{s[\varphi(z)]}{s(z)} f[\varphi(z)], \text{ if } z < z_d, \end{cases} \tag{100}$$

where ∂_z is the differential functional. The equilibrium density $f(z)$ has a jump at $z = z_d$, which captures the extra flow of the influx of the mass of agent with $z = z^\star$ after a liquidity shock. The jump is given by

$$f(z_-) = f(z) - \frac{s\left(z_-^\star\right)}{s(z)} f\left(z_-^\star\right), \text{ if } z = z_d. \tag{101}$$

Consider two regions of z: $[z_d, z^\star]$ and $[0, z_d)$. In the first region $[z_d, z^\star]$, the KFE (100) is just a standard ODE:

$$f'(z) = -\frac{\alpha + s'(z)}{s(z)} f(z), \text{ for all } z \in (z_d, z^\star) \tag{102}$$

Fix some arbitrary initial value, says $f(z_d) = 1$ (we will normalize the density function later), we can compute $f(z)$ in this region by integrating the ODE (100) from the initial condition $f(z_d) = 1$ up to the boundary $z = z^\star(\lambda_0, \Upsilon)$. If $s(z^\star_-) = 0$, then there is also a boundary condition for the KFE, which is given by

$$\lim_{z \uparrow z^\star} s(z) f(z) = 0. \tag{103}$$

On the other hand, if $s(z^\star_-) > 0$ (by the definition of $z^\star$, $s(z)$ cannot be non-positive for any $z < z^\star$, see the mathematics appendix), for example in the slack-labor equilibrium of the model with linear preferences, then there is a probability mass $F(z^\star) - F(z^\star_-)$ at $z = z^\star$, which is pinned down by a boundary condition

$$F(z^\star) - F(z^\star_-) = \frac{s(z^\star_-) f(z^\star_-)}{\alpha}. \tag{104}$$

Now consider the second region $[0, z_d)$. Transform $z = z_d - t$. and define $\phi(t) = f(z_d - t)$. Using (100), $\phi(t)$ also solves the following DDE

$$\phi'(t) = \left[\frac{\alpha - s'(z_d - t)}{s(z_d - t)}\right] \phi(t) - \alpha \frac{s[\varphi(z_d - t)]}{s(z_d - t)^2} \phi[z_d - \varphi(z_d - t)], \text{ for all } t \in (0, z_d) \tag{105}$$

The "delay" term is state-dependent and given by $\varphi(z_d - t)$, which is non-negative for all $t \in [-(z^\star - z_d), 0]$. Notice that, first, $t = z_d$ corresponds to $z = 0$ and $t = 0$ corresponds to $z = z_d$. Second, for all $t < 0$, the "history" is simply given by $\phi(t) = f(z_d - t)$, where f is given before in the region $(z_d, z^\star]$. So we can compute $\phi(t)$ by integrating the DDE (105) from $t = 0$ to $t = z_d$ given the initial value from (101)

$$\phi(0) = 1 - \frac{s(z^\star_-)}{s(z_d)} f(z^\star_-). \tag{106}$$

The (unnormalized) density function in this region can be obtained by having $f(z) = \phi(z_d - z)$ for all $z \in [0, z_d)$.

B.1.3. Computing the Stationary Equilibrium and Welfare Cost

Finally, we use the transversality condition and government's balanced budget condition to solve λ_0 and Υ. The transversality condition implies another boundary condition:

$$(r + \alpha + \pi) \zeta^\star(\lambda_0, \Upsilon) + \alpha \Omega[\zeta^\star(\lambda_0, \Upsilon)] = 0. \tag{107}$$

In the model of non-zero money growth with lump-sum injection, the government's balanced budget condition implies $\Upsilon = \mathbb{E}(z)$, which implies

$$\Upsilon = \frac{\int_0^{z^\star(\lambda_0,\Upsilon)} zfdF(z)}{\int_0^{z^\star(\lambda_0,\Upsilon)} dF(z)}, \tag{108}$$

where $F(z) \equiv \int_0^z f(z)\,dz$ is the cumulative density function. So we have two equations to solve for the two unknown λ_0 and Υ.

Recall that the welfare under inflation π is given by

$$\mathcal{W}_\pi = \int_0^{z_\pi^\star} \left\{u\left[c_\pi(z), \overline{h} - h_\pi(z)\right] + \alpha U\left[y_\pi(z)\right]\right\} dF_\pi(z).$$

Define the welfare cost of inflation Δ_π as the solution to

$$\mathcal{W}_\pi = \int_0^{z_\pi^\star} \left\{u\left[(1-\Delta_\pi)\,c_0(z), \overline{h} - h_0(z)\right] + \alpha U\left[(1-\Delta_\pi)\,y_0(z)\right]\right\} dF_0(z).$$

In other words, our measure of the welfare cost of inflation asks what percentage of households' consumption a social planner would be willing to give up to have inflation zero instead of π.

B.1.4. Special Case: Laissez-Faire

The time-elimination method is also convenient to construct the equilibrium under the special case of zero money growth and full depletion. We will show that this special case does not involve any fixed-point problem. The stationary point $(z^\star, \lambda^\star)$ is given by

$$h(\lambda^\star) = c(\lambda^\star) \tag{109}$$

$$z^\star = (U')^{-1}\left[\left(\frac{r+\alpha}{\alpha}\right)\lambda^\star\right], \tag{110}$$

The Jacobian of (94) and (95) at $z = z^\star$ and $\lambda = \lambda^\star$ is then given by

$$J = \begin{pmatrix} 0 & h'(\lambda^\star) - c'(\lambda^\star) \\ -\alpha U''(z^\star) & r+\alpha \end{pmatrix},$$

where we have used the fact of zero money growth and full depletion such that $V''(z^\star) = U''(z^\star)$. Then the negative eigenvalue of J (corresponding to the stable arm) is given by

$$\xi = -\frac{r+\alpha}{2}\left[\left[1 - \frac{4\alpha U''(z^\star)}{(r+\alpha)^2}\left[h'(\lambda^\star) - c'(\lambda^\star)\right]\right]^{1/2} - 1\right]. \tag{111}$$

Define p as the slope of the stable arm at $z = z^\star$ and $\lambda = \lambda^\star$, which is given by

$$p = \frac{\xi}{h'(\lambda^\star) - c'(\lambda^\star)}. \tag{112}$$

In the model with linear preferences, (109) is replaced with $h(\lambda^\star) = c(\lambda^\star) = 0$, $\lambda^\star = 1$ in (110), and $p = \frac{\alpha}{r+\alpha} U''(z^\star)$. Anyway, these equations solve $\lambda^\star$ and $z^\star$. Under full depletion and zero money growth, the dynamic system is reduced to the following ODEs of $\lambda(z)$ and $f(z)$:

$$\lambda'(z) = \frac{(r+\alpha)\lambda - \alpha U'(z)}{h(\lambda) - c(\lambda)}, \tag{113}$$

$$f'(z) = -\frac{\alpha + \lambda'(z)[h'(\lambda) - c'(\lambda)]}{h(\lambda) - c(\lambda)} f(z) \tag{114}$$

A convenient way to construct the stable arm is to: first, integrate (113) backward from $z = z^\star$ to $z = 0$ with initial value $\lambda(z^\star) = \lambda^\star$ and $\lambda'(z^\star) = p$ ($\lambda'(z^\star)$ involves zero dividing zero so it is pinned down by the eigenvector). After that we solve the stable arm $\lambda(z)$. Then we integrate (114) forward from $z = 0$ to $z = z^\star$ with initial value $f(0) = 1$ ($f(z^\star)$ may be infinite, which is plausible under the KFE boundary condition (103), so we cannot integrate backward like $\lambda(z)$). If $s(z^\star_-) > 0$ then we construct the probability mass $1 - F(z^\star_-)$ by the KFE boundary condition (104). After that we solve $f(z)$ (unnormalized). The initial values of λ_0 and Υ are set to $\lambda_0 = \lambda(0)$ and $\Upsilon = \int_0^{z^\star} z f(z)\, dz / \int_0^{z^\star} f(z)\, dz$.

B.1.5. Special Case: Linear Preferences

So far we need to solve a system of 2 DDE and 1 KFE. The system can be further simplified under linear preferences. Eliminating the time in (94) and (95), then using the fact that $\lambda = W'(z)$ and $V'(z) = U'[y(z)]$, we have

$$W''(z) = \frac{\dot{\lambda}}{\dot{z}} = \frac{(r+\alpha+\pi) W'(z) - \alpha U'[y(z)]}{\bar{h} - \pi(z - \Upsilon)}.$$

Shifting the state variable to $z - y(z)$, we have for all $z > (U')^{-1}(\lambda_0)$

$$W''[z - y(z)] = \frac{(r+\alpha+\pi) U'[y(z)] - \alpha U'[y(z - y(z))]}{\bar{h} - \pi[z - y(z) - \Upsilon]}, \tag{115}$$

where we have used the fact that $U'[y(z)] = V'(z) = W'[z - y(z)]$. Substituting (115) into (93), we have

$$y' = \begin{cases} 1 \text{ if } z \le (U')^{-1}(\lambda_0) \\ \left[1 + U''(y) \frac{h - \pi(z-y-\Upsilon)}{(r+\alpha+\pi)U'(y) - \alpha U'[y(z-y)]}\right]^{-1} \text{ if } z > (U')^{-1}(\lambda_0) \end{cases} \tag{116}$$

The equilibrium features full depletion if and only if $y'(z) = 1$ for all $z \in [0, z^\star]$. Notice that (93) is also a DDE but no longer depending on λ. Then now $z^\star$ is simply given by

$$z^\star = \min\left\{\frac{h}{\pi} + \Upsilon, y^{-1} \circ (U')^{-1}\left(1 + \frac{r+\pi}{\alpha}\right)\right\}. \tag{117}$$

The equilibrium features binding labor if $z^\star$ takes the first term on the right hand side of (117), otherwise the equilibrium features slack labor. Under linear preferences we have $s(z) = \overline{h} - \pi(z - \Upsilon)$, which is again independent to λ. Thus, the KFE (100) now is also independent to λ. In sum, the stationary equilibrium can be reduced to the system of 1 DDE (116) and 1 KFE (100).

The KFE can be further simplified under linear preferences. Notice that given $s(z) = \overline{h} - \pi(z - \Upsilon)$, the KFE (102) with initial value $f(z_d) = 1$ admits the following closed-form solution (unnormalized):

$$f(z) = \left[\frac{s(z)}{s(z_d)}\right]^{\frac{\alpha}{\pi}-1}, \text{ for all } z \in (z_d, z^\star). \tag{118}$$

If $s(z^\star_-) > 0$, then there is a probability mass at $z = z^\star$ given by the KFE boundary condition (104), which is simply (again unnormalized)

$$F(z^\star) - F(z^\star_-) = \frac{s(z_d)^{1-\frac{\alpha}{\pi}} s(z^\star_-)^{\frac{\alpha}{\pi}}}{\alpha}. \tag{119}$$

Finally, using the closed-form (118), the KFE (105) is reduced to an ODE given by

$$\phi'(t) = \frac{\alpha+\pi}{s(z_d - t)}\phi(t) - \alpha s(z_d)^{1-\frac{\alpha}{\pi}} \frac{s[\varphi(z_d - t)]^{\frac{\alpha}{\pi}}}{s(z_d - t)^2}, \text{ for all } t \in [0, z_d]. \tag{120}$$

The initial condition from (106) now becomes

$$\phi(0) = 1 - \left[\frac{s(z^\star_-)}{s(z_d)}\right]^{\frac{\alpha}{\pi}}. \tag{121}$$

B.2. Derivation of Kolmogorov Forward Equation

In this section we derive the KFE (100) used in the previous section. Denote the dynamics of real balances is given by

$$\dot{z} = s(z) \equiv h[\lambda(z)] - c[\lambda(z)] - \pi z + \tau(z), \text{ where } z \in [0, z^\star],$$

and agent's real balances reduces by $y(z)$ after a liquidity shock, which arrives at the Poisson rate α. Suppose $s(z)$ is continuous, bounded and positive for all $z \in [0, z^\star)$. Recall that $z_t = \int_0^t s(z_s)\,ds$, and let T denote the solution to $z^\star = \int_0^T s(z_s)\,ds$.

We use a discrete time, discrete state-space model to obtain the Kolmogorov forward equation and the boundary conditions for the density $f(z)$. Fix any integer n, then there exist $\Delta_n > 0$ and a sequence $\{z_i\}_{i=0}^n$ such that $z_0 = 0$, $z_n = z^\star$ and $z_i = z_{i-1} + s(z_{i-1})\Delta_n$ for any $i > 0$. To see it, notice that fix any n and $\Delta_n > 0$ and construct $z_i = z_{i-1} + s(z_{i-1})\Delta_n$, since $s(z)$ is bounded, we have $z_n \to \infty$ if $\Delta_n \to \infty$; $z_n \to 0$ if $\Delta_n \to 0$. So there must exists Δ_n such that $z_n = z^\star$. Divide $[0, z^\star]$ into $n+1$ discrete states $\{z_i\}_{i=0}^n$. Let $I(z)$ be the interval function such that $I(z) = [z_{i-1}, z_i]$ and $z \in (z_{i-1}, z_i]$. Let $D(z)$ be the correspondence such that $D(z) \subseteq \{z_i\}_{i=0}^n$ and $D(z) - y[D(z)] \subseteq I(z)$. As n goes to infinity, Δ_n converges to zero, $\{z_i\}_{i=0}^n$ converges to the continuous time process $z_t = \int_0^t s(z_\tau)\, d\tau$, $I(z)$ converges to z, and $D(z)$ converges to $\varphi(z)$.

Now let $f_n(z_i, t)$ denote the fraction of agents with real balances z_i at time t for fixed n. With a slight abuse of notation, let $f_n(z)$ be the stationary distribution. We are interested in characterizing the density $f(z) \equiv \lim_{n\to\infty} \frac{f_n(z)}{s(z)\Delta_n}$. For any $i \neq n$, the dynamics of z implies

$$f_n(z_i, t+\Delta_n) = (1-\alpha\Delta_n) f_n(z_{i-1}, t) + \alpha\Delta_n \sum_{z\in D(z_i)} f_n(z,t). \tag{122}$$

In any period of length Δ_n, a fraction $\alpha\Delta_n$ of agents are hit by a liquidity shock. Thus the fraction of agent with $z = z_i$ at $t+\Delta_n$ are a fraction $1-\alpha\Delta_n$ of those who were agents with $z = z_{i-1}$ at t but not hit by a liquidity shock, plus the sum of fraction $\alpha\Delta_n$ of those who were agents with $z \in D(z_i)$ at t and hit by a liquidity shock. Now impose stationarity of f_n. Dividing both side by Δ_n and rearrange terms, we have:

$$s(z_i)\frac{f_n(z_i)}{s(z_i)\Delta_n} - s(z_{i-1})\frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n} = -\alpha s(z_{i-1})\Delta_n \frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n} + \alpha \sum_{z\in D(z_i)} f_n(z).$$

Suppose $\varphi(z)$ has probability mass at some $z > 0$. Since $D(z)$ converges to $\varphi(z)$, $\sum_{z'\in D(z)} f_n(z')$ converges to $F[\varphi(z)] - F\left[\varphi(z)_-\right]$. Taking the limit as n goes to infinity and eliminating the term with Δ_n, we have:

$$s(z)\left[f(z) - f(z_-)\right] = \alpha\left\{F[\varphi(z)] - F\left[\varphi(z)_-\right]\right\}, \tag{123}$$

which implies (101) by taking $z = z^\star$. Suppose $\varphi(z)$ does not have probability mass (atomless) at some $z > 0$. Dividing both side by $s(z_{i-1})\Delta_n$ we have

$$\frac{s(z_i)\frac{f_n(z_i)}{s(z_i)\Delta_n} - s(z_{i-1})\frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n}}{s(z_{i-1})\Delta_n} = -\alpha\frac{f_n(z_{i-1})}{s(z_{i-1})\Delta_n} + \alpha\sum_{z\in D(z_i)}\frac{s(z)}{s(z_{i-1})}\frac{f_n(z)}{s(z)\Delta_n}.$$

Taking the limit as n converges to infinity, $D(z)$ converges to a function $\varphi(z)$ and $\sum_{z\in D(z_i)}\frac{s(z)}{s(z_{i-1})}\frac{f_n(z)}{s(z)\Delta_n}$ converges to $\frac{s[\varphi(z)]}{s(z)} f[\varphi(z)]$ if $\varphi(z) \leq z^\star$ (corresponding to $D(z_i) \neq \emptyset$), and 0 otherwise. Then we obtain Kolmogorov forward equations for all $z \in (0, z^\star)$

$$\partial_z\left[s(z) f(z)\right] = \begin{cases} -\alpha f(z) + \alpha\frac{s[\varphi(z)]}{s(z)} f[\varphi(z)], \text{ if } \varphi(z) \leq z^\star, \\ -\alpha f(z), \text{ otherwise.}\end{cases}, \tag{124}$$

where ∂_z is the differential functional defined as $\partial_z G(z) = \lim_{\Delta \to 0} \frac{G(z)-G(z-\Delta)}{\Delta}$. It implies (100).

To obtain boundary conditions, the dynamics of $z = z^{\star} = z_n$ implies

$$f_n(z_n, t+\Delta_n) = (1-\alpha\Delta_n) f_n(z_{n-1}, t) + (1-\alpha\Delta_n) f_n(z_n, t).$$

In any period of length Δ_n, a fraction $\alpha\Delta_n$ of agents are hit by a liquidity shock. Thus the fraction of agent with $z = z_n$ at $t + \Delta_n$ are a fraction $1 - \alpha\Delta_n$ of those who were agents with $z = z_{n-1}$ at t but not hit by a liquidity shock, plus the sum of fraction $1 - \alpha\Delta_n$ of those with $z = z_n$ at t not hit by a liquidity shock. Now impose stationarity of f_n. Rearranging terms we have

$$\alpha f_n(z_n) = (1-\alpha\Delta_n) s(z_{n-1}) \frac{f_n(z_{n-1})}{s(z_{n-1})\Delta_n}.$$

If $s(z_-^{\star}) = 0$, then we have the boundary condition

$$\lim_{z \uparrow z^{\star}} s(z) f(z) = 0, \tag{125}$$

which implies (103). If $s(z_-^{\star}) \neq 0$, then there is probability mass at $z = z^{\star}$, and $f_n(z_n)$ converges to $F(z^{\star}) - F(z_-^{\star})$. The boundary condition becomes

$$s(z_-^{\star}) f(z_-^{\star}) = \alpha\left[F(z^{\star}) - F(z_-^{\star})\right], \tag{126}$$

which implies (104). Finally, for another boundary $z = z_0 = 0$, the discrete time, discrete state-space KFE is given by

$$f_n(0, t+\Delta_n) = (1-\alpha\Delta_n) f_n(0, t) + \alpha\Delta_n \sum_{z \in D(0)} f_n(z, t). \tag{127}$$

Taking Δ_n to zero then we have both sides of (127) equal to $f(0)$, which does not impose any condition on $f(0)$.

B.3. Numerical Method to Compute a Stationary Equilibrium

Overview. In this section we provide a step-by-step numerical method to compute the stationary equilibrium with standard packages, for example Matlab. To efficiently solve the system we need to start from some initial values close to the solution. Step 1 suggests an efficient method to compute initial values of λ_0 and Υ: the solution to an economy with zero inflation and full depletion, which is close to the equilibrium if the money growth rate is not very large but $\overline{h}$ is not very low. Step 2 (or 2' under linear preferences) computes the system of DDE. Step 3 and 4 (or 4' under linear preferences) computes the KFE by partitioning into two regions such that the KFE is a forward

ODE in one region (Step 3) and becomes a backward DDE in another region (Step 4 or 4'). Step 5 solves λ_0 and Υ as fixed points.

Step 1a. Fix $y(z) = z$ and $\pi = 0$. Solve the following values for initiation:

$$\begin{aligned} h(\lambda^\star) &= c(\lambda^\star) \\ z^\star &= (U')^{-1}\left[\left(\frac{r+\alpha}{\alpha}\right)\lambda^\star\right], \\ p &= \frac{\xi}{h'(\lambda^\star) - c'(\lambda^\star)}, \end{aligned}$$

where ξ is the negative eigenvalue given by (111). Under linear preferences, we have $h(\lambda^\star) = c(\lambda^\star) = 0$, $\lambda^\star = 1$, $z^\star = (U')^{-1}\left(\frac{r+\alpha}{\alpha}\right)$ and $p = \frac{\alpha}{r+\alpha}U''(z^\star)$, as mentioned above.

Step 1b. Use ode45 routine of Matlab to solve $\lambda(z)$ from (113) backward from $z = z^\star$ to $z = 0$ with initial value $\lambda(z^\star) = \lambda^\star$ and $\lambda'(z^\star) = p$.

Step 1c. Having obtained $\lambda(z)$, use ode45 routine to solve $f(z)$ from (114) forward from $z = 0$ to $z = z^\star$ with initial value $f(0) = 1$. If $s\left(z^\star_-\right) > 0$ then we construct the probability mass $1 - F\left(z^\star_-\right)$ by the KFE boundary condition (104). It obtains $f(z)$.

Step 1d. The initial values of λ_0 and Υ are set to $\lambda_0 = \lambda(0)$ and $\Upsilon = \int_0^{z^\star} zf(z)\,dz / \int_0^{z^\star} f(z)\,dz$.

Step 2a. Jump to Step 2'a if under linear preferences. Given λ_0 and Υ (from Step 1 if it is the first time to run the iteration), use ddesd routine to integrate the system of DDE (96) and (98) from $\zeta = -\lambda_0$ with initial value $z(-\lambda_0) = 0$ and $\Omega(-\lambda_0) = U'(0)$ (or some arbitrary large value if $U'(0) = \infty$) until $h(-\zeta) - c(-\zeta) - \pi[z(\zeta) - \Upsilon] = 0$. Denote the stopping ζ and z as $\zeta^\star(\lambda_0, \Upsilon)$ and $z^\star(\lambda_0, \Upsilon)$. It obtains $z(\zeta)$ and $\Omega(\zeta)$.

Step 2b. Define

$$\begin{aligned} y(z) &\equiv (U')^{-1} \circ \Omega \circ z^{-1}(z), \\ z_d &\equiv z^\star - y(z^\star), \\ s(z) &\equiv h\left[-z^{-1}(z)\right] - c\left[-z^{-1}(z)\right] - \pi(z - \Upsilon). \end{aligned}$$

Jump to Step 3.

Step 2'a. Given λ_0 and Υ (from Step 1 if it is the first time to run the iteration), use ddesd routine to integrate the DDE (116) from $z = 0$ with initial value $y(0) = 0$ until either $z = \frac{h}{\pi} + \Upsilon$ or $U'[y(z)] = 1 + \frac{r+\pi}{\alpha}$. It obtains $y(z)$. Denote the stopping z as $z^\star(\lambda_0, \Upsilon)$. Define

$$\begin{aligned} z_d &\equiv z^\star - y(z^\star), \\ s(z) &\equiv \overline{h} - \pi(z - \Upsilon). \end{aligned}$$

Step 2'b. Use ode45 routine to integrate the following from $z = 0$ to $z^{\star}$

$$\lambda'(z) = \frac{(r + \alpha + \pi)\lambda(z) - \alpha U'[y(z)]}{\bar{h} - \pi(z - \Upsilon)},$$

with the initial condition $\lambda(0) = \lambda_0$. It obtains $\lambda(z)$. Define $\zeta^{\star}(\lambda_0, \Upsilon) = -\lambda(z^{\star})$.

Step 3. Define $\varphi(z)$ as the solution to $\varphi - y(\varphi) = z$. Consider the region $[z_d, z^{\star}]$, where the density function $f(z)$ is simply an ODE (102), which has closed-form solution (118) under linear preferences (and we need to construct the probability mass $F(z^{\star}) - F(z^{\star}_{-})$ by the KFE boundary condition (104)). Otherwise, use ode45 routine to solve $f(t)$ from (102) forward from $z = z_d$ to $z = z^{\star}$ with initial value $f(z_d) = 1$. It obtains $f(z)$ for all $z \in [z_d, z^{\star}]$.

Step 4. Jump to Step 4' if under linear preferences. Construct the "history" of $\phi(t)$ for all $t \in [-(z^{\star} - z_d), 0]$, by setting $\phi(t) = f(z_d - t)$, where $f(z)$ is given by Step 3. Use ddesd routine to solve $\phi(t)$ from (105) forward from $t = 0$ to $t = z_d$ with initial value $\phi(0)$ given by (106). Having obtained $\phi(t)$, set $f(z) = \phi(z_d - z)$ for all $z \in [0, z_d)$. Jump to Step 5.

Step 4'. Under linear preferences, use ode45 routine to solve $\phi(t)$ from (120) forward from $t = 0$ to $t = z_d$ with initial value $\phi(0)$ given by (121). Having obtained $\phi(t)$, set $f(z) = \phi(z_d - z)$ for all $z \in [0, z_d)$.

Step 5. Define a function $\Gamma(\lambda_0, \Upsilon) : \mathbb{R}^2_+ \to \mathbb{R}^2$, where the first and second coordinate are given by

$$\begin{aligned} \Gamma^{(1)}(\lambda_0, \Upsilon) &= (r + \alpha + \pi)\zeta^{\star} + \alpha U'[y(z^{\star})], \\ \Gamma^{(2)}(\lambda_0, \Upsilon) &= \frac{\int_0^{z^{\star}} z f(z)\, dz}{\int_0^{z^{\star}} f(z)\, dz} - \Upsilon, \end{aligned}$$

where $\zeta^{\star}$, $z^{\star}$, y and f are constructed given λ_0 and Υ from previous steps. Use fsolve routine to solve $\lambda_0^{\star}$ and $\Upsilon^{\star}$ such that $\Gamma(\lambda_0^{\star}, \Upsilon^{\star}) = 0$.

Step 6. Finally, the stationary equilibrium is given by the marginal value function $W'(z) = -z^{-1}(z; \lambda_0^{\star}, \Upsilon^{\star})$ and the density function $f(z; \lambda_0^{\star}, \Upsilon^{\star})$. Agents accumulate real balances according to $\dot{z} = s(z; \lambda_0^{\star}, \Upsilon^{\star})$, and the lumpy consumption is given by $y(z; \lambda_0^{\star}, \Upsilon^{\star})$. The above numerical algorithm works no matter the equilibrium features periodic full money depletion or periodic partial money depletion.

Appendix C. Search and bargaining with entry

We extend our model so that lumpy consumption opportunities take place in a decentralized market with search and bargaining, as in Shi (1995) and Trejos and Wright (1995). This extension will allow us to show that our model can easily be reinterpreted as a search-theoretic model of monetary exchange with a nondegenerate distribution of money holdings. Both our model and search-theoretic models assume random opportunities for lumpy consumption to generate a motive for precautionary saving in the form of liquid assets. We also endogenize the measure of producers in the decentralized market by introducing a free-entry condition along the lines of Pissarides (2000) and Rocheteau and Wright (2005). As a result, the frequency of lumpy-consumption opportunities, α, will be endogenous. We will show that our model is broadly consistent with existing models with degenerate distributions, and it yields new insights regarding the distribution of liquidity and inequalities across equilibria.

We assume that lumpy-consumption goods are produced by firms according to a linear technology that has labor as sole input. Upon finding a consumer in the decentralized goods market a firm hires labor instantly at the competitive wage (expressed in the numéraire good), w, and it uses the proceeds of the sale to pay its workers. As before, the numéraire good is produced according to a linear technology. As households must be indifferent between supplying their labor to firms in the decentralized goods market or producing the numéraire good according to the linear technology, it follows that $w = 1$.

We denote by n the measure of firms per households and $\alpha(n)$ the Poisson arrival rate at which a household meets a firm, with $\alpha' > 0$, $\alpha'' < 0$, $\alpha(0) = 0$, $\alpha(+\infty) = +\infty$, $\alpha'(0) = +\infty$, and $\alpha'(+\infty) = 0$. Because trades between households and firms are bilateral, the Poisson rate at which a firm meets a household is $\alpha(n)/n$. We consider the case where the household's utility for lumpy consumption is linear, $U(y) = Ay$, as it simplifies the bargaining problem, and we impose $u'(0) = +\infty$ so that $c > 0$.[24]

The lifetime expected utility of a household with z real balances solves the following flow Bellman equation:

$$rW(z) = \max_{c,h} \left\{ u(c) - v(h) + W'(z)(h - c) + \alpha(n) \left[Ay + W(z - py) - W(z) \right] \right\}, \qquad (128)$$

[24] In the case where $U(y)$ is not linear the value function, $W(z)$, is not linear either so that the household's surplus from a trade depends on the whole function, W, and not simply its derivative, λ. This makes the problem considerably harder.

where p denotes the price of one unit of lumpy consumption in terms of the numéraire. According to (128) the household chooses his flow consumption and hours of work to maximize his net flow utility, $u(c)-\upsilon(h)$, plus the continuation value of holding additional real balances, $W'(z)\dot{z}$ where $\dot{z}=h-c$. With Poisson arrival rate, $\alpha(n)$, the household is matched with a firm in the decentralized goods market, in which case it consumes y and transfers py real balances to the firm. In the following we conjecture that $W(z)$ is linear with $W'(z)=\lambda$.

Consider a match between a firm and a household with z units of real balances. The household and the firm negotiate the pair, (y,p). The surplus of the household is equal to the difference between the utility of consumption and the payment made to the firm, $S^H = Ay + W(z-py) - W(z) = Ay - \lambda py$ from the (assumed) linearity of W. The surplus of the firm is equal to its profits expressed in terms of the numéraire good, $S^F = (p-1)y$.[25] The equation for the Pareto frontier of the bargaining set is $S^F = z - (S^H + \lambda z)/A$. The Pareto frontier is linear and it shifts outward as z increases provided that $A > \lambda$.

We adopt the generalized Nash bargaining solution where the household's bargaining power is $\theta \in [0,1]$, i.e.,

$$(y,p) = \arg\max\,(Ay-\lambda py)^{\theta}\,(py-y)^{1-\theta} \quad \text{s.t.} \quad py \le z. \tag{129}$$

According to (129) the terms of trade maximize the weighted geometric average of the household's and firm's surpluses subject to the feasibility constraint that a household cannot transfer more real balances than what he currently holds. The Nash product can be rewritten as $(A-\lambda p)^{\theta}\,(p-1)^{1-\theta}y$, which implies $y = z/p$.[26] It is always optimal for the household to spend all his real balances as long as $A \ge \lambda$. Hence, the bargaining problem, (129), reduces to

$$p = \arg\max\left\{\frac{(A-\lambda p)^{\theta}\,(p-1)^{1-\theta}}{p}\right\}. \tag{130}$$

The first-order condition to this problem gives

$$p = \frac{A}{\theta A + (1-\theta)\lambda}. \tag{131}$$

In order to interpret (131) we can express the value of money in terms of lumpy consumption as $1/p = \theta + (1-\theta)\lambda/A$. So the price of money is a weighted average of the bid price of the firm, 1,

[25] It would be equivalent to express the firm's surplus in terms of the household's marginal utility by multiplying the profits by λ since the scaling of agents' payoff is irrelevant under Nash bargaining.

[26] It follows that our specification for $U(y)$ avoids issues related to the non-monotonicity of the Nash solution. For details about this issue, see Aruoba, Rocheteau, and Waller (2007). In our case both the household's surplus and the firm's profits are monotone increasing with the household's real balances.

and the ask price of the household, λ/a, where the weights reflect each agent's bargaining power. The markup of the price over the marginal cost is

$$p-1=\frac{(1-\theta)}{\theta A+(1-\theta)\lambda}\left(A-\lambda\right). \tag{132}$$

From (132) the markup is decreasing with the household's bargaining power: if $\theta=1$, then the markup is zero as in the competitive case studied so far; if $\theta=0$, then the markup is $A/\lambda-1$ and the household's surplus is zero.

Substituting the surplus of the household by its expression into (128) the flow Bellman equation can now be rewritten as

$$rW(z)=\max_{c,h}\left\{u(c)-v(h)+\lambda(h-c)\right\}+\alpha(n)\left(A-\lambda p\right)\frac{z}{p}. \tag{133}$$

Differentiating (133), the marginal value of money solves $r\lambda=\alpha(n)(A-\lambda p)/p$ or, equivalently,

$$\lambda(t)=\frac{\alpha(n)A}{p\left[r+\alpha(n)\right]},\ \text{ for all } t\in\mathbb{R}_{+}. \tag{134}$$

As conjectured the marginal value of money is independent of the household's real balances. It is simply equal to the expected discounted marginal utility of lumpy consumption divided by its price, $\lambda=\mathbb{E}\left[e^{-rT_1}A/p\right]$. From (134) if the measure of firms in the market goes to infinity, $\alpha(n)\rightarrow+\infty$, or if households are infinitely patient, $r\rightarrow 0$, the marginal value of real balances is maximum and equal to $\lambda=A/p$.

From (134) and (131) we can solve for the marginal value of real balances and the price of lumpy consumption in closed form,

$$\lambda = \frac{\theta\alpha A}{r+\theta\alpha} \tag{135}$$

$$p = \frac{r+\theta\alpha}{\theta\left(r+\alpha\right)}. \tag{136}$$

From (135) the marginal value of real balances increases with $\theta\alpha$. If households have more bargaining power, or if they receive more frequent trading opportunities, they increase their saving flow, $h-c$. From (136) the markup, $(1-\theta)r/\theta(r+\alpha)$, decreases with θ and α and it increases with r. So if households trade more frequently, or if they are more patient, the decentralized goods market becomes more "competitive".

In order to participate in the market a firm must incur a flow cost, $k>0$, in terms of the numéraire good.[27] This free-entry condition can be written as

$$k = \frac{\alpha(n)}{n}\left(\frac{p-1}{p}\right)\int[1-F(z)]\,dz. \tag{137}$$

The left side of (137) is the flow cost incurred by the firm to participate in the market. The right side is the expected profit of the firm: with Poisson rate, $\alpha(n)/n$, the firm is matched with a household, in which case it enjoys a profit per unit sold equal to $p-1$, and sells z/p units of lumpy consumption to a consumer drawn at random from F. From (**??**) the distribution of real balances is $F(z) = 1 - e^{-\alpha z/(h-c)}$. Substituting p and F by their expressions, (137) can be rewritten as

$$k = \frac{r(1-\theta)}{n\,[r+\theta\alpha(n)]}\,(h-c)\,. \tag{138}$$

The levels of consumption and hours are given by $u'(c) = v'(h) = \lambda$ (since $u'(0) = \infty$). A steady-state search equilibrium is a triple, (p, λ, n), that solves (135), (136), and (138).

In the following we characterize some properties of equilibria. First, for an equilibrium with entry to exist the household's bargaining power must be neither too high nor too low. From (137) if $\theta = 1$ then firms make not profits and hence they do not participate in the market, $n = 0$. From (131) if $\theta = 0$ then households get no surplus, $\lambda p = A$, and hence from (135) they have no incentive to accumulate real balances, $\lambda = 0$, i.e., the marginal value of money would be inconsistent with agents saving, $h - c > 0$.

Second, if an equilibrium with entry exists, then generically there are multiple steady-state equilibria. To see this, let $\underline{\lambda}$ be the solution to $c = h$ and $\underline{n}$ the value of n that solves (135) when $\lambda = \underline{\lambda}$. As $n \searrow \underline{n}$ the frequency of trading opportunities is so low that households have no incentive to save. As $n \to \infty$, households' outside options are so good that the markup is driven down to 0. In both cases the right side of (138) goes to 0. Therefore, if a solution to (138) exists, then there are an even number of solutions. This multiplicity arises because the profits are non-monotonic with the measure of firms in the market, as illustrated in Figure **??**. Indeed, for given p an increase in n induces households accumulate more real balances, which tends to raise firms' profits, but from (136) p decreases due to a competition effect. At the high equilibrium the measure of firms is higher, households have more frequent trading opportunities, and the price of lumpy consumption is lower. Across equilibria the distribution of nominal balances, $G(m) = 1 - e^{-m/M}$, is unchanged.

[27] The entry cost of firms is financed by households. The total profits of firms net of entry cost are zero, and hence they do not affect households' wealth. For a model where agents accumulate liquid wealth in the form of claims on firms' profits, see Rocheteau and Rodriguez (2013).

This implies that the distribution of real wealth has a higher mean and a higher variance at the high equilibrium: in an ex-ante sense households are better off, but ex-post inequalities are larger.