Aggregate Fluctuations and
The Network Structure of Intersectoral Trade

Vasco M. Carvalho
University of Chicago,
Department of Economics.
Job Market Paper.

Abstract:

This paper analyzes and models the flow of intermediate inputs across sectors by adopting a network perspective on sectoral interactions. I apply these tools to show how fluctuations in aggregate economic activity can be obtained from independent shocks to individual sectors. First, by interpreting data in detailed input-use matrices through this lens, I characterize the structure of input trade in the U.S.. On the demand side, a typical sector relies on a small number of key inputs and sectors are homogeneous in this respect. However, in their role as input-suppliers sectors do differ: many specialized input suppliers coexist alongside general purpose sectors that function as hubs to the economy. The paper then constructs a network model of input-use matrices that can reproduce these connectivity features in the data. In a standard multisector setup, I then use this model of input-use matrices to provide analytical expressions linking the variability in aggregates to the network structure of input trade. I show that the presence of sectoral hubs - by coupling production decisions across the economy - leads to fluctuations in aggregates. Furthermore, I show that this network approach provides a common framework for hitherto opposing arguments on how fast the volatility of aggregates decays with the number of sectoral technologies.

[0]This version: November 15th, 2007. For comments and contact, mail at carvalho@uchicago.edu. The latest version of this paper can be found at http://home.uchicago.edu/~carvalho. This research was made feasible by financial support from the Portuguese Ministry of Science and Technology, the Gulbenkian Foundation and The University of Chicago which I gratefully acknowledge. I thank my advisor, Lars Hansen, and my committee members, Robert Lucas and Timothy Conley. For comments and encouragement I thank Luis Amaral, Xavier Gabaix, Matthew Jackson, Marcin Peski, Hugo Sonnenschein, Randall Verbrugge and participants at the 2007 Econometric Society Summer Meetings, the Economic Dynamics and Economic Theory Working Groups at the University of Chicago and Luis Amaral's group at Northwestern University. All errors are mine.

1. Introduction

Comovement across sectors is a hallmark of cyclical fluctuations. A longstanding line of research in the business cycle literature asks whether trade in intermediate inputs can link otherwise independent technologies and generate such behavior. The intuition behind this hypothesis is clear: factor demand linkages can provide a source for comovement, as a shock to the production technology of a general purpose sector - say, petroleum refineries - is likely to propagate to the rest of the economy. In this way, cyclical fluctuations in aggregates are obtained as synchronized responses to changes in the productivity of narrowly defined but broadly used technologies.

Though intuitive, this hypothesis is faced with a strong challenge: by a standard diversification argument, as we disaggregate the economy into many sectors, independent sectoral disturbances will tend to average out, leaving aggregates unchanged and yielding a weak propagation mechanism; see the discussion in Lucas (1981) and the irrelevance theorems of Dupor (1999)[1].

In this paper, I take on this challenge by adopting a network perspective on sectoral interactions. From this vantage point, I provide answers to the following questions. First, given the availability of detailed input use data, can we identify the main features of the structure of linkages across sectors? Second, can we construct counterfactual models of input-output matrices that are able to mimic this connectivity structure and are amenable to use in standard multi-sector models? If so, can we use these models of connectivity to provide analytical results linking the variability of aggregates to the network structure of input flows? Finally, under what assumptions on the network structure can we render ineffective the shock diversification argument of the previous paragraph?

The argument linking the answers to these questions is the following: when determining whether a sectoral shock propagates or not, the number of sectoral connections originating from the source of the shock is the crucial variable to consider. Furthermore, if the number of connections varies widely across sectors, some shocks will propagate throughout the economy and persist through time while others will be short-lived and only propagate locally. As a consequence, economies where every sector relies heavily on only a few sectoral hubs - general purpose input suppliers - will show considerable conductance to shocks in those technologies. Conversely, as the structure of the economy is more diversified, different sectors will rely on different technologies and exhibit only loosely coupled dynamics. The answer to the law of large numbers arguments in Lucas and Dupor thus lies in

[1] For the most part, the answer to this challenge has been in the empirical vein. Long and Plosser (1983, 1987), Norrbin and Schlagenhauf (1990) and Horvath and Verbrugge (1996) document comovement of sectoral output growth series through vector autoregressions. They all add that the explanatory power of a common, aggregate shock is limited on its own and diminished once sector specific shocks are entertained. Shea (2002) and Conley and Dupor (2003) go further and devise ways of testing - and rejecting- the hypothesis that sectoral comovement is being driven by a common shock. Both emphasize positive cross-sectional covariance in sectoral productivity growth and show how this can be explained by the existence of sectors with similar input demand relations. Concurrently, the strategy of using actual input-output data in large scale multisector models generates aggregates that are quantitatively similar to data and to one-sector real business cycle models; see Horvath (2000).

understanding and modelling this tension between specialization and reliance on general purpose technologies.

I build on and extend this key notion of sectoral connectivity - first put forth in Horvath (1998) - by analyzing the structure of intersectoral linkages through graph-theoretical concepts and tools. The starting point for this analysis is to establish that an input-use matrix can be characterized as a network. That is, given a detailed list of production sectors and information on who trades with whom, I can map such trade interactions to a network of input trade where sectors become vertices and input-supply relations are represented by directed arcs. Given that these lists do exist in the form of input-use matrices, providing data at a fairly disaggregated level, one can ask questions related to the properties of such networks. In particular, from detailed input use matrices for the U.S. economy, I characterize heterogeneity across sectors along the input-demand and input-supply dimensions by exploiting well-defined measures of connectivity in a network.

Thus, along the demand side, I characterize sectors by the number of inputs used. This maps directly into the in-degree sequence of a network, giving for each sector the number of distinct inputs used in the production of the corresponding good. Reflecting specialization, narrowly defined production activities are found to rely only on a small number of inputs. Though different sectors use a different set of inputs, sectors can be characterized as homogeneous along the demand side in that the number of inputs used does not differ much across sectors.

This is to be contrasted with extensive heterogeneity across sectors in their role as input suppliers. In the data, highly specialized input suppliers - say, for example, optical lens manufacturing - coexist alongside general purpose inputs, such as iron and steel mills or petroleum refineries. Specifically, I characterize the empirical out-degree distribution of input-supply links - giving the number of sectors to which any given sector supplies inputs to - as a power law distribution. What makes this power law parameterization attractive is the following argument: the upshot of fat-tails, characteristic of power law degree distributions, is that a small, but non-vanishing, number of sectors will emerge as large input suppliers - or hubs - to the economy. As such, productivity fluctuations in these general purpose sectors can have a disproportionately larger effect in the aggregate economy.

I then show how to incorporate these network insights in multisector models by constructing a data generating process for input-use matrices. Informed by the analysis of input trade data, I show how one can construct and distinguish across models of input-use matrices by specifying three key parameters: the number of sectors under consideration, the average number of inputs used in a sector's production process and a parameter controlling the heterogeneity across sectors in their input-supply role. This enables me to build counterfactual connectivity structures for intersectoral trade. In particular, I show how to specify classes of input-use matrices that mimic the homogeneous-in-demand, heterogeneous-in-supply characterization of U.S. data.

Finally, I employ these models of input-use matrices in standard multisector economies and characterize analytically how the variability of aggregates is mediated by the network structure of these matrices. To achieve this I use a version of Long and Plosser's (1983) multisector setup- used by Horvath (1998) and Dupor (1999)[2]. I first demonstrate that the opposing conclusions reached by Horvath (1998) and Dupor (1999) can be traced to very particular restrictions on the intersectoral network structure, namely on its outdegree sequence. Thus, I show that the exact results in Horvath and Dupor - regarding the decay in the volatility of aggregates as a function of the total number of sectors - can be shown using known properties of simple network structures: complete regular networks in the case of Dupor and very particular sparse structures - star networks - in the case of Horvath.

More generally, I derive analytical expressions linking directly the degree of fat-tailness of the distribution of input-supply links with the strength of the propagation mechanism in a multisector economy. Taking the estimates obtained from input-use data as a guide to parameterize this class of matrices, I argue that the structure of intermediate input trade offers enough conductance of sectoral shocks to render the diversification argument of second order. In other words, in multisector economies, the presence of sectoral hubs facilitates the propagation of technological shocks and postpones the applicability of law of large numbers arguments.

The paper is closest in spirit to the contributions of Bak et al.(1993) and Scheinkman and Woodford (1994) by stressing the importance of the structure of input-supply chains in the transmission of shocks across sectors and, as a consequence, to aggregates. In comparison with these papers, by placing sectors on a network of input flows - rather than on a lattice - I allow for more general, and arguably more realistic, patterns of connections between sectors. Regarding the characterization of the decay behavior in the volatility of aggregates as a function of heterogeneity in the underlying production units, this paper is closely related to the recent analysis in Gabaix (2005). In comparison to the latter, I use a different model, a different aggregate statistic and different tools to approach the same problem. More importantly, in contrast to Gabaix (2005), the explanation here rests explicitly on the dynamic propagation of shocks through a network of technologies, rather than on careful total factor productivity accounting when some firms are large.

The idea of characterizing input-use relationships through graph-theoretical tools is not new, albeit it has merited only limited attention[3]. In the context of traditional input-output analysis Solow (1952) is, to the best of my knowledge, the first reference recognizing that an input-output matrix can be mapped into a network. These tools have resurfaced only sporadically in the analysis of static and dynamic input-output systems; see Rosenblatt (1957), Simon and Ando (1961) or Szydl

[2]To ensure comparability of results with Horvath (1998) and Dupor (1999) this paper preserves the model and the aggregate statistic considered therein and focuses on generalizing the set of admissible input-output matrices under consideration. The model is closely related to the original multisector real business cycle model of Long and Plosser (1983) and the myriad of extensions and applications since developed in the literature. Other setups have been explored: Cooper and Haltiwanger (1990), Bak et al (1993) and Scheinkman and Woodford (1994) all stress the role of inventories. Jovanovic (1987) instead focuses on the role of complementarities among sectors, as does the recent contribution of Nirei (2005), where this is coupled with indivisibilities in investment. Murphy et al. (1989) focus on aggregate demand spillovers.

[3]This stands in sharp contrast to the recent but burgeoning use of network tools in microeconomics; see Jackson (2005) for a comprehensive review or Vega-Redondo (2007) for a book length introduction.

(1985). Fisher and Vega-Redondo (2007) offer the only recent treatment of input-trade relations as a network and focus on identifying what are the "central sectors" in the US economy. However they do not address the implications of this notion for the business cycle literature[4].

In terms of tools this paper borrows heavily from recent work on networks and in particular, random graphs. Newman (2003) and Li et al (2006) offer good reviews mapping out recent theoretical advances and link them to a growing number of applications. Durrett (2006) and Chung and Lu (2006) provide book-length treatments of the tools used here. In particular, a model of random graphs with given expected degree sequences, set out in Chung and Lu (2006), forms the basis for my data-generating process for input-use matrices.

2. Overview in a Static Multisector Economy

Consider the following static multisector economy, a simplified version of the setup presented in Shea (2002). There is a representative household whose utility is affected by the levels of consumption of M goods, $\{C_j\}_{j=1}^{M}$, and total hours of work (L). Assume log preferences over M different goods, with weights given by $\{\theta_j\}_{j=1}^{M}$, and specify a time endowment of L, to be shared among the M production activities.

$$
\begin{aligned}
U(\{C_j, L_j\}_{j=1}^{M}) &= \sum_{j=1}^{M} \theta_j \log(C_j) - L, && (1)\\
\text{with } \sum_j \theta_j &= 1 \text{ and } \theta_j > 0, \forall_j, && (2)\\
\text{and } \sum_j L_j &\leq L && (3)
\end{aligned}
$$

The M productive units, or sectors, each produce a different good that can either be allocated to final consumption (by the household) or as intermediate goods to be used in the production of other goods. This is just a static version of the production technologies introduced in Long Plosser (1983). In particular, assume production functions are of the Cobb-Douglas, decreasing returns to scale variety:

$$
\begin{aligned}
Y_j &= Z_j L_j^{\beta_j} \prod_{i \in \check{S}_j} M_{ij}^{\gamma_{ij}} && (4)\\
1 &> \beta_j + \sum_{i \in \check{S}_j} \gamma_{ij} && (5)\\
Z_j &= \exp(\varepsilon_j),\ \varepsilon_j \sim N(0, \sigma_j^2) && (6)
\end{aligned}
$$

[4] Rauch (1999) is another recent exception in the macroeconomics literature. He adopts a network view to disaggregated world trade flows.

where M_{ij} is the amount of good i used as an intermediate input in the production of sector j. Z_j is a Hicks-neutral, log-normal, productivity shock to good j technology, to be drawn independently across sectors. The 'supply-to' set $\check{S}_j$ completes the description of technology in this simple economy. It gives, for every sector j, the list of goods that are necessary as inputs in the production of good i. Finally, market clearing implies that:

$$Y_j = C_j + \sum_{i:j\in\check{S}_i} M_{ji}, \;\; \forall j = 1, ..., M \tag{7}$$

It is a standard exercise to solve for the competitive equilibrium of this economy; see Shea (2002). Substituting the equilibrium input choices into the production function, simplifying and taking logarithms yields, in vector notation:

$$\mathbf{y} = \boldsymbol{\mu} + (I - \Gamma)^{-1'}\boldsymbol{\varepsilon} \tag{8}$$

where $\boldsymbol{\mu}$ is an M-dimensional vector of constants dependent on model parameters only. The pair of vectors M-dimensional vectors $(\mathbf{y}, \boldsymbol{\varepsilon})$ give, respectively, the log of equilibrium output and the log of the productivity shock for every sector in the economy.

Γ is a key object for the analysis of this paper. It is an $M \times M$ (non-negative) input-use matrix with typical element γ_{ij} where $(I - \Gamma)^{-1'}$ is well defined. Note that an entry γ_{ij} in this matrix will be zero whenever sector i's output is *not* a necessary input for the production of good j, that is whenever $i \notin \check{S}_j$. This simply means that there are no substitution possibilities between the inputs used in sector j's production and the i^{th} good. Notice also that $\gamma_j = \sum_i \gamma_{ij}$, the j^{th} column sum of Γ, gives the degree of returns to scale to intermediate inputs for sector j.

Independent technological shocks at the sectoral level propagate through the input-use matrix downstream[5], affecting the costs of input-using sectors and potentially influencing aggregate activity. The analysis of this paper focuses on the interplay between the structure of intermediate input use - the structure of the input output matrix, Γ- and the strength of this propagation mechanism as evaluated by the volatility of aggregate output. For analyzing the latter, and keeping in line with the literature (see Horvath, 1998, or Dupor, 1999), I will consider the following aggregate statistic:

$$\sigma_Y^2(\Gamma) \equiv E\left[\frac{\sum_{i=1}^{M}(y_i - \mu_i)}{M}\right]^2 \tag{9}$$

Note that $\sum_{i=1}^{M}(y_i - \mu_i)$ is the sum of log sectoral output (demeaned), or the log of the geometric sum of sectoral output. Dividing this by the number of sectors gives a log-linear approximation to the more obvious aggregate statistic, the log of total output. The difficulty with this latter statistic

[5] In general, technology shocks also have effects on upstream demand, by changing the demand of inputs necessary to produce output and changing sectoral output level. In the current setting, due to the Cobb-Douglas assumption on preferences and technology, these two effects cancel out exactly; see Shea (2002).

is that it involves a nonlinear function of the vector of shocks. The average of log sectoral output can therefore be taken as the log-linearization of this function[6]. Using this aggregate statistic will allow me to compare my results directly with those in Horvath (1998) and Dupor (1999).

We are now left with choosing a specification for the pattern of zeros in the input-output matrix Γ, or equivalently, announcing the necessary inputs for each of the M production activities (the $\check{S}_j$ lists). Consider the two following abstract, and rather extreme, cases. Fix an M and contrast an economy where only one sector is a material input supplier to all the other sectors with an economy where every sector supplies to all the other sectors in the economy. These two polar cases for the pattern of input-use relationships in an economy map exactly into very standard network representations, where the vertex set is given by the set of sectors in the economy and a directed arc from vertex (sector) i to vertex j represents a intermediate input supply link.

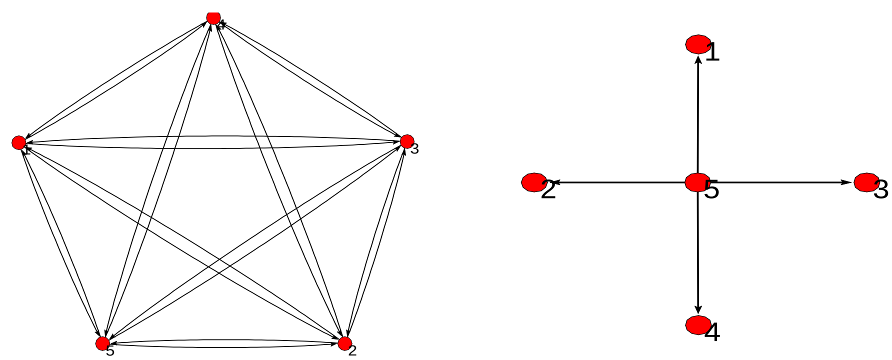


Figure 1: *Complete (l.h.s.) and Star (r.h.s.) input-supply structures for a 5 sector economy.*

Thus an economy where each sector is an input supplier to every other sector in the economy can be represented by a *complete* network, where for any two pair of vertices there is a directed arc from one to the other. Likewise, an economy where there is only one material input supplier maps directly into a *star* network, where one vertex acts as a hub with directed arcs from this vertex to all other vertices. An intermediate case is given by a $N-$*star* network, where N out of M, sectors in the economy act as material input suppliers to every sector and the remaining ones are solely devoted to final goods production. Figure 1 depicts intersectoral input relations under these two extreme cases - complete and star - for a five sector economy.

Notice also that I can map this connectivity structure thus: specify an $M \times M$ binary matrix A where $A_{ij} = 1$ if sector i supplies to sector j and $A_{ij} = 0$ otherwise. In this way the matrix A corresponding to a complete network will be given by a matrix of all ones (for every pair of sectors,

[6]This approximation is valid whenever the variances of the $\varepsilon_j' s$ are small and the mean of sectoral output does not differ much across sectors (Dupor, 1999).

i, j, $A_{ij} = 1$) and the one corresponding to an N-star network will have N rows of ones and $M - N$ rows of zeros.

The final step is going from the binary nature of network links to the input-output matrix Γ, giving elasticities of substitution for material inputs. The simplest working assumption is that every sector, regardless of what its particular input list is, uses its inputs in equal proportions. In short, all necessary inputs are assumed equally necessary for any given sector. In terms of model parameters this translates to:

Assumption 2.1. *For all sectors $j = 1, ..., M$, $\gamma_{ij} = \gamma_{kj}$ for all inputs i and k necessary to the production of output in sector j, that is for all pairs $\gamma_{ij}, \gamma_{kj} \neq 0$.*

This assumption will be used throughout the paper. It simplifies considerably the analysis by imposing homogeneity along the intensive margin of intersectoral trade - necessary inputs for any given sector have a symmetric role- while allowing for substantial heterogeneity along the extensive margin - sector can differ in the number of sectors they supply to. Under this assumption the analysis in Section 4 below, shows that the input-output matrix Γ conveniently factors into the product of two square M-dimensional matrices:.

$$\Gamma = A.D_\gamma \tag{10}$$

where D_γ is a diagonal matrix with the i^{th} diagonal element given by $\frac{\gamma_i}{d_i^{in}}$ where d_i^{in} is the number of inputs sector i uses (i.e. the cardinality of the $\check{S}_i$ list or the number of inlinks). A is the binary connectivity matrix with a typical element a_{ij} equal to one if sector i supplies to sector j and zero otherwise.

With this assumption and the resulting factorization of Γ in hand, I can derive an analytical expression for the aggregate volatility statistic, σ_Y, and compare the properties resulting from different, hypothetical, intersectoral trade structures. Proposition 1 below gives an expression for this statistic in complete network structure settings, $\sigma_Y^2(\Gamma_C)$, and that resulting of $N-$star structures, $\sigma_Y^2(\Gamma_{N-star})$

Proposition 2.2. *Assume that the share of material inputs, γ_j is the same across sectors, $\gamma_j = \gamma$, $j = 1, .., M$, and that sectoral volatility $\sigma_j^2 = \sigma^2$ for all sectors $j = 1, ..., M$. Then, for any M, a static $M-$sector economy aggregate volatility of equilibrium output, σ_Y^2 is given by:*

$$\sigma_Y^2(\Gamma_C) = \left(\frac{1}{1-\gamma}\right)^2 \frac{\sigma^2}{M} \tag{11}$$

for any complete intersectoral trade structure on M sectors, and

$$\sigma_Y^2(\Gamma_{N-Star}) = \left(\frac{N}{M} + \frac{2\gamma}{1-\gamma}\right)\frac{\sigma^2}{M} + \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\sigma^2}{N} \tag{12}$$

for any N star intersectoral trade structure on M sectors.

Notice that with the additional assumptions imposed in the proposition, sectoral technologies in these economies are symmetrical in all respects except, possibly, that some supply to more sectors than others. This is borne out in the expressions for aggregate volatility: they depend only on the share of material inputs, γ, sectoral volatility,σ^2, and the number of effective input suppliers in each case, M or N. The first two effects are standard. Thus, the higher the share of material inputs in production the more aggregate volatility will be affected by disturbances working through the input-output network[7]. Similarly, greater sectoral volatility translates mechanically into heightened volatility in aggregates.

Of interest to this paper is the dependence of aggregate volatility on the number of sectors. Thus, the expression for complete intersectoral structures of input trade is a particular case of the results in Dupor (1999). It echoes Lucas' (1981) law of large numbers argument: aggregate volatility scales with $1/M$. To understand how effective the shock diversification argument is in this case notice the following: holding sectoral productivity variance fixed as I move from a five sector economy to a five hundred sector economy, aggregate volatility will be a hundred times smaller. Conversely, to recover an aggregate σ_Y^2 of the order of two percent in a five hundred sector economy, would require stipulating sectoral volatilities, σ^2, to be five hundred times larger, an unreasonable magnitude at any time scale. From this, Dupor (1999) concludes that the input-output matrix provides a poor propagation mechanism for independent sectoral shocks.

The result for $N-$star sectoral networks offers a different, if somewhat predictable view. If there are only N sectors acting as intermediate input suppliers, the diversification of shocks argument underlying law of large number arguments only applies to those sectors[8]. Thus, in an economy where the effective number of input suppliers is small, the law of large numbers will be postponed relative to that of Dupor (1999): aggregate volatility now scales with N, the slowest decaying term in expression (12). This is Horvath's (1998) argument: limited sectoral interaction - of a very particular form - will give rise to greater aggregate volatility from sector specific shocks. The difficulty with this result is that the modeller is now left to specify, for each M, what is the number of input suppliers in an economy, N. From input-output data, Horvath (1998) argues that N - the number sectors with full rows in input-output matrices - grows slowly with M: Horvath argues for an N of order $\sqrt{M}$, which would slow down the rate of convergence. This would now yield a ten fold decrease in aggregate variability as we move from five to five hundred sectors.

In this way, two very particular assumptions on the connectivity structure of intersectoral trade generate predictions on the variability of aggregates that differ by an order of magnitude. This means that finding a better way to model networks of input trade can not only help solve this controversy but also has the potential of offering a theory where reasonable magnitudes of sectoral volatility yield

[7] This multiplier effect of $(1/1-\gamma)$ on aggregates is a standard feature of multisector economies; see for example the discussion in Jones (2007a, 2007b)

[8] Notice that for any $N < M$ the second term in the expression for $N-$Star networks dominates the rate of convergence (the first term converges to zero faster). As it should be, the two expressions in Proposition 2.2. will be equal for $N = M$. Finally, if N is fixed for any M the law of large numbers breaks down completely.

non-trivial aggregate volatility. Mechanically, we need only a theory of intersectoral connectivity that yields aggregate volatility decaying with M^{υ}, where υ is close to zero. The remainder of this paper does just this by going beyond these two extreme cases and building a model of sectoral interactions on a network. Figure 2 depicts the starting point of the analysis. It shows a considerably more intricate network of intersectoral input flows: that of the U.S. economy in 1997.

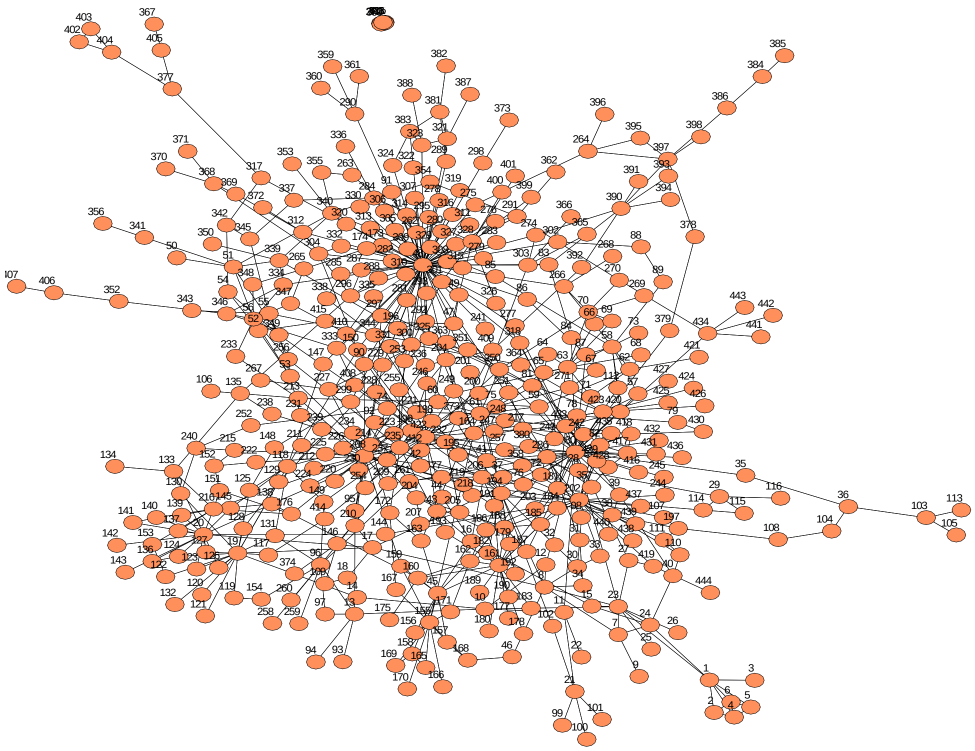


Figure 2: *Intermediate input flows between sectors in the U.S. economy in 1997. Each vertex corresponds to a sector in the 1997 benchmark detailed input-use matrix published by the BEA. For every input transaction above 5% of the total input purchases of the destination sector, a link between two vertices is drawn.*

Each dot - or vertex - corresponds to a sector defined at the NAICS 4-6 digit level of disaggregation in the BEA detailed input use tables, for a total of 471 sectors. Each link in the figure represents an input transaction between sector i to sector j, provided sector i supplies more than 5% of sector j total intermediate input purchases[9].

From this vantage point, Section 3 in the paper offers a two-pronged characterization of the structure of input flows by taking into consideration the direction in each of these links. Thus from by considering links from the perspective of the destination vertex I can analyze sectors in their role as input-demanders. I find that sectors are homogeneous along this dimension: the average of sectoral production technologies relies on a relatively small number of key inputs and sectors do not

[9] I exclude loops from the network for presentation purposes. Loops correspond to *intra*sectoral trade and are a well documented feature of detailed input use-matrices (see for example Jones, 2007b).

differ much in this respect. This is the upshot of specialization occurring at the level of narrowly defined production technologies.

However, looking at the source vertices of these links, another feature emerges. In their role as input-suppliers, I find sectors differing widely. Indeed, a first order feature of data is the presence of both hub-like sectors, supplying general purpose inputs to the rest of the economy and peripheral sectors, supplying specialized inputs to a limited number of sectors. In section 3 of this paper I strengthen this characterization by looking into every detailed input-use matrix available since 1972 and finding the same homogeneous-demand, heterogeneous-supply connectivity patterns.

In section 4 of the paper I build theoretical counterparts to detailed input-use matrices. This is achieved by approaching the problem of modelling input-use matrices in the same fashion as in the present section: splitting Γ into the product of a connectivity matrix - giving input supply relations between sectors - and a diagonal matrix setting the value of the corresponding elasticities of substitution. In particular, I offer a way if designing intersectoral connectivity structures that incorporate the two first order-features of the data: sparse and homogeneous input demand and strongly heterogenous input supply technologies.

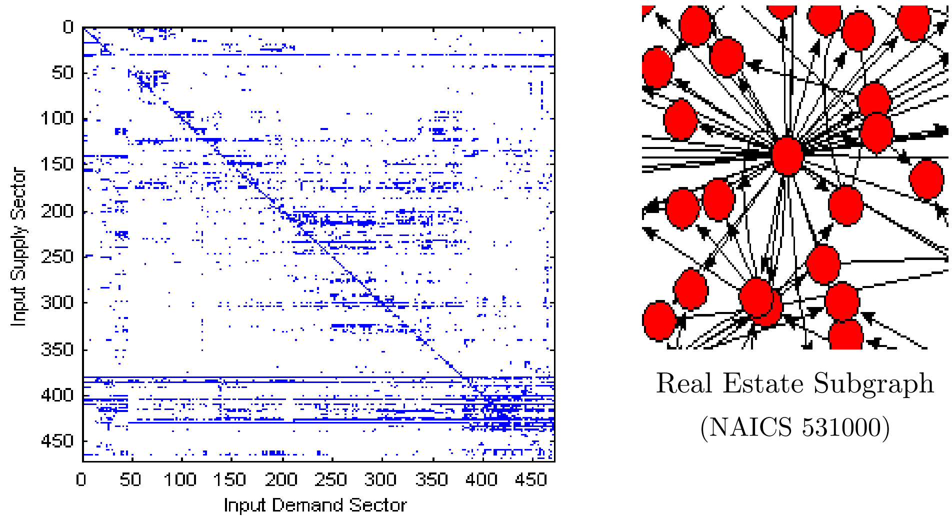
The remainder of the paper (Sections 4.3 and 5) is devoted to incorporating these models of input-demand relations in standard multisector setups. Thus in Section 4.3. I return to the simple static model presented in this section and show how the aggregate volatility statistic decays when I use my model of input-use matrices. The results generalize the characterization given in Proposition 2.2. and have a tight link with the recent contribution of Gabaix (2005). Namely, I derive analytical results showing that as we disaggregate the economy into many sectors, aggregate volatility decay is a function of how diversified the structure of input trade is. In Section 5, I show that exactly the same decay behavior obtains when I move to the dynamic multisectoral models considered in Horvath (1998) and Dupor (1999). In particular, I show that the decay characterization extends both to the variance and persistence of aggregate variables. Section 6 concludes.

3. Network Properties of Input Flows.

In this section I introduce some basic graph terminology and show how input-use matrices can be usefully described by these concepts. In particular, I introduce the notion of a degree sequence of a graph[10] and exploit it to characterize connectivity in inter-sectoral trade. Throughout I map these concepts to data from US input-use matrices. Namely I use benchmark detailed input-use tables available through the B.E.A.. The Figure below depicts the 1997 input-use matrix. It follows the 1997 NAICS definitions and yields a fine disaggregation of inter-sectoral trade at the 4-6 digit NAICS

[10] Following much of the literature I use the terms graph and network interchangibly.

definition level[11]. The use table gives the 1997 value (in millions of dollars), at producers' prices, of each commodity used as an intermediate input in the production each industry. I drop import, scrap, government industry and government demand, household and inventory valuation data. With this I get a detailed input-use matrix of dimension 471×471 sectors[12].



Real Estate Subgraph
(NAICS 531000)

Figure 3: *On the l.h.s. is the detailed input-use matrix for the US economy in 1997 (Source: BEA). Each dot corresponds to an input transaction from row (supplying) sector i to column (demanding) sector j. Only transactions above 1% of the total input purchases of a sector are displayed. On the r.h.s is a graph centered around the row sector Real estate (NAICS code 531000) where for each dot on the corresponding row of the l.h.s. matrix I draw a directed arc.*

On the left hand side of Figure 3 is a snapshot of the 1997 input-use matrix, dots corresponding to input-supply relations from row-sector i to column-sector j, provided sector i supplies 1% or more of the total input purchases of sector j. This is a sparse matrix in that the number of non-zero elements is small relative to the number of possible entries in the input-use matrix (under one tenth). This is as expected: at very fine disaggregation levels, most sectoral production processes are highly specific with respect to the intermediate inputs used. Thus, these limited substitution possibilities translate into sparse columns.

[11] To strengthen my analysis I also characterize the connectivity structure of every detailed input-use matrix available from 1972 to 1992. These are available on a 5 year interval and are based on a SIC classification, in contrast to the NAICS system adopted since 1997. While individual sectors are not immediately comparable between the two systems, the structure of zeros in these matrices - the object of analysis here - will be shown to be remarkably stable.

[12] I equate commodities with sectors as in the theoretical model where good i is produced exclusively by sector i. I am thus implicitly assuming that the make table in the input-output is diagonal. The same assumption is made in Horvath (1998) and Conley and Dupor (2003).

However, there are some full rows corresponding to sectors supplying inputs to many other sectors in the US economy. The right hand side of Figure 3, displays a graph corresponding to one of these sectoral-hubs: real estate. Indeed, one can ask what are the sectoral labels corresponding to the full rows in the input-use matrix above. For 1997, ranking rows by the number of non-zero elements gives, in descending order: wholesale trade, management of companies, truck transportation, electric power generation and distribution, real estate, advertising, iron and steel mills, paperboard and container manufacturing, plastic plumbing and fixtures, petroleum refining, telecommunications, semiconductors and architectural and engineering services. These can be termed general purpose sectors in that their output serves as a necessary input to production in almost all of the sectors of a modern economy[13]. Recalling the discussion in Section 2, the existence of these star-like sectors will be key in that it opens the possibility of a non-trivial propagation mechanism working through the hubs of an economy.

I now map this data on inter-sectoral input trade into standard graph theoretical notation. First, let the set of M sectors in an economy give the set of fixed labels for the vertex set $V \doteq \{v_1, ..., v_M\}$. Let E be a subset of the collection of all ordered pairs of vertices $\{v_i, v_j\}$, with $v_i, v_j \in V$. Define E by:

$$\{\ \{v_i, v_j\} \in V^2 : \{v_i, v_j\} \in E \text{ if } \textit{Sector } i \textit{ supplies Sector } j\}$$

That is, the edge set E, is given by an adjacency relation, $v_i \to v_j$ between elements of the set of all sectors where I allow reflexivity (a sector can be an input supplier of itself). With the collection V of sectors and input supply relations E, I define sectoral trade linkages as a directed graph G :

Definition 3.1. *$G = (V, E)$. G is a directed graph (digraph for short) with vertex set V and edge set E where each element of E is a directed arc from element i to j .*

A useful representation of a graph is its adjacency matrix, indicating which of the vertices are linked (adjacent). This will be a key object in the sections below and is defined by:

Definition 3.2. *For a digraph $G(V, E)$ define the adjacency matrix $A(G)$ to be an $M \times M$ matrix. If G is a directed graph define the a_{ij} element of $A(G)$ to be 1 if there is a directed edge from sector i to sector j (i.e. if sector i is a material input supplier of j).*

Thus, Figure 3 can be simply taken as the adjacency matrix representation of the 1997 intersectoral trade network where vertices are given by sectors and edges are given by input supply relations, the dots in the matrix; blank cells are the zero entries of the adjacency matrix. Notice that the actual input use tables provide more information than the directed arc structure. In a graph context the value of individual sector to sector transactions also provide the weights associated with each edge. Throughout this paper I will not exploit this information.

[13] See also the recent work in Jones (2007b) for a similar link between full rows of input-use matrices and general purpose sectors.

I focus instead on the extent of heterogeneity across sectors as given by two simple count measures: the number of different inputs a sector *demands* in order to produce- as measured by the columns sums of the adjacency matrix $A(G)$ - and the number of different sectors a sector *supplies* inputs to - as measured by the row sums of $A(G)$. These count measures can be mapped directly in two graphical objects, namely the indegree and outdegree sequences of an intersectoral graph G.

Definition 3.3. *The in-degree d_i^{in} of a vertex $v_i \in V$ is given by the cardinality of the set $\{v_j : v_i \rightarrow v_j\}$. The in-degree sequence of a graph $G(V, E)$ is given by $\{d_1^{in}, ..., d_M^{in}\}$..*

Figure 4 below, displays the empirical density of sectoral indegrees for every detailed input-use matrix available since 1972. I define the indegree of a sector i as the number of distinct input-demand transactions that exceed 1% of the total input purchases of that sector. Though arbitrary, this counting convention seems necessary as there is no way of distinguishing between, say, an input transaction from sector i to j in the order 10 million dollars and an input transaction from sector k to j two orders of magnitude above. Both get counted as one demand link of sector j. By only counting as links input transactions above 1% of a sector's total purchases, I am discarding very small transactions between sectors and focusing on the main components of the bill of goods necessary to the production of any given sector. Indeed, following this threshold rule, I account for 80% of the total value of intermediate input trade in the US economy in 1997. A similar number obtains for all the other years considered.

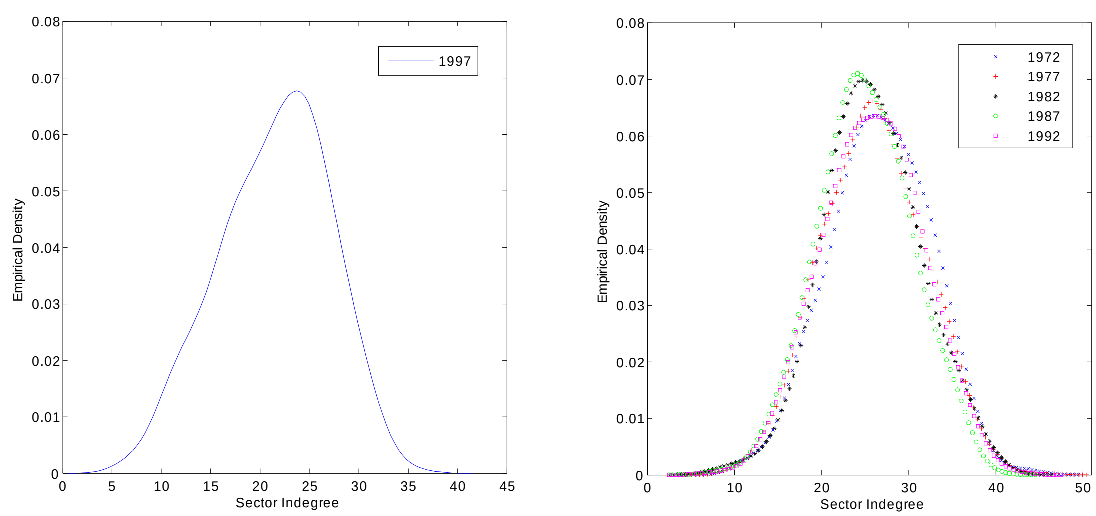
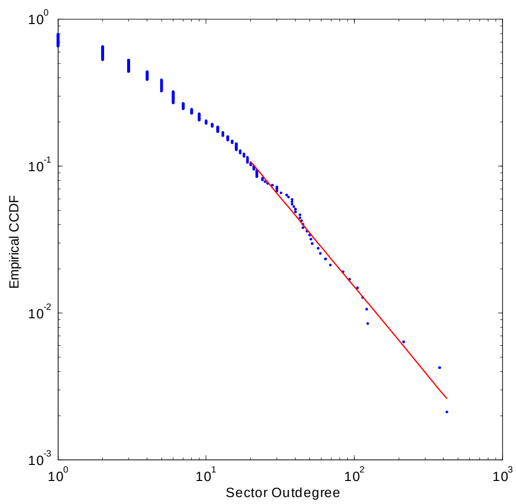


Figure 4: *Empirical density of sectoral indegrees. Only input demand transactions above 1% of the demanding sector's total input purchases are counted. On the l.h.s. is the indegree density for the 1997 detailed input-use matrix; on the r.h.s. are the empirical densities for the detailed input-use matrices from 1972 to 1992. Source: B.E.A..*

The demand side picture that emerges from Figure 4 is the following: the average sector in the US economy procures a non-trivial amount of inputs from only a small number of sectors ($\simeq 20$) when producing its good and sectors do not differ much along this demand margin. In other words, the average indegree is small relative to the total number of sectors and most sectors have an indegree that is close to the average indegree. Henceforth I'll dub this feature as homogeneity along the extensive margin of sectoral demand. This is to be contrasted with the extreme heterogeneity found along the supply side to which I now turn.

Definition 3.4. *The out-degree d_i^{out} of a vertex $v_i \in V$ is given by the cardinality of the set $\{v_i : v_i \to v_j\}$. The out-degree sequence of a graph $G(V, E)$ is given by $\{d_1^{out}, ..., d_M^{out}\}$.*

Figure 5 documents the heterogeneity in sectoral supply linkages by plotting the empirical out-degree distribution in the input-use data where again I use the 1% threshold to define a link. It gives a log-log rank-size plot constructed as follows: first, rank all sectors according to the total number of sectors they supply inputs to. Now plot the log of the out-degree of each sector (in the x-axis) against its log rank (in the y-axis). To interpret the plot it is useful to notice the following: if I rank sectors then, by definition, there are i sectors that supply inputs to a number of sectors that is greater or equal than that of the $i^{th}-$largest sector. Thus dividing the sector's rank i by the total number of sectors (M) gives the fraction of sectors larger than i. Figure 5 gives just this: a log-log plot of the empirical counter-cumulative distribution of the outdegrees, or the probability, $P(k)$, that a randomly selected sector supplies inputs to k or more sectors.



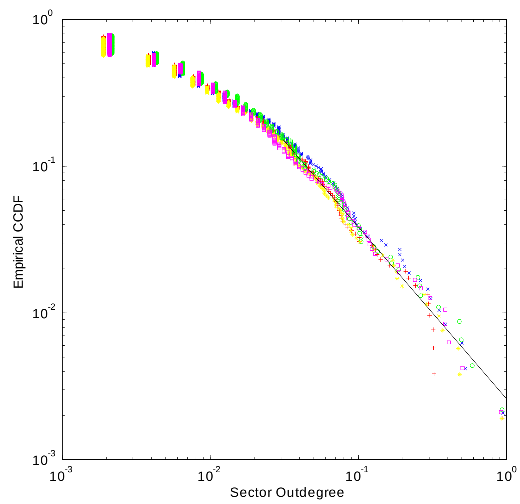


Figure 5: *Counter-cumulative outdegree distribution from input-use detailed tables. Only input demand transactions above 1% of the demanding sector's total input purchases are counted. On l.h.s is the 1997 data. The r.h.s. displays 1972, 1977, 1982 and 1992 data where I normalize the sectoral outdegree d_i^{out} by the total number of sectors in each year. Source: BEA.*

Given that every input-use matrix, from 1972 through 1992 differs slightly in its dimension (i.e. in the number of sectors considered), for each sector and for every year through 1992, I normalize sectoral outdegrees by the total number of sectors in the input-use matrix. This enables me to compare features of the distributions across different input-use matrices by standardizing the x-axis in the r.h.s of Figure 5.

The apparent linearity in the tail of the outdegree distribution in log scales is usually associated with a power law distribution. To see this formally, let $P(k) = \sum_{k'=k}^{M} p_{k'}$ be the countercumulative distribution of outdegrees, i.e. the probability that a sector selected at random from the population supplies to k or more sectors. We say that the number of sectors supplied (i.e. the outdegree), k, follows a power law distribution if, the p.d.f. p_k (giving the frequency of sectors that supply to exactly k sectors in the economy) is given by:

$$p_k = ck^{-\zeta} \text{ for } \zeta > 1, \text{ and } k \text{ integer, } k \geq 1$$

where c is a positive constant (from normalization) and ζ is the tail index. Well-known properties of this distribution are that for $2 \leq \zeta < 3$, k has diverging second (and above) moments[14] while for $1 < \zeta < 2$, k will have diverging mean as well (see Newman, 2003 and 2005 and Li et. al., 2006 for useful reviews and references therein). Given this expression for $p(k)$ and taking M to be large enough so that we can approximate the sum by an integral, the out-degree distribution yields $P(k) = c'k^{-\zeta+1}$, where c' is another constant. Now taking logs on both sides gives:

$$\log P(k) = \log c' - (\zeta - 1) \log k \tag{13}$$

yielding a linear relation between the log of the counter-cumulative distribution and the log of a sector's out-degree. Further, an estimate on the value of the tail parameter, ζ, can be obtained by running a simple least squares regression of the empirical log-CCDF on the log-outdegree sequence (or its normalized counterpart). Table 1 below shows the OLS estimates $\widehat{\zeta}$ for the right tail of the distribution (i.e. using all observations on or above the average degree) obtained for every year alongside with standard errors and the corresponding R^2.

	1972	1977	1982	1987	1992	1997
$\widehat{\zeta}$	2.073	2.118	2.092	2.056	2.110	2.104
s.e.	0.041	0.073	0.039	0.043	0.035	0.049
R^2	0.936	0.907	0.947	0.940	0.958	0.931

Table 1: *Least squares estimates of $\widehat{\zeta}$ in equation (13).Only the right tail of the counter-cumulative distribution of outdegrees in Figure 5 is used for estimation.*

[14] Though in any finite sample a finite variance can be computed, what this means is that the variance diverges to $+\infty$ as the total number of sectors grows larger.

The straight lines in Figure 5 show the OLS fit implied by $\widehat{\zeta} = 2.1$. While the usual caveats and pitfalls associated to OLS tail estimates apply[15], the evidence points to an average value of $\widehat{\zeta} \simeq 2.1$ that is remarkably stable across years. Notice that from the discussion above, this value of the tail parameter implies a strong fat tailed behavior where the variance is diverging with the number of sectors. This can be taken as a parametric characterization of a feature of input-use matrices already remarked in Horvath (1998) and discussed at the beginning of this section: as we disaggregate into finer definition of sectoral technologies, large input-supplying sectors do not vanish. In other words, at the most disaggregated level of sectoral input trade, the distribution of input-suppy links is fat tailed. The power law case reported for the 1% count rule can be seen as one particularly convenient parameterization of this heterogeneity[16].

4. Representing Intersectoral Input Flows as Networks

According to the analysis of the previous section one can begin to characterize the observed intersectoral input-trade pattern as a network. This subsection shows how to incorporate these network features in a model with Cobb-Douglas sectoral production functions. The analysis proceeds by first finding a way to incorporate the zeros usefully and then, following the characterization data, imposing homogeneity across sectors while input demanders and focusing on the implications of heterogeneity of sectors in their role as input suppliers.

4.1. Input-Output Matrices as Networks. In a multi-sector context, explicitly accounting for flows of inputs from one sector to another entails specifying both a list of intermediate inputs needed for the production of any given sector and the intensity of use of each particular intermediate input in that list. For the modeller this means not only specifying elasticity of substitution across intermediate inputs for a sectoral production function, but also setting to zero these parameters when a particular input is not required for the production that good. This is summarized by the input-use matrix defined as:

Definition 4.1.1. *The input-use matrix, Γ, is an $M \times M$ matrix with typical element $\gamma_{ij} \geq 0$. The j^{th} column sum of Γ gives the degree of returns to scale in material inputs for sector j:*

$$\gamma_j = \sum_{i=1}^{M} \gamma_{ij}$$

where $\gamma_j < 1$, such that the $M \times M$ matrix $(I - \Gamma)^{-1}$ is well defined.

[15] See, for example, the discussion in Gabaix and Ioannides (2004) or in Embrechts et al (1997). Brock (1999), Mitzenmacher (2003) and Durlauf (2005) provide further discussion on the difficulty of identifying power laws in data.

[16] Sensitivity results for other count rules to be added.

This definition simply states that the class of multisector models under consideration in this paper - and in most of the literature - imposes weak restrictions on the class of admissible input use-matrices. The only restriction is to insist that any sector in a M-sector economy exhibits decreasing returns to scale in material inputs, which translates to a strict unit bound on the column sums of the input-use matrix, Γ. This in turn ensures existence of $(I-\Gamma)^{-1}$, the Leontieff inverse[17]. Notice, in particular, that the modeller can introduce sparseness in Γ - zeros in a matrix of elasticities of substitution - at will as long as this column-sum condition is met.

The next Lemma is key to the paper in that it introduces a simple factorization of the input-use matrix Γ that will be used throughout.

Lemma 4.1.2. *Define a family of input-use matrices* $\Gamma(G)$ given by*:*

$$\Gamma(G) = A(G)D_{\gamma}$$

where $A(G)$ *is a binary adjacency matrix representation of the intersectoral trade digraph,* G*, and* D_{γ} *is a diagonal matrix with a typical element* $D_{kk} = \frac{\gamma_k}{d_k^{in}}$*, where* $\gamma_k < 1$ *and* d_k^{in} *is the number of sectors from which sector* i *purchases inputs. Then, for any* M *and any* G*, the columns sums of* $\Gamma(G)$ *are given by* $\gamma_k < 1$.

The proof of the last statement in the Lemma follows immediately by construction of $\Gamma(G)$. This Lemma offers a decomposition of the input-use matrix Γ into the product of two square $M-$dimensional: a binary matrix $A(G)$, giving the structure of connectivity in the economy by defining who trades with whom and a diagonal matrix D_{γ} setting the scale of input transactions between two sectors by defining the level of the elasticities of substitution for the non-zero elements of $\Gamma(G)$. Notice that Lemma 4.1.2. introduces a restriction on the class of admissible matrices: all non-zero elements of a given column of the input-use matrix are the same, as announced in Assumption 2.2..This is tantamount to imposing a symmetrical role for all the necessary inputs for a given sector, as the individual input elasticities will be the same.

For what follows, this simplifies considerably the description of a sectoral technology. I need only to specify two objects: a binary matrix announcing who supplies whom and a vector giving the returns to scale in material inputs for each sector. The individual elasticities of substitution are then given immediately by Lemma 4.1.2..I now focus on the problem of generating connectivity structures, $A(G)$, that approximate the homogeneous demand, heterogeneous supply of section 3.

4.2. Designing Families of Inter-Sectoral Trade. In this subsection I will be interested in generating connectivity structures that can encode the description of large-scale input-use matrices put forth in Section 3. To achieve this, I derive a sampling scheme for adjacency matrices $A(G)$

[17] $(I-\Gamma)^{-1}$ exists if every eigenvalue of Γ is less than one in absolute value. From the Frobenius theory of non-negative matrices, the maximal eigenvalue of Γ is bounded above by the largest column sum of Γ, $\max_k\{\gamma_k\}_{k=1}^{M}$, which is less than one.

that will then be used to construct input-use matrices $\Gamma(A)$ according to Lemma 4.1.2 above. The object of analysis is to construct families of random matrices, $\mathcal{A}$, from which individual members - intersectoral connectivity matrices, A - are drawn. These families of matrices should therefore be seen as a data generating process for the lists of intermediate inputs necessary for each sector or, put simply, families of intersectoral trade structures.

Below, I show how to specify these families of intersectoral trade according to three parameters: a parameter controlling the dimension of the problem - given by the number of sectors M; a demand side parameter $\overline{e}$, controlling the average connectivity in the economy - given by the number of inputs an average sector demands - and a supply side parameter ζ, controlling the heterogeneity across sectors in their role of input-suppliers.

To construct these families $\mathcal{A}(M, \overline{e}, \zeta)$ I use elements from the theory of random graphs. A random directed graph consists of family of directed graphs $\mathcal{G}$, indexed by the cardinality of the vertex set (M) and a probability distribution over $\mathcal{G}$. These constructions allow for statements, under some carefully chosen probability measure, about the probability of some particular property for any $M-$vertex digraph[18]. In particular, I'll make use of a particularly simple and useful construct recently introduced in the theory of random graphs. What follows is a simple digraph extension of Chung and Lu's (2002, 2006) model of undirected random graphs with given expected degree sequences.

Intuitively, I will be considering realizations of input-supply links (edge sets) in the following way: for a given number of sectors M, associate to the collection of all ordered pairs of sectors/vertices $\{v_i, v_j\}, v_i, v_j \in V$ an array of independent, Bernoulli random variables, X_{ij}, taking values 1 or 0 with probability p_{ij} and $1 - p_{ij}$ respectively. Finally, define a realization of the connectivity structure of intersectoral trade as an edge set E such that $\{v_i, v_j\}$ is an element of the edge set E, if $X_{ij} = 1$. In this way, one can compute the expected outdegree of any sector as $E(d_i^{out}) = E(\sum_j X_{ij}). = \sum_j p_{ij}$, given independent realizations of each supply-to link. Similarly the expected in-degree of a sector can be computed as $E(d_i^{in}) = E(\sum_i X_{ij}). = \sum_i p_{ij}$.

For this, for every M, associate a weight sequence $e \doteq \{e_1, ..., e_M\}$ to the collection of sectoral labels, such that $e_i \in [0, M]$ and let the average weight be denoted by $\overline{e} = \frac{\sum_{k=1}^{M} e_k}{M}$. Now, for each possible ordered pair of sectors $\{v_i, v_j\} \in V^2$ define the probability of having a directed arc from $v_i \longrightarrow v_j$ as

$$p_{ij} \doteq \frac{e_i}{M}, \forall j \in V \tag{14}$$

This encodes: *i)* a sector with higher weight, e_i, will have a higher probability to supply every sector in the economy and *ii)* for any given j, the probability of sector i being its input supplier depends only on the label of sector i and is thus not responsive to the label of j[19]. These are a

[18] The classical binomial model of Erdos and Renyi (see Bollobas (2001) for a texbook treatment) is the most commonly used construction for random graphs but there are many alternatives to this (see Durret, 2006, Newman, 2003, Bollobas and Riordan, 2003, or Chung and Lu, 2006).

[19] Chung and Lu's (2002, 2006) original model for undirected graphs gives $p_{ij} = \frac{e_i e_j}{\sum_{k=1}^{M} e_k}$ so that $p_{ij} = p_{ji}$ for all i, j.

strong assumptions in that describing whether an input trade relationship exists or not, both the identity of the supplying *and* that of demanding sector -label $j-$ should matter. Effectively this is reducing the problem of how to build theoretical counterparts to input-output matrices to a simpler problem of distinguishing sectors by how likely they are to be general purpose suppliers (i.e. sectors that have an e_i close to M). This is achieved at the cost of shutting down (at least in expectation) heterogeneity in the number of links along the demand side. The following Lemma elucidates further this construction and its implications:

Lemma 4.2.1. *For every M sector economy, associate a weight sequence $e \doteq \{e_1, ..., e_M\}$ to the collection of sectoral labels, such that $e_i \in [0, M]$ and define a realization of the intersectoral trade graph G as the realization of independent binary random variables $X_{ij} \in \{0,1\}$, $i, j = 1, ...M$ where $Pr(X_{ij} = 1) = p_{ij}$ given by [14]. Then,*

i) for any M, the expected out-degree of a sector will be given by:

$$E(d_i^{out}) = \sum_j p_{ij} = e_i, \quad i = 1, ..., M \tag{15}$$

ii) for any realization of the intersectoral trade graph G and for any sector i, if $e_i > \log M$, then its actual out-degree d_i^{out} will almost surely satisfy

$$\left|d_i^{out} - e_i\right| \leq 2 \log M$$

otherwise if $e_i \leq \log M$, then d_i^{out} almost surely satisfies

$$\left|d_i^{out} - e_i\right| \leq 2\sqrt{e_i \log M}$$

iii) for any sector i, its $E(d_i^{in})$ expected indegree is given by:

$$E(d_i^{in}) = \sum_i p_{ij} = \frac{\sum_i e_i}{M} \equiv \overline{e}, \forall i \tag{16}$$

Lemma 4.2.1 states that intersectoral connectivity structures drawn from the sampling scheme above will yield, on average, as much heterogeneity in sectors along their supply dimension as the modeller feeds it through the weights $\{e_i\}_{i=1}^{M}$. Conversely it will generate homogeneity in terms of the number of sectors a randomly chosen sector buys inputs from, i.e. it yields sectors that will be alike in terms of the number of inputs they demand. Part ii) of the Lemma states that actual (sampled) sequences of sectoral outdegrees will concentrate around its expected value and offers bounds that are tight for the larger sectors (in term of outdegrees).

When studying inter-sectoral supply links this symmetry is uncalled for: the fact that sector i has a high probability of supplying to j should not imply the converse.

What is left is to understand is how to specify the weight sequence $\{e_i\}_{i=1}^{M}$. From the discussion above notice that I am more likely to link vertices with a high expected degree, i.e. with a higher relative weight. In the remainder of this subsection, following Chung, Lu and Vu (2003) and Chung and Lu (2006) I construct weights e_i such that the expected outdegree sequence follows an exact power law sequence, this in order to capture the heterogeneity in sectoral supply linkages documented in Section 3.

Definition 4.2.2. *For every M sector economy define a realization of the intersectoral trade graph G as the realization of independent binary random variables $X_{ij} \in \{0,1\}$, $i, j = 1, ...M$ where $Pr(X_{ij} = 1) = p_{ij}$ given by [14] and the weight sequence is given by*

$$e_i = ci^{-\frac{1}{\zeta-1}} \quad \text{for } 1 \leq i \leq M \text{ and } \zeta > 2 \tag{17}$$

and

$$c = \frac{\zeta-2}{\zeta-1}\bar{e}M^{\frac{1}{\zeta-1}} \tag{18}$$

To see how this parameterization for link probabilities implies a power law sequence for expected out-degrees, notice that I can use expression (17) to solve for i and get:

$$i \propto E(d_i^{out})^{-\zeta+1} \tag{19}$$

Now suppose I rank sectors according to the expected number of sectors they supply inputs to $E(d_i^{out})$. The expression in (17) implies that they will be ranked according to i: $i = 1$ giving the largest sector, $i = 2$ the second largest and so forth. Notice also that, by definition there are i sectors that, in expectation, supply to at least the same number of sectors supplied by the $i^{th}-$largest sector. Thus a sector's rank i is proportional to the fraction of sectors larger than i. What expression (19) is stating is that the log of this fraction will scale linearly with the log expected out-degree of sector i, with parameter ζ controlling the scaling behavior. Thus, the expected outdegree sequence is an exact power law sequence[20]. Combined with result ii) in Lemma 4.2.1., this implies that sampling intersectoral trade structures with link probabilities formed according to (14), (17) and (18) will generate actual sectoral out-degrees sequences that concentrate around a power law sequence where ζ is the tail parameter[21].

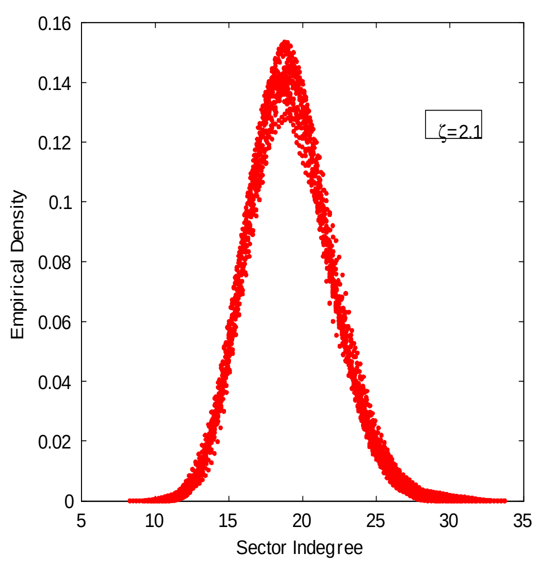
I now summarize the sampling procedure of intersectoral trade structures constructed in this section by defining the following family of binary matrices:

[20] This is a *deterministic* sequence with power-law like (or scaling) behavior in that it gives a finite sequence of real numbers, $E(d_1^{out}) \geq E(d_2^{out}) \geq ... \geq E(d_M^{out})$, such that $i = c\left[E(d_i^{out})\right]^{-\varphi}$ where c is a constant and φ is called the scaling index. See Li et al (2006) for a useful discussion on scaling sequences vs. power law distributions.

[21] Notice that the tail parameter ζ only controls the shape of the outdegree distribution - how skewed the distribution will be - but not the average degree, which is a free parameter. The presence of $\bar{e}$ in the weigth definition 4.2.2. is thus necessary in order to fix the average degree.

Definition 4.2.3. *Fix a triplet of parameters* $(M, \overline{e}, \zeta)$. *Let* $\mathcal{A}(M, \overline{e}, \zeta)$ *define a family of intersectoral trade matrices, elements of which are* $M \times M$ *binary matrices* A, *with entries* A_{ij} *equal to 1 with probability* p_{ij} *given by (14), (17) and (18) and zero otherwise.*

This is simply the adjacency matrix representation of the intersectoral trade graphs generated by the sampling scheme of Lemma 4.2.2. To understand what these matrices imply in terms of the discussion on connectivity properties of Section 3, Figure 6 below plots the model-based equivalent of Figures 4 and 5, the indegree density and the outdegree counter-cumulative distribution for a 500 sector economy.



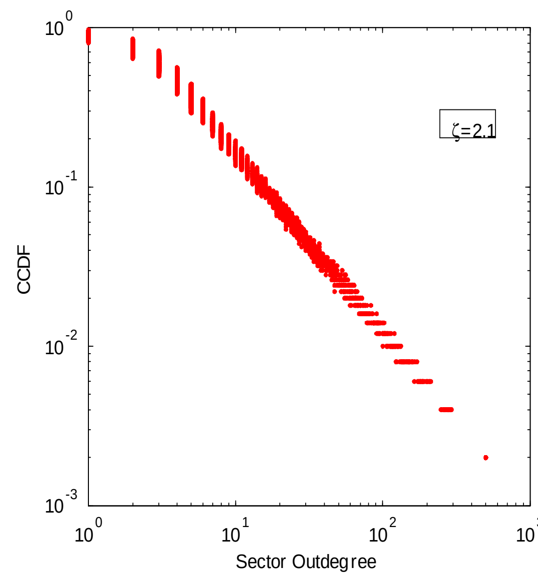


Figure 6: *Empirical indegree density (l.h.s.) and outdegree counter-cumulative distribution (r.h.s.) for 30 intersectoral trade structures drawn at random from* $\mathcal{A}(M, \overline{e}, \zeta)$ *with parameters* $M = 500$, $\overline{e} = 20$ *and* $\zeta = 2.1$.

More specifically, Figure 6 presents the sectoral demand-supply side breakdown for thirty A matrices drawn at random from a family of intersectoral digraphs, $\mathcal{A}(M, \overline{e}, \zeta)$ where I have picked the following parametrization: M is given by a 500 sector economy, where the average number of inputs needed per sector, $\overline{e}$, is set at 20, and the parameter controlling heterogeneity of sectors along the supply side, ζ, is set at 2.1. This parameterization is based on the corresponding objects computed from the B.E.A. detailed input-use matrices in Section 3.

While individual realizations of A are random objects, thus differing in the exact placement of zeros, the indegree and outdegree sequences implied by each intersectoral trade structure yield similar patterns. In other words, row and column sums will not differ much across realizations. By design, and as predicted theoretically by Lemma 4.2.1., each element of the family $\mathcal{A}(M, \overline{e}, \zeta)$, retains the features noted in Section 3: homogeneity along the demand side - for any member of a family $\mathcal{A}$, sectoral indegrees concentrate along the specified average degree, $\overline{e}$ - and heterogeneity along the supply side, where the number of sectors any given sector supplies can differ by orders of magnitude. Namely, the outdegree sequences implied by realizations of A display fat-tails in the form of a power law- as instructed by Definition 4.2.2.

Notice also that according to Definition 4.2.3. I can model formally the following thought experiment. Fix a number of sectors, M, and define a typical production technology by setting the average number of inputs ($\overline{e}$) a sector needs in order to produce its output. Now, entertain two different values of the tail parameter governing heterogeneity across sectors in their role as input suppliers, ζ_1 and ζ_2 such that $\zeta_1 < \zeta_2$. What this yields is two economies where sectoral production technologies differ in their degree of diversification. Thus ζ_1 economies will be less diversified in that more mass at the tail implies that a greater number of sectors rely on the same general purpose inputs. Conversely, ζ_2 economies, by having more mass at the center of the distribution of input supply links, will be more diversified: there will be a smaller number of hub-like sectors connecting all sectors in the economy and a greater number of specialized input suppliers, each supplying inputs to a smaller fraction of sectors. This intuition will be key to interpret the results in the rest of the paper.

Finally, I define a key matrix for the results that follow: the expected adjacency matrix, $E(A)$ for a family of random connectivity matrices $\mathcal{A}(M, \overline{e}, \zeta)$.

Definition 4.2.4. *The expected value of the adjacency matrix of a family of intersectoral trade structures $\mathcal{A}(M, \overline{e}, \zeta)$, denoted $E[A(G)]$, is an $M \times M$ matrix whose ij entry is given by $E[A_{ij}(G)] = p_{ij}$ with p_{ij} given by (14), (17) and (18).*

I now plug these tools to work in the context of the static multisector model of Section 2.

4.3. Key Results. This subsection uses the objects constructed in the previous two subsections to characterize the key object for the class of multi-sectoral models under consideration: the Leontieff inverse $(I_M - \Gamma)^{-1}$. Once this is derived I show, analytically, how the aggregate volatility statistic can be expressed as a function of γ, the degree of returns to scale in material inputs, M, the number of sectors and ζ, the tail parameter controlling the heterogeneity of the distribution of input-supply links for any member of the family connectivity matrices $\mathcal{A}(M, \overline{e}, \zeta)$.

I first focus on a simpler exercise: assume A is deterministic and given by the expected adjacency matrix of a family of intersectoral trade structures $E(A)$. I will then show how the characterization of the Leontieff inverse matrix and the resulting aggregate volatility statistic generalizes to any A sampled from a family $\mathcal{A}(M, \overline{e}, \zeta)$. Thus, to begin, the next Lemma shows how one can go from this expected adjacency matrix to an input-use matrix Γ that meets the column sum requirements of Definition 4.1.1..

Lemma 4.3.1. *Recall the expected adjacency matrix of a intersectoral digraph on M* sectors *as $E(A(G))$ and let the degree of returns to scale in material inputs be equal across sectors, $\gamma_j = \gamma < 1$, $\forall j = 1, ..., M$. Then the $M \times M$ matrix*

$$\overline{\Gamma}(A) \equiv E[(A(G))D_\gamma] = \frac{\gamma}{\overline{e}} E(A(G))$$

satisfies the column sums requirement for admissible input-use matrices in Definition 4.1.1.

The Lemma is trivially proven by noticing that the column sums of $E(A(G))$ are given by $\sum_{i=1}^{M} e_i/M$, which is the definition of $\overline{e}$, the expected average degree. Thus D_γ reduced to $\frac{\gamma}{\overline{e}}.I_M$. Further $\overline{\Gamma}$ has all column sums equal to $\gamma. < 1$ and, by the same reasoning of Lemma 4.1.2., $\left(I_M - \overline{\Gamma}\right)^{-1}$ is well defined.

Henceforth I will refer to this matrix $\overline{\Gamma}$, as the expected input-use matrix of a multisector economy. As the Lemma states it can be seen as an expectation over the families of input-use matrices defined by the decomposition in Lemma 4.1.2. and the construction in Lemma 4.2.3..The attractiveness of this, admittedly, very particular case resides in the fact that a matrix $\overline{\Gamma}$ thus constructed is a rank one matrix- all columns are equal. Thus we can decompose it into the product of two vectors. In particular let $\boldsymbol{\phi}$ be an $M \times 1$ vector with typical element $\frac{e_i}{\sum_{i=1}^{M} e_i}$, and let $\mathbf{1}_M$ is the unit vector of dimension $M \times 1$. It then follows that:

$$\overline{\Gamma} = \gamma \boldsymbol{\phi} \mathbf{1}_M'$$

This fact is useful in that we can derive a simple expression for the key object in multi-sectoral models:.

Proposition 4.3.2. *The Leontieff inverse of the expected input use matrix, $\left(I_M - \overline{\Gamma}\right)^{-1}$ is given by*

$$\left(I_M - \overline{\Gamma}\right)^{-1} = I_M + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}_M'$$

where I_M is the $M \times M$ identity matrix, $\boldsymbol{\phi}$ is a $M \times 1$ vector with typical element $\frac{e_i}{\sum_{i=1}^{M} e_i}$ and $\mathbf{1}_M$ is the unit vector of dimension $M \times 1$.

The idea of representing the Leontieff inverses as an identity plus a rank one matrix is already present in Horvath (1998) and Dupor (1999). In fact, the class of input-use matrices considered in Dupor (1999) is given by $\widehat{\boldsymbol{\phi}}\mathbf{1}_M'$ and restricting $\widehat{\boldsymbol{\phi}} = \left[\frac{M}{M^2}, ..., \frac{M}{M^2}\right]$ and the alternative provided by Horvath (1998) is simply $\widetilde{\boldsymbol{\phi}}\mathbf{1}'$ where $\widetilde{\boldsymbol{\phi}}$ is an $M-$dimensional vector where N elements are given by $\frac{M}{MN}$ and the remaining $M-N$ entries are fixed to zero[22].Thus, following the discussion in Section 2, Horvath, and Dupor's settings can be seen as particular subclasses of a general class of input-use matrices given by $\overline{\Gamma}$, where the vector $\boldsymbol{\phi}$ is fixed at ratio of the outdegree of each sector to the volume

[22]Recall that in Horvath (1998) N is the number of sectors supplying inputs to all M sectors and $M-N$ is the number of sectors dedicated solely to final good production. See the proof of Proposition 2.1. for further details.

of input-supply links in the economy: $\widehat{\boldsymbol{\phi}}$ and $\widetilde{\boldsymbol{\phi}}$ are exactly that object for a very particular bi-regular digraph in Horvath (1998) and for a complete digraph in Dupor (1998).

The following proposition describes what changes when instead of these simple structures for intersectoral trade, I posit Γ to be given by the expected input-use matrix of $\mathcal{A}(M, \bar{e}, \zeta)$.

Proposition 4.3.3. *Fix a triplet of parameters $(M, \bar{e}, \zeta)$. Assume that the input-use matrix is given by the expected input-use matrix $\overline{\Gamma}$ and that sectoral volatility $\sigma_j^2 = \sigma^2$ for all sectors j. Then, for a static $M-$sector economy (8) the aggregate volatility of equilibrium output, $\sigma_Y^2(\overline{\Gamma})$ is given by:*

$$\sigma_Y^2(\overline{\Gamma}) = \begin{cases} \kappa_1(\zeta)\left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\sigma^2}{M} + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta > 3 \\ \kappa_2(\zeta)\left(\frac{\gamma}{1-\gamma}\right)^2 \left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}} \sigma^2 + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta \in (2,3) \end{cases}$$

where the terms $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ are positive constants given a ζ.

In both regions of the tail parameter space, the first two terms of the expressions can be taken as constants given fixed parameters γ and ζ. However, the scaling of the aggregate volatility statistic with M is dependent which region of the parameter space ζ is set. In particular, Proposition 4.3.3. states that for an average input-use matrix,$\overline{\Gamma}$, aggregate volatility will depend on the tail of the size distribution of input supply links.

Thus, for thin tailed distributions of sectoral outdegrees $\zeta > 3$, aggregate volatility scales with the usual term of order $O(1/M)$. This means that the discussion in Section 2 regarding the decay rate in the special case of complete network structures assumed by Dupor, applies also to the current context. Intuitively, in economies with a large number of sectors that do not differ much in their role as input suppliers, aggregate volatility will be negligible.

However, once we consider the fat-tailed region for $\zeta \in (2,3)$ the decay behavior is altered: the aggregate volatility statistic now decays with M at a rate that is lowered significantly as we consider average input use matrices from more heterogeneous outdegree economies.Namely, Proposition 4.3.3. yields an analytical expression where the rate of decay in the volatility of aggregate output depends negatively on the degree of fat-tailness in the distribution of sectoral input-supply links. To see this notice that for $\zeta \in (2,3)$, the first term in the expression decays with M^{υ} where $\upsilon \equiv \frac{2\zeta-4}{\zeta-1} \in (0,1)$. This implies that the first term of the expression dictates the rate of convergence to zero. Namely, as ζ approaches its lower bound of 2, aggregate volatility, $\sigma_Y^2(\overline{\Gamma})$ will converge to zero arbitrarily slower. Taking, for example, the value of ζ of 2.1. estimated in Section 3, yields a much slower decay of order $\sqrt[6]{M}$ or $\sigma_Y^2(\overline{\Gamma}) \propto \frac{\sigma^2}{\sqrt[6]{M}}$. To have an idea of the magnitudes involved, this means that as I move from, say, a five sector economy to a five hundred sector economy I expect to find only a *two-fold* decrease in aggregate volatility. Thus, strong heterogeneity across input-supplying sectors opens the possibility of generating non-negligible aggregate fluctuations even in large scale multi-sectoral contexts[23].

[23] Notice also that Horvath (1998) conjecture of a $\sqrt{M}$ decay in aggregate volatility is obtained by fixing ζ at a very particular point: $\zeta = 2.333$.

However, the result in Proposition 4.3.4. is special in that it depends on a very particular input-use matrix: I am fixing the input-use matrix to be the expected input use matrix generated by a family $\mathcal{A}$. I now show that it is possible to generalize these findings to any matrix $\Gamma(A)$ drawn from $\mathcal{A}$. First, I show that for any such matrix, the basic result concerning the Leontieff inverse (Proposition 4.3.2) remains true, up to a random matrix with zero column sums. Then, using this result, I bound the effects that this extra random term might induce on aggregate volatility.

Proposition 4.3.5. *Fix a triplet of parameters* $(M, \overline{e}, \zeta)$ *and assume that* $\sum_{i \in \check{S}_j} \gamma_{ij} = \gamma$ *and that* $\sigma_j^2 = \sigma^2$ *for all sectors* $j = 1, ..., M$. *For any* $A(G)$ *sampled from the family of input-use graphs* $\mathcal{A}(M, \overline{e}, \zeta)$, *construct the input-use matrix according to Lemma 4.1.2. Then i) for any* $A(G)$

$$[I_M - \Gamma(A)]^{-1} = I_M + \frac{\gamma}{1-\gamma} \phi 1'_M + \Theta$$

where Θ *is an* $M \times M$ *random matrix with zero column sums. Further, ii) for any* $A(G)$ *of the family of input-use graphs* A, *and* $\zeta \in (2,3)$ *the following is a lower bound for aggregate volatility,* $\sigma_Y^2(\Gamma)$, *in a static* $M-$*sector economy (8) is given by:*

$$\sigma_Y^2(\Gamma) > \begin{cases} \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \kappa_1(\zeta) \frac{\sigma^2}{M} + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta > 3 \\ \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \kappa_2(\zeta) \left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}} \sigma^2 + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta \in (2,3) \end{cases}$$

where $\varkappa$ *is a constant strictly smaller than* $\left(\frac{\gamma}{1-\gamma}\right)^2$ *and* $\kappa_1(\zeta)$ *and* $\kappa_2(\zeta)$ *are positive constants given a* ζ.

Part i) of the proposition follows from the observation that one can always decompose any matrix $\Gamma(A)$ as the sum of $\overline{\Gamma} + (\Gamma(A) - \overline{\Gamma})$. The proof of part i) then follows along the same lines as that of Proposition 4.3.2., by using results on the inverse of the sum of two matrices. The only obstacle introduced by the random matrix Θ is that it induces the presence of a term in the expression for aggregate volatility that can be potentially be negative and dominate the rate of convergence. Part ii) of the Proposition states that this scenario can be ruled out. At worse the level of volatility in aggregates will be smaller than that derived in Proposition 4.3.4. but the rate of decay with M will be preserved. Hence the discussion following Proposition 4.3.4. applies in its entirety for any matrix A sampled from a family $\mathcal{A}(M, \overline{e}, \zeta)$.

This characterization of the decay in the volatility of aggregates with the number of sectors is closely related to results in Gabaix (2005, Proposition 2). As in Gabaix (2005), the fact that shocks do not average out according to standard law of large numbers arguments is due to heterogeneity at the level of the underlying production units. In contrast to Gabaix however, this is not the result of some firms accounting for a non-trivial share of aggregate output and thus, for a non-trivial share of aggregate volatility. Rather the argument here is based on the shock conductance implied by the interlocking of technologies in an economy. In other words, the emphasis here is on propagation rather than aggregation. These two approaches should therefore be seen as a complementary.

In short, and recalling the discussion at the end of the previous subsection, the results in Propositions 4.3.4. and 4.3.5. can be interpreted as follows. As ζ approaches its lower bound, sectors in an M sector economy will increasingly rely on the same general purpose inputs. Thus, considering economies with less diversified sectoral technologies corresponds to considering draws from connectivity structures with a lower value of ζ. Conversely, connectivity structures generated by higher values of ζ will yield greater technological diversification: sectoral technologies are relatively more reliant on specialized input-suppliers and less so on common, general purpose, inputs. Therefore, greater diversification in the form of less reliance on common inputs will yield only loosely coupled technologies and, as a result, lower aggregate volatility. Less diversification induces strongly coupled technologies and thus a stronger propagation mechanism[24]. The next section will show that this intuition carries through when we move to dynamic settings.

5. Dynamic Multi-Sector Economies

This section recalls a basic multi-sectoral production model, as introduced in Horvath (1998) and Dupor (1999). This is essentially a multi-sector version of a one-sector Brock-Mirman stochastic economy, where I make the necessary (and strong) assumptions to solve for the planner's solution analytically. More general, competitive equilibrium setups exist, such as Horvath (2000) or the closely related forerunner, Long and Plosser (1983). These are surely necessary for a careful quantification exercise. However, for the present purposes of analyzing how different structures of input-use imply distinct aggregate behavior, the simplicity of this setup is an asset and will allow me to extend the results of Section 4 while still addressing the results in Horvath (1998) and Dupor (1999).

5.1. Setup. A representative agent maximizes her expected discounted log utility from infinite vector valued sequences of M distinct goods.

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\sum_{j=1}^{M} \log(C_{jt}) \right] \tag{20}$$

where β is a time discount parameter in the $(0, 1)$ interval. Expectation is taken at time zero with respect to the infinite sequences of productivity levels in each sector, the only source of uncertainty in the economy. The production technology for each good/sector $j = 1, ..., M$ uses sector-specific capital and intermediate goods, i.e. production inputs bought from other sectors:

$$Y_{jt} = Z_{jt} K_{jt}^{\alpha_j} \prod_{i=1}^{M} M_{ijt}^{\gamma_{ij}} \tag{21}$$

[24]See Simon and Ando (1961) for a distant forerunner in analyzing the implications of loose vs. strong coupling across units.

where K_{jt} and Z_{jt} are the, time t, sector j, value of sector specific capital stock and its (neutral) productivity level. M_{ijt} gives the amount of good i used in sector j in period t. Further, define

$$\gamma_j = \sum_{i=1}^{M} \gamma_{ij}$$

with γ_{ij} denoting the cost-share of input from sector i in the total expenditure on intermediate inputs for sector j (allowed to take the value of zero). Again I can arrange the elasticities of substitution in a $M \times M$ input-use matrix, Γ.

I will assume productivity levels for sector j evolving stochastically according to an i.i.d. process, across sectors and across time, that is

$$\ln(Z_{jt}) = \varepsilon_{jt} \sim N(0,1) \tag{22}$$

Finally, assume that each sector exhibits decreasing returns to scale:

$$\begin{aligned} \alpha_j + \gamma_j &< 1, \ \forall_j \ \text{ and} \\ \alpha_j, \gamma_j &\geq 0, \ \forall_{i,j} \end{aligned} \tag{23}$$

and that sector specific capital depreciates fully in one period. The last two assumptions are necessary in order to derive an analytical solution to the Planner's problem in this economy[25].

5.2. The Social Planner's problem. The definition of the social planner's problem is given by

Definition 5.2.1. *The Social Planner's problem is to choose sequences of sector specific capital* $\{K_{jt+1}\}_{j,t}$*, intermediate input* $\{M_{ijt}\}_{i,j,t}$ *and consumption allocations*$\{C_{jt}\}_{j,t}$ *such that, given a vector of time zero capital stocks* $\{K_{j0}\}_j$ *and a sequence of sectoral productivity levels* $\{Z_{jt}\}_t$*, the following hold true:*

i)$\{C_{jt}\}_{j,t}$ *maximizes the representative consumer expected lifetime utility given by (20)*

ii) the sectoral resource constraint

$$Y_{jt} = C_{jt} + K_{jt+1} + \sum_{i=1}^{M} M_{ijt} \tag{24}$$

is satisfied, sector by sector, for all time periods and where Y_{jt} is given by (21).

Horvath and Dupor show that, under the assumption on technologies and preferences made above, an analytical solution to the Social Planner's problem is given by a first order vector autoregression:

[25] Assuming constant returns to scale would imply a stationary solution in growth rates. The characterization below then applies to the variance of growth rates rather than levels.

Proposition 5.2.2 (Horvath, 1998) *Given an $M \times 1$ vector of initial capital stocks, k_0, the vector sequence of sectoral capital given by:*

$$\mathbf{k}_{t+1} = \mathbf{h} + (I - \Gamma)^{-1\prime}\alpha_d \mathbf{k}_t + (I - \Gamma)^{-1\prime}\boldsymbol{\varepsilon}_{t+1} \tag{25}$$

solves the Social Planner's problem, where $\mathbf{h}$ *is a $M \times 1$ vector of constants (function of time-invariant model parameters), α_d is a $M \times M$ diagonal matrix with the vector α on its diagonal and I is the $M \times M$ identity matrix.*

For a proof, see Horvath's (1998) appendix where Howard's policy improvement algorithm is used to solve the recursive version of the Planner's Problem above. It is also easy to show that the Planner' solution implies a vector sequence of sectoral output differing from expression (25) for capital only by a constant. Hence all results below for sectoral and aggregate capital apply for output as well.

As in the simple static economy of Section 2, it is the Leontieff inverse $(I - \Gamma)^{-1}$ that mediates the propagation of independent technology shocks at the sectoral level. In this dynamic multisectoral model it rules not only the levels of capital and output but also the dynamics of these variables and hence of their aggregates. Now, in order to characterize the second moment properties of this economy I study the spectral density function for sectoral capital induced by expression (25) above. This is possible since, under the assumptions for technology made above, the $\{\mathbf{k}_t\}_t$ sequence (25) is stationary and thus admits an infinite moving average representation which, in turn, implies a frequency domain representation.

Proposition 5.2.3. (Horvath, 1998) *Under the assumptions above the population spectrum for sectoral output, k_i, for every sector $i = 1, ..., M$, at frequency ω is given by*

$$S_k(\omega) \doteq (2\pi)^{-1}(I - \alpha_d e^{-i\omega} - \Gamma')^{-1}(I - \alpha_d e^{i\omega} - \Gamma)^{-1} \tag{26}$$

Furthermore, given an $M \times 1$ vector $\mathbf{w}$*, of aggregation weights for log-aggregate capital stock, the spectrum for aggregate capital at frequency ω is given by*

$$S(\omega) \doteq \mathbf{w}' S_k(\omega)\mathbf{w} \tag{27}$$

The spectral density function is a useful object in that it provides a complete characterization of the autocovariance function for the average log of sectoral capital. Notice that by setting the elements of $\mathbf{w}$ to be equal and given by $1/M$, $S(\omega)$ is gives the dynamic counterpart to the aggregate statistic of the static model of Section 2; expression (9). In the next subsection I characterize the decay of the univariate spectral density expression (27) as I increase the level of disaggregation in these multisector models.

5.3. Characterizing the Decay of the Spectral Density. This subsection offers the dynamic counterpart to the results obtained in Section 4.3. I use the expression for the aggregate spectral density given in (26) and (27). The steps involved are the same as before: first I characterize the decay behavior of the aggregate spectral density with M for the particular case where $\Gamma = \overline{\Gamma}$ and then show that the resulting characterization extends for any $\Gamma(A)$ where A is drawn from the family of matrices $\mathcal{A}(M, \overline{e}, \zeta)$ and $\Gamma(A)$ is formed according to Lemma 4.1.2..

Thus, letting Γ be given by the expected input-use matrix, $\overline{\Gamma}$ expression [27] yields:

Proposition 5.3.1. *For fixed α and γ, aggregation weights $w= (1/M)1_M$ and $\Gamma = \overline{\Gamma}$, the spectral density for aggregate capital at frequency ω is given by:*

$$S(\omega, \overline{\Gamma}) = S(\omega, \overline{\Gamma}) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega) - \gamma^2) + \gamma^2\kappa_1(\zeta)\frac{1}{M}\right] if\ \zeta > 3$$

and

$$S(\omega, \overline{\Gamma}) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega) - \gamma^2) + \gamma^2\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\right] \quad if\ \zeta \in (2,3)$$

with $a(\omega) = \frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$, $b(\omega) = (1-\alpha e^{i\omega})(1-\alpha e^{-i\omega}).$, $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ *and* $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

As in Proposition 4.3.3. the expression for the volatility of aggregates differs according to the tail parameter governing heterogeneity across sectors in their role as input suppliers. Thus for $\zeta > 3$, i.e. thin tail distributions, or diversified economies, the expression again recovers the strong diversification of shocks argument given in Dupor. Volatility in aggregate variables decays at rate M as we expand the number of sectors, yielding negligible aggregate volatility for any moderate level of disaggregation. Conversely, for economies where large input-supplying hubs form the basis for input trade flows, this decay rate is slowed down arbitrarily as ζ approaches its lower bound. The more every sectoral technology in an economy relies on the same few key technologies the slower the law of large numbers applies.

More precisely, what the expression in the fat-tailed case is stating is the following. Let $\{\Gamma_M\}_{M=1}^{\infty}$ be the sequence of expected input-use matrices given by the construction of Lemma 4.3.1. with $\zeta \in (2,3)$. Then $S(\omega, \overline{\Gamma}) \to 0$ at rate M^{υ} for all ω, where $\upsilon \equiv \frac{2\zeta-4}{\zeta-1} \in (0,1)$. In this way for a ζ close to its lower bound, the law of large numbers is postponed.

This generalizes the results in Horvath (1998) and Dupor (1999). It is an easy exercise to show that results concerning the scaling of the variance with M, the number of sectors, are the same of the static model presented in section 2. However as the Proposition makes clear, this scaling now extends to the autocovariance function. Thus exactly the same decay description applies for persistence of aggregates in dynamic economies.

As in Section 4, I now extend this characterization for any $\Gamma(A)$ where A is drawn from the family of matrices $\mathcal{A}(M, \overline{e}, \zeta)$ and $\Gamma(A)$ is formed according to Lemma 4.1.2.

Proposition 5.3.2. *For fixed α and γ, aggregation weights $\mathbf{w} = (1/M)1_M$ and for any $\Gamma(A)$ where A is drawn from the family of matrices $\mathcal{A}(M, \overline{e}, \zeta)$ and $\Gamma(A)$ is formed according to Lemma 4.1.2., the spectral density for aggregate capital is bounded below by :*

$$S(\omega, \Gamma(A)) > \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega) - \gamma^2)\frac{1}{M} + \kappa_1(\zeta)(\gamma^2 - \varkappa)\frac{1}{M}\right] \; if \; \zeta > 3$$

and

$$S(\omega, \Gamma(A)) > \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega) - \gamma^2)\frac{1}{M} + \left(\gamma^2 - \varkappa\right)\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\right] \; if \; \zeta \in (2, 3)$$

with $\varkappa < \gamma^2$, $a(\omega) = \frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$, $b(\omega) = (1 - \alpha e^{i\omega})(1 - \alpha e^{-i\omega})$., $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ *and* $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

Proposition 5.3.2. states that as I consider any realization of $\Gamma(A)$ from $\mathcal{A}(M, \overline{e}, \zeta)$ - rather than its expectation, $\overline{\Gamma}$, the level of the spectral density for aggregate capital is, at worse, lower at every frequency, but preserves exactly the same decay behavior with M. What this means is that for any input-use matrix based on connectivity structures given by $\mathcal{A}(M, \overline{e}, \zeta)$, the link between volatility and persistence of aggregates and the network structure of the economy remains valid.

From a network perspective, common reliance on a few key input-supplying hubs will induce greater conductance to shocks in those sectors and this in turn generates less subdued and longer lived responses in aggregates. The next subsection illustrates this intuition by means of impulse response analysis.

5.4. Comovement in a Network Laboratory. The analysis above suggests that a shock to a large input-supplying sector, i.e. to a general purpose technology, will propagate throughout the economy as a large fraction of other technologies are dependent on it. This means that the structure of intermediate input trade renders the economy *vulnerable* to disturbances in particular sectors. However, one can also proceed to ask a related question: what is the response of the aggregate economy to an average shock? Here I translate an average shock as a shock to an average sector in terms of the number of sectors that it supplies inputs to. Intuition would indicate that the impact of this should be muted by the very fact that the output of an average sector is specialized and demanded only by a limited number of sectors. This in turn generates limited conductance to average shocks. This offers an alternative characterization of the structure of the economy as *robust* to typical shocks. This subsection illustrates these ideas through a simple impulse response analysis of the dynamic model of section 5.1.

The simplest way to proceed is to follow Long and Plosser (1983) and Horvath (1998) in assuming productivity shocks $Z_{j,t}$ follow a multiplicative random walk: $Z_{j,t+1} = Z_{j,t}\exp\{\varepsilon_{j,t+1}\}$ where $\varepsilon_{j,t+1}$ is a standard normal random variable. This renders the solution of the planner's problem stationary in first differences and given by:

$$\Delta \mathbf{y}_{t+1} = [I - \Gamma(A)]^{-1\prime} \alpha_d \Delta \mathbf{y}_t + [I - \Gamma(A)]^{-1\prime} \boldsymbol{\varepsilon}_{t+1} \tag{28}$$

where $\Delta \mathbf{y}_t$ is the log first difference of the vector of sectoral outputs and therefore can be seen as sectoral output growth rates[26]. In the simulations that follow, I assume decreasing returns to scale for all sectors, with $\gamma = 0.5$ and $\alpha = 0.45$, thus rendering all shocks transitory. Assuming constant returns to scale will induce a permanent change in the long run average growth rate following a shock but otherwise preserves the features described below.

Input-use matrices in these economies will be generated according to the construction in Section 4. Thus, I will be drawing connectivity structures from families of matrices $\mathcal{A}(M, \overline{e}, \zeta)$ and then constructing input-use matrices according to Lemma 4.1.2. Specifically, the simulations below trace impulse responses for five hundred sectors economies $(M = 500)$ where the average sector demands inputs from twenty other sectors ($\overline{e} = 20$). Given the results linking different volatility and persistence in aggregates with different levels of heterogeneity in input-supply links - or diversification of sectoral technologies - I consider sampling from two different families: $\zeta = 2.1$ and $\zeta = 3.1$.

Having drawn an input-use matrix, I simulate the growth rate response for each of the five hundred sectors to a one-standard deviation shock in the productivity of the largest input-supplying sector (i.e. the sector corrresponding to the largest row sum of the sampled A matrix). I also track what this implies for the average growth rate in these economies. I then follow the exact same procedure but instead give a unit pulse to an average sector. That is, for a sampled A matrix, I pick a sector that supplies to twenty other sectors. If none is found I pick the next largest sector. If more than one is found I shock at random one of the average degree sectors. Figures 7 and 8 below display the outcome of such experiments.

[26] Recall that the level of sectoral capital and that of sectoral output differ only by a constant in expression (25).

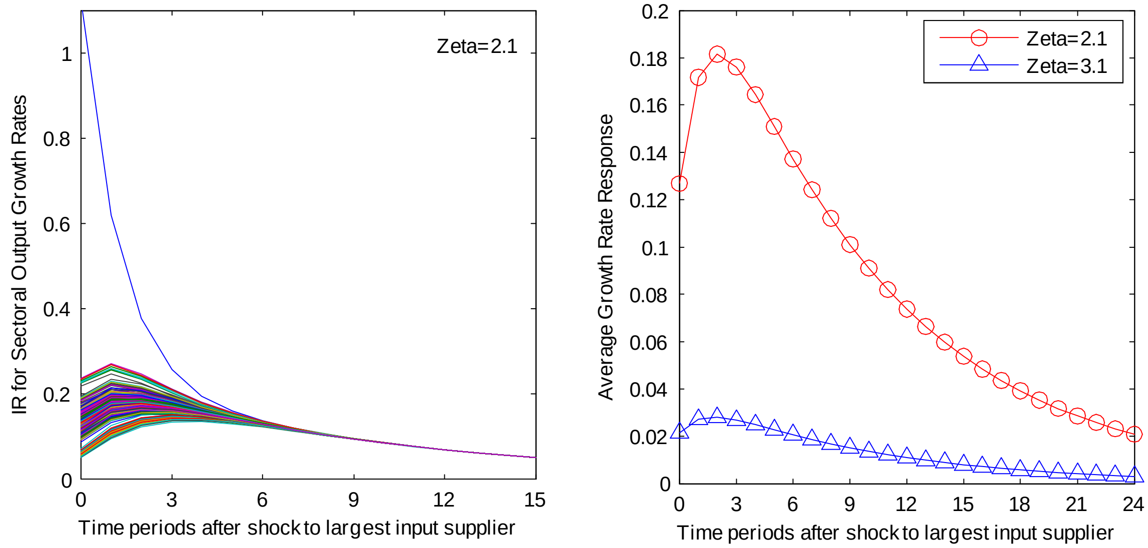


Figure 7: *Impulse responses (lhs) and mean growth rate response (rhs) following a shock to the largest input supplier.*

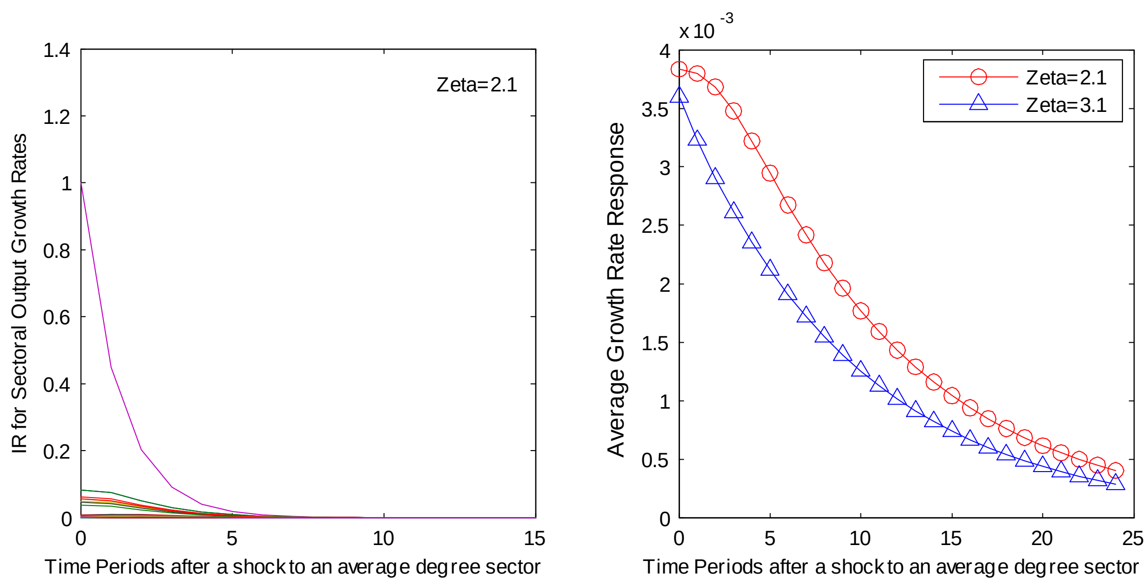


Figure 8: *Impulse responses (lhs) and mean growth rate (rhs) response following a shock to an average outdegree sector*

For a single A matrix, sampled from a family $\mathcal{A}(500, 20, 2.1)$, the left panel of Figures 7 and 8 shows five hundred growth rate responses following a one-standard deviation shock to the largest input-supplying sector (Figure 7) and to an average sector (Figure 8)[27]. The right panel shows the

[27] The top line in each panel gives the own-response of the sector given the unit pulse.

implied mean growth rate response averaged over thirty realizations of such A matrices and compares it to the corresponding object, averaged over thirty alternative A matrices, sampled from the less heterogeneous family of input-use matrices $\mathcal{A}(500, 20, 3.1)$.

What the left panels in Figures 7 and 8 display is precisely the robust yet vulnerable nature of heterogeneous connectivity economies. A shock to the largest sector induces broad comovement in the economy as disturbances in the production technology of a general purpose sector propagate to all sectors in the economy. Notice that this synchronized response induces a cycle-like behavior: as the shock propagates through the economy and over time as there is a gradual buildup in the growth rate of every sector followed by a gradual decay across the board. In contrast, a shock to an average connectivity sector induces responses in a small number of sectors. Its limited number of connections implies no synchronized movement and as a consequence, propagation is weak and short-lived.

The circle-line in the right panels of Figures 7 and 8, confirms that this is not the result of the particular A matrix sampled. The mean growth response averaged over thirty economies drawn from $\mathcal{A}(500, 20, 2.1)$ yields the same type of dynamics: it responds non-monotonically and persists through time in the case of a shock to the largest sector. In contrast, following a shock to an average sector, the mean growth rate displays a monotonic response that is two orders of magnitude smaller. The square-line in the right panels shows what happens when I sample from more diversified economies ($\zeta = 3.1$). The upshot of a thiner tail is that the largest sector sampled from a $\mathcal{A}(500, 20, 3.1)$ family will supply to relatively smaller number of sectors: as such propagation is weaker and the mean growth rate response is smaller by one order of magnitude. Interestingly, no such contrast obtains when I consider a shock to an average sector. This suggests that the difference between more and less diversified economies lies in their vulnerability to disturbances in large sectors and not in their robustness to an average shock.

7. Conclusion

Narrowly defined, the starting point of this paper was based on the following insight: setting to zero elasticities of substitution for particular intermediate inputs is tantamount to assuming particular network structures for sectoral linkages. From this, I have shown that it is possible to start characterizing sparseness in large-scale input-use matrices by using a network approach to data. More importantly, I have built models of input-use matrices that retain first-order connectivity characteristics of data. With this apparatus in hand, the paper employed these tools to solve a controversial question in the business cycle literature: can large-scale multisector models with independent productivity shocks generate non-negligible fluctuations in aggregates?

The answer that emerges from this paper is: yes, provided most sectors resort in large measure to the same general purpose inputs. In other words, aggregate fluctuations obtain in economies that are not too diversified in terms of the inputs required by different technologies. Further, input-use data seems to confirm that this is indeed the case, as most sectors rely on key, basic, technologies: oil, electricity, iron and steel, real estate, truck transportation and telecommunications. Sectors are therefore interconnected by their joint reliance on a limited number of general purpose technologies and differ only in the mix of remaining inputs each uses to produce its good.

From a network perspective this means that the linkage structure in the economy is dominated by a few sectoral hubs, supplying inputs to many different sectors. In this case, productivity fluctuations in these hub-like sectors propagate through the economy and affect aggregates, much in the same way as a shutdown at a major airport has a disruptive impact on all scheduled flights throughout a country. In either case, there are no close substitutes and every user is affected by disturbances at the source.

Once one starts to think about the fabric of input-trade in this way, other questions follow suit: can one characterize diversification in networks of sectoral technologies over time or across countries? Take, for example, a problem that has generated recent interest among macroeconomists: the decline in business cycle volatility over the past half century. The conjecture that follows from this paper is that reliance on traditional hubs has diminished as more specialized substitutes have developed. The response of the U.S. economy to past and present oil shocks seems to confirm this view: as alternative energy technologies develop and sectors diversify in their most preferred energy source, the role of oil as a hub to the economy has diminished. As such, oil shocks would likely have a smaller impact on aggregates. Concurrently, the I.T. revolution can be seen as having provided a wealth of alternatives to traditional means of communication and points of sale. The same network perspective can be taken across countries: do less developed economies rely relatively more on a limited number of key technologies? In this sense, can their technologies be characterized as less diversified? If so, the arguments in this paper would predict that less developed economies display more pronounced movements in aggregate output, as indeed seems to be the case in data.

To go beyond these conjectures necessarily implies more careful measurement of the network properties of input-use data and, most likely, more disaggregated data. Indeed, the particular network properties chosen in this paper - tail properties of degree sequences - are both hard to measure and special in that they pertain only to local features of a network. Other measures of connectivity exist and can be of use in characterizing properties of intersectoral trade flows.

At the same time, once one recognizes that network structure is linked to macroeconomic outcomes a more ambitious question emerges: what determines these structures? This requires developing a causal mechanism, i.e. a theory where the network of input-flows - or at least some of its properties - is the endogenous outcome of a well-specified economic model. Such a theory is surely necessary if one is to think rigourosly about the dynamic evolution of these complex objects and to make sense of the data patterns suggested in this paper. This paper falls short of this and makes the easier point that network structure matters. As such, this paper is a necessary starting point for a larger research agenda linking macroeconomic outcomes to the networked structure of modern economies.

References

Bak, P., Chen, K., Scheinkman, J. and M. Woodford, 1993, Aggregate Fluctuations for Independent Sectoral Shocks: Self Organized Criticality in a Model of Production and Inventory Dynamics, *Richerche Economiche*, 41, 3-30.

Bollobas, B., 2001, *Random Graphs*, Second Edition, Cambridge Mathematical Library Series, Cambridge University Press.

Bollobas, B. and O.M. Riordan, 2003, Mathematical Results on Scale-Free Random Graphs, in Bornholdt, S. and H.G. Schuster, eds., *Handbook of Graphs and Networks*, Wiley-VCH.

Brock, W.A., 1999, Scaling in Economics: a Reader's Guide, *Industrial and Corporate Change*, Vol. 8 (3).

Chung, F. and L. Lu, 2002, The Average Distance in Random Graphs with Given Expected Degrees, *Proceedings of the National Academy of Sciences.*

Chung, F., Lu, L. and V. Vu, 2003, Eigenvalues of Random Power Law Graphs, *Annals of Combinatorics*, 7, 21-33.

Chung, F., and L. Lu, 2006, *Complex Graphs and Networks*, CBMS, Regional Conference Series in Mathematics, 107.

Cochrane, J., 1994, Shocks, *Carnegie-Rochester Conference Series on Public Policy*, Vol. 41, pp. 295-364.

Conley, T.G. and B. Dupor, 2003, A Spatial Analysis of Sectoral Complementarity, *Journal of Political Economy*, 111, 2, 311-351.

Cooper, R. and J. Haltiwanger, 1990, Inventories and the Propagation of Sectoral Shocks, *American Economic Review*, 80, 170-90.

Embrechts, P., C. Kluppelberg, T. Mikosch, 1997, *Modelling Extremal Events for Insurance and Finance,* (Springer, New York).

Diestel, R., 1997, *Graph Theory,* Graduate Texts in Mathematics' Series #173, Springer-Verlag.

Dupor, B., 1999, Aggregation and Irrelevance in Multi-Sector Models, *Journal of Monetary Economics*, 43, 391-409.

Durlauf, S.N., 2005, Complexity and Empirical Economics, *Economic Journal*, 112, 225-243.

Durrett, R. 2006, *Random Graph Dynamics*, Cambridge University Press.

Fisher, E. and F. Vega-Redondo, 2007, *The Linchpins of a Modern Economy*, mimeo., paper presented at the 2007 annual meeting of the American Economic Association.

Gabaix, X., 2005, *The Granular Origins of Aggregate Fluctuations*, mimeo., NYU-Stern Business School.

Gabaix, X. and Y. Ioannides, 2004, The Evolution of City Size Distributions, *Handbook of Regional and Urban Economics*, volume 4, Chapter 53, V. Henderson and J-F. Thisse eds., North-Holland.

Golub, G. and C. Van Loan, 1996, *Matrix Computations*, Third edition, The Johns Hopkins University Press.

Henderson, H.V. and S.R. Searle, 1981, On Deriving the Inverse of a Sum of Matrices, *SIAM Review*, 23, 1, 53-60.

Horvath, M., 1998, Cyclicality and Sectoral Linkages: Aggregate Fluctuations from Independent Sectoral Shocks, *Review of Economic Dynamics*, 1, 781-808.

Horvath, M., 2000, Sectoral Shocks and Aggregate Fluctuations, *Journal of Monetary Economics*, 45, 69-106.

Horvath, M. and R. Verbrugge, 1996, *Shocks and Sectoral Interactions: an Empirical Investigation*, mimeo.

Jackson, M.O., 2005, The Economics of Social Networks, *Proceedings of the 9th World Congress of the Econometric Society*, eds. Richard Blundell, Whitney Newey and Torsten Persson, Cambridge University Press.

Jones, C.I., 2007a, Intermediate Goods and Weak Links: A Theory of Economic Development, mimeo., University of California - Berkeley.

Jones, C.I., 2007b, Input-Output Multipliers, General Purpose Technologies, and Economic Development, mimeo., University of California - Berkeley.

Jovanovic, B. 1987, Micro Shocks and Aggregate Risk, *Quarterly Journal of Economics*, 102, 395-409.

Li, L., Alderson, D., Doyle, J. and W. Willinger, 2006, Towards a Theory of Scale-Free Graphs: Definition, Properties and Implications., *Internet Mathematics*, forthcoming

LONG JR., J.B. AND C.I. PLOSSER, 1983, Real Business Cycles, *Journal of Political Economy*, 91, 1, 39-69.

LONG JR., J.B. AND C.I. PLOSSER, 1987, Sectoral versus Aggregate Shocks, *American Economic Review*, 77, 333-36.

LU, L.L., 2002, *Probabilistic Methods in Massive Graphs and Internet Computing*, PhD Dissertation, University of California, San Diego.

LUCAS, R.E., 1981, Understanding Business Cycles, in *Studies in Business Cycle Theory.* MIT Press.

MITZENMACHER, M. (2003), A Brief History of Generative Models for Power Law and Log Normal Distributions, *Internet Mathematics*, Vol 1(2), 226-251.

NIREI, M., 2005, Threshold Behavior and Aggregate Fluctuations, *Journal of Economic Theory*,

NEWMAN, M.E.J., 2003, The Structure and Function of Complex Networks, *SIAM Review*, 45, 2, 167-256.

NEWMAN, M.E.J., 2005, Power Laws, Pareto Distributions and Zipf's Law, *Contemporary Physics*, 46, 323-351

NORRBIN, S., AND D. SCHLAGENHAUF, 1990, Sources of Output Fluctuations in the United States during the Inter-War and Post-War Years, *Journal of Economic Dynamics and Control,* 14, 523-51.

RAUCH, J. E., 1999, Networks versus Markets in International Trade, *Journal of International Economics*, Vol. 48, 7-35.

ROSENBLATT, D., 1957, On Linear Models and the Graphs of Minkowski-Leontief Matrices, *Econometrica*, Vol. 25, No. 2.

SCHEINKMAN, J. AND M. WOODFORD, 1994, Self-Organized Criticality and Economic Fluctuations, *American Economic Review*, Vol. 84, No. 2, Papers and Proceedings, pp. 417-421.

SHEA, J., 2002, Complementarities and Comovements, *Journal of Money, Credit and Banking*, 34, 2, 412-433.

SIMON, H. A. AND A. ANDO, 1961, Aggregation of Variables in Dynamical Systems, *Econometrica*, Vol. 29, No. 2.

Solow, R., 1952, On the Structure of Linear Models, *Econometrica,* Vol. 20, No. 1, 29-46.

Summers, L. H. ,1986, Some Skeptical Observations on Real Business Cycle Theory, *Federal Reserve Bank Minneapolis Quarterly Review*, 10, 23–27.

Szydl, D.B., 1985, Conditions for the Existence of a Balance Growth Solution for the Leontief Dynamic Input-Output Model, *Econometrica*, Vol. 53, No. 6.

Vega-Redondo, F., 2007, *Complex Social Networks*, Econometric Society Monographs, Cambridge University Press.

A. Appendix

A.1. Some results on the inverse of a sum of two matrices.

Lemma A.1. Let A be a nonsingular $M-$dimensional matrix and let U, B and V be $M \times M$ matrices. Then,

$$(A + UBV)^{-1} = A^{-1} - (I + A^{-1}UBV)^{-1}A^{-1}UBVA^{-1}$$

Proof: See, for example, Henderson and Searle (1981).

A particular case of this is given by the Bartlett inverse

Lemma A.2. (Bartlett Inverse). *Let A be a square, invertible, $M-$dimensional matrix and u and v be $M-$dimensional vectors. Then:*

$$(A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$$

A.2. Proofs

Proofs of Section 2

Proposition 2.2. *Assume that $\sum_{i \in \check{S}_j} \gamma_{ij} = \gamma$ and that $\sigma_j^2 = \sigma^2$ for all sectors $j = 1$. Then*

$$\begin{aligned}
\sigma_Y^2(\Gamma_C) &= \left(\frac{1}{1-\gamma}\right)^2 \frac{\sigma^2}{M} \text{ for Complete Intersectoral Networks} \\
\sigma_Y^2(\Gamma_{Star}) &= \left(\frac{N}{M} + 2\frac{\gamma}{1-\gamma}\right)\frac{\sigma^2}{M} + \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\sigma^2}{N} \text{ for } N \text{ star networks}
\end{aligned}$$

Proof: First construct vectors u and v such that u is an $M \times 1$ vector with each entry u_i restricted $\in [0, M]$ and v an $M \times 1$ vector where each element v_i is given by $\frac{\gamma}{\sum_i u_i}$. Now let $\Gamma = uv'$ and apply Lemma A.1. to get

$$(I - \Gamma)^{-1} = (I - uv')^{-1} = I + \frac{uv'}{1-\gamma}$$

where the last equality follows from $v\prime u = \gamma$ (by constuction). Notice that for the static multi-sector economy deviations of sectoral output from its mean are then given, in vector form, by:

$$\mathbf{y} - \boldsymbol{\mu} = (\mathbf{I} + \frac{\mathbf{uv'}}{\mathbf{1-\gamma}})'\boldsymbol{\varepsilon}$$

Taking the complete case first:

$$\Gamma_C = \left(\frac{\gamma}{\overline{d_{G_c}}} A(G_c) \right)$$

where $A(G_c)$ is the adjacency matrix of a complete regular digraph of sectoral supply linkages and $.\overline{d_{G_c}}$ is its average out-degree $\frac{\sum_i d_{v_i}}{M}$.Notice that that the average out-degree of a complete digraph on M sectors is M. Hence I can rewrite Γ_C as

$$\Gamma_C = \gamma M \mathbf{1}_M \mathbf{1}_M^{'} \left(\frac{1}{\sum_i d_{v_i}} \right)$$

Now let u be given by $M\mathbf{1}_M$ (i.e. the degree sequence of G_c) and $v' = \mathbf{1}_M^{'} \left(\frac{\gamma}{\sum_i d_{v_i}} \right)$. Using this with the Bartlett inverse result:

$$\begin{aligned} \sum_{i=1}^{M} y_i - \mu_i &= \sum_{i=1}^{M} (1 + M \frac{u_i \gamma}{(1-\gamma)\sum_i d_{v_i}}) \varepsilon_i \\ &= \sum_{i=1}^{M} (1 + M \frac{M\gamma}{(1-\gamma)M^2}) \varepsilon_i \\ &= \sum_{i=1}^{M} \frac{1}{(1-\gamma)} \varepsilon_i \end{aligned}$$

Finally given the assumption of i.i.d. sectoral disturbances, $\sigma_Y^2 \equiv E\left[\frac{\sum_{i=1}^{M}(y_i - \mu_i)}{M}\right]^2$

$$\sigma_Y^2(\Gamma_C) = \left(\frac{1}{1-\gamma}\right)^2 \frac{\sigma^2}{M}$$

Now for the $N - Star$ case the input-use matrix is given by

$$\Gamma_{star} = \left(\frac{\gamma}{\overline{d_{G_{Star}}}} A(G_{Star}) \right)$$

where $A(G_{Star})$ is the binary matrix defined in the text and $\overline{d_{G_{Star}}}$ is its average out-degree $\frac{\sum_i d_{v_i}}{M}$.Without loss of generality, order sectors that the first $M - N$ vertices are not input suppliers and the remaining N sectors supply inputs to every sector in the economy. Then I can write Γ_{Star} as

$$\Gamma_{Star} = \gamma M \begin{bmatrix} \mathbf{0}_{M-N} \\ \mathbf{1}_N \end{bmatrix} \mathbf{1}_M^{'} \left(\frac{1}{\sum_i d_{v_i}} \right)$$

Where 0_{M-N} is an $M-N$ dimensional vector of zeros and 1_N be an N dimensional vector of ones. Now, take the Bartlett inverse, where u is given by $M \begin{bmatrix} \mathbf{0}_{M-N} \\ \mathbf{1}_N \end{bmatrix}$ (which is simply the out-degree sequence of the $N - star$ network) and $v' = 1_M^{'} \left(\frac{\gamma}{\sum_i d_{v_i}} \right)$. Then,

$$\begin{aligned}\sum_{i=1}^{M} y_i - \mu_i &= \sum_{i=M-N+1}^{M} (1 + M\frac{u_i\gamma}{(1-\gamma)\sum_i d_{v_i}})\varepsilon_i \\ &= \sum_{i=M-N+1}^{M} (1 + \frac{M^2\gamma}{(1-\gamma)\sum_i d_{v_i}})\varepsilon_i \\ &= \sum_{i=M-N+1}^{M} (1 + \frac{\gamma}{1-\gamma}\frac{M}{N})\varepsilon_i\end{aligned}$$

Then, given the assumption of i.i.d. sectoral disturbances

$$\sigma_Y^2(\Gamma_{Star}) = \left(\frac{N}{M} + 2\frac{\gamma}{1-\gamma}\right)\frac{\sigma^2}{M} + \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\sigma^2}{N}$$

As stated in the proposition. ■

Proofs of Section 4

Lemma 4.2.1. *For every M sector economy, associate a weight sequence $e \doteq \{e_1, ..., e_M\}$ to the collection of sectoral labels, such that $e_i \in (0, M)$ and define a realization of the intersectoral trade graph G as the realization of independent binary random variables $X_{ij} \in \{0,1\}$, $i, j = 1, ...M$ where $Pr(X_{ij} = 1) = p_{ij}$ given by [14]. Then,*

i) for any M, the expected out-degree of a sector will be given by:

$$E(d_i^{out}) = \sum_j p_{ij} = e_i, \quad i = 1, ..., M \tag{29}$$

ii) for any realization of the intersectoral trade graph G and for any sector i, if $e_i > \log M$, its actual out-degree d_i^{out} will almost surely satisfy

$$\left|d_i^{out} - e_i\right| \leq 2\log M$$

otherwise if $e_i \leq \log M$, then d_i^{out} almost surely satisfies

$$\left|d_i^{out} - e_i\right| \leq 2\sqrt{e_i \log M}$$

iii) for any sector i, its $E(d_i^{in})$ expected indegree is given by:

$$E(d_i^{in}) = \sum_i p_{ij} = \frac{\sum_i e_i}{M} \equiv \overline{e}, \forall i \tag{30}$$

Proof: The proof of parts i) and iii) are given in the Lemma. Part ii) is proven in Chung and Lu (2006, Lemma 5.7.), resorting to Chernoff bounds

Proposition 4.3.3.. *The Leontieff inverse of the expected input use matrix, $\left(I_M - \overline{\Gamma}\right)^{-1}$ is given by*

$$\left(I_M - \overline{\Gamma}\right)^{-1} = I_M + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'_M$$

where I_M is the $M \times M$ identity matix, ϕ is a $M \times 1$ vector with typical element $\frac{e_i}{\sum_{i=1}^{M} e_i}$ and 1_M is the unit vector of dimension $M \times 1$.

Proof: The proof follows along the same lines of the proof in Proposition 2.1., and is simply using Bartlett's formula in Lemma A.2. where $u = \boldsymbol{\phi}$ and $v = \gamma\mathbf{1}_M$. and noticing that $v\prime u = \gamma$ necessarily.

Proposition 4.3.4.. *Assume that the input-use matrix is given by the expected input-use matrix $\overline{\Gamma}$ and that sectoral volatility $\sigma_j = \sigma$ for all sectors j. Then, for a static $M-$sector economy the aggregate volatility of equilibrium output, σ_A is given by:*

$$\sigma_Y^2(\overline{\Gamma}) = \begin{cases} \left(\frac{\gamma}{1-\gamma}\right)^2 \kappa_1(\zeta)\frac{\sigma^2}{M} + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta > 3 \\ \left(\frac{\gamma}{1-\gamma}\right)^2 \kappa_2(\zeta)\left[\frac{1}{M}\right]^{\frac{4-2\zeta}{\zeta-1}} \sigma^2 + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta \in (2,3) \end{cases}$$

where terms $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ *are positive and constant given a ζ.*

Proof: Use the result in Proposition 4.3.3. to substitute in $(I - \overline{\Gamma})^{-1\prime}$ in expression (8) of the static model presented in section 2.

$$\mathbf{y} - \boldsymbol{\mu} = \left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\boldsymbol{\varepsilon}$$

Thus with $\mathbf{1}_M$ as the M-dimensional unit vector:

$$\begin{aligned} \sum_{i=1}^{M} y_i - \mu_i &= \mathbf{1}'\left[\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \Theta\right]'\boldsymbol{\varepsilon} \\ &= \mathbf{1}'I\boldsymbol{\varepsilon} + \frac{\gamma}{1-\gamma}\mathbf{1}'(\mathbf{1}\boldsymbol{\phi})\boldsymbol{\varepsilon} \\ &= \sum_{i=1}^{M}\varepsilon_i + \frac{\gamma}{1-\gamma}M\sum_{i=1}^{M}\phi_i\varepsilon_i \end{aligned}$$

Thus the aggregate statistic $\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i$

$$\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i = \frac{1}{M}\sum_{i=1}^{M}\varepsilon_i + \frac{\gamma}{1-\gamma}\sum_{i=1}^{M}\phi_i\varepsilon_i$$

which has expectation zero given independent technological disturbances ε_i. Now we're interested in $E\left[\left(\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i\right)^2\right]$

$$E\left[\left(\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i\right)^2\right] = \frac{1}{M^2}\sum_{i=1}^{M} E((\varepsilon_i)^2) + \left(\frac{\gamma}{1-\gamma}\right)^2 \sum_{i=1}^{M} \phi_i^2 E((\varepsilon_i)^2)$$

Now using the assumption $E((\varepsilon_i)^2) = \sigma^2$.

$$\sigma_Y^2 = \left(\frac{\gamma}{1-\gamma}\right)^2 \sigma^2 \sum_{i=1}^{M} \phi_i^2 + \frac{1}{M}\sigma^2$$

Finally notice that by definition of the vector $\boldsymbol{\phi}$ given in Proposition 4.3.3.

$$\begin{aligned}
\sum_{i=1}^{M} \phi_i^2 &\equiv \left[\sum_{i=1}^{M} e_i^2 / \left(\sum_{i=1}^{M} e_i\right)^2\right] \\
&= \frac{1}{M}\frac{\sum_{i=1}^{M} e_i^2 / \sum_{i=1}^{M} e_i}{\sum_{i=1}^{M} e_i / M} \\
&= \frac{1}{M}\frac{\widetilde{e}}{\overline{e}}
\end{aligned}$$

Where $\widetilde{e} \equiv \sum_{i=1}^{M} e_i^2 / \sum_{i=1}^{M} e_i$. Chung and Lu (2006, p.109) show that for large M and under the power law weight parameterization given in Definition 4.2.2., $\widetilde{e}$ is given by:

$$\widetilde{e} = \begin{cases} \overline{e}\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)} & \text{if} \quad \zeta > 3 \\ \overline{e}^{\zeta-2}\frac{(\zeta-2)^{\zeta-1} m^{3-\zeta}}{(\zeta-1)^{\zeta-2}(3-\zeta)} & \text{if} \quad \zeta \in (2,3) \end{cases}$$

where m is the maximum expected outdegree, e_1. Thus, according to Definition 4.2.2. $m = \frac{\zeta-2}{\zeta-1}\overline{e}M^{\frac{1}{\zeta-1}}$. Plugging in these expressions in the expression for σ_Y^2 above, I get:

$$\sigma_Y^2(\overline{\Gamma}) = \begin{cases} \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}\frac{\sigma^2}{M} + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta > 3 \\ \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}} \sigma^2 + \frac{1}{M}\sigma^2 & \text{if} \quad \zeta \in (2,3) \end{cases}$$

Letting $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ gives the claim. ■

Proposition 4.3.5. *Fix a triplet of parameters* $(M, \overline{e}, \zeta)$ *and assume that* $\sum_{i\in\check{S}_j}\gamma_{ij} = \gamma$ *and that* $\sigma_j = \sigma$ *for all sectors* $j = 1$. *For any* $A(G)$ *sampled from the family of input-use graphs* $\mathcal{A}(M, \overline{e}, \zeta)$, *construct the input-use matrix according to Lemma 4.1.2. Then i) for any* $A(G)$

$$[I_M - \Gamma(A)]^{-1} = I_M + \frac{\gamma}{1-\gamma}\phi 1'_M + \times$$

where $\times$ is an $M \times M$ random matrix with zero column sums. Further ii) for any $A(G)$ of the family of input-use graphs A, and $\zeta \in (2,3)$ the following is a lower bound for aggregate volatility

$$\sigma_A^2(\Gamma(A)) > \begin{cases} \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right]\kappa_1(\zeta)\frac{\sigma^2}{M} & \text{if} \quad \zeta > 3 \\ \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right]\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\sigma^2 & \text{if} \quad \zeta \in (2,3) \end{cases}$$

where $\varkappa$ is a constant strictly smaller than $\left(\frac{\gamma}{1-\gamma}\right)^2$.

Proof of claim i) in the Proposition: First notice that for any realization of the intersectoral trade digraph $A(G)$ one can always decompose $A(G)D$ as

$$A(G)D = E(A(G)D) + [A(G)D - E(A(G)D)]$$

where E is the expectation operator. Therefore the matrix Z, i.e. the Leontieff inverse $[I - \gamma A(G)D]^{-1}$ can be expressed as:

$$[I - \gamma A(G)D]^{-1} = \{I - \gamma E(A(G))E(D) - \gamma[A(G)D - E(A(G))E(D)]\}^{-1}$$

Now, to apply the formula for the inverse of a sum of matrices in Lemma A.1., let

$$\begin{aligned} I - \gamma E(A(G))E(D) &\equiv C \\ -\gamma[A(G)D - E(A(G))E(D)] &\equiv U \end{aligned}$$

to express the problem as an inverse of a sum of matrices:

$$[I - \gamma A(G)D]^{-1} = [C + I_M U I_M]^{-1}$$

so that the formula for the inverse in A.1. yields

$$[I - \gamma A(G)D]^{-1} = C^{-1} - C^{-1}U[I + C^{-1}U]^{-1}C^{-1}$$

To calculate C^{-1} notice that $E(A(G))E(D)$ can be expressed as a rank one matrix:

$$E(A(G))E(D) = \boldsymbol{\phi}\mathbf{1}'_M$$

where $\boldsymbol{\phi} = \left[\frac{e_1}{\sum_{i=1}^M e_i}, ..., \frac{e_M}{\sum_{i=1}^M e_i}\right]$ and $\mathbf{1}' \equiv [1, ..., 1]$. Thus applying Barttlet formula for C^{-1} yields

$$[I - \gamma\boldsymbol{\phi}\mathbf{1}']^{-1} = I + \frac{\gamma\boldsymbol{\phi}\mathbf{1}'}{1 - \gamma\mathbf{1}'I\boldsymbol{\phi}}$$

Since $\mathbf{1}'\phi = 1$ we get for C^{-1}

$$[I - \gamma\boldsymbol{\phi}\mathbf{1}']^{-1} = I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'$$

Now to solve for $[I + C^{-1}U]^{-1}$ substitute in C^{-1} to get

$$\begin{aligned}
[I + C^{-1}U]^{-1} &= \left[I + \left(I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right) U \right]^{-1} \\
&= \left[I + U + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' U \right]^{-1}
\end{aligned}$$

Notice that for any realization of $A(G)D$,the matrix $U = -\gamma[A(G)D - E(A(G))E(D)]$, will have zero column sums. This is so since, by construction, $A(G)D$ and $E(A(G))E(D)$ have the same column sums (and equal to 1 for every column). Hence the difference will yield zero column sums. Thus $\mathbf{1}'U = \mathbf{0}'$ where $\mathbf{0}$ is an $M \times 1$ vector of zeros and $\phi \mathbf{1}'U$ is a $M \times M$ matrix of zeros. This implies that:

$$[I + C^{-1}U]^{-1} = [I + U]^{-1}$$

Collecting results

$$[I - \gamma A(G)D]^{-1} = \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right] - \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right] U[I + U]^{-1} \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right]$$

This expression can be further simplified by again using the fact that $\phi \mathbf{1}'U$ is a matrix of zeros. Thus:

$$[I - \gamma A(G)D]^{-1} = \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right] - U[I + U]^{-1} \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right]$$

or

$$[I - \gamma A(G)D]^{-1} = \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right] + \Theta$$

with Θ defined as

$$\Theta \equiv \gamma[A(G)D - E(A(G)D)][I - \gamma[A(G)D - E(A(G)D)]]^{-1} \left[I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right]$$

Finally, for the zero column sum claim on Θ I use the following result (see Golub and van Loan, Lemma 2.3.3.)

$$if \quad \|\Gamma\|_1 < 1 \text{ then } \left\| (I - \Gamma)^{-1} \right\|_1 \leq \frac{1}{1 - \|\Gamma\|_1}$$

where $\|\Gamma\|_1$ is the maximum absolute column sum of Γ. Notice that by construction for any $A(G)$, all column sums of Γ are positive and given by $\|\Gamma(A)\|_1 = \gamma < 1$ and therefore $\left\| (I - \Gamma)^{-1} \right\|_1 \leq \frac{1}{1-\gamma}$. Now notice that $I + \frac{\gamma}{1-\gamma} \phi \mathbf{1}'$ is a $M \times M$ matrix with all columns sums equal to $\frac{1}{1-\gamma}$ and thus $\left\| I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}' \right\|_1 = \frac{1}{1-\gamma}$. This therefore implies that Θ has all column sums equal to zero.since otherwise $\left\| (I - \Gamma)^{-1} \right\|_1$ would be greater than $\frac{1}{1-\gamma}$.

Proof of part ii) of the Proposition. Recall $\Theta \equiv -U[I+U]^{-1}\left[I+\frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$. Then the static multisectoral model gives:

$$\mathbf{y}-\boldsymbol{\mu}=\left[\left[I+\frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]+\Theta\right]'\boldsymbol{\varepsilon}$$

Thus with $\mathbf{1}' \equiv [1,...,1]$

$$\begin{aligned}\sum_{i=1}^{M} y_i-\mu_i &= \mathbf{1}'\left[\left[I+\frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]+\Theta\right]'\boldsymbol{\varepsilon} \\ &= \mathbf{1}'I\boldsymbol{\varepsilon}+\frac{\gamma}{1-\gamma}\mathbf{1}'(\mathbf{1}\boldsymbol{\phi})\boldsymbol{\varepsilon}+\mathbf{1}'\Theta'\boldsymbol{\varepsilon} \\ &= \sum_{i=1}^{M}\varepsilon_i+\frac{\gamma}{1-\gamma}M\sum_{i=1}^{M}\phi_i\varepsilon_i+\sum_{j=1}^{M}\sum_{i=1}^{M}\theta_{ij}\varepsilon_i\end{aligned}$$

Thus the aggregate statistic $\frac{1}{M}\sum_{i=1}^{M} y_i-\mu_i$

$$\frac{1}{M}\sum_{i=1}^{M} y_i-\mu_i=\frac{1}{M}\sum_{i=1}^{M}\varepsilon_i+\frac{\gamma}{1-\gamma}\sum_{i=1}^{M}\boldsymbol{\phi}_i\varepsilon_i+\frac{1}{M}\sum_{j=1}^{M}\sum_{i=1}^{M}\theta_{ij}\varepsilon_i$$

which has expectation zero given independent technological disturbances. Now we're interested in $E\left[\left(\frac{1}{M}\sum_{i=1}^{M} y_i-\mu_i\right)^2\right]$

$$\begin{aligned}E\left[\left(\frac{1}{M}\sum_{i=1}^{M} y_i-\mu_i\right)^2\right] &= \frac{1}{M^2}\sum_{i=1}^{M}E((\varepsilon_i)^2)+\left(\frac{\gamma}{1-\gamma}\right)^2\sum_{i=1}^{M}\phi_i^2E((\varepsilon_i)^2)+\frac{1}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2E((\varepsilon_i)^2) \\ &\quad +\frac{2}{M}\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_iE((\varepsilon_i)^2)+\frac{2}{M}\sum_{i=1}^{M}\sum_{j=1}^{M}\theta_{ij}E((\varepsilon_i)^2) \\ &\quad +\frac{2}{M}\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)E((\varepsilon_i)^2)\end{aligned}$$

Notice that $\sum_{i=1}^{M}\sum_{j=1}^{M}\theta_{ij}=\sum_{j=1}^{M}\sum_{i=1}^{M}\theta_{ij}=0$ (since Θ is a zero column-sum matrix), that $\sum_{i=1}^{M}\phi_i=1$ and that $E((\varepsilon_i)^2)=\sigma^2$. Using these facts I can simplify the expression to:

$$\sigma_A^2=\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{1}{M}\sigma^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2$$

Now the second to last term in the expression can potentially be negative and/or of higher order than the first term. To show that the latter is never the case, group the first and the last two terms in this formula:

$$\left(\frac{\gamma}{1-\gamma}\right)^2 \sigma^2 \sum_{i=1}^{M} \phi_i^2 + \frac{2}{M}\sigma^2 \left(\frac{\gamma}{1-\gamma}\right) \sum_{i=1}^{M} \phi_i \left(\sum_{j=1}^{M} \theta_{ij}\right) + \frac{\sigma^2}{M^2} \sum_{i=1}^{M} \left(\sum_{j=1}^{M} \theta_{ij}\right)^2$$
$$= \sigma^2 \sum_{i=1}^{M} \left[\frac{\gamma}{1-\gamma}\phi_i + \frac{1}{M}\sum_{j=1}^{M} \theta_{ij}\right]^2$$

Thus, since the resulting value of this sum in i has to be non-negative it has to be the case that, for every M:

$$\left(\frac{\gamma}{1-\gamma}\right)^2 \sum_{i=1}^{M} \phi_i^2 \geq \left|\frac{2}{M}\left(\frac{\gamma}{1-\gamma}\right) \sum_{i=1}^{M} \phi_i \left(\sum_{j=1}^{M} \theta_{ij}\right) + \frac{1}{M^2} \sum_{i=1}^{M} \left(\sum_{j=1}^{M} \theta_{ij}\right)^2\right|$$

Further, since $\sum_{i=1}^{M}\sum_{j=1}^{M} \theta_{ij} = 0$, if for some i $\sum_{j=1}^{M} \theta_{ij} < 0$ then it has to be the case that there exists at least one index k such that $\sum_{j=1}^{M} \theta_{kj} > 0$. Therefore, the inequality above is strict for all M. In particular, this implies that the right hand side in the inequality cannot have a slower rate of convergence to zero than the term on the left hand side. Thus I have shown that $\left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \sum_{i=1}^{M} \phi_i^2$ where $\varkappa$ is strictly lower than $\left(\frac{\gamma}{1-\gamma}\right)^2$ and independent of M, is a worst case bound for the $i-$sum of these three terms. Therefore

$$\begin{aligned}
\sigma_A^2(A) &> \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \sigma^2 \sum_{i=1}^{M} \phi_i^2 + \frac{1}{M}\sigma^2 + \frac{2}{M}\sigma^2 \left(\frac{\gamma}{1-\gamma}\right) \\
&> \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \sigma^2 \sum_{i=1}^{M} \phi_i^2
\end{aligned}$$

for any M. Inserting the expression for $\sum_{i=1}^{M} \phi_i^2$ derived in the Proposition 4.3.4. yields,

$$\sigma_A^2(A) > \begin{cases} \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)} \frac{\sigma^2}{M} & \text{if } \zeta > 3 \\ \left[\left(\frac{\gamma}{1-\gamma}\right)^2 - \varkappa\right] \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)} \left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}} \sigma^2 & \text{if } \zeta \in (2,3) \end{cases}$$

Letting $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ gives the claimed expression.■

Proofs of Section 5

Proposition 5.3.1. *For fixed α and γ, aggregation weights $w = (1/M)1_M$ and $\Gamma = \overline{\Gamma}$ give the univariate spectral density*

$$S(\omega,\overline{\Gamma}) = S(\omega,\overline{\Gamma}) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega)-\gamma^2)+\gamma^2\kappa_1(\zeta)\frac{1}{M}\right] \, if \; \zeta > 3$$

and

$$S(\omega,\overline{\Gamma}) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega)-\gamma^2)+\gamma^2\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\right] \quad if \; \zeta \in (2,3)$$

with $a(\omega) = \frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$, $b(\omega) = (1-\alpha e^{i\omega})(1-\alpha e^{-i\omega})$., $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ *and* $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

Proof: Let $\Phi(e^{i\omega}) \doteq (1-\alpha e^{i\omega})I - \overline{\Gamma}$.. Following the same reasoning as in the proof of Proposition 4.3.2. notice that $\overline{\Gamma} = \gamma\boldsymbol{\phi}\mathbf{1}'_M$. This

$$\Phi(e^{i\omega}) = (1-\alpha e^{i\omega})I - \gamma\boldsymbol{\phi}\mathbf{1}'_M$$

Then using Bartlett formula in Lemma A.2.

$$\begin{aligned}
\Phi(e^{i\omega})^{-1} &= (1-\alpha e^{i\omega})^{-1}I + (1-\alpha e^{i\omega})^{-1}\frac{\gamma\boldsymbol{\phi}\mathbf{1}'_M}{1-\mathbf{1}_M(1-\alpha e^{i\omega})^{-1}I\gamma\boldsymbol{\phi}}(1-\alpha e^{i\omega})^{-1} \\
&= (1-\alpha e^{i\omega})^{-1}[I + \frac{\gamma\boldsymbol{\phi}\mathbf{1}'_M}{(1-\alpha e^{i\omega})-\mathbf{1}_M\gamma\boldsymbol{\phi}'}] \\
&= (1-\alpha e^{i\omega})^{-1}[I + \frac{\gamma\boldsymbol{\phi}\mathbf{1}'_M}{1-\alpha e^{i\omega}-\gamma}]
\end{aligned}$$

Now, recalling the expression for the spectrum for aggregate capital for a given frequency

$$S(\omega) \doteq (2\pi)^{-1}\mathbf{w}'\Phi(e^{i\omega})^{-1\prime}\Phi(e^{i\omega})^{-1}\mathbf{w}$$

and w= $(1/M)\mathbf{1}_M$ where again, $\mathbf{1}_M$ is a $M\times 1$ vector of ones. Thus, letting $\kappa = 1/(1-\alpha e^{i\omega}-\gamma)$

$$\begin{aligned}
S(\omega) &= \frac{1}{(2\pi)M^2}\mathbf{1}'_M\Phi(e^{i\omega})^{-1\prime}\Phi(e^{i\omega})^{-1}\mathbf{1}_M \\
&= \frac{(1-\alpha e^{i\omega})^{-1}(1-\alpha e^{-i\omega})^{-1}}{(2\pi)M^2}\mathbf{1}'_M(I+\kappa'\mathbf{1}_M\boldsymbol{\phi}')(I+\kappa\boldsymbol{\phi}\mathbf{1}'_M)\mathbf{1}_M \\
&= \frac{(1-\alpha e^{i\omega})^{-1}(1-\alpha e^{-i\omega})^{-1}}{(2\pi)M^2}[M+\kappa\mathbf{1}'_M\boldsymbol{\phi}\mathbf{1}'_M\mathbf{1}_M+\kappa'\mathbf{1}'_M\mathbf{1}_M\boldsymbol{\phi}'\mathbf{1}_M+\kappa'\kappa\mathbf{1}'_M\mathbf{1}_M\boldsymbol{\phi}'\boldsymbol{\phi}\mathbf{1}'_{\mathbf{M}}\mathbf{1}_M]
\end{aligned}$$

Now, term by term, on the RHS square brackets:

$$i)\kappa\mathbf{1}'_{\mathbf{M}}\boldsymbol{\phi}\mathbf{1}'_M\mathbf{1}_M = \kappa\sum_{i=1}^{M}\phi_i\frac{M\gamma}{\sum_{i=1}^{M}\phi_i} = (1-\alpha e^{i\omega}-\gamma)M\gamma$$

$$ii)\kappa'\mathbf{1}'_M\mathbf{1}_M\boldsymbol{\phi}'\mathbf{1}_{\mathbf{M}} = \kappa'\frac{M\gamma}{\sum_{i=1}^{M}\phi_i}\sum_{i=1}^{M}\phi_i = (1-\alpha e^{-i\omega}-\gamma)M\gamma$$

$$iii)\kappa'\kappa\mathbf{1_M'}\mathbf{1}_M\boldsymbol{\phi}'\boldsymbol{\phi}\mathbf{1}_M'\mathbf{1_M} = \kappa'\kappa M\gamma(\sum_{i=1}^{M}\phi_i^2)M\gamma$$

Thus:

$$\begin{aligned} S(\omega) &= \frac{(1-\alpha e^{i\omega})^{-1}(1-\alpha e^{-i\omega})^{-1}}{(2\pi)}\left(\frac{1}{M}+\kappa\frac{\gamma}{M}+\kappa'\frac{\gamma}{M}++\kappa'\kappa\frac{\gamma^2}{M^2}\sum_{i=1}^{M}\phi_i^2\right) \\ &= \frac{(1-\alpha e^{i\omega})^{-1}(1-\alpha e^{-i\omega})^{-1}}{(2\pi)(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}\left[\frac{1}{M}[(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)-\gamma^2]+\frac{\gamma^2}{M^2}\sum_{i=1}^{M}\phi_i^2\right] \end{aligned}$$

Thus, defining $a(\omega) = \frac{(1-\alpha e^{i\omega})^{-1}(1-\alpha e^{-i\omega})^{-1}}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$ and $b(\omega) = (1-\alpha e^{i\omega})(1-\alpha e^{-i\omega})$

$$S(\omega) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega)-\gamma^2)+\frac{\gamma^2}{M^2}\sum_{i=1}^{M}\phi_i^2\right]$$

Now for $\boldsymbol{\phi}_i = \frac{e_i}{\sum_i e_i}$ I can rewrite this expression in terms of the expected degree, $\overline{e}$, and the second order expected degree $\widetilde{e}$.

$$S(\omega) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega)-\gamma^2)+\frac{\gamma^2}{M}\left(\frac{\widetilde{e}}{\overline{e}}\right)\right]$$

Substitution in for the second order expected degree $\widetilde{e}$ and simplifying gives

$$S(\omega,\overline{\Gamma}) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega)-\gamma^2)+\gamma^2\frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\right] \quad if\ \zeta\in(2,3)$$

and

$$S(\omega,\overline{\Gamma}) = \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[\frac{1}{M}(b(\omega)-\gamma^2)+\gamma^2\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}\frac{1}{M}\right] if\ \zeta>3$$

Letting $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ gives the claim.
■

Proposition 5.3.2. *For fixed α and γ, aggregation weights $\mathbf{w} = (1/M)1_M$ and for any $\Gamma(A)$ where A is drawn from the family of matrices $A(M,\overline{e},\zeta)$ and $\Gamma(A)$ is formed according to Lemma 4.1.2. the spectral density for aggregate capital is bounded below by:*

$$S(\omega,\Gamma(A)) > \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega)-\gamma^2)\frac{1}{M}+\kappa_1(\zeta)(\gamma^2-\varkappa)\frac{1}{M}\right]\ if\ \zeta>3$$

and

$$S(\omega,\Gamma(A)) > \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega)-\gamma^2)\frac{1}{M}+\left(\gamma^2-\varkappa\right)\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\right]\ if\ \zeta\in(2,3)$$

with $\varkappa < \gamma^2$, $a(\omega) = \frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$, $b(\omega) = (1-\alpha e^{i\omega})(1-\alpha e^{-i\omega})$., $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ *and* $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

Proof: Let $\lambda = 1 - \alpha e^{-i\omega}$. By same argument as in Proposition 4.3.5. one can always decompose

$$[(1-\alpha e^{i\omega})I - \Gamma]^{-1} = [\lambda I - \gamma A(G)D]^{-1} = \{\lambda I - \gamma E(A(G))E(D) - \gamma[A(G)D - E(A(G)(D))]\}^{-1}$$

Now let

$$\begin{aligned} \lambda I - \gamma E(A(G))E(D) &\equiv C_a \\ -\gamma[A(G)D - E(A(G))E(D)] &\equiv U \end{aligned}$$

and apply the Sherman-Morrison formulas to get:

$$[\lambda I - \gamma A(G)D]^{-1} = C_\lambda^{-1} - C_\lambda^{-1}U[I + C_\lambda^{-1}U]^{-1}C_\lambda^{-1}$$

To solve C_λ^{-1} through the Barttlet inverse formula

$$\begin{aligned} C_\lambda^{-1} &= \lambda^{-1}I + \frac{\lambda^{-1}\gamma\boldsymbol{\phi}\mathbf{1}'\lambda^{-1}}{1-\gamma\lambda^{-1}\mathbf{1}'\boldsymbol{\phi}} \\ &= \lambda^{-1}\left[I + \frac{\lambda^{-1}\gamma\boldsymbol{\phi}\mathbf{1}'}{1-\gamma\lambda^{-1}\mathbf{1}'\boldsymbol{\phi}}\right] \\ &= \lambda^{-1}\left[I + \frac{\gamma\boldsymbol{\phi}\mathbf{1}'}{\lambda-\gamma\mathbf{1}'\boldsymbol{\phi}}\right] \\ &= \lambda^{-1}\left[I + \frac{\gamma\boldsymbol{\phi}\mathbf{1}'}{\lambda-\gamma}\right] \end{aligned}$$

For $[I + C_\lambda^{-1}U]^{-1}$ substitute in C_λ^{-1} to get

$$\begin{aligned} [I + C_\lambda^{-1}U]^{-1} &= \left[I + \lambda^{-1}\left(I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right)U\right]^{-1} \\ &= \left[I + \lambda^{-1}U + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'U\right]^{-1} \end{aligned}$$

where again $\phi\mathbf{1}'U$ is a $M \times M$ matrix of zeros. Thus:

$$[I + C_\lambda^{-1}U]^{-1} = I + \lambda^{-1}U$$

Collecting results

$$\begin{aligned} [\lambda I - \gamma A(G)D]^{-1} &= C_\lambda^{-1} - C_\lambda^{-1}U[I + C_\lambda^{-1}U]^{-1}C_\lambda^{-1} \\ &= \lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] - \lambda^{-2}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]U[I + \lambda^{-1}U]^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] \\ &= \lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] - \lambda^{-2}U[I + \lambda^{-1}U]^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] \end{aligned}$$

where the last line uses the fact that $\phi\mathbf{1}'U$ is a $M \times M$ matrix of zeros.

Again defining Θ defining as

$$\Theta \equiv \gamma[A(G)D - E(A(G)D)][I - \gamma[A(G)D - E(A(G)D)]]^{-1}\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$$

I can rewrite the expression above as

$$[\lambda I - \gamma A(G)D]^{-1} = \lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \lambda^{-2}\Theta$$

Also, by the exact same argument as in the proof of Proposition 4.3.5. the column sums of Θ have to be zero for any realization of $A(G)$.

Now using the formula for the aggregate spectrum I get:

$$\begin{aligned}
S(\omega) &= \frac{1}{2\pi}\frac{1}{M^2}\mathbf{1}'[(1-\alpha e^{i\omega})I - \Gamma]^{-1\prime}[(1-\alpha e^{i\omega})I - \Gamma]^{-1}\mathbf{1} \\
&= \frac{1}{2\pi}\frac{1}{M^2}\mathbf{1}'\left[\lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \lambda^{-2}\Theta\right]'\left[\lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \lambda^{-2}\Theta\right]\mathbf{1} \\
&= \frac{1}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{M^2}\left[\begin{array}{c}
\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} \\
+\mathbf{1}'\lambda^{-1\prime}\Theta'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} \\
+\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\lambda^{-1}\Theta\mathbf{1} \\
+\mathbf{1}'\lambda'^{-1}\lambda^{-1}\Theta'\Theta\mathbf{1}
\end{array}\right]
\end{aligned}$$

Now term by term in the expression in square brackets. First,

$$\begin{aligned}
\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} &= \mathbf{1}'\left[I + \frac{\gamma}{\lambda'-\gamma}\mathbf{1}\boldsymbol{\phi}' + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}' + \frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\mathbf{1}\boldsymbol{\phi}'\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} \\
&= M + M\frac{\gamma}{\lambda'-\gamma}\sum_{i=1}^{M}\phi_i + M\frac{\gamma}{\lambda-\gamma}\sum_{i=1}^{M}\phi_i + \frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}M^2\sum_{i=1}^{M}\phi_i^2
\end{aligned}$$

where λ' is the complex conjugate of λ and is given by $1 - \alpha e^{-i\omega}$.

Second

$$\begin{aligned}
\mathbf{1}'\lambda^{-1\prime}\Theta'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} &= \lambda^{-1\prime}\mathbf{1}'\Theta'\mathbf{1} + \lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}\mathbf{1}'\Theta'\boldsymbol{\phi}\mathbf{1}'\mathbf{1} \\
&= \lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}M\mathbf{1}'\Theta'\boldsymbol{\phi} \\
&= \lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}M\sum_j\sum_i\theta_{ij}\phi_i
\end{aligned}$$

Where $\Theta'\mathbf{1}$ is a $M \times 1$ vector of zeros (since the columns of Θ sum to zero), so the first term disappears.

Third, and applying same reasoning,

$$\mathbf{1}'\left[I+\frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\lambda^{-1}\Theta\mathbf{1}=\mathbf{1}\qquad{}'\lambda^{-1}\Theta\mathbf{1}+\lambda^{-1}\mathbf{1}'\frac{\gamma}{\lambda'-\gamma}\mathbf{1}\boldsymbol{\phi}'\Theta\mathbf{1}$$
$$=\lambda^{-1}\frac{\gamma}{\lambda'-\gamma}M\boldsymbol{\phi}'\Theta\mathbf{1}$$
$$=\lambda^{-1}\frac{\gamma}{\lambda'-\gamma}M\sum_j\sum_i\theta_{ij}\phi_i$$

Finally,

$$\mathbf{1}'\lambda'^{-1}\lambda^{-1}\Theta'\Theta\mathbf{1}=\lambda'^{-1}\lambda^{-1}\sum_i\left(\sum_j\theta_{ij}\right)^2$$

Collecting results:

$$\begin{aligned}
S(\omega) &= \frac{1}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{M^2}\left[\begin{array}{c} M+M\frac{\gamma}{\lambda'-\gamma}+M\frac{\gamma}{\lambda-\gamma}+\frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}M^2\sum_{i=1}^{M}\phi_i^2 \\ +\left(\lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}+\lambda^{-1}\frac{\gamma}{\lambda'-\gamma}\right)M\sum_j\sum_i\theta_{ij}\phi_i+\lambda'^{-1}\lambda^{-1}\sum_i\left(\sum_j\theta_{ij}\right)^2\end{array}\right] \\
&= \frac{1}{2\pi}\lambda'^{-1}\lambda^{-1}\left[\begin{array}{c}\frac{\lambda'\lambda-\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\frac{1}{M}+\frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\sum_{i=1}^{M}\phi_i^2+ \\ \left(\frac{2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\right)\frac{1}{M}\sum_j\sum_i\theta_{ij}\phi_i+\frac{1}{M^2}\lambda'^{-1}\lambda^{-1}\sum_i\left(\sum_j\theta_{ij}\right)^2\end{array}\right] \\
&= \frac{1}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{(\lambda-\gamma)(\lambda'-\gamma)}\left[\begin{array}{c}[\lambda'\lambda-\gamma^2]\frac{1}{M}+\gamma^2\sum_{i=1}^{M}\phi_i^2+ \\ \left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_j\sum_i\theta_{ij}\phi_i+\frac{1}{M^2}\lambda'^{-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\sum_i\left(\sum_j\theta_{ij}\right)^2\end{array}\right]
\end{aligned}$$

To establish the lower bound I follow the same strategy as in the proof of Proposition 4.3.4. Again the situation I am interested in ruling out is one where $\frac{1}{M}\sum_j\sum_i\theta_{ij}\phi_i$ is negative and has a decay rate with M that is slower than that of $\sum_{i=1}^{M}\phi_i^2$. For this not to be the case start by grouping the last three terms in the expression above thus:

$$\sum_{i=1}^{M}\left[\gamma^2\phi_i^2+\left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_j\theta_{ij}\phi_i+\frac{1}{M^2}\lambda'^{-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\left(\sum_j\theta_{ij}\right)^2\right]$$

and notice this is simply the i sum of the product of conjugate pairs dexed by i:

$$\sum_{i=1}^{M}\left[\gamma\phi_i+\frac{1}{M}\lambda^{-1}(\lambda-\gamma)\sum_j\theta_{ij}\right]\left[\gamma\phi_i+\frac{1}{M}\lambda^{-1\prime}(\lambda'-\gamma)\sum_j\theta_{ij}\right]$$

Now we know that the product of conjugate pairs is always real and nonnegative. Hence it must be the case for all M and each i

$$\gamma^2\phi_i^2 \geq \left| \left[2\gamma - (\lambda^{-1\prime} + \lambda^{-1})\gamma^2\right] \frac{1}{M} \sum_j \theta_{ij}\phi_i + \frac{1}{M^2}\lambda'^{-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\left(\sum_j \theta_{ij}\right)^2 \right|$$

which implies that the term on the r.h.s. of the inequality cannot decay at a slower rate than $\gamma^2\phi_i^2$. Since this holds for all i and all M:

$$\sum_{i=1}^{M} \gamma^2\phi_i^2 \geq \sum_{i=1}^{M} \left| \left[2\gamma - (\lambda^{-1\prime} + \lambda^{-1})\gamma^2\right] \frac{1}{M} \sum_j \theta_{ij}\phi_i + \frac{1}{M^2}\lambda'^{-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\left(\sum_j \theta_{ij}\right)^2 \right|$$

Thus, at worse, the r.h.s is a term that decays at the same rate as $\gamma^2\phi_i^2$ but is never larger in absolute value. Further, by the fact that Θ is always a zero column sum matrix, we know that at least for some i, $\sum_j \theta_{ij} > 0.$, which implies that the inequality above is strict. Thus it has to be the case that there exists a real and positive $\varkappa$, that does not depend on M and is strictly smaller than γ^2 such that $\left[\gamma^2 - \varkappa\right] \sum_{i=1}^{M} \phi_i^2$ is a strict lower bound for the sum of these three terms in i.

Using this in the expression for $S(\omega)$ it as to be the case that for any M and any frequency ω

$$S(\omega) > \frac{1}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{(\lambda-\gamma)(\lambda'-\gamma)}\left[(\lambda'\lambda - \gamma^2)\frac{1}{M} + (\gamma^2 - \varkappa)\sum_{i=1}^{M}\phi_i^2\right]$$

Substituting in for $\lambda = 1 - \alpha e^{-i\omega}$ and $\sum_{i=1}^{M}\phi_i^2$ yields:

$$S(\omega, \Gamma(A)) > \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega) - \gamma^2)\frac{1}{M} + \kappa_1(\zeta)(\gamma^2 - \varkappa)\frac{1}{M}\right] \; if \; \zeta > 3$$

and

$$S(\omega, \Gamma(A)) > \frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega) - \gamma^2)\frac{1}{M} + \left(\gamma^2 - \varkappa\right)\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\right] \; if \; \zeta \in (2,3)$$

where $\varkappa < \gamma^2$, $a(\omega) = \frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$,$b(\omega) = (1-\alpha e^{i\omega})(1-\alpha e^{-i\omega})$,.$\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ as claimed ■