Technology, Population and Natural Resources: A Theory of Take-off and Convergence to Sustainable Growth

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Overview

This project develops a Schumpeterian approach to the analysis of the interactions of technology, population and exhaustible natural resources in shaping the economy's transition to a sustainable growth path driven by endogenous technological change.

Motivation

I'm curious...

And, perhaps surprisingly, on the crucial issue of sustainability the literature has not made much progress since the classic papers by Solow and Stiglitz in 1974.

Brock and Taylor's chapter in 2005 Handbook of Economic Growth confirms this impression.

When they introduce natural resources in the standard Solow model to discuss sustainability they cite just two papers: Solow (1974) and Stiglitz (1974).

When they push harder for new results, like many others, they only use Romer (1990). Hence, in trying to bring to bear the power of modern theory of endogenous technological change on crucial issues that are at the forefront of the current debate, the literature is 18 years behind the frontier.

Why?

I don't have an answer...

However, understanding endogenous evolution of technology is crucial in making intelligent projections about the future and in assessing the dynamic costs/benefits of proposed interventions.

The Model

Final producers: Homogeneous good that be consumed, used to produce intermediate goods, or invested in R&D. (Basically, one-sector structure.)

Intermediate producers: Develop new goods and set up operations to serve market (variety innovation or entry) and, when already in operation, invest in R&D internal to firm (quality innovation).

Households: Consume, save and, in extension that I might not have time to discuss in detail, set optimal path of population growth and/or resource exhaustion.

Final producers

Technology:

$$Y = \int_0^N X_i^\theta \left(Z_i^\alpha Z^{1-\alpha} L_i^\gamma O_i^{1-\gamma} \right)^{1-\theta} di,$$

where $0 < \theta, \alpha, \gamma < 1$, $P_Y \equiv 1$, $Z = \int_0^N \frac{1}{N} Z_j dj$.

FOC yield:

$$X_i = \left(\frac{\theta}{P_i} \right)^{\frac{1}{1-\theta}} Z_i^\alpha Z^{1-\alpha} L_i^\gamma O_i^{1-\gamma}.$$

Also:

$$N \cdot PX = \int_0^N P_i X_i di = \theta Y;$$

$$wL = w \int_0^N L_i di = \gamma \left(1 - \theta\right) Y;$$

$$pO = p \int_0^N O_i di = \left(1 - \gamma\right) \left(1 - \theta\right) Y.$$

Intermediate producers

Technologies:

$$C_i = 1 \cdot X_i + \phi Z;$$

$$\dot{Z}_i = R_i.$$

Firm's objective:

$$V_i\,(0) = \int_0^\infty e^{-\int_0^t r(s)ds}\left[\Pi_i(t) - R_i\,(t)\right]dt,$$

where

$$\Pi_i = X_i\,(P_i - 1) - \phi Z.$$

In symmetric equilibrium:

$$\max_{P_i, R_i} V_i \Rightarrow r = \alpha\frac{X\left(\frac{1}{\theta} - 1\right)}{Z};$$

$$V_i^{\max} = \beta X_i \Rightarrow r = \frac{1}{\beta}\frac{X\left(\frac{1}{\theta} - 1\right) - \phi Z - R}{X} + \frac{\dot{X}}{X}.$$

Households:

$$U(0) = \int_0^\infty e^{-\rho t} \log\left(\frac{C(t)}{M(t)}\right) dt, \quad \rho > 0;$$

$$\dot{A} = rA + wM + pS - C.$$

Important:

$$M(t) = M_0 e^{mt};$$

$$S(t) = S_0 e^{-st}.$$

Note:

i. flow utility depends only on consumption per capita;

ii. both population and natural resource stock follow exponential processes.

$s > 0$ implies *depletion;* $s = 0$ implies that the natural resource is non-exhaustible (e.g., Ricardian land).

In today's presentation I set $s = 0$; I'm still working on exhaustion...

Optimal plan:

$$L(t) = M(t);$$

$$O(t) = S_0.$$

And:

$$r = \rho + \frac{\dot{C}}{C}.$$

General Equilibrium

Symmetry in final output production plus factor markets equilibrium yields:

$$Y = \kappa Z M^{\gamma} S_0^{1-\gamma}, \quad \kappa \equiv \theta^{\frac{2\theta}{1-\theta}}.$$

Output per capita:

$$y \equiv \frac{Y}{M} = \kappa Z \left(\frac{S_0}{M}\right)^{1-\gamma}.$$

Accordingly,

$$g \equiv \frac{\dot{y}}{y} = z - (1-\gamma)(m+s),$$

where

$$z \equiv \frac{R}{Z} = \frac{\dot{Z}}{Z}.$$

In words, output per capita growth is TFP growth minus *growth drag* due to essential natural resource. Of course, if $\gamma = 1$ drag disappears.

Define *market growth factor*

$$\eta \equiv \gamma m - (1 - \gamma)\, s > 0.$$

In words: growth of aggregate market for intermediate goods. (Recall I focus on case $s = 0$.)

Note: $\eta > 0$ is wholly consistent with growth drag in output per capita growth.

Intuition: population growth net of resource exhaustion drives growth of aggregate market for intermediate goods and thus agents' investment decisions in variety and quality dimensions of technology. These decisions support positive output per capita growth if resulting TFP growth rate is larger than the growth drag; this is nothing else than the famous Stiglitz (1974) condition for sustainability, with the difference that here TFP growth is endogenous and not necessarily positive.

Next, observe that

$$r = \alpha \frac{X\left(\frac{1}{\theta} - 1\right)}{Z} = \theta\left(1 - \theta\right)\kappa \frac{M^{\gamma} S_0^{1-\gamma}}{N}.$$

Define

$$x \equiv \frac{X\left(P - 1\right)}{Z} = \theta\left(1 - \theta\right)\kappa \frac{M^{\gamma} S_0^{1-\gamma}}{N}.$$

In words: key measure of firm size is quality-adjusted gross cash flow. It drives incentives to innovation in both the vertical and horizontal dimensions of technology.

Interesting results produced by this model stem from corner solutions where one or both of the two R&D activities shut down. Simplicity of structure allows us to compute in closed form the dates when the economy turns on each one of the two innovation engines.

Equilibrium when entrants are active, $\dot{N} > 0$.

Assets market equilibrium requires

$$A = NV = N \cdot \beta X = \beta \theta^2 Y.$$

Define $c \equiv \frac{C}{Y}$ and rewrite household budget:

$$0 = \rho + \frac{\dot{c}}{c} + \frac{1 - \theta - c}{\beta \theta^2}.$$

Thus, c jumps to

$$c^* = \beta \theta^2 \rho + 1 - \theta.$$

We have Solow-like property that throughout transition consumption ratio is constant. This simplifies dramatically analysis of dynamics.

Equilibrium when entrants are not active, $\dot{N} = 0$.

Assets market equilibrium still requires $A = NV$ but it is no longer true that $V = \beta X$ since by definition free-entry condition does not hold.

However, arbitrage condition on return to equity still holds. Use it to rewrite household budget:

$$0 = N\left(\frac{1-\theta}{\theta}X - \phi - R\right) + (1-\theta)\,Y - C.$$

Definitions of c and x, R&D technology plus fact that $NX = \theta^2 Y$, then yield

$$c = \theta\,(1-\theta)\left(1 - \frac{\phi + z}{x}\right) + 1 - \theta.$$

In words: In equilibrium with no entry, incumbents earn rents that as a fraction of GDP are increasing in our measure of firm size, x, and decreasing in the R&D intensity of the firm, $z = \frac{R}{Z}$, if positive.

The steady state

Before embarking on a detailed characterization of dynamics, it is useful to look at the endpoint of the transition. Reveals restrictions needed to satisfy sustainability condition. Such restrictions, of course, have implications for the shape of the transition path.

In general, definition of c, saving schedule and reduced-form production function yield:

$$r = \rho + \frac{\dot{c}}{c} + \gamma m + z.$$

Observe that if we are looking at a situation where both quality and variety R&D take place – the former to drive endogenous growth of output per capita, the latter to sterilize the scale effect – then we have $c = c^*$, so that $\dot{c} = 0$, while x constant implies $\frac{\dot{X}}{X} = z$.

Accordingly:

$$\rho + \gamma m = \frac{1-\theta}{\beta\theta}\left[1 - \alpha\frac{\phi + z}{\rho + \gamma m + z}\right],$$

which yields

$$z^* = \frac{\alpha\phi - (\rho + \gamma m)\left[1 - \frac{\beta\theta}{1-\theta}(\rho + \gamma m)\right]}{1 - \alpha - \frac{\beta\theta}{1-\theta}(\rho + \gamma m)},$$

which is positive if:

$$\alpha\phi > (\rho + \gamma m)\left[1 - \frac{\beta\theta}{1-\theta}(\rho + \gamma m)\right];$$

$$1 - \alpha > \frac{\beta\theta}{1-\theta}(\rho + \gamma m).$$

Moreover:

$$g^* = z^* - (1 - \gamma)\, m.$$

Hence, *sustainability condition*:

$$\frac{\alpha\phi - (\rho + \gamma m)\left[1 - \frac{\beta\theta}{1-\theta}(\rho + \gamma m)\right]}{1 - \alpha - \frac{\beta\theta}{1-\theta}(\rho + \gamma m)} > (1 - \gamma)\, m.$$

Transition dynamics

As mentioned above, a key feature of this model is that there are threshold values of firm size x below which investment in either quality or variety, or both, shuts down and the economy does not grow in per capita terms. I have already discussed how equilibria with entry differ from equilibria with no entry in terms of the consumption-saving decisions of households. I now work out in detail how these thresholds shape the transition path.

Observe first that:

$$\dot{Z} > 0 \Leftrightarrow r > \rho + \gamma m + \frac{\dot{c}}{c};$$

$$\dot{Z} = 0 \Leftrightarrow r \leq \rho + \gamma m + \frac{\dot{c}}{c}.$$

When firms do undertake R&D

$$r = \alpha x.$$

Hence, identify threshold of firms size, x_Z. Rewrite return to entry as

$$\frac{\dot{x}}{x} = \begin{cases} \frac{1-\theta}{\beta\theta}\left[1 - \alpha - \frac{\phi-\rho-\gamma m}{x}\right] - \rho - \gamma m & x > x_Z \\ \frac{1-\theta}{\beta\theta}\left[1 - \frac{\phi}{x}\right] - \rho - \gamma m & x \leq x_Z \end{cases}.$$

The specific value of x_Z depends on whether entrants are active or not.

If $\dot{N} > 0$, $c = c^*$ so that $\dot{c} = 0$ and $\dot{Z} > 0$ for $x > \frac{\rho + \gamma m}{\alpha}$;

if $\dot{N} = 0$, calculating the threshold is slightly more complicated. The Appendix provides the details; the end result is that $\dot{Z} > 0$ for

$$x > \text{arg solve}\left\{\alpha x = \rho + \frac{(1+\theta)\,x}{(1+\theta)\,x - \phi\theta}\gamma m\right\}.$$

Intuition: The former is the threshold for quality R&D given that the market *already* supports entry of new firms; the latter is the threshold for quality R&D given that the market *does not yet* support entry of new firms.

In former case, firms undertaking quality R&D compete for resources with entrepreneurs that are setting up new firms and face a constant reservation interest rate demanded by savers; in the latter they do not compete for resources with entrepreneurs but, because the free entry condition does not hold and they distribute to shareholders rents that grow with the size of the market, they face a reservation interest rate that reflects the associated growing consumption ratio.

Reasoning that led us to identify threshold x_Z applies to investment in entry as well. Definition of x allows us to $\frac{\dot{x}}{x}$ equation as

$$\frac{\dot{N}}{N} = \begin{cases} \frac{1-\theta}{\beta\theta}\left[1-\alpha-\frac{\phi-\rho-\gamma m}{x}\right]-\rho & x > x_Z \\ \frac{1-\theta}{\beta\theta}\left[1-\frac{\phi}{x}\right]-\rho & x \leq x_Z \end{cases}.$$

This says that when $x > x_Z$ and firm do R&D, $\dot{N} > 0$ for

$$x > \frac{\phi-\rho-\gamma m}{1-\alpha-\frac{\rho\beta\theta}{1-\theta}}$$

and $\dot{N} = 0$ otherwise.

Similarly, when $x < x_Z$ and firms do not do R&D, $\dot{N} > 0$ for

$$x > \frac{\phi}{1-\frac{\rho\beta\theta}{1-\theta}}$$

and $\dot{N} = 0$ otherwise.

We can then identify two cases. We plot the expressions

$$\frac{1-\theta}{\beta\theta}\left[1-\frac{\phi}{x}\right]-\frac{1-\theta}{\beta\theta}\left[\alpha-\frac{\rho+\gamma m}{x}\right]$$

and

$$\frac{1-\theta}{\beta\theta}\left[1-\frac{\phi}{x}\right].$$

By construction these intersect exactly at $x = x_Z$. There is value $\bar{m}$ such that the two curves evaluated at their intersection point yield exactly ρ. This is value of market growth factor such that the thresholds for $\dot{Z} = 0$ and $\dot{N} = 0$ are identical:

$$\frac{\phi-\rho-\gamma\bar{m}}{1-\alpha-\frac{\rho\beta\theta}{1-\theta}}=\frac{\phi}{1-\frac{\rho\beta\theta}{1-\theta}}=\frac{\rho+\gamma\bar{m}}{\alpha}$$

$$\Rightarrow \bar{m} \equiv \left(\frac{\alpha\phi}{1-\frac{\rho\beta\theta}{1-\theta}}-\rho\right)\frac{1}{\gamma}.$$

Now, as $m > \bar{m}$ the line describing the rate of entry with positive R&D shifts up while the line describing the rate of entry with no R&D stays still. We then have a configuration whereby the threshold for $\dot{Z} > 0$ is smaller than the threshold for $\dot{N} > 0$.

As $m < \bar{m}$, instead, the line describing the rate of entry with positive R&D shifts down and we have a configuration whereby the threshold for $\dot{Z} > 0$ is larger than the threshold for $\dot{N} > 0$.

The most interesting consequence of this feature for the economy's dynamics is that the *sequence* in which society turns on the two innovation engines determines the shape of the transition path.

The case of fast market growth, $m > \bar{m}$

We have:

$$\dot{x} = \begin{cases} \gamma m x & x \leq x_N \\ \varphi_1 - \varphi_2 x & x_N < x \leq x_Z \\ \varphi_3 - \varphi_4 x & x > x_Z \end{cases},$$

where

$$\varphi_1 \equiv \frac{\phi(1-\theta)}{\beta\theta};$$

$$\varphi_2 \equiv \left[\frac{1-\theta}{\beta\theta} - \rho - \gamma m\right];$$

$$\varphi_3 \equiv \frac{(\phi - \rho - \gamma m)(1-\theta)}{\beta\theta};$$

$$\varphi_4 \equiv \left[\frac{(1-\theta)(1-\alpha)}{\beta\theta} - \rho - \gamma m\right];$$

$$x_N \equiv \frac{\phi}{1 - \frac{\rho\beta\theta}{1-\theta}} < x_Z \equiv \frac{\rho + \gamma m}{\alpha}.$$

Associated to this, we have:

$$g(t) = \begin{cases} -(1-\gamma)m & x < x_N \\ -(1-\gamma)m & x_N < x < x_Z \\ \alpha x(t) - \rho - m & x > x_Z \end{cases},$$

since $z = r - \rho - \gamma m$ for $x > x_Z > x_N$.

Denote:

$$\nu \equiv \frac{(1-\theta)(1-\alpha)}{\beta\theta} - \rho - \gamma m;$$

$$\bar{\nu} \equiv \frac{1-\theta}{\beta\theta} - \rho - \gamma m.$$

Also:

$$x^* \equiv \frac{(\phi - \rho - \gamma m)(1-\theta)}{\beta\theta\nu};$$

$$\bar{x}^* \equiv \frac{\phi(1-\theta)}{\beta\theta\bar{\nu}}.$$

These are, respectively, the convergence coefficients and steady-state values of x in the region where $\dot{Z} > 0$ and in the region where $\dot{Z} = 0$.

We construct solution as follows.

Let

$$x_0 = \frac{M_0^{\gamma} S_0^{1-\gamma}}{N_0} < x_N.$$

Integrate first line of $\dot{x}$ equation between 0 and t to obtain

$$x(t) = x_0 e^{\gamma m t}.$$

Since x grows exponentially, there exists a date T_N such that

$$x(T_N) = x_0 e^{\gamma m T_N} = x_N,$$

which yields

$$T_N = \frac{1}{\gamma m} \log\left(\frac{x_N}{x_0}\right).$$

Thereafter, economy follows second line of $\dot{x}$ equation. Integrating between T_N and t yields

$$x(t) = x_N e^{\bar{\nu}(T_N - t)} + \bar{x}^* \left(1 - e^{\bar{\nu}(T_N - t)}\right).$$

Assuming that $\bar{x}^* > x_Z$, which must be true since $g^* > 0$, there exists a date T_Z such that

$$x(T_Z) = x_N e^{\bar{\nu}(T_N - T_Z)} + \bar{x}^* \left(1 - e^{\bar{\nu}(T_N - T_Z)}\right) = x_Z,$$

which yields

$$T_Z = T_N + \frac{1}{\bar{\nu}} \log \left(\frac{\bar{x}^* - x_N}{\bar{x}^* - x_Z} \right).$$

Thereafter economy follows third line of $\dot{x}$ equation. Integrating between T_Z and t yields

$$x(t) = x_Z e^{\nu(T_Z - t)} + x^* \left(1 - e^{\nu(T_Z - t)}\right),$$

which converges to x^* as $t \to \infty$.

Time-path of output per capita growth:

$$g(t) = -(1 - \gamma) m$$

for $0 \leq t < T_N$ and $T_N < t < T_Z$;

$$g(t) = \alpha \left[x_Z e^{\nu(T_Z - t)} + x^* \left(1 - e^{\nu(T_Z - t)}\right)\right] - \rho - m$$

for $t > T_Z$.

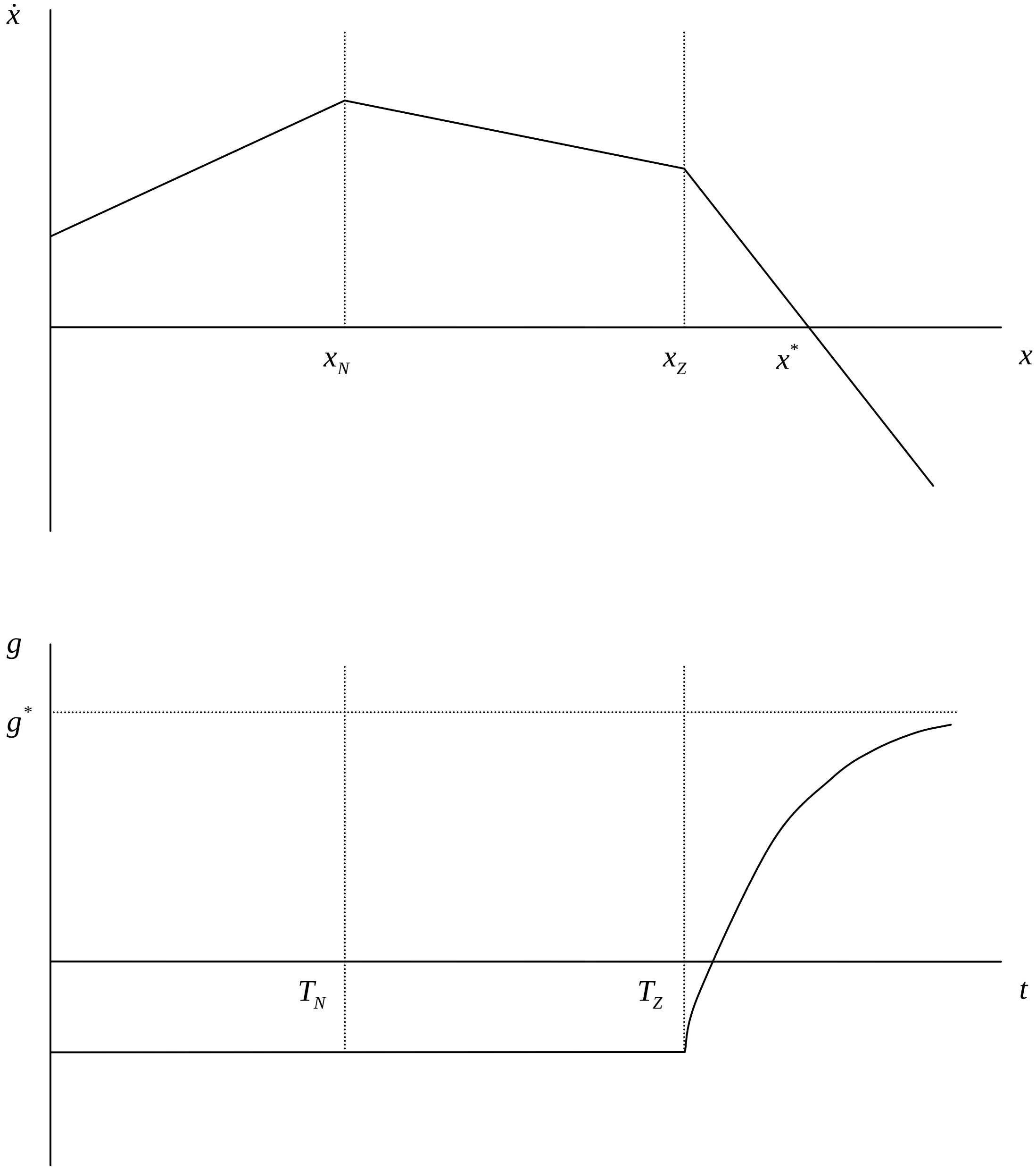

Figure 1: The case of fast population growth

As one can see, the economy starts out in a situation where there is no entry and firms earn rents. These rents grow with the size of the market and fuel consumption growth in excess of output growth. Consequently, negative output per capita growth does not necessarily imply falling consumption per capita, C/M. In fact, it is straightforward to verify that for $z = 0$, we have

$$\frac{\dot{C}}{C} - m = \frac{\dot{c}}{c} + \frac{\dot{Y}}{Y} - m = \frac{\dot{c}}{c} - (1 - \gamma)\, m \gtrless 0$$

if

$$\frac{\gamma m}{\frac{1+\theta}{\theta}\frac{x}{\phi} - 1} \gtrless (1 - \gamma)\, m.$$

The case of slow market growth, $m < \bar{m}$

We have

$$\dot{x} = \begin{cases} \gamma m x & x \leq x_Z \\ \gamma m x & x_Z < x \leq x_N \\ \varphi_5 - \varphi_6 x & x > x_N \end{cases},$$

where:

$$\varphi_5 \equiv \frac{(\phi - \rho - \gamma m)(1-\theta)}{\beta\theta};$$

$$\varphi_6 \equiv \left[\frac{(1-\theta)(1-\alpha)}{\beta\theta} - \rho - \gamma m\right];$$

$$x_N \equiv \frac{\phi - \rho - \gamma m}{1 - \alpha - \frac{\rho\beta\theta}{1-\theta}};$$

$$x_Z \equiv \text{arg solve}\left\{\alpha x - \gamma m \frac{(1+\theta)x}{(1+\theta)x - \theta\phi} = \rho\right\}.$$

Also:

$$g(t) = \begin{cases} -(1-\gamma)m & x < x_Z \\ \alpha x(t) - \frac{\dot{c}}{c} - \rho - m & x_Z < x < x_N \\ \alpha x(t) - \rho - m & x > x_N \end{cases}.$$

Term $\frac{\dot{c}}{c}$ in second line complicates things a bit. It picks up the role of consumption smoothing in driving reservation interest rate of savers in a situation where free entry does not yet apply and existing firm earn growing rents.

Important: When firms turn on quality innovation engine but free entry does not yet apply, we need to characterize path of an endogenous growth model driven by vertical innovation with a fixed number of products and exponential population growth.

This is precisely *explosive* situation that has been long considered problematic in first-generation models and that prompted the development of the class of models that exploits product variety expansion to sterilize the scale effect.

Difference is that while we are working with a model of that very class, we are not imposing that product variety expansion be operational all the time so that a *finite period of faster than exponential growth* is possible.

Let $x_0 < x_Z$ and integrate the first line of $\dot{x}$ equation between 0 and t to obtain

$$x(t) = x_0 e^{\gamma m t}.$$

Then, there exists a date T_Z such that

$$x(T_Z) = x_0 e^{\gamma m T_Z} = x_Z,$$

which yields

$$T_Z = \frac{1}{\gamma m} \log\left(\frac{x_Z}{x_0}\right).$$

After crossing this threshold economy keeps following $\dot{x} = \gamma m x$ and therefore *must* cross entry threshold x_N in finite time. Integrating between T_Z and t yields

$$x(t) = x_Z e^{\gamma m (t - T_Z)}.$$

There thus exists a date T_N such that

$$x(T_N) = x_Z e^{\gamma m (T_N - T_Z)} = x_N,$$

which yields

$$T_N = T_Z + \frac{1}{\gamma m} \log\left(\frac{x_N}{x_Z}\right) = \frac{1}{\gamma m} \log\left(\frac{x_N}{x_0}\right).$$

Thereafter economy follows third line of $\dot{x}$ equation. Integrating between T_Z and t, we have:

$$x(t) = x_N e^{\nu(T_N - t)} + x^* \left(1 - e^{\nu(T_N - t)}\right),$$

which converges to x^* as $t \to \infty$.

Time-path of output per capita growth:

$$g(t) = -(1 - \gamma) m$$

for $0 \leq t < T_Z$;

$$g(t) = z(t) - (1 - \gamma) m$$

for $T_Z < t < T_N$;

$g(t) = \alpha \left[x_N e^{\nu(T_N - t)} + x^* \left(1 - e^{\nu(T_N - t)}\right)\right] - \rho - m,$

for $t > T_N$.

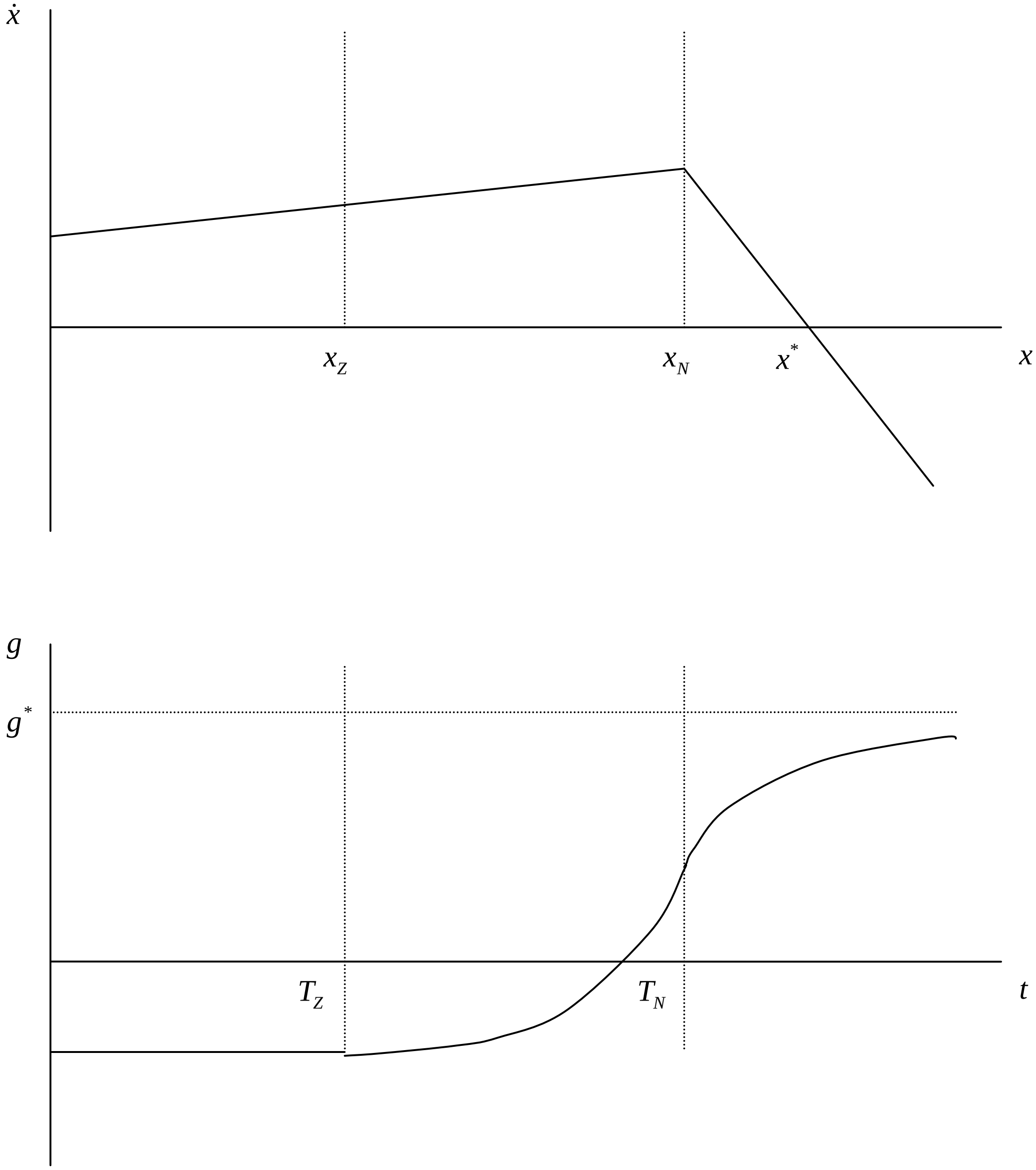

Figure 2: The case of slow population growth

An important feature of this equilibrium path is that society turns on the quality innovation engine of growth before the free-entry condition applies. This means that firms start undertaking R&D when they are still earning escalating rents driven by aggregate market growth due to population growth. As in the previous case, these rents fuel consumption growth in excess of output growth.

We now have:

$$\frac{\dot{C}}{C} - m = \frac{\gamma m}{\frac{1+\theta}{\theta}\frac{x}{\phi} - 1} - (1 - \gamma)\, m,$$

for $\phi \leq x \leq x_Z$;

$$\frac{\dot{C}}{C} - m = \frac{\gamma m}{\frac{1+\theta}{\theta}\frac{x}{\phi} - 1} - (1 - \gamma)\, m + z$$

for $x_Z < x \leq x_N$.

The most important difference with the previous case is that for $x > x_Z$ we have $z = \alpha x - \rho - \gamma m - \frac{\dot{c}}{c}$ so that the second line of this equation reduces to

$$\alpha x - \rho - m.$$

It follows that consumption per capita growth hits its minimum at $x = x_Z$. It is then possible to choose parameter values such that

$$\alpha x_Z - \rho - m > 0$$

and consequently have a growth path whereby consumption per capita grows all the time despite the fact that output per capita growth is initially negative.

Extension: Endogenous m

I modify preferences as follows:

$$U_0 = \int_0^{\infty} e^{-\rho t} \left[\log \left(\frac{C(t)}{M(t)} \right) + \mu \log \left(b(t) M(t) \right) \right] dt,$$

where μ measures preference for children and replace exogenous exponential process for population growth with

$$\dot{M} = M(b - \delta),$$

where b is the fertility rate.

To keep things as simple as possible, I assume that reproduction entails a cost of ψ units of time per child. Therefore, flow budget constraint is

$$\dot{A} = rA + wM(1 - \psi b) + pO - C.$$

The first order conditions for control variables C, b are:

$$1 = \lambda_A C; \quad \frac{\mu}{b} = \lambda_A w M \psi - \lambda_M M.$$

Conditions for state variables A, M are:

$$r + \frac{\dot{\lambda}_A}{\lambda_A} = \rho;$$

$$\frac{\mu - 1 + \lambda_A w M \left(1 - \psi b\right)}{\lambda_M M} + \left(\frac{\dot{\lambda}_M}{\lambda_M} + \frac{\dot{M}}{M}\right) = \rho.$$

The first order conditions for consumption C and financial wealth A yield the usual Euler equation

$$r = \rho + \frac{\dot{C}}{C}.$$

Also, in equilibrium $wL = wM(1-\psi b) = \gamma(1-\theta)Y$ and $pO = (1-\gamma)(1-\theta)Y$.

Then, if entrants are active, i.e., if $\dot{N} > 0$, we have $A = \beta\theta^2 Y$. Hence, define $c \equiv \frac{C}{Y}$ and rewrite budget constraint as

$$0 = \beta\theta^2\left(\rho + \frac{\dot{c}}{c}\right) + 1 - \theta - c,$$

which again says that the consumption ratio jumps to

$$c^* = \beta\theta^2\rho + 1 - \theta.$$

Eliminating λ_A and labor income wL reduces first order condition for population M to

$$\frac{\mu - 1 + \frac{\gamma(1-\theta)}{c^*}}{\lambda_M M} + \left(\frac{\dot{\lambda}_M}{\lambda_M} + \frac{\dot{M}}{M}\right) = \rho.$$

This is an unstable differential equation in the control-like variable $\lambda_M M$ that, accordingly, jumps to

$$(\lambda_M M)^* = \frac{\mu - 1 + \frac{\gamma(1-\theta)}{c^*}}{\rho}.$$

We can rewrite the condition for the fertility rate b as

$$\frac{\mu}{b} = \frac{\gamma\,(1-\theta)}{c^*}\frac{\psi}{1-\psi b} - \frac{1}{\rho}\left[\mu - 1 + \frac{\gamma\,(1-\theta)}{c^*}\right],$$

which yields b^* *constant*.

So, when entrants are active we replace constant, but arbitrary, exogenous growth rate of population m used in the previous section with $m^* = b^* - \delta$, which is still constant but no longer arbitrary since it is endogenous.

When entrants not active, i.e., $\dot{N} = 0$, we have more complicated story...

The gist is that there we can start economy at $x = \phi$:

- profits are zero, there is no tech. progress, population is stable;

- but there is incentive to reproduce (shadow value of children is high).

Hence, households jump on path with $\delta < b(t) < b^*$.

Important:

$$m(t) > 0 \Rightarrow \dot{x}(t) > \gamma m(t) x(t).$$

Also,

$$\dot{b}(t) > 0.$$

Hence, firm size grows at accelerating rate. Eventually, hits threshold x_N. Thereafter we are in $\dot{N} > 0$ case.

Generate path whereby g becomes negative as pop. growth takes economy to region where tech. progress drives g positive.

Additional twist. If $\psi(z)$ for $z > 0$ with $\psi' < 0$, then can get

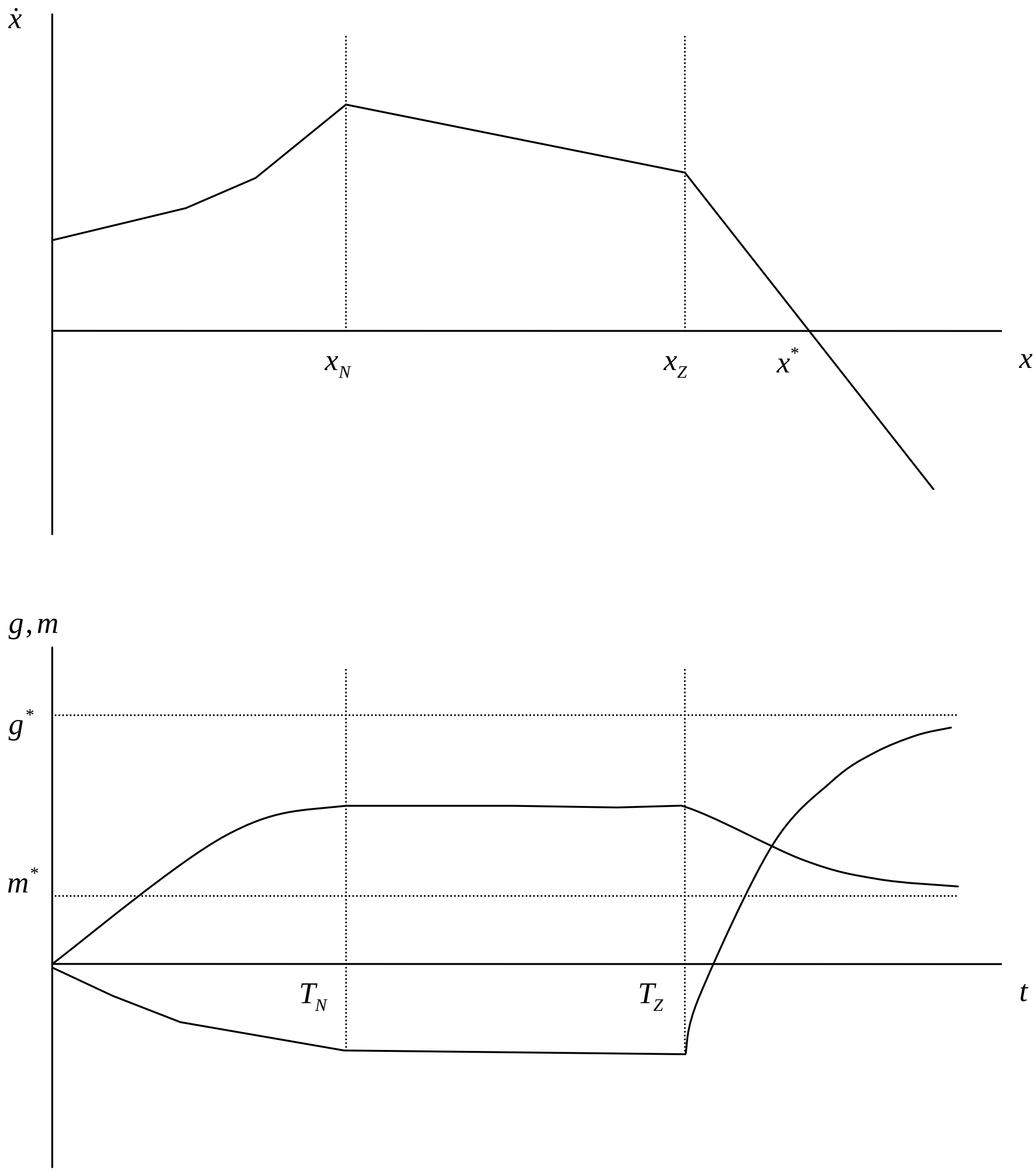

Figure 3: The general case