Search and Rest Unemployment*

Fernando Alvarez
University of Chicago
f-alvarez1@uchicago.edu

Robert Shimer
University of Chicago
shimer@uchicago.edu

January 20, 2008

Abstract

This paper extends Lucas and Prescott's (1974) search model to develop a notion of rest unemployment. The economy consists of a continuum of labor markets, each of which produces a heterogeneous good. There is a constant returns to scale production technology in each labor market, but labor productivity is continually hit by idiosyncratic shocks, inducing the costly reallocation of workers across labor markets. Under some conditions, some workers may be rest-unemployed, waiting for local labor market conditions to improve, rather than engaged in time consuming search. The model has distinct notions of unemployment (moving to a new labor market or waiting for labor market conditions to improve) and inactivity (enjoying leisure while disconnected from the labor market). We obtain closed-form expressions for key aggregate variables and use them to evaluate the model. Quantitatively, we find that in the U.S. economy many more people may be in rest unemployment than in search unemployment.

*We are grateful for comments by Daron Acemoglu, Gadi Barlevy, Morten Ravn, Marcelo Veracierto, and numerous seminar participants, and for research assistance by Lorenzo Caliendo. Shimer's research is supported by a grant from the National Science Foundation.

1 Introduction

This paper distinguishes "search" and "rest" unemployment. Search unemployment is a costly reallocation activity in which workers look for the best available employment opportunities. Rest unemployment is a less costly activity where a worker waits for her current labor market conditions to improve. While one might naturally think of temporary layoffs as the empirical counterpart of rest unemployment, we believe it corresponds to a more common phenomenon. We view anyone who is not currently working but still loosely attached to an industry where she previously worked as rest-unemployed.

We construct a model where there is a role for both search and rest unemployment. We then use the model to ask whether economies with different amounts of search and rest unemployment would behave differently and whether the life of a worker in search unemployment is substantially different from that of one in rest unemployment. We use the answers to evaluate the importance of search and rest unemployment in the U.S. economy.

Our model is an extension of Lucas and Prescott's (1974) search model. The economy consists of a continuum of sectors, each of which produces a heterogeneous intermediate good which aggregates into the final consumption good using a Dixit-Stiglitz technology with constant elasticity of substitution θ. Each intermediate good is produced with a constant returns to scale technology using labor only. Labor productivity is continually hit by idiosyncratic shocks whose growth rate has a constant expected value and a constant variance per unit of time. Households have standard time additive preferences. In any time period, households can use their time endowment to engage in four mutually exclusive activities, from which they derive different amounts of leisure: work, search unemployment, rest unemployment, and inactivity, i.e. out of the labor force.

We assume that the reallocation of workers across intermediate good sectors requires search unemployment. Because of that we refer interchangeably to intermediate good sectors as labor markets. A worker in a given labor market can either work, engage in rest unemployment, or leave her current labor market. A rest-unemployed worker is available to return to work in that labor market, and that labor market only, at no cost. If a worker leaves her labor market she can either be inactive or engage in search unemployment. A search-unemployed worker finds a job after a random, exponentially distributed amount of time, upon which she can locate in the market of her choosing. Thus, search is directed as in Lucas and Prescott (1974). Finally, workers can costlessly move between search unemployment and inactivity.

We study stationary competitive equilibria with complete markets. Equivalently, we assume that the household is composed of a large number of members. This implies that a household values the contribution of the earnings of their members in terms of their expected discounted values. Firms producing a given intermediate good take as given the aggregate output of the final good and the price of all intermediate goods. Labor demand in each market has elasticity θ, due to the effect of sectoral output on its relative price. Idiosyncratic productivity shocks shift the demand for labor. Wages are determined competitively in each labor market and so depend on the number of workers in the labor market and on labor productivity.

To characterize the equilibrium, let ω denote the log of the wage that would prevail in a particular labor market if all workers in the market were employed, i.e. if there were no rest unemployment; we measure wages in utility-equivalent units. The behavior of workers in different labor markets is characterized by three threshold values $\underline{\omega} \leq \hat{\omega} < \bar{\omega}$. Workers who have successfully concluded their search process arrive in the best labor markets, which keeps ω below $\bar{\omega}$ in all labor markets. Workers in depressed labor markets leave to become search unemployed, which keeps ω above $\underline{\omega}$. In markets with $\omega > \hat{\omega}$ there is full employment and the log wage is ω. For $\omega < \hat{\omega}$, wages stay at $\hat{\omega}$ and the rest unemployment rate in the market increases. Workers engaged in rest unemployment stay in depressed labor markets waiting for conditions to improve. If conditions get bad enough, $\omega = \underline{\omega}$, they leave the market. Depending on parameter values, there may be no rest unemployment, $\hat{\omega} = \underline{\omega}$.

We solve our model in continuous time, with log productivity following a Brownian motion. Workers' decision on when to enter and exit labor markets implies that ω is a regulated Brownian motion, with barriers given by the endogenously determined thresholds $\underline{\omega}$ and $\bar{\omega}$. Any barriers $\underline{\omega}$ and $\bar{\omega}$ imply an invariant distribution of ω across workers. Aggregating across workers, we then determine the value of final output and the employment and unemployment rates. We obtain simple characterizations of key endogenous variables, including a pair of equations for the two thresholds $\underline{\omega}$ and $\bar{\omega}$ and closed-form solutions for the labor force participation rate, unemployment rate, and share of searchers in the unemployment pool.

The closed form solution facilitates comparative statics and a quantitative evaluation of the model. We find a tight relationship between the search unemployment rate and the autocorrelation of wages at the labor market level. Using data for five-digit North American Industry Classification System (NAICS) industries, we show that annual average weekly earnings at the industry level are essentially a random walk. According to our model, this implies that labor markets rarely hit the barriers $\bar{\omega}$ and $\underline{\omega}$ that regulate wages. But since a labor market must move from the upper barrier to the lower barrier in order for a newly hired worker to enter search unemployment, it follows that the model cannot simultaneously generate strongly autocorrelated wages and significant amounts of search unemployment. Our calibrations suggest that the search unemployment rate—the ratio of search unemployment to the labor force—is less than 0.5%. One tension with our conclusion that the search unemployment rate is small is that it requires a relatively large cost of search. We conjecture that the introduction of labor market-specific human capital accumulation will substantially reduce the necessary search cost, a topic that we leave for future research.

Rest provides a complementary source of unemployment. We find that the model is able to generate significant levels of rest unemployment with plausible parameters, while still being consistent with the autocorrelation and standard deviation of wages in the data. In particular, although rest provides almost as much leisure as inactivity, the transition rate from rest unemployment back to employment is high, approximately $1/2t$ per unit of time at short unemployment durations t. We also find that rest unemployment can explain why measures of job creation and job destruction are a concave function of the time horizon. For labor markets with rest unemployment, creation and destruction are frequently reversed, inducing the observed concavity. With only search unemployment, creation and destruction would be nearly linear functions of elapsed time.

Our model is closely related to Lucas and Prescott (1974). There are three significant differences between the models. First, we introduce rest unemployment to the framework. Second, we make particular assumptions on the stochastic process for productivity which enable us to obtain closed-form solutions; however, we believe our insights, e.g. on the link between search unemployment and the autocorrelation of wages and on the role of rest unemployment, carry over to alternative productivity processes. Finally, in Lucas and Prescott (1974), all labor markets produce a homogeneous good but there are diminishing returns to scale in each labor market. In our model, each labor market produce a heterogeneous good and has constant returns to scale. We believe this approach is more attractive because the extent of diminishing returns is determined by the elasticity of substitution between goods, which is potentially more easily measurable than the degree of decreasing returns on variable inputs (Atkeson, Khan, and Ohanian, 1996). An online Appendix B.2 tightens the connections between these models by solving a market social planner's problem and proving that the equilibrium is efficient.

Our concept of rest unemployment is closely related to the one used in Jovanovic (1987), from whom we borrow the term.[1] While in both his model and ours search and rest unemployment coexist, the aims of both papers and hence the setup of the models are different. Jovanovic (1987) focuses on the cyclical behavior of unemployment and productivity, and so allows for both idiosyncratic and aggregate productivity shocks. But to be able to analyze the

[1] Our decision to use this name was not easy. An obvious alternative is "wait unemployment," but the literature uses this to refer to workers who wait for a job in a high wage primary labor market rather than accept a readily available job in a low wage secondary labor market. Although workers in rest unemployment are waiting for a job, their behavior is quite different from those referred to in this literature. Our concept of rest unemployment corresponds closely to one notion of structural unemployment; see, for example Abel and Bernanke (2001, p. 95).

model with aggregate shocks, Jovanovic (1987) assumes that at the end of each period, there is exactly one worker in each location. This implies that search unemployment is socially wasteful. Our model illustrates how search unemployment may play an important role in reallocating workers away from severely depressed labor markets, while rest unemployment may be an efficient use of workers' time in marginal labor markets.

King (1990) and Gouge and King (1997) also develop models of rest unemployment in the Lucas and Prescott (1974) framework. King (1990) focuses on comparative statics of rest and search unemployment, while Gouge and King (1997) reexamine the business cycle issues in Jovanovic (1987), especially the finding that rest unemployment is likely to be countercyclical, consistent with empirical evidence but not with many models of reallocation. Both papers assume that the idiosyncratic productivity shock follows a two-state Markov process. This coarse parameterization has the advantage of making the analysis tractable but complicates the mapping from model to data. Instead, we find that when log productivity follows a Brownian motion, the model is still tractable and the mapping from model to data is direct.

In Section 2, we describe the economic environment. We analyze a special case where workers can immediately move to the best labor market in Section 3. Without any search cost, there no rest unemployment, since either working in the best labor market or dropping out of the labor force dominates this activity. Instead, idiosyncratic productivity shocks lead to a continual reallocation of workers across labor markets.

Section 4 characterizes the stationary equilibrium of the economy. Although the microeconomic structure of our model is rich, we can characterize the equilibrium as the solution to a system of two equations in two endogenous variables and various model parameters. The system is simple in the sense that we can express the equations in closed form. We prove that the equilibrium is unique and show that it is easy to compute and perform comparative statics. In particular, we find that there is rest unemployment only if the cost in terms of foregone leisure is low. We also provide closed form expressions for the employment, search unemployment, and rest unemployment rates. While the unemployment rates depend a comparison of the relative advantage of different leisure activities—search, rest, and inactivity—the employment rate depends on a comparison of market versus nonmarket activity.

Section 5 uses our model to understand the extent of search and rest unemployment in the U.S. economy. The model has trouble simultaneously generating significant levels of search unemployment and strongly autocorrelated wages; however, there is no tension between rest unemployment and wages. We also show how the presence of rest unemployment affects the behavior of job creation and destruction and the hazard rate of finding a job. The model with rest unemployment is broadly in line with the data, while the model with only search is not.

Section 6 uses our model to ask whether economies with different amounts of search and rest unemployment can be distinguished and whether the life of a worker in search unemployment is substantially different from that of one in rest unemployment. Our preliminary conclusions are affirmative and support our finding that rest unemployment may be an empirically important phenomenon.

2 Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete.

2.1 Intermediate Goods

There is a continuum of intermediate goods indexed by $j \in [0,1]$. Each good is produced in a separate labor market with a constant returns to scale technology that uses only labor. In a typical labor market j at time t, there is a measure $l(j,t)$ workers. Of these, $e(j,t)$ are employed, each producing $Ax(j,t)$ units of good j, while the remaining $l(j,t) - e(j,t)$ are rest-unemployed. The price of good j, $p(j,t)$, and the wage in labor market j, $w(j,t)$, are determined competitively at each instant t. We use the final good as the numeraire.

A is the aggregate component in productivity while $x(j,t)$ is an idiosyncratic shock that follows a geometric random walk,

$$d \log x(j,t) = \mu_x dt + \sigma_x dz(j,t), \tag{1}$$

where μ_x measures the drift of log productivity, $\sigma_x > 0$ measures the standard deviation, and $z(j,t)$ is a standard Wiener process, independent across labor markets.

To keep a well-behaved distribution of labor productivity, we assume that labor market j shuts down according to a Poisson process with arrival rate δ, independent across labor markets and independent of labor market j's productivity. When this shock hits, all the workers are forced out of the market. A new labor market, also named j, enters with positive initial productivity $x \sim F(x)$, keeping the total measure of labor markets constant. We assume a law of large numbers, so the share of labor markets experiencing any particular sequence of shocks is deterministic.

2.2 Final Goods

A competitive final goods producing sector combines the intermediate goods using the constant returns to scale technology

$$Y(t) = \left(\int_0^1 y(j,t)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \tag{2}$$

where $y(j,t)$ is the input of good j at time t and $\theta > 0$ is the elasticity of substitution across goods. We assume $\theta \neq 1$ throughout the paper and comment in Section 3 on the role of this assumption. The final goods sector takes the price of the intermediate goods $\{p(j,t)\}$ as given and chooses $y(j,t)$ to maximize profits. It follows that

$$y(j,t) = \frac{Y(t)}{p(j,t)^{\theta}}. \tag{3}$$

2.3 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for studying complete markets allocations.

At each moment in time t, each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the intermediate goods (or equivalently labor) markets.
  - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed and get leisure b_r.
- $U_s(t)$ household members are search-unemployed, looking for a new labor market and getting leisure b_s.
- The remaining $1 - E(t) - U_r(t) - U_s(t)$ household members are inactive, getting leisure b_i.

We assume $b_i > b_s$. Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch intermediate goods markets without going through a spell of search unemployment. Workers exit their intermediate goods market for inactivity or search in three circumstances: first, they may do so endogenously at any time at not cost; second, they must do when their market shuts down, which happens at rate δ; and third, they must do so when they are hit by an idiosyncratic shock, according to a Poisson process with arrival rate q, independent across individuals and independent of their labor market's productivity. We introduce the idiosyncratic "quit" shock q to account for separations that are unrelated to the state of the industry. Finally, a worker in search unemployment finds a job according to a Poisson process with arrival rate α. When this happens, she may enter the intermediate goods market of her choice.

We can represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t}\big(u(C(t)) + b_i\big(1 - E(t) - U_r(t) - U_s(t)\big) + b_r U_r(t) + b_s U_s(t)\big)dt, \tag{4}$$

where $\rho > 0$ is the discount rate, u is increasing, differentiable, strictly concave, and satisfies the Inada conditions $u'(0) = \infty$ and $\lim_{C\to\infty} u'(C) = 0$, and $C(t)$ is the household's consumption of the final good. The household finances its consumption using its labor income.

2.4 Equilibrium

We look for a competitive equilibrium of this economy. At each instant, each household chooses how much to consume and how to allocate its members between employment in each labor market, rest unemployment in each labor market, search unemployment, and inactivity, in order to maximize utility subject to technological constraints on reallocating members across labor markets, taking as given the stochastic process for wages in each labor market; each final goods producer maximizes profits by choosing inputs taking as given the price for all the intermediate goods; and each intermediate goods producer j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Moreover, the demand for labor from intermediate goods producers is equal to the supply from households in each intermediate goods market; the demand for intermediate goods from the final goods producers is equal to the supply from intermediate goods producers; and the demand for final goods from the households is equal to the supply from the final goods producers.

Standard arguments imply that for given initial conditions, there is at most one competitive equilibrium of this economy.[2] We look for a stationary equilibrium where all aggregate quantities and the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets are constant. With identical households and complete markets, consumption is equal to current labor income and hence we ignore financial markets

[2]The first welfare theorem implies that any equilibrium is Pareto optimal. Since there is only one type of household, if there were multiple equilibria, household utility would be equal in each. But a convex combination of the equilibrium allocations would be feasible and Pareto superior, a contradiction.

in the remainder of this paper.

3 Costless Mobility

To understand the mechanics of the model, we start with a version where nonworkers can instantaneously become workers; formally, this is equivalent to the limit of the model when $\alpha \to \infty$. In this limit, the household does not need to devote any workers to search unemployment. Moreover, for $b_i > b_r$ there is no rest unemployment, since with costless mobility, resting is dominated by inactivity. Thus the household divides its time between employment and inactivity. Finally, with costless mobility all workers must earn a common, constant wage w.

The household therefore solves

$$\max_{E(t)} \int_0^\infty e^{-\rho t}\big(u(wE(t)) + b_i(1 - E(t))\big)dt,$$

The first order conditions imply that at each date t, $E \leq 1$ and $b_i \leq wu'(Ew)$ with complementary slackness; note that the Inada condition $u'(0) = \infty$ rules out the possibility of zero employment.

To close the model, we compute the equilibrium wage. Consider an intermediate goods market with productivity x and l workers. Output is $Q(l, x) = lAx$ and so equation (3) implies the price of the good is $P(l, x) = \left(\frac{Y}{lAx}\right)^{1/\theta}$. Then since workers are paid their marginal revenue product, the wage is

$$W(l, x) = \left(\frac{Y(Ax)^{\theta-1}}{l}\right)^{\frac{1}{\theta}}. \tag{5}$$

Since wages are equalized across markets, $W(l, x) = w$, this pins down the relationship between productivity and employment across labor markets. When $\theta > 1$, more productive labor markets employ more workers, while if goods are poor substitutes, $\theta < 1$, an increase in labor productivity lowers employment so as to keep output relatively constant. In the special case of $\theta = 1$, employment is constant and equal to Y/w in all labor markets. This is a simple but uninteresting case and so we omit its analysis from the rest of the paper.

Using equation (5), substitute $Q(l, x) = lAx = \frac{Y(Ax)^\theta}{w^\theta}$ into equation (2) and simplify to show that the wage is a weighted average of productivity across markets,

$$w = A\left(\int_0^1 x(j, t)^{\theta-1}dj\right)^{\frac{1}{\theta-1}}. \tag{6}$$

With an invariant distribution for x, we can rewrite equation (6) by integrating across that distribution. Appendix A.1 finds an expression for the invariant distribution, which requires $\delta > 0$. Using this, we prove in the appendix that the wage solves

$$w = AX_0 \left(\frac{\delta}{\delta - (\theta - 1)\left(\mu_x + (\theta - 1)\frac{\sigma_x^2}{2}\right)} \right)^{\frac{1}{\theta-1}}. \tag{7}$$

if

$$\delta > (\theta - 1)\left(\mu_x + (\theta - 1)\frac{\sigma_x^2}{2}\right) \tag{8}$$

and

$$X_0 = \left(\int_0^\infty x_0^{\theta-1} dF(x_0) \right)^{\frac{1}{\theta-1}} \in (0, \infty) \tag{9}$$

Condition (7) imposes that the exit rate of markets must be sufficiently large, the drift and variance of productivity sufficiently small, or that the elasticity of substitution sufficiently close to 1. If this parameter restriction were to fail with $\theta > 1$, the wage w would be infinite because extremely productive firms would produce an enormous amount of easily-substitutable goods; with $\theta < 1$, the wage would be zero because very unproductive firms would require a huge amount of labor to produce any of the poorly-substitutable goods. Condition (9) restricts the distribution of productivity in new markets for the same reasons.

For future reference, we note two important properties of the frictionless wage. First, if $\mu_x + (\theta - 1)\sigma_x^2/2 = 0$, $w = AX_0$, independent of δ. We introduced the assumption that $\delta > 0$ mainly for technical convenience, so imposing the restriction $\mu_x + (\theta - 1)\sigma_x^2/2 = 0$ allows to focus on the limit as δ converges to zero; we do so in Section 5. The condition is equivalent to imposing that $x^{\theta-1}$ is a martingale which, by equation (5), implies employment l is a martingale. If employment were a supermartingale, output would converge to zero for small δ, while if it were a submartingale output would converge to infinity. Second, because of constant returns to scale, the wage depends on technology but not on preference parameters. This implies that an increase in the leisure value of inactivity b_i raises the marginal utility of consumption $u'(C)$ by the same proportion, while employment E decreases in proportion to C. Similar properties hold in the frictional model, to which we turn now.

4 Characterization of Equilibrium

We now return to the model where it takes time to find a new labor market, $\alpha < \infty$. We look for a steady state equilibrium where the household maintains constant consumption, obtains a constant income stream, and keeps a positive and constant fraction of its workers in each of the activities, employment, rest unemployment, search unemployment, and inactivity. In equilibrium, in each labor market, which is characterized by productivity x and the number of workers l, the ratio $x^{\theta-1}/l$ follows a Markov process. Workers enter labor markets when the ratio exceeds a threshold and exit labor markets when it falls below a strictly smaller threshold.

4.1 The Marginal Value of Household Members

We start by computing the marginal value of an additional household member engaged in each of the three activities. These are related by the possibility of reallocating household members.

Consider first a household member who is permanently inactive. It is immediate from equation (4) that he contributes

$$\underline{v} = \frac{b_i}{\rho} \tag{10}$$

to household utility. Since the household may freely shift workers between inactivity and search unemployment, this must also be the incremental value of a searcher, assuming some members are engaged in each activity. A searcher gets flow utility b_s and the possibility of finding a labor market at rate α, giving capital $\bar{v} - \underline{v}$, where $\bar{v}$ is the value to the household of having a worker in the best labor market. This implies $\rho\underline{v} = b_s + \alpha\left(\bar{v} - \underline{v}\right)$ or

$$\bar{v} = b_i\left(\frac{1}{\rho} + \kappa\right), \text{ where } \kappa \equiv \frac{b_i - b_s}{b_i\alpha} \tag{11}$$

is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment $1/\alpha$. Conversely, a worker may freely exit her labor market, and so the lower bound on the value of a household member in a labor market, either employed or search unemployed, is $\underline{v}$. If the household values a worker at some intermediate amount, it will be willing to keep her in her labor market rather than having her search for a new one.

Finally, consider the margin between employment and resting for a worker in a labor market paying a wage w. An employed worker generates income valued at $u'(C)w$, while a resting worker generates b_r utils. Since switching between employment and resting is costless, all workers are employed in any labor market with $w > b_r/u'(C)$, and all workers are resting in any market with $w < b_r/u'(C)$. In the intermediate case, some may be employed and some resting.

4.2 Wage and Labor Force Dynamics

Consider a labor market with l workers and productivity x. Let $P(l,x)$ denote the price of its good, $Q(l,x)$ denote the amount of the good produced, $W(l,x)$ denote the wage rate, and $E(l,x)$ denote the number of workers who are employed. Competition ensures that the wage is equal to the marginal product of labor, $W(l,x) = P(l,x)Ax$, while the production function implies $Q(l,x) = E(l,x)Ax$. Combining these conditions with the intermediate good demand curve from equation (3) and the rest-work decision gives

$$W(l,x) = \frac{1}{u'(C)} \max\{b_r, e^{\omega}\} \tag{12}$$

$$Q(l,x) = lAx \min\{1, e^{\omega}/b_r\}^{\theta} \tag{13}$$

where

$$\omega \equiv \frac{\log Y + (\theta-1)\log(Ax) - \log l}{\theta} + \log u'(C) \tag{14}$$

is the logarithm of the "full-employment wage" measured in utils, the wage that would prevail if there were full employment in the labor market.[3] When $e^{\omega} > b_r$, the wage exceeds the value of leisure and so there is no rest unemployment. Otherwise, enough workers rest to the raise the wage to $b_r/u'(C)$.

Since the wage only depends on ω, we look for an equilibrium in which workers immediately enter any labor market with $\omega > \bar{\omega}$ and exit any labor market with $\omega < \underline{\omega}$, where the thresholds $\underline{\omega} \le \bar{\omega}$ are determined endogenously. There is neither entry nor endogenous exits from labor markets with $\omega \in (\underline{\omega}, \bar{\omega})$, although a fraction of the workers $q\,dt$ quit during an interval of time dt. We allow for the possibility that $\underline{\omega} = -\infty$ so workers never exit labor markets. When a positive shock hits a labor market j with $\omega(j,t) = \bar{\omega}$, ω stays constant and the labor force l increases. Conversely, negative shocks reduce ω, with l falling as workers exogenously quit the market. At $\underline{\omega} < \omega(j,t) < \bar{\omega}$, both positive and negative shocks affect ω, while l falls deterministically at rate q. When $\omega(j,t) = \underline{\omega}$, a negative shock reduces l without affecting ω, while a positive shock raises ω, with l falling due to quits.

If there is an equilibrium with this property, its definition in equation (14) implies $\omega(j,t)$ is a regulated Brownian motion in each market j. When $\omega(j,t) \in (\underline{\omega}, \bar{\omega})$, only productivity shocks change ω, so

$$d\omega(j,t) = \frac{\theta-1}{\theta} d\log x(j,t) + \frac{q}{\theta}dt = \mu dt + \sigma_{\omega} dz(j,t), \tag{15}$$

[3] Note that e^{ω} is analogous to $R(Ax,l)$ in Lucas and Prescott's (1974) notation. Their production technology implies that Y does not affect R, while risk-neutrality ensures that $u'(C)$ is constant. Lucas and Prescott (1974) also assume that $R_x > 0$—see their equation (1)—which in our set-up is equivalent to $\theta > 1$.

where

$$\mu \equiv \frac{\theta-1}{\theta}\mu_x + \frac{q}{\theta}, \quad \sigma_\omega \equiv \frac{\theta-1}{\theta}\sigma_x, \quad \text{and} \quad \sigma \equiv |\sigma_\omega|,$$

i.e., in this range $\omega(j,t)$ has drift μ and instantaneous standard deviation σ. When the thresholds $\underline{\omega}$ and $\bar{\omega}$ are finite, they act as reflecting barriers, since productivity shocks that would move ω outside the boundaries are offset by the entry and exit of workers. If we allowed $\theta = 1$, productivity shocks would be offset by price changes in a way that leaves the wage unchanged and so $\mu = q$ and $\sigma = 0$, an uninteresting case that would require a separate analysis.

4.3 Labor Force Participant Value Function

We return now to the value of a worker in a labor market with productivity x and with l workers. Since the wage only depends on ω, a regulated Brownian motion, we look for an equilibrium where the worker's value depends only on ω. Given arbitrary values for the barriers $\underline{\omega} \leq \bar{\omega}$, we can define the incremental value to the household of having a worker in a market with current productivity ω_0 as

$$v(\omega_0; \underline{\omega}, \bar{\omega}) = \mathbb{E}\left(\int_0^\infty e^{-(\rho+\lambda)t}\left(\max\{b_r, e^{\omega(t)}\} + \lambda \underline{v}\right) dt \,\middle|\, \omega(0) = \omega_0\right), \tag{16}$$

where expectations are taken with respect to future values of the random variable $\omega(t)$ and

$$\lambda \equiv q + \delta$$

is the rate at which workers exogenously exit markets. The discount rate $\rho + \lambda$ accounts both for impatience and for the possibility that the labor market ends exogenously. The time-t payoff is the prevailing wage; this holds whether the worker is employed or rest-unemployed because when there is rest unemployment, the worker is indifferent between the two states. In addition, if the worker exogenously leaves the market, which happens with hazard rate λ, the household gets a terminal value $\underline{v}$.

The utility of a worker in any market must be between $\underline{v}$ and $\bar{v}$. If she is in the best possible market, her utility must be $\bar{v}$ so searchers are willing to take a job there. If she is in the worst possible market, her utility must be $\underline{v}$ so she is indifferent about exiting her labor market; such a market exists only if the lower threshold is finite.

$$\begin{aligned} &v(\omega;\underline{\omega},\bar{\omega}) \in [\underline{v},\bar{v}] \text{ for all } \omega \\ &v(\bar{\omega};\underline{\omega},\bar{\omega}) = \bar{v} \\ &v(\underline{\omega};\underline{\omega},\bar{\omega}) = \underline{v} \text{ if } \underline{\omega} > -\infty. \end{aligned} \tag{17}$$

To characterize the thresholds further, define

$$\Pi(\omega;\omega_0;\underline{\omega},\bar{\omega}) \equiv \mathbb{E}\left(\int_0^\infty e^{-(\rho+\lambda)t} I_\omega(\omega(t))dt \;\middle|\; \omega(0)=\omega_0 \right),$$

where $I_\omega(\omega(t))$ is an indicator function, equal to 1 if $\omega(t) < \omega$ and equal to zero otherwise. This *discounted occupancy function* evaluates to zero at $\omega \leq \underline{\omega}$ and to $\frac{1}{\rho+\lambda}$ at $\omega \geq \bar{\omega}$. We use $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for the density of ω or the *discounted local time function*, where the subscript denotes the partial derivative with respect to the first argument. Then switching the order of integration in equation (16), which is permissible since for $-\infty \leq \underline{\omega} \leq \bar{\omega} < \infty$ and $\rho+\lambda > 0$, the function $\max\{b_r, e^\omega\} + \lambda\underline{v}$ is integrable, we get

$$v(\omega_0;\underline{\omega},\bar{\omega}) = \int_{\underline{\omega}}^{\bar{\omega}} \left(\max\{b_r, e^\omega\} + \lambda\underline{v}\right) \Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) d\omega. \tag{18}$$

The value of being in a market with current log full-employment wage ω_0 is equal to the expected value of future ω weighted by the appropriate discounted local time function. Equation (18) is convenient because $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ is a known function (Stokey, 2006).

It is worth noting that equations (17) and (18) imply some familiar conditions:

$$(\rho+\lambda)v(\omega_0;\underline{\omega},\bar{\omega}) = \max\{b_r, e^\omega\} + \lambda\underline{v} + \mu v_{\omega_0}(\omega_0;\underline{\omega},\bar{\omega}) + \frac{\sigma^2}{2} v_{\omega_0,\omega_0}(\omega_0;\underline{\omega},\bar{\omega}), \tag{19}$$

$$\text{and } v_{\omega_0}(\bar{\omega};\underline{\omega},\bar{\omega}) = v_{\omega_0}(\underline{\omega};\underline{\omega},\bar{\omega}) = 0, \tag{20}$$

where subscripts denote partial derivatives with respect to the first argument.[4] Together with the "value-matching" conditions $v(\bar{\omega};\underline{\omega},\bar{\omega}) = \bar{v}$ and $v(\underline{\omega};\underline{\omega},\bar{\omega}) = \underline{v}$ in equation (17), this is an equivalent representation of the labor force participant's value function.

Using monotonicity of the period payoff function in equation (18) and monotonicity in the sense of first order stochastic dominance of Π_ω, we can now prove

[4] The first condition, the Hamilton-Jacobi-Bellman equation, can be verified directly by differentiating equation (18) using the definition of Π_ω in equation (44). The interested reader can consult the online Appendix B.1 for the details of the algebra. The second pair of conditions, "smooth-pasting," follow from equation (18) because equation (44) implies $\frac{\partial \Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})}{\partial \omega_0} = 0$ when $\omega_0 = \underline{\omega}$ or $\omega_0 = \bar{\omega}$.

Lemma 1. *v is continuous and nondecreasing in ω_0, $\underline{\omega}$, and $\bar{\omega}$. It is strictly increasing in each argument if $\omega_0 \in (\underline{\omega}, \bar{\omega})$ and $\bar{\omega} > \log b_r$.*

The proof is in Appendix A.2. Building on this, we characterize the thresholds:

Proposition 1. *Equations (17) and (18) uniquely define $\underline{\omega}$ and $\bar{\omega}$ as functions of model parameters. A proportional increase in b_i, b_r, and b_s raises $e^{\underline{\omega}}$ and $e^{\bar{\omega}}$ by the same proportion. Moreover, $\underline{\omega} < \log b_i < \bar{\omega} < \infty$, with $\underline{\omega} > -\infty$ if and only if $b_r < b_i$.*

The proof of the Proposition in Appendix A.3 defines two objects, $\bar{\Omega}(\underline{\omega})$ solving $v(\bar{\omega}; \underline{\omega}, \bar{\omega}) \equiv \bar{v}$ and $\underline{\Omega}(\bar{\omega})$ solving $v(\underline{\omega}; \underline{\omega}, \bar{\omega}) \equiv \underline{v}$. That is $\bar{\Omega}(\underline{\omega})$ gives the value of the upper threshold such that if workers enter and exit the market to keep $\bar{\omega} \geq \omega \geq \underline{\omega}$, a market at the upper threshold in fact delivers value $\bar{v}$. Similarly, $\underline{\Omega}(\bar{\omega})$ gives the value of the lower threshold such that under the same condition, a market at the lower threshold delivers value $\underline{v}$. An equilibrium is then defined by a fixed point $\bar{\omega} = \bar{\Omega}(\underline{\omega})$ and $\underline{\omega} = \underline{\Omega}(\bar{\omega})$. Exploiting a single-crossing of the functions $\underline{\Omega}$ and $\bar{\Omega}$, we prove that there exists a unique solution to this pair of equations. The single-crossing property relies on two key features of the model: the period return function is monotonic; and an increase in the upper (lower) bound affects the discounted occupancy function more when ω_0 is closer to the upper (lower) bound. Although our proof relies on the exact functional form of the discounted occupancy function, the properties are likely to hold more generally when ω is persistent. One can also construct an alternative proof relying on solving the "island planner's problem" developed in the online Appendix B.2.

The Proposition establishes that $\underline{\omega}$ is finite when $b_r < b_i$. Intuitively, if a market is hit by sufficiently adverse shocks, workers will leave since rest unemployment is costly and has low expected payoffs. In contrast, when $b_r \geq b_i$, rest unemployment is costless and hence workers only leave labor markets when they shut down. Moreover, if $b_r \leq b_i$, there is no rest unemployment in the best labor markets, $\bar{\omega} > \log b_r$. The next proposition addresses whether there is rest unemployment in the worst labor markets, $\underline{\omega} \gtrless \log b_r$.

Proposition 2. *There exists a $\bar{b}_r$ such that in an equilibrium, $b_r \gtreqless e^{\underline{\omega}}$ if and only if $b_r \gtreqless \bar{b}_r$, with $\bar{b}_r = B(\kappa, \rho + \lambda, \mu, \sigma) b_i$ for some function B, positive-valued and decreasing in κ with $B(0, \rho + \lambda, \mu, \sigma) = 1$.*

The proof is in Appendix A.4. This Proposition implies that there is rest unemployment if search costs κ are sufficiently high given any $b_r > 0$, or equivalently if the leisure value of resting b_r is sufficiently close to the leisure of inactivity b_i given any $\kappa > 0$. If a searcher finds a job sufficiently fast (so κ is small) or resting gives too little leisure (so b_r is small), there is no reason to wait for labor market conditions to improve, and so B is monotone in κ.

4.4 Equilibrium

We have solved for the values $\underline{\omega}$ and $\bar{\omega}$ that describe workers' incentive to enter and exit labor markets as functions of model parameters. Taking these thresholds as given, we now find the remaining variables and equations that determine an equilibrium and establish that they have a unique solution by solving them explicitly. This section is about mechanics and aggregation given our characterization of optimizing behavior.

The first equilibrium condition is that the final goods market clears, $Y = C$.

Next, a key object for us is the stationary distribution of the L workers across log full-employment wages ω; we denote its density by f defined on $[\underline{\omega}, \bar{\omega}]$. Since f is a density,

$$\int_{\underline{\omega}}^{\bar{\omega}} f(\omega) d\omega = 1. \tag{21}$$

By taking the limit of a discrete time, discrete state-space analog of our model, we prove in Appendix A.5 that this density has to satisfy three conditions, equations (22)–(24) below. First, in the interior of its support, it must solve a Kolmogorov forward equation,

$$\lambda f(\omega) = -\mu f'(\omega) + \frac{\sigma^2}{2} f''(\omega) \text{ for all } \omega \in (\underline{\omega}, \bar{\omega}). \tag{22}$$

This captures the requirement that inflows and outflows balance at each point in the support of the density. Workers exit markets either because of quits or shutdowns at rate λ, while otherwise ω is a Brownian motion with drift μ and standard deviation σ. Workers whose ω changes leave this point in the density for higher or lower values of ω, while the density picks up mass from points above and below when they are hit by appropriate shocks. In a short period of time, this relates the density of f at nearby points, i.e. it relates the level of f and its derivatives.

Second, at the lower bound $\underline{\omega}$,

$$\frac{\sigma^2}{2} f'(\underline{\omega}) - \left(\mu + \frac{\theta \sigma^2}{2}\right) f(\underline{\omega}) = 0. \tag{23}$$

The elasticity of substitution θ appears in this equation because it determines how many workers must exit from depressed markets required to regulate ω above $\underline{\omega}$. The exogenous separation rate λ does not appear in this equation because the ratio of endogenous to exogenous exits is infinite in a short time interval for a market at the lower bound. Since by definition there are no markets with smaller ω, $f(\underline{\omega})$ is not fed from below, which explains the difference between equations (22) and (23). Finally, at the upper bound $\bar{\omega}$,

$$\frac{\sigma^2}{2} f'(\bar{\omega}) - \left(\mu + \frac{\theta\sigma^2}{2}\right) f(\bar{\omega}) = \delta \frac{L_0}{L}, \tag{24}$$

where L_0 is the (endogenous) average number of workers in a new labor market. The logic for the left hand side of this equation parallels the logic behind equation (23). There is an extra inflow at $\bar{\omega}$ coming from newly-formed markets, which absorb δL_0 workers per unit of time; dividing by L expresses this inflow as a percentage of the workers located in labor markets.

In a new labor market with productivity x_0, equation (14) implies that the number of workers is $Yu'(Y)^\theta (Ax_0)^{\theta-1} e^{-\theta\bar{\omega}}$ to ensure a log full-employment wage $\bar{\omega}$. Integrating this across markets gives

$$L_0 = \int_0^\infty Yu'(Y)^\theta (Ax_0)^{\theta-1} e^{-\theta\bar{\omega}} dF(x_0) = Yu'(Y)^\theta (AX_0)^{\theta-1} e^{-\theta\bar{\omega}}, \tag{25}$$

where the second equation uses the definition of X_0 in equation (9).

Our last condition relates intermediate and final goods output. It is convenient to first define the productivity of a location x consistent with l workers present in the location, a log full-employment wage ω, and aggregate output and consumption Y. From equation (14), this solves

$$x = \xi(l, \omega, Y) \equiv \frac{1}{A} \left(\frac{l e^{\theta\omega}}{Yu'(Y)^\theta}\right)^{\frac{1}{\theta-1}}. \tag{26}$$

Then from equation (13), output in a market with l workers and log full-employment wage ω is

$$Q(l, \xi(l, \omega, Y)) = Y^{\frac{-1}{\theta-1}} \left(\frac{e^\omega l}{u'(Y)}\right)^{\frac{\theta}{\theta-1}} \min\{1, e^\omega / b_r\}^\theta. \tag{27}$$

Using this notation, we can write equation (2) as

$$Y = \left(\int_0^1 Q\big(l(j,t), \xi(l(j,t), \omega(j,t), Y)\big)^{\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}} = \left(\int_0^1 Q\big(L, \xi(L, \omega(j,t), Y)\big)^{\frac{\theta-1}{\theta}} \frac{l(j,t)}{L} dj\right)^{\frac{\theta}{\theta-1}}$$

The second equation follows because $Q(\cdot, \xi(\cdot, \omega, Y))^{\frac{\theta-1}{\theta}}$ is linear (equation 27). To solve this, we change the variable of integration from the name of the market j to its log full-employment wage ω and number of workers l. Let $\tilde{f}(\omega, l)$ be the density of the joint distribution of workers in markets (ω, l). Then

$$Y = \int_{\underline{\omega}}^{\bar{\omega}} \int_0^\infty Q(L, \xi(L, \omega, Y))^{\frac{\theta-1}{\theta}} \frac{l}{L} \tilde{f}(\omega, l)\, dl\, d\omega.$$

Since $f(\omega) = \int_0^\infty \frac{l}{L} \tilde{f}(\omega, l)\, dl$, we can solve the inner integral to obtain

$$Y = \left(\int_{\underline{\omega}}^{\bar{\omega}} Q\big(L, \xi(L, \omega, Y)\big)^{\frac{\theta-1}{\theta}} f(\omega) d\omega \right)^{\frac{\theta}{\theta-1}}, \tag{28}$$

without characterizing the joint density $\tilde{f}$.

To summarize, given the thresholds $\underline{\omega}$ and $\bar{\omega}$, an equilibrium is a list $\{Y, L_0, L, f(\cdot)\}$ solving equations (21)–(25) and (28). We have the following result:

Proposition 3. There exists a unique equilibrium. The steady state density of workers across labor markets is

$$f(\omega) = \frac{\sum_{i=1}^2 |\eta_i + \theta| e^{\eta_i(\omega - \underline{\omega})}}{\sum_{i=1}^2 |\eta_i + \theta| \frac{e^{\eta_i(\bar{\omega} - \underline{\omega})} - 1}{\eta_i}}; \tag{29}$$

output solves

$$u'(Y)^{1-\theta} = \frac{-2\delta (AX_0)^{\theta-1} e^{-(\theta\bar{\omega} - \hat{\omega})}}{\sigma^2(\theta + \eta_1)(\theta + \eta_2)\big(e^{\eta_2(\bar{\omega} - \underline{\omega})} - e^{\eta_1(\bar{\omega} - \underline{\omega})}\big)} \times \sum_{i=1}^2 |\theta + \eta_i| \left(\frac{e^{\eta_i(\hat{\omega} - \underline{\omega})} - e^{-\theta(\hat{\omega} - \underline{\omega})}}{\theta + \eta_i} + e^{\eta_i(\bar{\omega} - \underline{\omega})} \frac{e^{(\bar{\omega} - \hat{\omega})} - e^{-\eta_i(\bar{\omega} - \hat{\omega})}}{1 + \eta_i} \right); \tag{30}$$

and the number of workers in labor markets is

$$L = \frac{Y u'(Y) \sum_{i=1}^2 |\theta + \eta_i| \frac{e^{\eta_i(\bar{\omega} - \underline{\omega})} - 1}{\eta_i}}{e^{\hat{\omega}} \sum_{i=1}^2 |\theta + \eta_i| \left(\frac{e^{\eta_i(\hat{\omega} - \underline{\omega})} - e^{-\theta(\hat{\omega} - \underline{\omega})}}{\theta + \eta_i} + e^{\eta_i(\bar{\omega} - \underline{\omega})} \frac{e^{(\bar{\omega} - \hat{\omega})} - e^{-\eta_i(\bar{\omega} - \hat{\omega})}}{1 + \eta_i} \right)}, \tag{31}$$

where $\hat{\omega} \equiv \max\{\underline{\omega}, \log b_r\}$ and $\eta_1 < 0 < \eta_2$ solve the characteristic equation $\lambda = -\mu\eta + \frac{\sigma^2}{2}\eta^2$.

Proof. We solve the system of equations defining an equilibrium recursively. The second order differential equation (22) and the boundary conditions equations (21) and (23) yields equation (29) using standard calculations.

Using the expression for f we solve equations (24), (25), and (28) for L_0, L, and Y. First eliminate L_0 between equations (24) and (25) and evaluate $f(\bar{\omega})$ using equation (29) to get

$$L = \frac{-2\delta Y u'(Y)^{\theta}(AX_0)^{\theta-1}e^{-\theta\bar{\omega}}}{\sigma^2(\theta+\eta_1)(\theta+\eta_2)\big(e^{\eta_2(\bar{\omega}-\underline{\omega})}-e^{\eta_1(\bar{\omega}-\underline{\omega})}\big)}\sum_{i=1}^{2}|\theta+\eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}. \tag{32}$$

Substitute equation (27) into equation (28) to get

$$\begin{aligned} Y &= \frac{L}{u'(Y)}\int_{\underline{\omega}}^{\bar{\omega}} e^{\omega}\min\{1, e^{\omega}/b_r\}^{\theta-1}f(\omega)d\omega \\ &= \frac{Le^{\bar{\omega}}}{u'(Y)}\frac{\sum_{i=1}^{2}|\theta+\eta_i|e^{-(\bar{\omega}-\hat{\omega})}\left(\frac{e^{\eta_i(\hat{\omega}-\underline{\omega})}-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta+\eta_i}+e^{\eta_i(\bar{\omega}-\underline{\omega})}\frac{e^{(\bar{\omega}-\hat{\omega})}-e^{-\eta_i(\bar{\omega}-\hat{\omega})}}{1+\eta_i}\right)}{\sum_{i=1}^{2}|\theta+\eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}}, \end{aligned} \tag{33}$$

where we solve the integral using the expression for f in equation (29). Eliminating L between these equations and solving for $u'(Y)$ gives equation (30). Since the left hand side of this equation is monotone, there exists a unique Y solving this equation. Finally, solve equation (33) for L to complete the proof. □

4.5 Measurement of Unemployment

Once we have found an equilibrium, we can measure the rest and search unemployment rates. Recall that U_r is the fraction of household members who are rest-unemployed. If $\log b_r \le \underline{\omega}$, this is zero. Otherwise, in a market with $\omega \in [\underline{\omega}, \hat{\omega}]$, the rest unemployment rate is $1-e^{\theta(\omega-\hat{\omega})}$. Integrating across such markets using equation (29) gives

$$\frac{U_r}{L} = \int_{\underline{\omega}}^{\hat{\omega}}\left(1-e^{\theta(\omega-\hat{\omega})}\right)f(\omega)\,d\omega = \theta\frac{\frac{e^{\eta_2(\hat{\omega}-\underline{\omega})}-1}{\eta_2}-\frac{e^{\eta_1(\hat{\omega}-\underline{\omega})}-1}{\eta_1}}{\sum_{i=1}^{2}|\theta+\eta_i|\left(\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}\right)}. \tag{34}$$

The remaining household members who are in labor markets are employed, $E = L - U_r$.

Now we turn to the search unemployed. Let N_s be the number of workers among L that leave their labor market per unit of time, either because conditions are sufficiently bad or because their labor market has exogenously shut down. Appendix A.6 takes limits of the discrete time, discrete state space model to show that this rate is given by

$$N_s = \frac{\theta\sigma^2}{2}f(\underline{\omega})L + \lambda L. \tag{35}$$

The first term gives the fraction of workers who leave their labor market to keep ω above $\underline{\omega}$. The second term is the fraction of workers who exogenously leave their market. In steady state, the fraction of workers who leave labor markets must balance the fraction of workers who arrive in labor markets. The latter is given by the fraction of workers engaged in search unemployment U_s, times the rate at which they arrive to the labor market α, so $\alpha U_s = N_s$. Solve equation (35) using equation (29) to obtain an expression for the ratio of search unemployment to workers in labor markets:

$$\frac{U_s}{L} = \frac{1}{\alpha}\left(\frac{\theta\sigma^2}{2}\frac{\eta_2 - \eta_1}{\sum_{i=1}^{2}|\theta + \eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}} + \lambda\right) \tag{36}$$

To have an interior equilibrium we require that $U_s + U_r + E \leq 1$ so that the labor force is smaller than the total population.[5]

We deliberately leave the expressions for unemployment as a function of the thresholds $\underline{\omega}$, $\hat{\omega}$, and $\bar{\omega}$ in order disentangle optimization—the choice of thresholds—from the mechanics of aggregation. This has two advantages. First, we find it useful to exploit this dichotomy in our numerical evaluation of the model in Section 5. Second, the expressions for rest and search unemployment as a function of the thresholds are identical in other variants of the model, including the original Lucas and Prescott (1974) model. For example, suppose the curvature of labor demand comes from diminishing returns at the island level, due to a fixed factor, rather than imperfect substitutability (see footnote 3). Then the analog of the elasticity of substitution is the reciprocal of the elasticity of revenue with respect to the fixed factor, while the expressions for unemployment are otherwise unchanged.

We close this section by noting some homogeneity properties of employment, rest unemployment, search unemployment, and consumption.

Proposition 4. *Let $b_r = \beta\bar{b}_r$, $b_s = \beta\bar{b}_s$, $b_i = \beta\bar{b}_i$ for fixed $\bar{b}_r$, $\bar{b}_s$, and $\bar{b}_i$. The equilibrium value of the unemployment rate $\frac{U_s+U_r}{U_s+U_r+E}$ and the share of rest-unemployed $\frac{U_r}{U_r+U_s}$ do not depend on β, the level of productivity A, the distribution of productivity in new labor markets F, or the utility function u. The equilibrium value of $u'(Y)$ is proportional to $\frac{\beta}{AX_0}$.*

Proof. By inspection, the unemployment rate and share of rest-unemployed are functions of the difference in thresholds $\bar{\omega} - \underline{\omega}$ and $\bar{\omega} - \hat{\omega}$ and the parameters α, δ, q, θ, μ (or μ_x), and σ (or σ_x), either directly or indirectly through the roots η_i. From Proposition 1, the thresholds depend on the same parameters and on the discount rate ρ. This completes the first part of the proof.

[5] If this condition fails, all household members participate. The equilibrium is equivalent to one with a higher leisure value of inactivity, the value of b_i such that $U_s + U_r + E = 1$. In any case, Proposition 4 implies that for b_r, b_s, and b_i large enough, the equilibrium has $U_s + U_r + E < 1$.

Next, recall from Proposition 1 that $e^{\underline{\omega}}$ and $e^{\bar{\omega}}$ are proportional to β. Then equation (30) implies $u'(Y)$ inherits the same proportionality. On the other hand, Proposition 1 implies AX_0 does not affect any of the thresholds and so equation (30) implies $u'(Y)$ is inversely proportional to AX_0. □

This proposition shows that the unemployment rate and composition of unemployment is determined by the relative advantage of different leisure activities, while output, and hence consumption and employment, depends on an absolute comparison of leisure versus market production. Indeed, the finding that $u'(Y)$ is proportional to β/AX_0 holds in the frictionless benchmark, where an interior solution for the employment rate requires $b_i = u'(Y)w$, while the wage is proportional to AX_0 (see equation 7). Whether higher productivity lowers or raises equilibrium employment depends on whether income or substitution effects dominate in labor supply. With $u(Y) = \log Y$, an increase in productivity raises consumption proportionately without affecting employment or labor force participation.

4.6 The Limiting Economy

This section discusses an important limit of the model, when the exogenous shut-down rate of markets δ is zero. We introduced the assumption that intermediate goods markets shut down for technical reasons, to ensure an invariant distribution of productivity and employment. Still, with the parameter restriction, $\mu_x + (\theta - 1)\sigma_x^2/2 = 0$, discussed previously in Section 3, the economy is well behaved even when δ limits to zero. It is clear from Proposition 1 that $\underline{\omega}$ and $\bar{\omega}$ converge nicely for any value of μ_x as long as the discount rate ρ is positive. More problematic is whether aggregate employment, unemployment, and output converge. We now show that the same parameter restriction yields a well-behaved limit of the frictional economy.

When $\mu_x = -(\theta-1)\sigma_x^2/2$ and $\delta \to 0$, the roots of the characteristic equation in Proposition 3 converge to $\eta_1 = -\theta$ and $\eta_2 = 2q/\theta\sigma^2$. Substituting into equation (29), we find

$$f(\omega) = \frac{\eta_2 e^{\eta_2(\omega - \underline{\omega})}}{e^{\eta_2(\bar{\omega} - \underline{\omega})} - 1}.$$

If $q = 0$ as well, this simplifies further to $f(\omega) = 1/(\bar{\omega} - \underline{\omega})$, i.e. f is uniform on its support, while for positive q the density is increasing in ω. We can also confirm from equation (30) that output is positive and finite in the limiting economy:

$$u'(Y)^{1-\theta} = \frac{(AX_0)^{\theta-1}(\theta + \eta_2)e^{-(\theta\bar{\omega} - \hat{\omega})}}{e^{\eta_2(\bar{\omega} - \underline{\omega})} - e^{-\theta(\bar{\omega} - \underline{\omega})}} \left(\frac{e^{\eta_2(\hat{\omega} - \underline{\omega})} - e^{-\theta(\hat{\omega} - \underline{\omega})}}{\theta + \eta_2} + e^{\eta_2(\bar{\omega} - \underline{\omega})} \frac{e^{(\bar{\omega} - \hat{\omega})} - e^{-\eta_2(\bar{\omega} - \hat{\omega})}}{1 + \eta_2} \right).$$

We can similarly compute limits of the key measures of employment and unemployment. From equation (31), the fraction of household members in labor markets is

$$L = \frac{Yu'(Y)e^{-\hat{\omega}}\frac{e^{\eta_2(\bar{\omega}-\underline{\omega})}-1}{\eta_2}}{\frac{e^{\eta_2(\hat{\omega}-\underline{\omega})}-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta+\eta_2}+e^{\eta_2(\bar{\omega}-\underline{\omega})}\frac{e^{(\bar{\omega}-\hat{\omega})}-e^{-\eta_2(\bar{\omega}-\hat{\omega})}}{1+\eta_2}}.$$

Finally, from equations (34) and (36), rest and search unemployment converge to

$$\frac{U_r}{L} = \frac{\theta\frac{e^{\eta_2(\hat{\omega}-\underline{\omega})}-1}{\eta_2}+e^{-\theta(\hat{\omega}-\underline{\omega})}-1}{(\theta+\eta_2)\left(\frac{e^{\eta_2(\bar{\omega}-\underline{\omega})}-1}{\eta_2}\right)} \quad \text{and} \quad \frac{U_s}{L} = \frac{q}{\alpha\left(1-e^{-\eta_2(\bar{\omega}-\underline{\omega})}\right)}. \tag{37}$$

Each of these expressions simplifies further when there are no quits, $q = 0$ and so $\eta_2 \to 0$.[6] But the important point is that, although productivity does not have a well-behaved limiting distribution when δ converges to 0, aggregate output, employment, and rest and search unemployment are well-behaved in this limit.

5 Quantitative Evaluation

The goal of this section is to use our model to understand the role of search and rest unemployment in the U.S. economy. We focus on the limit of the economy discussed above, i.e. we assume that $\mu_x = -(\theta-1)\sigma_x^2/2$ and that $\delta \to 0$. Because we have closed-form expressions for the unemployment and labor force participation rates, comparative statics are relatively straightforward. It is also straightforward to see how various parameters affect other variables of interest, including the stochastic process for wages, measures of job creation and destruction, and the hazard rate of exiting unemployment.

5.1 No Rest Unemployment

To connect the model to the data, we find it useful first to examine the reduced-form relationships in equation (37), without worrying about the determinants of the three thresholds $\underline{\omega}$, $\hat{\omega}$, and $\bar{\omega}$.[7] Focus first on U_s/L and assume there is no rest unemployment, $\hat{\omega} = \underline{\omega}$. One can think of the search unemployment rate as depending on four forces. $1/\alpha$ is the mean duration of a spell of search unemployment and so U_s/L is decreasing in α. q is the exogenous quit rate and so U_s/L is increasing in q. $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ determines the average time it takes a labor market to move from the hiring threshold to the firing threshold, and so is related to the

[6] The order of convergence of δ and q to zero does not affect these results.

[7] The thresholds are determined by the discount rate ρ and the three leisure values, b_i, b_r, and b_s, in addition to the parameters that directly enter equation (37), θ, σ, q, and α.

duration of employment. And $\theta\sigma$ determines how many workers must exit a labor market at the lower threshold following a one standard deviation productivity shock, and so is also important for the duration of employment.

To determine reasonable values for these variables, we map our model to the data. From 1990 to 2006, the unemployment rate in the United States averaged 5.5 percent. The mean duration of an in-progress unemployment spell was 0.31 years, which is implied by $\alpha = 3.2$.[8] For now, we set $q = 0$ so as to examine the amount of search unemployment that the model can generate endogenously.

To pin down the remaining parameters, we need to take a stand on the nature of a labor market. A labor market has two defining characteristics. First, the goods produced within a labor market are homogeneous while the goods produced in different labor markets are heterogeneous, as captured by the elasticity of substitution θ. This suggests modeling a labor market as an industry. Second, workers are free to move within a labor market but not between labor markets, presumably both because of some specificity of human capital and because of geographic mobility costs. To the extent that human capital is occupation, not industry, specific (Kambourov and Manovskii, 2007), this suggests that a labor market may be a cross between an occupation and a geographic location. In the end, our definition of a labor market is governed by data availability: we measure a labor market as a five-digit NAICS industry. Using international trade data, Broda and Weinstein (2006) report median estimates of the elasticity of substitution between goods at the five-digit SITC level of about 2.8 (see their Table IV). We therefore set θ to 3.[9]

Rather than take a stand on values of σ and $\bar{\omega} - \underline{\omega}$ directly, we observe that σ is critical for the volatility and $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ for the autocorrelation of wages at the industry level. That σ affects the volatility is immediate from equation (15). When $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ is large, wages are nearly a random walk and so this year's wage level is very informative about next year's, i.e. the autocorrelation of wages is large. For small values of $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$, wages hit the bounds frequently within a short period of time and so are nearly uncorrelated.[10] We measure industry-level average weekly earnings for 312 five-digit industries from 1990 to 2006 from the Current

[8] The empirical duration numbers were constructed by the Bureau of Labor Statistics from the Current Population Survey and may be obtained from `http://www.bls.gov/cps/`. Our choice of years is governed by the availability of industry wage data.

[9] This elasticity is in line with the one used in much of the literature that quantitatively evaluates the Lucas and Prescott (1974) model. Recall that the analog of θ in a model with diminishing returns at the labor market level due to a fixed factor is the reciprocal of the elasticity of revenue with respect to the fixed factor. If the fixed factor is capital, then a capital share of $\frac{1}{3}$ is empirically reasonable. Alvarez and Veracierto (1999) set the elasticity of fixed factor to 0.36, Alvarez and Veracierto (2001) set it to 0.23, and Kambourov and Manovskii (2007) set it to 0.32, in line with values of θ between 2.8 and 4.3.

[10] Ball and Roma (1998) find an exact formula for the autocorrelation of annual observations from a reflected Brownian motion without drift and prove it depends only on $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$.

Employment Statistics (http://www.bls.gov/ces/), all the industries with available data. We deflate the nominal annual average of industry earnings by the nominal annual average of private sector earnings $\bar{w}_t$ and take logs to construct $\omega_{j,t}$, log relative average weekly earnings in industry j and year t.[11] We then measure the standard deviation of the growth rate of earnings in the J industries and T years as

$$SD = \frac{1}{J}\sum_{j=1}^{J}\left(\frac{1}{T-1}\sum_{t=2}^{T}\left(\omega_{j,t}-\omega_{j,t-1}\right)^2\right)^{1/2}$$

and the autocorrelation of earnings as

$$AC = \frac{1}{J}\sum_{j=1}^{J}\frac{\sum_{t=2}^{T}\left(\omega_{j,t}\omega_{j,t-1}\right)^2}{\left(\sum_{t=1}^{T-1}\omega_{j,t}^2\sum_{t=2}^{T}\omega_{j,t}^2\right)^{1/2}}.$$

Note that neither of these formulas subtracts industry-specific means. Consistent with our model, we assume we know industry-specific mean log average weekly earnings deflated by private sector earnings: it is equal to zero.

To compare the model with U.S. data, we generate data from a discrete time version of the model, where a time period is 1 week. Each week, we add a normal innovation to log wages and adjust employment to keep log wages in the appropriate bounds. At the end of each year, we average the model-generated data, deflate by the theoretical average earnings

$$\bar{w} = e^{\hat{\omega}}\int_{\underline{\omega}}^{\hat{\omega}} f(\omega)d\omega + \int_{\hat{\omega}}^{\bar{\omega}} e^{\omega} f(\omega)d\omega,$$

and take logs to construct $\omega_{j,t}$. We repeat for $T = 17$ years and $J = 10,000$ industries to obtain accurate estimates of the two moments SD and AC.

To summarize, we match the empirical search unemployment rate of $\frac{U_s}{U_s+L} = 0.055$ and set $\alpha = 3.2$, $q = 0$, and $\theta = 3$. This is consistent with many different values of σ and $\bar{\omega} - \underline{\omega}$, each associated with a different standard deviation of the growth rate and autocorrelation of the level of earnings. Figure 1 shows that it is impossible to simultaneously match the two targets. The empirical average autocorrelation of the level of earnings is more than 0.97. To match such a high autocorrelation, the model requires a standard deviation of growth rates about eight times as high as the empirical value of 0.03.[12] Intuitively, if the difference $\bar{\omega} - \underline{\omega}$

[11] Fluctuations in productivity A may cause fluctuations in average earnings. With log utility, such fluctuations cause proportional changes in wages but do not affect the unemployment rate. Deflating by $\bar{w}_t$ therefore perfectly controls for aggregate fluctuations.

[12] Other moments contain similar information to the autocorrelation of wages, notably the kurtosis of wage growth. When $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ is small, wage changes are frequently truncated and so the distribution of wage growth

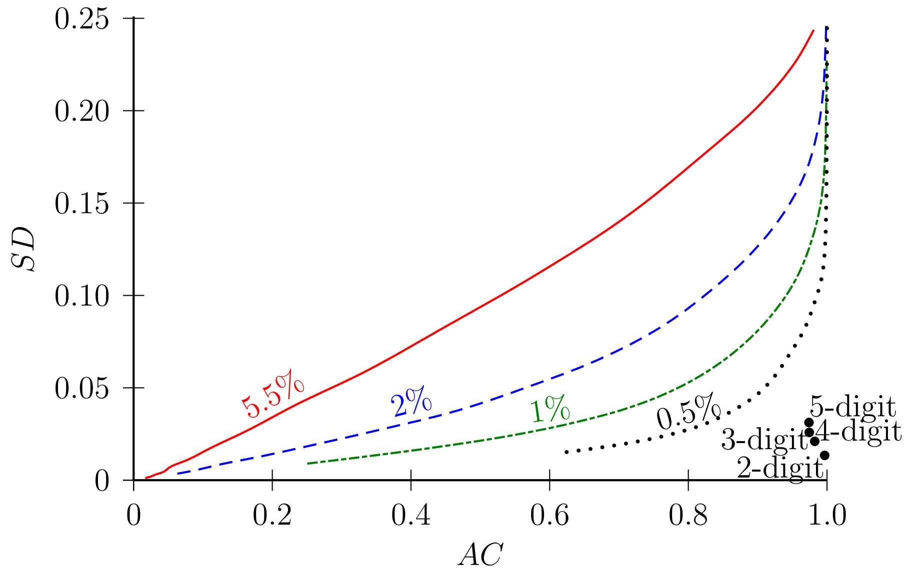

Figure 1: Variation in search unemployment rate. AC is the autocorrelation of earnings and SD is the standard deviation of the growth rate of earnings. The solid red line shows the correlation between $w_{j,t}$ and $w_{j,t-1}$ and the standard deviation of $w_{j,t} - w_{j,t-1}$ consistent with $\alpha = 3.2$, $\theta = 3$, $q = 0$, and a 5.5% unemployment rate with no rest unemployment. The dashed blue line shows the combinations consistent with 2% search unemployment; the dash-dotted green line shows 1% search unemployment; and the dotted black line shows 0.5% search unemployment. The four larger dots show the empirical averages at the two- to five-digit NAICS levels.

is large, wages are nearly a random walk; however, when this difference is large, the incidence of unemployment is small because it takes labor markets a long time to move from upper to the lower threshold.

Looking at more disaggregated data might mitigate but is unlikely to eliminate this problem. First, the additional dots in Figure 1 shows that at higher levels of industry aggregation, the autocorrelation in wages is slightly higher and the standard deviation of wages slightly lower. Although data are not available, it is reasonable to expect similarly modest changes in the autocorrelation and standard deviation in more disaggregated data. Second, one can prove that changing the elasticity from θ to $\theta' > \theta$ shifts the curve down by a factor θ'/θ. In other words, raising the elasticity of substitution from $\theta = 3$ to $\theta' = 6$ implies that the model-generated value of the standard deviation SD is four times the empirical value (rather than eight times, as discussed in the previous paragraph) when the model matches the empirical value of the autocorrelation AC. Broda and Weinstein (2006) find that at the seven digit (TSUSA/HTS) level, the median elasticity of substitution is still below 4, so

is platykurtic. For large $\frac{\bar{\omega}-\omega}{\sigma}$, time aggregation leads to a slightly leptokurtic wage growth distribution. For example, at values of $\frac{\bar{\omega}-\omega}{\sigma}$ that generate $AC = 0.97$, the kurtosis of wage growth is about 3.3. In the data, wage growth is slightly more leptokurtic, with a kurtosis of 3.7.

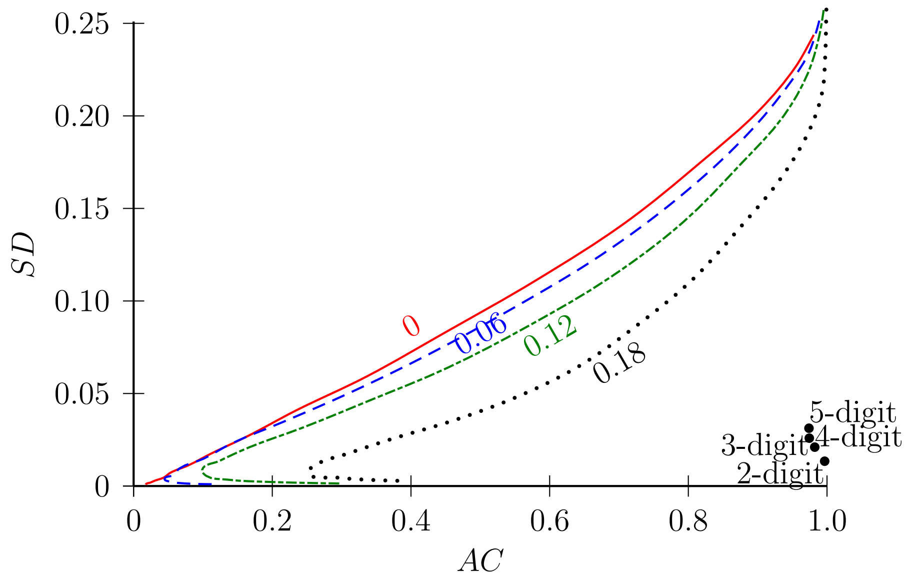

Figure 2: Variation in quit rate. AC is the autocorrelation of earnings and SD is the standard deviation of the growth rate of earnings. The solid red line shows the correlation between $w_{j,t}$ and $w_{j,t-1}$ and the standard deviation of $w_{j,t} - w_{j,t-1}$ consistent with $\alpha = 3.2$, $\theta = 3$, $q = 0$, and a 5.5% unemployment rate with no rest unemployment. The dashed blue line shows the combinations consistent with $q = 0.06$; the dash-dotted green line shows $q = 0.12$; and the dotted black line shows $q = 0.18$ search unemployment. The four larger dots show the empirical averages at the two- to five-digit NAICS levels.

we view the values of the elasticity needed to bring the model in line with the data to be empirical implausible at any level of disaggregation. Alternatively, we can reduce the amount of search unemployment that we ask the model to generate. This alone does not seem to be a promising resolution: the remaining three lines in Figure 1 show that even reducing the search unemployment rate to 0.5% reduces but does not eliminate the gap between the model and data.

Allowing for exogenous quits also improves the fit of the model to data, but again only marginally. Keep the same values of the other parameters as in the baseline, $\alpha = 3.2$, $\theta = 3$, and a 5.5% search unemployment rate. Figure 2 shows that by raising the exogenous quit rate to 0.18, we again close part of the gap between model and data. However, with these parameters, exogenous quits account for 98 percent of all unemployment. In any case, we find a model where exogenous parameters govern both unemployment incidence (q) and duration (α) to be uninteresting.

5.2 Reintroducing Rest Unemployment

Reintroducing rest unemployment improves the model's fit for several reasons: it creates another source of unemployment; it reduces the standard deviation of wages by creating an interval $[\underline{\omega}, \hat{\omega}]$ where wages are constant from year-to-year; and it raises the autocorrelation of wages because of the persistence generated by labor markets that spend time in this interval. The last two forces imply that the presence of rest unemployment may permit more search unemployment without generating unrealistically low autocorrelations or high standard deviations of growth rates of average weekly earnings.

To be concrete, again set $\alpha = 3.2$, $\theta = 3$, and $q = 0$, and choose σ, $\bar{\omega} - \underline{\omega}$, and $\hat{\omega} - \underline{\omega}$ to generate a 1.3 percent search unemployment rate and 4.2 percent rest unemployment rate. This breakdown between search and rest unemployment is consistent with evidence in Murphy and Topel (1987) from the March Current Population Survey and in Loungani and Rogerson (1989) from the Panel Study of Income Dynamics (PSID). The two papers classify switchers and stayers in a similar way, based on whether the workers switched two-digit industry, and find that workers who switch two-digit industries account for about a quarter of all unemployment spells, while stayers account for the remaining three-quarters. According to our model, all stayers experienced a spell of rest unemployment, while switchers' unemployment spell ended in search.[13]

The solid red line in Figure 3—which has a more compressed scale than the preceding figures—shows that higher values of σ allow for a higher autocorrelation AC without substantially increasing the standard deviation SD; indeed, in the limit, increases in σ raise the fraction of industries with rest unemployment by enough to drive AC to 1 and SD to 0. Although a gap between model and data remains, the figure shows that this can be reduced by introducing exogenous quits. Still fixing $\alpha = 3.2$ and $\theta = 3$, let $q = 0.04$, $\sigma = 0.12$, $\bar{\omega} - \underline{\omega} = 1.61$, and $\hat{\omega} - \underline{\omega} = 0.55$. This generates the target search and rest unemployment rates and matches the autocorrelation of wages (AC) at the industry level.[14] The standard deviation of wage growth (SD) is still higher in the model (0.046) than the data (0.031 at the five-digit level), as can be seen in Figure 3. We can close this gap either by a moderate increase in the elasticity of substitution, to approximately $\theta = 4.5$, or by a further reduction in the target for the search unemployment rate matched by an increase in the target for the rest unemployment rate.

[13]In the model without rest unemployment, we chose α to match the mean duration of unemployment. In Section 5.4, we compute the mean duration of unemployment in the full model and show that it is no longer equal to $1/\alpha$. Still, we keep α fixed at 3.2 for two reasons: it simplifies the comparison of the models with and without rest unemployment; and Loungani and Rogerson (1989) find that switchers account for about a a quarter of all unemployment spells and a third of all weeks of unemployment, which implies that the duration of unemployment spell for a switcher is only slightly longer than for the average unemployment spell.

[14]The kurtosis of wage growth is 3.4, slightly higher than the values with only search unemployment and the same AC (see footnote 12). Rest unemployment raises the kurtosis by effectively creating two sets of industries, some with instantaneous standard deviation of wage growth σ and some—those with rest unemployment—with zero standard deviation.

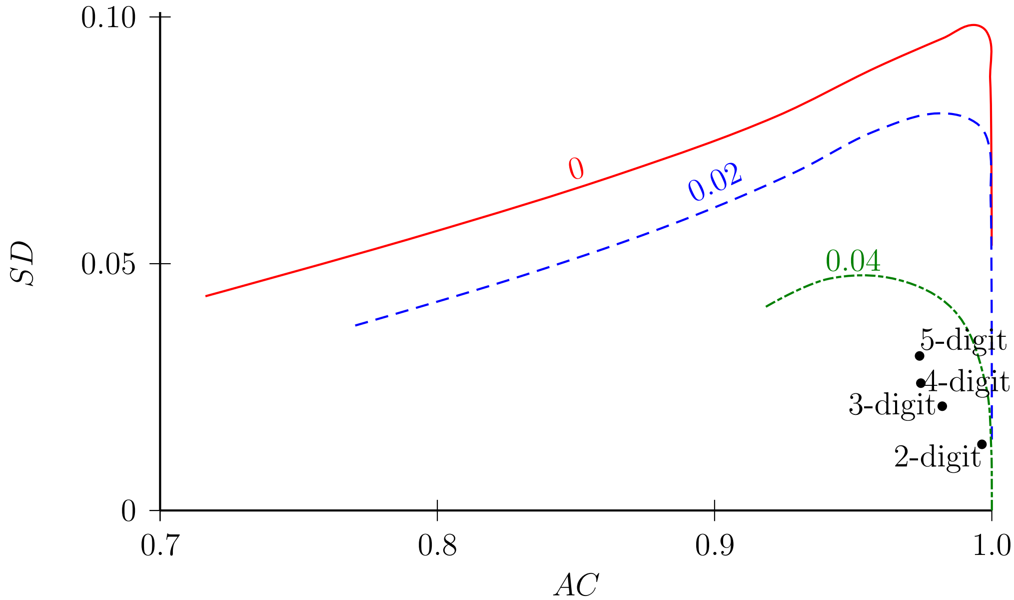

Figure 3: Variation in quit rate with rest unemployment. AC is the autocorrelation of earnings and SD is the standard deviation of the growth rate of earnings. The solid red line shows the correlation between $w_{j,t}$ and $w_{j,t-1}$ and the standard deviation of $w_{j,t} - w_{j,t-1}$ consistent with $\alpha = 3.2$, $\theta = 3$, $q = 0$, a 1.3% search unemployment rate, and a 4.2% rest unemployment rate. The dashed blue line shows the combinations consistent with $q = 0.02$; and the dash-dotted green line shows $q = 0.04$. The four larger dots show the empirical averages at the two- to five-digit NAICS levels.

We can back out the structural parameters consistent with this search and rest unemployment rate. To be consistent with balanced growth, we assume $u(\cdot) = \log(\cdot)$. Still thinking of a unit of a time as a year, we set $\rho = 0.05$. From Proposition 4, the search and rest unemployment rates, $\frac{U_r}{U_s+L} = 0.042$ and $\frac{U_s}{U_s+L} = 0.013$, then depend only on ratios of the three leisure values, which pins down $b_r/b_i = 0.97$ and $\kappa \equiv \frac{b_i - b_s}{\alpha b_i} = 12.3$.

These numbers reveal two regularities. First, in order to generate rest unemployment, b_r/b_i must be close to 1. Given all the other structural parameters, we require $b_r/b_i > 0.76$ to have any rest unemployment (see Proposition 2). This suggests that, while the rest unemployed must pay some cost to remain in contact with their labor market, the cost is small. Put differently, rest unemployment and inactivity may look quite similar to an outsider who observes individuals' time use, even though the rest unemployed may be much more likely to return to work.

Second, to generate a strong autocorrelation in wages, we need the search cost κ to be large, here equivalent to 12.3 years of inactivity.[15] Indeed since $\kappa > 1/\alpha$, the leisure value of

[15]This is mainly due to the high autocorrelation in wages, not rest unemployment. If we calibrate the model to match a 5.5% search unemployment rate with no rest unemployment, $\theta = 3$, $\alpha = 3.2$, no exogenous quits, and $AC = 0.97$ (so $\sigma = 0.55$, $\bar{\omega} - \underline{\omega} = 2.44$, and $SD = 0.24$), the required search cost is $\kappa = 9.4$. Calibrating the model with exogenous quits, it is higher still.

rest unemployment must actually be negative. A strong autocorrelation in wages requires a large region of inaction $\bar{\omega} - \underline{\omega}$, but this implies that the wage in the most productive labor markets is much higher than the wage in less productive markets. In order for workers to be willing to endure such an unproductive market, the cost of moving must be large. This is essentially the contrapositive of Hornstein, Krusell, and Violante's (2006) finding that when search costs are small, search models cannot generate much wage dispersion. Introducing other mobility costs, such as market-specific human capital, may alleviate this issue.

5.3 Job Creation and Destruction

We can use the model to measure gross job creation and job destruction rate at the industry level. Following Davis, Haltiwanger, and Schuh's (1996) analysis of firms, we define the number of jobs destroyed in a labor market between t_0 and t_1 as the decrease in the number of employed workers in that labor market between those dates, or zero if the number of employed workers increased. Job creation is defined symmetrically. The gross job destruction (creation) rate is then defined as the total number of jobs destroyed (created) across all labor markets divided by employment E.

The job creation and destruction rates are easily computed numerically. Consider a labor market that has a log full-employment wage ω at t_0. Using Monte Carlo, we find the job creation and destruction rates at t_1; these rates depend on ω but are independent of the number of workers in the market. We then take a weighted average of job creation and destruction rates, weighting by the fraction of employed workers at each value of ω, $e^{\theta \min\{\omega - \hat{\omega}, 0\}} f(\omega)$, where $e^{\theta \min\{\omega - \hat{\omega}, 0\}}$ accounts for rest unemployment in markets with $\omega < \hat{\omega}$.

When there is no rest unemployment, jobs are destroyed only to keep ω regulated above $\underline{\omega}$. The instantaneous job destruction rate is then given by N_s in equation (35). Moreover, once a job is destroyed, it is only recreated if the market experiences a series of shocks that brings ω back to $\bar{\omega}$. Over annual frequencies, this probability is negligible if $\frac{\bar{\omega} - \underline{\omega}}{\sigma}$ is large enough to give a plausible autocorrelation of earnings. Thus we would expect the job destruction rate over a time horizon t to be roughly equal to $N_s t$. Unreported simulations support this intuition.

In the full model, jobs are also destroyed when workers become rest unemployed, which is easily reversed. This makes the job destruction rate a concave function of the amount of elapsed time. The red line shows job destruction (and job creation) in the calibrated model with search and rest unemployment; the concavity is clearly visible.

We can compare this finding with data on job creation and job destruction at the industry level. We use the Current Employment Statistics measure of monthly employment for 387 five-digit NAICS industries from 1990 to 2006 and adjust the data to eliminate multiplicative

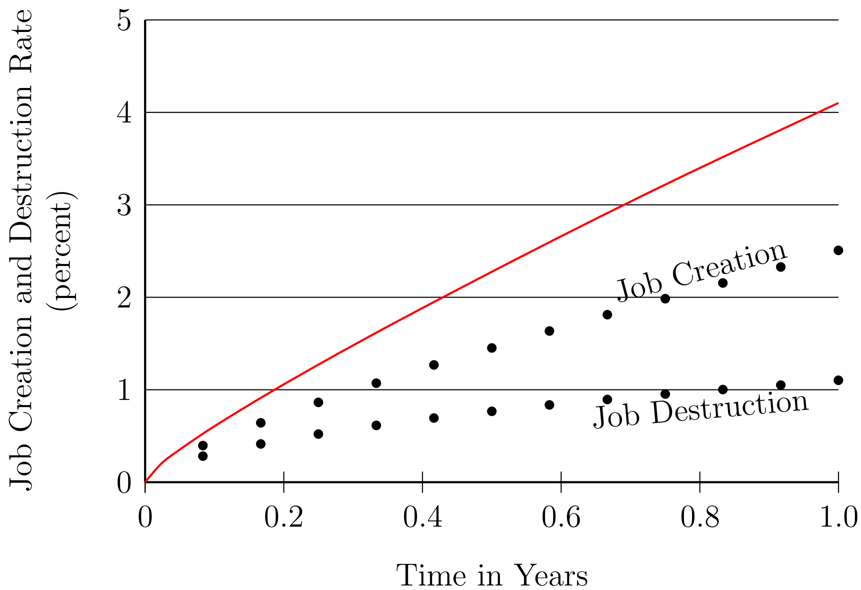

Figure 4: The solid red line shows the job creation and destruction rate from the model. The parameter values are in the text. The dots show show job creation and destruction for five-digit industries at different frequencies.

seasonal factors.[16] Figure 4 shows a clear concave pattern in these measures as well. In the data, the ratio of job destruction at annual and quarterly frequencies is 2.1. For job creation, the ratio is somewhat higher, 2.9. According to the calibrated model, the ratio of job creation or destruction at annual and quarterly frequencies should be 3.2.

It is worth noting that the model significantly overstates the incidence of job creation and destruction at the industry level. It is unclear how to reconcile model and data along this dimension. For example, reducing the elasticity of substitution to $\theta = 2$ reduces the annual creation and destruction rates only marginally below 4 percent. One possibility is lowering the target for search unemployment and raising the target for rest unemployment, keeping the total unemployment rate constant. Since rest unemployment is rapidly reversed, it contributes to unemployment incidence but has little effect on longer-term measures of creation and destruction. For example, set the target for the search unemployment rate to 0.7 percent and for the rest unemployment rate to 4.8 percent. Also reduce the quit rate to $q = 0.02$, since higher values are inconsistent with such a low search unemployment rate. Then the annual job creation and destruction rates are about 2.6 percent, with little effect on the curvature in Figure 4.

[16]The data show net employment growth, which is easily introduced to the model by making households increase in size over time. We find the existence of significant seasonal factors intriguing. It would be hard to get workers moving in and out of the labor force seasonally when doing so entails time-consuming search. It is easier instead to move between employment and rest unemployment. Incorporating seasonal fluctuations in the value of leisure into the model goes beyond the scope of this paper.

The finding that job creation and destruction increase less than linearly with time is consistent with other datasets. Davis, Haltiwanger, and Schuh (1996) report a quarterly job destruction rate for manufacturing establishments of 5.5% and an annual destruction rate of 10.3%.[17] Using the same data set, Schuh and Triest (2000) report 27.4% job destruction at a five year horizon. Faberman (2003) studies all private sector establishments in 53 Metropolitan Statistical Areas in five states. He reports 6.7% job destruction at quarterly frequencies and 11.4% at annual frequencies. Using microdata from the Job Openings and Labor Turnover Survey, Davis, Faberman, and Haltiwanger (2006) report a monthly job destruction rate of 1.5% and quarterly job destruction rate of 3.1%. From the Business Employment Dynamics survey of all private establishments, they report 7.6% quarterly job destruction and 13.7% annual. Each of these papers finds similar curvature for job creation. The strong reversibility of job creation and job destruction, consistent with firms having easy access to a pool of rest-unemployed workers, appears to be a robust empirical fact.

5.4 Hazard Rate of Exiting Unemployment

When there is no rest unemployment, the hazard of exiting unemployment is simply α. This section characterizes the hazard of exiting unemployment when there is rest unemployment, $\hat{\omega} > \underline{\omega}$. A worker who just switched between employment and rest unemployment is at the margin between the two states. A small shock will move him back. But the longer a worker remains unemployed, the more likely her labor market has suffered a series of adverse shocks, reducing the hazard of finding a job. The low hazard rate of exiting long-term unemployment may be important for understanding the coexistence of many workers who move easily between jobs and a relatively small number of workers who suffer extended unemployment spells (Juhn, Murphy, and Topel, 1991).

Observe that our model determines the number of workers who are employed and rest unemployed in each labor market j and the number of workers who are in search unemployment. It does not determine which of the workers in a labor market with rest unemployment is employed, nor who exits a market when ω hits $\underline{\omega}$, nor which workers are in search unemployment rather than inactivity. To discuss labor market flows, we need to determine these.

First, we impose that some workers are permanently inactive. This means that workers who leave their labor market become search unemployed. This can be interpreted as the outcome of heterogeneity in preference for leisure, in which case our analysis considers the

[17] Our model is inconsistent with annual job destruction less than half as big as quarterly job destruction. There is a significant seasonal component in Davis, Haltiwanger, and Schuh's (1996) data which may explain this finding.

limit as the heterogeneity vanishes.

To determine who works and who is unemployed, we suppose there is a small amount of industry-specific learning-by-doing. Then the least-skilled worker—the one with the shortest tenure in the industry—will always be the first to enter rest unemployment and the first to exit the labor market. Again, our analysis corresponds to the limit as learning-by-doing vanishes. The important implication of this assumption is that the hazard rate of exiting unemployment conditional on unemployment duration is the same for all workers.

Let $H(t, \Delta)$ be the probability that a worker who has had a (rest or search) unemployment spell of duration t will transit from unemployment to employment at least once in the next Δ units of time and let $h(t) = \lim_{\Delta \to 0} H(t, \Delta)/\Delta$ be the associated hazard rate. Given values for the thresholds $\underline{\omega}$, $\hat{\omega}$, and $\bar{\omega}$, we can compute the hazard rate $h(t)$ as the weighted average of two hazard rates:

$$h(t) = \hat{h}_r(t)\frac{u_r(t)}{u_r(t) + u_s(t)} + \alpha \frac{u_s(t)}{u_r(t) + u_s(t)}$$

where $\frac{u_r(t)}{u_r(t)+u_s(t)}$ is the probability that a worker with unemployment duration t is rest-unemployed. For a search-unemployed worker, spells end at rate α, independent of the duration of the spell. For a rest-unemployed worker, her spell ends when local labor market conditions improve enough for her to reenter employment. We let $\hat{h}_r(t)$ denote that hazard rate of this event. It is also useful to let $\underline{h}_r(t)$ denote the hazard of endogenously exiting rest unemployment for search unemployment. We prove in Appendix A.7 that

$$\hat{h}_r(t) = \frac{\sum_{n=1}^{\infty} n^2 e^{-t\psi_n}}{\sum_{n=1}^{\infty} \frac{n^2}{\psi_n} e^{-t\psi_n} \left(1 - (-1)^n e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}\right)} \tag{38}$$

$$\underline{h}_r(t) = \frac{-\sum_{n=1}^{\infty} n^2 e^{-t\psi_n} (-1)^n e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}}{\sum_{n=1}^{\infty} \frac{n^2}{\psi_n} e^{-t\psi_n} \left(1 - (-1)^n e^{-\frac{\mu(\hat{\omega} - \underline{\omega})}{\sigma^2}}\right)},$$

where

$$\psi_n \equiv \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{n^2 \pi^2 \sigma^2}{(\hat{\omega} - \underline{\omega})^2}\right).$$

These sums are easily calculated numerically.

We then compute the duration-contingent unemployment rates by solving a system of two ordinary differential equations with time-varying coefficients:

$$\dot{u}_r(t) = -u_r(t)(\delta + q + \underline{h}_r(t) + \hat{h}_r(t)) \text{ and } \dot{u}_s(t) = -u_s(t)\alpha + u_r(t)(\delta + q + \underline{h}_r(t)) \tag{39}$$

for all $t > 0$. The number of workers in rest unemployment falls as markets shut down and workers exogenously quit, as they exit the market for search unemployment, and as they reenter employment. In the first three events, they become search unemployed, while search unemployment falls at rate α as these workers find jobs. To solve these differential equations, we require two boundary conditions; however, to compute the share of rest unemployed in the unemployed population with duration t, $\frac{u_r(t)}{u_r(t)+u_s(t)}$, we need only a single boundary condition,

$$\frac{\int_0^\infty u_r(t)dt}{\int_0^\infty u_s(t)dt} = \frac{U_r}{U_s}, \tag{40}$$

where U_r and U_s are given in equations (34) and (36).

The hazard rate is particularly easy to characterize both at short and long durations. When t is small, we find that $\hat{h}_r(t) \approx \frac{1}{2t}$. Intuitively, consider a market with $\omega = \hat{\omega}$. After a short time interval—short enough that the variance of the Brownian motion dominates the drift—there is a $\frac{1}{2}$ probability that ω has increased, so the worker is reemployed, and a $\frac{1}{2}$ chance it has fallen. But a one-half probability over any horizon t implies a hazard rate $1/2t$.

When t is large, the first term of the partial sum in equation (38) dominates,

$$\lim_{t\to\infty} \hat{h}_r(t) = \frac{\psi_1}{1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}} \text{ and } \lim_{t\to\infty} \underline{h}_r(t) = \frac{\psi_1 e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}.$$

In addition, if $\alpha > \delta + q + \psi_1$,

$$\lim_{t\to\infty} \frac{u_r(t)}{u_s(t)} = \frac{(\alpha-\psi_1-\delta-q)\left(1+e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)}{\delta+q+(\delta+q+\psi_1)e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}},$$

while otherwise the limiting ratio is zero. Together this implies $\lim_{t\to\infty} h(t) = \min\{\alpha, \psi_1 + \delta + q\}$, a function only of the slower exit rate. In our baseline calibration, $\alpha = 3.2$, $\delta = 0$, $q = 0.04$, and $\psi_1 = 0.24$, so the exit rate from long-term unemployment is governed by the behavior of the rest unemployed. The efficiency of search affects the hazard of exiting long-term unemployment only indirectly, through its influence on the distance between the rest unemployment boundaries $\hat{\omega} - \bar{\omega}$.

Figure 5 shows the annual hazard rate of finding a job in our baseline calibration, including a 4.2 percent rest unemployment rate and 1.3 percent search unemployment rate. The overall hazard rate roughly mimics the behavior of $\hat{h}_r(t)$, especially at short unemployment durations, when most unemployed workers are in rest unemployment. Since the rest unemployed find jobs so quickly at the start of an unemployment spell, the share of searchers among the unemployed grows rapidly (Figure 6), peaking at about 54 percent of unemployment after two months duration. After this point, however, the hazard of exiting rest unemployment

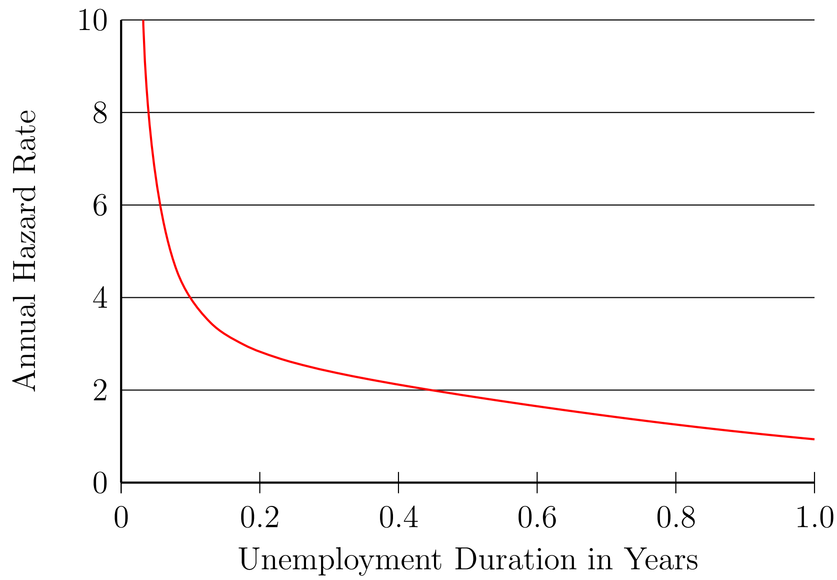

Figure 5: Hazard rate of finding a job as a function of unemployment duration. The parameter values are in the text.

falls below the hazard of exiting search unemployment and so the share of searchers starts to decline, asymptoting to just 4 percent of unemployment at very long durations.

Our finding of a constant hazard rate for workers in search unemployment and a decreasing hazard rate for workers in rest unemployment is qualitatively consistent with Katz and Meyer (1990) and Starr-McCluer (1993). Katz and Meyer (1990) show that the empirical decline in the job finding hazard rate is concentrated among workers on temporary layoff. Moreover, they find that workers who expect to be recalled to a past employer and are not—in the parlance of our model, workers who end a spell of rest unemployment by searching for a new labor market, at hazard $\underline{h}_r(t) + \delta + q$—experience longer unemployment duration than observationally equivalent workers who immediately entered search unemployment. In our model, this last group would correspond to workers experiencing a δ or q shock. Starr-McCluer (1993) finds that the hazard of exiting unemployment is decreasing for workers who move to a job that is similar to their previous one (rest unemployed) while it is actually increasing for workers who move to a different type of job (search unemployed).

In contrast to Murphy and Topel (1987) and Loungani and Rogerson (1989), Starr-McCluer (1993) finds relatively few stayers and finds that stayers experience relatively short unemployment spells. The behavior of hazard rates in our model can reconcile these findings. To understand why, note that in Starr-McCluer's (1993) data, many unemployment spells are censored. She finds that stayers account for half of all unemployment spells but a quarter of all weeks of unemployment, switchers account for a quarter of all unemployment spells

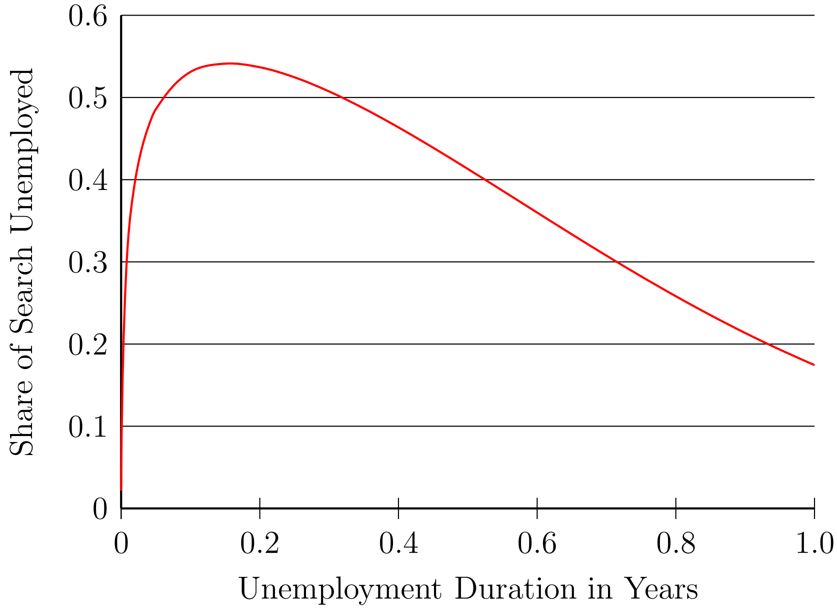

Figure 6: Fraction of searchers among the unemployed by duration, $\frac{u_s(t)}{u_r(t)+u_s(t)}$. The parameter values are in the text.

and a quarter of all weeks of unemployment, and censored observations account for a quarter of all unemployment spells and and a half of all weeks of unemployment. That is, stayers find jobs quickly while censored observations have the longest unemployment spells. Because of the relative duration of stayers' and switchers' spells, she guesses that most of the censored observations would ultimately be classified as switchers if the data were available. Our model's declining hazard rate for the rest-unemployed instead suggests that many of them would ultimately be classified as stayers. Using the classification suggested by our model, Starr-McCluer's (1993) data imply a breakdown between stayers and switchers and a pattern for their unemployment duration consistent with Loungani and Rogerson (1989).

Finally, in the baseline calibration of our model, the mean duration of an in progress unemployment spell is 2.5 years while the median duration is 1.1 years. Both of these numbers are far larger than in the data, reflecting infrequent episodes of very long term unemployment. Unfortunately, we believe it is not possible to parameterize the model to match simultaneously the level of unemployment, the division between search and rest, the behavior of wages, and the mean duration of unemployment.

6 Discussion

We finish by answering the two questions posed in the introduction: Can we distinguish between economies with different amounts of search and rest unemployment? And is the life of a worker in search unemployment substantially different from that of one in rest unemployment?

Our analysis in Section 5 provides a partial answer to the first question. Rest unemployment helps explain why wages are so persistent yet some workers cycle frequently between jobs. It helps explain why job creation and destruction are such concave functions of elapsed time. And it helps explain why the hazard of exiting unemployment declines with unemployment duration, leading to the coexistence of a large number of very short unemployment spells with a small number of workers who stay unemployed for years. In each case, this is a consequence of workers' ability to cycle costlessly between rest unemployment and work.

To answer the second question, we talk with a worker in rest unemployment. She might tell us that she routinely moves in and out of jobs, and perhaps is classified as working part time.[18] When we ask her why she does not search for a job in a different industry, she explains all the costs of doing so, including gathering the necessary information, retraining, and moving to a new city. Things really are not that bad and might get a lot better.

When we catch up with our worker a year later, she tells us that she has not worked since we last met. Again we ask her why she doesn't look for a better job. She tells us that she is thinking about moving, but in fact many of her former coworkers have already left town. If labor market conditions should improve, she could easily find a job paying a high wage. We ask her, with the benefit of hindsight, if she should have looked for a new job after our previous conversation. She says of course, but stresses that there is no way she could have known how badly things would turn out. When we try to find her after another year has passed, her neighbor tells us she has left town and is now working at a higher wage in a new job. In the end, he says, it took her less than four months to find the new job.

[18] From 1994 to 2007, the U.S. unemployment rate averaged 5.1 percent. Including discouraged workers and other marginally attached workers raises this to 6.1 percent, while also adding those who work "part time for economic reasons" raises the average to 8.9 percent. See `http://www.bls.gov/webapps/legacy/cpsatab12.htm`.

A Appendix

A.1 Density of Productivity x

Consider a labor market with initial productivity x_0. Let $f_x(\tilde{x}; x_0)$ denote the steady state density of log productivity, $\tilde{x} \equiv \log x$, across all such labor markets. This solves a Kolmogorov forward equation:

$$\delta f_x(\tilde{x}; x_0) = -\mu_x f_x'(\tilde{x}; x_0) + \frac{\sigma_x^2}{2} f_x''(\tilde{x}; x_0)$$

at all $\tilde{x} \neq \tilde{x}_0 \equiv \log x_0$. The solution to this equation takes the form

$$f_x(\tilde{x}; x_0) = \begin{cases} D_1^1(x_0) e^{\tilde{\eta}_1 \tilde{x}} + D_2^1(x_0) e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} < \log x_0 \\ D_1^2(x_0) e^{\tilde{\eta}_1 \tilde{x}} + D_2^2(x_0) e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} > \log x_0, \end{cases}$$

where $\tilde{\eta}_1 < 0 < \tilde{\eta}_2$ are the two real roots of the characteristic equation

$$\delta = -\mu_x \tilde{\eta} + \frac{\sigma_x^2}{2} \tilde{\eta}^2. \tag{41}$$

For this to be a well-defined density, integrating to 1 on $(-\infty, \infty)$, we require that $D_1^1(x_0) = D_2^2(x_0) = 0$. To pin down the remaining constants, we use two more conditions: the density is continuous at $\tilde{x} = \log x_0$; and it integrates to 1. Imposing these boundary conditions delivers

$$f_x(\tilde{x}; x_0) = \begin{cases} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\tilde{\eta}_1 - \tilde{\eta}_2} e^{\tilde{\eta}_2 (\tilde{x} - \log x_0)} & \text{if } \tilde{x} < \log x_0 \\ \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\tilde{\eta}_1 - \tilde{\eta}_2} e^{\tilde{\eta}_1 (\tilde{x} - \log x_0)} & \text{if } \tilde{x} > \log x_0. \end{cases} \tag{42}$$

With this notation, we can rewrite equation (6) as

$$w = A \left(\int_0^\infty \int_{-\infty}^\infty e^{(\theta - 1)\tilde{x}} f_x(\tilde{x}; x_0) d\tilde{x} dF(x_0) \right)^{\frac{1}{\theta - 1}}. \tag{43}$$

The interior integral converges if $\tilde{\eta}_1 + \theta - 1 < 0 < \tilde{\eta}_2 + \theta - 1$. The definition of $\tilde{\eta}_i$ in equation (41) implies these inequalities are equivalent to condition (8). With this restriction, equation (43) reduces to equation (7).

A.2 Proof of Lemma 1

Stokey (2006) proves in Proposition 10.4 that for all $\omega_0 \in [\underline{\omega}, \bar{\omega}]$,

$$\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \begin{cases} \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\bar{\omega}} - \zeta_1 e^{\zeta_1\bar{\omega}+\zeta_2\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \underline{\omega} \le \omega < \omega_0 \\[2ex] \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\underline{\omega}} - \zeta_1 e^{\zeta_1\underline{\omega}+\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\bar{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\bar{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \omega_0 \le \omega \le \bar{\omega}, \end{cases} \tag{44}$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation

$$\rho + \lambda = \mu\zeta + \frac{\sigma^2}{2}\zeta^2. \tag{45}$$

For $\omega_0 < \underline{\omega}$, $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \Pi_\omega(\omega;\underline{\omega};\underline{\omega},\bar{\omega})$ and for $\omega_0 > \bar{\omega}$, $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \Pi_\omega(\omega;\bar{\omega};\underline{\omega},\bar{\omega})$. That v is continuous follows immediately from equations (18) and (44). In particular, the latter equation defines Π_ω as a continuous function.

We next prove that the distribution $\Pi(\cdot;\omega_0;\underline{\omega},\bar{\omega})$ is increasing in each of ω_0, $\underline{\omega}$, and $\bar{\omega}$ in the sense of first order stochastic dominance. This follows from differentiating equation (44) with respect to each variable and using simple algebra. One can verify that an increase in $\underline{\omega}$ strictly increases $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for all $\omega \in (\underline{\omega},\bar{\omega})$. This therefore strictly reduces $\Pi(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for $\omega \in (\underline{\omega},\bar{\omega})$. Similarly, an increase in $\bar{\omega}$ strictly reduces $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for all $\omega \in (\underline{\omega},\bar{\omega})$, which also strictly reduces $\Pi(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for $\omega \in (\underline{\omega},\bar{\omega})$. Finally, an increase in ω_0 when $\omega_0 \in (\underline{\omega},\bar{\omega})$ reduces $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for $\omega \in (\underline{\omega},\omega_0)$ and raises it for $\omega \in (\omega_0,\bar{\omega})$. Once again, this implies a stochastic dominating shift in Π.

Since the return function $\max\{b_r, e^\omega\} + \lambda\underline{v}$ is nondecreasing in ω, weak monotonicity of v in each argument follows immediately from equation (18). In addition, the return function is strictly increasing when $\omega > \log b_r$, and so we obtain strict monotonicity when the support of the integral includes some $\omega > \log b_r$, i.e. when $\bar{\omega} > \log b_r$.

A.3 Proof of Proposition 1

We start by proving the result when $b_r < b_i$ and defer $b_r \ge b_i$ until the end.

First, define $\bar{\omega}^*$ to solve $v(\bar{\omega}^*;\bar{\omega}^*,\bar{\omega}^*) = \bar{v}$. When ω is regulated at the point $\bar{\omega}^*$, it is trivial to solve equation (16) to obtain

$$\frac{e^{\bar{\omega}^*} + \lambda\underline{v}}{\rho+\lambda} = \bar{v}.$$

This point is depicted along the 45° line in Figure 7. Lemma 1 ensures v is continuous and strictly increasing in its first three arguments. Moreover, for any $\underline{\omega} < \bar{\omega}^*$, we can

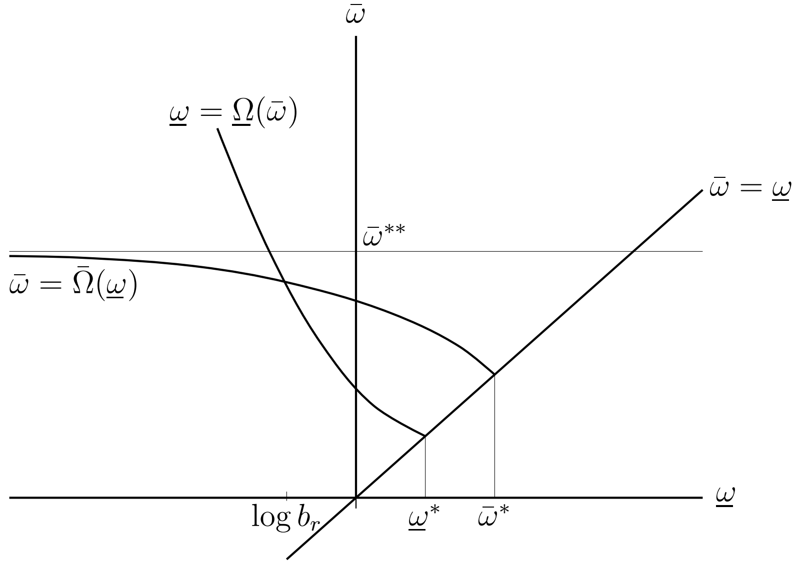

Figure 7: Illustration of the proof of Proposition 1 when $b_r < b_i$.

make $v(\bar{\omega}; \underline{\omega}, \bar{\omega})$ unboundedly large by increasing $\bar{\omega}$, while we can make it smaller than $\bar{v}$ by setting $\bar{\omega} = \bar{\omega}^*$. Then by the intermediate value theorem, for any $\underline{\omega} < \bar{\omega}^*$, there exists a $\bar{\Omega}(\underline{\omega}) > \bar{\omega}^*$ solving $v(\bar{\Omega}(\underline{\omega}); \underline{\omega}, \bar{\Omega}(\underline{\omega})) \equiv \bar{v}$. Continuity of v ensures $\bar{\Omega}$ is continuous while monotonicity of v ensures it is decreasing. In addition, because the period return function $s(\omega) \equiv \max\{b_r, e^{\omega}\} + \lambda \underline{v}$ is bounded below but not above, $\bar{\omega}^{**} \equiv \lim_{\underline{\omega} \to -\infty} \bar{\Omega}(\underline{\omega})$ is finite. Thus $\bar{\Omega}(\underline{\omega}) \in (\bar{\omega}^*, \bar{\omega}^{**})$ for any $\underline{\omega} < \bar{\omega}^*$. Figure 7 illustrates this function.

Similarly, define $\underline{\omega}^*$ to solve $v(\underline{\omega}^*; \underline{\omega}^*, \underline{\omega}^*) = \underline{v}$. Again solve equation (16) to obtain

$$\frac{e^{\underline{\omega}^*} + \lambda \underline{v}}{\rho + \lambda} = \underline{v}.$$

Since $\underline{v} < \bar{v}$, $\underline{\omega}^* < \bar{\omega}^*$, while equation (10) implies $\underline{\omega}^* = \log b_i$. For any $\bar{\omega} > \underline{\omega}^*$, we can make $v(\underline{\omega}; \underline{\omega}, \bar{\omega})$ approach $\frac{\rho b_r + \lambda b_i}{\rho(\rho+\lambda)} < \underline{v}$ by making $\underline{\omega}$ arbitrarily small, while we can make it bigger than $\underline{v}$ by setting $\underline{\omega} = \underline{\omega}^*$. Then by the intermediate value theorem, for any $\bar{\omega} > \underline{\omega}^*$, there exists a $\underline{\Omega}(\bar{\omega}) < \underline{\omega}^*$ solving $v(\underline{\Omega}(\bar{\omega}); \underline{\Omega}(\bar{\omega}), \bar{\omega}) \equiv \underline{v}$. Continuity of v ensures $\underline{\Omega}$ is continuous while monotonicity of v ensures it is decreasing. Thus $\underline{\Omega}(\bar{\omega}) < \underline{\omega}^*$ for any $\bar{\omega} > \underline{\omega}^*$.

An equilibrium is simply a fixed point $\bar{\omega}$ of the composition of the functions $\bar{\Omega} \circ \underline{\Omega}$. The preceding argument implies that this composition maps $[\bar{\omega}^*, \bar{\omega}^{**}]$ into itself and is continuous, and hence has a fixed point.

To prove the uniqueness of the fixed point when $b_r < b_i$, we prove that the composition of the two functions has a slope less than 1, i.e. $\bar{\Omega}'(\underline{\Omega}(\bar{\omega}))\underline{\Omega}'(\bar{\omega}) < 1$. To start, simple transformations of equation (44) imply that the cross partial derivatives of the discounted occupancy function satisfy

$$\Pi_{\omega_0,\bar{\omega}}(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \frac{\zeta_1\zeta_2 e^{(\zeta_1+\zeta_2)\bar{\omega}}\left(e^{-\zeta_1(\omega-\underline{\omega})} - e^{-\zeta_2(\omega-\underline{\omega})}\right)\left(e^{\zeta_1\underline{\omega}+\zeta_2\omega_0} - e^{\zeta_1\omega_0+\zeta_2\underline{\omega}}\right)}{\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)^2(\rho+\lambda)} < 0$$
$$\Pi_{\omega_0,\underline{\omega}}(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \frac{-\zeta_1\zeta_2 e^{(\zeta_1+\zeta_2)\underline{\omega}}\left(e^{\zeta_2(\bar{\omega}-\omega)} - e^{\zeta_1(\bar{\omega}-\omega)}\right)\left(e^{\zeta_1\omega_0+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\omega_0}\right)}{\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)^2(\rho+\lambda)} > 0,$$

where the inequalities use the fact that all the terms in parenthesis are positive. Then use integration-by-parts on equation (18) to write

$$v(\omega_0;\underline{\omega},\bar{\omega}) = \frac{s(\bar{\omega})}{\rho+\lambda} - \int_{\underline{\omega}}^{\bar{\omega}} s'(\omega)\Pi(\omega;\omega_0;\underline{\omega},\bar{\omega})d\omega,$$

where the period return function $s(\omega)$ is nondecreasing and strictly increasing for $\omega > \log b_r$, and Π is the discounted occupancy function. Taking the cross partial derivatives of this expression gives $v_{\omega_0,\bar{\omega}}(\omega_0;\underline{\omega},\bar{\omega}) > 0 > v_{\omega_0,\underline{\omega}}(\omega_0;\underline{\omega},\bar{\omega})$. In particular,

$$v_{\bar{\omega}}(\bar{\omega};\underline{\omega},\bar{\omega}) > v_{\bar{\omega}}(\underline{\omega};\underline{\omega},\bar{\omega}) \text{ and } v_{\underline{\omega}}(\underline{\omega};\underline{\omega},\bar{\omega}) > v_{\underline{\omega}}(\bar{\omega};\underline{\omega},\bar{\omega}).$$

Now since $v_{\omega_0}(\omega_0;\underline{\omega},\bar{\omega}) > 0$ from Lemma 1, these inequalities imply

$$\frac{v_{\underline{\omega}}(\bar{\omega};\underline{\omega},\bar{\omega})}{v_{\omega_0}(\bar{\omega};\underline{\omega},\bar{\omega}) + v_{\bar{\omega}}(\bar{\omega};\underline{\omega},\bar{\omega})}\frac{v_{\bar{\omega}}(\underline{\omega};\underline{\omega},\bar{\omega})}{v_{\omega_0}(\underline{\omega};\underline{\omega},\bar{\omega}) + v_{\underline{\omega}}(\underline{\omega};\underline{\omega},\bar{\omega})} < 1.$$

In particular, this is true when evaluated at any point $\{\underline{\omega},\bar{\omega}\}$ where $\bar{\omega} = \bar{\Omega}(\underline{\omega})$ and $\underline{\omega} = \underline{\Omega}(\bar{\omega})$. Implicit differentiation of the definitions of these functions shows that the first term in the above inequality is $-\bar{\Omega}'(\underline{\omega})$ and the second term is $-\underline{\Omega}'(\bar{\omega})$, which proves $\bar{\Omega}'(\underline{\Omega}(\bar{\omega}))\underline{\Omega}'(\bar{\omega}) < 1$.

Next we prove proportionality of the thresholds $e^{\bar{\omega}}$ and $e^{\underline{\omega}}$ to the leisure values b_r, b_i, and b_s. From equations (10) and (11), $\underline{v}$ and $\bar{v}$ are homogeneous of degree one in the three leisure values. The function $\max\{b_r, e^{\omega}\} + \lambda\underline{v}$ is also homogeneous of degree 1 in the leisure values and e^{ω}. By inspection of equation (44), Π_ω is unaffected by an equal absolute increase in each of its arguments. Then the integral in equation (18) is homogeneous of degree one in the b's and $e^{\bar{\omega}}$ and $e^{\underline{\omega}}$. The result follows from equation (17).

Finally we consider $b_r \geq b_i$, so the period return function $s(\omega) \geq b_i + \lambda\underline{v}$ for all ω. This implies $v(\omega;-\infty,\bar{\omega}) \geq \underline{v}$ for all ω and $\bar{\omega}$. Then an equilibrium is defined by $v(\bar{\omega};-\infty,\bar{\omega}) = \bar{v}$. As discussed above, the solution of this equation is $\bar{\omega}^{**} \in (\bar{\omega}^*, \infty)$.

A.4 Proof of Proposition 2

First, set $b_r = 0$. By Proposition 1, there exists a unique equilibrium characterized by thresholds $\underline{\omega}_0$ and $\bar{\omega}_0$. We now prove that $\bar{b}_r \equiv e^{\underline{\omega}_0}$. To see why, observe that for all $b_r \leq \bar{b}_r$, the equations characterizing equilibrium are unchanged from the case of $b_r = 0$ because $\log b_r \leq \underline{\omega}_0$, and hence the equilibrium is unchanged. Conversely, for all $b_r > \bar{b}_r$, the equations characterizing equilibrium necessarily are changed, and so the equilibrium must have $\log b_r > \underline{\omega}_{b_r}$.

Next we prove that $\bar{b}_r/b_i = B(\kappa, \rho+\lambda, \mu, \sigma)$. Again with $b_r = 0$, combine equations (17) and (18), noting the discounted local time function Π_ω integrates to $\frac{1}{\rho+\lambda}$, and use the definitions of $\underline{v}$ and $\bar{v}$ in equations (10) and (11):

$$\frac{b_i}{\rho+\lambda} = \int_{\underline{\omega}_0}^{\bar{\omega}_0} e^{\omega} \Pi_\omega(\omega; \underline{\omega}_0; \underline{\omega}_0, \bar{\omega}_0) d\omega \text{ and}$$
$$b_i \left(\frac{1}{\rho+\lambda} + \kappa \right) = \int_{\underline{\omega}_0}^{\bar{\omega}_0} e^{\omega} \Pi_\omega(\omega; \bar{\omega}_0; \underline{\omega}_0, \bar{\omega}_0) d\omega.$$

Since Π_ω is homogeneous of degree zero in the exponentials of its arguments (see equation 44), this implies $e^{\underline{\omega}_0}$ and $e^{\bar{\omega}_0}$ are homogeneous of degree 1 in b_i. Moreover, ζ_i depends on $\rho + \lambda$, μ, and σ by equation (45) and so the density Π_ω in equation (44) depends on these same parameters. It follows that the solution to these equations can depend only on these parameters and the parameters on the left hand side of the above equations. In particular, this proves

$$e^{\underline{\omega}_0} = b_i B(\kappa, \rho+\lambda, \mu, \sigma).$$

Since $\bar{b}_r = e^{\underline{\omega}_0}$, that establishes the dependence of $\bar{b}_r$ on this limited set of parameters.

Obviously B is positive-valued. By Proposition 1, $\underline{\omega}_0 < \log b_i$ and so $B < 1$. We finally prove it is decreasing in κ. Since κ affects $\underline{\omega}$ and $\bar{\omega}$ only through $\bar{v}$, to establish that B is decreasing in κ it suffices to show that the $\underline{\omega}$ and $\bar{\omega}$ that solve equations (17) and (18) is decreasing in $\bar{v}$. This follows because $\bar{\Omega}(\underline{\omega})$ is increasing in $\bar{v}$ and $\underline{\Omega}(\bar{\omega})$ is unaffected, where these functions are defined in the proof of Proposition 1. A decrease in $\bar{v}$ then reduces the composition $\bar{\Omega} \circ \underline{\Omega}$. Since the slope of this function is less than 1, it reduces the location of the fixed point $\bar{\omega}$ and hence raises $\underline{\omega} = \underline{\Omega}(\bar{\omega})$.

A.5 Derivation of the Density f

We use a discrete time, discrete state space model to obtain the Kolmogorov forward equations and boundary conditions for the density f. Divide $[\underline{\omega}, \bar{\omega}]$ into n intervals of length $\Delta\omega = (\bar{\omega} - \underline{\omega})/n$. Let the time period be $\Delta t = (\Delta\omega/\sigma)^2$ and assume that when $\omega < \bar{\omega}$, it decreases with probability $\frac{1}{2}(1+\Delta p)$ where $\Delta p = \mu \Delta \omega / \sigma^2$; when $\omega > \underline{\omega}$, it increases with probability $\frac{1}{2}(1-\Delta p)$; and otherwise ω stays constant. Note that for $\underline{\omega} < \omega(t) < \bar{\omega}$, the expected value of $\omega(t+\Delta t) - \omega(t)$ is $\mu \Delta t$ and the second moment is $\sigma^2 \Delta t$. As n goes to infinity, this converges to a regulated Brownian motion with drift μ and standard deviation σ.

Now let $f_n(\omega, t)$ denote the fraction of workers in markets with log full employment wage ω at time t for fixed n. With a slight abuse of notation, let $f_n(\omega)$ be the stationary distribution. We are interested in characterizing the density $f(\omega) = \lim_{n\to\infty} \frac{f_n(\omega)}{\Delta \omega}$. For $\omega \in [\underline{\omega} + \Delta\omega, \bar{\omega} - \Delta\omega]$, the dynamics of ω imply

$$f_n(\omega, t+\Delta t) = (1-\lambda \Delta t)\Big(\tfrac{1}{2}(1+\Delta p) f_n(\omega - \Delta\omega, t) + \tfrac{1}{2}(1-\Delta p) f_n(\omega + \Delta \omega, t)\Big). \tag{46}$$

In any period of length Δt, a fraction $\lambda \Delta t$ of workers leave due to market shut downs and idiosyncratic quits. Thus the workers in markets with ω at $t+\Delta t$ are a fraction $1-\lambda\Delta t$ of those who were in markets at $\omega - \Delta\omega$ at t and had a positive shock, plus the same fraction of those who were in markets at $\omega + \Delta\omega$ at t and had a negative shock. Now impose stationarity on f_n. Take a second order approximation to $f_n(\omega + \Delta\omega)$ and $f_n(\omega - \Delta\omega)$ around ω, substituting Δt and Δp by the expressions above:

$$f_n(\omega) = \left(1 - \lambda \frac{\Delta\omega^2}{\sigma^2}\right)\left(f_n(\omega) - \mu \frac{\Delta\omega^2}{\sigma^2} f_n'(\omega) + \frac{\Delta\omega^2}{2} f_n''(\omega)\right)$$
$$\Rightarrow \lambda f_n(\omega) = \left(1 - \lambda \frac{\Delta\omega^2}{\sigma^2}\right)\left(-\mu f_n'(\omega) + \frac{\sigma^2}{2} f_n''(\omega)\right)$$

Taking the limit as n converges to infinity, $\frac{f_n(\omega)}{\Delta\omega} \to f(\omega)$ solving equation (22).

Now consider the behavior of f_n at the lower threshold $\underline{\omega}$. A similar logic implies

$$f_n(\underline{\omega}, t+\Delta t) = (1-\lambda\Delta t)\tfrac{1}{2}(1-\Delta p)\big(f_n(\underline{\omega} + \Delta\omega, t) + f_n(\underline{\omega}, t)(1-\Delta \tilde{l})\big).$$

The workers at $\underline{\omega}$ at $t+\Delta t$ either were at $\underline{\omega}+\Delta\omega$ or at $\underline{\omega}$ at t; in both cases, they had a negative shock. Moreover, in the latter case, a fraction $\Delta\tilde{l} \equiv \theta\Delta\omega$ of the workers exited the market to keep ω above $\underline{\omega}$. Again impose stationarity but now take a first order approximation to $f_n(\underline{\omega} + \Delta\omega)$ at $\underline{\omega}$; the higher order terms will drop out later in any case. Replacing Δt, Δp, and $\Delta\tilde{l}$ with the expressions described above gives

$$f_n(\underline{\omega}) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(1 - \frac{\mu\Delta\omega}{\sigma^2}\right)\left(f_n(\underline{\omega})\left(1 - \frac{\theta\Delta\omega}{2}\right) + \frac{\Delta\omega}{2} f_n'(\underline{\omega})\right)$$

Again eliminating terms in $f_n(\underline{\omega})$ and taking the limit as $n \to \infty$, we obtain $\frac{f_n(\underline{\omega})}{\Delta\omega} \to f(\underline{\omega})$

solving equation (23).

Now consider the behavior of f_n at the upper threshold $\bar{\omega}$:

$$f_n(\bar{\omega}, t+\Delta t) = (1-\lambda\Delta t)\tfrac{1}{2}(1+\Delta p)\big(f_n(\bar{\omega}-\Delta\omega, t) + f_n(\bar{\omega}, t)(1+\Delta\tilde{l})\big) + \delta\Delta t L_0/L.$$

Compared to the equation at the lower threshold, the only significant change is the last term, which reflects the fact that on average a fraction L_0/L workers enter at the upper threshold when a new market is created. Recall also that markets are destroyed at rate δ per unit of time and hence $\delta\Delta t L_0/L$ is the fraction of workers added to the upper threshold due to newly created markets. Impose stationarity and take limits to get

$$f_n(\bar{\omega}) = \left(1-\lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(1+\frac{\mu\Delta\omega}{\sigma^2}\right)\left(f_n(\bar{\omega})\left(1+\frac{\theta\Delta\omega}{2}\right) - \frac{\Delta\omega}{2}f_n'(\bar{\omega})\right) + \delta\frac{\Delta\omega^2}{\sigma^2}L_0/L$$

Eliminate terms in $f_n(\bar{\omega})$ and take the limit as $n \to \infty$ to obtain $\frac{f_n(\bar{\omega})}{\Delta\omega} \to f(\bar{\omega})$ solving equation (24).

A.6 Exit Rates from Labor Markets

A worker exits her labor market if the log full-employment wage is $\underline{\omega}$ and the market is hit by an adverse shock, if the labor market closes, or if she quits. In the discrete time, discrete state space model, the first event hits a fraction $\frac{1}{2}\Delta\tilde{l}(1-\Delta p)$ of the workers who survive in a labor market with $\omega = \underline{\omega}$:

$$N_s\Delta t \equiv (1-\lambda\Delta t)\tfrac{1}{2}(1-\Delta p)\Delta\tilde{l} f_n(\underline{\omega})L + \lambda\Delta t L$$

Reexpress $\Delta\omega$, $\Delta\tilde{l}$, and Δp in terms of Δt, take the limit as $n \to \infty$, and use $\frac{f_n(\omega)}{\Delta\omega} \to f(\omega)$, to get equation (35).

A.7 Hazard Rates

Consider a Brownian motion with initial $\omega \in (\underline{\omega}, \hat{\omega})$. Let $\hat{G}(t;\cdot;\cdot)$ and $\underline{G}(t;\cdot;\cdot)$ denote the cumulative distribution function for the times until each of the barriers is hit, conditional on the initial value of ω:

$$\hat{G}(t;\hat{\omega},\underline{\omega};\omega) = \Pr\{t \le T_{\hat{\omega}}, T_{\hat{\omega}} < T_{\underline{\omega}}\,|\omega(0)=\omega\}$$
$$\underline{G}(t;\hat{\omega},\underline{\omega};\omega) = \Pr\{t \le T_{\underline{\omega}}, T_{\underline{\omega}} < T_{\hat{\omega}}\,|\omega(0)=\omega\},$$

with associated densities $\hat{g}$ and $\underline{g}$. Kolkiewicz (2002, pp. 17–18) proves

$$\hat{g}(t;\hat{\omega},\underline{\omega};\omega) = \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\hat{\omega}-\underline{\omega}}\right) e^{\frac{\mu(2(\hat{\omega}-\omega)-\mu t)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}}$$

$$\underline{g}(t;\hat{\omega},\underline{\omega};\omega) = \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\hat{\omega}-\omega)}{\hat{\omega}-\underline{\omega}}\right) e^{\frac{-\mu(2(\omega-\underline{\omega})-\mu t)}{2\sigma^2}-\frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}}.$$

The hazard rate of the first hitting time, conditional on a rest unemployment spell starting at time 0, i.e conditional on $\omega = \hat{\omega}$, is

$$\hat{h}_r(t) \equiv \lim_{\omega\uparrow\hat{\omega}} \frac{\hat{g}(t;\hat{\omega},\underline{\omega},\omega)}{1-\hat{G}(t;\hat{\omega},\underline{\omega},\omega)-\underline{G}(t;\hat{\omega},\underline{\omega},\omega)} \text{ and } \underline{h}_r(t) \equiv \lim_{\omega\uparrow\hat{\omega}} \frac{\underline{g}(t;\hat{\omega},\underline{\omega},\omega)}{1-\hat{G}(t;\hat{\omega},\underline{\omega},\omega)-\underline{G}(t;\hat{\omega},\underline{\omega},\omega)}.$$

Equation (38) follows using L'Hopital's rule.

References

Abel, Andrew B., and Ben S. Bernanke, 2001. *Macroeconomics*, Addison Wesley Longman, Boston, 4th edn.

Abel, Andrew B., and Janice C. Eberly, 1996. "Optimal Investment with Costly Reversibility." *The Review of Economic Studies*. 63 (4): 581–593.

Alvarez, Fernando, and Marcelo Veracierto, 1999. "Labor-Market Policies in an Equilibrium Search Model." *NBER Macroeconomics Annual*. 14: 265–304.

Alvarez, Fernando, and Marcelo Veracierto, 2001. "Severance Payments in an Economy with Frictions." *Journal of Monetary Economics*. 47 (3): 477–498.

Atkeson, Andrew, Aubhik Khan, and Lee Ohanian, 1996. "Are Data on Industry Evolution and Gross Job Turnover Relevant for Macroeconomics?." *Carnegie-Rochester Conference Series on Public Policy*. 44: 216–250.

Ball, Clifford A., and Antonio Roma, 1998. "Detecting Mean Reversion Within Reflecting Barriers: Application to the European Exchange Rate Mechanism." *Applied Mathematical Finance*. 5 (1): 1–15.

Bentolila, Samuel, and Giuseppe Bertola, 1990. "Firing Costs and Labour Demand: How Bad is Eurosclerosis?." *The Review of Economic Studies*. 47 (3): 381–402.

Broda, Christian, and David E. Weinstein, 2006. "Globalization and the Gains from Variety." *The Quarterly Journal of Economics*. 121 (2): 541–585.

Davis, Steven J., R. Jason Faberman, and John Haltiwanger, 2006. "The Flow Approach to Labor Markets: New Data Sources and Micro-Macro Links." NBER Working Paper 12167.

Davis, Steven J., John C. Haltiwanger, and Scott Schuh, 1996. *Job Creation and Job Destruction*, MIT Press, Cambridge, MA.

Faberman, Jason R., 2003. "Job Flows and Establishment Characteristics: Variations Across U.S. Metropolitan Areas." William Davidson Institute Working Paper 609.

Fleming, Wendell H., and Mete H. Soner, 1993. *Controlled Markov Processes and Viscosity Solutions*, Springer-Verelag, New York, NY.

Gouge, Randall, and Ian King, 1997. "A Competitive Theory of Employment Dynamics." *Review of Economic Studies*. 64 (1): 1–22.

Hornstein, Andreas, Per Krusell, and Giovanni L. Violante, 2006. “Frictional Wage Dispersion in Search Models: A Quantitative Assessment.” Federal Reserve Bank of Richmond Working Paper 2006-07.

Jovanovic, Boyan, 1987. “Work, Rest, and Search: Unemployment, Turnover, and the Cycle.” *Journal of Labor Economics*. 5 (2): 131–148.

Juhn, Chinhui, Kevin M. Murphy, and Robert H. Topel, 1991. “Why has the Natural Rate of Unemployment Increased over Time?.” *Brookings Papers on Economic Activity*. 1991 (2): 75–142.

Kambourov, Gueorgui, and Iourii Manovskii, 2007. “Occupational Specificity of Human Capital.” Mimeo.

Katz, Lawrence F., and Bruce D. Meyer, 1990. “Unemployment Insurance, Recall Expectations, and Unemployment Outcomes.” *The Quarterly Journal of Economics*. 105 (4): 973–1002.

King, Ian P., 1990. “A Natural Rate Model of Frictional and Long-term Unemployment.” *Canadian Journal of Economics*. 23 (3): 523–545.

Kolkiewicz, Adam W., 2002. “Pricing and Hedging More General Double-Barrier Options.” *Journal of Computational Finance*. 5 (3): 1–26.

Loungani, Prakash, and Richard Rogerson, 1989. “Cyclical Fluctuations and Sectoral Reallocation: Evidence from the PSID.” *Journal of Monetary Economics*. 23 (2): 259–273.

Lucas, Robert E. Jr., and Edward C. Prescott, 1974. “Equilibrium Search and Unemployment.” *Journal of Economic Theory*. 7: 188–209.

Murphy, Kevin M., and Robert Topel, 1987. “The Evolution of Unemployment in the United States: 1968–85.” in Stanley Fischer (ed.), *NBER Macroeconomics Annual*pp. 11–58. MIT Press, Cambridge, MA.

Schuh, Scott, and Robert K. Triest, 2000. “The Role of Firms in Job Creation and Destruction in U.S. Manufacturing.” *New England Economic Review*. pp. 29–44.

Starr-McCluer, Martha, 1993. “Cyclical Fluctuations and Sectoral Reallocation: A Reexamination.” *Journal of Monetary Economics*. 31 (3): 417–425.

Stokey, Nancy L., 2006. “Brownian Models in Economics.” book manuscript, November 10.

B Additional Appendixes not for Publication

B.1 Derivation Hamilton-Jacobi-Bellman

This appendix proves that if $v(\omega_0)$ is given by:

$$v(\omega_0) = \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\Pi_\omega(\omega;\omega_0)d\omega \tag{47}$$

for an arbitrary continuous function $R(\cdot)$ and where the local time function $\Pi_\omega(\cdot)$ is given as in Stokey (2006) Proposition 10.4:

$$\Pi_\omega(\omega;\omega_0) = \begin{cases} \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\bar{\omega}} - \zeta_1 e^{\zeta_1\bar{\omega}+\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \underline{\omega} \le \omega < \omega_0 \\[2ex] \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\underline{\omega}} - \zeta_1 e^{\zeta_1\underline{\omega}+\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\bar{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\bar{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \omega_0 \le \omega \le \bar{\omega}, \end{cases} \tag{48}$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation $\rho + \lambda = \mu\zeta + \frac{\sigma^2}{2}\zeta^2$, then

$$(\rho+\lambda)v(\omega_0) = R(\omega_0) + \mu v'(\omega_0) + \frac{\sigma^2}{2}v''(\omega_0).$$

Proof. Differentiating v with respect to ω_0 we get

$$\begin{aligned} v'(\omega_0) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\Pi_{\omega\omega_0}(\omega;\omega_0)d\omega \\ v''(\omega_0) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0)d\omega + R(\omega_0)\big(\lim_{\omega\uparrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0) - \lim_{\omega\downarrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0)\big) \end{aligned}$$

where we use that Π_ω is continuous but $\Pi_{\omega\omega_0}$ has a jump at $\omega = \omega_0$. Then

$$\begin{aligned} &(\rho+\lambda)v(\omega_0) - \mu v'(\omega_0) - \frac{\sigma^2}{2}v''(\omega_0) \\ &\qquad = \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\left((\rho+\lambda)\Pi_\omega(\omega;\omega_0) - \mu\Pi_{\omega\omega_0}(\omega;\omega_0) - \frac{\sigma^2}{2}\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0)\right)d\omega \\ &\qquad\qquad - \frac{\sigma^2}{2}R(\omega_0)\left(\lim_{\omega\uparrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0) - \lim_{\omega\downarrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0)\right). \end{aligned}$$

Using the functional form of Π_ω we have, for $\omega < \omega_0$:

$$\Pi_\omega(\omega;\omega_0) = e^{\zeta_1\omega_0}\tilde{h}_1(\omega) - e^{\zeta_2\omega_0}\tilde{h}_2(\omega)$$

where

$$\tilde{h}_1(\omega) = \frac{\zeta_2 e^{\zeta_2 \bar{\omega}} \left(\zeta_2 e^{\zeta_2 (\underline{\omega}-\omega)} - \zeta_1 e^{\zeta_1 (\underline{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1 \underline{\omega}+\zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega}+\zeta_2 \underline{\omega}}\right)}$$

$$\text{and } \tilde{h}_2(\omega) = \frac{\zeta_1 e^{\zeta_1 \bar{\omega}} \left(\zeta_2 e^{\zeta_2 (\underline{\omega}-\omega)} - \zeta_1 e^{\zeta_1 (\underline{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1 \underline{\omega}+\zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega}+\zeta_2 \underline{\omega}}\right)}.$$

Thus for all $\omega < \omega_0$:

$$\begin{aligned}
&(\rho+\lambda)\Pi_\omega(\omega;\omega_0) - \mu \Pi_{\omega\omega_0}(\omega;\omega_0) - \frac{\sigma^2}{2}\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0) \\
&\quad = \left[(\rho+\lambda) - \zeta_1 \mu - (\zeta_1)^2 \frac{\sigma^2}{2}\right] e^{\zeta_1 \omega_0} \tilde{h}_1(\omega) - \left[(\rho+\lambda) - \zeta_2 \mu - (\zeta_2)^2 \frac{\sigma^2}{2}\right] e^{\zeta_2 \omega_0} \tilde{h}_2(\omega) = 0
\end{aligned}$$

where the last equality follow from the definition of the roots ζ_i . Hence

$$\int_{\underline{\omega}}^{\omega_0} R(\omega) \left((\rho+\lambda)\Pi_\omega(\omega;\omega_0) - \mu \Pi_{\omega\omega_0}(\omega;\omega_0) - \frac{\sigma^2}{2}\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0)\right) d\omega = 0.$$

Using a symmetric calculation for $\omega > \omega_0$ we have:

$$\int_{\omega_0}^{\bar{\omega}} R(\omega) \left((\rho+\lambda)\Pi_\omega(\omega;\omega_0) - \mu \Pi_{\omega\omega_0}(\omega;\omega_0) - \frac{\sigma^2}{2}\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0)\right) d\omega = 0.$$

Next, differentiating $\Pi_\omega(\omega;\omega_0)$ when $\omega < \omega_0$ and when $\omega > \omega_0$ and let $\omega \to \omega_0$ from below and from above, tedious—but straightforward—algebra, gives:

$$\lim_{\omega \uparrow \omega_0} \Pi_{\omega\omega_0}(\omega;\omega_0) - \lim_{\omega \downarrow \omega_0} \Pi_{\omega\omega_0}(\omega;\omega_0) = -\frac{\zeta_1 \zeta_2}{\rho+\lambda}.$$

Then use the expression for the roots: $\zeta_1\zeta_2 = -(\rho+\lambda)/(\sigma^2/2)$. Putting this together proves the result. □

B.2 Market Social Planner's Problem

In this section we introduce a dynamic programming problem whose solution gives the equilibrium value for the thresholds $\underline{\omega}, \bar{\omega}$. This problem has the interpretation of a fictitious social planner located in a given market who maximizes net consumer surplus by deciding how many of the agents currently located in the market work and how many rest and whether to adjust the number of workers in the market. The equivalence of the solution of this problem with the equilibrium value of the labor market participant has the following implications. First, it establishes that our market decentralization is rich enough to attain an efficient equilibrium, despite the presence of search frictions. Second, it gives an alternative argument to establish the uniqueness of the equilibrium values for the thresholds $\underline{\omega}$ and $\bar{\omega}$. Third, it connects our results with the decision theoretic literature analyzing investment and labor demand model with costly reversibility.

The market planner maximizes the net surplus from the production of the final good in a market with current log productivity $\tilde{x}$ and l workers, taking as given aggregate consumption C and aggregate output Y. The choices for this planner are to increase the number of workers located in this market (hire), paying $\bar{v}$ to the households for each or them, or to decrease the number of workers located at the market number (fire), receiving a payment $\underline{v}$ for each. Increases and decreases are non-negative, and the prices associated with them have the dimension of an asset value, as opposed to a rental. We let $M(\tilde{x}, l)$ be the value function of this planner, hence:

$$M(\tilde{x}, l) = \max_{l_h, l_f} \mathbb{E}\left(\int_0^\infty e^{-(\rho+\delta)t} \big(\big(S(\tilde{x}(t), l(t)) + \underline{v} q l(t)\big) dt \, - \bar{v} dl_h(t) + \underline{v} dl_f(t) \big) \Big| \, \tilde{x}(0) = \tilde{x}, l(0) = l \right)$$

$$\text{subject to } dl(t) = -ql(t)dt + dl_h(t) - dl_f(t) \;\text{ and }\; d\tilde{x} = \mu_x dt + \sigma_x dz. \tag{49}$$

The $l_h(t)$ and $l_f(t)$ are increasing processes describing the cumulative amount of "hiring" and "firing" and hence $dl_h(t)$ and $dl_f(t)$ intuitively have the interpretation of hiring and firing during period t. The term $ql(t)dt$ represent the exogenous quits that happens in a period of length dt. The planner discounts at rate $\rho + \delta$, accounting both for the discount rate of households and for the rate at which her labor market disappears.

The function $S(\tilde{x}, l)$ denotes the return function of the market social planner per unit of time and is given by

$$S(\tilde{x}, l) = \max_{E \in [0,l]} u'(C) \int_0^{EAe^{\tilde{x}}} \left(\frac{Y}{y}\right)^{\frac{1}{\theta}} dy + b_r(l - E) + \delta l \underline{v}.$$

The first term is the consumer's surplus associated with the particular good, obtained by the output produced by E workers with log productivity $\tilde{x}$. The second term is value of the workers that the planner chooses to send back to the household, receiving $\underline{v}$ for each. The third term is the value of the "sale" of all the workers if the market shuts down. Setting $q = \delta = b_r = 0$ our problem is formally equivalent to Bentolila and Bertola's (1990) model of a firm deciding employment subject to a hiring and firing cost and to Abel and Eberly's (1996) model of optimal investment subject to costly irreversibility, i.e. a different buying and selling price for capital.

Using the envelope theorem, we find that the marginal value of an additional worker is:

$$S_l(\tilde{x}, l) = \max\left\{u'(C)\left(\frac{Y(Ae^{\tilde{x}})^{\theta-1}}{l}\right)^{\frac{1}{\theta}}, b_r\right\} + \delta\underline{v} \tag{50}$$
$$\equiv s\left(\frac{(\theta-1)(\tilde{x}+\log A)+\log Y - \log l}{\theta} + \log u'(C)\right)$$

where the function $s(\cdot)$ is given by $s(\omega) = \max\{e^{\omega}, b_r\} + \delta\underline{v}$ and is identical to the expression for the per-period value of a labor market participant in our equilibrium, except that $\delta\underline{v}$ is in place of $\lambda\underline{v}$. This is critical to the equivalence between the two problems.

To prove this equivalence, we write the market social planner's Hamilton-Jacobi-Bellman equation. For each $\tilde{x}$, there are two thresholds, $\underline{l}(\tilde{x})$ and $\bar{l}(\tilde{x})$ defining the range of inaction. The value function $M(\cdot)$ and thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ solve the Hamilton-Jacobi-Bellman equation if the following two conditions are met:

1. For all $\tilde{x}$, and $l \in (\underline{l}(\tilde{x}),\ \bar{l}(\tilde{x}))$ employment decays exponentially with the quits at rate q and hence the value function M solves

$$(\rho+\delta)M(\tilde{x}, l) = S(\tilde{x}, l) - qM_l(\tilde{x}, l) + \mu_x M_{\tilde{x}}(\tilde{x}, l) + \frac{\sigma_x^2}{2}M_{\tilde{x}\tilde{x}}(\tilde{x}, l). \tag{51}$$

2. For all $(\tilde{x}, l)$ outside the interior of the range of inaction,

$$(\rho+\delta)M(\tilde{x}, l) + qlM_l(\tilde{x}, l) - \mu_x M_{\tilde{x}}(\tilde{x}, l) - \frac{\sigma_x^2}{2}M_{\tilde{x}\tilde{x}}(\tilde{x}, l) \leq S(\tilde{x}, l), \tag{52}$$
$$\underline{v} = M_l(\tilde{x}, l)\ \forall l \geq \bar{l}(\tilde{x}), \text{ and } \bar{v} = M_l(\tilde{x}, l)\ \forall l \leq \underline{l}(\tilde{x}) \tag{53}$$

Equation (53) is also referred to as *smooth pasting*. Since $M(\tilde{x}, \cdot)$ is linear outside the range of inaction, a twice-continuously differentiable solution implies *super-contact*, or that for all $\tilde{x}$:

$$0 = M_{ll}(\tilde{x}, \bar{l}(\tilde{x})) = M_{ll}(\tilde{x}, \underline{l}(\tilde{x})). \tag{54}$$

According to Verification Theorem 4.1, Section VIII in Fleming and Soner (1993), a twice-continuously differentiable function $M(\tilde{x}, l)$ satisfying equations (51), (53), and (54) solves the market social planner's problem.

If M is sufficiently smooth, finding the optimal thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ can be stated as a boundary problem in terms of the function $M_l(\tilde{x}, l)$ and its derivatives. To see this, differentiate both sides of equation (51) with respect to l and replace S_l using equation (50):

$$(\rho+\delta+q)M_l(\tilde{x},l) = s\left(\frac{(\theta-1)(\tilde{x}+\log A)+\log Y-\log l}{\theta}+\log u'(C)\right) \\ - qlM_{ll}(\tilde{x},l)+\mu_x M_{\tilde{x}l}(\tilde{x},l)+\frac{\sigma_x^2}{2}M_{\tilde{x}\tilde{x}l}(\tilde{x},l). \quad (55)$$

If the required partial derivatives exist, any solution to the market social planner's problem must solve equations (53)–(55). Moreover, there is a clear relationship between the value function $v(\omega)$ in the decentralized problem and the marginal value of a worker M_l in the market social planner's problem:

LEMMA 2. Assume that $\theta \neq 1$ and that the functions $M_l(\cdot)$ and $v(\cdot)$ satisfy

$$M_l(\tilde{x},l) = v(\omega), \text{ where } \omega = \frac{\log Y+(\theta-1)(\log A+\tilde{x})-\log l}{\theta}+\log u'(C) \quad (56)$$

and that thresholds functions $\{\underline{l}(\cdot),\bar{l}(\cdot)\}$ and the thresholds levels $\{\underline{\omega},\bar{\omega}\}$ satisfy

$$\log \bar{l}(\tilde{x}) = \log Y+(\theta-1)(\tilde{x}+\log A)-\theta(\underline{\omega}-\log u'(C)) \quad (57)$$
$$\log \underline{l}(\tilde{x}) = \log Y+(\theta-1)(\tilde{x}+\log A)-\theta(\bar{\omega}-\log u'(C)). \quad (58)$$

Then, $M_l(\cdot)$ and $\{\underline{l}(\cdot),\bar{l}(\cdot)\}$ solve equations (53)–(55) for all $\tilde{x}$ and $l \in [\underline{l}(\tilde{x}),\bar{l}(\tilde{x})]$ if and only if $v(\cdot)$ and $\{\underline{\omega},\bar{\omega}\}$ solve equations (17).

Proof. Differentiate equation (56) with respect to $\tilde{x}$ and l to get

$$M_{l\tilde{x}}(\tilde{x},l) = v'(\omega)\,\frac{\theta-1}{\theta}\ ,\ M_{l\tilde{x}\tilde{x}}(\tilde{x},l) = v''(\omega)\left(\frac{\theta-1}{\theta}\right)^2 \text{ and } M_{ll}(\tilde{x},l) = -v'(\omega)\,\frac{1}{\theta}.$$

Recall that a solution of equation (17) is equivalent to a solution to equations (19), (20), and $v(\bar{\omega}) = \bar{v}$ and $v(\underline{\omega}) = \underline{v}$. The equivalence between equation (17) and equations (53)–(55) is immediate, recalling that $\lambda = q+\delta$, and the definitions of μ and σ. □

This lemma has important implications. First, it establishes, not surprisingly, that the equilibrium allocation is Pareto Optimal. Second, since the market social planner's problem is a maximization problem, the solution is easy to characterize. For instance, since the problem is convex, it has at most one solution and hence the equilibrium value of a labor market participant is uniquely defined, for given $u'(C)$ and Y. The fact that v is increasing is then equivalent to the concavity of $S(\tilde{x},\cdot)$. Finally, notice that Proposition 1 in Section

4.3 we establish existence and uniqueness of the solution to equation (17) only under mild conditions on $s(\cdot)$, i.e. that it was weakly increasing and bounded below. Proposition 1 can be used to extend the uniqueness and existence results of the literature of costly irreversible investment to a wider class of production functions. Currently the literature uses that the production function is of the form $x^{a_x} l^{a_l}$ for some constants a_x and a_l, with $0 < a_l < 1$, as in Abel and Eberly (1996). Proposition 1 shows that the only assumption required is that the production function be concave in l, and that the marginal productivity of the factor l can be written as a function of the ratio of the quantity of the input l to (a power of) the productivity shock x.