Estimation of Vector Error Correction Models with Mixed-Frequency Data

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Draft Date: October 2, 2008

ABSTRACT

We develop a method for directly modeling cointegrated multivariate time series that are observed in mixed frequencies. We regard lower-frequency data as regularly (or irregularly) missing and treat them with higher-frequency data by adopting a state-space model. This utilizes the structure of multivariate data as well as the available sample information more fully than the methods of transformation to a single frequency, and enables us to estimate parameters including cointegrating vectors and the missing observations of low-frequency data and to construct forecasts for future values. For the maximum likelihood estimation of the parameters in the model, we use an expectation maximization algorithm based on the state-space representation of the error correction model. The statistical efficiency of the developed method is investigated through a Monte Carlo study. We apply the method to a mixed-frequency data set that consists of the quarterly real gross domestic product and the monthly consumer price index.

JEL classification: C13, C22, C32.

Keywords: missing data, cointegration, Kalman filter, expectation maximization algorithm, smoothing

1. INTRODUCTION

Multivariate time series that arise in economics and business are often observed in mixed frequencies. For example, data available from the database of the Bureau of Economic Analysis, the Bureau of Labor Statistics, and the Bureau of the Census are often in mixed frequencies, mostly with quarterly, monthly or weekly sampling intervals. Although cointegration, which represents a long-run equilibrium among the components of nonstationary multivariate time series, has been one of the most extensively investigated research topics, especially, in economics and business during the past two decades since Engle and Granger (1987), studies of cointegration have been limited to the case where all the components of a multivariate series are observed at the same frequency.

Data observed in mixed frequencies are usually transformed to a single frequency by temporally aggregating higher-frequency data to lower frequencies, or by interpolating lower-frequency data to higher frequencies. However, temporal aggregation destroys sample information (Zadrozny 1990). Granger and Siklos (1995) examined the misinterpretation of the long-run component of a time series constructed by temporal aggregation. Marcellino (1999) theoretically derived the effects of temporal aggregation on cointegration such as the asymptotic invariance of cointegrating (CI) rank and vectors. However, he illustrated some possibilities of the loss of power of cointegration tests due to a decline in the number of available observations. Haug (2002) showed in Monte Carlo experiments with various cointegration tests and data generating processes (DGPs) that these power losses indeed occur and assessed their extents in samples of typical size used in empirical work.

For the case of interpolation of mixed-frequency data, Chow and Lin (1971) used the conventional regression approach, and Ghysels and Valkanov (2006) used the projection of low-frequency data on high-frequency data. Other researchers, for example, Bernanke *et al.* (1997), Cuche and Hess (2000), and Liu and Hall (2001), used the state-space framework suggested by Harvey and Pierse (1984) where interpolation is based on univariate regression. When the goal is to estimate a multivariate model for forecasting or other purposes, this kind of interpolation is at best an intermediate nuisance and at worst a source of distortion in the data to be used for estimation (Zadrozny 1990, p. 2).

Recently, Mariano and Murasawa (henceforth, MM) (2003, 2004) considered multivariate models for constructing a new index of economic indicators using mixed-frequency data, which overcome the drawbacks of a univariate approach and exploit the cross-frequency sample information. However, they used differenced data instead of levels data. This causes a loss of information on the long-run dynamics among the variables.

In this paper we develop a method for directly modeling the cointegrated multivariate time series with mixed-frequency data, which is based on the state-space representation of the error correction model (ECM). We use the state-space formulation for mixed-frequency data in our development by fully utilizing the structure of multivariate data as well as the available sample information. For the analysis, we exploit an expectation maximization (EM) algorithm. The method is applied to estimate the parameters and the missing observations of the low-frequency variables and to construct forecasts of their future values.

The structure of the paper is as follows. In Section 2, we describe preliminary concepts for mixed-frequency data and set up the state-space representation for the ECM of cointegration. In Section 3, we develop the procedures for estimating the parameters of the cointegrated model using the EM algorithm and discuss initial conditions for the procedures. In Section 4, we comment on smoothing, forecasting, logarithmic transformation, and models with different types of deterministic terms. In Section 5, we conduct Monte Carlo experiments for the investigation of the performance of the developed method. In Section 6, we consider a numerical example to illustrate the method and we conclude the paper in Section 7.

2. MODEL WITH MIXED-FREQUENCY DATA

In this section, we define mixed-frequency data and the ECM for multivariate cointegrated time series.

2.1 Mixed-frequency data

We define high-frequency variables as those observed at the shortest sampling interval and define low-frequency variables as those observed at longer sampling intervals, either as temporal aggregates or as skip samples of their high-frequency values. A variable is skip sampled when, for example, it is generated every month but is sampled every third month, say, in the last month of every quarter. We assume that the underlying data generating process of a multivariate time series of mixed-frequency data, composed of both the high and the low-frequency variables, operates at the highest frequency, as in Zadrozny (1990), Ghysels and Valkanov (2006), and MM

(2003, 2004) among others. All variables are assumed to be produced at the highest frequency, but some variables are not observed at the highest frequency. For example, we consider a bivariate time series of the consumer price index (CPI) observed monthly and the gross domestic product (GDP) observed quarterly. CPI is the high-frequency variable and GDP is the low-frequency variable. The highest frequency is monthly and GDP is in principle 'produced' monthly but is observed only quarterly. Variables like GDP observed as temporal aggregates are often called flows, while variables observed at the high frequency are often called stocks.

2.2 Error correction model and state-space representation

Let u_t be an *n*-dimensional vector autoregressive process of order *p*, $\mathrm{VAR}(p)$, which operates at the highest frequency with CI rank h, and consider the corresponding error correction form

$$\Delta u_t = ab' u_{t-1} + \sum_{j=1}^{p-1} \Psi_j \Delta u_{t-j} + \varepsilon_t , \tag{1}$$

where a and b are $n \times h$ matrices with $0 < h < n$ and $h = n - d$, Ψ_j, for $j = 1, \cdots, p-1$, are $n \times n$ matrices, and ε_t is an independent $N_n(0, \Omega)$ random vector. We assume that the elements of u_t are ordered such that the last $d = n - h$ elements are not cointegrated. This assumption permits the normalization $b = [I_h \;\; \beta_0']'$ for identification, as in Ahn and Reinsel (1990), where I_h denotes an $h \times h$ identity matrix and β_0 is a $d \times h$ matrix. The characteristic equation of model (1) has exactly d roots equal to one and all other roots are assumed to be outside the unit circle, so that u_t is cointegrated of order (1,1) (Engle and Granger 1987).

We reorder the elements of u_t to form $z_t = (z_{1t}', \, z_{2t}')'$ such that an $n_1 \times 1$ vector

z_{1t} corresponds to high-frequency variables and an $n_2 \times 1$ vector z_{2t} corresponds to low-frequency variables, where $n = n_1 + n_2$. For brevity, we assume that the low-frequency variables are observed as temporal aggregates because the method can be easily modified to accommodate the alternative case of skip-sampled data. The reordering implies $z_t = Vu_t$, where V is an $n \times n$ permutation matrix. Substituting u_t with $V^{-1}z_t$ and pre-multiplying both sides of equation (1) by V, we can rewrite model (1) as

$$\Delta z_t = \alpha\beta' z_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta z_{t-j} + e_t, \tag{2}$$

where $\alpha = Va$, $\beta = Vb$, $\Gamma_j = V\Psi_j V'$, $e_t = V\varepsilon_t$, $e_t \sim N(0, \Sigma)$, and $\Sigma = V\Omega V'$. Also, note that $V^{-1} = V'$. For convenience of estimation, we write $\beta = V_1 I_h + V_2 \beta_0$, where V_1 and V_2 are $n \times h$ and $n \times d$ matrices, such that $V = [V_1 \quad V_2]$. The VAR(p) representation of model (2) is

$$z_t = \sum_{j=1}^{p} \Phi_j z_{t-j} + e_t, \tag{3}$$

where $\Phi_1 = I_n + \alpha\beta' + \Gamma_1$, $\Phi_j = \Gamma_j - \Gamma_{j-1}$, for $j = 2, \cdots, p-1$, and $\Phi_p = -\Gamma_{p-1}$.

In practice, as mentioned above, the low-frequency variables, z_{2t}, are not observed directly at the highest frequency, but either as temporally-aggregated flows or as skip-sampled stocks. Flows can be expressed as $y_{2t} = \sum_{j=0}^{v} C_j z_{2t-j}$, where v denotes the maximum degree of aggregation and the C_j's are $n_2 \times n_2$ diagonal indicator matrices with zeros and ones on the principal diagonal. By adjusting the diagonal elements of C_j, Zadrozny (1990) suggested a way to treat a variable which is observed directly as a stock, but with a delay.

We now construct a state-space representation of model (3). Let the $s \times 1$ state vector $x_t = (z_t', \cdots, z_{t-r+1}')'$, where $r = \max(p, v+1)$ and $s = nr$ for $t = 1, \cdots, T$. Then, we define the state equation, as in Zadrozny (1990), by

$$x_t = Fx_{t-1} + Ge_t, \tag{4}$$

where the initial state, x_0, is assumed to be a normal random vector with mean vector λ and $s \times s$ covariance matrix Λ. Here, F and G denote $s \times s$ and $s \times n$ matrices defined by

$$F = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_r \\ I_n & O_n & \cdots & O_n \\ O_n & I_n & \cdots & O_n \\ \vdots & \vdots & \cdots & \vdots \\ O_n & O_n & \cdots & O_n \end{bmatrix}, \quad G = \begin{bmatrix} I_n \\ O_{(s-n)\times n} \end{bmatrix}, \quad \Phi_j = O_n \text{ if } j > p,$$

where O_n and $O_{m\times n}$ denote $n \times n$ and $m \times n$ zero matrices, respectively.

Next, assuming no observation errors, we define the measurement equation

$$y_t \equiv \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} x_t, \tag{5}$$

where

$$y_{1t} = z_{1t}, \quad H_1 \equiv \lfloor I_{n_1} \quad O_{n_1\times(s-n_1)} \rfloor,$$

$$H_2 \equiv \left[O_{n_2\times n_1} \quad C_0 \quad O_{n_2\times n_1} \quad C_1 \quad \cdots \quad O_{n_2\times n_1} \quad C_{r-1} \right], \text{ and } C_j = O_{n_2} \text{ if } j > v.$$

The matrix H_1 picks out the high-frequency variables from state vector x_t and the matrix H_2 picks out the low-frequency variables from the state vector as temporal aggregates. We introduce a new series, y_{2t}^+, which is observed only at lower frequencies, in order to deal with the missing observations in y_{2t}. As in Brockwell and Davis (1991), we fill in missing observations of y_{2t} with random vectors which are independent of y_t and are distributed independently of the parameters in model (2). Accordingly, we modify measurement equation (5) as

$$y_t^+ \equiv \begin{pmatrix} y_{1t} \\ y_{2t}^+ \end{pmatrix} = \begin{pmatrix} H_1 \\ H_{2t} \end{pmatrix} x_t + \begin{pmatrix} O_{n_1 \times n_2} \\ Q_{2t} \end{pmatrix} w_t, \tag{6}$$

$$H_{2t} = \begin{Bmatrix} H_2 & \text{if } y_{2t} \text{ is observable} \\ O_{n_2 \times s} & \text{otherwise} \end{Bmatrix}, \quad Q_{2t} = \begin{Bmatrix} O_{n_2} & \text{if } y_{2t} \text{ is observable} \\ I_{n_2} & \text{otherwise} \end{Bmatrix},$$

and w_t is an independent random vector distributed $N_{n_2}(0, I_{n_2})$. We also define $H_t = [H_1', H_{2t}']'$ and $Q_t = [O_{n_1 \times n_2}', Q_{2t}']'$, which will appear in equation (16) and in Appendix. In the implementation, because the realization of w_t is independent of y_t, setting $w_t = 0$ is the preferred simple choice (Brockwell and Davis 1991; MM 2003). Instead of using w_t, a selection matrix may be used for constructing a measurement equation in order to control the mixed-frequency data, as in Zadrozny (1990).

We note that when the low-frequency variables are stocks, with missing data attributable to skip-sampling, we redefine $r = \max(p, 2)$, and $H_2 \equiv \lfloor O_{n_2 \times n_1} \quad I_{n_2} \quad O_{n_2 \times (s-n)} \rfloor$. We need $r = 2$ in order to construct the state equation when the autoregressive order is one. Similar adjustments can be applied to more complicated cases, where the low-frequency variables are observed as both stocks and flows.

3. MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS

In this section, we consider maximum likelihood estimation (MLE) of parameters in error correction model (2) in the state-space form (4) and (6).

3.1 EM algorithm

Dempster *et al*. (1977), Shumway and Stoffer (1982), and Watson and Engle (1983) developed and illustrated the EM algorithm for estimating a model in state-space form, when some variables are partly or completely unobserved (latent).

Let $X_s = (x_t;\, 0 \le t \le s)$ and $Y_s^+ = (y_t^+;\, 1 \le t \le s)$ be information sets. In order to develop an EM algorithm for estimating the parameters of the state-space model in (4) and (6), we consider several transformations for forming the likelihood function with respect to the complete data X_T and Y_T^+. Let

$$\Gamma = \left[\alpha \;\; \Gamma_1 \;\; \cdots \;\; \Gamma_{p-1}\right], \;\; \Gamma^* = \left[\alpha V_1' \;\; \Gamma_1 \;\; \cdots \;\; \Gamma_{p-1}\right],$$

$$A = [I_n \quad -I_n \quad O_{n\times(s-2n)}]\text{, and } D = \left[V_2' \;\; O_{d\times(s-n)}\right],$$

where $Dx_{t-1} = V_2' z_{t-1}$. Define

$$B = \begin{bmatrix} \beta' & O_{h\times n} & O_{h\times n} & \cdots \\ I_n & -I_n & O_n & \cdots \\ O_n & I_n & -I_n & \cdots \\ O_n & O_n & I_n & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}, \;\; B^* = \begin{bmatrix} I_n & O_n & O_n & \cdots \\ I_n & -I_n & O_n & \cdots \\ O_n & I_n & -I_n & \cdots \\ O_n & O_n & I_n & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix},$$

where $Bx_{t-1} = (z_{t-1}'\beta,\, \Delta z_{t-1}',\, \cdots,\, \Delta z_{t-p+1}')'$, $B^* x_{t-1} = (z_{t-1}',\, \Delta z_{t-1}',\, \cdots,\, \Delta z_{t-p+1}')'$, and B and B^* are $\{h + (p-1)n\}\times s$ and $np \times s$ matrices. We consider two transformations for forming the likelihood function and estimating Γ, Σ, and β_0.

First,

$$Ax_t = \Delta z_t = \alpha\beta' z_{t-1} + \Gamma_1 \Delta z_{t-1} + \cdots + \Gamma_{p-1}\Delta z_{t-p+1} + e_t = \Gamma B x_{t-1} + e_t, \tag{7}$$

$$\begin{aligned} Ax_t &= \alpha\beta_0' V_2' z_{t-1} + \alpha V_1' z_{t-1} + \Gamma_1 \Delta z_{t-1} + \cdots + \Gamma_{p-1}\Delta z_{t-p+1} + e_t \\ &= \alpha\beta_0' D x_{t-1} + \Gamma^* B^* x_{t-1} + e_t \\ &= \left\{\alpha \otimes (Dx_{t-1})'\right\}\text{vec}(\beta_0) + \Gamma^* B^* x_{t-1} + e_t, \end{aligned} \tag{8}$$

because $\alpha\beta_0' D x_{t-1} = \text{vec}\{(\alpha\beta_0' D x_{t-1})'\}$ and $\text{vec}\{(Dx_{t-1})'\beta_0\alpha'\} = \left\{\alpha \otimes (Dx_{t-1})'\right\}\text{vec}(\beta_0)$, which are obtained using the vectorization rule $\text{vec}(ABC) = (C' \otimes A)\text{vec}(B)$, where

$\text{vec}(\cdot)$ vectorizes a matrix columnwise from left to right and the symbol $\otimes$ denotes the Kronecker product.

Then, we express the log-likelihood function as

$$\log L(\theta; X_T, Y_T^+) = -\frac{1}{2}\log|\Lambda| - \frac{1}{2}(x_0 - \lambda)'\Lambda^{-1}(x_0 - \lambda)$$
$$-\frac{T}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=1}^{T}(Ax_t - \Gamma B x_{t-1})'\Sigma^{-1}(Ax_t - \Gamma B x_{t-1}), \tag{9}$$

or as

$$\log L(\theta; X_T, Y_T^+) = -\frac{1}{2}\log|\Lambda| - \frac{1}{2}(x_0 - \lambda)'\Lambda^{-1}(x_0 - \lambda)$$
$$-\frac{T}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=1}^{T}[Ax_t - \Gamma^* B^* x_{t-1} - \{\alpha \otimes (Dx_{t-1})'\}\text{vec}(\beta_0)]' \times$$
$$\Sigma^{-1}[Ax_t - \Gamma^* B^* x_{t-1} - \{\alpha \otimes (Dx_{t-1})'\}\text{vec}(\beta_0)], \tag{10}$$

where terms that do not contain the parameters, $\theta \equiv (\text{vec}(\beta_0)'$, $\text{vec}(\Gamma)'$, $\text{vech}(\Sigma)')'$, are omitted and $\text{vech}(\cdot)$ vectorizes the lower triangular part of a matrix columnwise. Version (9) of the log-likelihood function is used to estimate the "stationary" parameters in Γ and Σ. Version (10) of the log-likelihood function is used to estimate the remaining "nonstationary" parameters in β_0. The distribution of w_t in the measurement equation does not have any effect on equations (9) and (10). Because the log likelihood function depends on the unobserved information, X_T, the EM algorithm is applicable, conditional on the observed information, Y_T^+. Specifically, we define the estimated parameters at iteration $l+1$ as the value of θ which maximizes

$$\text{Q}(\theta \mid \theta^{(l)}) = \text{E}_l\left\{\log L(\theta; X_T, Y_T^+) \mid Y_T^+\right\}, \tag{11}$$

where $\theta^{(l)}$ denotes estimated θ after l iterations and $E_l\{\cdot \mid Y_T^+\}$ denotes the conditional expectation with respect to a density containing $\theta^{(l)}$, given Y_T^+.

Calculating (11) constitutes the expectation step and maximizing (11) with respect to θ constitutes the maximization step. Using the derivatives of (11) with respect to θ which are in Appendix A of Seong *et al*. (2007), we obtain the following equations for updating $\theta^{(l+1)}$ at the end of *l* iterations,

$$\beta_0^{(l+1)} = \left[DM_{11}D'\right]^{-1}\left[D\left\{M_{10}A' - M_{11}(\Gamma^{*(l)}B^{*})'\right\}\Sigma^{(l)-1}\alpha^{(l)}\right]\left[\alpha'^{(l)}\Sigma^{(l)-1}\alpha^{(l)}\right]^{-1}, \tag{12}$$

$$\Gamma^{(l+1)} = \left(AM_{01}B'^{(l+1)}\right)\left(B^{(l+1)}M_{11}B'^{(l+1)}\right)^{-1}, \tag{13}$$

$$\Sigma^{(l+1)} = T^{-1}\left(AM_{00}A' - \Gamma^{(l+1)}B^{(l+1)}M_{10}A'\right), \tag{14}$$

where, in equation (12), $\alpha^{(l)}$, $\Sigma^{(l)}$, and $\Gamma^{*(l)}$ are given by the previous iteration; in equation (13), $B^{(l+1)}$ is given by $\beta_0^{(l+1)}$, according to equation (12); and, in equation (14), $\Gamma^{(l+1)}$ is given by equation (13). In equations (12) to (14), M_{00}, M_{01}, M_{10}, and M_{11} are given by

$$M_{jk} = \sum_{t=1}^{T} \mathrm{E}_l(x_{t-j}x'_{t-k} \mid Y_T^{+}) = \sum_{t=1}^{T}\left(P_{t-j,t-k}^{T} + x_{t-j}^{T}x_{t-k}^{T}\right), \tag{15}$$

for $j, k = 0, 1$, where $P_{t-j,t-k}^{T}$ and x_{t-j}^{T} are produced by the prediction and updating recursions of the Kalman filter. In Appendix, we adopt the fixed-interval smoothing algorithm of De Jong (1989), which avoids inversion of large matrices, and, hence, is computationally more efficient than the classical smoothing equations (Durbin and Koopman 2001). We note that the mean vector, λ, and the covariance matrix, Λ, of the initial state vector, x_0, cannot be estimated simultaneously. Following Shumway and Stoffer (1982), we preset the covariance matrix and estimate the mean vector as $\lambda^{(l+1)} = x_0^{T}$ by maximizing (11).

We summarize the iterative EM procedure as follows:

(1) Calculate M_{jk} for $j,k = 0,1$ using equations (A.1) to (A.6) in the Appendix,

with the initial values $\lambda^{(0)}$, Λ, and $\theta^{(0)}$.

(2) Estimate $\lambda^{(1)} = x_0^T$ and calculate $\theta^{(1)}$ using equations (12) to (14).

(3) Iterate on steps (1) and (2) above until the parameter estimates or the likelihood values converge. At each iteration, we calculate the innovations form of the log-likelihood function (Schweppe 1965),

$$\log L(\theta; Y_T^+) = -\frac{1}{2}\sum_{t=1}^{T} \log | H_t P_t^{t-1} H_t' + Q_t Q_t' | \\ -\frac{1}{2}\sum_{t=1}^{T} (y_t^+ - H_t x_t^{t-1})'(H_t P_t^{t-1} H_t' + Q_t Q_t')^{-1}(y_t^+ - H_t x_t^{t-1}), \quad (16)$$

and stop when the difference between $\log L(\theta^{(l+1)}; Y_T^+)$ and $\log L(\theta^{(l)}; Y_T^+)$ is less than a predetermined small value.

The Newton-Raphson (NR) method is known to converge faster, compared with the EM. However, the NR method is more likely to fail because it is very sensitive to initial values; see Shumway and Stoffer (1982) for further details. Thus, we use the EM algorithm, especially, because in cointegration with mixed-frequency data initial values of parameters are difficult to obtain. Since it is known that the EM algorithm slows down near the maximum, one may switch to the NR method near the maximum for faster convergence; see Watson and Engle (1983) and MM (2004).

3.2 Initialization

To start the EM algorithm with the Kalman filter, we need to specify the initial values, $\lambda^{(0)}$, Λ, and $\theta^{(0)}$. When the state equation is nonstationary, the unconditional distribution of the state vector is not defined. Usually, the initial distribution of x_0 must be specified by a diffuse or noninformative prior because genuine prior information is generally not available (Harvey 1991). Therefore, we set $\lambda^{(0)} = 0$ and $\Lambda = \delta I$, where δ is a large value, for example, $\delta = 10^8$.

Regarding initial values of parameters, we first obtain an estimate of the nonstationary parameters by using single-frequency data that are usually obtained by transforming the high-frequency variables, y_{1t}, to match the low frequency of y_{2t}. Then, we use the relationship between the nonstationary parameters in mixed-frequency non-temporally-aggregated and single-frequency temporally-aggregated models, estimated using mixed- and single-frequency data, in order to obtain an initial estimate of the nonstationary parameters in $\beta_0^{(0)}$. For example, see Marcellino (1999) and Pons and Sansó (2005) for the explicit formulas for the relationship in several cases. For the remaining stationary parameters, however, it is not easy to obtain an explicit formula to describe the relationship between the parameters of models of mixed- and single-frequency data. Therefore, treating the initial estimate $\beta_0^{(0)}$ as known, and, thus, fixed in equation (2), we obtain initial estimates of the stationary parameters, $\Gamma^{(0)}$ and $\Sigma^{(0)}$, using the iterative EM algorithm.

In order to prespecify the CI rank in the analysis of cointegration with mixed-frequency data, we use the fact stated by Marcellino (1999) that the CI rank is invariant to temporal aggregation. Then, we can use the CI rank, obtained by applying the CI rank test to the temporally-aggregated single-frequency data.

We select as "best" the VAR(p) model whose MLE yields the lowest values of Akaike's information criterion (AIC) and Schwartz's Bayesian information criterion (SBC),

$$\mathrm{AIC} = T^{-1}\{-2\log L(\hat{\theta};Y_T^+) + 2\dim(\theta)\},$$

$$\mathrm{SBC} = T^{-1}\{-2\log L(\hat{\theta};Y_T^+) + \dim(\theta)\log T\},$$

where $\log L(\hat{\theta};Y_T^+)$ is given by equation (16), $\hat{\theta}$ is the MLE of θ, obtained using the proposed EM algorithm, and $\dim(\theta)$ is the dimension of θ.

4. COMMENTS

One of the important purposes of the paper is to estimate missing or unobserved low-frequency variables, y_{2t}, which satisfy the structure of the cointegrated multivariate time-series model. This can be done by estimating x_t^T, together with its covariance matrix P_t^T, using smoothing equations (A.5) and (A.6) in Appendix. One of the advantages of the proposed method is that we can use it to forecast low-frequency variables which are generated jointly with variables observed at higher frequencies. For example, with quarterly GDP and other monthly variables, we can forecast monthly GDP even if GDP is observed only quarterly.

Usually, logarithms of variables are taken before fitting a multivariate time-series model, especially to stabilize the variances of series. This creates no difficulties for high-frequency series (Harvey and Pierce 1984). However, for a temporally-aggregated low-frequency series, although the sum of the original variables is observed, the logarithm of a sum is not equal to the sum of the logarithms. In such a case, we assume that the logarithms of the original variables are integrated of order 1. There are three ways to handle the issue on the logarithm transformation. First, a temporally-aggregated variable is treated as the geometric sum (mean) of an unobserved high-frequency variable, as in MM (2003, 2004). Mitchell *et al*. (2005) say that this is a very good first-order approximation in some cases such as constant price GDP, although the aggregated variable is the arithmetic sum (mean) under the common accounting identity that links high- and low-frequency values. Second, the following first-order Taylor approximation by Aadland (2000),

$$\sum_{j=1}^{v} \log(z_{2,t-j+1}) \cong v \log\left(\sum_{j=1}^{v} z_{2,t-j+1}\right) - v\log(v), \tag{17}$$

can be used to linearize the measurement equation. Third, a nonlinear state-space model with the extended Kalman filter, as in Anderson and Moore (1979, pp. 193-195), may be employed, see also Harvey and Pierce (1984).

We include various combinations of deterministic terms in model (1), as in Johansen (1996). The explicit state-space representations that can accommodate such deterministic terms are in Appendix C of Seong *et al*. (2007).

5. MONTE CARLO EXPERIMENTS

Monte Carlo experiments are conducted to investigate the performance of the proposed method. The data generating process we consider is similar to the one in Ahn and Reinsel (1990), except that we consider a 3-dimensional process, specifically,

$$\Delta u_t = (\Delta u_{1t}, \Delta u_{2t}, \Delta u_{3t})' = \gamma_1 + ab'u_{t-1} + \varepsilon_t, \tag{18}$$

where $\varepsilon_t \sim \text{i.i.d.} N_3(0,\Omega)$, for $t = 1, \cdots, T$, and γ_1 denotes an unrestricted constant term. The parameters are set at the following values:

$$\gamma_1 = (\gamma_{11}, \gamma_{21}, \gamma_{31})' = (-0.2, 0.1, 0.3)', \quad a = (a_1, a_2, a_3)' = (0.6, 1, 0.4)',$$

$$b = (b_1, b_2, b_3)' = (1, -2, 3)' \text{ and } \Omega = \begin{pmatrix} 25 & 7.5 & 2.5 \\ 7.5 & 9 & 1.5 \\ 2.5 & 1.5 & 1 \end{pmatrix},$$

so that u_t is cointegrated with rank one.

For convenience, we assume that permutation matrix V is an identity matrix, that is, u_{2t} and u_{3t} are not cointegrated. Then, $u_t = z_t$ and the parameters in models (17) and (18) are identical. After generating u_t, we set $y_{1t} = u_{1t}$ and $y_{2t} = u_{2t}$, making them high-frequency variables, and, then, generate the low-frequency variable, y_{3t}, by $y_{3t} = u_{3t} + u_{3t-1} + u_{3t-2}$, for $t = 3, 6, 9, \cdots$, which makes y_{3t} a temporally-aggregated flow variable. We generate 1,000 replications of the series for sample sizes $T = 120$ and $T = 240$, which represent 10 and 20 years of monthly data, such that the first 50 values are discarded in order to reduce dependence on starting values. We estimate a VAR(1) with an unrestricted constant.

Because, to our best knowledge, there is no other method available to analyze mixed-frequency data with cointegration, we evaluate the performance of the proposed method against the case in which all variables are observed at the highest frequency, that is, there are no missing data. In addition, we compare the interpolation ability of the proposed method with that of MM (2004), which does not use long-run information but mainly uses short-run dynamics. We expect our method to perform nearly as well with intermittently-missing mixed-frequency data as with complete high-frequency data. The same issue arose in Chen and Zadrozny (1998), where a similar Monte Carlo experiment was conducted to evaluate the performance of their method for estimating a stationary VAR model using mixed-frequency data, relative to using complete high-frequency data.

The method of Ahn and Reinsel (1990) is used for estimating with the complete high-frequency data. Table 1 contains the simulation result for the performance of the proposed method against the complete data case. We observe that the differences in the table are generally fairly small and that the proposed method performs well with the mixed-frequency data, compared with the complete data, in terms of biases and root mean-squared errors (RMSEs) of estimated parameters. As expected, the proposed method performs worse using mixed-frequency data than using complete high-frequency data.

It is interesting to compare table 1 with tables 2 to 7 in Chen and Zadrozny

(1998), which show how MLE and extended Yule-Walker (XYW) parameter estimates deteriorate, in terms of RMSE, when going from complete to mixed-frequency data. In their table 5, the RMSEs of MLE decline about 44%, whereas the best RMSEs of XYW decline about 77% or more. Consider how the RMSEs change in table 1, for a's, b's, and γ's when $T = 240$, as we move from using complete data to using mixed-frequency data. The RMSEs increase, ranging between 5.7% (=0.002/0.035) for a_1, by 65.1% (=0.151/0.232) for γ_{31}, on average by 40.3%. We conclude that MLEs of parameters in both stationary and nonstationary (cointegrated) VAR processes lose a similar amount of RMSE accuracy when going from complete to mixed-frequency data. However, we should be cautious in drawing a general conclusion about these numbers because they surely also depend on whether the VAR process is stationary or not, on the dimension of the process, and on the sample size.

For longer series, biases and RMSEs of the proposed method are smaller. In table 1, we also report decline rates of RMSEs for estimated parameters, when T doubles from 120 to 240. For stationary parameters, a and Ω_{ij}, and nonstationary parameters, b, the RMSEs are consistent with the respective convergence rates of $O_p(T^{-1/2})$ and $O_p(T^{-1})$. RMSEs of stationary parameters generally decline by 29% ($=1-\sqrt{120/240}$) or more when T doubles from 120 to 240 and RMSEs of nonstationary parameters generally decline by 50% ($=1-120/240$) or more when T doubles. However, changes in the RMSEs of stationary parameters, γ, are ambiguous, because their RMSEs decline faster than is predicted for stationary parameters but slower than is predicted for nonstationary parameters. Most importantly, this also occurs when we apply the method of Johansen (1996) or Ahn and Reinsel (1990) to the analysis of complete data and the RMSEs of the stationary parameters, γ, do not decline more than those of the nonstationary parameters.

Since the parameters in the vector error correction model can not be estimated using the method in MM (2004), we compare our method with MM's through the interpolation capabilities. The method of MM (2004) is run by their Ox program available on the website (www.eco.osakafu-u.ac.jp/~murasawa). Since the model of the first differenced series generated by (18) is a non-invertible vector autoregressive moving average process of order (1,1) or VARMA(1,1), we fit a VAR(2) as an approximation to this VARMA(1,1) for the investigation of the performances of MM.

Figures 1 and 2 show RMSEs of estimates of missing values of y_{3t}, $t \neq 3, 6, 9, \cdots$, when $T = 120$ and 240, respectively. Four outlying values in starting and ending points are omitted, because they are spuriously large due to endpoint effects, which arise as a result of the backward smoothing recursions (A.3) to (A.6), with $r_{T+1} = 0$ and $R_{T+1} = 0$ being a sort of initialization.

As noted in the figures, the average of the RMSE of the proposed method is about 39% of that of MM's for both sample sizes considered. Therefore, the proposed method is better in terms of interpolation capability. We note that, from Figure 1 and 2, as T doubles from 120 to 240, the RMSEs decline by 8.70% and 7.97% in the proposed method and MM's, respectively. This indicates that the RMSEs of the estimates of the missing values may decline more slowly than the rate of $O_p(T^{-1/2})$.

6. EXAMPLES

We illustrate the proposed method using real mixed-frequency U.S. data of monthly CPI and quarterly real GDP, from December 1959 to December 2003, which comprises

176 quarters or 529 months of observations. The CPI and GDP data are from the Bureau of Labor Statistics (www.bls.gov) and from the Bureau of Economic Analysis (www.bea.gov), are seasonally adjusted, have a base CPI value of 100 in 1982-1984, and have a unit of GDP in billions of chained 2000 dollars. The original data, denoted CPI_t and GDP_t in month *t*, were transformed to cpi_t and gdp_t by taking natural logs and subtracting from them the natural logs at the starting dates, as

$$y_t = \begin{pmatrix} cpi_t \\ gdp_t \end{pmatrix} = \begin{pmatrix} \log(CPI_t) - \log(CPI_{1959:12}) \\ \log(GDP_t) - \log(GDP_{1959:12}) \end{pmatrix}. \tag{19}$$

Henceforth, the lower-case variable names, *cpi* and *gdp*, will refer to these transformations of *CPI* and *GDP*.

As mentioned in section 3.2, we first estimate the CI rank and the CI vector by using a quarterly single-frequency sample, obtained by picking one monthly value of *cpi* per quarter and keeping quarterly *gdp* as is (we call this "skip-sampling"). The quarterly single-frequency data indicated that the estimated model may have a constant term. The model selection criteria, minimum AIC and SBC, both indicated choosing a VAR(4) model. Then, applying Johansen's trace test to the VAR(4) model, estimated using the single-frequency data, we obtained a p-value less than 0.001, which indicated a CI rank of one. The corresponding estimate of the CI vector was $(1, -2.063)'$. For the mixed-frequency observations on $y_t = (cpi_t, gdp_t)'$, we obtained $(1, -6.189)'$ as the estimate of the cointegrating vector by using the relationship in Pons and Sansó (2005), which we could because GDP is temporally aggregated of order 3.

We used minimum AIC and SBC criteria to select the best monthly (highest frequency) model, and chose the VAR(4) model

$$\Delta y_t = \begin{pmatrix} 0.0900 \\ 0.3434 \end{pmatrix} + \begin{pmatrix} 0.0005 \\ 0.0018 \end{pmatrix} \begin{pmatrix} 1, & -5.0580 \end{pmatrix} y_{t-1} + \begin{pmatrix} 0.4561 & -0.0008 \\ -0.0993 & -0.9010 \end{pmatrix} \Delta y_{t-1}$$
$$+ \begin{pmatrix} 0.2664 & 0.0054 \\ 0.0066 & -0.2126 \end{pmatrix} \Delta y_{t-2} + \begin{pmatrix} 0.0600 & 0.0012 \\ -0.1575 & -0.0235 \end{pmatrix} \Delta y_{t-3} + \varepsilon_t ,$$
$$\hat{\text{var}}(\varepsilon_t) = \begin{pmatrix} 0.0432 & 0.0052 \\ 0.0052 & 0.0663 \end{pmatrix}.$$

as the best VAR(p) model. For this model, we computed monthly smoothed estimates of gdp_t and monthly forecasts of cpi_t and gdp_t.

Table 2 shows monthly smoothed estimates of gdp_t, from 2003:1 to 2003:12, as examples of monthly smoothed estimates of gdp_t in the sample period. The estimates are of high-frequency monthly disaggregated gdp_t and of low-frequency quarterly aggregated gdp_t for 2003. Tables 6 and 7 state out-of-sample (that is, out of model estimation sample) forecasts of cpi_t and gdp_t for 2004 and compare these with true values. The tables show smaller forecast errors when using mixed-frequency data than when using single-frequency data. Specifically, the absolute values of forecast errors are smaller by about 7% and 82% on average.

8. CONCLUSION

We have developed and illustrated a method, for estimating a multivariate cointegrated VAR model with mixed-frequency time-series data, by using a state-space representation of an error correction model. The method allows us not only to estimate such a model using mixed-frequency data, but also to estimate missing or unobserved high-frequency values of the low-frequency variables. Monte Carlo experiments, applied to mixed-frequency data, with missing observations, and to single-frequency data, with complete observations, indicate that the proposed method performs well with missing data due to mixed frequencies.

ACKNOWLEDGEMENT

B. Seong's research was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-331-C00060). S. K. Ahn's work was supported by the Korea Research Foundation Grant (KRF-2005-070-C00022) funded by the Korean Government (MOEHRD). P. Zadrozny is affiliated as a Research Fellow with the Center for Financial Studies, Goethe University, Frankfurt, Germany, and the Center for Economic Studies and Ifo Institute for Economic Research (CESifo), Munich, Germany. The paper represents the authors' views and does not necessarily represent any official positions of the Bureau of Labor Statistics.

APPENDIX

Let $x_t^s = \mathrm{E}_l(x_t \mid Y_s^+)$, $P_t^s = \mathrm{cov}_l(x_t \mid Y_s^+)$ and $P_{t,t-1}^s = \mathrm{cov}_l(x_t, x_{t-1} \mid Y_s^+)$, where $\mathrm{E}_l(\cdot \mid Y_s^+)$ and $\mathrm{cov}_l(\cdot \mid Y_s^+)$ denote the conditional expectation and conditional covariance with respect to the density based on $\theta^{(l)}$.

We calculate the prediction and updating recursions using the following equations (for example, Shumway and Stoffer 1982). For $t = 1, \cdots, T$,

$$x_t^{t-1} = F x_{t-1}^{t-1}, \quad P_t^{t-1} = F P_{t-1}^{t-1} F' + GG', \tag{A.1}$$

$$x_t^t = x_t^{t-1} + K_t(y_t^+ - H_t x_t^{t-1}), \quad P_t^t = P_t^{t-1} - K_t H_t P_t^{t-1}, \tag{A.2}$$

where $K_t = P_t^{t-1} H_t' (H_t P_t^{t-1} H_t' + Q_t Q_t')^{-1}$. We start iterations (A.1) and (A.2) by setting $x_0^0 = \lambda$ and $P_0^0 = \Lambda$. In order to calculate x_t^T, P_t^T and $P_{t,t-1}^T$ using equations (A.1) and (A.2), for $t = T, \cdots, 1$, we iterate over the backwards recursions

$$r_t = H_t'(H_t P_t^{t-1} H_t' + Q_t Q_t')^{-1}(y_t^+ - H_t x_t^{t-1}) + L_t' r_{t+1} \tag{A.3}$$

$$R_t = H_t'(H_t P_t^{t-1} H_t' + Q_t Q_t')^{-1} H_t + L_t' R_{t+1} L_t \tag{A.4}$$

where $r_{T+1} = 0$, $R_{T+1} = 0$, and $L_t = F(I_s - K_t H_t)$. Following Durbin and Koopman (2001), we obtain the smoothing equations

$$x_t^T = x_t^{t-1} + P_t^{t-1} r_t, \quad P_t^T = P_t^{t-1} - P_t^{t-1} R_t P_t^{t-1}, \tag{A.5}$$

for $t = 1, \cdots, T$, and

$$P_{t+1,\,t}^T = (I_s - P_{t+1}^t R_{t+1}) L_t P_t^{t-1}, \tag{A.6}$$

for $t = 1, \cdots, T-1$.

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Table 1. Performance of EM algorithm in complete versus mixed-frequency data and decline rates of RMSE as T goes from 120 to 240

		$T=120$		$T=240$		Decline rate	
		Complete data	Mixed-freq. data	Complete data	Mixed-freq. data	Complete data	Mixed-freq. data
$a_1=0.6$	Mean	0.585	0.578	0.593	0.588		
	RMSE	0.049	0.056	0.035	0.037	29%	34%
$a_2=1$	Mean	1.007	0.995	1.002	0.993		
	RMSE	0.032	0.040	0.020	0.025	38%	38%
$a_3=0.4$	Mean	0.399	0.396	0.399	0.397		
	RMSE	0.011	0.016	0.007	0.010	36%	38%
$b_1=-2$	Mean	-1.999	-1.990	-2.001	-2.001		
	RMSE	0.030	0.050	0.011	0.015	63%	70%
$b_2=3$	Mean	2.997	2.980	3.002	3.003		
	RMSE	0.066	0.112	0.023	0.033	65%	71%
$\gamma_{11}=-0.2$	Mean	-0.148	-0.199	-0.203	-0.276		
	RMSE	0.860	1.650	0.466	0.665	46%	60%
$\gamma_{21}=0.1$	Mean	0.114	0.019	0.071	-0.052		
	RMSE	1.229	2.652	0.590	0.953	52%	64%
$\gamma_{31}=0.3$	Mean	0.314	0.271	0.294	0.240		
	RMSE	0.480	1.065	0.232	0.383	52%	64%
$\Omega_{11}=25$	Mean	24.470	25.891	24.843	25.483		
	RMSE	3.114	3.874	2.240	2.537	28%	35%
$\Omega_{12}=7.5$	Mean	7.374	9.816	7.469	8.642		
	RMSE	1.508	3.999	1.074	2.226	29%	44%
$\Omega_{13}=2.5$	Mean	2.461	3.773	2.486	3.090		
	RMSE	0.511	2.175	0.352	1.173	31%	46%
$\Omega_{22}=9$	Mean	8.806	12.950	8.931	11.063		
	RMSE	1.177	5.957	0.863	3.352	27%	44%
$\Omega_{23}=1.5$	Mean	1.461	3.590	1.474	2.572		
	RMSE	0.307	3.034	0.216	1.692	30%	44%
$\Omega_{33}=1$	Mean	0.976	1.931	0.984	1.461		
	RMSE	0.132	1.400	0.092	0.758	30%	46%

Note: Decline rate = 1 – (RMSE for T=240) / (RMSE for T=120)

Table 2. Monthly smoothed estimates of in-sample quarterly gdp_t, 2003:1 to 2003:12

Year: month	Observed	Temp. agg.	Skip-sampled	Year: month	Observed	Temp. agg.	Skip-sampled
2003:1		141.58	141.67	2003:7		143.65	144.40
2003:2		141.81	142.11	2003:8		144.20	144.71
2003:3	141.96	141.96	142.12	2003:9	144.76	144.76	145.16
2003:4		142.20	142.37	2003:10		145.06	145.32
2003:5		142.52	143.07	2003:11		145.40	145.72
2003:6	142.97	142.97	143.48	2003:12	145.78	145.78	146.30

Note: Reported "aggregated" and "skip-sampled" gdp_t in columns 3 and 4, respectively, reflect quarterly sums of monthly values ending in the indicated month and monthly values for that month multiplied by three (in order to be in quarterly form comparable to observed quarterly gdp_t in column 2).

Table 3. Monthly out-of-sample forecasts of cpi_t

Year: month	Observed	Single-frequency		Mixed-frequency	
		Forecast	Error	Forecast	Error
2004:1	184.39			184.00	0.39
2004:2	184.71			184.13	0.58
2004:3	185.14	184.05	1.08	184.24	0.90
2004:4	185.35			184.39	0.96
2004:5	185.94			184.51	1.43
2004:6	186.20	184.64	1.56	184.65	1.55
2004:7	186.15			184.79	1.36
2004:8	186.20			184.93	1.27
2004:9	186.36	185.01	1.35	185.07	1.29
2004:10	186.94			185.21	1.73
2004:11	187.20			185.35	1.85
2004:12	187.20	185.39	1.81	185.49	1.71

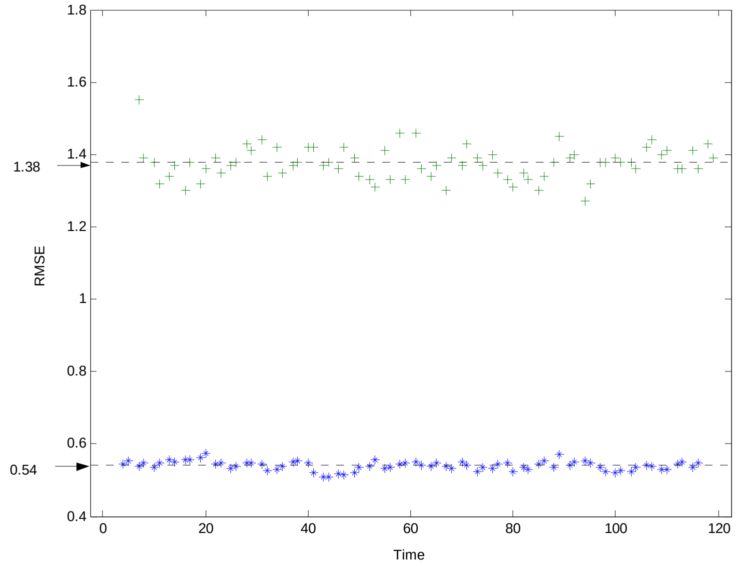
Note: "Out-of-sample" means for months beyond earlier months, from 1959:12 to 2003:12, used to estimate the model which was used to produce the forecasts.

Table 4. Monthly out-of-sample forecasts of quarterly gdp_t

Year: month	Observed	Single-frequency		Mixed-frequency		
		Forecast	Error	Forecast Low Freq.	Error	Forecast High Freq.
2004:1				146.133		146.376
2004:2				146.515		146.868
2004:3	146.88	146.51	0.37	146.775	0.10	147.081
2004:4				147.139		147.468
2004:5				147.431		147.744
2004:6	147.69	147.43	0.26	147.767	-0.08	148.089
2004:7				148.074		148.389
2004:8				148.398		148.716
2004:9	148.67	148.17	0.50	148.71	-0.04	149.025
2004:10				149.028		149.343
2004:11				149.34		149.652
2004:12	149.61	148.82	0.79	149.654	-0.05	149.967

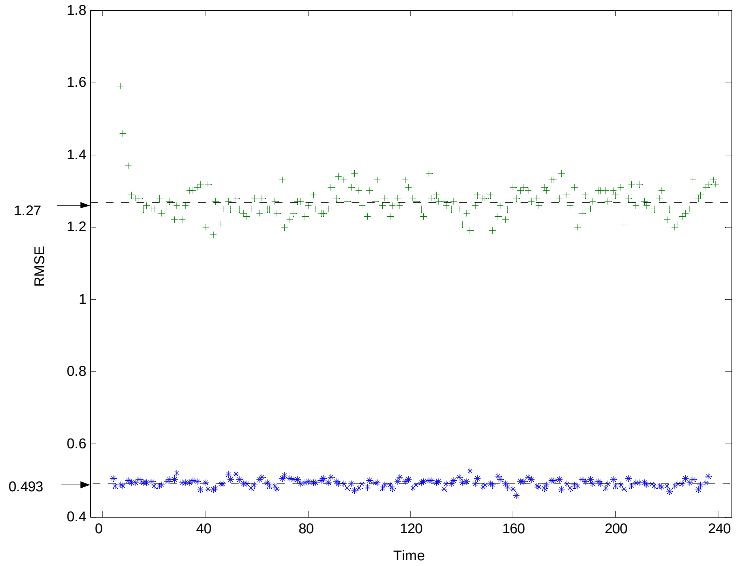
Note: Same as in Table 3.

Figure 1. RMSEs of estimated y_{3t}, when $T = 120$



Note: The plus signs and asterisk plot the RMSEs by MM (2004) and the proposed EM, respectively. The dotted lines denote average values of the corresponding RMSEs.

Figure 2. RMSEs of estimated y_{3t}, when $T = 240$



Note: Same as in Figure 1.