Uncertainty and Leveraged Lucas Trees: The Cross Section of Equilibrium Volatility Risk Premia

JOBMARKET PAPER

Andrea Vedolin*

Abstract

Volatility risk premia compensate agents for holding assets whose payoffs correlate with times of high return variation. This paper takes a structural approach to explain the cross-section of volatility risk premia of stocks using a Lucas orchard with heterogeneous beliefs, stochastic macro-economic uncertainty, and default risk. I study two manifestations of uncertainty, namely (i) agents' disagreement and (ii) time-varying volatility of fundamental growth rates. The paper shows that while the former source of risk accounts for the level of the risk premia, the latter mainly affects the higher order moments of the risk premium distribution. Together with uncertainty, default risk associated with levered trees implies a non-monotonic equilibrium link between stock returns and volatility which allows for positive or negative risk premia. Calibrating the economy, I show that the model accounts for predictability of excess stock returns and corporate credit spreads. I construct volatility risk premia from option and stock prices and document that in the time-series, volatility risk premia of individual stocks can be positive or negative, and switch sign rather often. In the cross-section, they are only weakly related to traditional risk factors. I then test the model predictions and find that empirical proxies for investors' uncertainty about expected growth rates and macro-economic uncertainty are priced risk factors that convey information over and above those contained in other standard factors to explain these risk premia. In line with the model predictions, I present predictability evidence of individual volatility risk premia for stock excess returns and corporate credit spreads.

JEL Classification Codes: D80, G12, G13.

Keywords: Disagreement, Filtering, Skewness, Uncertainty, Volatility Risk Premium

First Version: November 24, 2009

This Version: November 24, 2009

*Andrea Vedolin (andrea.vedolin@usi.ch) is at the University of Lugano. I thank Peter Kondor, Filippos Papakonstantinou, Pietro Veronesi, and seminar participants at Imperial College London for helpful comments. For wit, intellect, and smiles I thank Andrea Buraschi, Philippe Mueller, Lukas Schmid, and Fabio Trojani. This paper was written while visiting the Financial Markets Group at the London School of Economics. All errors are obviously mine.

Volatility risk premia compensate investors for holding assets whose payoffs are correlated with times of high return variation. While most of the recent literature has focused on the index volatility risk premium, this paper studies individual volatility risk premia in the cross-section of stocks both theoretically and empirically. I develop an economy with multiple assets where each firm is subject to default. The profitability of each firm is unknown and has to be learned. I allow two rational manifestations of uncertainty, (i) agents' disagreement about the future profitability of firms and a business-cycle indicator, and (ii) time-varying volatility of fundamental growth rates to have a bearing on risk premia in equilibrium. My results show that the interplay between leverage and the two sources of uncertainty add a crucial component to the understanding of both the time-series and cross-section of individual volatility risk premia.

Empirically, volatility risk premia of firms with low leverage display a different behavior than those of firms with high leverage. Figure 1 documents this fact. The upper left panel plots the average volatility risk premium, proxied by the difference between the model-free implied volatility calculated from option prices and the realized volatility calculated from stock returns, for the 5th leverage percentile of all firms in the S&P 500 index. The lower left panel depicts the average volatility risk premium for the 95th leverage percentile. There are several observations. First, both volatility risk premia display a counter-cyclical behavior and the volatility risk premium of the high leverage firms tends to be more counter-cyclical than the one of the low leverage firms. This is indicated by the larger increase of the risk premium in periods of financial or economic crisis. For instance, the average change in the volatility risk premium of high leverage firms in crisis periods is 30%, opposed to a 11% increase for low leverage firms. Second, the average volatility risk premium of low leverage firms turns negative more often than high leverage firms.[1] This is confirmed in the right panel of Figure 1 where I depict the boxplot of volatility risk premia for low and high leverage firms: Low leverage firms display a more negatively skewed distribution than high leverage firms. The skewness is -1.32 which is more than 50% more negative than for the high leverage firms. Moreover, the median volatility risk premium of the low and high leverage firms is 0.0049 and 0.0192, respectively. Running a simple t-test reveals that I can reject the null hypothesis of zero difference between the two means.

[Insert Figure 1 approximately here.]

The counter-cyclical nature of the market volatility risk premium has prompted a number of researchers to examine the impact of economic uncertainty on the volatility risk premium. Drechsler and Yaron (2008), Drechsler (2008), and Bollerslev, Tauchen, and Zhou (2009) study time-varying economic uncertainty under the assumption of recursive preferences in a consumption-based model. In such an economy, absent an explicit link between the innovations in consumption and volatility, one must endow agents with a preference for early resolution of uncertainty and a stochastic volatility of consumption growth volatility, in order to replicate the distinct empirical features of the volatility risk premium and its higher order moments. In such a framework, however, the volatility risk premium can only be positive, as agents require a higher premium for holding the risk. While the volatility risk premium for the

[1] In the time-series, I find that in 18% of all observations the volatility risk premium turns negative, while for high leverage firms the volatility risk premium is negative 10% of the time.

market is indeed positive most of the time, it does switch signs over time and interestingly, these periods of negative risk premia tend to coincide with times of economic or financial crises.[2] On the individual stock level, as outlined before, I find that volatility risk premia tend to be negative more often.[3]

Surprisingly little research has been undertaken to understand the economic drivers of volatility risk premia of individual stocks. This paper seeks to fill this gap and studies the cross-section of volatility risk premia from an asset pricing perspective. In a partial information economy with multiple assets and agents, I propose a structural explanation for the differential behavior of volatility risk premia in the cross-section of stocks, based on the role of stochastic macro-economic uncertainty and heterogeneity in beliefs. I study a Lucas' orchard economy, in which rational investors update their beliefs according to Bayes' Law and where claims to trees are subject to default risk. The growth rate of the firms' cash flow streams and a business cycle indicator are unknown to all agents in the economy. In order to implement optimal portfolio choices, these growth rates have to be estimated and investors disagree on them. In addition, I assume that these expected growth rates carry a volatility which follows a two-state Markov switching process. The intuition borrows from the long-run risk literature, which emphasizes the importance of a time-varying conditional volatility of consumption growth. In equilibrium, when belief heterogeneity and macro-economic uncertainty vary over time, they drive a substantial fraction of the volatility of asset returns. The interaction between leverage, belief heterogeneity, and uncertainty allows for both positive and negative volatility risk premia. For low leverage firms with high uncertainty, the volatility risk premium can be negative, whereas for high leverage firms, the risk premium tends to be positive. I study these model implications theoretically within a calibrated model and link them directly to the differential empirical pricing patterns of individual options.

My approach departs from the existing literature in two important aspects. First, I study the cross-section of individual volatility risk premia in a multi tree economy, which poses a more challenging quest than understanding the index volatility risk premium itself due to the large dispersion of the volatility risk premia in the cross-section of stocks. Second, I distinguish between two manifestations of uncertainty: (i) The uncertainty induced by fundamental values (the stochastic volatility of the expected growth rate of cash flows and a business cycle indicator) and (ii) its heterogeneous perception by the agents. While both carry the label uncertainty, I allow for two distinct channels through which uncertainty is reflected in risk premia. The former enters into the stochastic discount factor by affecting the moments of the distribution of the disagreement. The higher the macro-economic uncertainty, the higher the first and second moments of the disagreement. In addition, I show that for explaining the cross-sectional pattern of volatility risk premia, both specifications contribute distinct parts. While the subjective uncertainty quantitatively fits well the level, the macro-economic uncertainty accounts for higher order moments of the volatility risk premia such as volatility, skewness, and kurtosis.

[2]In the period from January 1996 to September 2008, the volatility risk premium of the S&P 500 was negative five times. Note that the points of a strongly negative market volatility risk premium usually occur during times of economic or financial crises. For instance, one striking drop in the index volatility risk premium was in spring 2000, right after the NASDAQ bubble burst. Similarly, since the start of the economic crisis in 2007, the market volatility risk premium has turned negative three times.

[3]This finding is inline with Bakshi, Kapadia, and Madan (2003) and Carr and Wu (2009) who document a large dispersion in variance risk premia of individual stocks and more importantly, a more positively skewed return distribution.

The more specific features of the model are as follows: The growth volatility of fundamentals and agents' disagreement are time-varying and are allowed to affect the volatility risk premia via the stochastic discount factor. A higher disagreement on future cash flows induces a higher volatility and negative skewness for medium to high leveraged firms. For firms with low leverage, it is possible that the risk-neutral skewness turns positive. The economic reason is due to the different sensitivities of the default option embedded in stock returns: For low leverage companies, this option is far out-of-the-money and its value is more sensitive to changes in skewness. For high leverage companies, the default option is closer to be in-the-money and the opposite holds. In my economy, agents have time-separable utility and the diffusions of the fundamental process and the consumption growth volatility are assumed independent. In an economy without disagreement, under these assumptions, the difference between the risk-neutral and physical expected volatility would be equal to zero. If disagreement is a priced risk factor, the wedge is entirely driven by agents' uncertainty and the time-varying macro-economic risk. Calibrating the model, I confirm several of the empirical findings. Both disagreement and realized conditional consumption growth volatility matter in their implications for volatility risk premia, and I find the first source to dominate. Shutting down the stochastic volatility of cash flow growth, the level of volatility risk premia is mainly explained by the difference in beliefs. However, higher order moments, such as the persistence, skewness, and kurtosis of the risk premia are driven, to a large extent, by economic uncertainty. A simulation exercise shows that the model is able to replicate the predictability of stock excess returns and corporate credit spreads. Moreover, the estimated coefficients correspond in sign and size to their empirical counterparts. This paper is a first step towards a unified study of agents' (disagreement) and macro-economic uncertainty, and I hope to give credence to the hypothesis that both subjective agents' uncertainty and realized objective uncertainty could matter for volatility risk premia of stocks.

To empirically test the model predictions, I construct model-free measures of volatility risk premia (see Carr and Madan, 1998 and Britten-Jones and Neuberger, 2000) and risk-neutral skewness (Bakshi, Kapadia, and Madan, 2003) from stock option prices using data from January 1996 to September 2008. Following Hamilton (1989), I estimate a Markov chain process for the second moment of consumption growth to get a measure of the conditional volatility of fundamentals. In addition, I build a forward-looking measure of disagreement about future cash flows of firms based on forecasts of corporate earnings. The separation of the two uncertainty channels has an empirical appeal. By its forward-looking construction, disagreement captures perceptions of risk looming on investors' horizon. Thus, it conveys information over and above the one contained in the conditional volatility of consumption which is estimated using historical data.

I document two results which are, to the best of my knowledge, new to the literature. First, simple panel regressions reveal that both disagreement and the conditional volatility of consumption matter for volatility risk premia and outperform the explanatory content of other factors known to affect these risk premia. For instance, a one standard deviation change in firm-specific disagreement increases the volatility risk premium by 0.5%, which corresponds to one third of the average volatility risk premium in the cross-section of stocks. A slightly smaller change is induced by the conditional volatility of consumption growth. Disagreement and time-varying consumption growth volatility account for 12% of the variation of volatility risk premia in the cross-section. The empirical assessment also reveals that the impact of uncertainty on the volatility risk premia can be positive or negative depending on the leverage level. For low leverage firms, the impact is negative, whereas for medium to high leverage firms the estimated coefficients are positive. The stability and strength of these results is particularly noteworthy given the sample period I consider: Using data from 1996 to 2008, I capture interesting periods during which volatility risk premia reverse signs and as such turn out to be more difficult to explain. I also find that the volatility risk premia of cyclical stocks are more exposed to uncertainty than non-cyclical stocks. For instance, the estimated coefficient for uncertainty on cyclical stocks is twice as large as for non-cyclical ones.

Second, I present evidence that the volatility risk premium contains forecasting power for both the cross-section of stock excess returns and corporate credit spreads. Similar to previous results in the literature for the index volatility risk premium, I find that predictability of the stock excess returns is strongest in the short run: The predictive power of the individual volatility risk premia is strongest at a horizon of eight months and then steadily declines. An analogue pattern is found for corporate credit spreads.

Understanding the differential pricing of options and their embedded volatility risk premia in the cross-section is important, as the joint behavior of derivative and equity markets may help to improve our general understanding of the dynamics of asset prices. This has become particularly apparent in the past few months: While almost all asset classes have experienced major drawdowns, trading volume of VIX futures has increased fivefold during the crisis relative to its volume at inception in 2006.[4]

A traditional way to lock-in the implicit volatility risk premium is by forming delta-hedged short straddle or strangle portfolios. These strategies are becoming increasingly popular, especially in the aftermath of the 2008 credit crisis as volatility increased to exceptionally high levels.[5] In Figure 2, I plot the annualized returns and Sharpe ratios for five common option trading strategies and three benchmark indices in the period January 1996 to September 2007.[6] Naked call overwriting and the short index put strategy outperform the equity indices in terms of risk-return relationship: The average Sharpe ratio of the indices is around 0.4 while the call overwriting (short put) yields a Sharpe ratio of 0.55 (0.7). The most successful strategy is the short variance swap strategy on the index, which yields an average Sharpe ratio of 1.64.

Now, imagine a naïve investor who writes at-the-money, 1% out-of-the money, and 5% out-of-the-money straddles on all options on stocks in the S&P 500, independent of their cheapness. During this period, the investor would have earned an annualized return of 8.4% with a Sharpe ratio of 0.5. If one were confident about the existence of a large volatility risk premium, then an alternative strategy can be developed by forming a long-short factor mimicking portfolio. In this portfolio, the agent is long the (cheapest) stocks which have the largest negative volatility risk

[4]Short volatility option strategies not only represent an attractive diversification strategy in bear markets due to the high negative correlation between the market index and any plain vanilla option strategy, but empirical studies also show these strategies perform exceptionally well in crisis periods. For example, after the tech bubble burst in 2000, the S&P 500 declined by 45% over the years September 2000 to September 2002. At the same time, a short variance swap strategy on the same index went up by more than 40% with a lower standard deviation (see e.g. Grant, Gregory, and Lui, 2007).

[5]Other papers that study trading strategies which use the volatility risk premium as a trading signal are Goyal and Saretto (2009) (plain vanilla option strategies) and DeMiguel, Plyakha, Uppal, and Vilkov (2009) (hedge fund strategies).

[6]I acknowledge that Sharpe ratios can be highly misleading when analyzing derivatives (see Goetzmann, Ingersoll, Spiegel, and Welch, 2007). To comply with the majority of the most recent derivatives literature, however, I rely on Sharpe ratios as a performance measure.

premium and short the stocks with the largest positive (most expensive) volatility risk premium. With respect to the previous strategy, a potential advantage of this portfolio is that, since it is long-short, it can be constructed not to be exposed to average changes in volatility but only to cross-sectional differences in the risk premium. Applying this method, I find that this zero cost trading strategy yields a return of 21% and a Sharpe ratio of 1.66. I also find that for a CRRA investor, independent of the level of risk aversion, the certainty equivalent of the sophisticated strategy is always higher than the one of the naïve short straddle strategy.

[Insert Figure 2 approximately here.]

The remainder of the paper is organized as follows. Section I. introduces a general equilibrium model with multiple trees and agents, which is solved for prices and risk premia, and investigates quantitatively the implications of the model for the cross-section of volatility risk premia. Section II. describes the data. Section III. tests the model predictions. Finally, Section IV. concludes the paper.

Literature Review:

The paper contributes to several strands of the literature. First, I give an economic rationale for the non-monotonic relationship between leverage and the implied volatility skew of options: The default option in stocks coupled with the uncertainty renders the implied volatility smile positive or negative depending on the level of leverage. Second, I study two different sources of uncertainty: Agents' disagreement and macro-economic uncertainty as proxied by the stochastic growth rate volatility of cash flows. While both sources of risk have been shown to play an important role in explaining volatility risk premia in the literature, this is the first paper to unify both approaches. Third, the solution technique for the multi-asset economy is different from the literature. The conditional density of the state variables is recovered through Fourier inversion of the conditional characteristic function, which is in closed-form. This differs from an approach of replacing the moment-generating function with an integral of exponential functions or a modeling via the share process. Lastly, my paper contributes to the literature studying heterogeneous beliefs and asset prices. In my economy, learning on the expected growth rate of cash flows is through a nonlinear filter due to the Markov switching growth volatility. One attractive feature of non-linear filtering is that the a posteriori uncertainty remains stochastic, unlike in models with deterministic decay. While the literature has focused mainly on a Markovian structure for the expected growth rate of dividends itself, this is the first paper to study a switching growth rate volatility.

The notion that leverage should impact on the implied volatility of options goes back to Geske (1979) and Toft and Prucyk (1997). The idea is that as the value of the firm's assets declines, the firm becomes more levered and the volatility of equity increases. Their models feature firm's assets with a constant variance processes, but explicitly account for the impact of risky debt on the dynamics of the firm's equity. By incorporating the option-like characteristics of levered equity, this branch of literature introduces a natural negative relationship between stock prices and volatility.[7] More recently, Geske and Zhou (2008) study the impact of leverage on the pricing of individual

[7] The relationship between leverage and stock returns has prompted a plethora of literature using dynamic capital structure models. The empirical evidence, however, is mixed. See Gomes and Schmid (2008) for an excellent summary and extension of the literature.

options. They find that leverage significantly impacts on the return distribution. Moreover, leverage reduces pricing errors by 60% with respect to other option pricing models without leverage. Engle and Mistry (2008) and Hong, Wang, and Yu (2008) find that financially unconstrained firms have more positively skewed daily returns and that more importantly, lower leverage implies a more positively skewed return distribution.

Equilibrium models with multiple trees have been studied in the literature before. Menzly, Santos, and Veronesi (2004) and Santos and Veronesi (2009) study a multi-asset economy with external habit and derive closed-form solutions for prices in equilibrium. Pavlova and Rigobon (2007) study a two country two good economy with demand shocks and log-linear preferences. Cochrane, Longstaff, and Santa-Clara (2008) study a Lucas (1978) economy with two trees and its implications for stock returns, correlations, and the equity risk premium. Both papers provide closed-form solutions for the case of i.i.d. trees and log utility. A different approach is taken by Martin (2009) and Chen and Joslin (2009), who replace the moment generating function with an integral of exponential functions and show that under certain forward measures (Esscher transforms), closed-form solutions can be obtained. Martin (2009) then extends the analysis of Cochrane, Longstaff, and Santa-Clara (2008) to multiple trees, Poisson jumps in the dividend process, and general power utility. He shows that his model is able to replicate many salient features of asset returns, such as momentum, mean-reversion, contagion, fight-to-quality, the value-growth effect, and excess volatility. Chen and Joslin (2009) generalize the method in Martin (2009) and allow for a general class of non-linear transforms of affine jump-diffusions. They apply their method to a series of applications like credit risk, multiple trees, and heterogeneous beliefs economies. My paper differs from the previous ones in several aspects. First, none of these papers studies the impact of uncertainty or disagreement in a multi-tree economy on volatility risk premia. Second, the solution technique taken here is different from the previous ones. Pavlova and Rigobon (2007) and Cochrane, Longstaff, and Santa-Clara (2008) can derive solutions for the case of i.i.d. dividends and log utility as the density function for other cases is not known. The solution technique proceeds in two steps: I first solve for the posterior density of the state variables via a nonlinear filter and then can derive the Fourier transform. The conditional density of the state variables is then recovered through Fourier inversion of the conditional characteristic function which can be found in closed-form. In particular, I show that the parameters driving the state variables can be summarized as a system of matrix Riccati equations, which in turn can be linearized and solved very conveniently. This circumvents the curse of dimensionality problem usually encountered in the literature, see e.g. Dumas, Kurshev, and Uppal (2009). Martin (2009) and Chen and Joslin (2009) solve very elegantly for asset prices by replacing the moment generating function with an integral of exponential functions for which closed-form solutions are known. In particular, this method allows the authors to solve for a high dimensional state space.

The literature aiming at giving a structural explanation for the emergence of volatility risk premia is sparse. Motivated by the empirical results in Bollen and Whaley (2004), who show that changes in implied volatility are related to signed option volume, Gârleanu, Pedersen, and Poteshman (2009) study the link between the level of end user option demand and the level and overall shape of implied volatility curves. They document that end users tend to have a net long index option position and a short equity-option position, thus helping to explain the relative expensiveness of index options.[8] They also show a strong downward skew in the net demand of index but not equity options, which helps to explain the difference in the shapes of their overall implied volatility curves. Their framework is effective in explaining the steeper slope of index options due to the excess demand for out-of-the-money puts, but less so in differentiating the pricing of individual options in the cross-section.

Previous papers have studied the impact of uncertainty on the market volatility risk premium, mostly in settings with recursive preferences. Bansal and Shaliastovich (2008) introduce learning into a long-run risk model, in which asset prices require a premium for jump risk. Shaliastovich (2008) sets up a similar economy, where the unobservable expected growth of consumption has to be learned from a cross-section of signals. The quality of these signals generates an uncertainty which is modeled to contain large positive shocks. Fluctuating confidence risk affects the equilibrium asset prices and since out-of-the-money puts hedge jump risk in confidence, they appear expensive relative to models with no jump risk. Further, endogenous negative jumps in equilibrium prices due to the positive jumps in uncertainty generate a negatively skewed and heavy-tailed unconditional distribution of returns. Eraker (2008) studies an equilibrium with long-run risk and a highly persistent volatility process. Drechsler and Yaron (2008) add infrequent but potentially large spikes in the level of volatility together with infrequent jumps in the small, persistent component of consumption and dividend growth. While volatility shocks from a standard long-run risk model have a sufficiently large market price of risk to generate a variance risk premium, second and third moments of the variance risk premium together with the short-horizon predictability of stock returns is generated in a setting with non-Gaussian shocks. Zhou (2009) links the time-variation of the market volatility risk premium to the stochastic volatility of volatility of consumption growth in a model with recursive preferences. The literature studying the volatility risk premium with recursive preferences typically interprets the wedge between the risk-neutral expected and physical volatility as a proxy for the aggregate risk aversion (see e.g. Bollerslev, Gibson, and Zhou, 2009) or economic uncertainty (see e.g. Drechsler, 2008, Drechsler and Yaron, 2008, and Bollerslev, Tauchen, and Zhou, 2009). It is challenging to extend this reasoning to a multi asset framework in such a setting. First, in these model, stochastic risk aversion or economic uncertainty is a latent process whereas my model allows me to empirically pin down uncertainty. Second, it is difficult to align a negative risk premium with aggregate risk aversion or economic uncertainty as in this case it lacks an intuition.

Finally, previous papers have studied learning through a non-linear filter in an endowment economy. David (2008) constructs a general equilibrium economy with two groups of investors who have heterogeneous beliefs about the state of the economy. Imperfect risk-sharing implies that less risk averse agents trade more aggressively and demand a higher risk premium. A calibrated model fits the equity premium and replicates the distinct features of the time-variation in consumption growth moments. Chen and Pakoš (2008) and Pakoš (2008) model an economy with a representative agent who has recursive preferences. The expected growth rate of consumption follows a two-state Markov chain and using observations on aggregate consumption, dividend, and a signal, the investor builds beliefs about the growth rates being in one state or another. The authors then show that persistency of the consumption growth rate, together with a preference for early resolution generates an uncertainty premium which is large enough

[8] These findings are also complemented by the recent work of Lakonishok, Lee, Pearson, and Poteshman (2007) who document that for both individual equity calls and puts end users are more short than long.

to account for various features of asset prices. My paper departs from these papers along several dimensions. First, in my economy, agents do not build beliefs about the probability of switching itself but investors learn about the expected growth rate. Second, none of the papers distinguishes between a subjective (agent-driven) uncertainty and objective (macro-economic) uncertainty. Third, these papers do not study the cross-section of volatility risk premia.

I. The Economy with Macroeconomic Uncertainty and Heterogeneous Beliefs

A. *The Model*

I extend the standard single-asset Lucas-tree pure-exchange framework to the case with multiple assets and two investors. The economy has infinite horizon $[0, \infty)$ and uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ on which is defined a standard Brownian motion:
$W = \left(W_{A_1}, W_{A_2}, W_z, W_{\mu_{A_1}}, W_{\mu_{A_2}}, W_{\mu_z}, W_{\sigma_{\mu_{A_1}}}, W_{\sigma_{\mu_{A_2}}}, W_{\sigma_{\mu_z}}\right)'$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, \infty)\}$, the augmented filtration generated by the Brownian motion W. There are two firms in the economy, which produce their perishable good. Cash flows of firm $i = 1, 2$ have the following dynamics:

$$d \log A_i(t) = \mu_{A_i}(t)dt + \sigma_{A_i} dW_{A_i}(t),$$

where $\sigma_{A_i} \in \mathbb{R}^+$ is the cash flow volatility of firm i. Cash flows are observable, but their expected growth rate $\mu_{A_i}(t)$ is not and has to be estimated given the available information. The dynamics of $\mu_{A_i}(t)$ is given by:

$$d\mu_{A_i}(t) = \left(a_{0A_i} + a_{1A_i}\mu_{A_i}(t)\right) dt + \sigma_{\mu_{A_i}} dW_{\mu_{A_i}}(t),$$

where $a_{0A_i} \in \mathbb{R}$ is the growth rate of expected cash flow growth, $a_{1A_i} < 0$ is the mean-reversion parameter. Parameter $\sigma_{\mu_{A_i}}(t) \in \mathbb{R}^+$ measures the uncertainty about the individual growth rate of firm i cash flows. The dynamics of $\sigma_{\mu_{A_i}}$ follow a hidden Markov chain with two states $\overline{\sigma}_{\mu_{A_i}} > \underline{\sigma}_{\mu_{A_i}}$, with a transition matrix for the time interval $(t, t + dt)$ given by:

$$\mathbb{P}(dt) = \begin{bmatrix} 1 - \lambda dt & \lambda dt \\ \mu dt & 1 - \mu dt \end{bmatrix}. \tag{1}$$

The Brownians W_{A_i} and $W_{\mu_{A_i}}$ are assumed to be independent. This explicitly rules out any correlation between the return and the volatility due to statistical correlation.

In order to estimate the dividend growth rates, investors make also use of information produced by a market-wide indicator $z(t)$ of the aggregate growth rate in the economy. The market-wide indicator $z(t)$ has the following dynamics:

$$\begin{aligned} dz(t) &= \left(\alpha_{A_1}\mu_{A_1}(t) + \alpha_{A_2}\mu_{A_2}(t) + \beta\mu_z(t)\right) dt + \sigma_z dW_z(t), \\ d\mu_z(t) &= \left(a_{0z} + a_{1z}\mu_z(t)\right) dt + \sigma_{\mu_z}(t) dW_{\mu_z}(t), \end{aligned}$$

where $\sigma_z(t) \in \mathbb{R}^+$ is a signal precision parameter and is given b The market-wide signal $z(t)$ is an unbiased estimator of $\alpha_{A_1}\mu_{A_1}(t) + \alpha_{A_2}\mu_{A_2}(t) + \beta\mu_z(t)$ and is therefore linked to the growth rate of both firms in the economy. For $\beta = 0$, it produces unbiased estimates of a linear combination of firms' growth rates. If additionally $\alpha_{A_1} = \alpha_{A_2}$, it is an unbiased indicator of the actual aggregate growth rate in the economy. When $\beta \neq 0$, signal $z(t)$ is biased by another unobservable variable $\mu_z(t)$, which is completely independent of cash flows. Hence, the relative importance of parameters α_{A_1}, α_{A_2}, and β determines to which extent the information provided by $z(t)$ is contaminated by an orthogonal market-wide component not directly related to expected dividend growth. Parameter $\sigma_{\mu_z} \in \mathbb{R}^+$ measures the market-wide uncertainty about $\mu_z(t)$ itself, and it further controls the degree of uncertainty with which $z(t)$ can be interpreted as an aggregate indicator of economic growth. If $\beta \neq 0$ and σ_{μ_z} is large, shocks to $z(t)$ are interpreted as an evidence of a change in expected cash flows and the highly volatile orthogonal market-wide component $\mu_z(t)$. Therefore, on average, a change in $z(t)$ will tend to generate large revisions of agents' estimates of the market-wide component $\mu_z(t)$ and a larger degree of disagreement across investors. If $\sigma_{\mu_z} = 0$, any change in the expected growth rate of $z(t)$ derives from a change in dividend growth rates and $\mu_z(t)$ can be treated by investors as deterministic. Finally, a_{0z} is the long-term growth rate of the expected change in $\mu_z(t)$ and $a_{1z} < 0$ its mean-reversion parameter.

B. Macroeconomic Uncertainty and Disagreement

I consider a simple specification for the uncertainty and disagreement in my economy. Investors update their beliefs based on the available information using Bayes' Law. Differences in their posteriors can arise either from a difference in agents' priors, a difference in some subjective parameter of the dynamics of cash flows and the market-wide indicator or a stochastic volatility of the cash flow growth. In the latter situation, agents learn through a nonlinear filter which renders the disagreement truly stochastic even asymptotically. I follow this assumption when I derive the testable predictions of my model. The assumption of independent Brownian motions W_{A_i} and $W_{\mu_{A_i}}$ and the independence of the Markov chain with the fundamentals makes the learning dynamics tractable as in this case, the posterior distribution of the expected growth rates is a mixture of the Markov chain and the conditionally Gaussian growth rate dynamics.

Given the filtered volatilities, the model implied state dynamics is conditionally Gaussian at the steady-state of the posterior mean. Therefore, in this case, the Bayesian updating rule of each agent can be derived by standard filtering methods and the heterogeneity in beliefs is fully summarized by the differences in posterior means $m^i(t)$ and variance-covariance matrices $\gamma(t)$ across agents.

Let $m^n(t) := \left(m^n_{A_1}(t), m^n_{A_2}(t), m^n_z(t)\right)' := E^n\left(\left(\mu_{A_1}(t), \mu_{A_2}(t), \mu_z(t)\right)' | \mathcal{F}^Y_t\right)$ where $\mathcal{F}^Y_t := \mathcal{F}^{A_1,A_2,z}_t$ is the information generated by cash flows and the market-wide signal up to time t, and $E^n(\cdot)$ denotes expectation with respect to the subjective probability of investor $n = A, B$. $\gamma^n(t) := E^n\left(\left(\mu(t) - m^n(t)\right)\left(\mu(t) - m^n(t)\right)' | \mathcal{F}^Y_t\right)$ is the posterior variance-covariance matrix of agent n.

To specify the disagreement process in my economy, let the state vector be $Y(t) = (\log A_1(t), \log A_2(t), z(t))$, $b(t) = \mathrm{diag}(\sigma_{\mu_{A_1}}(t), \sigma_{\mu_{A_2}}(t), \sigma_{\mu_z}(t))$ and the following parameters be fixed: $a_0 = (a_{0A_1}, a_{0A_2}, a_{0z})'$, $a_1 = \mathrm{diag}\,(a_{1A_1}, a_{1A_2}, a_{1z})$, $B = \mathrm{diag}(\sigma_{A_1}, \sigma_{A_2}, \sigma_z)$, and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha_{A_1} & \alpha_{A_2} & \beta \end{pmatrix}.$$

The time-dependence and boundedness of $b(t)$ leads to a non-linear filtering problem. Stratonovich (1959, 1960) and Kushner (1967) present solutions to the non-linear case. However, except for some special cases (see Beneš, 1981), the evolution of the conditional distribution cannot be captures by that of a finite set of parameters. Conditions under which one can recapture the Markovian structure concern the efficiency of the representation of the historical information through the conditional moments. If the posterior distributions are fully described by a finite number of moments and if these moments can be updated recursively, one can identify a state vector solution. When the filter is not compact, that is the posterior distribution cannot be described by a finite number of moments but rather by infinitely many moments one might want to use an approximation. Since the underlying dynamics of the growth rate volatilities follow a finite state Markov chain, non-linear filtering becomes efficient via the Wonham (1964) filter which has been used in the literature before, see e.g. David (2008), Pakoš (2008), and Chen and Pakoš (2008).

The posterior beliefs of agent A are as follows:

$$dm^A(t) = (a_0 + a_1 m^A(t))dt + \gamma^A(t) A' B^{-1} dW_Y^A(t), \tag{2}$$

$$d\gamma^A(t)/dt = a_1 \gamma^A(t) + \gamma^A(t) a_1' + b(t) b(t)'(t) A'(BB')^{-1} A \gamma^A(t), \tag{3}$$

with initial conditions $m^A(0) = m_0^A$ and $\gamma^A(0)v = \gamma_0^A$, where $dW_Y^A(t) := B^{-1}\left(\left(dY(t) - Am^A(t)\right)dt\right)$ is the innovation process induced by investor A's belief and filtration.[9] The time-varying parameter $b(t)$ impacts on the distribution of $m^A(t)$ indirectly, by influencing the Riccati differential equation for $\gamma^A(t)$.

Note that since cash flow growth rates μ_{A_i} and the signal growth rate μ_z are unobservable, the true parameter value for $b(t)$ cannot be recovered from the quadratic variations of observable variables, even if one would sample at asymptotically high frequency. This implies that, at least in this setting, disagreement does not necessarily entail irrational agents.

Finally, to complete the specification of the disagreement structure in my economy, I finally define the disagreement dynamics implied by the learning dynamics of agent B. This process is the key state variable driving all equilibrium quantities. In my economy, it is defined by the following three dimensional process:

$$\Psi(t) := \begin{pmatrix} \Psi_{A_1}(t) \\ \Psi_{A_2}(t) \\ \Psi_z(t) \end{pmatrix} = \begin{pmatrix} \left(m_{A_1}^A(t) - m_{A_1}^B(t)\right)/\sigma_{A_1} \\ \left(m_{A_2}^A(t) - m_{A_2}^B(t)\right)/\sigma_{A_2} \\ \left(m_z^A(t) - m_z^B(t)\right)/\sigma_z \end{pmatrix}.$$

[9] A formal proof of this result can be found in Liptser and Shiryaev (2000).

The first two components of $\Psi(t)$ measure the disagreement about the expected growth rates of future cash flows. The third component captures the disagreement about the market-wide indicator $z(t)$. Since the market-wide uncertainty parameter σ_{μ_z} influences the subjective dynamics of each individual belief, it also has implications for the stochastic properties of the disagreement process itself. The dynamics for $\Psi(t)$ follows after a standard application of Itô's Lemma:

$$d\Psi(t) = B^{-1}\left(a_1 B + \gamma^B(t)A'B^{-1}\right)\Psi(t)dt + B^{-1}(\gamma^A(t) - \gamma^B(t))A'B^{-1}dW_Y^A(t), \tag{4}$$

with initial conditions $\Psi(0) = \Psi_0$ and $\gamma^B(0) = \gamma_0^B$. The average level and the heterogeneity of the subjective uncertainty parameters across agents are linked to the steady state distribution of the joint disagreement dynamics (4). This feature implies a natural link between the latent market-wide uncertainty and the stochastic properties of the heterogeneity in beliefs, like for instance, the level of the conditional dependence between the dividend disagreement processes $\Psi_{A_1}(t)$ and $\Psi_{A_2}(t)$. A sufficient degree of dependence between $\Psi_{A_1}(t)$ and $\Psi_{A_2}(t)$ can naturally motivate a large degree of commonality in beliefs as the one found in my data.[10]

C. Investors' Preferences and Equilibrium

There are two investors in the economy with different subjective beliefs, but identical in all other aspects, such as preferences, endowments, and risk aversion. They maximize the life-time expected power utility subject to the relevant budget constraint:

$$V^n = \sup_{c^n_{A_1}, c^n_{A_2}} E^n\left(\int_0^\infty e^{-\delta t}\left(\frac{c^n_{A_1}(t)^{1-\gamma}}{1-\gamma} + \frac{c^n_{A_2}(t)^{1-\gamma}}{1-\gamma}\right)dt \,\middle|\, \mathcal{F}_0^Y\right), \tag{5}$$

where $c^n_{A_i}(t)$ is the consumption of agent n of good i, $\gamma > 0$ is the relative risk aversion coefficient, and $\delta \geq 0$ is the time preference parameter. I assume time-separable utility functions. This not only simplifies the computation of the equilibrium, but also interpretations, since I can sum over individual beliefs without making any further assumptions on aggregation. Agents can trade in the risk-free bond, the firms' stocks, and additionally on options written on the stocks. I denote by $r(t)$ the risk-free rate of the zero-coupon bond, assumed in zero net supply, by $S_i(t)$ the stock price of firm i, assumed in positive net supply, by $O_i(t)$ the price of a European option on the stock i, assumed in zero net supply, and by $B_i^d(t)$ the price of the defaultable bond, also assumed in zero net supply. $V_i(t)$ denotes the firm value of firm i in my economy.

Definition 1 (Equilibrium). *An equilibrium consists of a unique stochastic discount factor such that (I) given equilibrium prices, all agents in the economy solve the optimization problem (5), subject to their budget constraint. (II) Good and financial markets clear.*

The equilibrium is solved using the martingale approach, originally developed by Cox and Huang (1989). The extension to the case with heterogenous beliefs is due, among others, to Cuoco and He (1994), Karatzas and Shreve

[10] Buraschi, Trojani, and Vedolin (2009) study in more detail co-movement patterns of the belief processes.

(1998), and Basak and Cuoco (1998). In this extension, the utility function of the representative agent is a weighted average of the utility functions of the individual agents. Different than in a standard setting, however, the relative weight $\lambda(t)$ is stochastic and a function of the heterogeneity in beliefs across agents. Thus, differences in beliefs affect real allocation of resources and equilibrium prices. Let $\xi^n(t)$ be the stochastic discount factor of agent n, the price of any contingent claim in equilibrium can be computed from the expectation of the contingent claim payoffs weighted by $\xi^n(t)$. In my economy, from the market clearing assumption and the optimality conditions (5), closed form expressions for $\xi^A(t)$ and $\xi^B(t)$ in terms of exogenous variables follow with standard methods:

$$\xi^A(t) = \frac{e^{-\delta t}}{y_A} A_1(t)^{-\gamma} \left(1 + \lambda(t)^{1/\gamma}\right)^{\gamma}, \quad \xi^B(t) = \frac{e^{-\delta t}}{y_B} A_1(t)^{-\gamma} \left(1 + \lambda(t)^{1/\gamma}\right)^{\gamma} \lambda(t)^{-1}, \tag{6}$$

where y_A and y_B are the Lagrange multipliers in the (static) budget constraint of agent A and B, respectively. The stochastic discount factor of each agent is equal to the product of two terms, the first is the standard homogeneous economy term which is proportional to the equilibrium marginal utility of consumption, the second is a function of weighting process $\lambda(t)$. The dynamics of the weighting process $\lambda(t) := y_A \xi^A(t) / \left(y_B \xi^B(t)\right)$ is:

$$\frac{d\lambda(t)}{\lambda(t)} = -\left(\sum_{i=1}^{2} \Psi_{A_i}(t) dW_{A_i}^A(t) + \left(\sum_{i=1}^{2} \alpha_{A_i} \Psi_{A_i}(t) \frac{\sigma_{A_i}}{\sigma_z} + \beta \Psi_z(t)\right) dW_z^A(t)\right). \tag{7}$$

Due to the separability assumption on agents' utility function individual state prices $\xi^n(t)$ can be expressed as functions only of $A_1(t)$ and $\lambda(t)$ (or equivalently $A_2(t)$ and $\lambda(t)$). The equilibrium relative price of good 1 and good 2 is $p(t) = (A_2(t)/A_1(t))^{-\gamma}$.

The dynamics of the weighting factor $\lambda(t)$ depends on perceived shocks to cash flows ($W_{A_1}^A(t)$ and $W_{A_2}^A(t)$, respectively) and the market-wide indicator ($W_z(t)$). The market-wide shocks impact on $\lambda(t)$ proportionally to the disagreement about cash flows and the market-wide signal ($\Psi_{A_1}(t)$, $\Psi_{A_2}(t)$, $\Psi_z(t)$), the relative precision of market-wide and firm-specific shocks (σ_{A_i}/σ_z, $i = 1, 2$), and the informativeness of the market-wide signal for estimating dividend growth (coefficients α_{A_1} and α_{A_2}). The state price volatility is stochastic and increasing in both $\Psi_{A_i}(t)$, $i = 1, 2$, and $\Psi_z(t)$. The difference in the agent-specific $\xi^n(t)$ reflects the different consumption plans of the two agents in the economy which are necessary to induce market clearing ex-ante. Assume, for illustration purposes, that investor A is optimistic about future cash flows of both firms. Then, in equilibrium, investor B will select a relatively higher consumption in states of low cash flows of either firm 1 or 2. Therefore, the relative consumption share in this economy is stochastic and its cyclical behavior is reflected by the the stochastic weight $\lambda(t)$.

To finance her consumption plan, the pessimistic investor needs to buy financial protection, e.g., put options, from the more optimistic agent. This excess demand increases the price of securities with negative exposure to cash flow shocks. In this way, part of the risk embedded in bad cash flow states is transferred from the pessimist to the optimist. Ex post, if a negative state occurs, the more optimistic agent is hit twice: First, because the aggregate endowment is lower, second, as a consequence of the protection agreement which makes her consumption share lower in those states. Ex ante, the more optimistic agent is compensated by a premium for having entered the insurance contract with the pessimist. This premium is increasing in the degree of disagreement among agents about the probability of future bad cash flow states.

D. Pricing of Financial Assets

Given the expressions for the individual state price densities $\xi^A(t)$ and $\xi^B(t)$, it is straightforward to price any contingent claim in the economy by computing expectations of its contingent payoffs weighted by state price densities. For simplicity, I give the relevant pricing expressions from the perspective of agent A. The equilibrium firm value and the price of a default-free zero coupon with maturity T of firm i are given by:

$$V_i(t) = A_i(t)E_t^A\left(\int_t^{\infty} e^{-\rho(u-t)}\frac{\xi^A(u)}{\xi^A(t)}\frac{A_i(u)}{A_i(t)}du\right) \ , \ B_i(t,T) = E_t^A\left(\frac{\xi^A(T)}{\xi^A(t)}\right) \ .$$

As in the Merton (1974) model, default occurs only at maturity of the corporate bonds, if the value of the assets of the firm is less than the face value of the bond. I assume zero-bankruptcy costs. Therefore, in the event of default, equity holders are left with a zero price of equity and the corporate bond holders share the residual firm value. To focus on the implications of disagreement and macro-economic uncertainty, I keep the default structure as simple as possible and do not explore more general default rules or more flexible default and liquidation procedures. In this setting, it follows that the price of the defaultable bond is the sum of the prices of the zero-coupon bond and the price of a short put option on the firm value and the equity is the firm value residual in excess of the price of the corporate debt or a call option on the firm value:

$$B_i^D(t,T) = K_{B_i}B_i(t,T) - E_t^A\left(\frac{\xi^A(T)}{\xi^A(t)}\left(K_{B_i} - V_i(T)\right)^+\right), \ S_i(t) = E_t^A\left(\frac{\xi^A(T)}{\xi^A(t)}\left(V_i(T) - K_{B_i}\right)^+\right), \tag{8}$$

where K_{B_i} is the face value of the bond of firm i. To compute the expectations in the pricing expressions, I need the joint density of $A_1(t), A_2(t)$, $\lambda(t)$, and the contingent claim payoff, since the stochastic discount factor, $\xi_A(t)$, is a function of both $A_1(t), A_2(t)$ and $\lambda(t)$. Unfortunately, the joint distribution of $A_i(t), A_j(t)$ and $\lambda(t)$ is typically unavailable in closed-form.

In general, there exist several methods to compute the moments of the posterior distributions. The first one includes either Monte Carlo simulations or numerical solutions of the partial differential equation which arises from the expectations via the Feynman-Kač methodology. However, this method not only lacks accuracy but is also highly time-consuming, especially when dealing with multi-asset economies where the state space is high dimensional. Another one is to recover the conditional density of the state variables through Fourier inversion of the conditional characteristic function, which is often available in closed-form. Then one can evaluate the moments directly using this density. This is the approach taken by Heston (1993), Bates (1996), Bakshi, Cao, and Chen (1997), and Dumas, Kurshev, and Uppal (2009). More recently, Chen and Joslin (2009) propose to take the Fourier transform of the moment function and to replace the moment function with an integral of exponential functions, for which the expectations can be computed analytically through the Duffie, Pan, and Singleton (2000) approach.[11]

While the second method can run into curse of dimensionality due to multiple numerical integrals (see e.g. Dumas, Kurshev, and Uppal, 2009), I can calculate the joint Laplace transform in closed-form with only one single integral left. In order to obtain closed-form expressions, I reduce the system of ordinary differential equations for functions A_Ψ, B_Ψ and C_Ψ in Lemma 1 to a system of matrix Riccati equations, which can be linearized using Radon's Lemma and explicitly solved. This Laplace transform can be used, in a second step, to price all securities in my economy by Fourier Transform methods more efficiently. In this way, I can avoid a pricing approach that relies exclusively on simulation methods, which would be highly computationally intensive if not infeasible.[12] Note, however, that I compute the joint Laplace transform of $A_1(t), A_2(t)$, and $\lambda(t)$ in the case $\sigma^A_{\mu_z} \neq \sigma^B_{\mu_z}$, which admits a steady state distribution for the relevant state variables in my model.

Lemma 1. *Under the steady state distribution, the joint Laplace transform of $A_1(t), A_2(t)$ and $\lambda(t)$ with respect to the belief of agent A is given by:*

$$E^A_t\left(\left(\frac{A_1(T)}{A_1(t)}\right)^{\epsilon_{A_1}}\left(\frac{A_2(T)}{A_2(t)}\right)^{\epsilon_{A_2}}\left(\frac{\lambda(T)}{\lambda(t)}\right)^{\chi}\right) = F_{m^A}\left(m^A, t, T; \epsilon_{A_1}, \epsilon_{A_2}\right) \times F_\Psi\left(\Psi, t, T; \epsilon_{A_1}, \epsilon_{A_2}, \chi\right), \tag{9}$$

where

$$F_{m^A}(m^A, t, T; \epsilon_{A_1}, \epsilon_{A_2}) \quad = \quad \exp\left(A_{m^A}(\tau) + B_{m^A}(\tau) m^A\right), \tag{10}$$

with $\tau = T - t$ and

$$F_\Psi\left(\Psi, t, \epsilon_{A_1}, \epsilon_{A_2}, \chi, u\right) = \exp\left(A_\Psi(\tau) + B_\Psi(\tau)\Psi + \Psi' C_\Psi(\tau)\Psi\right).$$

for functions $A_{m^A}, B_{m^A}, A_\Psi, B_\Psi$ and C_Ψ detailed in the proof in the Appendix.

The Laplace transform in Lemma 1 is a function of $m^A(t)$ and $\Psi(t)$. The dependence on $m^A(t)$ is exponentially affine. The dependence on $\Psi(t)$ is exponentially quadratic. By computing the closed–form characteristic function of $A_1(t), A_2(t)$, and $\lambda(t)$ I can now price the contingent claims in the economy more by Fourier inversion methods. The spirit of this approach is similar to the one used to price derivatives in stochastic volatility models, such as Heston (1993), Duffie, Pan, and Singleton (2000), and Carr, Geman, Madan, and Yor (2001), or in interest-rate models, such as Chacko and Das (2002). The pricing expressions implied by the Fourier Transform approach for all contingent claims in my economy can now be summarized. Let

$$G(t, T, x_{A_1}, x_{A_2}; \Psi) \equiv \int_0^\infty \left(\frac{1+\lambda(T)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^\gamma \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\lambda(T)}{\lambda(t)}\right)^{-i\chi} F_\Psi\left(\Psi, t, T; x, i\chi\right) d\chi\right] \frac{d\lambda(T)}{\lambda(T)}.$$

[11] For example, Martin (2009) takes the Fourier transform of a nonlinear pricing kernel where the characteristic function of the Fourier transform is known in closed form.

[12] The approach taken by Chen and Joslin (2009) is, however, the way to proceed for future research, especially for estimation proposes.

The equilibrium firm value of firm 1 is:

$$V_1(t) := V_1(A_1, m_A, \Psi) = A_1(t)\int_t^{\infty} e^{-\rho(u-t)} F_{m^A}(m_A, t, u; 1-\gamma, 0)\, G(t, u, 1-\gamma; \Psi)\, du.$$

Similarly, the equilibrium firm value of firm 2 is:

$$\begin{aligned} V_2(t) &:= V_2\left(A_1, A_2, m^A, \Psi\right), \\ &= A_2(t)\int_t^{\infty} e^{-\delta(u-t)} F_{m^A}\left(m^A, t, u; -2\gamma, 1+\gamma\right) G(t, u, -2\gamma, 1+\gamma; \Psi)\, du. \end{aligned}$$

The equilibrium price of the corporate zero-coupon bond is:

$$B_i(t,T) := B_i(t,T; m^A, \Psi) = e^{-\rho(T-t)} F_{m^A}\left(m^A, t, T; -\gamma\right) G(t, T, -\gamma; \Psi).$$

The equilibrium price of the defaultable bond of firm i is:

$$B_i^D(t,T) := B^D\left(t, T; A_i, m^A, \Psi\right) = B(t,T) - E_t^A\left(e^{-\rho(T-t)}\left(\frac{A_i(t)}{A_i(T)}\frac{1+\lambda(T)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^{\gamma}(K_{B_i} - V_i(T))^+\right).$$

The equilibrium price of equity of firm i is:

$$\begin{aligned} S_i(t) &:= S_i\left(t, T; A_i, m^A, \Psi\right), \\ &= E_t^A\left(e^{-\delta(T-t)}\left(\frac{A_i(t)}{A_i(T)}\frac{1+\lambda(T)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^{\gamma}(V_i(T) - K_{B_i})^+\right), \end{aligned}$$

From the above formulas, I obtain a semi-explicit description for the dependence of the prices of corporate bonds and equity on economic uncertainty and the degree of disagreement about cash flows and the market-wide signal. In the standard Merton (1974) model, the firm value volatility is constant and the risk–neutral skewness is zero. It follows that the price of equity is increasing in the firm value volatility parameter and co-moves positively with the value of the firm. In my economy with uncertainty and leverage, the volatility and risk-neutral skewness of the firm value are stochastic. For firms with different degrees of leverage, I find that the price of equity can either increase or decrease with disagreement. This is also documented in Buraschi, Trojani, and Vedolin (2008). To give an intuition why these features can arise, I shortly sketch the main intuition. Note that the price of equity can be represented as a portfolio consisting of a long position in the firm value $V_i(t)$, a short position in K_1 risk–less zero bonds with price $ZCB(t)$, and a long position in an out-of-the-money put on the firm value, with strike K_1 and price $P(t)$:

$$S_i(t) = V_i(t) - K_1 \cdot ZCB(t) + P_i(t, K_1)\ .$$

The first term, V_i, is independent of leverage and is decreasing in disagreement. The price of the zero coupon bond can be shown to be decreasing in disagreement for a relative risk aversion parameter greater than one. Thus, the effects of the first two components of the price of equity tend to offset each other, with the second component increasing proportionally to firm leverage. The last term – i.e. the price of the put option $P(t, K_1)$ – has a positive impact on the price of equity, but the size of the effect depends significantly and in a non monotonic way on firm's leverage. For some regions of leverage, I find that this effect can be large enough to reverse the negative impact of the change in the value of the firm:

$$\underset{+/-}{\frac{dS}{d\Psi}} = \underset{-}{\frac{dV}{d\Psi}} - K_1 \cdot \underset{-}{\frac{dZCB}{d\Psi}} + \Big[\overbrace{\underset{-}{\frac{dP}{dV}} \cdot \underset{-}{\frac{dV}{d\Psi}}}^{\text{Delta: } +} + \overbrace{\underset{+}{\frac{dP}{d\sigma_V}} \cdot \underset{+}{\frac{d\sigma_V}{d\Psi}}}^{\text{Vega: } +} + \overbrace{\underset{-}{\frac{dP}{dSk_V}} \cdot \underset{-}{\frac{dSk_V}{d\Psi}}}^{\text{Skewness: } +} \Big]. \tag{11}$$

When leverage is high, the dominating effect on the price of equity comes from the first two terms in (11), as the Delta, Vega, and Skewness effects on the put price are all small in relative terms. For very low leverage values, the values of the put option and the position in the zero bond are a small fraction of firm value. Therefore, the price of equity is dominated by the first term in (11). It follows that for high and very low leverage the value of equity is decreasing with disagreement at the calibrated model parameters. For the intermediate leverage region, however, the price of the embedded out-of-the-money put option can be a non–negligible fraction of the firm value, and its sensitivity to increases in negative skewness (the last term in square brackets) is high. In particular, I find that the last effect can be high enough to compensate the negative change of the firm value and make the price of equity increase. Buraschi, Trojani, and Vedolin (2008) present calibrated evidence of these effects in an economy with one firm. Similar to the findings for the firm value, the endogenous stochastic co–movement between the price and the volatility of equity generates an asymmetric physical stock price density. However, in contrast to the unambiguously negative sign of the skewness of firm value, the skewness of stock returns can be both positive and negative in my model: The positive (negative) co–movement between the price and the volatility of equity tends to generate stock returns that are positively (negatively) skewed.

E. Volatility Risk Premia in the Cross-Section

In the models of Bollerslev, Tauchen, and Zhou (2009) and Drechsler and Yaron (2008) a high variance in returns occurs when uncertainty is high. Thus, any asset that pays off a lot in those states with high realized variance of returns is a hedge for the uncertainty risk, which explains why the variance premium is positive in their economy. In my economy, as a function of leverage, there exists assets that do not pay off a lot in bad states of the world. As we have seen in the previous section, this applies for low leverage firms. The volatility risk premium in my economy can be expressed as follows:

$$vp(t, t+1) = \mathrm{Cov}_t \left(\frac{\xi^A(t+1)}{\xi^A(t)}, RV_i\,(t, t+1) \right) \lesseqgtr 0.$$

To calculate the conditional covariance between the stochastic discount factor and the realized volatility, note that the diffusion of the stochastic discount factor can be written as follows:

$$\begin{aligned}\frac{d\xi^A}{\xi^A(t)} - E_t\left(\frac{d\xi^A(t)}{\xi^A(t)}\right) &= -\left(\gamma\sigma_{A_1} + (1-s_i(t))\,\Psi_{A_1}(t)\right) dW^A_{A_1}(t) \\ &\quad - (1-s_i(t))\left(\alpha_{A_1}\Psi_{A_1}(t)\frac{\sigma_{A_1}}{\sigma_z} + \alpha_{A_2}\Psi_{A_2}(t)\frac{\sigma_{A_2}}{\sigma_z} + \beta\Psi_z\right) dW^A_z(t).\end{aligned}$$

The realized volatility is given in Appendix A. It is obvious that an analytic closed-form expression is infeasible. However, it is straightforward, to borrow from the intuition of the risk-neutral skewness. Bakshi, Kapadia, and Madan (2003) relate the risk-neutral skewness of the return distribution to the steepness of the implied volatility smile and in particular find that the more negative the risk-neutral skewness, the steeper the implied volatility smile. The optimal risk sharing between optimistic and pessimistic agents in my economy implies a lower price for contingent claims with positive exposure to future cash flows. This price is proportional to a stochastically weighted marginal utility of optimistic and pessimistic agents in the economy, with a weight that is a function of agents' relative consumption share. Since agents have decreasing marginal utility and the equilibrium consumption share of the pessimist (optimist) is larger in low (high) dividend states, the price of all states tends to be lower in the economy with heterogeneity in beliefs. However, note that the equilibrium state price adjustment associated with low dividend states tends to be larger than the one associated with high dividend states, which yields an endogenous negative risk-neutral skewness of stock returns for high levered firms. In case of a high disagreement or high cash flow growth volatility and low leverage, the opposite can occur which renders an endogenous positive risk-neutral skewness. In my model, this asymmetry in the equilibrium risk-neutral distribution follows from the fact that the agents' marginal utility in the economy tends to be larger in low dividend states. It then follows that in order to reallocate a given amount of consumption across agents in bad dividend states, a larger state price adjustment is needed. Due to the assumption that agents have power utility, the marginal utility is convex, and hence these features also imply that the state price of a bad aggregate dividend state is proportionally lower that the average state price of a bad dividend of either one of the two firms in the economy.

[Insert Figure 3 approximately here.]

Figure 3 plots the volatility risk premia for different levels of leverage at different option moneyness and levels of macro-economic uncertainty. The three different leverage bins correspond to the lower tercile (leverage below 0.05), median tercile (leverage at 0.10), and high tercile (leverage above 0.14). The disagreement is set to 0.5 in the high disagreement state (i.e. $\Psi_{A_1} = \Psi_{A_2} = \Psi_z = 0.5$) and 0.1 in the low disagreement state (i.e. $\Psi_{A_1} = \Psi_{A_2} = \Psi_z = 0.1$). The cash flow growth volatility is set to 0.0029 in the low uncertainty case and 0.0041 in the high uncertainty case, which corresponds to a two standard deviations change from the unconditional mean of the consumption growth volatility. In the left panel, I plot the volatility risk premium for a high leverage firm. The figure displays the usual left skewness of stocks. Increasing the disagreement from 0.1 (gray line) to 0.5 (black line) increases the volatility risk premium of the individual firm by almost 0.5% which corresponds to a 50% increase of the at-the-money volatility risk premium. Interestingly, an increase in the volatility of the cash flow growth rate not only implies a shift in the level of the volatility risk premium but also in the steepness of the smile. For instance, increasing the cash flow growth volatility from 0.0029 (gray line) to 0.0041 (red line), induces an ascent in the slope coefficient especially in the out-of-the-money put region. For intermediate leverage, the repercussions of a rise in both disagreement and uncertainty are larger than for the high leverage firm. Increasing the disagreement implies a 0.5% boost in the volatility risk premia across the whole moneyness spectrum. I note, however, that the slope is slightly smaller than for the high leverage firm. Finally, in the right panel, I plot the volatility risk premia for a low leverage firm. As expected from the previous results, the volatility risk premium turns negative at-the-money for almost all specifications except the low disagreement and low uncertainty case.

F. Simulated Regressions

Since the volatility risk premium rises or decreases with uncertainty and this is directly linked to the stochastic discount factor, volatility risk premia predict stock returns or corporate credit spreads. Firstly, I show that both subjective and macro-economic uncertainty affect significantly the volatility risk premia in the cross-section of stocks. To this end, I run panel regressions from the volatility risk premia of the individual firms onto the firm-specific, common disagreement and the consumption growth volatility. The results are presented in Table 4.

[Insert Table 4 approximately here.]

All three measures of uncertainty positively impact on the volatility risk premia. In terms of adjusted R^2, the uncertainty proxies explain almost 20% of the variation in the volatility risk premia. In the second and third column, I present regression results for low and high leverage firms. The low leverage firms are derived by fixing the leverage to 0.05 and the high leverage firms are fixed to have a debt to equity ratio of 0.2. While the estimated coefficient for all three uncertainty proxies is indeed negative while for the high leverage firms, the impact was consistently positive across all leverage bins.

The volatility risk premium in my economy predicts stock returns and corporate credit spreads. The predictive regression is:

$$r_i(t+1) = \beta_0 + \beta_1 \times \left(vol_t^{\mathbb{Q}}(r_i(t+1)) - vol_t^{\mathbb{P}}(r_i(t+1)) \right) + \epsilon(t),$$

where

$$\beta_1 = \frac{\text{Cov}\left(r_i(t+1) + \epsilon(t), vol_t^{\mathbb{Q}}(r_i(t+1)) - vol_t^{\mathbb{P}}(r_i(t+1)) \right)}{\text{Var}\left(vol_t^{\mathbb{Q}}(r_i(t+1)) - vol_t^{\mathbb{P}}(r_i(t+1)) \right)}.$$

And a similar expression arises from the regression of the corporate credit spread. From the previous section, we know that the expected stock return is positively related to the volatility increase: For high leverage firms, the expected stock return and volatility increases with a rise in uncertainty. For low leverage firms, both the expected stock returns and volatility decrease. To test this conjecture, I present in Table 5 the predictability regressions for a horizon of 1, 6, and 12 months.

[Insert Table 5 approximately here.]

The volatility risk premium loads positively on the expected stock returns and corporate credit spreads at all horizons. For the stock returns, the size of the coefficient decreases by more than 50% while for the corporate credit spreads the size increases by almost 50%. In line with the empirical findings , the adjusted R^2 for the stock return regression display the decreasing pattern at longer horizons. For the corporate credit spreads, the adjusted R^2 increases which is contrary to what I found in the data.

II. Data

I firstly present some new empirical evidence that underscores the importance of including agents' disagreement and macroeconomic uncertainty when studying volatility risk premia. I then study the predictive power of the volatility risk premium for expected stock excess returns and corporate credit spreads. The firm-specific, common disagreement, and volatility of consumption growth explain significant portions of the volatility risk premium in the cross-section. The three measures together explain 12% of the time variation in the volatility risk premia. Similarly, all uncertainty measures explain on average around 8% of the time variation in the risk-neutral skewness. Volatility risk premia predict stock excess returns and corporate credit spreads in the cross-section. The predictive power is concentrated at shorter horizons (8 months) and then drops considerably for stock returns and remains fairly stable for corporate credit spreads.

I merge a panel of option, stock and bond prices and firm-specific information with analysts' forecasts of future earnings. The time period covers January 1996 to September 2008. Appendix B summarizes the data construction in detail.

A. Option Data

I use option information from the OptionMetrics Ivy DB database, which is the most comprehensive database available. Individual stock options are American style and usually expire on the Saturday following the third Friday of the contract month. Therefore, time to maturity is defined as the number of calendar days between the last trading date and expiration date. I apply a number of data filters to circumvent the problem of large outliers. First, I eliminate prices that violate arbitrage bounds, i.e. call prices are required not to fall outside the interval $\left(Se^{-rd} - Ke^{-\tau r}, Se^{-\tau d}\right)$, where S is the value of the underlying asset, K is the strike price, d is the dividend yield, r is the risk-free rate, and τ is the time to maturity. Second, I eliminate all observations for which (i) the ask is lower than the bid price, (ii) the bid is equal to zero, or (iii) the spread is lower than the minimum tick size (equal to USD 0.05 for options trading below USD 3 and USD 0.10 in any other cases). Importantly, to mitigate the impact of stale quotes I eliminate all observations for which both the bid and the ask are equal to the one on the previous day. I focus on short-term options which are known to be the most liquid with a time to maturity between 14 and 31 days.

The volatility risk premium is defined as the difference between the risk-neutral and physical expectation of the return variation. Thus, the one-month volatility risk premium, $vp(t, t+1)$ is defined as $E_t^{\mathbb{Q}}\left(\int_t^{t+1} dr_i\right) - E_t^{\mathbb{P}}\left(\int_t^{t+1} dr_i\right)$ where

$\mathbb{Q}$ denotes the risk-neutral measure and r_i is the return of firm i. In this paper, I specify a model-free implied volatility using a continuum of European call options.[13] Let $C(t,T,K)$ denote the price of a European call option maturing at T with strike price K at time t, and $B(t,T)$ is the price of a time t zero-coupon bond with maturity T. Carr and Madan (1998) and Britten-Jones and Neuberger (2000) show that the risk-neutral expectation of the return variance between time t and $t+1$ can be expressed in a model-free fashion by the following expression:

$$IV(t,t+1) \equiv E_t^{\mathbb{Q}}\left(Var(t,t+1)\right) = 2\int_0^{\infty} \frac{C\left(t,t+1,K\right) - \max\left(S(t)-K,0\right)}{K^2} dK,$$

which takes advantage of a continuum of calls with strikes from zero to infinity. In practice, the implied variance must be constructed from a finite number of strikes, which turns out to be a fairly accurate approximation to the true risk-neutral expectation of the future variance under reasonable assumptions (see Jiang and Tian, 2005). To construct a realized variance under the physical measure, I approximate the realized variance for month t as:

$$RV(t,t+1) \equiv \sum_{j=1}^{n} r_i(j)^2,$$

where $r_i(j)$ is the stock return of firm i at time j. To measure the a model-free skewness, I use a model-free and ex-ante measure of risk-neutral skewness given by Bakshi, Kapadia, and Madan (2003). For each stock on date t, the skewness of the risk-neutral density of the stock return over the period $[t,t+1]$ can be inferred from the contemporaneous prices of out-of-the-money call options and put options as follows:

$$Skew(t,\tau) = \frac{e^{r\tau}W(t,\tau) - 3\mu(t,\tau)e^{r\tau}V(t,\tau) + 2\mu(t,\tau)^3}{\left(e^{r\tau}V(t,\tau) - \mu(t,\tau)^2\right)^{3/2}},$$

where

$$\mu(t,\tau) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}V(t,\tau) - \frac{e^{r\tau}}{6}W(t,\tau) - \frac{e^{r\tau}}{24}X(t,\tau),$$

and $V(t,\tau), W(t,\tau)$, and $X(t,\tau)$ are the weighted sums of out-of-the-money call option prices $C(t,\tau,K)$ and put option prices $P(t,\tau,K)$ and expressions given in the Appendix A.4. The computation of the intrinsic value of higher-moment payoffs requires options with constant maturity and I thus fix it to 28 days. Details on the Riemann integral approximation with other related implementation issues are addressed in Jiang and Tian (2005) and Bakshi and Madan (2006). Moreover, implementation with a finite grid of out-of-the-money calls and puts is reasonable accurate with small approximation errors (see Dennis and Mayhew, 2002).

[13] There are several reasons to use the model-free implied volatility opposed to one which comes from a option pricing model, say, the Black and Scholes (1973) model. First, the approximation seems to be fairly accurate of the true (unobservable) risk-neutral expectation of the future return volatility than an inversion of the Black and Scholes formula for at-the-money options, see e.g. Jiang and Tian (2005). Second, Bollerslev, Tauchen, and Zhou (2009) report much higher R^2 when regressing excess market returns on the volatility risk premium constructed from model-free measures rather than the Black and Scholes implied volatility.

B. Stock Returns Data

Stock data is retrieved from the CRSP database. To calculate the realized volatility, I use daily stock returns from CRSP. I calculate the realized volatility over 21-day windows, requiring that the stock has at least 15 non-zero return observations.

C. Bond Data

The bond data is obtained from the Fixed Income Securities Database (FISD) on corporate bond characteristics and the National Association of Insurance Commissioners (NAIC) database on bond transactions. The FISD database contains issue and issuer-specific information for all U.S. corporate bonds. The NAIC data set contains all transactions on these bonds by life insurance, property and casualty insurance, and health maintenance companies, as distributed by Warga (2000). This database is an alternative to the no longer available database used by Duffee (1998), Elton, Gruber, Agrawal, and Mann (2001), and Collin-Dufresne, Goldstein, and Martin (2001). U.S. regulations stipulate that insurance companies must report all changes in their fixed income portfolios, including prices at which fixed income instruments were bought and sold. Insurance companies are major investors in the fixed income market and, according to Campbell and Taksler (2003), they hold about one-third of outstanding corporate bonds. These data represent actual transaction data and not trader quotes or matrix prices. Initially, I eliminate all bonds with embedded optionalities, such as callable, putable, exchangeable, convertible securities, bonds with sinking fund provisions, non-fixed coupon bonds, and asset-backed issues. The data set contains information on the seniority level of the bonds. I am thus able to divide our data sample into senior secured and junior subordinated bonds. I manually delete all data entry errors. Moreover, to control for the possibility of residual errors, I windsorize our database at the 1% and 99% level. Finally, to compute corporate bond credit spreads, I use zero-coupon yields available from the Center for Research in Security Prices (CRSP).

D. Difference in Beliefs Proxy

To obtain a proxy of firm-specific belief disagreement, I follow the procedure in Buraschi, Trojani, and Vedolin (2008). I use analysts' forecasts of earnings per share from the Institutional Brokers Estimate System (I/B/E/S) database and compute for each firm the mean absolute difference of analysts' earning forecasts scaled by an indicator of earnings uncertainty. In order to get a common belief disagreement factor for the index, I estimate a dynamic component using factor analysis for the analysts' earning forecasts. Factor analysis has mainly been implemented for forecasting measures of macroeconomic activity and inflation (see, e.g., Stock and Watson 2002a, 2002b, 2004) and more recently in financial applications (see Ludvigson and Ng, 2007 and 2009). Dynamic factor models allows us to escape the limitations of existing empirical analyzes in several dimensions. If comovement between individual difference in beliefs is strong, it makes sense to represent the overall belief disagreement in the economy by an index or a few factors, which describe the common behavior of these variables. Using dynamic factors instead of static principal components has two reasons. First, I want it to be dynamic. Static factor analysis allows only for a contemporaneous relation in the cross-section. However, empirical evidence points towards lagged effects of increases in uncertainty across different industries. Second, I want to allow for cross-correlation among the idiosyncratic components, because orthogonality is an unrealistic assumption in our setting. I estimate the common belief disagreement process according to Forni, Hallin, Lippi, and Reichlin (2000). Generalized dynamic factor models are usually characterized by a large time series dimension and a small cross section. Forni, Hallin, Lippi, and Reichlin (2000) generalize the dynamic factor models by allowing for a limited amount of cross-correlation among the idiosyncratic components. The main idea is that the panel of belief disagreement processes can be decomposed into the sum of two non-observable, mutually orthogonal parts, namely the common component and the idiosyncratic component. I disregard the idiosyncratic component, because it is the common component I am interested in. In the dynamic factor approach, the common component results from the interaction of a small number of unobservable shocks. Since the common component is unobservable, it has to be estimated from aggregating the disagreement proxies in the panel. The intuition is that the aggregation will eliminate the idiosyncratic component.

I now shortly outline the estimation procedure, however, for a more thorough description the reader is referred to the Appendix B and further technicalities can be found in Forni, Hallin, Lippi, and Reichlin (2000). The estimation consists of two steps. In the first step, I estimate the spectral density matrix of the common and idiosyncratic components by means of a dynamic principal component procedure. From the spectral density matrices, I can derive the covariances of the common component by applying the Fourier transform. In a second step, I estimate the factor space. From the covariances estimated in the first step, I estimate the generalized principal components as linear combinations of the observable belief disagreement processes. The key idea is that these generalized principal components[14] have the smallest idiosyncratic-common variance ratio (for further technical details, see Forni, Hallin, Lippi, and Reichlin, 2000). To estimate the common component, I weight the individual belief disagreement processes of each firm by its market capitalization.[15]

E. Other Control Variables

Leverage is a natural variable to include in the regressions and it is defined as total debt divided by total asset value. Fama and French (1992) suggest that size is a potential risk factor in the cross-section of stock returns and it is reasonable to assume that size impacts on the cross-section of option prices. Firm size is defined as the natural log of the firm's market value of equity, in thousands of dollars. Firms which are highly correlated with the market are potentially useful to hedge market risk. I therefore expect firms which are more exposed to market risk (as proxied by the market beta) to have a more negative risk-neutral skewness. The firm's market beta is calculated from a regression from the stock excess returns on the market excess returns.

Following earlier work (see Hansen and Singleton, 1983), aggregate consumption is measured as the seasonally adjusted real consumption of nondurables plus services. Consumption volatility is then estimated from a Markov Chain as in Hamilton (1989). The state conditional volatilities are 0.423 and 0.182 for the high and low state,

[14]For a formal treatment of generalized eigenvalues and eigenvectors, see Wilkinson (1965).

[15]For robustness, I also used equal weights for the common component. The results remain quantitatively the same.

respectively. The volatility regimes are persistent given that the probability of remaining in one state is 0.992 for the high state and 0.973 for the low state. The filtered volatility is then calculated as:

$$\hat{\sigma}_c = \pi(t)\overline{\sigma} + (1 - \pi(t))\,\underline{\sigma},$$

where $\pi(t)$ is the filtered probability of a high volatility state, $\overline{\sigma}$ is set to 0.423 and $\underline{\sigma}$ is equal to 0.182. Corradi, Distaso, and Mele (2009) document a significant impact of macro-economic variables on the index volatility risk premium. While the consumption volatility captures the macro-economic uncertainty, I construct a macroeconomic level factor by extracting the first principal component using industrial production, housing start number, the producer price index, non-farm employment, and the S&P 500 P/E ratio. I retrieve S&P 500 price-earnings data from the S&P webpage, and the other macro variables I get from FRED.

Summary statistics of the main variables and different uncertainty proxies are presented in Table 1. In Panel A, I present the 5th, median, and 95th percentile of the implied volatility, realized volatility, and risk-neutral skewness for the years 1996 to 2008. A clear pattern is apparent. The implied and realized volatilities seem to move in lock step. Both peak around 2000, during the dot-com bubble, then descent and regain momentum in 2007 and 2008.[16] The median risk-neutral skewness is negative, which indicates a left-skewed distribution of average. It reaches its lowest level in 2000, when the 5th percentile is around -5.2. There are some periods, where the median volatility risk premium turns negative. For instance in the year 1998, the volatility risk premium is almost -1%. Panel B presents the average summary statistics for the main variables together with their unconditional correlations with the volatility risk premium and the risk-neutral skewness. To calculate the unconditional correlation, I first estimate the correlation firm-by-firm and then average in the time-series. The average firm-specific disagreement is almost 30%, the common disagreement is 9%, and the consumption volatility is 0.35%. The consumption growth volatility is highly persistent with a autocorrelation coefficient of 0.96. The firm-specific and common disagreement proxies are less persistent with values of 0.45 and 0.6, respectively. The macro factor displays a very high autocorrelation of 0.96.

[Insert Table 1 approximately here.]

III. Empirical Analysis

A. Determinants of Volatility Risk Premia and Skewness in the Cross-Section

To test the impact of the volatility of consumption growth and disagreement on the volatility risk premium and skewness, I run the following set of panel regressions:

$$y_i(t) = \beta_0 + \beta_1 DIB_{i,t} + \beta_2 \overline{DIB}_t + \sum_{j=3}^{6} \beta_j \text{Control}(j)_{i,t} + \sum_{k=1}^{3} \gamma_k \text{Control}_t + \epsilon_{i,t},$$

[16] Using a different measure of the risk-neutral volatility, Conrad, Dittmar, and Ghysels (2008) find a very similar result.

where $y_i(t)$ is the volatility risk premium and skewness of firm i, $DIB_{i,t}$ is the proxy of belief disagreement of each individual firm i at time t, $\overline{DIB}_t$ the common disagreement estimated from the cross-section of individual disagreement proxies, Control$_{i,t}$ are leverage, firm size, and market beta of each firm i at time t, and Control$_t$ is the macro factor. Table 6 reports the results.

[Insert Table 6 approximately here.]

All uncertainty measures load positively on the volatility risk premium and the risk-neutral skewness and are economically and statistically significant. For instance, a one standard deviation change in the firm specific (common) disagreement induces a 1% (0.5%) change in the volatility risk premium, similarly, a 1% standard deviation change in the consumption growth volatility yields a 0.4% change in the volatility risk premium. Leverage is positively correlated with the volatility risk premium and it is statistically significant for all specifications. The other firm-specific factors such as the firm size or the market beta are not significant. The aggregate macro factor is significant and loads negatively on the volatility risk premium which points towards the counter-cyclical nature of the volatility risk premia. As already noted in the example in the introduction, leverage could be an important variable when studying the cross-section of volatility risk premia. As outlined in the example, the low leverage firms, on average, change their sign more often than the high leverage firms. Moreover, the average volatility risk premium of the low leverage firms is 30% lower than for the high leverage firms. To better understand the impact of leverage on the volatility risk premia, I stratify all firms into three different leverage bins. The results are presented in the lower panel of Table 6. Low, medium, and high indicate the lower, medium, and higher tercile of the leverage distribution across all firms. For the low leverage firms, I notice that the sign of the slope coefficient of the firm-specific disagreement and consumption volatility is negative, meaning that a higher idiosyncratic disagreement and consumption growth volatility implies a lower volatility risk premium. The estimated coefficient for the common disagreement proxy remains positive across all leverage bins, however, for the median bin, the estimated coefficient is statistically significant at the 10% level only.

B. Predictability Regressions

Bollerslev, Tauchen, and Zhou (2009), and Zhou (2009) document the strong predictive power of the market volatility risk premium for excess market returns and the default premium, defined as the difference between the Moody's BAA and AAA corporate bond spreads. Cast in a model with long-run risk, the authors find that the entire time-variation in the variance risk premium comes from the assumption of a stochastic volatility of consumption growth volatility and hence a higher persistency of the stochastic volatility will induce predictability in returns.

In the following, I seek to examine the cross-sectional relationship between individual volatility risk premia/risk-neutral skewness and the cross-section of stock returns and credit spreads. To this end, I regress monthly equity excess returns on the individual volatility risk premia:

$$xr_i(t+h) = \alpha + \beta_1 VolRP_i(t) + \epsilon(t+h),$$

where xr_i is the excess return of stock i or the credit spread of firm i and h goes to horizon 12 months. The results are reported in Table 7.

[Insert Table 7 approximately here.]

The degree of predictability, as measured by the adjusted R^2, starts out rather low but reaches a peak at eight months with an adjusted R^2 of 8% and then drops by 50% at a horizon of one year. The size of the estimated coefficient remains remarkably robust across all horizons. On average, a 1% deviation change in the volatility risk premium leads to a 0.6% change in the excess stock return. The pattern for the individual volatility risk premia and stock excess returns is similar to the ones found in Bollerslev, Tauchen, and Zhou (2009) for the index. However, the estimated coefficients in my regressions are on average more significant than in their regression. While for the index, the volatility risk premium has no predictive power after a horizon of six months, the estimated coefficients for the individual volatility risk premia remain statistically significant up to one year with a t-statistic of 2.02 at the 12 month horizon. Panel B presents estimated coefficients for the credit spread predictability regressions. The estimated coefficients are statistically significant across all horizons with t-statistics ranging from 2.02 to 5.29. Interestingly, the predictive power of the volatility risk premium displays a very similar pattern as for the stock excess returns: The adjusted R^2 starts at 0.1 and then peaks around the eight month horizon and then tends to taper off for the longer horizons.

Summarizing, the regressions reveal that the volatility risk premium or the difference between the risk-neutral and physical expected variance captures an important component not only in stock but also in credit markets. The model suggests a structural explanation for this reduced-form result.

B.1. Cyclical versus Non-Cyclical Firms

The example in the introduction has hinted that the effect of macro-economic uncertainty on the volatility risk premium could possibly a function of the business cycle. In line with this supposition, Beber and Brandt (2008) empirically document that cyclical stocks are more exposed to macro-economic uncertainty than non-cyclical stocks. In the economy I posit, a higher macro-economic uncertainty should lead to a higher volatility risk premia and therefore it is natural to expect that firms that are more exposed to the overall business cycle should also be more affected by uncertainty. To verify this hypothesis more formally, I first test the impact of uncertainty on the volatility risk premium for cyclical versus non-cyclical firms. I do this by running separate regressions for the two sets of firms. I then re-run the predictability regressions for both cyclical and non-cyclical firms. To this end, I sort each firm in industries and industries I divide into cyclical and non-cyclical ones. Industry classifications are based on Boudoukh, Richardson, and Whitelaw (1994), who sort industries by their correlation between industry level output growth and aggregate output growth. I label the five industries with the highest output growth beta as cyclical and the five industries with the lowest output growth beta as non-cyclical. The results for the volatility risk premia are presented in Table 8.

[Insert Table 8 approximately here.]

The estimated coefficient for the uncertainty measures are highly significant for both the cyclical and non-cyclical firms, the size of the coefficients, however, is more than twice as large for the cyclical firms than for the non-cyclical ones. This indicates that the impact of uncertainty on the cyclical firms is larger. Within the model I study, a higher exposure of cyclical stocks to uncertainty is natural as those firms which correlate more with the aggregate output are more prone to higher volatility risk and therefore should also carry a higher risk premium to compensate for the uncertainty. In Table 9, I present the estimated coefficients for the predictability regressions for both cyclical and non-cyclical firms.

[Insert Table 9 approximately here.]

The results reveal that the predictive power of the volatility risk premium for both the stock excess returns and corporate credit spreads is higher for cyclical stocks than for non-cyclical stocks. The estimated coefficients are larger in absolute size and the t-statistics are on average more than twice as large. In terms of adjusted R^2, the volatility risk premium of cyclical stocks is 10% at a one year horizon, whereas for the non-cyclical stocks, the adjusted R^2 amounts to only 4%.

IV. Conclusions

This paper develops an equilibrium model with heterogeneous beliefs and stochastic macroeconomic uncertainty to explain individual volatility risk premia in the cross-section of stocks. I develop a multi-asset Lucas (1978) economy in which uncertainty about firms growth opportunities induces agents to disagree about expected cash flows and a market-wide business cycle signal. The volatility of cash flow growth rates and the signal is modeled as a two-state Markov switching process which allows time-varying macroeconomic uncertainty to have a bearing on agents' disagreement. These features generate two sources of risk in equilibrium: Priced disagreement and macroeconomic uncertainty risk. In equilibrium, when belief heterogeneity and macro-economic uncertainty vary over time, they drive a substantial fraction of the volatility of asset returns. I study these model implications theoretically within a calibrated model and link them directly to the differential empirical pricing patterns of individual options.

The more specific characteristics of my model are as follows: Consumption growth volatility and disagreement are time-varying and are allowed to impact on the volatility risk premia via the stochastic discount factor. A higher disagreement on future cash flows induces a higher volatility and negative skewness for medium to high leveraged firms. For firms with low leverage, it is possible that the risk-neutral skewness turns positive. The economic reason is due to the different sensitivities of the default option embedded in stock returns: For low leverage companies, this option is far out-of-the-money and its value is more sensitive to changes in skewness. For high leverage companies, the default option is closer to be in-the-money and the opposite holds. In my economy, agents have time-separable utility and the diffusions of the fundamental process and the consumption growth volatility are assumed independent. In this case, in an economy without disagreement, the difference between the risk-neutral and physical expected volatility would be equal to zero. If disagreement is a priced risk factor, the wedge is entirely driven by agents' uncertainty and the time-varying macro-economic risk. Calibrating the model, I confirm several of the empirical findings. Both disagreement and realized conditional consumption growth volatility matter in their implications for volatility risk premia, and I find the first source to dominate. Shutting down the stochastic volatility of cash flow growth, I find that the level of volatility risk premia is mainly explained by the difference in beliefs. However, higher order moments, such as the persistence, skewness, and kurtosis of the risk premia are driven, to a large extent, by economic uncertainty. A simulation exercise shows that the model is able to replicate the predictability of stock excess returns and corporate credit spreads. Moreover, the estimated coefficients correspond in sign and size approximately to their empirical counterparts. This paper is a first step towards a unified study of agents' (disagreement) and macro-economic uncertainty, and I hope to give credence to the hypothesis that both subjective agents' uncertainty and realized objective uncertainty could matter for volatility risk premia of stocks.

To empirically test the model predictions, I construct model-free measures of volatility risk premia (see Carr and Madan, 1998 and Britten-Jones and Neuberger, 2000) and risk-neutral skewness (Bakshi, Kapadia, and Madan, 2003) from stock option prices using data from January 1996 to September 2008. Following Hamilton (1989), I estimate a Markov chain process for the second moment of consumption growth to get a measure of the conditional volatility of fundamentals. In addition, I build a forward-looking measure of disagreement about future cash flows of firms based on forecasts of corporate earnings. The separation of the two uncertainty channels has an empirical appeal. By its forward-looking construction, disagreement captures perceptions of risk looming on investors' horizon. Thus, it conveys information over and above the one contained in the conditional volatility of consumption which is estimated using historical data. In addition to the conditional volatility of consumption growth, I construct a forward-looking measure of disagreement about future cash flows of firms based on forecasts of corporate earnings. Simple panel regressions reveal that both disagreement and the conditional volatility of consumption matter for volatility risk premia and outperform the explanatory content of other factors known to affect these risk premia. For instance, a one standard deviation change in firm-specific disagreement increases the volatility risk premium by 0.5%, which corresponds to one third of the average volatility risk premium in the cross-section of stocks. A slightly smaller change is induced by the conditional volatility of consumption growth. Disagreement and time-varying consumption growth account for 12% of the variation of volatility risk premia in the cross-section. I also find that the volatility risk premia of cyclical stocks are more exposed to uncertainty than non-cyclical stocks. For instance, the estimated coefficient for uncertainty on cyclical stocks is twice as large as for non-cyclical ones. The empirical assessment also reveals that the impact of uncertainty on the volatility risk premia can be positive or negative depending on the leverage level. For low leverage firms, the uncertainty proxies induce a decrease in the volatility risk premia, whereas for high leverage firms, the effect is the opposite. I also present evidence that the volatility risk premium contains forecasting power for both the cross-section of stock excess returns and corporate credit spreads. Similar to previous results in the literature for the index volatility risk premium, I find that predictability of the stock excess returns is strongest in the short run: The predictive power of the individual volatility risk premia is strongest at a horizon of eight months and then steadily declines. An analogue pattern is found for corporate credit spreads.

References

Alessi, L., M. Barigozzi, and M. Capasso (2008): "A Robust Criterion for Determining the Number of Static Factors in Approximate Factor Models," European Central Bank, Working Paper Series, No. 903.

Bai, J., and S. Ng (2001): "Determining the Number of Factors in Approximate Factor Models," *Econometrica*, 70, p. 191 – 221.

Bakshi, G., C. Cao, and Z. Chen (1997): "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52, p. 2003 – 2049.

Bakshi, G., and D. Madan (2006): "A Theory of Volatility Spreads," *Management Science*, 52, p. 1945 – 1956.

Bakshi, G., N.Kapadia, and D. Madan (2003): "Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options," *Review of Financial Studies*, 16, p. 101 – 143.

Bansal, R., and I. Shaliastovich (2008): "Confidence Risk and Asset Prices," Working Paper, Duke University.

Basak, S., and D. Cuoco (1998): "An Equilibrium Model with Restricted Stock Market Participation," *Review of Financial Studies*, 11, p. 309 – 341.

Bates, D. (1996): "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options," *Review of Financial Studies*, 9, p. 69 – 107.

Beber, A., and M. W. Brandt (2008): "Resolving Macroeconomic Uncertainty in Stock and Bond Markets," Working Paper, HEC, University of Lausanne.

Beneš, V. E. (1981): "Exact Finite-Dimensional Filters for Certain Diffusions with Nonlinear Drift," *Stochastics*, 5, p. 65 – 92.

Black, F., and M. Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, p. 637–659.

Bollen, N. P. B., and R. E. Whaley (2004): "Does Net Buying Pressure Affect the Shape of Implied Volatility Functions?," *Journal of Finance*, 59, p. 711–753.

Bollerslev, T., M. Gibson, and H. Zhou (forthcoming, 2009): "Dynamic Estimation of Volatility Risk Premia and Investor Risk Aversion from Option-Implied and Realized Volatilities," *Journal of Econometrics*.

Bollerslev, T., G. Tauchen, and H. Zhou (2009): "Expected Stock Returns and Variance Risk Premia," *Review of Financial Studies*, 22, p. 4463 – 4492.

Boudoukh, J., M. Richardson, and R. F. Whitelaw (1994): "Industry Returns and the Fisher Effect," *Journal of Finance*, 49, p. 1595 – 1615.

Brillinger, D. R. (2001): *Time Series: Data Analysis and Theory*. Society for Industrial and Applied Mathematics.

Britten-Jones, M., and A. Neuberger (2000): "Option Prices, Implied Price Processes, and Stochastic Volatility," *Journal of Finance*, 55, p. 839 – 866.

Buraschi, A., F. Trojani, and A. Vedolin (2008): "Economic Uncertainty, Disagreement, and Credit Markets," Working Paper, University of Lugano.

Campbell, J. Y., and G. B. Taksler (2003): "Equity Volatility and Corporate Bond Yields," *Journal of Finance*, 58, p. 2321 – 2349.

Carr, P., H. Geman, D. B. Madan, and M. Yor (2001): "Stochastic Volatility for Lévy Processes," Working Paper, University of Paris IX Dauphine and ESSEC.

Carr, P., and D. Madan (1998): *Volatility: New Estimation Techniques for Pricing Derivatives*chap. Towards a Theory of Volatility Trading, pp. 417 – 427. RISK Publications.

Carr, P., and L. Wu (2009): "Variance Risk Premiums," *Review of Financial Studies*, 22, p. 1311 – 1341.

Chacko, G., and S. Das (2002): "Pricing Interest Rate Derivatives: A General Approach," *Review of Financial Studies*, 15, p. 195 – 241.

Chamberlain, G., and M. Rothschild (1983): "Arbitrage, Factor Structure and Mean-Variance Analysis in Large Asset Markets," *Econometrica*, 51, p. 1305 – 1324.

Chen, H., and S. Joslin (2009): "Generalized Transform Analysis of Affine Processes And Asset Pricing Applications," Working Paper, MIT Sloan School of Management.

Chen, H., and M. Pakoš (2008): "Asset Pricing With Uncertainty About the Long Run," Working Paper, MIT.

Cochrane, J. H., F. A. Longstaff, and P. Santa-Clara (2008): "Two Trees," *Review of Financial Studies*, 21, p. 347 – 385.

Collin-Dufresne, P., R. S. Goldstein, and J. S. Martin (2001): "The Determinants of Credit Spread Changes," *Journal of Finance*, 56, p. 2177 – 2207.

Connor, G., and R. Korajczyk (1986): "Performance Measurement with the Arbitrage Pricing Theory: A New Framework for Analysis," *Journal of Financial Economics*, 15, p. 373 – 394.

Conrad, J., R. F. Dittmar, and E. Ghysels (2008): "Ex Ante Skewness and Expected Stock Returns," Working Paper, University of North Carolina at Chapel Hill.

Corradi, V., W. Distaso, and A. Mele (2009): "Macroeconomic Determinants of Stock Market Volatility and Volatility Risk-Premiums," Working Paper, University of Warwick.

Cox, J., and C.-F. Huang (1989): "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, 49, p. 33–83.

Cristadoro, R., M. Forni, L. Reichlin, and G. Veronese (2005): "A Core Inflation Index for the Euro Area," CEPR Working Paper,.

Cuoco, D., and H. He (1994): "Dynamic Equilibrium in Infinite-Economies with Incomplete Financial Markets," Working Paper, The Wharton School, University of Pennsylvania.

David, A. (2008): "Heterogeneous Beliefs, Speculation, and the Equity Premium," *Journal of Finance*, 63, p. 41 – 83.

DeMiguel, V., Y. Plyakha, R. Uppal, and G. Vilkov (2009): "Improving Portfolio Selection Using Option-Implied Volatility and Skewness," Working Paper, London Business School.

Dennis, P., and S. Mayhew (2002): "Risk-Neutral Skewness: Evidence from Stock Options," *Journal of Financial and Quantitative Analysis*, 37, p. 471 – 493.

Drechsler, I. (2008): "Uncertainty, Time-Varying Fear, and Asset Prices," Working Paper, University of Pennsylvania.

Drechsler, I., and B. Yaron (2008): "What's Vol Got to Do With It," Working Paper, University of Pennsylvania.

Duffee, G. R. (1998): "Estimating the Price of Default Risk," *Review of Financial Studies*, 12, p. 197–226.

Duffie, D., J. Pan, and K. J. Singleton (2000): "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, p. 1343 – 1376.

Dumas, B., A. Kurshev, and R. Uppal (2009): "Equilibrium Portfolio Strategies in the Presence of Sentiment Risk and Excess Volatility," *Journal of Finance*, 64, p. 579 – 629.

Elton, E. J., M. J. Gruber, D. Agrawal, and C. Mann (2001): "Explaining the Rate Spread on Corporate Bonds," *Journal of Finance*, 56, p. 247–277.

Engle, R., and A. Mistry (2008): "Priced Risk and Asymmetric Volatility in the Cross-Section of Skewness," Working Paper, NYU.

Eraker, B. (2008): "The Volatility Premium," Working Paper, Duke University.

Fama, E. F., and K. R. French (1992): "The Cross-Section of Expected Stock Returns," *Journal of Finance*, 47, p. 427 – 486.

Forni, M., M. Hallin, M. Lippi, and L. Reichlin (2000): "The Generalized Dynamic-Factor Model: Identification and Estimation," *The Review of Economics and Statistics*, 82, p. 540 – 554.

Gârleanu, N., L. Pedersen, and A. M. Poteshman (2009): "Demand Based Option Pricing," *forthcoming, Review of Financial Studies*.

Geske, R. (1979): "The Valuation of Compound Options," *Journal of Financial Economics*, 7, p. 63 – 81.

Geske, R., and Y. Zhou (2008): "Pricing Individual Stock Options On Firms with Leverage," Working Paper, UCLA.

Goetzmann, W. N., J. E. Ingersoll, M. Spiegel, and I. Welch (2007): "Portfolio Performance Manipulation and Manipulation-Proof Performance Measures," *Review of Financial Studies*, 20, p. 1503 – 1546.

Gomes, J. F., and L. Schmid (2008): "Levered Returns," *forthcoming, Journal of Finance*.

Goyal, A., and A. Saretto (forthcoming, 2009): "Cross-Section of Option Returns and Volatility," *Journal of Financial Economics*.

Granger, C. W. J., and P. Newbold (1986): *Forecasting Economic Time Series*. Academic Press, 2nd edition.

Grant, M., K. Gregory, and J. Lui (2007): "Volatility As An Asset," Goldman Sachs, Options Research.

Hallin, M., and R. Liška (2007): "Determining the Number of Factors in the General Dynamic Factor Model," *Journal of the American Statistical Association*, 102, p. 603 – 617.

Hamilton, J. D. (1989): "A New Approach to the Economic Analysis of Nonstationary Time-Series and the Business Cycle," *Econometrica*, 57, p. 357 – 384.

Hansen, L. P., and K. Singleton (1983): "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns temporal behavior of asset returns," *Journal of Political Economy*, 91, p. 249 – 265.

Heston, S. L. (1993): "A Closed-Form Solution for Options with Stochstic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, p. 327 – 343.

Hong, H., J. Wang, and J. Yu (2008): "Firms as Buyers of Last Resort," *Journal of Financial Economics*, 88, p. 119 – 145.

Jiang, G. J., and Y. S. Tian (2005): "The Model-Free Implied Volatility and Its Information Content," *Review of Financial Studies*, 18, p. 1305 – 1342.

Karatzas, I., and S. E. Shreve (1998): *Methods of Mathematical Finance*. Springer Verlag.

Kushner, H. (1967): "Nonlinear Filtering: The Exact Dynamical Equations Satisfied by the Conditional Mode," *Automatic Control*, 12, p. 262 – 267.

Lakonishok, J., I. Lee, N. D. Pearson, and A. M. Poteshman (2007): "Option Market Activity," *Review of Financial Studies*, 20, p. 813 – 857.

Lewellen, J., and S. Nagel (2006): "The Conditional CAPM does not explain Asset-Pricing Anomalies," *Journal of Financial Economics*, 82, p. 289 – 314.

Liptser, R., and A. Shiryaev (2000): *Statistics of Random Processes I: General Theory*. Springer Verlag.

Lucas, R. E. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, p. 1429–1446.

Ludvigson, S., and S. Ng (2009): "Macro Factors in Bond Risk Premia," *forthcoming, Review of Financial Studies*.

Ludvigson, S. C., and S. Ng (2007): "The Empirical-Risk Return Relationship: A Factor Analysis Approach," *Journal of Financial Econometrics*, 83, p. 171 – 222.

Martin, I. (2009): "The Lucas Orchard," Working Paper, Stanford University.

Menzly, L., T. Santos, and P. Veronesi (2004): "Understanding Predictability," *Journal of Political Economy*, 112, p. 1 – 47.

Merton, R. C. (1974): "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 29, p. 449 – 470.

Newey, W., and K. West (1987): "A Simple, Positive-Definite, Heteroskedasticity and Autocorrelation Consisten Covariance Matrix," *Econometrica*, 55, p. 703 – 708.

Ni, S. X. (2007): "Stock Option Returns: A Puzzle," Working Paper, Hong Kong University of Science and Technology.

Pakoš, M. (2008): "Equity Prices Under Bayesian Doubt About Macroeconomic Fundamentals," Working Paper, Carnegie Mellon University.

Pavlova, A., and R. Rigobon (2007): "Asset Prices and Exchange Rates," *Review of Financial Studies*, 20, p. 1139 – 1181.

Santos, T., and P. Veronesi (2009): "Habit Formation, the Cross Section of Stock Returns and the Cash-Flow Risk Puzzle," *forthcomin, Journal of Financial Economics*.

Shaliastovich, I. (2008): "Learning, Confidence, and Option Prices," Working Paper, University of Pennsylvania.

Stock, J. H., and M. W. Watson (2002a): "Forecasting using Principal components from a Large Number of Predictors," *Journal of the American Statistical Association*, 97, p. 1167 – 1179.

——— (2002b): "Macroeconomic Forecasting unsing Diffusion Indexes," *Journal of Business and Economic Statistics*, 20, p. 147 – 162.

——— (2004): "Forecasting with Many Predictors," Working Paper, Princeton University.

Stratonovich, R. L. (1959): "Optimum Nonlinear Systems which bring about a Separation of a Signal with Constant Parameters from Noise," *Radiofizika*, 2:6, p. 892 – 901.

——— (1960): "Application of the Markov Processes Theory to Optimal Filtering," *Radio Engineering and Electronic Physics*, 5:11, p. 1 – 19.

Toft, K. B., and B. Prucyk (1997): "Options on Leveraged Equity: Theory and Empirical Tests," *Journal of Finance*, 52, p. 1151 – 1180.

Warga, A. D. (2000): "National Association of Insurance Commissioners Database," University of Houston, Houston, Texas.

Wilkinson, J. H. (1965): *The Algebraic Eigenvalue Problem*. Oxford University Press.

Wonham, W. M. (1964): "Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering," *SIAM Journal of Control*, 2, p. 347 – 369.

Zhou, H. (2009): "Variance Risk Premia, Asset Predictability Puzzles, and Macroeconomic Uncertainty," Working Paper, Federal Reserve Board.

A Proofs

A.1 Learning

The disagreement dynamics in our economy follows in a straightforward way, once I can condition on the steady-state volatility of the filtered growth rate dynamics. The dynamics of the individual beliefs are:

$$\begin{aligned} dm^A(t) &= (a_0 + a_1 m^A(t))dt + \gamma^1(t)A'B^{-1}dW_Y^1(t), \\ dm^B(t) &= (a_0 + a_1 m^B(t))dt + \gamma^2(t)A'(BB')^{-1}(m^A(t) - m^B(t))dt + \gamma^2(t)A'B^{-1}dW_Y^A(t) \ , \end{aligned}$$

using the symmetry of B. The dynamics of $\Psi(t) = B^{-1}(m^A(t) - m^B(t))$ follows, as:

$$\begin{aligned} d\Psi(t) &= B^{-1}\left(a_1 + \gamma^2(t)A'(BB')^{-1}\right)(m^A(t) - m^B(t))dt + B^{-1}\left(\gamma^1(t) - \gamma^2(t)\right)A'B^{-1}dW_Y^A(t) \\ &= B^{-1}\left(a_1 B + \gamma^2(t)A'B^{-1}\right)\Psi(t)dt + B^{-1}\left(\gamma^1(t) - \gamma^2(t)\right)A'B^{-1}dW_Y^A(t), \end{aligned}$$

with initial condition $\Psi(0) = B^{-1}(m^A(0) - m^B(0))$. The solution of this stochastic differential condition is:

$$\Psi(t) = \exp\left\{\int_0^t M(s)ds\right\}\Psi(0) + \int_0^t \exp\left\{\int_0^s M(u)du\right\}B^{-1}\left(\gamma^1(s) - \gamma^2(s)\right)A'B^{-1}dW_Y^1(s), \tag{12}$$

where $M(s) = B^{-1}\left(a_1 B + \gamma^B(s)A'B^{-1}\right)$. It follows that $\Psi(t)$ is normally distributed as:

$$\Psi(t) \sim \mathcal{N}\left(e^{\int_0^t M(s)ds}\Psi(0), \int_0^t e^{\int_0^s M(u)du}B^{-1}(\gamma^1(s) - \gamma^B(s))A'(BB')^{-1}A\left(\gamma^1(s) - \gamma^B(s)\right)'B^{-1}e^{\int_0^s M(u)du}ds\right).$$

The parameter $b = diag(\sigma_{\mu_{A_1}}, \sigma_{\mu_{A_2}}, \sigma_{\mu_z})$ in the dynamics for $\gamma(t)$ impacts the distribution of $m(t)$ directly via the Markovian structure.

A.2 Equilibrium Quantities

For completeness, I derive all equilibrium quantities in this Appendix. The proofs follow grossly Basak (2005)?.
(i) Dynamics of the stochastic weighting process λ: Itô's Lemma applied to $\eta(t) = \xi^A(t)/\xi^B(t)$ gives:

$$d\eta(t) = \frac{d\xi^A(t)}{\xi^B(t)} - \frac{\xi^A(t)}{(\xi^A(t))^2}d\xi^B(t) + \frac{1}{2}\frac{2\xi^A(t)}{(\xi^B(t))^3}\left(d\xi^B(t)\right)^2 - \frac{1}{(\xi^B(t))^2}d\xi^B(t)d\xi^A(t).$$

Since markets are complete, there exists a unique stochastic discount factor for each agent. Absence of arbitrage implies for $i = 1, 2$:

$$\frac{d\xi^i(t)}{\xi^i(t)} = -r(t)dt - \theta^i(A_1(t), A_2(t), z(t))'dW_Y^i,$$

where $\theta^i = (\theta^i_{A_1}(t), \theta^i_{A_2}(t), \theta^i_z(t))'$ is the vector of market prices of risk perceived by agent i. It then follows,

$$\begin{aligned} d\eta(t) &= \frac{\xi^A(t)}{\xi^B(t)}\frac{d\xi^A(t)}{\xi^A(t)} - \frac{\xi^A(t)}{\xi^B(t)}\frac{d\xi^B(t)}{\xi^B(t)} + \frac{\xi^A(t)}{\xi^B(t)}\left(\frac{d\xi^B(t)}{\xi^B(t)}\right)^2 - \frac{1}{(\xi^B(t))^2}d\xi^B(t)d\xi^A(t), \\ &= \eta(t)\Bigg(- r(t)dt - \theta^1_A(t)dW_A(t) - \theta^1_z(t)dW_z(t) - \left(-r(t)dt - \theta^2_A(t)dW_A(t) - \theta^2_z(t)dW_z(t)\right) \\ &\quad + \left(\left(\theta^2_A(t)\right)^2 + \left(\theta^2_z(t)\right)^2 - \theta^1_A(t)\theta^2_A(t) - \theta^1_z(t)\theta^2_z(t)\right)dt\Bigg). \end{aligned} \tag{13}$$

The prices of the stock and the defaultable bond in my economy follow the dynamics:

$$dS_i(t) = S_i(t)\left(\mu_{S_i}(t)dt + \sigma_{S_iA_i}dW_{A_i}(t) + \sigma_{S_jA_j}dW_{A_k}(t) + \sigma_{S_iz}dW_z(t)\right), \tag{14}$$

$$dB_i^d(t) = B_i^{S_i}(t)\left(\mu_{B_i^{S_i}}(t)dt + \sigma_{B_i^{S_i}A_i}dW_{A_i}(t) + \sigma_{B_j^{S_j}A_j}dW_{A_j}(t) + \sigma_{B_i^{S_i}z}dW_z(t)\right), \tag{15}$$

where $S_i(t)$ is the price of equity and $B_i^d(t)$ the price of the defaultable bond, and the expected growth rates $\mu_{S_i}(t)$ and $\mu_{B_i^d}(t)$ and the volatility coefficients $\sigma_{S_iA_i}, \sigma_{B_i^{S_i}A_i}, \sigma_{S_iz}$ and $\sigma_{B_i^d z}$ are determined in equilibrium and $i \neq j$. It is easily shown that the difference in the perceived rates of return have to satisfy the consistency condition:

$$\mu_n^A(t) - \mu_n^B(t) = \sigma_n \left(\Psi_{A_1}(t), \Psi_{A_2}(t) \quad \alpha_{A_1}\Psi_{A_1}(t)\frac{\sigma_{A_1}}{\sigma_z} + \alpha_{A_2}\Psi_{A_2}(t)\frac{\sigma_{A_2}}{\sigma_z} + \beta\Psi_z(t)\right)',$$

where n denotes security n. The definition of market price of risk yields:

$$\sigma_{nA_1}\theta_{A_1}^i(t) + \sigma_{nA_2}\theta_{A_2}^i(t) + \sigma_{nz}\theta_z^i(t) = \mu_n^i(t) - r(t).$$

After some simple algebra, I obtain:

$$\begin{aligned}&\sigma_{nA_1}(t)\left(\theta_{A_1}^A(t) - \theta_{A_1}^B\right) + \sigma_{nA_2}(t)\left(\theta_{A_2}^A(t) - \theta_{A_2}^B\right) + \sigma_{nz}(t)\left(\theta_z^A(t) - \theta_z^B(t)\right) = \sigma_{nA_1}(t)\Psi_{A_1}(t) + \sigma_{nA_2}(t)\Psi_{A_2}(t)\\ &+\sigma_{nz}(t)\left(\alpha_{A_1}\Psi_{A_1}(t)\frac{\sigma_{A_1}}{\sigma_z} + \alpha_{A_2}\Psi_{A_2}(t)\frac{\sigma_{A_2}}{\sigma_z} + \beta\Psi_z(t)\right).\end{aligned}$$

Since this equation has to hold for any $\sigma_{nA}(t)$ and $\sigma_{nz}(t)$, it follows:

$$\begin{aligned}\theta_A^1(t) - \theta_A^2(t) &= \Psi_A(t),\\ \theta_z^1(t) - \theta_z^2(t) &= \left(\alpha\Psi_A(t)\frac{\sigma_A}{\sigma_z} + \beta\Psi_z(t)\right).\end{aligned}$$

By construction, I also have:

$$dW_A(t) = \frac{m_A^i(t) - \mu_A(t)}{\sigma_A}dt + dW_A^i(t), \quad dW_z(t) = \left(\alpha\frac{m_A^i(t) - \mu_A(t)}{\sigma_z} + \beta\frac{m_z^i(t) - \mu_z(t)}{\sigma_z} + dW_z^i(t)\right).$$

Therefore, after substituting in equation (13), I get:

$$\begin{aligned}\frac{d\eta(t)}{\eta(t)} &= -dW_A^1(t)\Psi_A(t) - \theta_z^1(t)dW_z^1(t) + \theta_z^2(t)\left(dW_z^1(t) + \alpha\Psi_A(t)\frac{\sigma_A}{\sigma_z} + \beta\Psi_z(t)\right)\\ &\quad + \left(\left(\theta_A^1(t) - \Psi_A(t)\right)^2 + \theta_z^2(t)\left(\theta_z^2(t) - \theta_z^1(t)\right) - \theta_A^1(t)\left(\theta_A^1(t) - \Psi_A(t)\right)\right)dt,\\ &= -dW_A^1(t)\Psi_A(t) - dW_z^1(t)\left(\alpha\Psi_A(t)\frac{\sigma_A}{\sigma_z} + \beta\Psi_z(t)\right).\end{aligned}$$

(ii) Representative investor optimization and optimal consumption policies: The representative agent in the economy faces the following optimization problem:

$$\sup_{c_1(t)+c_2(t)=A(t)} U(c_1(t), c_2(t), \lambda(t)) = \frac{c_1(t)^{1-\gamma}}{1-\gamma} + \lambda(t)\frac{c_2(t)^{1-\gamma}}{1-\gamma}, \tag{16}$$

where $\lambda(t) > 0$. Optimality of individual consumption plans implies that the stochastic weight takes the following form:

$$\lambda(t) = u'(c_1(t))/u'(c_2(t)) = y_1\xi^A(t)/y_2\xi^B(t),$$

where $u'(c(t)) = c(t)^{-1/\gamma}$ is the marginal utility function, which is assumed identical across agents. The first order condition for agent one is:

$$e^{-\rho t}c_1(t)^{-\gamma} = y_1\xi^A(t).$$

The first order condition for agent two is:

$$\eta(t)e^{-\rho t}c_2(t)^{-\gamma} = y_2\xi^A(t).$$

The aggregate resource constraint can now be easily derived as:

$$\left(\frac{y_2\xi^A(t)e^{\rho t}}{\eta(t)}\right)^{-1/\gamma} + \left(y_1\xi^A(t)e^{\rho t}\right)^{-1/\gamma} = A(t).$$

Thus, the solutions for the individual state price densities are:

$$\xi^A(t) = e^{-\rho t}\frac{1}{y_1}A(t)^{-\gamma}\left(1+\lambda(t)^{1/\gamma}\right)^{\gamma}, \quad \xi^B(t) = e^{-\rho t}\frac{1}{y_2}A(t)^{-\gamma}\left(1+\lambda(t)^{1/\gamma}\right)^{\gamma}\lambda(t)^{-1}.$$

To solve for the optimal consumption policy of each agent, I plug in the functional forms for the individual state price densities:

$$c_1(t) = (y_1\xi^A(t)e^{\rho t})^{-1/\gamma} = A(t)\left(1+\lambda(t)^{1/\gamma}\right)^{-1}.$$

Good's market clearing, finally implies:

$$c_2(t) = A_i(t) - c_1(t) = A_i(t)\lambda(t)^{1/\gamma}\left(1+\lambda(t)^{1/\gamma}\right)^{-1}.$$

A.3 Security Prices

For convenience, I express everything in terms of firm 1. However, the formulas can easily be extended to firm 2, by making the corresponding changes and adjusting for the relative price. By definition, the risk-less zero coupon bond price is given by:

$$B_1(t,T) = \frac{1}{\xi^A(t)}E_t^A\left(e^{-\rho(T-t)}\xi^A(T)\right).$$

Using the expression for $\xi^A(t)$, I get:

$$B_1(t,T) = E_t^A\left(e^{-\rho(T-t)}\left(\frac{A_1(T)}{A_1(t)}\right)^{-\gamma}\left(\frac{1+\lambda(T)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^{\gamma}\right). \tag{17}$$

Let

$$G(t,T,x;\Psi_{A_1},\Psi_{A_2},\Psi_z) \equiv \int_0^{\infty}\left(\frac{1+\lambda(T)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^{\gamma}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\lambda(T)}{\lambda(t)}\right)^{-i\chi}F_{\Psi_A,\Psi_z}(\Psi_{A_1},\Psi_{A_2},\Psi_z,t,T;-\gamma,i\chi)d\chi\right]\frac{d\lambda(T)}{\lambda(T)}.$$

By Fourier inversion, it then follows:

$$B_1(t,T) = e^{-\rho(T-t)}F_{m_A^1}(m_A^1,t,T;-\gamma)G(t,T,-\gamma;\Psi_{A_1},\Psi_{A_2},\Psi_z).$$

In a similar way, the firm value is:

$$\begin{aligned}
V_1(t) &= E_t^A\left(\int_t^{\infty} e^{-\rho(u-t)}\frac{\xi^A(u)}{\xi^A(t)}A(u)du\right),\\
&= A_i(t)E_t^A\left(\int_t^{\infty} e^{-\rho(u-t)}\left(\frac{1+\lambda(t)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^{\gamma}\left(\frac{A_1(u)}{A_1(t)}\right)^{1-\gamma}du\right),\\
&= A_1(t)\int_t^{\infty}\left(e^{-\rho(u-t)}F_{m_A^1}(m_A^1,t,u;1-\gamma)G(u,T,1-\gamma;\Psi_{A_1},\Psi_{A_2},\Psi_z)\right)du.
\end{aligned}$$

The price of the defaultable bond is:

$$\begin{aligned}
B_1^d(t,T) &= K_1 B_1(t,T) - E_t^A\left(e^{-\rho(T-t)}\frac{\xi^A(T)}{\xi^A(t)}(K_1 - V_1(T))^+\right), \\
&= K_1 B_1(t,T) - E_t^A\left(e^{-\rho(T-t)}\left(\frac{A_1(T)}{A_1(t)}\right)^{-\gamma}\left(\frac{1+\lambda(T)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\right)^{\gamma}(K_1 - V_1(T))^+\right), \\
&= K_1 B_i(t,T) - P(t,T,K_1),
\end{aligned}$$

where $P_1(t,T,K_1)$ is the price of the put option on the firm value. Equity in my economy is a call option on the firm value. Therefore:

$$S_1(t) = E_t^A\left(e^{-\rho(T-t)}\frac{\xi^A(T)}{\xi^A(t)}\left(V_1(T) - (K_1+K_2)\right)^+\right) = C_1(t,T,K_1+K_2).$$

A European call option on the equity value is derived in the following way:

$$O_1(t,T) = E_t^A\left(e^{-\rho(T-t)}\frac{\xi^A(T)}{\xi^A(t)}\left(S_1(T) - K_e\right)^+\right).$$

A.4 Risk-Neutral Skewness

Bakshi, Kapadia, and Madan (2003) show that one can express the τ maturity price of a security that pays the quadratic, cubic, and quartic return on the base security as:

$$\begin{aligned}
V(t,\tau) &= \int_{S(t)}^{\infty}\frac{2\left(1 - ln\left(K/S(t)\right)\right)}{K^2}C(t,\tau;K)dK \\
&\quad + \int_0^{S(t)}\frac{2\left(1 + ln\left(K/S(t)\right)\right)}{K^2}P\left(t,\tau;K\right)dK, \\
W(t,\tau) &= \int_{S(t)}^{\infty}\frac{6ln\left(K/S(t)\right) - 3\left(ln\left(K/S(t)\right)\right)^2}{K^2}C\left(t,\tau;K\right)dK \\
&\quad + \int_0^{S(t)}\frac{6ln\left(K/S(t)\right) + 3\left(ln\left(K/S(t)\right)\right)^2}{K^2}P\left(t,\tau;K\right)dK, \\
X(t,\tau) &= \int_{S(t)}^{\infty}\frac{12ln\left(K/S(t)\right) - 4\left(ln\left(K/S(t)\right)\right)^2}{K^2}C\left(t,\tau;K\right)dK \\
&\quad + \int_0^{S(t)}\frac{12ln\left(K/S(t)\right) + 4\left(ln\left(K/S(t)\right)\right)^2}{K^2}P\left(t,\tau;K\right)dK,
\end{aligned}$$

where $V(t,\tau), W(t,\tau)$, and $X(t,\tau)$ are the quadratic, cubic, and quadratic contracts, respectively, and $C(t,\tau;K)$ and $P(t,\tau;K)$ are the prices of European calls and puts written on the underlying stock $S(t)$ with strike price K and maturity τ.

A.5 Stock Price Volatility and Volatility Risk Premia

The price of the stock satisfies a diffusion process which is given by:

$$\frac{dS_1}{S_1} = \mu_{S_1}^A(t)dt + \sigma_{S_1A_1}(t)dW_{A_1}^A(t) + \sigma_{S_1A_2}(t)dW_{A_2}^A(t) + \sigma_{S_1z}(t)dW_z^A(t).$$

The diffusion term is characterized by:

$$
\begin{aligned}
dS_1(t) - S_1(t)\mu_{S_1}^A(t)dt &= \frac{\partial S_1}{\partial A_1}\left(dA_1(t) - E_t^A\left(dA_1(t)\right)\right) + \frac{\partial S_1}{\partial m_{A_1}^A}\left(dm_{A_1}^A(t) - E_t^A\left(dm_{A_1}^A\right)\right) \\
&\quad + \frac{\partial S_1}{\partial \Psi_{A_1}}\left(d\Psi_{A_1}(t) - E_t^A\left(d\Psi_{A_1}(t)\right)\right) \\
&\quad + \frac{\partial S_1}{\partial \Psi_{A_2}}\left(d\Psi_{A_2}(t) - E_t^A\left(d\Psi_{A_2}(t)\right)\right) + \frac{\partial S_1}{\partial \Psi_z}\left(d\Psi_z - E_t^A\left(d\Psi_z(t)\right)\right), \\
&= \frac{\partial S_1}{\partial A_1} A_1 \sigma_{A_1} dW_{A_1}^A(t) \\
&\quad + \frac{\partial S_1}{\partial m_{A_1}^A}\left(\frac{\gamma_{A_1}^A}{\sigma_{A_1}} dW_{A_1}^A(t) + \frac{\gamma_{A_1A_2}^A}{\sigma_{A_2}} dW_{A_2}^A(t) + \left(\frac{\alpha_{A_1}\gamma_{A_1}^A + \alpha_{A_2}\gamma_{A_1A_2}^A + \beta\gamma_{A_1z}^A}{\sigma_z}\right) dW_z^A(t)\right) \\
&\quad + \frac{\partial S_1}{\partial \Psi_{A_1}}\Bigg(\left(\frac{\gamma_{A_1}^A - \gamma_{A_1}^B}{\sigma_{A_1}^2}\right) dW_{A_1}^A(t) + \left(\frac{\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^A}{\sigma_{A_1}\sigma_{A_2}}\right) dW_{A_2}^A(t) \\
&\quad + \left(\frac{\alpha_{A_1}\left(\gamma_{A_1}^A - \gamma_{A_1}^B\right) + \alpha_{A_2}\left(\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B\right) + \beta\left(\gamma_{A_1z}^A - \gamma_{A_1z}^B\right)}{\sigma_{A_1}\sigma_z}\right) dW_z^A(t)\Bigg) \\
&\quad + \frac{\partial S_1}{\partial \Psi_{A_2}}\Bigg(\left(\frac{\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B}{\sigma_{A_1}\sigma_{A_2}}\right) dW_{A_1}^A(t) + \left(\frac{\gamma_{A_2}^A - \gamma_{A_2}^B}{\sigma_{A_2}^2}\right) dW_{A_2}^A(t) \\
&\quad + \left(\frac{\alpha_{A_1}\left(\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B\right) + \alpha_{A_2}\left(\gamma_{A_2}^A - \gamma_{A_2}^B\right) + \beta\left(\gamma_{A_2z}^A - \gamma_{A_2z}^B\right)}{\sigma_{A_2}\sigma_z}\right) dW_z^A(t)\Bigg) \\
&\quad + \frac{\partial S_1}{\partial \Psi_z}\Bigg(\left(\frac{\gamma_{A_1z}^A - \gamma_{A_1z}^B}{\sigma_{A_1}\sigma_z}\right) dW_{A_1}^A(t) + \left(\frac{\gamma_{A_2z}^A - \gamma_{A_2z}^B}{\sigma_{A_2}\sigma_z}\right) dW_{A_2}^A(t) \\
&\quad + \left(\frac{\alpha_{A_1}\left(\gamma_{A_1z}^A - \gamma_{A_1z}^B\right) + \alpha_{A_2}\left(\gamma_{A_2z}^A - \gamma_{A_2z}^B\right) + \beta\left(\gamma_z^A - \gamma_z^B\right)}{\sigma_z^2}\right) dW_z^A(t)\Bigg)
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{S_1A_1}(t) &= \frac{1}{S_1(t)}\left(\frac{\partial S_1}{\partial A_1} A_1\sigma_{A_1} + \frac{\partial S_1}{\partial m_{A_1}^A}\frac{\gamma_{A_1}^A}{\sigma_{A_1}} + \frac{\partial S_1}{\partial \Psi_{A_1}}\left(\frac{\gamma_{A_1}^A - \gamma_{A_1}^B}{\sigma_{A_1}^2}\right) + \frac{\partial S_1}{\partial \Psi_{A_2}}\left(\frac{\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B}{\sigma_{A_1}\sigma_{A_2}}\right) + \frac{\partial S_1}{\partial \Psi_z}\left(\frac{\gamma_{A_1z}^A - \gamma_{A_1z}^B}{\sigma_{A_1}\sigma_z}\right)\right), \\
\sigma_{S_1A_2}(t) &= \frac{1}{S_1(t)}\left(\frac{\partial S_1}{\partial m_{A_1}^A}\frac{\gamma_{A_1A_2}^A}{\sigma_{A_2}} + \frac{\partial S_1}{\partial \Psi_{A_1}}\left(\frac{\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B}{\sigma_{A_1}\sigma_{A_2}}\right) + \frac{\partial S_1}{\partial \Psi_{A_2}}\left(\frac{\gamma_{A_2}^A - \gamma_{A_2}^B}{\sigma_{A_2}^2}\right) + \frac{\partial S_1}{\partial \Psi_z}\left(\frac{\gamma_{A_2z}^A - \gamma_{A_2z}^B}{\sigma_{A_2}\sigma_z}\right)\right), \\
\sigma_{S_1z}(t) &= \frac{1}{S_1(t)}\Bigg(\frac{\partial S_1}{\partial m_{A_1}^A}\left(\frac{\alpha_{A_1}\gamma_{A_1}^A + \alpha_{A_2}\gamma_{A_1A_2}^A + \beta\gamma_{A_1z}^A}{\sigma_z}\right) \\
&\quad + \frac{\partial S_1}{\partial \Psi_{A_1}}\left(\frac{\alpha_{A_1}\left(\gamma_{A_1}^A - \gamma_{A_1}^B\right) + \alpha_{A_2}\left(\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B\right) + \beta\left(\gamma_{A_1z}^A - \gamma_{A_1z}^B\right)}{\sigma_{A_1}\sigma_z}\right) \\
&\quad + \frac{\partial S_1}{\partial \Psi_{A_2}}\left(\frac{\alpha_{A_1}\left(\gamma_{A_1A_2}^A - \gamma_{A_1A_2}^B\right) + \alpha_{A_2}\left(\gamma_{A_2}^A - \gamma_{A_2}^B\right) + \beta\left(\gamma_{A_2z}^A - \gamma_{A_2z}^B\right)}{\sigma_{A_2}\sigma_z}\right) \\
&\quad + \frac{\partial S_1}{\partial \Psi_z}\left(\frac{\alpha_{A_1}\left(\gamma_{A_1z}^A - \gamma_{A_1z}^B\right) + \alpha_{A_2}\left(\gamma_{A_2z}^A - \gamma_{A_2z}^B\right) + \beta\left(\gamma_z^A - \gamma_z^B\right)}{\sigma_z^2}\right)\Bigg).
\end{aligned}
$$

Using the following the derivatives,

$$\begin{aligned}
\frac{\partial S_1}{\partial A_1} &= \int_t^{\infty} e^{-\delta(u-t)} F_{m^A}\left(m^A, t, T; \epsilon_{A_1}, \epsilon_{A_2}\right) G\left(t, u, 1-\gamma; \Psi\right) du, \\
\frac{\partial S_1}{\partial m^A_{A_1}} &= A_1 \int_t^{\infty} e^{-\delta(u-t)} A\left(u-t\right) F_{m^A}\left(m^A, t, T; \epsilon_{A_1}, \epsilon_{A_2}\right) G\left(t, u, 1-\gamma; \Psi\right) du, \\
\frac{\partial S_1}{\partial \Psi} &= A_1 \int_t^{\infty} e^{-\delta(u-t)} \left(B_\Psi + 2C_\Psi\right) F_{m^A}\left(m^A, t, T; \epsilon_{A_1}, \epsilon_{A_2}\right) G\left(t, u, 1-\gamma; \Psi\right) du,
\end{aligned}$$

I can easily compute the stock volatility which is given by $\left(\sigma^2_{S_1 A_1} + \sigma^2_{S_1 A_2} + \sigma^2_{S_1 z}\right)^{1/2}$. The corresponding coefficients for the volatility of stock 2 are:

$$\begin{aligned}
\sigma_{S_2 A_1}(t) &= \frac{1}{S_2(t)}\left(\frac{\partial S_2}{\partial m^A_{A_2}}\left(\frac{\gamma^A_{A_1 A_2}}{\sigma_{A_1}}\right) + \frac{\partial S_2}{\partial \Psi_{A_1}}\left(\frac{\gamma^A_{A_1} - \gamma^B_{A_1}}{\sigma^2_{A_1}}\right) + \frac{\partial S_2}{\partial \Psi_{A_2}}\left(\frac{\gamma^A_{A_1 A_2} - \gamma^B_{A_1 A_2}}{\sigma_{A_1}\sigma_{A_2}}\right) + \frac{\partial S_2}{\partial \Psi_z}\left(\frac{\gamma^A_{A_1 z} - \gamma^B_{A_1 z}}{\sigma_{A_1}\sigma_z}\right)\right), \\
\sigma_{S_2 A_2}(t) &= \frac{1}{S_2(t)}\left(\frac{\partial S_2}{\partial A_2} A_2 \sigma_{A_2} + \frac{\partial S_2}{\partial m^A_{A_2}}\frac{\gamma^A_{A_2}}{\sigma_{A_2}} + \frac{\partial S_2}{\partial \Psi_{A_1}}\left(\frac{\gamma^A_{A_1 A_2} - \gamma^B_{A_1 A_2}}{\sigma_{A_1}\sigma_{A_2}}\right) + \frac{\partial S_2}{\partial \Psi_{A_2}}\left(\frac{\gamma^A_{A_2} - \gamma^B_{A_2}}{\sigma^2_{A_2}}\right) + \frac{\partial S_2}{\partial \Psi_z}\left(\frac{\gamma^A_{A_2 z} - \gamma^B_{A_2 z}}{\sigma_{A_2}\sigma_z}\right)\right), \\
\sigma_{S_2 z}(t) &= \frac{1}{S_2(t)}\left(\frac{\partial S_2}{\partial m^A_{A_2}}\left(\frac{\alpha_{A_1}\gamma^A_{A_1 A_2} + \alpha_{A_2}\gamma^A_{A_2} + \beta\gamma^A_{A_2 z}}{\sigma_z}\right)\right. \\
&\quad + \frac{\partial S_2}{\partial \Psi_{A_1}}\left(\frac{\alpha_{A_1}\left(\gamma^A_{A_1} - \gamma^B_{A_1}\right) + \alpha_{A_2}\left(\gamma^A_{A_1 A_2} - \gamma^B_{A_1 A_2}\right) + \beta\left(\gamma^A_{A_1 z} - \gamma^B_{A_1 z}\right)}{\sigma_{A_1}\sigma_z}\right) \\
&\quad + \frac{\partial S_2}{\partial \Psi_{A_2}}\left(\frac{\alpha_{A_1}\left(\gamma^A_{A_1 A_2} - \gamma^B_{A_1 A_2}\right) + \beta\left(\gamma^A_{A_2 z} - \gamma^B_{A_2 z}\right)}{\sigma_{A_2}\sigma_z}\right) \\
&\quad \left. + \frac{\partial S_2}{\partial \Psi_z}\left(\frac{\alpha_{A_1}\left(\gamma^A_{A_1 z} - \gamma^B_{A_1 z}\right) + \alpha_{A_2}\left(\gamma^A_{A_2 z} - \gamma^B_{A_2 z}\right) + \beta\left(\gamma^A_z - \gamma^B_z\right)}{\sigma^2_z}\right)\right).
\end{aligned}$$

A.6 Proof of Laplace Transform

I first summarize the most important filtered dynamics needed to compute the Laplace transform. Disagreement Dynamics of firm i are:

$$\begin{aligned}
d\Psi_i(t) &= \left(\underbrace{\left(a_{1i} + \frac{\gamma^B_i}{\sigma^2_i}\right)}_{K_{1i}}\Psi_i(t) + \underbrace{\frac{\gamma^B_{ij}}{\sigma_j\sigma_i}}_{K_{1j}}\Psi_j(t) + \underbrace{\frac{\left(\alpha_i\gamma^B_i + \alpha_j\gamma^B_{ij} + \beta\gamma^B_{iz}\right)}{\sigma_i\sigma_z}}_{K_{1z}}\Psi_z(t)\right) dt \\
&\quad + \underbrace{\left(\frac{\gamma^A_i - \gamma^B_i}{\sigma^2_i}\right)}_{\sigma_{1i}} dW^A_i(t) + \underbrace{\left(\frac{\gamma^A_{ij} - \gamma^B_{ij}}{\sigma_i\sigma_j}\right)}_{\sigma_{1j}} dW^A_j(t) + \underbrace{\frac{1}{\sigma_i\sigma_z}\left(\alpha_i\left(\gamma^A_i - \gamma^B_i\right) + \alpha_j\left(\gamma^A_{ij} - \gamma^B_{ij}\right) + \beta\left(\gamma^A_{iz} - \gamma^B_{iz}\right)\right)}_{\sigma_{1z}} dW^A_z(t).
\end{aligned}$$

Similarly, the disagreement dynamics of firm j are given by:

$$
\begin{aligned}
d\Psi_j(t) \quad = \quad & \left(\underbrace{\frac{\gamma_{ij}^B}{\sigma_j \sigma_i}}_{K_{2i}} \Psi_i(t) + \underbrace{\left(a_{1j} + \frac{\gamma_j^B}{\sigma_j^2} \right)}_{K_{2j}} \Psi_j(t) + \underbrace{\frac{\alpha_i \gamma_{ij}^B + \alpha_j \gamma_j^B + \beta \gamma_{jz}^B}{\sigma_j \sigma_z}}_{K_{2z}} \Psi_z(t) \right) dt \\
& + \underbrace{\left(\frac{\gamma_{ij}^A - \gamma_{ij}^B}{\sigma_i \sigma_j} \right)}_{\sigma_{2i}} dW_i^A(t) + \underbrace{\left(\frac{\gamma_j^1 - \gamma_j^B}{\sigma_j^2} \right)}_{\sigma_{2j}} dW_j^A(t) + \underbrace{\frac{1}{\sigma_j \sigma_z} \left(\alpha_i \left(\gamma_{ij}^A - \gamma_{ij}^B \right) + \alpha_j \left(\gamma_j^A - \gamma_j^B \right) + \beta \left(\gamma_{jz}^A - \gamma_{jz}^B \right) \right)}_{\sigma_{2z}} dW_z^A(t).
\end{aligned}
$$

The disagreement about the signal growth rate is given by:

$$
\begin{aligned}
d\Psi_z(t) \quad = \quad & \left(\underbrace{\frac{\gamma_{iz}^B}{\sigma_i \sigma_z}}_{K_{3i}} \Psi_i(t) + \underbrace{\frac{\gamma_{jz}^2}{\sigma_z \sigma_j}}_{K_{3j}} \Psi_j(t) + \underbrace{\left(a_{1z} + \frac{1}{\sigma_z^2} \left(\alpha_i \gamma_{iz}^B + \alpha_j \gamma_{jz}^B + \beta \gamma_z^B \right) \right)}_{K_{3z}} \Psi_z(t) \right) dt \\
& + \underbrace{\left(\frac{\gamma_{iz}^A - \gamma_{iz}^B}{\sigma_i \sigma_z} \right)}_{\sigma_{3i}} dW_i^A(t) + \underbrace{\left(\frac{\gamma_{jz}^A - \gamma_{jz}^B}{\sigma_j \sigma_z} \right)}_{\sigma_{3j}} dW_j^A(t) + \underbrace{\frac{1}{\sigma_z^2} \left(\alpha_i \left(\gamma_{iz}^A - \gamma_{iz}^B \right) + \alpha_j \left(\gamma_{jz}^A - \gamma_{jz}^B \right) + \beta \left(\gamma_z^A - \gamma_z^B \right) \right)}_{\sigma_{3z}} dW_z^A(t).
\end{aligned}
$$

The growth rate of firm i is given by:

$$
dm_i^A(t) = \left(a_{0i} + a_{1i} m_i^A(t) \right) dt + \underbrace{\frac{\gamma_i^A}{\sigma_i}}_{\sigma_{4i}} dW_i^A(t) + \underbrace{\frac{\gamma_{ij}^A}{\sigma_j}}_{\sigma_{4j}} dW_j^A(t) + \underbrace{\frac{\alpha_i \gamma_i^A + \alpha_i \gamma_{ij}^A + \beta \gamma_{iz}^A}{\sigma_z}}_{\sigma_{4z}} dW_z^A(t).
$$

Similarly, the growth rate of firm j is given by:

$$
dm_j^A(t) = \left(a_{0j} + a_{1j} m_j^A(t) \right) dt + \underbrace{\frac{\gamma_{ij}^A}{\sigma_i}}_{\sigma_{5i}} dW_i^A(t) + \underbrace{\frac{\gamma_j^A}{\sigma_j}}_{\sigma_{5j}} dW_j^A(t) + \underbrace{\frac{\alpha_i \gamma_{ij}^A + \alpha_j \gamma_j^A + \beta \gamma_{jz}^A}{\sigma_z}}_{\sigma_{5z}} dW_z^A(t).
$$

Finally, the fundamentals have the following dynamics:

$$
\begin{aligned}
dA_i(t)/A_i(t) &= m_i^A(t) dt + \underbrace{\sigma_i}_{\sigma_{7i}} dW_i(t)^A \\
dA_j(t)/A_j(t) &= m_j^A(t) dt + \underbrace{\sigma_j}_{\sigma_{7j}} dW_j(t)^A
\end{aligned}
$$

Let me collect all variables in the vector $X = (\Psi_i, \Psi_j, \Psi_z, m_i, m_j, \eta, A_i, A_j)'$, and all shocks in the vector $W^A = \left(W_i^A, W_j^A, W_z^A\right)'$. I can now compactly write the dynamics of X as:

$$dX = \left[\underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ a_{0i} \\ a_{0j} \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{K_0} + \underbrace{\begin{pmatrix} K_{1i} & K_{1j} & K_{1z} & 0 & 0 & 0 & 0 & 0 \\ K_{2i} & K_{2j} & K_{2z} & 0 & 0 & 0 & 0 & 0 \\ K_{3i} & K_{3j} & K_{3z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{1i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{1j} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_j & 0 & 0 \end{pmatrix}}_{K_1} X(t) \right] dt + \underbrace{\begin{pmatrix} \sigma_{1i} & \sigma_{1j} & \sigma_{1z} \\ \sigma_{2i} & \sigma_{2j} & \sigma_{2z} \\ \sigma_{3i} & \sigma_{3j} & \sigma_{3z} \\ \sigma_{4i} & \sigma_{4j} & \sigma_{4z} \\ \sigma_{5i} & \sigma_{5j} & \sigma_{5z} \\ -\eta\hat{\sigma}_{6i} & -\eta\hat{\sigma}_{6j} & -\eta\hat{\sigma}_{6z} \\ A_i\sigma_{7i} & 0 & 0 \\ 0 & A_j\sigma_{8j} & 0 \end{pmatrix}}_{S} dW_i^A(t),$$

where hats in $\hat{\sigma}_{6i}, \hat{\sigma}_{6j}$ and $\hat{\sigma}_{6z}$ shall remind us that diffusion of $\frac{d\eta}{\eta}$ is a function of $\Psi(t)$. Let me also denote those sub-blocks (yellow ones) of K_0, K_1, S, which are related to $\Psi(t)$ by:

$$\kappa_0 := K_{0(1:3)} = 0_{3\times 1}$$
$$\kappa_1 := K_{1(1:3,1:3)}$$
$$\Sigma := S_{(1:3,1:3)}.$$

Thus, the dynamics of $\Psi(t)$ can be compactly written as:

$$d\Psi(t) = (\kappa_0 + \kappa_1\Psi(t))\, dt + \Sigma dW_i^A(t).$$

Let $m^A(t) = (m_{A_1}^A(t) m_{A_2}^A(t) m_z^A(t))$, then the joint Laplace transform of $X(t)$ is given by:

$$F_X = A_i^{\varepsilon_{A_i}} A_j^{\varepsilon_{A_j}} \eta^{\chi} F_{m^A} F_{\Psi}, \tag{18}$$

where F_{m^A} is affine in m_i and m_j:

$$F_{m^A} = \exp\left(A_{m^A} + \bar{A}_i(\tau)m_i + \bar{A}_j(\tau)m_j\right),$$

and F_{Ψ_i,Ψ_j,Ψ_z} is affine-quadratic in $\Psi = (\Psi_i, \Psi_j, \Psi_z)$:

$$F_{\Psi_i,\Psi_j,\Psi_z} = \exp(A(\tau) + B(\tau)'\Psi + \Psi' C(\tau)\Psi).$$

In matrix notation, Feynman-Kač implies:

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X'}(K_0 + K_1 X) + \frac{1}{2} Tr\left(\frac{\partial^2 F}{\partial X \partial X'} SS'\right) = 0$$

or

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{11} \frac{\partial F}{\partial X_i}(K_0(i) + K_1(i,:)X) + \frac{1}{2}\sum_{i,j=1}^{11} C_{ij}\frac{\partial^2 F}{\partial X_i \partial X_j} = 0,$$

where C_{ij} is the $ij-$th element of the matrix $C = SS'$. My goal is to reduce the problem of solving the PDE in F_X, to solving simply the PDE in F_{Ψ_i,Ψ_j,Ψ_z}, which can be done virtually in closed form. This requires several adjustments to the parameters of Ψ dynamics and the constants in the respective ODEs. Specifically, I will show that $A(\tau), B(\tau), C(\tau)$ solve the following system:

$$
\begin{aligned}
\frac{\partial A}{\partial \tau} &= B(\tau)'\bar{\kappa}_0 + Tr\left[C(\tau)\Sigma\Sigma'\right] + \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + A_c \qquad \text{(ODE.1)} \\
\frac{\partial B'}{\partial \tau}\Psi &= B(\tau)'\bar{\kappa}_1\Psi + 2\bar{\kappa}_0'C(\tau)\Psi + 2Tr\left[B'(\tau)\Sigma\Sigma'C(\tau)\Psi\right] + B_c'\Psi \qquad \text{(ODE.2)} \\
Tr\left[\frac{\partial C'}{\partial \tau}\Psi\Psi'\right] &= Tr\left[\left(C(\tau)\bar{\kappa}_1 + \bar{\kappa}_1'C(\tau)\right)\Psi\Psi'\right] + 2Tr\left[C(\tau)\Sigma\Sigma'C(\tau)\Psi\Psi'\right] + Tr\left(C_c\Psi\Psi'\right) \qquad \text{(ODE.3)},
\end{aligned}
\tag{19}
$$

where $\bar{\kappa}_0, \bar{\kappa}_1$ denote appropriately adjusted coefficients κ_0, κ_1, and A_c, B_c, C_c are the adjusted constants.

Since the adjustments involve the cross-derivatives of F with respect to X, I spell out the Hessian matrix for completeness:

$$
\frac{\partial^2}{\partial X \partial X'} = \begin{pmatrix}
\frac{\partial^2}{\partial \Psi_i^2} & \frac{\partial^2}{\partial \Psi_i \partial \Psi_j} & \frac{\partial^2}{\partial \Psi_i \partial \Psi_z} & \frac{\partial^2}{\partial \Psi_i \partial m_{A_2}} & \frac{\partial^2}{\partial \Psi_i \partial m_{A_2}} & \frac{\partial^2}{\partial \Psi_i \partial \eta} & \frac{\partial^2}{\partial \Psi_i \partial A_1} & \frac{\partial^2}{\partial \Psi_i \partial A_2} \\
 & \frac{\partial^2}{\partial \Psi_j^2} & \frac{\partial^2}{\partial \Psi_j \partial \Psi_z} & \frac{\partial^2}{\partial \Psi_j \partial m_{A_2}} & \frac{\partial^2}{\partial \Psi_j \partial m_{A_2}} & \frac{\partial^2}{\partial \Psi_{A_2} \partial \eta} & \frac{\partial^2}{\partial \Psi_j \partial A_1} & \frac{\partial^2}{\partial \Psi_j \partial A_2} \\
 & & \frac{\partial^2}{\partial \Psi_z^2} & \frac{\partial^2}{\partial \Psi_z \partial m_{A_1}} & \frac{\partial^2}{\partial \Psi_z \partial m_{A_2}} & \frac{\partial^2}{\partial \Psi_z \partial \eta} & \frac{\partial^2}{\partial \Psi_z \partial A_1} & \frac{\partial^2}{\partial \Psi_z \partial A_2} \\
 & & & \frac{\partial^2}{\partial m_{A_2}^2} & \frac{\partial^2}{\partial m_{A_2} \partial m_{A_2}} & \frac{\partial^2}{\partial m_{A_2} \partial \eta} & \frac{\partial^2}{\partial m_{A_2} \partial A_1} & \frac{\partial^2}{\partial m_{A_2} \partial G} \\
 & & & & \frac{\partial^2}{\partial m_{A_2}^2} & \frac{\partial^2}{\partial m_{A_2} \partial \eta} & \frac{\partial^2}{\partial m_{A_2} \partial A_1} & \frac{\partial^2}{\partial m_{A_2} \partial A_2} \\
 & & & & & \frac{\partial^2}{\partial \eta^2} & \frac{\partial^2}{\partial \eta \partial A_1} & \frac{\partial^2}{\partial \eta \partial A_2} \\
 & & & & & & \frac{\partial^2}{\partial A_1^2} & \frac{\partial^2}{\partial A_1 \partial A_2} \\
 & & & & & & & \frac{\partial^2}{\partial A_2^2}
\end{pmatrix}
$$

Types of adjustments:

1. Terms involving derivatives of the type $\frac{\partial^2}{\partial \Psi_i \partial m_j}, i = \{A_1, A_2, z\}, j = \{V, G, M\}$ adjust the vector κ_0 in the drift of Ψ.
2. Terms involving derivatives of the type $\frac{\partial^2}{\partial \Psi_i \partial A_1}, \frac{\partial^2}{\partial \Psi_i \partial A_2}, i = \{A_1, A_2, z\}$ adjust the vector κ_0 in the drift of Ψ.
3. Terms involving derivatives of the type $\frac{\partial^2}{\partial \Psi_i \partial \eta}, i = \{A_1, A_2, z\}$ adjust the mean reversion matrix κ_1 in the drift of Ψ.
4. Terms involving derivatives of the type $\frac{\partial^2}{\partial m_i \partial \eta}, i = \{A_1, A_2\}$ adjust the constant vector B_c in the ODE.2.
5. Terms involving derivatives of the type $\frac{\partial^2}{\partial \eta \partial A_1}, \frac{\partial^2}{\partial \eta \partial A_2}$ adjust the constant vector B_c in the ODE.2.
6. Terms involving derivatives of the type $\frac{\partial^2}{\partial \eta^2}$ adjust the constant matrix C_c in the ODE.3.

For all the cases above, I need to consider the structure of $\frac{1}{dt}\frac{\partial^2}{\partial X_i \partial X_j}\langle dX_i, dX_j\rangle$. Below I neglect the division by $\frac{1}{dt}$ for brevity. Let us start with the adjustments of the constant drift matrix κ_0.

Adjustment 1: $\frac{\partial^2 F}{\partial \Psi_i \partial m_j}$ where $i = \{A_1, A_2, z\}$ and $j = \{A_1, A_2\}$

$$
\begin{array}{ccc}
\frac{\partial^2 F}{\partial \Psi_i \partial m_{A_2}} = \frac{\partial F}{\partial \Psi_i}\bar{A}_{A_1} & \text{and} & \langle d\Psi_i, dm_{A_2}\rangle = \sum_i \sigma_{1i}\sigma_{5i} \\
\frac{\partial^2 F}{\partial \Psi_i \partial m_{A_2}} = \frac{\partial F}{\partial \Psi_i}\bar{A}_{A_2} & \text{and} & \langle d\Psi_i, dm_{A_2}\rangle = \sum_i \sigma_{1i}\sigma_{6i} \\
... & \text{and} & ...
\end{array}
$$

I need to add the following vector to κ_0 :

$$
\kappa_0^{adj1} = \begin{pmatrix}
\bar{A}_{A_1}\sum_i \sigma_{1i}\sigma_{5i} + \bar{A}_{A_2}\sum_i \sigma_{1i}\sigma_{6i} + \bar{A}_M\sum_i \sigma_{1i}\sigma_{7i} \\
\bar{A}_{A_1}\sum_i \sigma_{2i}\sigma_{5i} + \bar{A}_{A_2}\sum_i \sigma_{2i}\sigma_{6i} + \bar{A}_M\sum_i \sigma_{2i}\sigma_{7i} \\
\bar{A}_{A_1}\sum_i \sigma_{3i}\sigma_{5i} + \bar{A}_{A_2}\sum_i \sigma_{3i}\sigma_{6i} + \bar{A}_M\sum_i \sigma_{3i}\sigma_{7i} \\
\bar{A}_{A_1}\sum_i \sigma_{4i}\sigma_{5i} + \bar{A}_{A_2}\sum_i \sigma_{4i}\sigma_{6i} + \bar{A}_M\sum_i \sigma_{4i}\sigma_{7i}
\end{pmatrix} = \underbrace{\begin{pmatrix} \sigma_{(1,:)} \\ \sigma_{(2,:)} \\ \sigma_{(3,:)} \\ \sigma_{(4,:)} \end{pmatrix}}_{4\times4} \underbrace{\left(\bar{A}_{A_1}\sigma'_{(5,:)} + \bar{A}_{A_2}\sigma'_{(6,:)} + \bar{A}_M\sigma'_{(7,:)}\right)}_{4\times1}.
$$

Adjustment 2: $\frac{\partial^2 F}{\partial \Psi_i \partial I}, i = \{A_1, A_2, z\}, I = \{V, G, M\}$

$$\begin{array}{rclcc}
\frac{\partial^2 F}{\partial \Psi_i \partial V} & = & \frac{\partial F}{\partial \Psi_i}\frac{\varepsilon_V}{V} & \text{and} & \langle d\Psi_i, dV\rangle = V\sigma_{1i}\sigma_{9V} \\
\frac{\partial^2 F}{\partial \Psi_i \partial G} & = & \frac{\partial F}{\partial \Psi_i}\frac{\varepsilon_G}{G} & \text{and} & \langle d\Psi_i, dV\rangle = G\sigma_{1j}\sigma_{10G} \\
\frac{\partial^2 F}{\partial \Psi_i \partial M} & = & \frac{\partial F}{\partial \Psi_M}\frac{\varepsilon_M}{M} & \text{and} & \langle d\Psi_i, dV\rangle = M\sigma_{1M}\sigma_{11M} \\
 & \ldots & & \text{and} & \ldots
\end{array}$$

Adjustment 2 adds the following vector to κ_0:

$$\kappa_0^{adj2} = \underbrace{\begin{pmatrix} \varepsilon_V \sigma_{1i}\sigma_{9A_1} + \varepsilon_G \sigma_{1j}\sigma_{10G} + \varepsilon_M \sigma_{1M}\sigma_{11M} \\ \varepsilon_V \sigma_{2V}\sigma_{9A_1} + \varepsilon_G \sigma_{2G}\sigma_{10G} + \varepsilon_M \sigma_{2M}\sigma_{11M} \\ \varepsilon_V \sigma_{3V}\sigma_{9A_1} + \varepsilon_G \sigma_{3G}\sigma_{10G} + \varepsilon_M \sigma_{3M}\sigma_{11M} \\ 0 \end{pmatrix}}_{4\times 1},$$

where 0 in the last row of κ_0^{adj2} follows from the independence between the disagreement on the signal and V, G and M dynamics.

Summarizing, Adj 1 and **Adj 2** imply that the adjusted vector $\bar{\kappa}_0$ in the drift of Ψ is:

$$\boxed{\bar{\kappa}_0 = \kappa_0 + \kappa_0^{adj1} + \kappa_0^{adj2}.}$$

Adjustment 3 Next consider the terms that can be absorbed into the mean reversion matrix κ_1. Those involve $\frac{\partial^2 F}{\partial \Psi_i \partial \eta}, i = \{A_1, A_2, z\}$.

$$\begin{array}{rclcl}
\frac{\partial^2 F}{\partial \Psi_i \partial \eta} & = & \frac{\partial F}{\partial \Psi_i}\frac{\chi}{\eta} & \text{and} & \langle d\Psi_V, d\eta\rangle = -\eta\left[(\sigma_{1i} + \sigma_{1z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{1j} + \sigma_{1z}\bar{\alpha}_G)\Psi_j + (\sigma_{1M} + \sigma_{1z}\bar{\alpha}_M)\Psi_M + \sigma_{1z}\beta\Psi_z\right] \\
 & & & \Rightarrow & \kappa_{1(1,:)}^{adj} = -\chi\begin{pmatrix} (\sigma_{1i} + \sigma_{1z}\bar{\alpha}_{A_1}), & (\sigma_{1j} + \sigma_{1z}\bar{\alpha}_{A_2}), & (\sigma_{1M} + \sigma_{1z}\bar{\alpha}_M), & \sigma_{1z}\beta \end{pmatrix}_{1\times 4} \\
\frac{\partial^2 F}{\partial \Psi_j \partial \eta} & = & \frac{\partial F}{\partial \Psi_j}\frac{\chi}{\eta} & \text{and} & \langle d\Psi_G, d\eta\rangle = -\eta\left[(\sigma_{2V} + \sigma_{2z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{2G} + \sigma_{2z}\bar{\alpha}_G)\Psi_j + (\sigma_{2M} + \sigma_{2z}\bar{\alpha}_M)\Psi_M + \sigma_{2z}\beta\Psi_z\right] \\
 & & & \Rightarrow & \kappa_{1(2,:)}^{adj} = -\chi\begin{pmatrix} (\sigma_{2V} + \sigma_{2z}\bar{\alpha}_{A_1}), & (\sigma_{2G} + \sigma_{2z}\bar{\alpha}_{A_2}), & (\sigma_{2M} + \sigma_{2z}\bar{\alpha}_M), & \sigma_{2z}\beta \end{pmatrix}_{1\times 4} \\
\frac{\partial^2 F}{\partial \Psi_M \partial \eta} & = & \frac{\partial F}{\partial \Psi_M}\frac{\chi}{\eta} & \text{and} & \langle d\Psi_M, d\eta\rangle = -\eta\left[(\sigma_{3V} + \sigma_{3z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{3G} + \sigma_{3z}\bar{\alpha}_G)\Psi_j + (\sigma_{3M} + \sigma_{3z}\bar{\alpha}_M)\Psi_M + \sigma_{3z}\beta\Psi_z\right] \\
 & & & \Rightarrow & \kappa_{1(3,:)}^{adj} = -\chi\begin{pmatrix} (\sigma_{3V} + \sigma_{3z}\bar{\alpha}_{A_1}), & (\sigma_{3G} + \sigma_{3z}\bar{\alpha}_{A_2}), & (\sigma_{3M} + \sigma_{3z}\bar{\alpha}_M), & \sigma_{3z}\beta \end{pmatrix}_{1\times 4} \\
\frac{\partial^2 F}{\partial \Psi_z \partial \eta} & = & \frac{\partial F}{\partial \Psi_z}\frac{\chi}{\eta} & \text{and} & \langle d\Psi_z, d\eta\rangle = -\eta\left[(\sigma_{4V} + \sigma_{4z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{4G} + \sigma_{4z}\bar{\alpha}_G)\Psi_j + (\sigma_{4M} + \sigma_{4z}\bar{\alpha}_M)\Psi_M + \sigma_{4z}\beta\Psi_z\right] \\
 & & & \Rightarrow & \kappa_{1(4,:)}^{adj} = -\chi\begin{pmatrix} (\sigma_{4V} + \sigma_{4z}\bar{\alpha}_{A_1}), & (\sigma_{4G} + \sigma_{4z}\bar{\alpha}_{A_2}), & (\sigma_{4M} + \sigma_{4z}\bar{\alpha}_M), & \sigma_{4z}\beta \end{pmatrix}_{1\times 4}
\end{array}$$

Thus,

$$\kappa_1^{adj} = -\chi\underbrace{\begin{bmatrix} (\sigma_{1i} + \sigma_{1z}\bar{\alpha}_{A_1}) & (\sigma_{1G} + \sigma_{1z}\bar{\alpha}_{A_2}) & (\sigma_{1M} + \sigma_{1z}\bar{\alpha}_M) & \sigma_{1z}\beta \\ (\sigma_{2V} + \sigma_{2z}\bar{\alpha}_{A_1}) & (\sigma_{2G} + \sigma_{2z}\bar{\alpha}_{A_2}) & (\sigma_{2M} + \sigma_{2z}\bar{\alpha}_M) & \sigma_{2z}\beta \\ (\sigma_{3V} + \sigma_{3z}\bar{\alpha}_{A_1}) & (\sigma_{3G} + \sigma_{3z}\bar{\alpha}_{A_2}) & (\sigma_{3M} + \sigma_{3z}\bar{\alpha}_M) & \sigma_{3z}\beta \\ (\sigma_{4V} + \sigma_{4z}\bar{\alpha}_{A_1}) & (\sigma_{4G} + \sigma_{4z}\bar{\alpha}_{A_2}) & (\sigma_{4M} + \sigma_{4z}\bar{\alpha}_M) & \sigma_{4z}\beta \end{bmatrix}}_{4\times 4}$$

Summarizing, the adjusted mean reversion matrix in the dynamics of Ψ is given by:

$$\boxed{\bar{\kappa}_1 = \kappa_1 + \kappa_1^{adj}.}$$

Adjustment 4: $\frac{\partial^2 F}{\partial m_i \partial \eta}, i = \{V, G, M\}$

$$\begin{array}{rclcl}
\frac{\partial^2 F}{\partial m_{A_2} \partial \eta} & = & \frac{\chi \bar{A}_{A_1}}{\eta}F & \text{and} & \langle dm_{A_2}, d\eta\rangle = -\eta\left[(\sigma_{4i} + \sigma_{5z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{5G} + \sigma_{5z}\bar{\alpha}_{A_2})\Psi_j + (\sigma_{5M} + \sigma_{5z}\bar{\alpha}_M)\Psi_M + (\sigma_{5z}\beta)\Psi_z\right] \\
\frac{\partial^2 F}{\partial m_{A_2} \partial \eta} & = & \frac{\chi \bar{A}_{A_2}}{\eta}F & \text{and} & \langle dm_{A_2}, d\eta\rangle = -\eta\left[(\sigma_{6V} + \sigma_{6z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{6G} + \sigma_{6z}\bar{\alpha}_{A_2})\Psi_j + (\sigma_{6M} + \sigma_{6z}\bar{\alpha}_M)\Psi_M + (\sigma_{6z}\beta)\Psi_z\right] \\
\frac{\partial^2 F}{\partial m_M \partial \eta} & = & \frac{\chi \bar{A}_M}{\eta}F & \text{and} & \langle dm_M, d\eta\rangle = -\eta\left[(\sigma_{7V} + \sigma_{7z}\bar{\alpha}_{A_1})\Psi_i + (\sigma_{7G} + \sigma_{7z}\bar{\alpha}_{A_2})\Psi_j + (\sigma_{7M} + \sigma_{7z}\bar{\alpha}_M)\Psi_M + (\sigma_{7z}\beta)\Psi_z\right]
\end{array}$$

Since these equations do not involve any derivatives of the type $\frac{\partial}{\partial \Psi_i}$, but do involve linear terms in Ψ_i, I realize that they adjust the constant B_c in ODE.2:

$$B_c^{adj1} = -\chi \begin{pmatrix} \left[\bar{A}_{A_1}(\sigma_{4i} + \sigma_{5z}\bar{\alpha}_{A_1}) + \bar{A}_{A_2}(\sigma_{6V} + \sigma_{6z}\bar{\alpha}_{A_1}) + \bar{A}_M(\sigma_{7V} + \sigma_{7z}\bar{\alpha}_{A_1})\right] \\ \left[\bar{A}_{A_1}(\sigma_{5G} + \sigma_{5z}\bar{\alpha}_{A_2}) + \bar{A}_{A_2}(\sigma_{6G} + \sigma_{6z}\bar{\alpha}_{A_2}) + \bar{A}_M(\sigma_{7G} + \sigma_{7z}\bar{\alpha}_{A_2})\right] \\ \left[\bar{A}_{A_1}(\sigma_{5M} + \sigma_{5z}\bar{\alpha}_M) + \bar{A}_{A_2}(\sigma_{6M} + \sigma_{6z}\bar{\alpha}_M) + \bar{A}_M(\sigma_{7M} + \sigma_{7z}\bar{\alpha}_M)\right] \\ \left[\bar{A}_{A_1}(\sigma_{5z}\beta) + \bar{A}_{A_2}(\sigma_{6z}\beta) + \bar{A}_M(\sigma_{7z}\beta)\right] \end{pmatrix}$$

Adjustment 5: The same logic applies to the derivatives of the form: $\frac{\partial^2 F}{\partial \eta \partial I}, I = \{V, G, M\}$

$$\begin{aligned} \frac{\partial^2 F}{\partial \eta \partial V} &= \frac{\chi \varepsilon_V}{\eta V} F \quad \text{and} \quad \langle d\eta, dV \rangle = -\eta V \sigma_{9A_1} \Psi_i \\ \frac{\partial^2 F}{\partial \eta \partial G} &= \frac{\chi \varepsilon_G}{\eta G} F \quad \text{and} \quad \langle d\eta, dG \rangle = -\eta G \sigma_{10G} \Psi_j \end{aligned}$$

Thus the second adjustment to B_c is:

$$B_c^{adj2} = -\chi \begin{pmatrix} \varepsilon_V \sigma_{9A_1} \\ \varepsilon_G \sigma_{10G} \\ \varepsilon_M \sigma_{11M} \\ 0 \end{pmatrix}$$

Summarizing, the constant vector in ODE.2 is given by:

$$\boxed{B_c = B_c^{adj1} + B_c^{adj2}}$$

Adjustment 6:

$$\frac{1}{2}\frac{\partial^2 F}{\partial \eta^2} = \frac{1}{2}\frac{\chi(\chi-1)F}{\eta^2}$$

$$\langle d\eta, d\eta \rangle = \eta^2 \left(\hat{\sigma}_{8V}^2 + \hat{\sigma}_{8G}^2 + \hat{\sigma}_{8M}^2 + \hat{\sigma}_{8z}^2\right) = Tr \left[\begin{pmatrix} (1 + \bar{\alpha}_{A_1}^2) & \bar{\alpha}_{A_1}\bar{\alpha}_{A_2} & \bar{\alpha}_V \bar{\alpha}_M & \bar{\alpha}_{A_1}\beta \\ \bar{\alpha}_{A_1}\bar{\alpha}_{A_2} & (1 + \bar{\alpha}_{A_2}^2) & \bar{\alpha}_G \bar{\alpha}_M & \bar{\alpha}_{A_2}\beta \\ \bar{\alpha}_{A_1}\bar{\alpha}_M & \bar{\alpha}_{A_2}\bar{\alpha}_M & (1 + \bar{\alpha}_M^2) & \bar{\alpha}_M \beta \\ \bar{\alpha}_{A_1}\beta & \bar{\alpha}_{A_2}\beta & \bar{\alpha}_M \beta & \beta^2 \end{pmatrix} \Psi \Psi' \right]$$

Thus, the **adjustment** of the constant matrix in ODE.3 is given by:

$$\boxed{C_c = \frac{1}{2}\chi(\chi - 1) \begin{pmatrix} (1 + \bar{\alpha}_{A_1}^2) & \bar{\alpha}_{A_1}\bar{\alpha}_{A_2} & \bar{\alpha}_{A_1}\bar{\alpha}_M & \bar{\alpha}_{A_1}\beta \\ \bar{\alpha}_{A_1}\bar{\alpha}_{A_2} & (1 + \bar{\alpha}_{A_2}^2) & \bar{\alpha}_{A_2}\bar{\alpha}_M & \bar{\alpha}_{A_2}\beta \\ \bar{\alpha}_{A_1}\bar{\alpha}_M & \bar{\alpha}_{A_2}\bar{\alpha}_M & (1 + \bar{\alpha}_M^2) & \bar{\alpha}_M \beta \\ \bar{\alpha}_{A_1}\beta & \bar{\alpha}_{A_2}\beta & \bar{\alpha}_M \beta & \beta^2 \end{pmatrix}}$$

I can now obtain the coefficients $A(\tau), B(\tau), C(\tau)$ by solving the system:

$$\frac{\partial A}{\partial \tau} = B(\tau)' \bar{\kappa}_0 + Tr\left[C(\tau)\Sigma\Sigma'\right] + \frac{1}{2}B'(\tau)\Sigma\Sigma' B(\tau) + A_c \qquad \text{(ODE.1)} \tag{20}$$

$$\frac{\partial B'}{\partial \tau} = B(\tau)' \bar{\kappa}_1 + 2\bar{\kappa}_0' C(\tau) + 2B'(\tau)\Sigma\Sigma' C(\tau) + B_c' \qquad \text{(ODE.2)} \tag{21}$$

$$\frac{\partial C}{\partial \tau} = C(\tau)\bar{\kappa}_1 + \bar{\kappa}_1' C(\tau) + 2C(\tau)\Sigma\Sigma' C(\tau) + C_c \qquad \text{(ODE.3)},$$

where $A(0) = 0, B(0) = 0, C(0) = 0$.

Last equation (ODE.3) can be solved in closed form by matrix Riccati linearization:

$$C(\tau) = F_{22}(\tau)^{-1} F_{21}(\tau),$$

where

$$\left(\begin{array}{cc} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{array}\right) = \exp\left[\tau\left(\begin{array}{cc} \bar{\kappa}_1 & -2\Sigma\Sigma' \\ C_c & -\bar{\kappa}_1 \end{array}\right)\right].$$

Given $C(\tau)$, I can obtain the solution for ODE.2 in closed form. First, for tractability, I re-write ODE.2 as:

$$\frac{\partial B}{\partial \tau} = \widetilde{\kappa}_1 B(\tau) + \widetilde{\kappa}_0,$$

where

$$\widetilde{\kappa}_1 = \bar{\kappa}_1 + 2C(\tau)\Sigma\Sigma'$$
$$\widetilde{\kappa}_0 = 2C(\tau)\bar{\kappa}_0 + B_c.$$

The solution for $B(\tau)$ is follows as (see e.g. Laub, 2005):

$$B(\tau) = \int_0^\tau e^{\widetilde{\kappa}_1(\tau - s)}\widetilde{\kappa}_0 ds.$$

This matrix integral admits an explicit solution (see Thm.1 in van Loan, 1978). Defining the following auxiliary matrix

$$C_{aux} = \left(\begin{array}{cc} \widetilde{\kappa}_1 & \widetilde{\kappa}_0 \\ 0_{4\times 1} & 0_{1\times 1} \end{array}\right),$$

$B(\tau)$ is represented as the upper right subblock of the following matrix exponential

$$\exp\left(\tau C_{aux}\right) = \left(\begin{array}{cc} F_1(\tau) & B(\tau) \\ 0 & F_2(\tau) \end{array}\right).$$

This is the correct solution if $\widetilde{\kappa}_0, \widetilde{\kappa}_1$ are not time dependent. However, alternatively, the system can be solved numerically (and efficiently) using, e.g., the Runge-Kutta method.

I can proceed in an analogous way to obtain the coefficients in

$$F_{m_{A_2}, m_{A_2}, m_M} = e^{\bar{C}(\tau) + \bar{A}_{A_1}(\tau) m_{A_2} + \bar{A}_{A_2}(\tau) m_{A_2} + \bar{A}_M(\tau) m_M} = e^{\bar{C}(\tau) + \bar{A}(\tau)' m}$$

where $m = (m_{A_2}, m_{A_2}, m_M)'$. Let $J = (V, G, M)', \bar{W} = (W_V, W_G, W_M)$.

$$dm = \left[\underbrace{\left(\begin{array}{c} a_{0i} \\ a_{0j} \\ a_{0M} \end{array}\right)}_{l_0} + \underbrace{\left(\begin{array}{ccc} a_{1i} & 0 & 0 \\ 0 & a_{1j} & 0 \\ 0 & 0 & a_{1M} \end{array}\right)}_{l_1} m\right] dt + \underbrace{\left(\begin{array}{ccc} \sigma_{4i} & \sigma_{5G} & \sigma_{5M} \\ \sigma_{6V} & \sigma_{6G} & \sigma_{6M} \\ \sigma_{7V} & \sigma_{7G} & \sigma_{7M} \end{array}\right)}_{L} d\bar{W}$$

$$dJ = \left[\underbrace{\left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right)}_{\lambda_0} + \underbrace{\left(\begin{array}{ccc} V & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & M \end{array}\right)}_{\lambda_1} J\right] dt + \underbrace{\left(\begin{array}{ccc} V\sigma_{9A_1} & 0 & 0 \\ 0 & G\sigma_{10G} & 0 \\ 0 & 0 & M\sigma_{11M} \end{array}\right)}_{\Lambda} d\bar{W}$$

Our goal is to reduce the problem of solving PDE in F_X, to solving just the PDE in F_m, which can be done virtually in closed form:

$$\frac{\partial F_m}{\partial t} + \frac{\partial F_m}{\partial m'}\underbrace{(l_0 + l_1 m)}_{\mu_m} + \frac{1}{2}Tr\left[\frac{\partial^2 F_m}{\partial m \partial m'} LL'\right] = 0.$$

Which can be solved in two ODEs:

$$\frac{\partial \bar{A}'}{\partial \tau} m = \bar{A}(\tau)'\bar{l}_1 m + \bar{A}_c m \qquad \text{(ODE.1.1)}$$

$$\frac{\partial \bar{C}}{\partial \tau} = \bar{A}(\tau)'\bar{l}_0 + \frac{1}{2}\bar{A}(\tau)LL'\bar{A}(\tau) + \bar{C}_c \qquad \text{(ODE.1.2)}$$

To this end, I apply the following adjustments:

1. Terms involving derivatives of the type $\frac{\partial}{\partial J}, J = \{A_1, A_2\}$ adjust the constant vector in ODE.1.1.
2. Terms involving derivatives of the type $\frac{\partial^2}{\partial m_i \partial I}, i = \{A_1, A_2\}, I = \{A_1, A_2\}$ adjust the vector l_0 in the drift of m.
3. Terms involving derivatives of the type $\frac{\partial^2}{\partial J_i \partial J_j}, J = \{A_1, A_2\}$ adjust the constant in ODE.1.2.

Adjustment 1: $\frac{\partial F}{\partial J}, J = \{A_1, A_2\}$

$$\begin{array}{lll} \frac{\partial F}{\partial V} = \frac{\varepsilon_V}{V} F & \text{and} & \mu_V = V m_{A_2} \\ \frac{\partial F}{\partial G} = \frac{\varepsilon_G}{G} F & \text{and} & \mu_G = G m_{A_2} \\ \frac{\partial F}{\partial G} = \frac{\varepsilon_G}{G} F & \text{and} & \mu_M = M m_M \end{array}$$

Adjustment 1 adds the following vector to the constant $\bar{A}_c$ in ODE.1.1:

$$\bar{A}_c^{adj1} = \begin{pmatrix} \varepsilon_V \\ \varepsilon_G \\ \varepsilon_M \end{pmatrix}.$$

$$\bar{A}_c = \bar{A}_c^{adj1}.$$

Adjustment 2: $\frac{\partial^2 F}{\partial m_i \partial J_j}$

$$\begin{array}{lll} \frac{\partial^2 F}{\partial m_{A_2} \partial V} = \frac{\partial F}{\partial m_{A_2}} \frac{\varepsilon_V}{V} = \bar{A}_{A_1} \frac{\varepsilon_V}{V} F & \text{and} & \langle dm_{A_2}, dV \rangle = V \sigma_{9A_1} \sigma_{4i} \\ \frac{\partial^2 F}{\partial m_{A_2} \partial G} = \frac{\partial F}{\partial m_{A_2}} \frac{\varepsilon_G}{G} = \bar{A}_{A_1} \frac{\varepsilon_G}{G} F & \text{and} & \langle dm_{A_2}, dG \rangle = V \sigma_{10G} \sigma_{4i} \\ \frac{\partial^2 F}{\partial m_{A_2} \partial M} = \frac{\partial F}{\partial m_{A_2}} \frac{\varepsilon_M}{M} = \bar{A}_{A_1} \frac{\varepsilon_M}{M} F & \text{and} & \langle dm_{A_2}, dM \rangle = V \sigma_{11M} \sigma_{4i} \\ \dots & & \dots \end{array}$$

Adjustment 2 adds the following vector to l_0

$$\begin{aligned} l_0^{adj1} &= \begin{pmatrix} \varepsilon_V \sigma_{9A_1} \sigma_{4i} + \varepsilon_G \sigma_{10G} \sigma_{5G} + \varepsilon_M \sigma_{11M} \sigma_{5M} \\ \varepsilon_V \sigma_{9A_1} \sigma_{6V} + \varepsilon_G \sigma_{10G} \sigma_{6G} + \varepsilon_M \sigma_{11M} \sigma_{6M} \\ \varepsilon_V \sigma_{9A_1} \sigma_{7V} + \varepsilon_G \sigma_{10G} \sigma_{7G} + \varepsilon_M \sigma_{11M} \sigma_{7M} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_i^A \varepsilon_V + \gamma_{ij}^A \varepsilon_G \\ \gamma_{ij}^A \varepsilon_V + \gamma_G^1 \varepsilon_G \end{pmatrix} \\ \bar{l}_0 &= l_0 + l_0^{adj1}. \end{aligned}$$

Adjustment 3 $\frac{\partial^2 F}{\partial J_i \partial J_j}, J = \{A_1, A_2\}$, i.e.$\frac{1}{2}\sum_{ij} \frac{\partial^2 F}{\partial J_i \partial J_j} C_{ij} = \frac{1}{2} Tr\left(\frac{\partial^2 F}{\partial J \partial J'} \Lambda\Lambda'\right)$

$$\begin{array}{lll} \frac{\partial^2 F}{\partial V^2} = \frac{\varepsilon_V(\varepsilon_V - 1)}{V^2} & \text{and} & \langle dV \rangle = V^2 \sigma_{9A_1}^2 \\ \frac{\partial^2 F}{\partial V \partial G} = \frac{\varepsilon_V \varepsilon_G}{VG} & \text{and} & \langle dV, dG \rangle = 0 \\ \frac{\partial^2 F}{\partial V \partial M} = \frac{\varepsilon_V \varepsilon_M}{VM} & \text{and} & \langle dV, dM \rangle = 0 \\ \dots & & \dots \end{array}$$

$$
\begin{aligned}
\bar{C}_c &= \frac{1}{2} Tr \left[\begin{pmatrix} \varepsilon_{A_1}(\varepsilon_{A_1}-1) & \varepsilon_{A_1}\varepsilon_G \\ \varepsilon_{A_1}\varepsilon_{A_2} & \varepsilon_{A_2}(\varepsilon_{A_2}-1) \end{pmatrix} \begin{pmatrix} \sigma_{9A_1}^2 & 0 \\ 0 & \sigma_{10G}^2 \end{pmatrix} \right] \\
&= \frac{1}{2} \left[\varepsilon_{A_1}(\varepsilon_{A_1}-1)\sigma_{9A_1}^2 + \varepsilon_{A_2}(\varepsilon_{A_2}-1)\sigma_{10G}^2 \right]
\end{aligned}
$$

The system of ODEs (1.1–1.2) can now be solved explicitly:

$$
\begin{aligned}
\frac{\partial \bar{A}'}{\partial \tau} &= \bar{A}(\tau)'\bar{l}_1 + \bar{A}_c && \text{(ODE.1.1)} \\
\frac{\partial \bar{C}}{\partial \tau} &= \bar{A}(\tau)'\bar{l}_0 + \frac{1}{2}\bar{A}(\tau)LL'\bar{A}(\tau) + \bar{C}_c && \text{(ODE.1.2)}
\end{aligned}
$$

It is easy to see that since l_1 is a diagonal matrix, the equations in ODE.1.1 are uncoupled, i.e. can be solved explicitly element-by-element.

$$
\begin{aligned}
\frac{\partial \bar{A}_i}{\partial \tau} &= \bar{A}_i(\tau) l_{1,ii} + \bar{A}_{c,i}, \qquad \bar{A}_i(0) = 0 \\
\bar{A}_i(\tau) &= -\frac{\bar{A}_{c,i}}{l_{1,ii}} \left(1 - e^{l_{1,ii}\tau}\right).
\end{aligned}
$$

Given this solution ODE.1.2. is obtained by direct integration.

B Data Appendix

The data runs from January 1996 to September 2008 (153 observations). I use a monthly frequency throughout the paper.

B.1 Options Data

Options data is from OptionMetrics, LLC. I apply a series of filters

1. I eliminate prices that violate arbitrage bounds, i.e. call prices are required not to fall outside the interval $\left(Se^{-rd} - Ke^{-\tau r}, Se^{-\tau d}\right)$, where S is the spot price, K is the strike price, d is the dividend yield, r is the risk-free rate, and τ is the time to maturity.
2. I eliminate all observations for which (i) the ask is lower than the bid price, (ii) the bid is equal to zero, or (iii) the spread is lower than the minimum tick size (equal to USD 0.05 for options trading below USD 3 and USD 0.10 in any other cases).
3. To mitigate the impact of stale quotes I eliminate from the sample all observations for which both the bid and the ask are equal to the one on the previous day.
4. I focus on short-term options which are known to be the most liquid with a time to maturity between 14 and 31 days.

Moneyness of a call (put) option is defined as the ratio of strike price to stock price (stock price to strike price).[17]

B.2 Stock Data

Monthly stock data is from CRSP.

B.3 Corporate Bond Data

Bond data is from the Fixed Income Securities Database (FISD) on corporate bond characteristics and the National Association of Insurance Commissioners (NAIC) database on bond transactions. Corporate bond spreads are calculated using the corresponding zero-coupon yield available from CRSP.

B.4 Firm-Specific Information

Leverage is defined as book value of debt (Compustat item 9 and 34) divided by total asset value (Compustat item 6). The stock's **market beta** is calculated from a regression of the stocks excess return on the market excess return using historical returns over a 180 day window[18]. **Firm size** is defined as the natural log of the firm's market value of equity, in thousands of dollars. Data is from Compustat.

B.5 Macroeconomic Variables:

Monthly data on real nondurable and services **consumption** is from the Bureau of Economic Analysis. Expenditures are in 2000 dollars. Monthly consumption growth is defined as the log-differenced real consumption per capita. Following Granger and Newbold (1986), I construct a smoothed consumption series by using an exponential moving average. Consumption volatility is constructed by estimating a GARCH(1,1) on the consumption growth series. I retrieve **S&P 500 price-earnings** data from the S&P webpage, and the other macro variables we get from FRED.

[17] There are several alternative ways to measure moneyness. For instance, Bollen and Whaley (2004) use the Black and Scholes delta to measure moneyness, Ni (2007) and Carr and Wu (2009) use the total volatility adjusted strike to stock price ratio.

[18] The choice of a relatively short window size is in line with Lewellen and Nagel (2006).

B.6 Disagreement Measures

Firm Specific Disagreement:

To construct a proxy of disagreement about future cash flows, I collect analysts' earning forecasts from the Institutional Brokers Estimate System (I/B/E/S) database. In particular, I use the unadjusted Forecast Detail database which provides forecast data unadjusted for stock splits. I then calculate for each firm, for each month, the mean absolute difference of all forecasts made.

Common Disagreement:

When the number of belief disagreement processes, N is very large, an estimation of the common factor via the Likelihood function is computational infeasible (see Ludvigson and Ng, 2007). This high dimensionality has motivated the work on alternative methods to estimate dynamic factor models. Chamberlain and Rothschild (1983) and Connor and Korajczyk (1986) show that if n goes to infinity, the factors are estimated consistently using principal components, see Brillinger's (2001) dynamic principal components. The theory of applying this theory, based on frequency domain methods is developed by Forni, Hallin, Lippi, and Reichlin (2000). In the following, we briefly summarize the estimation procedure, for technical details and a more rigorous presentation the reader is referred to Forni, Hallin, Lippi, and Reichlin (2000).

The estimation procedure is derived in two steps. The first step consists of estimating the spectral density matrix and the covariances of the common components. We start by estimating the spectral density matrix $\Sigma(\omega)$ of $D(t) = (D_{1t}, D_{2t}, \cdots, D_{Nt})'$. The estimation of $\Sigma(\omega)$ is obtained by using a Bartlett lag-window size of $M = 2$. We use a heuristic rule, which sets the window size equal to $round(\sqrt{T}/4)$, see Forni, Hallin, Lippi, and Reichlin (2000). The estimation is done via the discrete Fourier transform:

$$\hat{\Sigma}_D(\omega) = \frac{1}{2\pi} \sum_{k=-M}^{M} w_k \hat{\Gamma}_k \exp(-i\omega k),$$

where $\hat{\Gamma}_k$ denotes the sample autocovariance matrices, and $w_k = 1 - |k|/(M+1)$ the weights.

We then perform the dynamic principal component decomposition (see Brillinger, 2001). For each frequency of the grid, we compute the eigenvalues and eigenvectors of $\hat{\Sigma}(\omega)$. By ordering the eigenvalues in descending order for each frequency and collecting values correspondingly to different frequencies, the eigenvalues, $\hat{\lambda}_j$ and eigenvectors, $U_j(\omega)$, are obtained. The eigenvalue $\hat{\lambda}_j$ can be interpreted as the spectral density of the j-the principal component. To determine the optimal number of common factors, we study the contribution of the j-th principal component to the total variance:

$$c_j = \int_{-\pi}^{\pi} \hat{\lambda}_j(\omega) d\omega / \sum_{j=1}^{N} \int_{-\pi}^{\pi} \hat{\lambda}_j(\omega) d\omega.$$

Forni, Hallin, Lippi, and Reichlin (2000) show that there exist a linkage between the number of common factors and the eigenvalues of the spectral density matrix. In practice, however, there does not exist a formal testing procedure to distinguish between very slowly diverging eigenvalue and a bounded one, therefore, we follow the heuristic procedure applied by Cristadoro, Forni, Reichlin, and Veronese (2005), by imposing the criteria that the dynamic common factors should account for a certain percentage of the total variability in the data across all frequencies, and the number of dynamic common factors is set equal to the number of largest dynamic eigenvalues that together capture this variance ratio. In our case, the number of common components is set to $q = 2$. Let $\Lambda_q(\omega)$ be a diagonal matrix, having as elements the eigenvalues, $\lambda_1(\omega), \cdots, \lambda_q(\omega)$ and let $U(\omega)$ be the $[n \times q]$ matrix of the eigenvectors, $U_1(\omega), \cdots, U_q(\omega)$. Then the estimate of the spectral density matrix of the common components, $\varsigma(t) = (\varsigma_{1t} \cdots \varsigma_{Nt})'$:

$$\hat{\Sigma}_\varsigma(\omega) = U(\omega)\Lambda(\omega)\tilde{U}(\omega),$$

where the tilde denotes a conjugate. The spectral density of the idiosyncratic component is obtained as the difference of the spectral density matrices of the common component and the D.

The second step consists of estimating the factor space. Given the estimated covariance matrices in the first step, we can now estimate the factors as a linear combination of the observable variables, $D_{jt}, j = 1, \cdots, N$. To this end, we take the first r generalized principal components of the estimated spectral density matrix of the common components, $\hat{\Gamma}_\varsigma$ with respect to the diagonal matrix having on the diagonal the variances of the idiosyncratic components. Estimates of the common components are derived by projecting the common components on the space spanned by the first r generalized principal components. To this end, we compute the generalized eigenvalues, μ_j, solving $det(\hat{\Gamma}_D(0) - z\hat{\Gamma}_\varsigma = 0$ along with the generalized eigenvectors, V_j, satisfying $V_j\hat{\Gamma}_D(0) = \mu_j V_j \hat{\Gamma}_\varsigma(0)$. Then we take the eigenvectors corresponding to the largest r eigenvalues. The estimtaed factors are then $V_j t = V_j' \Psi_j t$. To determine the number of static factors, r, we rely on Forni, Hallin, Lippi, and Reichlin (2005), who propose to use $r = q(k+1)$.[19] Using the generalized principal components and the covariances estimated previously, we can now estimate the common component $\bar{\Psi}_t$. The estimate of $\bar{\Psi}_t$ is given by:

$$\bar{\Psi}_{t+h} = \hat{V}_D(h)V\left(V'\hat{\Gamma}(0)V\right)^{-1}V'\Psi_t.$$

[19]More formal information criterions include Bai and Ng (2001), Hallin and Liška (2007), and Alessi, Barigozzi, and Capasso (2008).

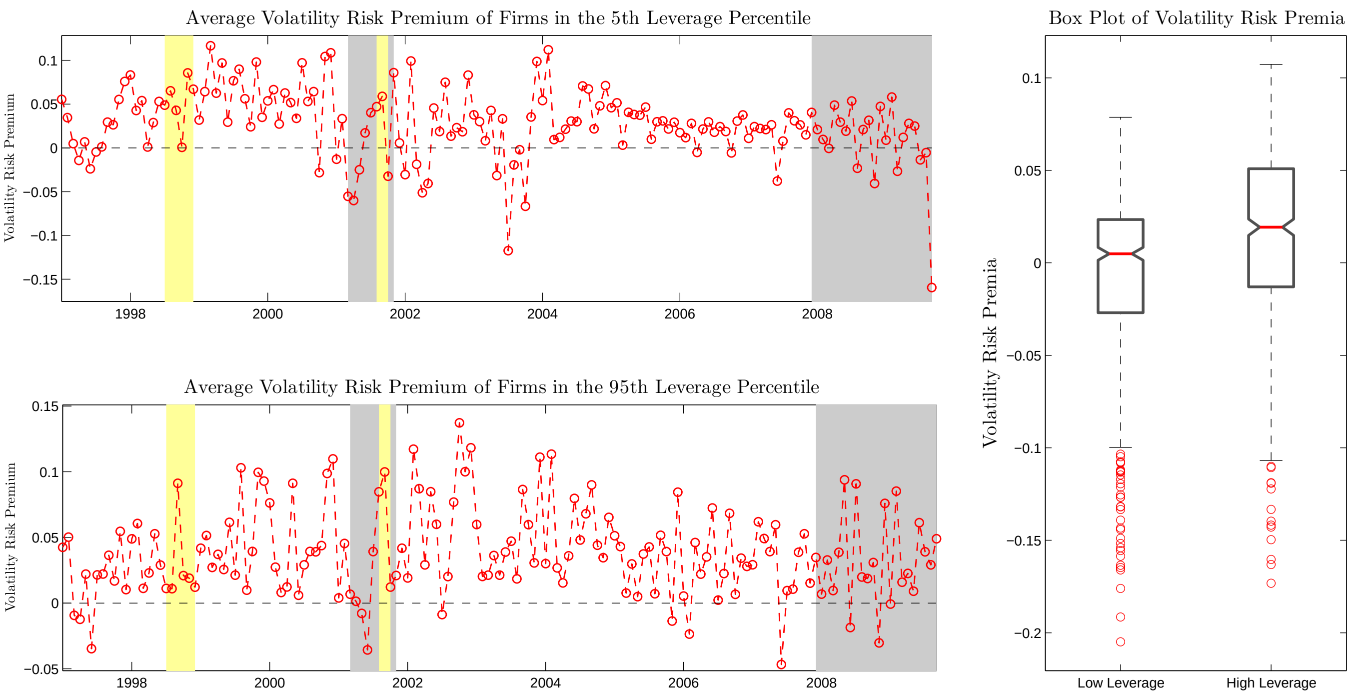

Figure 1. Volatility Risk Premium for Low and High Leverage Firms

The upper (lower) left panel plots the average volatility risk premium defined as $E_t^{\mathbb{Q}}\left(\int_t^{t+1} dr_i\right) - E_t^{\mathbb{P}}\left(\int_t^{t+1} dr_i\right)$ of all firms in the S&P500 in the 5th (95th) leverage percentile. The integrated variance under the risk-neutral measure $\mathbb{Q}$ is proxied using the method in Carr and Madan (1998) and Britten-Jones and Neuberger (2000). It is defined as:

$$E_t^{\mathbb{Q}}\left(\int_t^{t+1} dr_i^2\right) = 2\int_0^{\infty} \frac{C\left(t, t+1, K\right) - \max\left(S(t) - K, 0\right)}{K^2} dK,$$

and the model-free implied volatility is defined as the square root of this expression. The integrated realized volatility under the physical measure, $\mathbb{P}$, is the square root of the sum of squared daily log continuously compounded returns over the month. Both volatility measures are of monthly basis and are available at the end of each observation month. Leverage is defined as the ratio of the book value of debt and the total asset value. The average is market capitalization weighted. The shaded areas represent financial or economic crises defined according to the NBER. The right panel depicts the boxplots of the volatility risk premia summary of low (left) and high (right) leverage firms. The upper, middle, and lower lines indicate the upper quartile, median, and lower quartile, respectively.

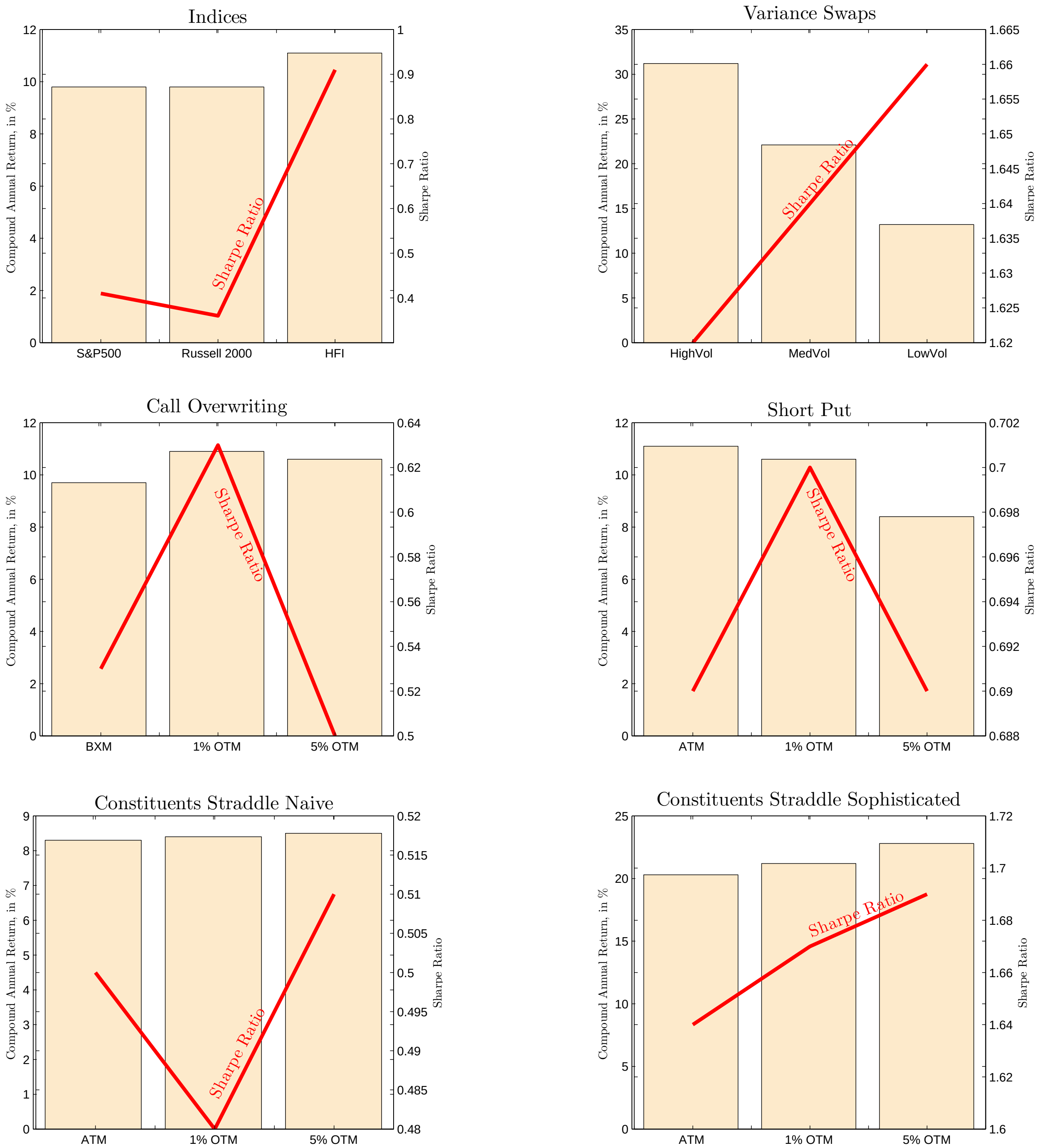

Figure 2. Trading Strategies

This figure presents annual returns and Sharpe ratios for six different option trading strategies. HFI refers to the Credit Suisse/Tremont Hedge Fund Index. The variance swap strategies are so called Risk Equivalent Portfolios (REPs). They usually contain one risk-free asset (LIBOR) and one short variance swap, where the amount of variance sold is such that it meets a certain risk/return objective. LowVol stands for 5% volatility per annum, MedVol for 10% volatility per annum, and HighVol for 15% volatility per annum. BXM stands for the CBOE S&P 500 BuyWrite Index. The data is monthly and runs from January 1996 to September 2007.

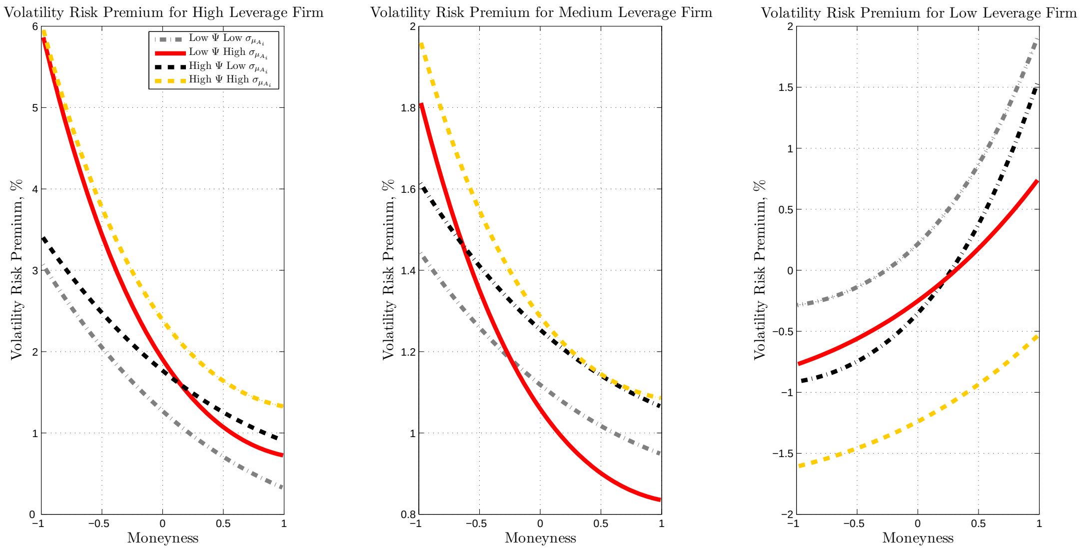

Figure 3. Volatility Risk Premia for Different Leverage Ratios

This Figure plots the volatility risk premia, defined as the difference between the expected integrated volatility under the risk-neutral and physical distribution for three different leverage ratios and two different levels of macro-economic uncertainty ($\sigma_{\mu_{A_i}}$). Low (high) Ψ indicates that the disagreement is set to 0.5 (0.1), i.e. $\Psi_{A_1} = \Psi_{A_2} = \Psi_z = 0.5$ ($\Psi_{A_1} = \Psi_{A_2} = \Psi_z = 0.1$). Low and high growth rate volatility are set to 0.0029 and 0.0041, respectively. Low leverage is fixed at 0.05, medium at 0.1, and high leverage is fixed at 0.14.

Table 1
Summary Statistics

This table reports the 5th percentile, median, and 95th percentiles of the model-free implied volatility, realized volatility, and skewness across securities per year. The risk-neutral implied volatility is calculated as in Carr and Madan (1998) and Britten-Jones and Neuberger (2000). The realized volatility is calculated from stock returns and finally, the risk-neutral skewness is calculated as in Bakshi, Kapadia, and Madan (2003) using data on out-of-the-money calls and puts. The Firm Specific Disagreement is calculated from analysts' earning forecasts, the Common Disagreement is a dynamic factor estimated from the cross-section of firm-specific disagreement, Consumption Growth Volatility is estimated from a two state Markov switching process on consumption growth data. Leverage is total debt over total asset value, the Macro Factor is the first principal component using industrial production, housing start number, the producer price index, non-farm employment, and the S&P 500 P/E ratio, the Market Beta is the slope coefficient from a regression of the firm excess return on the market excess return, finally, Firm Size is the market value of equity. AC(1) indicates the first autocorrelation coefficient. The time period is from January 1996 to September 2008.

Year	Implied Volatility			Realized Volatility			Skewness		
	0.05	0.5	0.95	0.05	0.5	0.95	0.05	0.5	0.95
1996	10.938	25.098	45.923	9.292	23.988	44.110	-4.198	-0.583	1.193
1997	11.022	29.912	46.387	10.283	29.011	45.182	-3.138	-0.420	0.998
1998	12.928	28.726	46.373	13.288	28.928	46.927	-4.293	-0.625	0.872
1999	13.938	27.826	47.993	13.283	27.837	47.837	-4.382	-0.477	0.937
2000	15.119	25.883	48.837	16.012	26.282	48.928	-5.192	-0.638	0.682
2001	13.873	24.193	45.773	12.018	23.444	43.737	-5.018	-0.544	0.766
2002	12.728	23.877	44.298	11.327	21.336	42.311	-4.876	-0.651	0.972
2003	10.834	23.293	43.399	9.828	20.218	41.299	-4.119	-0.837	0.880
2004	11.639	23.847	42.982	10.002	20.292	40.192	-4.288	-0.921	1.018
2005	12.938	24.636	41.828	21.993	23.339	42.774	-3.930	-0.762	0.872
2006	13.938	24.474	43.938	13.382	23.933	43.998	-4.498	-0.482	1.013
2007	14.866	25.837	45.943	14.948	26.887	46.994	-4.872	-0.293	0.917
2008	18.928	28.912	47.198	19.287	27.182	46.248	-3.938	-0.318	1.198

	Mean	StDev	AC(1)	Corr VolRP	Corr Skewness
Volatility Risk Premium $\times 10^2$	1.4644	1.3753	0.49	1.00	0.62
Risk-Neutral Skewness	-0.2808	0.1880	0.54	0.62	1.00
Idiosyncratic Disagreement	0.2981	0.4192	0.45	0.46	0.37
Common Disagreement	0.0928	0.0384	0.65	0.44	0.21
Consumption Growth Volatility	0.4211	0.1192	0.87	0.35	0.41
Leverage	0.1024	0.1907	0.32	0.38	0.42
Macro Factor	0.0148	0.1814	0.96	0.34	0.47
Market Beta	1.0098	0.6239	0.67	0.19	0.08
Firm Size	17.9283	2.9388	0.20	-0.29	0.10

Table 2

Choice of Parameter Values and Benchmark Values of State Variables

This table lists the parameter values used for all figures in the paper. We calibrate the model to the mean and volatility of the dividends on the S&P 500. The average growth rate for the period 1996-2006 is 5.93% and the volatility is 3.52%. The initial values for the conditional variances are set to their steady-state variances.

PARAMETERS FOR FUNDAMENTALS		
Long-term growth rate of cash flow growth	a_{0A_i}	0.01
Mean-reversion parameter of cash flow growth	a_{1A_i}	-0.01
Volatility of cash flow	σ_{A_i}	0.07
Volatility of cash flow growth	$\sigma_{\mu_{A_i}}$	0.0035
Initial level of cash flow	A_i	1.00
Initial level of cash flow growth	$m^A_{A_i}$	0.01
PARAMETERS FOR SIGNAL		
Long-term growth rate of signal	a_{0z}	0.01
Mean-reversion parameter of signal	a_{1z}	-0.03
Volatility of signal	σ_z	0.06
AGENT SPECIFIC PARAMETERS		
Relative risk aversion for both agents	γ	4.00
Time Preference Parameter	ρ	0.02

Table 3
Calibrated Moments

This table presents moments based on 250 simulations each with 153 months of observation. The parameters for used are given in Table 2. No $\sigma_{\mu_{A_i}}$ indicates that the time-varying cash flow growth volatility is shut down by setting it to a constant value. No DiB means that all the disagreement processes are set equal to zero, i.e. $\Psi_{A_i} = \Psi_z = 0$. 0.05, 0.5, and 0.95 represent the 5th, median, and 95th quantile, respectively. $\sigma^{\mathbb{P}}$ ($\sigma^{\mathbb{Q}}$) indicates the expected volatility under the physical (risk-neutral) measure.

Statistic	Full Model			No $\sigma_{\mu_{A_i}}$			No DiB		
	0.05	0.5	0.95	0.05	0.5	0.95	0.05	0.5	0.95
$\sigma^{\mathbb{P}}$	2.14	10.27	17.93	2.64	8.72	10.30	2.38	8.12	9.62
$\sigma^{\mathbb{Q}}$	3.28	13.31	19.28	2.92	8.98	11.84	2.39	8.13	9.64
AC(1) — $\sigma^{\mathbb{P}}$	0.78	0.89	0.98	0.69	0.73	0.84	0.67	0.70	0.84
AC(1) — $\sigma^{\mathbb{Q}}$	0.69	0.78	0.83	0.65	0.69	0.86	0.69	0.70	0.84
Mean	1.12	1.20	1.39	0.27	0.42	1.02	0.03	0.10	0.12
StDev	0.47	0.84	1.23	0.25	0.56	0.74	0.10	0.18	0.22
Skewness	0.48	1.29	3.92	0.22	0.87	2.01	0.01	0.45	1.01
Kurtosis	7.92	9.29	10.29	4.28	6.39	8.82	3.91	4.32	5.01

Table 4
Simulated Volatility Risk Premia Regressions

This table presents estimated coefficient of running the following panel regression:

$$VolRP_i(t) = \beta_0 + \beta_1 DIB_{i,t} + \beta_2 \overline{DIB}_t + \beta_3 \sigma_{\mu_c} + \epsilon_{i,t},$$

where $VolRP_i(t)$ is the volatility risk premium of firm i, $DIB_{i,t}$ is the proxy of belief disagreement of each individual firm i at time t, $\overline{DIB}_t$ the common disagreement which I proxy as the equally weighted average from the individual $DIB_{i,t}$, and σ_{μ_c}, the consumption growth volatility, which is estimated from a two-state Markov switching process on the equilibrium consumption. The volatility risk premium is calculated from simulated stock returns and option prices over 1,000 simulations of 153 months length. The second and third column present the estimated coefficient for a low (high) leverage firm, respectively. A low (high) leverage firm is defined to have a fixed asset to debt ratio of 0.05 (0.2). The R^2 is adjusted for degrees of freedom.

	Volatility Risk Premium	Low Leverage	High Leverage
Constant	0.001	0.002	0.002
	(3.98)	(4.10)	(5.92)
Firm-specific DiB	0.029	-0.038	0.037
	(15.98)	(-1.83)	(17.93)
Common DiB	0.092	-0.084	0.044
	(5.93)	(-4.92)	(6.09)
Consumption Growth Volatility	0.363	0.482	0.519
	(7.93)	(6.32)	(8.88)
Adjusted R^2	0.34	0.23	0.41

Table 5

Simulated Stock Excess Returns and Credit Spreads: Predictability

This table presents estimated coefficients from predictability regressions, using simulated data. The statistics are based on 250 simulations each with the corresponding sample size as its data counterpart (153 months). The parameters for calibrating the model are given in Table 2. Panel A reports estimated coefficients of the following regression:

$$ExRet_i(t+h) = \alpha + \beta_1 VolRP_i(t) + \epsilon(t+h),$$

where $ExRet_i$ is the return of stock i in excess of the risk-free rate and $VolRP_i$ is the volatility risk premium of firm i, defined as the difference between the model-free implied volatility and realized volatility. Panel B reports estimated coefficients of the following regression:

$$CS_i(t+h) = \alpha + \beta_1 VolRP_i(t) + \epsilon(t+h),$$

where CS_i is the credit spread of firm i defined as the difference between the corporate bond credit spread and the zero-coupon yield. Robust t-Statistics following Newey and West (1987) are reported in brackets below the estimate calculated with 24 lags. The R^2 is adjusted for degrees of freedom.

	Stock Excess Returns			Corporate Credit Spreads		
Horizon	1	6	12	1	6	12
Constant	0.021	0.023	0.019	109.983	110.323	112.938
	(3.94)	(2.38)	(2.09)	(4.48)	(5.69)	(5.48)
VolRP	0.291***	0.191***	0.102***	56.938**	67.987***	72.394***
	(5.93)	(6.83)	(7.10)	(5.98)	(6.00)	(7.37)
Adjusted R^2	0.04	0.07	0.08	0.14	0.17	0.22

Table 6
Panel Regression Analysis: Volatility Risk Premia and Skewness

This table presents estimated coefficients from the following panel regression:

$$y_i(t) = \beta_0 + \beta_1 DIB_{i,t} + \beta_2 \overline{DIB}_t + \sum_{j=3}^{6} \beta_j \text{Control}(j)_{i,t} + \sum_{k=1}^{3} \gamma_k \text{Control}_t + \epsilon_{i,t},$$

where $y_i(t)$ is the volatility risk premium and skewness of firm i, $DIB_{i,t}$ is the proxy of belief disagreement of each individual firm i at time t, $\overline{DIB}_t$ the common disagreement estimated from the cross-section of individual disagreement proxies, $\text{Control}_{i,t}$ are leverage, firm size, and market beta of each firm i at time t, and Control_t is the macro factor. The data has monthly frequency and covers the period January 1996 to September 2008. The R^2 is adjusted for degrees of freedom.

	Volatility Risk Premium			Risk-Neutral Skewness		
Constant	0.001	0.001	0.002	-0.210	-0.339	-0.417
	(5.29)	(4.98)	(5.38)	(-2.13)	(-2.02)	(-1.99)
Firm-Specific DiB	0.041	0.042	0.045	0.014	0.013	0.014
	(3.29)	(2.98)	(2.02)	(4.95)	(5.93)	(2.93)
Common DiB	0.108	0.090	0.180	0.098	0.083	0.096
	(2.22)	(1.98)	(2.73)	(2.94)	(2.84)	(3.01)
Consumption Volatility	0.293	0.321	0.421	0.539	0.874	0.476
	(3.02)	(3.98)	(2.49)	(2.39)	(1.98)	(2.17)
Leverage		0.087	0.091		0.411	0.431
		(2.03)	(2.22)		(4.02)	(3.77)
Firm Size		-0.001	0.001		0.001	0.003
		(1.02)	(1.08)		(0.83)	(0.54)
Market Beta		-0.039	-0.031		-0.129	-0.102
		(-1.45)	(-1.21)		(-1.10)	(-0.97)
Macro Factor			-0.219			-0.389
			(-3.10)			(-2.01)
Adjusted R^2	0.12	0.13	0.20	0.10	0.13	0.24

	Volatility Risk Premium			Risk-Neutral Skewness		
	Low	Medium	High	Low	Medium	High
Constant	0.001	0.001	-0.001	0.002	-0.398	-0.182
	(3.18)	(2.17)	(-3.82)	(2.19)	(-2.48)	(-3.01)
Firm-Specific DiB	-0.055	0.009	0.039	-0.008	0.001	0.029
	(-2.01)	(1.67)	(3.08)	(-1.93)	(2.99)	(2.48)
Common DiB	0.022	0.019	0.029	0.003	0.006	0.007
	(2.02)	(1.64)	(1.99)	(3.94)	(2.90)	(2.74)
Consumption Volatility	-1.294	1.329	1.702	-0.103	0.304	0.410
	(-3.92)	(4.02)	(2.49)	(-0.93)	(1.78)	(2.05)
Firm Size	-0.001	0.001	0.001	-0.001	0.001	0.001
	(-0.83)	(0.56)	(1.07)	(-0.98)	(0.56)	(1.02)
Market Beta	-0.021	-0.031	-0.042	-0.302	-0.102	-0.001
	(-0.78)	(-1.98)	(-0.96)	(-1.09)	(-1.45)	(-1.00)
Macro Factor	-0.098	-0.293	-0.342	-0.412	-0.341	-0.356
	(-1.49)	(-2.02)	(-3.77)	(-2.01)	(-2.93)	(-2.09)
Adjusted R^2	0.14	0.14	0.18	0.18	0.19	0.20

Table 7
Stock Excess Returns and Credit Spreads: Predictability

Panel A reports estimated coefficients of the following regression:

$$ExRet_i(t+h) = \alpha + \beta_1 VolRP_i(t) + \epsilon(t+h),$$

where $ExRet_i$ is the return of stock i in excess of the 1 month Libor and $VolRP_i$ is the volatility risk premium of firm i, defined as the difference between the model-free implied volatility and realized volatility. Panel B reports estimated coefficients of the following regression:

$$CS_i(t+h) = \alpha + \beta_1 VolRP_i(t) + \epsilon(t+h),$$

where CS_i is the credit spread of firm i defined as the difference between the corporate bond credit spread and the zero-coupon yield. Robust t-Statistics following Newey and West (1987) are reported in parentheses below the estimate calculated with 24 lags. The data has monthly frequency and covers the period January 1996 to December 2007. The R^2 is adjusted for degrees of freedom.

Panel A: Stock Excess Returns

Horizon	1	2	3	4	5	6	7	8	9	10	11	12
Constant	0.024	0.014	0.019	0.023	0.027	0.020	0.022	0.024	0.028	0.019	0.025	0.031
	(1.23)	(1.01)	(1.03)	(1.28)	(0.98)	(0.83)	(0.91)	(1.02)	(1.29)	(0.84)	(0.97)	(1.02)
VolRP	0.018	0.024	0.031	0.027	0.022	0.019	0.025	0.011	0.023	0.028	0.029	0.034
	(2.01)	(1.87)	(1.98)	(1.99)	(2.14)	(2.48)	(2.53)	(2.61)	(2.02)	(1.77)	(1.85)	(2.02)
Adjusted R^2	0.01	0.02	0.04	0.05	0.06	0.07	0.04	0.08	0.07	0.06	0.05	0.04

Panel B: Corporate Credit Spreads

Horizon	1	2	3	4	5	6	7	8	9	10	11	12
Constant	120.632	110.934	118.736	114.183	98.734	103.283	117.837	109.827	110.990	103.811	119.098	100.987
	(5.24)	(4.29)	(3.94)	(4.20)	(2.99)	(4.76)	(5.07)	(4.82)	(3.91)	(4.01)	(4.92)	(5.38)
VolRP	47.943	51.832	62.420	43.394	44.998	42.038	48.912	51.039	48.783	45.921	50.293	58.912
	(2.48)	(3.92)	(4.10)	(4.92)	(5.01)	(5.29)	(4.09)	(5.31)	(2.02)	(3.93)	(3.03)	(4.91)
Adjusted R^2	0.10	0.09	0.12	0.14	0.16	0.20	0.21	0.22	0.19	0.20	0.19	0.18

Table 8
Volatility Risk Premia: Cyclical versus Non-Cyclical Firms

This table presents estimated coefficient of running the following panel regression once for cyclical firms and non-cyclical firms:

$$VolRP_i(t) = \beta_0 + \beta_1 DIB_{i,t} + \beta_2 \overline{DIB}_t + \sum_{j=3}^{5} \beta_j \text{Control}(j)_{i,t} + \sum_{k=1}^{3} \gamma_k \text{Control}_t + \epsilon_{i,t},$$

where $VolRP_i(t)$ is the volatility risk premium of firm i, $DIB_{i,t}$ is the proxy of belief disagreement of each individual firm i at time t, $\overline{DIB}_t$ the common disagreement estimated from the cross-section of individual disagreement proxies, $\text{Control}_{i,t}$ are leverage, firm size, and market beta of each firm i at time t, and Control_t is the macro factor. Industry classifications are based on Boudoukh, Richardson, and Whitelaw (1994), who sort industries by their correlation between industry level output growth and aggregate output growth. I label the five industries with the highest output growth beta as cyclical and the five industries with the lowest output growth beta as non-cyclical. The data has monthly frequency and covers the period January 1996 to September 2008. The R^2 is adjusted for degrees of freedom.

	Cyclical	Non-Cyclical
Constant	0.001	0.001
	(1.39)	(1.48)
Firm-Specific DiB	0.045	0.020
	(3.83)	(2.01)
Common DiB	0.207	0.112
	(1.84)	(1.73)
Consumption Growth Volatility	2.867	1.028
	(4.28)	(1.64)
Firm Size	-0.001	0.001
	(1.37)	(1.42)
Market Beta	-0.023	-0.031
	(-1.01)	(-0.65)
Macro Factor	-0.421	-0.227
	(-4.11)	(-3.24)
Adjusted R^2	0.19	0.16

Table 9
Stock Excess Returns and Credit Spreads: Predictability of Cyclical versus Non-Cyclical Firms

Panel A reports estimated coefficients for cyclical firms of the following regression:

$$xr_i(t+h) = \alpha + \beta_1 VolRP_i(t) + \epsilon(t+h),$$

where xr_i is the excess return of stock i or the credit spread of firm i and h and $VolRP_i$ is the volatility risk premium of firm i, defined as the difference between the model-free implied volatility and realized volatility. Panel B reports estimated coefficients for the non-cyclical firms. Industry classifications are based on Boudoukh, Richardson, and Whitelaw (1994), who sort industries by their correlation between industry level output growth and aggregate output growth. I label the five industries with the highest output growth beta as cyclical and the five industries with the lowest output growth beta as non-cyclical. Robust t-Statistics following Newey and West (1987) are reported in brackets below the estimate calculated with 24 lags. $\star$ denotes significance at the 10% level, $\star\star$ denotes significance at the 5% level and $\star\star\star$ denotes significance at the 1% level. The data has monthly frequency and covers the period January 1996 to December 2007. The R^2 is adjusted for degrees of freedom.

Panel A: Cyclical Stocks

	Stock Excess Returns					Corporate Credit Spreads				
Horizon	1	3	6	9	12	1	3	6	9	12
Constant	0.009	0.013	0.014	0.011	0.015	105.012	102.372	106.182	112.290	117.176
	(1.03)	(1.27)	(1.37)	(1.67)	(1.37)	(2.81)	(2.38)	(3.98)	(4.22)	(3.87)
VolRP	0.039	0.045	0.042	0.040	0.041	45.193	44.918	50.474	51.228	50.991
	(2.38)	(3.94)	(4.29)	(5.37)	(5.92)	(4.93)	(4.09)	(5.82)	(5.29)	(5.92)
Adjusted R^2	0.01	0.03	0.04	0.06	0.10	0.21	0.24	0.25	0.20	0.21

Panel B: Non-Cyclical Stocks

	Stock Excess Returns					Corporate Credit Spreads				
Horizon	1	3	6	9	12	1	3	6	9	12
Constant	0.010	0.012	0.019	0.017	0.019	108.932	109.282	110.838	119.726	113.321
	(1.21)	(1.45)	(1.03)	(1.27)	(1.37)	(2.83)	(4.27)	(5.38)	(4.99)	(3.83)
VolRP	0.022	0.019	0.020	0.019	0.022	40.913	39.932	43.938	39.712	40.110
	(1.45)	(1.73)	(1.65)	(1.67)	(1.82)	(3.28)	(1.87)	(2.09)	(2.73)	(2.44)
Adjusted R^2	0.01	0.01	0.03	0.03	0.04	0.12	0.13	0.14	0.17	0.14