Unawareness in Dynamic Psychological Games*

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Abstract

Building on Battigalli and Dufwenberg (2009)'s framework of dynamic psychological games and the recent progress in the modeling of dynamic unawareness, we provide a general framework that allows for 'unawareness' in the strategic interaction of players motivated by belief-dependent psychological preferences like reciprocity and guilt. We show that unawareness has a pervasive impact on the strategic interaction of psychologically motivated players. Intuitively, unawareness influences players' beliefs concerning, for example, the intentions and expectations of others which in turn impacts their behavior. Moreover, we highlight the strategic role of communication concerning feasible paths of play in these environments.

Keywords: Unawareness; Extensive-form games; Communication; Belief-dependent preferences; Sequential equilibrium.

JEL-Classifications: C72, C73, D80

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1 Introduction

Recent lab and field evidence suggests that people not only care about the monetary consequences of their actions, but that their behavior is also driven by belief-dependent psychological preferences [see e.g. Fehr *et al.* (1993), Charness and Dufwenberg (2006), Falk *et al.* (2008), Bellemare *et al.* (2010)]. Two prominent examples of belief-dependent preferences in the hitherto existing literature are reciprocity [see e.g. Rabin (1993), Dufwenberg and Kirchsteiger (2004), Falk and Fischbacher (2006)] and guilt aversion [see e.g. Charness and Dufwenberg (2006), Battigalli and Dufwenberg (2007b)]. Departing from the strictly consequentialist tradition in economics Geanakoplos *et al.* (1989) and Battigalli and Dufwenberg (2009) present general frameworks for analyzing the strategic interaction of people with belief-dependent psychological preferences: 'psychological games'. Roughly speaking, psychological games are games in which players' preferences depend upon players' beliefs about the strategies that are being played, players' beliefs about the beliefs of others about the strategies that are being played, and so on *ad infinitum*.

A widely unspoken assumption that is underlying all psychological as well as standard (i.e. non-psychological) game-theoretic analyses is that players are aware of the complete structure of the strategic environment they are in. Bluntly speaking, it is assumed that players are aware of everything. In many real life situations this is not the case–people often have asymmetric awareness levels concerning their own as well as others' feasible choices although they are part of the same strategic environment. Players are frequently 'surprised' in the sense that they become aware of new strategic alternatives by e.g. observing actions they had previously been unaware of or through communication. It has been shown that any non-trivial notion of unawareness is precluded in classical game-theoretic models [see e.g. Dekel *et al.* (1998), Modica and Rustichini (1999)]. Standard game-theoretic models preclude surprise by implicitly assuming that players know all states of the world (the axiom of awareness) and know all states they do not know (the axiom of wisdom) [see Samuelson (2004)]. In other words, players can assign probabilities to all states of the world and, hence, cannot be truly surprised.

However, it is not only in standard games that unawareness is important. As we will show here, asymmetric awareness also has a profound impact on the strategic interaction of players with belief-dependent psychological preferences. To see this consider the following intuitive example: Imagine two friends, *Ann* and *Bob*. Assume it is *Bob*'s birthday, he is planning a party and would be very happy, if *Ann* could come. Unfortunately, *Ann* has an important exam the next day and therefore cannot make it. Obviously *Ann* is certain that

Bob would feel let down, if she were to cancel his party without having a very good excuse. Quite intuitively, in this situation *Ann* does not experience any guilt towards *Bob* for not coming to his party. She knows that the important exam is a good excuse and that *Bob* is not let down as he does not expect her to come. In contrast, consider now the following variant of the same example: *Ann* is aware of the fact that the exam is postponed, meaning that it is feasible for her to attend *Bob*'s party. However, she has studied so hard for days and nights that she feels too tired to go to *Bob*'s party. Quite intuitively, in this situation *Ann* does not feel guilty towards *Bob* as long as she believes that *Bob* is unaware of the fact that the exam is postponed. As long as she believes that *Bob* is unaware of the fact that she actually has the possibility/time to come, she might not feel guilty towards him as she believes that he does not expect her to come and, hence, is not let down. In fact, if she were sure that *Bob* would never become aware of the fact that her exam is postponed, she probably had a strong emotional incentive in this situation to stick to the original story and leave him unaware in order not to raise his expectations. In other words, she had a strong incentive not to make him aware of the fact that she actually has the time to come to his party, but is too tired. Interestingly, if *Ann* were only interested in her own payoff in this strategic situation with unawareness, she would not care whether *Bob* is or will become aware of the postponement. She would simply not attend his party irrespective of his awareness. Only her belief-dependent feeling of guilt towards *Bob* creates the strong emotional incentive not to make him aware.

Bob's unawareness concerning *Ann*'s possibility to come to his party and, connectedly, *Ann*'s incentive not to tell him about the postponement of her exam intuitively highlight the focus of our analysis here. We analyze the influence and importance of asymmetric awareness and communication concerning feasible paths of play for the strategic interaction of players with belief-dependent preferences. This means, building on Battigalli and Dufwenberg (2009)'s framework of dynamic psychological games and the recent progress in the modeling of dynamic unawareness, we first provide a general framework that allows for unawareness and communication in the strategic interaction of players motivated by belief-dependent psychological preferences like reciprocity and guilt. Second, we provide a solution concept which can be used in our class of games with unawareness, communication and belief-dependent preferences and, third, we discuss an application to exemplify the influence of unawareness and communication using a specific type of belief-dependent preference: reciprocity.

More specifically, to allow for unawareness we extend the existing multi-stage framework along two dimensions. First, we divide extensive forms into subtrees consisting of paths of play, each describing a level of awareness. A player confined to a given subtree is aware of other subtrees which can be embedded in the one he is confined to, and unaware of all other subtrees. Second, as our setting is dynamic, players may become aware of more by learning from actions taken by others. However, as our analysis concentrates on the influence of asymmetric awareness on the strategic interactions of players with belief-dependent preferences, we abstract from the question how players become aware of new paths of play. We simply assume that whenever they observe an action that they had previously been unaware of they become aware of some 'larger' subtree which is consistent with the observed actions. Many different ways of modeling unawareness have been suggested in recent years both from a logic, an epistemic and a game theoretic perspective [see e.g. Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001), Heifetz *et al.* (2006), Halpern and Rêgo (2008), Heifetz *et al.* (2008), Li (2009) and Mengel *et al.* (2009)]. Closely related to our class of extensive forms with unawareness are the dynamic models by Halpern and Rêgo (2008), Feinberg (2009) and Heifetz *et al.* (2010). We show in Appendix A that in every stage our multi-stage framework adheres to the unawareness properties of Dekel *et al.* (1998).

In the spirit of our example above, we also allow for communication in our framework. We model such communication by assuming that players can choose to send 'awareness messages' containing feasible paths of play (i.e. subtrees) they are aware of, or they can choose not to communicate. Note that this is different from the communication allowed for in the experimental setting of Charness and Dufwenberg (2006). In their setting players are aware of everything and can send messages e.g. concerning intended play. In contrast, a message in our setting contains information concerning a set of feasible paths of play. Communicating feasible paths of play is obviously meaningless in strategic environments without unawareness. It is the asymmetric awareness of players which makes communication an important integral part of the strategic environment with unawareness. If a player observes a message containing information about paths of play that he was previously unaware of, he will update his level of awareness by taking this information into account.

Having defined our class of extensive forms with unawareness and communication, we formally characterize belief-dependent preferences in this structure. In synthesis, for each player confined to a certain awareness level, his pure strategy is defined on the extensive form he is confined to and his beliefs concerning the other players' strategies defined on each of the extensive forms induced by all subtrees he is aware of. A behavioral strategy profile is thus an independent probability distribution over these pure strategies each specifying a definite choice. Beliefs about others' pure strategies (first-order beliefs), beliefs about their beliefs about others' pure strategies (second-order beliefs), and so on, are shown to exist for all possible hierarchies. We use these hierarchies of beliefs for the general specification of belief-dependent psychological preferences. As mentioned above, specific types of belief-dependent preferences that can be embedded in our general setting with unawareness and communication are among others reciprocity and guilt aversion. In both of these examples belief-dependent psychological preferences depend on first- and second order beliefs. In contrast to Battigalli and Dufwenberg (2009), in our setting such psychological preferences will be limited by the awareness of each player who plays 'partial games'. A partial game is a description of the strategic situation a player is aware of, and the strategic situation he believes others might be aware of. As players may be aware of different partial games at different stages, we define a dynamic psychological game with unawareness and communication as the 'modelers' game which entails all relevant partial games.

Given the characterization of dynamic psychological games with unawareness and communication, we propose a sequential psychological equilibrium solution concept and prove its existence. We assume that a profile of conjectures (first-order beliefs) in a partial game is derived from a behavioral strategy profile in the same game. This implies, that in equilibrium any two players confined to the same partial game will independently hold the same conjectures about any third player. An assessment in our structure, a behavioral strategy profile and a profile of infinite hierarchies of beliefs, is consistent if the profile of first-order beliefs is derived from the behavioral strategy profile and each higher-order belief assigns probability one to lower-order beliefs. Intuitively, players aware of the same must in equilibrium hold common, correct beliefs about each others infinite belief hierarchies. A consistent assessment and sequential rationality (based on belief-dependent preferences) induce a sequential psychological equilibrium in the partial game. As players are unaware of any situation in which other players are aware of more than themselves, they believe that the game they are confined to is the most expressive. This implies that there exists an equilibrium strategy in which players confined to a partial game fix the equilibrium strategies of other players, whom they believe are confined to 'smaller' partial games, and then choose an equilibrium strategy based on this belief.

After defining our class of extended psychological games and characterizing our solution concept, we use an application to demonstrate the influence and importance of unawareness on the strategic interaction of agents with belief-dependent preferences. That is, we use the sequential prisoners dilemma also analyzed by Dufwenberg and Kirchsteiger (2004) to show the impact of unawareness and communication on the strategic interaction of reciprocal agents. As a benchmark we start from their results and subsequently discuss two scenarios in which players have asymmetric awareness levels. Importantly, the application shows how asymmetric awareness levels of players concerning feasible paths of play can give rise to equilibrium predictions that are distinct from predictions using Dufwenberg and Kirchsteiger (2004)'s setting without unawareness and a standard setting in which people are only concerned about the monetary consequences of their actions.

The organization of the paper is as follows: In section 2 we introduce a class of extensive forms with unawareness and communication. Following this, in section 3 we define hierarchies of conditional beliefs and belief-dependent preferences in our class of extensive forms. Section 4 contains the definition of our equilibrium concept: psychological sequential equilibrium. In section 5 we discuss a specific application. Sections 6 and 7 respectively contain extensions and a discussion of some of our assumptions as well as a conclusion.

2 Framework

In this section we introduce a class of extensive forms with unawareness and communication. We first define an extensive form with unawareness and without communication (2.1) and consider how players learn from actions taken by others (2.2). Following this we augment the extensive form with unawareness to include strategic messages (2.3) and (2.4), and show how players also learn from messages sent by others (2.5).

2.1 Extensive forms with unawareness

We extend Battigalli and Dufwenberg (2009)'s setting of finite extensive forms with observable actions, no chance moves, and complete information, to include the possibility that players are unaware of parts of the extensive form. We assume that players simultaneously move in every decision node. Note that simultaneous moves do not exclude games where players move in alternation, as we allow for the possibility that players have singleton action sets meaning they are 'passive'. The restrictions made by observable actions, no chance moves, and complete information can be removed, at the cost of additional notational complexity.[1]

Finite extensive forms with observable actions: A finite extensive form with observable actions, no chance moves, and complete information is a tuple $\langle I, N \rangle$ where I is the finite set of players, and N is the finite set of decision nodes. A decision node of length $l \in L$ is a sequence of actions $n = (a^1, \ldots, a^l)$ where each $a^t = (a_i^t)_{i \in I}$ represents the profile of actions taken at stage t $(1 \leq t \leq l)$. The decision node $\tilde{n} = (\tilde{a}^1, \ldots, \tilde{a}^k)$ precedes $n = (a^1, \ldots, a^l)$,

[1]Different extensions of our general framework are discussed in section 6

written $\tilde{n} \prec n$, if $\tilde{n}$ is a prefix of n (i.e., $k < l$ and $(\tilde{a}^1, \ldots, \tilde{a}^k) = (a^1, \ldots, a^k)$). The initial empty node, denoted by n^0, is an element of N. We denote the finite set of feasible actions for player i at node n by $A_{i,n}$ and the set of action profiles by $A_n = \prod_{i \in I} A_{i,n}$. Typical elements of $A_{i,n}$ and A_n are respectively denoted as $a_{i,n}$ and a. $A_{i,n}$ is empty if and only if n is a terminal node. Let Y denote the set of terminal nodes.

Awareness subtrees: Consider now a family $\boldsymbol{T}$ of subtrees of N, ordered by the inclusion of histories. That is,

$$\boldsymbol{T} = \{T \subseteq N : \exists D \in 2^Y \backslash \{\varnothing\}, T = \{n : \exists y \in D : n \leq y\}\},$$

where $n \leq y$ means that n is y or a predecessor of y.

Each subtree $T \in \boldsymbol{T}$ represents a set of feasible paths of play. The 'largest' of these trees is the set N itself. To further clarify the structure of each $T \in \boldsymbol{T}$ we state the following definition for awareness subtrees:

Definition 1 (Awareness subtrees)**.** A set of nodes $T \in \boldsymbol{T}$ is an awareness subtree if there is some nonempty subset of terminal nodes $D \subseteq Y$ such that

$$T = \{n \in N : n \leq y \text{ for some } y \in D\}.$$

Note that such a construction of subtrees ensures that any $T \in \boldsymbol{T}$ starts at the root n^0, that it is naturally ordered by proper subhistories, and implies that each terminal history of each subtree $y \in D$ is associated with a well defined terminal history in Y.

Example 1: The construction of the family $\boldsymbol{T}$ can be demonstrated by a simple example. Consider the extensive form underlying the sequential prisoners dilemma also analyzed by Dufwenberg and Kirchsteiger (2004).

[Figures 1]

It is an extensive form without communication $\langle I, N \rangle$ with $I = \{Ann, Bob\}$ and $N = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$.[2] In the initial node n^0 *Ann* can choose between *cooperate* and *defect* and *Bob* is passive. In nodes n^1 and n^2 *Bob* can respectively choose between *cooperate* and *defect* and *Ann* is passive. Histories n^3, n^4, n^5 and n^6 are terminal nodes.

[2] We will draw on this example in the subsequent sections and develop it further along the lines of our analysis.

The cardinality of the family of subtrees $\boldsymbol{T}$ is $|\boldsymbol{T}| = 2^{|Y|} - 1$. In the context of our example this means $|Y| = 4$ and $|\boldsymbol{T}| = 15$:

[Figures 2]

Awareness: To model that players may have different views on the set of feasible paths of play at different decision nodes n we assume that there exists an 'awareness correspondence':

Definition 2 (Awareness)**.** For each player $i \in I$ there exists a to the player unknown correspondence

$$\varphi_i : N \to \boldsymbol{T},$$

which assigns to player i his level of awareness at node $n \in N$.

A given player i with awareness correspondence $\varphi_i(n) = T$ at n is confined to the sequences of actions associated with the subtree T.

2.2 Learning from others' actions

Each player may become aware of more by learning from actions taken by others. Consider any two nodes $\tilde{n}$ and n with $n = (\tilde{n}, a)$, i.e. $\tilde{n}$ directly precedes n. Furthermore, let $\varphi_i(\tilde{n}) = T$ be the subtree player i is confined to at $\tilde{n}$. At node n player i only learns if the actions taken by others contain information he was previously unaware of, i.e. $n \notin T$. If a player observes that the actions taken by others are different from what he had foreseen, he will have an 'enlightening moment' and discover some subtree T' which is consistent with the actions just observed.

Definition 3 (Learning from actions)**.** All players $i \in I$ with awareness level $\varphi_i(\tilde{n}) = T$ update their view of the game at any history $n = (\tilde{n}, a)$ after observing the profile of actions $a \in A_{\tilde{n}}$ at $\tilde{n}$:

$$\varphi_i(n) = \begin{cases} \{T \cup T'\} : T' \in \boldsymbol{T}, n \in T' & \text{if } n \notin T \\ T & \text{otherwise} \end{cases}$$

As also hinted at in the introduction, for simplicity we abstract from the question how players become aware of new paths of play. We simply assume that observing some actions that a player was not previously aware of, makes him aware of some subtree that is consistent with the actions just observed. Furthermore, we implicitly assume that players are always 'confident' concerning their awareness level. This means, they are always–even after an enlightening moment in which they discover something more–unaware of the fact that they might still be unaware of something: they are unaware of their unawareness.

This characterization of learning implies that players cannot become unaware during the game. More formally:

Remark 1. Awareness may only increase along the path: if there is a path $n, \dots, n' \in N$, and $\varphi_i(n) = T$ while $\varphi_i(n') = T'$ then $T \subseteq T'$.

Furthermore, note that we assume that perfect recall holds.

As argued in the introduction, when players have asymmetric awareness levels communication concerning the feasible paths of play becomes an integral part of the strategic environment. Therefore we next define the set of messages that players can send concerning the feasible paths of play and then augment our extensive form with unawareness to allow for communication.

2.3 Messages about feasible paths of play

Communication is an integral and important part of strategic interactions in situations in which players might be unaware concerning feasible paths of play. Assume that players can either choose to communicate some set of feasible paths of play they are aware of or choose not to communicate which we denote by sending the empty message $\{\varnothing\}$. This means, the set of possible messages of a player confined to some subtree T is defined as:

$$M^T = \{\{T'\}_{T' \subseteq T} \cup \{\varnothing\}\}\,.$$

The set of possible messages for all other players and subtrees is defined analogously. The set of messages which is associated with the largest tree in the family $\boldsymbol{T}$ is denoted by M^N.

Each of these messages only reveals information about the structure of the game, i.e. feasible paths of play. Therefore, over and above a potential role as coordination devices, our messages are irrelevant in settings with full awareness since they contain no new information. However, in settings with asymmetric awareness such messages are an important part of the strategic environment. By construction our messages can only be informative.

2.4 Extensive forms with unawareness and communication

We are now ready to define extensive forms with unawareness and communication.

Finite extensive forms with observable actions and communication: A finite extensive form with observable actions and communication is a tuple $\langle I, H\rangle$ where H is the finite set of histories. Note that we speak of histories and choices rather than decision nodes and actions in our extended setting with communication in order to clearly highlight the distinction between the setting with and without communication. A history of length $l \in L$ is a sequence of choices $h = (c^1, \dots, c^l)$ whereby the sequence of choices consists of the Schur product [Schur (1911)] of the sequence of actions taken and messages sent at each stage $t \leq l$. This means, a history $h \in H$ of length l is a sequence of actions augmented by a sequence of messages $h = (n, m)$ where $(n, m) = ((a^1, m^1), \dots, (a^l, m^l))$. Each $m^t = (m_i^t)_{i \in I}$ with $m_i^t \in M^N$ represents the profile of messages sent at stage t $(1 < t < l)$. The finite set of feasible choices for player i at history $h = (n, m)$ is denoted by $C_{i,h} = A_{i,n} \times M_{i,h}$ where $A_{i,n}$ and $M_{i,h} = M^N$ respectively are the sets of actions and messages of player i in history h. Furthermore, we denote the set of terminal histories by Z.

Example 2: Consider again the extensive form in Figure 1. Let's concentrate on the initial node n^0. In our extensive form with communication *Ann*'s set of feasible choices in the initial history $h^0 \in H$ is, $C_{A,h^0} = \{(C, T_1), \cdots, (C, T_{15}), (C, \{\varnothing\}), (R, T_1), \cdots, (R, T_{15}), (R, \{\varnothing\})\}$. On the other hand, Bob who is passive, P, in h^0 can only communicate, i.e. $C_{B,h^0} = \{(P, T_1), \cdots, (P, T_{15}), (P, \{\varnothing\})\}$.

Augmented Awareness: To model that players may have asymmetric awareness in our extended setting with communication we also have to augment Definition 2 by extending the domain of φ_i from N to H.

Definition 4 (Augmented Awareness)**.** For each player $i \in I$ in our extensive form with communication there exists a to the player unknown correspondence

$$\varphi_i : H \to \boldsymbol{T},$$

which assigns to player i his level of awareness at history $h \in H$.

Analogue to the setting without communication, in any history $h = (n, m)$ a given player i with $\varphi_i(h) = T$ is confined to the set of histories H_T defined as:

$$H_T = \{(n, ((m_i^t)_{i \in I})_{t \leq l}) \in H : n \in T, m_i^t \in M^T, \forall l \in L\},$$

where $((m_i^t)_{i \in I})_{t \leq l}$ is the sequence of all messages he has observed up to stage l, and Z_T is the set of relevant terminal histories.

More intuitively, any given player i with awareness correspondence $\varphi_i(h) = T$ at h is confined to the sequences of actions and messages associated with the subtree T. Wherever necessary we speak of the copy of h in the set H_T as h_T. Given this, the set of choices player i confined to subtree T is aware of at some history $h \in H$ is denoted by C_{i,h_T}. However, we will avoid the subscript T when no confusion may arise.

2.5 Learning from others' choices

Besides learning from actions taken by other players, a player may also learn from messages sent containing new information about feasible paths of play. Analogue to the setting without communication, if a player observes that the choices taken by others are different from what he had foreseen, he will have an enlightening moment and discover some subtree T' which is consistent with the choices just taken. Consider any two histories $\tilde{h}$ and h with $h = (\tilde{h}, c)$. Furthermore, let $\varphi_i(\tilde{h}) = T$ and H_T be the subtree player i is confined to at history $\tilde{h}$. At history h player i only learns if the choices c taken at history $\tilde{h}$ contain information he was previously unaware of.

Definition 5 (Learning from choices)**.** All players $i \in I$ with awareness level $\varphi_i(\tilde{h}) = T$ update their view of the game at any history $h = (\tilde{h}, c)$ after observing the profile of actions c at $\tilde{h}$ such that:

$$\varphi_i(h) = \begin{cases} \{T \cup T'\} : T' \in \boldsymbol{T}, h \in H_{T'} & \text{if } h \notin H_T \\ T & \text{otherwise} \end{cases}$$

This kind of learning thus implies that player i, by constructing a new subtree to which he is confined, updates his current awareness. He does so by aggregating information about paths of play gained from either unforeseen actions taken by others, or messages containing new information. Remark 1 also holds in our setting with communication.

2.6 Strategies

For the extensive form with observable actions and communication $\langle I, H\rangle$, we let S_i denote the set of (pure) strategies of player i. A typical strategy is denoted $s_i = (s_{i,h})_{h \in H \setminus Z}$, where $s_{i,h}$ is the choice that would be selected by s_i if history h obtained. The set of i's strategies that does not prevent h from being reached is denoted by $S_i(h)$.

Strategies cannot in our framework be interpreted as an ex-ante plan of choices since players might be unaware of some of the histories in H. A strategy should therefore rather be viewed as a list of answers to the hypothetical question: 'what would the player do if h where the history he considered possible?' However, there is no guarantee that such a question is meaningful to the player at histories he is unaware of. The answer should therefore be interpreted as given by the modeler, as part of the description of the situation.

For strategy $s_i \in S_i$ and a set of copies $H_T \subseteq H$, we denote by s_i^T the strategy based on some of these copies. Let $S_i^{H_T} \subseteq S_i$ denote the set of such strategies. Define $S_{-i}^{H_T} = \prod_{j \neq i} S_j^{H_T}$. A player might think that others are aware of less than himself. He therefore takes into account histories that he believes they base their strategies on. The possible histories that a player believes others can be aware of is in the set of copies $\boldsymbol{H}_T = \bigcup_{T' \subseteq T} H_{T'}$ with terminal histories in $\boldsymbol{Z}_T = \bigcup_{T' \subseteq T} Z_{T'}$. The subset of copies $h_T \in \boldsymbol{H}_T$ induced by the same history h is denoted by $\boldsymbol{h}_T$, and the sequence of directly succeeding copies is given by $(\boldsymbol{h}_T, c) = (h_{T'}, c)_{T' \subseteq T}$. A player thus considers the set of others' strategies $S_{-i}^{\boldsymbol{H}_T} = \bigcup_{T' \subseteq T} S_{-i}^{H_{T'}}$ as possible. Slightly abusing notation, we let $S_{-i}^{\boldsymbol{H}_T}(\boldsymbol{h}_T) = \bigcup_{h_T \in \boldsymbol{h}_T} S_{-i}^{H_T}(h_T)$ denote the set of strategies that allow for elements in $\boldsymbol{h}_T$. Let a strategy profile be denoted by $(s_i^T, s_{-i}^{T'}) \in S_i^{H_T} \times S_{-i}^{\boldsymbol{H}_T}$, and the the path function which defines the terminal history z_T induced by $(s_i^T, s_{-i}^{T'})$ be denoted by $\zeta(s_i^T, s_{-i}^{T'}) \in Z_T$.[3] That is, a player evaluates others' strategies in the subtree he is confined to.

This concludes the definition of our class of extensive forms with observable actions, messages and unawareness. In the next section we define dynamic psychological games in the context of our class of extensive forms.

3 Dynamic psychological games with unawareness

We now develop our notion of dynamic psychological games with unawareness. We start by considering some mathematical preliminaries (3.1). Hierarchies of conditional beliefs and coherency are characterized in section 3.2, games with unawareness and belief-dependent preferences are defined in section 3.3.

3.1 Mathematical preliminaries

A topological space is deemed Polish if it is separable and completely measurable. The countable product of Polish spaces, endowed with the product topology, is Polish. For a given Polish space X and associated Borel sigma-algebra $\mathscr{B}$, let $\Delta(X)$ be the set of Borel probability measures $\mu : \mathscr{B} \to [0,1]$ on $(X, \mathscr{B})$. A class $\mathscr{B}$ of subsets of X is a Borel sigma-algebra if it contains X itself and is closed under the formation of complements and

[3]The path function $\zeta : S_i^{H_T} \times S_{-i}^{\boldsymbol{H}_T} \to Z_T$ is defined such that $z_T = (c^1, \ldots, c^L) = \zeta(s_i^T, s_{-i}^{T'})$ if and only if $c^1 = (s_{i,h_T^0}^T, s_{-i,h_{T'}^0}^{T'})$ and $c^{t+1} = (s_{i,(c^1,\ldots,c^t)}^T, s_{-i,(c^1,\ldots,c^t)}^{T'})$ for all $t \in \{1, \ldots, L-1\}$.

countable unions. An element $\mu \in \Delta(X)$ satisfies $\mu(\varnothing) = 0$, $\mu(X) = 1$, $\mu(A) \in [0,1]$ for $E \in \mathscr{B}$, and if $E^1, E^2, \ldots$ is a disjoint sequence of sets in $\mathscr{B}$ then $\mu(\bigcup_{\alpha=1}^{\infty} E^\alpha) = \sum_{\alpha=1}^{\infty} \mu(E^\alpha)$. The triplet $(X, \mathscr{B}, \mu)$ is a probability space. The support $\text{supp}(\mu) = \{x \in X : \mu(x) > 0\}$ is a set of points with positive probability, and the marginal of a measure μ on some set $A^\alpha \in \mathscr{B}$ is denoted $\text{marg}_{A^\alpha}\mu = \mu^\alpha$. Finally, if the topology on X is Polish, then the weak topology is also Polish. A sequence $\{\mu^k\}_{k=1}^{\infty}$ in $\Delta(X)$ converges in a weak sense to a measure $\mu \in \Delta(X)$, written $\mu^k \overset{w}{\to} \mu$, if and only if, for every bounded, continuous function $\psi : X \to \mathbb{R}$, $\int_X \psi(x) d\mu^k = \int_X \psi(x) d\mu$.

Consider some player who is uncertain about which element in a set X is true. Assume X is a compact Polish space. Players assign probabilities to events $E, F, \ldots$ in the Borel sigma-algebra $\mathscr{B}$ of X according to some (countably additive) probability measure. Let $\Delta(X)$ denote the set of all probability measures on $(X, \mathscr{B})$. As events unfold players update their beliefs. The actual and/or potential beliefs of a player are described by a conditional probability system. Let $\mathscr{C} \subset \mathscr{B}$ be a non-empty, finite or countable collection such that each $F \in \mathscr{C}$ is both closed and open. The interpretation is that any given player i is uncertain about the element $x \in X$, and $\mathscr{C}$ represents a collection of 'relevant hypotheses'.

Definition 6. A conditional probability system (*cps*) is a function $\mu(\cdot|\cdot) : \mathscr{B} \times \mathscr{C} \to [0,1]$ defined on $(X, \mathscr{B}, \mathscr{C})$ such that for all $E \in \mathscr{B}$ and $F', F \in \mathscr{C}$:

1. $\mu(\cdot|\cdot) \in \Delta(X)$,
2. $\mu(F|F) = 1$,
3. $E \subseteq F' \subseteq F$ implies $\mu(E|F) = \mu(E|F')\,\mu(F'|F)$.[4]

The set of cps' on $(X, \mathscr{B}, \mathscr{C})$ is a subset of the topological space $[\Delta(X)]^{\mathscr{C}}$ (the set of mappings from $\mathscr{C}$ to $\Delta(X)$) and it is denoted $\Delta^{\mathscr{C}}(X)$. Accordingly, we often write $\mu = (\mu(\cdot|F))_{F \in \mathscr{C}} \in \Delta^{\mathscr{C}}(X)$. The topology on X and $\mathscr{B}$ are alway understood and need not be explicit in our notation. Thus we simply say 'conditional probability system (or cps) on $(X, \mathscr{C})$.' We endow $\Delta(X)$ with the topology of weak convergence of measures, and $[\Delta(X)]^{\mathscr{C}}$ with the product topology. The set $\Delta^{\mathscr{C}}(X)$ of cps' on $(X, \mathscr{C})$ is a closed subset of $[\Delta(X)]^{\mathscr{C}}$. Therefore $\Delta^{\mathscr{C}}(X)$ (endowed with the relative topology inherited from $[\Delta(X)]^{\mathscr{C}}$) and $X \times \Delta^{\mathscr{C}}(X)$ (endowed with the product topology) are compact Polish spaces.

[4]The tuple $(X, \mathscr{B}, \mathscr{C}, \mu)$ is called a conditional probability space by Renyi (1955). When $\mathscr{B} = 2^X$, $\mathscr{C} = 2^X \backslash \{\varnothing\}$ and X is finite we obtain Myerson (1986)'s cps'. Battigalli and Siniscalchi (1999) shows how to construct cps' when X is σ-additive.

3.2 Hierarchies of coherent conditional beliefs

Let $(X, \mathscr{B}, \mathscr{C})$ be defined by $X = S_{-i}^{\boldsymbol{H}_T}$ (a finite set) or $X = S_{-i}^{\boldsymbol{H}_T} \times V$ where V is a compact Polish space representing a set of other players beliefs. We are interested in players' (mutual) conditional beliefs at each history $h_T \in \boldsymbol{H}_T$. Thus, the relevant hypothesis in this context is $\mathscr{C} = \{F \subseteq S_{-i}^{\boldsymbol{H}_T} \times V : \exists h_T \in \boldsymbol{H}_T, F = S_{-i}^{\boldsymbol{H}_T}(h_T) \times V\}$ (or $\mathscr{C} = \{F \subseteq S_{-i}^{\boldsymbol{H}_T} : \exists h_T \in \boldsymbol{H}_T, F = S_{-i}^{\boldsymbol{H}_T}(h_T)\}$). The Borel sigma-algebra $\mathscr{B}$ on $S_{-i}^{\boldsymbol{H}_T}$ is implicitly understood.[5] Since each element of $\mathscr{C}$ represents the event that some history $h_T \in \boldsymbol{H}_T$ obtains, we simplify our notation for cps' on $S_{-i}^{\boldsymbol{H}_T} \times V$ (or $S_{-i}^{\boldsymbol{H}_T}$) and replace $\mathscr{C}$ with $\boldsymbol{H}_T$. Indeed, we shall denote strategic form events $F \in \mathscr{C}$ by $h_T \in \boldsymbol{H}_T$ whenever needed. The event that $\boldsymbol{h}_T$ obtains will be defined as the union of the strategic form events that define each elements h_T in $\boldsymbol{h}_T$.

Each player knows the strategies he is aware of and holds cps' about others' strategies given his awareness. A player's conditional first-order cps is then an element of $\Delta^{\boldsymbol{H}_T}(S_{-i}^{\boldsymbol{H}_T})$. Since a player may not know the cps' of other players, he must have second-order beliefs. That is, a player's second-order cps is an element of $\Delta^{\boldsymbol{H}_T}(S_{-i}^{\boldsymbol{H}_T} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(S_{-j}^{\boldsymbol{H}_T}))$. Formally, we define all orders of beliefs inductively by the spaces:

$$
\begin{aligned}
&X_{-i}^0 = S_{-i}^{\boldsymbol{H}_T};\\
&\text{for all } k \geq 1,\\
&X_{-i}^k = X_{-i}^{k-1} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(X_{-j}^{k-1}).
\end{aligned}
$$

An element $\mu_{i,T}^k \in \Delta^{\boldsymbol{H}_T}(X_{-i}^{k-1})$ is a k-order cps. That is, player i confined to H_T evaluates opponents strategies conditional on his awareness. Similar for the other $-i$ players.

The assumed topology implies that for all $k \geq 1$, X_{-i}^k and $\Delta^{\boldsymbol{H}_T}(X_{-i}^k)$ are compact Polish spaces. Since each X_{-i}^k is a cross-product of compact Polish spaces, it is compact Polish itself. Player i's hierarchy of cps' is an infinite sequence of cps' $\mu_{i,T} = (\mu_{i,T}^1, \mu_{i,T}^2, \dots) \in \prod_{k=0}^{\infty} \Delta^{\boldsymbol{H}_T}(X_{-i}^k)$.

Intuitively, player i's cps $\mu_{i,T}$ defines his conditional belief about the set of others' strategies he is aware of (first-order cps), his belief about each of his opponents' beliefs (second-order cps), his belief about each of his opponents' beliefs about others' beliefs (third-order cps), and so on *ad infinitum.*

[5] $\mathscr{B}$ obtains from the product of the discrete topology on $S_{-i}^{\boldsymbol{H}_T}$

However, the conditional belief system just defined may not be meaningful: a player's belief may fail to uniquely specify her own cps. For example, for i's first-order cps $\mu^1_{i,T} \in \Delta^{\boldsymbol{H}_T}(X^0_{-i})$ and her second-order cps $\mu^2_{i,T} \in \Delta^{\boldsymbol{H}_T}(X^1_{-i})$ to be meaningful beliefs, the marginal distribution of $\mu^2_{i,T}$ on X^0_{-i} must coincide with $\mu^1_{i,T}$. We therefore impose that the various levels of beliefs of a player cannot contradict each other. In other words beliefs should be coherent, i.e.,

$$\mu^k_{i,T}(\cdot|\boldsymbol{h}_T) = \operatorname{marg}_{X^{k-1}} \mu^{k+1}_{i,T}(\cdot|\boldsymbol{h}_T) \text{ for all } k \geq 1 \text{ and } \boldsymbol{h}_T \subset \boldsymbol{H}_T.$$ [6]

This does however not imply that a given player is certain that his opponents' conditional beliefs about his beliefs are coherent. In particular, he might believe one or more of his opponents' beliefs are incoherent, or that they may believe that one or more of their opponents may have incoherent beliefs, and so on *an infinitum*. Therefore we impose common knowledge of coherency. The set of cps' $\mu_{i,T}$ for player i in which he is certain that coherency is common knowledge is denoted $B_{i,T}$.[7] We let $B^k_{i,T}$ denote the set of k-order beliefs consistent with collective coherency, i.e., the projection of $B_{i,T}$ on $\Delta^{\boldsymbol{H}_T}(X^{k-1}_{-i})$, and let $B^k_{-i,T} = \prod_{j\neq i} B^k_{j,T}$, $B_{-i,T} = \prod_{j\neq i} B_{j,T}$, and $B_T = \prod_{i\in N} B_{i,T}$.

We have seen that i's k-order conditional beliefs induce beliefs about the set $S^{\boldsymbol{H}_T}_{-i}$ and opponents $(k-1)$-order beliefs, but that does not guarantee that there exists a hierarchy wherein i has beliefs about opponents beliefs in the limit (i.e., $k \to \infty$). Thus, a model that specifies only finite k-order beliefs is not closed. The following Lemma states that i's coherent infinite hierarchy induces beliefs about $S^{\boldsymbol{H}_T}_{-i}$ and the infinite hierarchy of her opponents.

Lemma 1. For each $i \in I$ there is a homeomorphism

$$f_{i,T} = (f_{i,\boldsymbol{h}_T})_{\boldsymbol{h}_T\subset\boldsymbol{H}_T} : B_{i,T} \to \Delta^{\boldsymbol{H}_T}(S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T}).$$ [8]

[6]It can be show by a version of Kolmogorov's Existence Theorem [see e.g. Brandenburger and Dekel (1993) and Battigalli and Siniscalchi (1999) for the proof] that coherent infinite cps' exists and are unique measures in the limit. If we let $\bar{B}_{i,T}$ be the set of i's coherent infinite cps' $\mu_{i,T}$, then there is a 'canonical' homeomorphism

$$g_i = (g_{i,\boldsymbol{h}_T})_{\boldsymbol{h}_T\subset\boldsymbol{H}_T} : \bar{B}_{i,T} \to \Delta^{\boldsymbol{H}_T}(S^{\boldsymbol{H}_T}_{-i} \times \textstyle\prod_{j\neq i}\prod_{k=0}^{\infty} \Delta^{\boldsymbol{H}_T}(X^k_{-j})).$$

[7]Collective coherency can be defined as follows: Again let $\bar{B}_{i,T}$ be the set of i's coherent infinite cps' $\mu_{i,T}$. We will say that i knows some event $E \subseteq S^{\boldsymbol{H}_T}_{-i} \times \prod_{j\neq i} \bar{B}_{j,T}$ at $\boldsymbol{h}_T \subset \boldsymbol{H}_T$ if $g_{i,\boldsymbol{h}_T}(\mu_{i,T})(E) = 1$. For every $i \in N$ and $\boldsymbol{h}_T \subset \boldsymbol{H}_T$ inductively define, for $m = 1, 2, \ldots$, the sets

$\bar{B}_{i,T}(1) = \bar{B}_{i,T}$, and $\forall m \geq 2$

$\bar{B}_{i,T}(m) = \{\mu_{i,T} \in \bar{B}_{i,T}(m-1) :$ for all $\boldsymbol{h}_T \subset \boldsymbol{H}_T$ and $g_{i,\boldsymbol{h}_T}(\mu_{i,T})(S^{\boldsymbol{H}_T}_{-i} \times \prod_{j\neq i} \bar{B}_{j,T}(m-1)) = 1\}$

Then $B_{i,T} = \bigcap_{m=1}^{\infty} \bar{B}_{i,T}(m)$ is i's set of collectively coherent cps'.

[8]See proof in Battigalli and Siniscalchi (1999, Proposition 2).

One might be concerned as to why the homeomorphism g is 'natural'. The reason is that the marginal probability assigned by each $f_{i,\boldsymbol{h}_T}(\mu^1_{i,T}, \mu^2_{i,T}, \dots)$ to a given event in X^{k-1}_{-i} is equal to the probability that $\mu^k_{i,T}$ assigns to that same event. That is, in deriving probabilities on the product space $S^{\boldsymbol{H}_T}_{-i} \times B_{-i,T} = X^0_{-i} \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(X^0_{-j}) \times \prod_{j \neq i} \Delta^{\boldsymbol{H}_T}(X^1_{-j}) \times \cdots$ from $(\mu^1_{i,T}, \mu^2_{i,T}, \dots)$, the function $f_{i,h_{T'}}$ preserves the probabilities specified by $\mu^k_{i,T}$ on each X^{k-1}_{-i}.

Definition 7. A k-order cps of players $i \in I$ at $\boldsymbol{h}_T \subset \boldsymbol{H}_T$ is such that for all $\mu_{i,T} = (\mu^1_{i,T}, \mu^2_{i,T}, \dots) \in B_{i,T}$, $k \geq 1$,

$$\mu^k_{i,T}(\cdot|\boldsymbol{h}_T) = \mathrm{marg}_{S^{\boldsymbol{H}_T}_{-i} \times B^1_{-i,T} \times \cdots \times B^{k-1}_{-i,T}} f_{i,\boldsymbol{h}_T}(\mu_{i,T}).$$

3.3 Dynamic psychological games with unawareness

We are now ready to state our definition of a dynamic psychological game with unawareness:

Definition 8. A dynamic psychological game with unawareness and belief-dependent preferences is a tuple

$$\Gamma = \langle I, H, (\varphi_i)_{i \in I}, (u_i)_{i \in I} \rangle,$$

where $u_i = (u_{i,T})_{T \in \boldsymbol{T}}$ and $u_{i,T} : Z_T \times B_{i,T} \to \mathbb{R}$ is the psychological payoff function of any given player i for whom $\varphi_i(\cdot) = T$.

Contrary to an ordinary game, a game with unawareness is not known to the players, and should therefore be interpreted from the modelers' point of view. At some $h \in H$, players $i \in I$ for whom $\varphi_i(h) = T$ can only conceive the 'partial game' defined as follows:

Definition 9. A T-partial game is a tuple

$$G_T = \langle I, \boldsymbol{H}_T, (u_{i,T})_{i \in I} \rangle.$$

Being confined to a T-partial game a player is aware of the set of copies in $\boldsymbol{H}_T$, the strategies that are based on these copies and his corresponding beliefs. He is also aware of any T'-partial game for which $T \supset T'$ since the set of copies and set of domains of the payoff functions are strict subsets, T'-partial games for which $T' \supset T''$ since their sets of copies and domains are strict subsets, and so on in finitely many steps. In fact, for any sequence $T, T', \dots, T''$ satisfying $T \supset T' \supset \cdots \supset T''$ he will be aware of the transitive closure of the partial games. In a T-partial game, any given player i may be interpreted as being unaware of the T'-partial game if it is not in the transitive closure.[9]

To highlight the role of mutual unawareness in T-partial games consider the following variant of our introductory example, in which *Ann*'s exam was postponed and she could have gone to *Bob*'s party: assume now that *Ann* is aware of the possibility that the exam change-of-date is posted on the instructor's website. Furthermore, assume that *Ann* is certain that *Bob* is also aware of that possibility, but is certain that *Ann* is unaware. That is, *Ann* is certain that *Bob* is unaware of the possibility that she could be aware that he revealed her lie. This situation would be modeled by having *Ann* being confined to some partial game in which she is certain that, (i) *Bob* is confined to some subset of her game, and (ii) *Bob* is certain that she is confined some subset of his game. *Bob* might in this situation be either generous enough not to reveal that he caught her lying, or reveal everything because he is furious that she lied to him.

4 Sequential psychological equilibrium

In the following we will propose a version of Kreps and Wilson (1982)'s sequential equilibrium concept for dynamic psychological games with unawareness. We will define and interpret consistent assessments (4.1), state the main definition of equilibrium and provide an existence theorem (4.2). Lengthy mathematical proofs are relegated to Appendix (B).

4.1 Consistent assessments

We will assume that players consider behavioral strategies. A behavioral strategy is an independent probability distribution over pure strategies each specifying a definite choice at each history the player is aware of. Formally, let $\sigma_{i,T}(\cdot|\boldsymbol{h}_T) = (\sigma_{i,T}(\cdot|h_T))_{h_T \in \boldsymbol{H}_T}$ where $\sigma_{i,T}(\cdot|h_T) \in \Delta(C_{i,h_T})$ be the profile of behavioral strategies conditioned on the set of copies $\boldsymbol{h}_T$. We denote the behavioral strategy profile of players $i \in I$ by $\sigma_{i,T} = (\sigma_{i,T}(\cdot|\boldsymbol{h}_T))_{\boldsymbol{h}_T \subset \boldsymbol{H}_T}$, and the behavioral strategy profile by $\sigma_T = (\sigma_{i,T})_{i \in N}$. The notion reflects that a player plans a collection of randomizations, one for each of the points at which he thinks choices are made. However, in our interpretation we exclude actual randomizations. Rather, we assume that players do not know the pure strategies of others, and the randomization of these players

[9] Such a characterization of unawareness can be shown to comply with the properties that any non-trivial concept of unawareness should satisfy, as suggested by Dekel *et al.* (1998). That is, attractive properties of unawareness obtain in our framework, including; strong plausibility, KU introspection, AU introspection, and weak necessitation [see Appendix (A) for the formal proof]. Especially, it can be shown that the state space implied by the proposed structure is not standard – players need not to know all tautologies.

represents their uncertainty, their conjecture (independent first-order cps) about others' pure strategies (Aumann and Brandenburger, 1995).

Consider any given T-partial game G_T. Let $\Pr_{\sigma_{j,T}}(\cdot|\hat{\boldsymbol{h}}_T) \in \Delta(S_j^{\boldsymbol{H}_T}(\hat{\boldsymbol{h}}_T))$ denote the probability measure over j's strategies conditional on $\hat{\boldsymbol{h}}_T \subset \boldsymbol{H}_T$ and derived from behavioral strategy $\sigma_{j,T}$ under the assumption of independence across histories such that for all $s_j^{T'} \in S_j^{\boldsymbol{H}_T}(\hat{\boldsymbol{h}}_T)$:

$$\Pr_{\sigma_{j,T}}(s_j^{T'}|\hat{\boldsymbol{h}}_T) := \prod_{\boldsymbol{h}_T \subset \boldsymbol{H}_T \setminus \boldsymbol{Z}_T : \boldsymbol{h}_T \nprec \hat{\boldsymbol{h}}_T} \sigma_{j,T}(s_{j,h_{T'}}^{T'}|\boldsymbol{h}_T)$$

where $\boldsymbol{h}_T \nprec \hat{\boldsymbol{h}}_T$ means that $h_{T'}$ does not precede $\hat{h}_{T'}$ for $h_{T'} \in \boldsymbol{h}_T$ and $\hat{h}_{T'} \in \hat{\boldsymbol{h}}_T$. Intuitively, player i who is confined to the T-partial game G_T evaluates the condition probability that some player j makes a certain choice if the future path of play reaches histories $h_{T'} \in \boldsymbol{h}_T \nprec \hat{\boldsymbol{h}}_T$.

A profile of conjectures (first-order cps') $\mu_T^1 = (\mu_{i,T}^1)_{i \in N}$ in the T-partial game G_T is derived from a behavioral strategy profile $\sigma_T = (\sigma_{i,T})_{i \in N}$ if for all $i \in N$, $s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}$, $\hat{\boldsymbol{h}}_T \subset \boldsymbol{H}_T$

$$\mu_{i,T}^1(s_{-i}^{T'}|\hat{\boldsymbol{h}}_T) = \prod_{j \neq i} \Pr_{\sigma_{j,T}}(s_j^{T'}|\hat{\boldsymbol{h}}_T).$$

This implies that for any three players i, j, k, the conjectures of i and j about k coincide. That is, for all $\hat{\boldsymbol{h}}_T \subset \boldsymbol{H}_T$ and $s_k^{T'} \in S_k^{\boldsymbol{H}_T}(\hat{\boldsymbol{h}}_T)$:

$$\mathrm{marg}_{S_k^{\boldsymbol{H}_T}} \mu_{i,T}^1(s_k^{T'}|\hat{\boldsymbol{h}}_T) = \Pr_{\sigma_{k,T}}(s_k^{T'}|\hat{\boldsymbol{h}}_T) = \mathrm{marg}_{S_k^{\boldsymbol{H}_T}} \mu_{j,T}^1(s_k^{T'}|\hat{\boldsymbol{h}}_T).$$

Since we assume that behavioral strategies are independent the conjectures will also be independent.

We are now ready to define consistent assessments:

Definition 10. An assessment (σ_T, μ_T) in the T-partial game G_T is consistent if

(i) μ_T^1 is derived from σ_T,

(ii) and higher order beliefs in μ_T assign probability 1 to the lower order beliefs, such that for all $i \in N$, $k > 1$, $\boldsymbol{h}_T \subseteq \boldsymbol{H}_T$

$$\mu_{i,T}^k(\cdot|\boldsymbol{h}_T) = \mu_{i,T}^{k-1}(\cdot|\boldsymbol{h}_T) \times \delta_{\mu_{-i,T}^{k-1}}$$

where δ_x is the Dirac measure which assigns probability 1 to singleton $\{x\}$.

Players with the same awareness must hold common correct beliefs about each others' belief hierarchies $\mu_{i,T} = (\mu^1_{i,T}, \mu^2_{i,T}, \dots)$. The interpretation is that beliefs are the end-product of transparent reasoning by rational players: any two players with the same awareness must share the same initial first-order cps about any other player. In addition, they come to a correct conclusion about the belief hierarchies of others they are aware of because they are able to replicate their hierarchical reasoning.

At any given copy h_T players of the same 'awareness level' are certain that at every subsequent history they will be confined to the same partial game, that is, they are unaware of the possibility that their current awareness and beliefs may be wrong. As long as players only observe choices they where previously aware of, they will (Bayesian) update their beliefs in a consistent manner. If they observe some choices they were previously unaware of, then they will become aware of some new and larger subtree which they are then confined to. This implies that players will reconsider the situation and come to new common correct beliefs. They will then again continue to update their beliefs in a consistent manner until they (possibly) observe some new choices they were previously unaware of.

4.2 Equilibrium concept

Let some player i be confined to the T-partial game G_T, fix a hierarchy of cps' $\mu_{i,T}$, and a strategy $s_i^T \in S_i^{\boldsymbol{H}_T}(h_T)$. Note that the implicit assumption is that each player is confident in their own plan of choices (and there is common certainty of this). The expectation of $u_{i,T}$ conditional on a non terminal history $\boldsymbol{h}_T \subset \boldsymbol{H}_T \backslash \boldsymbol{Z}_T$ and $\mu_{i,T}$ is

$$\mathbb{E}_{s_i^T, \mu_{i,T}}[u_{i,T} | \boldsymbol{h}_T] := \sum_{h_{T'} \in \boldsymbol{h}_T} \mu^1_{i,T}(h_{T'} | \boldsymbol{h}_T) \times \sum_{s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}(h_{T'})} \mu^1_{i,T}(s_{-i}^{T'} | h_{T'}) u_{i,T}(\zeta(s_i^T, s_{-i}^{T'}), \mu_{i,T}), \tag{1}$$

where $u_{i,T}(\cdot)$ is the psychological payoff of player i. This term gives the payoff from the strategies of others he is aware of. However, player i confined to the T-partial game does–in general–not know the awareness and strategies of the others and thus evaluates his payoff with respect to his conjecture $\mu^1_{i,T}(\cdot|\boldsymbol{h}_T) = \mu^1_{i,T}(\cdot|h_{T'})\mu^1_{i,T}(h_{T'}|\boldsymbol{h}_T)$[10]. Similar for the $-i$ other players.

[10] Here we use the idea that $h_{T'}$ and $\boldsymbol{h}_T$ are strategic form events $F', F \in \mathscr{C}$, respectively, such that $\mu_i(\cdot|F) = \mu_i(\cdot|F')\mu_i(F'|F)$.

Definition 11. An assessment (σ_T, μ_T) in the T-partial game G_T is a sequential equilibrium if it is consistent and for all $i \in I$, $T' \subseteq T$, $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_T \backslash \boldsymbol{Z}_T$,

$$\forall j \neq i, \text{ supp marg}_{S_i^{\boldsymbol{H}_{T'}}} \mu^1_{j,T'}(\cdot|\boldsymbol{h}_{T'}) \subseteq \arg \max_{s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'})} \mathbb{E}_{s_i^{T'}, \mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}].$$

A sequential equilibrium in the T-partial game specifies strategies which are best responses for each player at each embeddable partial game. The idea is that players fictitiously put themselves in the positions of other players by playing a fictitious partial game corresponding to their awareness levels. However, being confined to the T-partial game players will treat their equilibrium strategies in less descriptive partial games as imaginary constructs.

It is often convenient to take the point of view of an 'agent' (i, h_T) in charge of the move, who seeks to maximize i's conditional expected utility given the consistent assessment (σ_T, μ_T). The expected payoff of i conditional on $\boldsymbol{h}_T \subset \boldsymbol{H}_T$ and $c_i \in C_{i,h_T}$ given (σ_T, μ_T) can be expressed as

$$\begin{aligned}\mathbb{E}_{\sigma_T,\mu_T}[u_{i,T}|\boldsymbol{h}_T, c_i] := \sum_{h_{T'} \in \boldsymbol{h}_T} \Pr_{\sigma_{i,T}}(h_{T'}|\boldsymbol{h}_T) & \sum_{s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}(h_{T'})} \prod_{j \neq i} \Pr_{\sigma_{j,T}}(s_{-i}^{T'}|h_{T'}) \times \\ & \sum_{s_i^T \in S_i^{\boldsymbol{H}_T}(h_T, c_i)} \Pr_{\sigma_{i,T}}(s_i^T|h_T, c_i) u_{i,T}(\zeta(s_i^T, s_{-i}^{T'}), \mu_{i,T}),\end{aligned}$$

where $\Pr_{\sigma_{i,T}}(s_i^T|h_T, c_i) := \prod_{\tilde{h}_T \in H_T \backslash Z_T : \tilde{h}_T \npreceq h_T} \sigma_{i,T}(s_{i,h_T}^T|\tilde{h}_T)$ ($\tilde{h}_T \npreceq h_T$ means that $\tilde{h}_T$ is not h_T nor a predecessor of h_T) is player i's own probability measure over strategy s_i^T conditional on the history which he believes follows his choice. Each player i is confident in his own awareness and therefore only considers his own strategies based on histories in the largest subtree he is aware of. Here we use $\Pr_{\sigma_{i,T}}(\cdot|\boldsymbol{h}_T) = \Pr_{\sigma_{i,T}}(\cdot|h_{T'}) \Pr_{\sigma_{i,T}}(h_{T'}|\boldsymbol{h}_T)$ instead of i's conjecture to highlight the relationship with behavioral strategies. Similar for the $-i$ other players.

Definition 12. An assessment (σ_T, μ_T) in the T-partial game G_T is a sequential equilibrium if it is consistent and for all $i \in I$, $T' \subseteq T$, $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_T$,

$$\text{supp}(\sigma_{i,T'}(\cdot|h_{T'})) \subseteq \arg \max_{c_i \in C_{i,h_{T'}}} \mathbb{E}_{\sigma_{T'},\mu_{T'}}[u_{i,T'}|\boldsymbol{h}_{T'}, c_i].$$

Now each player fictitiously puts himself in the position of other players by considering the optimal choices of all players (including himself) in the fictitious partial games he believes possible. He then fixes the optimal choices of other players at each of these partial games and best responds himself by making an optimal choice in the partial game he is confined to. A version of the One-Shot-Deviation principle holds in this framework:

Proposition 1. An optimal strategy of players confined to the T-partial game G_T satisfies the One-Shot-Deviation property since it holds for all $i \in I$, $T' \subseteq T$, $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_{T'} \backslash \boldsymbol{Z}_{T'}$, that

$$\max_{c_i \in C_{i,h_{T'}}} \mathbb{E}_{\sigma_{T'},\mu_{T'}}[u_{i,T'}|\boldsymbol{h}_{T'}, c_i] = \max_{s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'})} \mathbb{E}_{s_i^{T'},\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}].$$

Proof. See Appendix (B).

The following existence theorem obtains:

Theorem 1. If the belief-dependent payoffs are continuous, then there exists at least one sequential equilibrium assessment in the T-partial game G_T.

Proof. See Appendix (B).

The proof of existence basically relies on the trembling-hand perfect equilibrium concept [due to Selten (1975)]: no matter how close to being rational players are, they will never be perfectly rational. There will always be some chance that a player will make a mistake. This idea can be used to approximate a candidate equilibrium behavioral strategy profile by a nearby completely mixed strategy profile (tremble) and require that any deliberately made choices, i.e. those given positive probability in the candidate strategy profile be optimal–not only against the candidate strategy profile, but also against the nearby mixed strategy profile. More formally, any profile of behavioral strategies $\sigma_{T'}$ is a perfect equilibrium if there is a sequence of completely mixed strategy profiles $\{\epsilon^k\}$ such that at each history and for each ϵ^k, the behavior of $\sigma_{T'}$ at the history is optimal against ϵ^k, i.e. is optimal when behavior at all other histories is given by ϵ^k. It is shown by Kakutani's fixed point theorem that in each ϵ^k-perturbed game there exists at least one ϵ^k-equilibrium strategy profile $\sigma^k_{T'}$, implying that there exist an assessment $(\sigma^k_{T'}, \beta(\sigma^k_{T'}))$ where $\beta(\sigma^k_{T'})) = \mu_{T'}$.[11] As $\epsilon^k \to 0$ the corresponding strategy $\sigma^k_{T'}$ has an accumulation point $\sigma^*_{T'}$, such that $(\sigma^*_{T'}, \beta(\sigma^*_{T'}))$. For each 'agent' $(i, h_{T'})$,

[11]Let $\beta^1(\sigma_{T'}) = (\beta^1(\sigma_{T'}))_{i \in N}$ denote the profile of first-order beliefs derived from $\sigma_{T'}$ according to condition (i) in Definition 10. The profile of infinite belief hierarchies $\mu_{T'} = \beta(\sigma_{T'})$ is obtained by applying condition (ii) in the same definition.

$\sigma^*_{i,T'}(\cdot|h_{T'})$ assigns positive probability only to choices that are best responses to $(\sigma^*_{T'}, \beta(\sigma^*_{T'}))$ at $h_{T'}$. By Definition 12 each $(\sigma^*_{T'}, \beta(\sigma^*_{T'}))$ is a sequential equilibrium assessment.

Corollary 1. For any embeddable T'-partial game $G_{T'}$ suppose that there exists an equilibrium. For such an equilibrium of $G_{T'}$, there is an equilibrium of the T-partial game G_T in which players confined to $G_{T'}$ (due to unawareness) play their equilibrium assessments in $G_{T'}$.

Proof. See Appendix (B).

This proposition suggests a procedure for constructing equilibria in a dynamic psychological game with unawareness and communication. First, fix the T-partial game to which the player is initially confined. Then, start from the last stage: any copy such that all feasible choices at each copy terminate the game. If there is more than one player who can move then look for an equilibrium in each subgame, by: (i) calculating the best responses of other players at the copy in the smallest partial game, and (ii) extend the equilibrium step-by-step to copies in larger partial games by finding a fixed point taking the choices of the other players at copies in smaller partial games as given. If there is one player in charge of the move, then we only need to fix his best response at each copy. Now go backward and look at histories and copies thereof in the second-to-last stage. The best responses has already been calculated for all copies $(h_{T'}, c) \in (\boldsymbol{h}_{T'}, c)$, because such copies correspond to the last stage of the game. We assume that each active player at the second-to-last stage makes feasible choices that maximizes his expected payoff given the best responses in the last stage, because he expects that the other players will also best respond in the last stage. We continue to go backwards in this ways until we reach the initial stage. A sequential equilibrium in the T-partial game rules out any profitable deviations given the player's awareness level. However the definition does not exclude the possibility that deviations at successive stages might increase (or decrease) his belief-dependent utility as he may become aware of more paths of play. This implies that in analyzing a dynamic psychological game with unawareness, one has to consider each player's awareness at each stage. If a player at some history becomes aware of more, then he re-evaluates the strategic situation and starts over by backward inducting until the initial stage.

5 Application

In the following we will use a sequential prisoners dilemma to highlight the impact and importance of unawareness in strategic interactions of agents with belief-dependent preferences.

The specific belief-dependent motivation that we concentrate on is a modified version of Dufwenberg and Kirchsteiger (2004)'s 'theory of sequential reciprocity' (5.1). A full description of the strategic interaction with all possible awareness levels and equilibria is beyond the scope of this paper. Therefore, we limit the analysis to three different awareness scenarios and the respective characterization of only one equilibrium (5.2). Results and intuitions are presented in this section, lengthy mathematical proofs are relegated to the Appendix (C).

5.1 A sequential prisoners dilemma with reciprocity

Consider the following sequential prisoners dilemma also analyzed by Dufwenberg and Kirchsteiger (2004):

[Figure 3]

Figure 3 is an extensive form game without communication, where $I = \{Ann, Bob\}$, $N = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$, augmented with material payoffs associated with each joint strategy profile. *Ann* can in the initial history n^0 choose between $Cooperate$ and $Defect$ and *Bob* is passive. While in history n^1 and n^2 *Bob* can choose between $cooperate$ and $defect$, respectively, and *Ann* is passive. If the path $(Cooperate, cooperate)$ is chosen both players get a material payoff of 1, if $(Cooperate, defect)$ is chosen *Ann* gets -1 and *Bob* gets 2. Furthermore, if path $(Defect, defect)$ is chosen both players get a material payoff of 0 and if $(Defect, cooperate)$ is chosen *Ann* gets 2 and *Bob* gets -1.

Following section (2), we now consider the extensive form with unawareness and communication associated with the just described game.

For simplicity we assume that only *Bob* is motivated by belief-dependent reciprocity.[12] More specifically, for any $T \in \boldsymbol{T}$ *Bob*'s utility is given by:

$$u_{B,T}(\zeta(s_B^T, s_A^{T'}), \mu_B) = \pi_B(\cdot) + Y \times \kappa_{BA}(\cdot) \times \lambda_{BAB}(\cdot),$$

where $s_B^T \in S_B^{H_T}$ and $s_A^{T'} \in S_A^{\boldsymbol{H}_T}$. $\pi_B(\cdot)$ is *Bob*'s expected monetary payoff which depends on his first-order belief concerning *Ann*'s strategy $(\mu_{B,T}^1(s_A^{T'}))$ and his own strategy (s_B^T). That means, at $\boldsymbol{h}_T$ *Bob*'s expected monetary payoff is given by $\pi_B(\mu_{B,T}^1\left(s_A^{T'}|\boldsymbol{h}_T\right), s_B^T)$. $Y > 0$ is a constant that captures his sensitivity to reciprocity towards *Ann*. *Bob*'s belief about his kindness towards *Ann* is $\kappa_{BA}(\cdot)$ and *Bob*'s perception of *Ann*'s kindness towards him is $\lambda_{BAB}(\cdot)$.

[12]It is assumed that *Ann* is only interested in her own monetary payoff.

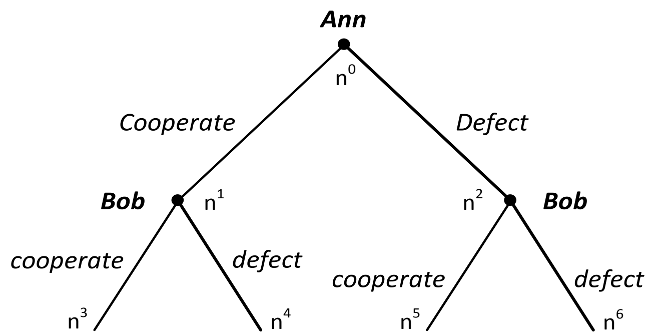

Figure 1: An extensive form without communication

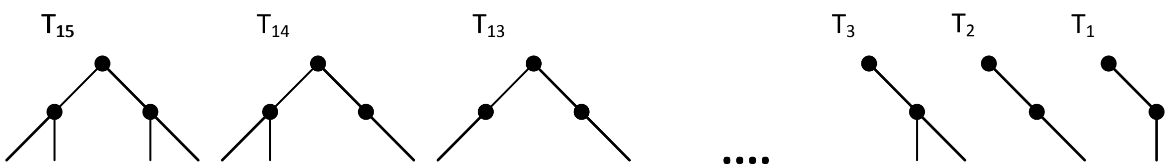

Figure 2: The family of subtrees $\boldsymbol{T}$

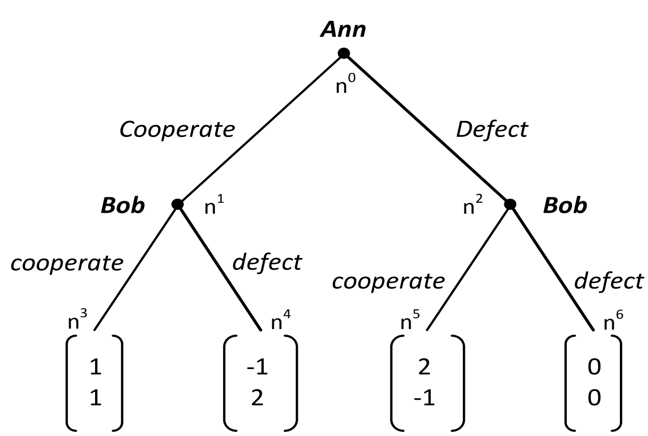

Figure 3: 'Sequential Prisoners Dilemma' without communication

Formally, *Bob*'s perception of *Ann*'s kindness towards him at $\boldsymbol{h}_T$ is:

$$\lambda_{BAB}(\cdot) = \pi_B\left(\mu^1_{B,T}\left(s^{T'}_A|h_T\right), \mu^2_{B,T}\left(\cdot|h_T\right)\right) - \pi^{e_A}_B\left(\mu^1_{B,T}\left(s^{T'}_A|h_T\right), \mu^2_{B,T}\left(\cdot|h_T\right)\right),$$

where $\mu^1_{B,T}\left(s^{T'}_A|h_T\right)$ and $\mu^2_{B,T}\left(\cdot|h_T\right)$ respectively are *Bob*'s (updated) first- and second-order beliefs conditional on $h_T \in \boldsymbol{h}_T$. Of course the domain of $\lambda_{BAB}(\cdot)$ is $\boldsymbol{h}_T$. However, we assume *Bob* only cares about *Ann*'s strategies allowed for by the history h_T in his evaluation of *Ann*'s kindness towards him. That implies, in his evaluation of *Ann*'s kindness towards him, *Bob* basically assigns probability 0 to every strategy $s^{T'}_A \notin S^{\boldsymbol{H}_T}_A(h_T)$. Intuitively, these beliefs describe what *Bob* believes *Ann* would do and believe had she the same awareness level as him. Given this, $\pi_B(\cdot)$ and $\pi^{e_A}_B(\cdot)$ respectively describe what *Bob* believes *Ann* would intend for him and the average that *Ann* would be able to give had she the same awareness level as *Bob*. The equitable payoff is formally defined as follows:

$$\begin{aligned}\pi^{e_A}_B(\cdot) = \frac{1}{2}\Big[&\max\left\{\pi_B\left(\mu^1_{B,T}\left(s^{T'}_A|h_T\right), \mu^2_{B,T}\left(\cdot|h_T\right)\right), s^{T'}_A \in S^{\boldsymbol{H}_T}_A\right\} \\ &+ \min\left\{\pi_B\left(\mu^1_{B,T}\left(s^{T'}_A|h_T\right), \mu^2_{B,T}\left(\cdot|h_T\right)\right), s^{T'}_A \in S^{\boldsymbol{H}_T}_A\right\}\Big]. \end{aligned} \tag{2}$$

The first term in the brackets, $\max\{\pi_B(\mu^1_{B,T}(s^{T'}_A|h_T), \mu^2_{B,T}(\cdot|h_T))\}$, describes *Bob*'s belief about *Ann*'s belief about the maximum that she could have given to him. On the other hand, $\min\{\pi_B(\mu^1_{B,T}(s^{T'}_A|h_T), \mu^2_{B,T}(\cdot|h_T))\}$ describes *Bob*'s belief about *Ann*'s belief concerning the minimum she could have given to him. Intuitively *Bob* does not blame *Ann* for being unaware of some paths of play. He just forms a belief about what *Ann* would and could do were she of the same awareness level as he is.

Note that in Dufwenberg and Kirchsteiger (2004) the set of joint strategy profiles is commonly known. However, in our setting with unawareness kindness perceptions take into account the fact that others might be aware of less. Furthermore, full awareness implies, that the basis upon which the others' kindness is evaluated remains unchanged. In contrast, in our setting the basis upon which the own as well as the kindness of others is judged changes as players become aware of more feasible paths of play.

Bob's kindness towards *Ann* at $\boldsymbol{h}_T$ can be described as:

$$\kappa_{BA}(\cdot) = \pi_A\left(\mu^1_{B,T}\left(s^{T'}_A|\boldsymbol{h}_T\right), s^T_B\right) - \pi^{e_B}_A\left(\mu^1_{B,T}\left(s^{T'}_A|\boldsymbol{h}_T\right), s^T_B\right),$$

where $s^{T'}_A \in S^{\boldsymbol{H}_T}_A$ and $\pi^{e_B}_A(\cdot)$ is defined in an analogous fashion to Equation 2.

Ann's expected material payoff $\pi_A(\cdot)$ describes what *Bob* believes *Ann* gets, given his beliefs concerning her strategy $s_A^{T'} \in S_A^{\boldsymbol{H}_T}$ and his own strategy $s_B^T \in S_B^{H_T}$ where $S_B^{H_T}$ is the set of own strategies that *Bob* is aware of in copies $\boldsymbol{h}_T$. Furthermore, $\pi_A^{e_B}(\cdot)$ is *Bob*'s belief about the average that he can give to *Ann*.

This concludes the definition of our sequential prisoners dilemma with reciprocity.

5.2 Three Different Awareness Scenarios

As said before, for simplicity we concentrate on three different awareness scenarios and the respective characterization of one equilibrium. By considering these three awareness scenarios we limit our attention to a subset of all possible equilibria. The first scenario represents the benchmark case without unawareness also analyzed in Dufwenberg and Kirchsteiger (2004). Scenarios 2 and 3, on the other hand, include asymmetric awareness. For simplicity, both have the following characteristics:

(i) one player is initially aware of more than the other,

(ii) the player that is initially aware of more is certain of the other player's awareness and about the impact of his choices on the other player's awareness,

(iii) the player that is initially aware of less is certain that the other player is of the same awareness level as himself,

These simplifying assumptions imply that we can check for equilibria in our sequential prisoners dilemma in the normal way, i.e. by looking at the second mover following all possible choices of the first mover. Analyze his optimal behavior given his awareness. Go one step backward and analyze the optimal behavior of the first mover given the optimal choices of the second mover.

Scenario 1: As a first awareness scenario consider the benchmark case in which *Ann* and *Bob* are aware of everything. That is, there is no unawareness. Obviously, in such an environment messages that contain feasible paths of play are irrelevant because everyone is aware of all feasible paths of play. Given this we can abstract from messages in our benchmark case and concentrate on the actions of *Ann* and *Bob*. From Dufwenberg and Kirchsteiger (2004) we know that:

Result 1. If *Ann* chooses $Defect$, *Bob* also chooses $defect$ in equilibrium independent of his sensitivity to reciprocity Y. Furthermore, if *Ann* chooses $Cooperate$, *Bob* chooses $cooperate$ in equilibrium if his sensitivity to reciprocity is $Y \geq 1$.

Proof. See Dufwenberg and Kirchsteiger (2004), p. 293.

Given *Bob*'s behavior following *Ann*'s action, it also holds in our benchmark case that:

Result 2. If *Bob*'s sensitivity to reciprocity is $Y \geq 1$, *Ann* chooses *Cooperate* in equilibrium.

Proof. It is easy to see that *Ann* chooses *Cooperate* given *Bob*'s equilibrium behavior, as this gives her 1 in monetary payoffs, rather than 0 which she would get by choosing $Defect$. ■

This shows that without unawareness and a reciprocal *Bob* ($Y \geq 1$), *Ann* can trigger a cooperative reaction from *Bob* by choosing to cooperate. Note that this very intuitive result stands in contrast to the result we would obtain with traditional assumptions about human behavior, i.e. egoistic preferences. If both players are only interested in their own monetary payoff, then *Ann* and *Bob* defecting would be part of the only pure strategy sequential equilibrium.

Scenario 2: As a second simple awareness scenario consider now the following:

- *Bob* is aware of everything, i.e. $T_{15} = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$.[13]
- *Ann* is initially only aware of $T_3 = \{n^0, n^2, n^5, n^6\}$.
- *Bob* is certain that *Ann* is initially only aware of the subtree $T_3 = \{n^0, n^2, n^5, n^6\}$.
- Wherever *Ann* finds herself, she will be certain that *Bob* has the same awareness level.

Different to the previous scenario without unawareness, in this scenario *Ann* is initially unaware of her action *Cooperate* and *Bob*'s actions *cooperate* and $defect$ following it. As before, we start by looking at the optimal behavior of *Bob*. That is, we start to look at all possible partial games *Bob* can find himself in after *Ann*'s choice. We fix his optimal behavior in these worlds and then go one step back to analyze *Ann*'s optimal choice given the optimal choice of *Bob*.

Result 3. If *Ann* chooses $Defect$, then *Bob* chooses *cooperate* and sends any message if his sensitivity to reciprocity is $Y \geq 1$.

Proof. See Appendix (C).

[13] Note that subtrees in our application are indexed in line with the subtrees in Figure 2.

The reason why *Bob* nevertheless cooperates even after the seemingly unkind action $Defect$ of *Ann* is the following: *Bob* is aware of the fact that *Ann* is not aware of her action *Cooperate* and his actions *cooperate* and $defect$ following it. However, *Bob* evaluates *Ann*'s kindness on the basis of what he is aware of. *Bob* holds the equilibrium belief that *Ann* would have cooperated had she been aware of what he is aware of. In equilibrium *Bob* believes that *Ann* would have played *Cooperate* and, hence, would have acted kind, had she been aware of what he is aware of. As he is the last to choose in this situation, his choice is independent of the specific message that he sends, i.e. any of his messages is part of this equilibrium.

Concerning the behavior of *Ann* it is easy to see that her equilibrium behavior is

Result 4. In any sequential equilibria *Ann* chooses $Defect$ and sends any message.

Obviously *Ann* chooses $Defect$ in scenario 2 because this is the only feasible action that she is initially aware of. Furthermore, as she is certain that *Bob* is aware of what she is aware of messages do not play any strategic role for her, and, therefore, any message is part of this sequential equilibrium. This completes the second awareness scenario.

Different to the setting without unawareness by Dufwenberg and Kirchsteiger (2004), in our setting with unawareness *Bob* still cooperates even after the seemingly unkind action $Defect$. *Bob* simply takes into account that *Ann* was unaware of her action *Cooperate* and his subsequent actions $defect$ and *cooperate* and, hence, evaluates her kindness on what she would have done had she been aware of what he is aware of. Importantly, $(Defect, cooperate)$ is neither part of an equilibrium given classical assumptions about human behavior, nor is it part of an equilibrium given reciprocal preferences and full awareness. It is the asymmetric awareness of *Bob* and *Ann* that produces this prediction. This demonstrates how allowing for asymmetric awareness influences our equilibrium predictions.

This scenario practically demonstrates how one can solve for sequential equilibria in our class of psychological games with unawareness and communication. One first has to look at the optimal behavior of all players active in the last non-terminal histories in all their partial games and then go backward history by history repeating the same procedure until the initial history.

Scenario 3: To furthermore see the importance of messages assume now the following awareness scenario:

- *Ann* is aware of everything, i.e. $T_{15} = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$.
- *Bob* is initially only aware of the subtree $T_4 = \{n^0, n^1, n^3, n^4\}$

- *Ann* is certain that *Bob* is initially only aware of the subtree $T_4 = \{n^0, n^1, n^3, n^4\}$.
- *Ann* is certain that, wherever *Bob* finds himself, he will believe that *Ann* has the same awareness level.
- *Ann* is certain that *Bob* will become aware of everything, if she chooses $Defect$.

We start again by analyzing this situation by looking at *Bob*'s choices in all the partial games that he can be in following all possible choices of *Ann*.

Result 5. If *Ann* chooses $Defect$ and any message, *Bob* chooses $defect$ and sends any message in all sequential equilibria.

Proof. See Appendix (C).

To see this, remember that if *Ann* chooses $Defect$, *Bob* becomes aware of everything independent of the message that *Ann* sends in addition to her action. This means, in any history following *Ann*'s action $Defect$ *Bob* re-evaluates *Ann*'s kindness towards him on the basis of $T_{15} = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$. Doing this, *Bob* perceives *Ann*'s choice as unkind independent of the message that she sends. Therefore, *Bob* chooses $defect$ out of reciprocity as well as own monetary considerations. Note, our result 5 is analog to Dufwenberg and Kirchsteiger (2004, p. 282)'s observation 1 in the context of their sequential prisoners dilemma.

Next, consider *Bob*'s behavior following *Ann*'s action $Cooperate$:

Result 6. If *Ann* chooses $Cooperate$ and sends

(i) a message that does not contain any new information on the feasible paths of play, then *Bob* chooses $defect$ in equilibrium and sends any message independent of his sensitivity to reciprocity.

(ii) a message which contains $T_3 = \{n^0, n^2, n^5, n^6\}$ as new paths of play, then *Bob* chooses *cooperate* in equilibrium and sends any message, if his sensitivity to reciprocity is $Y \geq 1$.

(iii) a message which contains only $T_2 = \{n^0, n^2, n^6\}$ as new paths of play, then *Bob* chooses *cooperate* in equilibrium and sends any message, if his sensitivity to reciprocity is $Y \geq 1$.

(iv) a message which contains only $T_1 = \{n^0, n^2, n^5\}$ as new paths of play, then *Bob* chooses *cooperate* in equilibrium and sends any message, if his sensitivity to reciprocity is $Y \geq \frac{1}{2}$.

Proof. See Appendix (C).

Result 6 gives a first impression of how messages about feasible paths of play influence the strategic interaction of reciprocal players. Different to Dufwenberg and Kirchsteiger (2004, p. 282) in the context of their sequential prisoners dilemma with full awareness, our result 6 depends on *Ann*'s message to *Bob*. By sending a message *Ann* can influence the basis on which *Bob* evaluates her kindness. That is, she can influence the partial game that *Bob* will find himself in. If he is unaware of *Ann*'s action $Defect$ and all of his own subsequent actions, *Bob* evaluates the kindness of *Ann* following her choice *Cooperate* on the basis of $T_4 = \{n^0, n^1, n^3, n^4\}$. This implies that he perceives a kindness $\lambda_{BAB} = 0$. This in turn means that *Bob* only takes into account his own monetary payoff when optimizing his choice. Only when *Ann* sends a message that contains some new information, i.e. a subtree consistent with her action $Defect$, *Bob*'s awareness and, hence, the partial game he plays as well as the basis upon which he evaluates *Ann*'s kindness changes.

By sending a message which contains $T_1 = \{n^0, n^2, n^5\}$ as new information, *Bob* becomes aware of $T_{12} = \{n^0, n^1, n^2, n^3, n^4, n^5\}$ (case (iv) of result 6). Hence, *Bob* finds himself in a new partial game and has a new basis upon which he evaluates the kindness of *Ann*. Now *Bob* is aware of the fact that *Ann* could have chosen $Defect$ which would have implied (according to his awareness) a material payoff of -1 for him. Given this, he perceives *Ann*'s choice *Cooperate* as kind because independent of his choice following *Ann*'s choice *Cooperate*, his material payoff is higher than -1. He reciprocates this kindness in equilibrium if his sensitivity to reciprocity is $Y \geq \frac{1}{2}$. Following the same kind of reasoning in cases (*ii*) and (*iii*) implies that *Bob* reciprocates by choosing *cooperate*, if his sensitivity to reciprocity is $Y \geq 1$.

As can easily be seen, if *Ann* had no possibility to send a message to *Bob*, i.e. to make *Bob* aware of what else she could have done, *Ann* would be unable to induce *Bob* to cooperate. *Bob* would simply remain aware of what he was aware of before and continue to evaluate *Ann*'s kindness on this basis.

This brings us to the equilibrium behavior of *Ann*

Result 7. *Ann's equilibrium behavior depends on Bob's sensitivity to reciprocity Y:*

(i) *If Bob's sensitivity to reciprocity is $Y < \frac{1}{2}$, Ann chooses $Defect$ in equilibrium and sends any message.*

(ii) If *Bob*'s sensitivity to reciprocity is $\frac{1}{2} \leq Y \leq 1$, *Ann* chooses *Cooperate* in equilibrium and sends a message which contains *only* $T_1 = \{n^0, n^2, n^5\}$ as new paths of play.

(iii) If *Bob*'s sensitivity to reciprocity is $Y \geq 1$, *Ann* chooses *Cooperate* in equilibrium and sends a message which contains *at least* $T_1 = \{n^0, n^2, n^5\}$ as new paths of play.

Proof. See Appendix (C).

Intuitively, if *Bob*'s sensitivity to reciprocity is low, i.e. $Y < \frac{1}{2}$, *Ann* knows that whatever she makes *Bob* aware of, he will always choose $defect$. Given this, she prefers to choose $Defect$ to get 0 in monetary payoffs, rather than $Cooperate$ which would give her -1. Now, if *Bob* has a sensitivity to reciprocity $Y \geq \frac{1}{2}$, *Ann* can induce *Bob* to cooperate by choosing $Cooperate$ and making him aware of her action $Defect$ and *Bob*'s subsequent possibility $cooperate$ (case (ii) of result 7). Making *Bob* aware changes the basis on which he evaluates the kindness of *Ann* towards him. Aware of *Ann*'s action $Defect$ and *Bob*'s action $cooperate$, *Bob* realizes that *Ann*'s action $Cooperate$ was actually kind. This is something he would not have realized had he remained unaware of $Defect$ and his subsequent action $cooperate$. By choosing action $Cooperate$ and communicating either $T_3 = \{n^0, n^2, n^5, n^6\}$ or $T_2 = \{n^0, n^2, n^6\}$ *Ann* also induces a positive perception of her action, but less than in case (ii). Hence, *Ann* only chooses $Cooperate$ and one of these messages in equilibrium if *Bob*'s sensitivity is higher $Y \geq 1$.

The bottom line: awareness messages are important in the interaction of players with reciprocal preferences as they influence their perceptions about their own as well as others' kindness.

These three simple awareness scenarios demonstrate how unawareness influences the strategic interaction of players with belief-dependent preferences. Furthermore, they show the important role of awareness messages through which players can influence other players' awareness. By influencing awareness levels players influence equilibrium behavior. To put it differently, taking into account asymmetric awareness levels of players when analyzing strategic interactions leads to new and intuitive equilibrium predictions.

6 Extensions and discussion

In this section we first consider some relevant extensions of our model, namely guilt aversion (6.1), moves by nature (6.2), initial asymmetric information (6.3), and strategic information transmission (6.4). We then go on to discuss how to interpret hierarchies of beliefs (6.5), whether unawareness in any meaningful way can be modeled as zero probability events (6.6), and finally consider the relevance of non-equilibrium solution concepts in out setting (6.7).

6.1 Guilt aversion and unawareness

In Section (5) we focused on reciprocity, however our framework is general implying that it can be used to analyze how unawareness affects other forms of belief-dependent motivation such as guilt and regret. In the following we will consider a simple two player example highlighting how unawareness might influence guilt aversion.

We will say that *Ann* 'lets down' *Bob* if his actual material payoff from *Ann*'s strategy, denoted $\pi_B(s_A^T)$, is lower than the payoff *Ann* believes he expects to get, $\pi_B(\mu_A^2(\cdot|\boldsymbol{h}_T), \mu_A^1(s_B^{T'}|\boldsymbol{h}_T))$. This can be measured by the following expression:

$$\max\{0, (\pi_B(\mu_A^2(\cdot|\boldsymbol{h}_T), \mu_A^1(s_B^{T'}|\boldsymbol{h}_T)) - \pi_B(s_A^T))\}.$$

Taking *Ann*'s belief concerning *Bob*'s disappointment into account, we obtain the following utility function exhibiting guilt aversion:

$$u_A(\zeta(s_A^T, s_B^{T'}), \mu_A) = \pi_A(\boldsymbol{z}_T) - Y \times \max\{0, (\pi_B(\mu_A^2(\cdot|\boldsymbol{h}_T), \mu_A^1(s_B^{T'}|\boldsymbol{h}_T)) - \pi_B(s_A^T))\},$$

where $Y \geq 0$ is some psychological sensitivity parameter of *Ann*.

Now consider the example considered in the introduction, in which *Ann*'s exam was postponed and she could have gone to *Bob*'s party. Remember, *Ann* would rather not go to the party because she is tired. Now imagine that *Ann* correctly believes that *Bob* is unaware of the postponement: *Ann* will in equilibrium be certain that *Bob* will be certain that she cannot come, and *Ann* will therefore feel no guilt if she stays away. In a game with full awareness this would however not be a unique equilibrium. *Ann* could also be certain that *Bob* expects her to come because her exam was canceled. If *Ann*'s sensitivity to disappointing *Bob* in this situation is high enough, she would come to his party.

The two forms of belief-dependent motivation we have considered up to now (reciprocity and guilt) have relied on first- and second-order beliefs. However, our model is not restricted to only looking at these forms of beliefs – Definition 7 allows for higher-order belief-dependence. An example involving dependence on third-order beliefs is Battigalli and Dufwenberg (2007b)'s 'guilt from blame,' which assumes that a player cares about the other player's inferences regarding the extend to which he is willing to let him down. Intuitively, *Ann* experiences guilt to the extent that *Bob*'s beliefs indicate that *Ann* intended to disappoint him.

6.2 Moves by nature

Moves by nature is an important extension for applications. For example, Sebald (2010) shows that the strategic interactions of reciprocal players may be influenced by the possibility that material payoffs are affected by moves of nature rather than players. One could easily imagine that such considerations might be amplified (or mitigated) by unawareness.

Let $I^0 = \{0, 1, \ldots, n\}$ where index 0 denotes nature, and $\sigma_{0,T} := \sigma_{0,T}(\cdot|h_{T'}) \in \prod_{h_{T'} \in \boldsymbol{H}_T \setminus \boldsymbol{Z}_T} \Delta^0(A_{0,h_{T'}} \times \{\varnothing\})$ be the awareness restricted strictly positive objective plan of moves by nature. Note that given some awareness level, a 'real' player could never imagine that nature would send messages from which he could learn some new paths of play. We do therefore not consider messages send by nature.

An assessment $(\sigma_T, \mu_T) = (\sigma_{i,T}, \mu_{i,T})_{i \in I^0}$ is consistent if there is a sequence of strictly positive behavioral strategy profiles $\sigma^k \to \sigma$ such that for all $i \in I$, $s_{-i}^{T'} \in S_{-i}^{\boldsymbol{H}_T}$, $\boldsymbol{h}_T \subset \boldsymbol{H}_T$

$$\mu_{i,T}^1(s_{-i}^{T'}|\boldsymbol{h}_T) = \lim_{k \to \infty} \frac{\Pr_{\sigma_{0,T}}(s_0^{T'}) \prod_{j \neq 0,i} \Pr_{\sigma_{j,T}^k}(s_j^{T'})}{\sum_{s_{-i}'^{T'} \in S_{-i}^{T'}(\boldsymbol{h}_T)} \Pr_{\sigma_{0,T}}(s_0'^{T'}) \prod_{j \neq 0,i} \Pr_{\sigma_{j,T}^k}(s_j^{T'})}$$

Kreps and Wilson (1982, Section 5) have a similar condition that refers to cps' of histories (or nodes), and further more for all $l > 1$, $\mu_{i,T}^l$ assigns probability 1 to $\mu_{-i,T}^{l-1}$. (σ_T, μ_T) is a sequential equilibrium if it is consistent and for all $i \in I$, $T' \subseteq T$, $\boldsymbol{h}_T \subset \boldsymbol{H_T \setminus Z_T}$,

$$\forall j \neq i, \text{supp } \text{marg}_{S_i^{\boldsymbol{H}_T}} \mu_{i,T}(\cdot|\boldsymbol{h_T}) \subseteq \arg \max_{s_i^{T'} \in S_i^{\boldsymbol{H}_T}(h_{T'})} \mathbb{E}_{s_i^{T'}, \mu_{i,T'}}[u_{i,T'}|h_{T'}],$$

where $\mathbb{E}_{s_i^T, \mu_{i,T}}[u_{i,T}|h_T]$ is the obvious modification of Equation 1. It can easily be proven that that the existence theorem also holds when we add nature as a player (if the payoff functions are continuous).

6.3 Initial asymmetric information

One might well argue that it is unrealistic to assume that players know each psychological propensity, unless one models interaction within a family or amongst friends. This observation motivates the following extension.

If we want to model asymmetric information about initial moves by nature, we should assume that at the initial history h^0_T (or copy thereof) the only active player is 0 (nature), $A_{0,h^0_T} = \Theta$, where $\Theta \subseteq \Theta_1 \times \cdots \times \Theta_n$ is a set of exogenous payoff relevant parameters. Each player i observes only coordinate θ_i of $\theta = (\theta_1, \ldots, \theta_n)$; θ may affect payoffs, or choice sets, or the probability of future moves by nature. Note that by defining asymmetric information in this way one introduces fictitious ex ante beliefs.

A full blown generalization of information in our model would also include imperfectly observable choices. However, such an extension is non-trivial: the information sets of players $i \neq 0$ need to apply to some consistency requirements. For example, an information set may not be such that some histories are indistinguishable at some subtrees while not at others. A full characterization of imperfect information in our model is beyond the scope of this paper.

6.4 Strategic information transmission

Strategic information transmission has been studied in economic theory for over a quarter of a century. Traditionally this has been done via signaling, whereby a player can influence the beliefs of other players by his actions (e.g., choice of education). To highlight the difference between influencing players' perceptions through signals and awareness messages, we will focus solely on the updating of players' beliefs. The discussion is therefore relevant for, among others, costly market signaling [Spence (1973), Rothschild and Stiglitz (1976), Wilson (1977)], cheap talk [Crawford and Sobel (1982), Farrell (1993)], and observational learning [Banerjee (1992), Bikhchandani *et al.* (1992), Smith and Sørensen (2000)].

The canonical signaling game for our class of unawareness games is basically a Bayesian extensive form with observable actions. We will say that nature selects types independently for the players and refer to player i after he receives information θ_i as type θ_i and $\theta = (\theta_1 \times \cdots \times \theta_n)$ as the state of nature. We assume that there exists a common prior $p \in \Delta(\Theta)$ with the properties that for all i, θ_i and θ_{-i}, $p(\{\theta_i\} \times \Theta_{-i}) > 0$ (type θ_i has positive 'prior' probability) and $p(\theta_{-i}|\theta_i) = p((\theta_i, \theta_{-i})|\{\theta_i\} \times \Theta_{-i})$ (i.e., $p(\theta_{-i}|\theta_i)$ is the conditional probability of θ_{-i} given θ_i). Since types are independent we have that the product measures $p = (p_1 \times \cdots \times p_n)$ is a common prior, where $p_i \in \Delta(\Theta_i)$ is the marginal probability on $\Theta_1 \times \cdots \Theta_n$ for some $i \in I$; equivalently, $p(\theta_{-i}|\theta_i) = \prod_{j \neq i} p_j(\theta_j)$ for all i and θ. We can now associate a signaling game with the set of histories $\boldsymbol{H}_T \times \Theta$. Each information set of each player j takes the form $I(h_{T'}, \theta_j) = \{(h_{T'}, (\theta_j, \theta'_{-j})) : \theta'_{-j} \in \Theta_{-j}\}$ for $\theta_j \in \Theta_j$. Player j's behavioral strategies is denoted by $\sigma_{j,T'}(\cdot|(h_{T'}, \theta_j))) \in [\Delta(A_j(h_{T'}))]^{\Theta}$. We interpret $\sigma_{j,T'}$ as a common array of common conditional first-order beliefs $\mu^1_{-j,T'}$ held by j's opponents. As is standard in signaling we assume that beliefs are determined by actions, which implies that: (i) if player j does not have to move then the actions taken do not affect the other players' belief about player j's type and (ii) if player j is one of the players who takes an action then the other players' beliefs about j's type depend only on the action taken by j, not on the other players' actions. (This is consistent with behavioral strategies being independent.) If $p_j(\theta_j|h^0_{T'}) = p_j(\theta_j)$ and a_j is in the support of $\mu^1_{-j,T'}(\cdot|(h_{T'},\theta_j))$ then for any $\theta_j \in \Theta_j$ we have

$$p_j(\theta_j|h_{T'},a) = \frac{\mu^1_{-j,T'}(a_j|(h_{T'},\theta_j)) \cdot p_j(\theta_j|h_{T'})}{\sum_{\theta'_j \in \Theta_j} \mu^1_{-j,T'}(a_j|(h_{T'},\theta'_j)) \cdot p_j(\theta'_j|h_{T'})}.$$

Upon observing the signal from player j the other players update their beliefs about player j's exogenous type using Bayes' rule until his behavior contradicts the other players' common belief $\mu^1_{-j,T'}$, at which point they form a new conjecture about player i's type that is the basis for future Bayesian updating until there is another conflict with $\mu^1_{-j,T'}$. Such influencing of others' beliefs through signalling does not exists when there is complete information (i.e., Θ is a singleton).

Taking actions or sending messages that other players are unaware of can in our class of games (with complete information) also be interpreted as strategic information transmission. Since each of these actions/messages only reveals information about the structure of the game, and not about the probability of other players being of certain exogenous types, the information transmission we allow for is different from that known from signalling. Remember, in equilibrium player i confined to some subtree forms beliefs about some other player j's equilibrium beliefs at each subtree he might be confined to (which can be embedded in the subtree i is confined to). By strategically revealing paths of play, player i can exclude the subtrees player j can be confined to which does not allow for the revealed paths. This means that our information revealing actions/messages are irrelevant in settings with full awareness. However, in games with asymmetric awareness such information transmission becomes an important part of the strategic interaction.

6.5 Hierarchy representation of beliefs

The hierarchy representation of beliefs plays a prominent role in belief-dependent preferences. The interpretation of such a representation has been discussed a great deal in the literature, and it is therefore important to clarify how one should interpret such hierarchies in our framework. By using a hierarchy representation, we implicitly assume that the game is analyzed at a 'point in time' subsequent to the player knowing his beliefs. That is, there exist no beliefs at a 'prior' point in time, nor is there any information about what the players would have believed had their information been 'less' or 'more' than what it in fact is. The hierarchy of beliefs therefore offers no meaningful argument for identifying beliefs at a prior point in time. When considering unawareness any interpretation of beliefs at a prior point in time becomes nonsensical: one would have to imagine that each player had been aware of all relevant paths of play at some prior point and then become unaware of some of the paths ex-ante, while nevertheless having received more information about the paths they are aware of. Insisting that priors be common does in this setting not reflect where differences in beliefs may come from, but rather constitutes a complex and unintuitive restriction on each hierarchy of beliefs. Even if we were to impose common priors this would not render a prior point in time relevant, nor would it render the prior distribution meaningful.[14]

6.6 Unawareness as zero probability events

One may also wonder to what extend unawareness of paths of play can be modeled as zero probability events. First, assigning probability zero to an event is still compatible with realizing what could happen if the probability zero event were nevertheless to obtain. This is conceptually different from being completely unaware of the event. Second, if one nevertheless wants to model unawareness as zero probability events, then it is impossible in the standard framework Savage (1954). According to Dekel *et al.* (1998), a player should be unaware of an event if and only if he is unaware of being aware of it. So a player being unaware of an event would have to assign probability zero both to the event and its negation. Because of additivity, a probability measure in the standard framework can never assign both zero to an event and its complement.

6.7 Non-equilibrium solution concepts

Our solution concept ideally involves interpreting hierarchies beliefs as a rest-point of a transparent reasoning process, one could argue that it is difficult to carry over such interpretations to a setting in which every increase of awareness is by definition a shock or surprise. Once the player's view of the game itself is challenged in the course of play, some may find it difficult to justify the idea that a new set of equilibrium hierarchy beliefs for the continuation of the game are readily available. One could, for example, consider some version of extensive-form rationalizability (Battigalli, 1997) since it embodies forward inductive reasoning. If somebody makes a player aware of some relevant paths of play, it seems like a strong assumption to dismiss the increased level of awareness as an unintended consequence of others' behavior.

[14] The plausibility and justification of the ex-ante versus the interim view of beliefs has been extensively discussed in the literature, see Harsanyi (1967–68), Dekel and Gul (1997), Gul (1998), and Aumann (1998).

Rather, the player should try to infer from others' choices, re-interpret others' past behavior, and try to infer from it their future moves. In psychological games payoffs are affected by hierarchical beliefs, so rationalizability has to be defined as a property of the whole structure the player is aware of rather than of strategies, and one therefore has to consider players' belief revision processes (Battigalli and Siniscalchi, 2002).

In order to facilitate comparison, and highlight common features, with the existing literature on psychological games with sequential moves, we have chosen to adopt Kreps and Wilson (1982)'s sequential equilibrium concept which has become a benchmark for the analysis of such games (see for example, Dufwenberg and Kirchsteiger, 2004 and Battigalli and Dufwenberg, 2007b).

7 Conclusion

In our analysis we have shown that unawareness has a profound impact on the strategic interaction of agents with belief-dependent preferences. That means, taking account of asymmetric awareness levels leads to intuitive and distinct equilibrium predictions. Furthermore, we have demonstrated that communication concerning feasible paths of play is an important integral part of the strategic environment when players have asymmetric awareness levels–a type of communication that is meaningless in environments without unawareness. In our analysis we have first formalized a general framework with unawareness, communication and belief-dependent psychological preferences. Second, we have presented a solution concept and shown that all dynamic psychological games with continuous utility functions have at least one sequential psychological equilibrium. Third, we have analyzed a specific application to demonstrate the impact of unawareness and communication in a specific context with reciprocal agents. The application has highlighted the fact that any analysis of strategic interactions with asymmetric awareness levels has to start with a description of what players are aware of and what they become aware of when play unravels. Finally, the application has also practically demonstrated how sequential psychological equilibria can be found in specific strategic settings.

Summarizing, unawareness has a profound impact on the strategic interaction of players with belief-dependent psychological preferences. Thus, it should not be neglected and assumed away, but rather taken into account as an integral part of strategic environments.

A Appendix

A.1 Static unawareness properties of Dekel *et al.* (1998)

We will in this section show that the form of unawareness we are considering is non-trivial in the sense of Dekel *et al.* (1998). To do so, we first need to develop a formal (epistemic) language of events and operators.

- Let Ω be the set of states, elements $\omega \in \Omega$ corresponds to a complete description of all the relevant aspects of the strategic situation, including what each player believes. The state of a player is therefore given by his strategy and his hierarchy of cps (s_i, μ_i). The set of states for player i is $\Omega_i = S_i^{\boldsymbol{H}_N} \times B_{i,N}$, and the set of states of the world is $\Omega = \prod_{i \in N} \Omega_i$. We let $\Omega_{-i} = \prod_{j \neq i} \Omega_j$ and with a slight abuse of notation we also write $\omega = (\omega_i, \omega_{-i}) \in \Omega = \Omega_i \times \Omega_{-i}$.
- Let $\mathscr{B}$ denote the Borel sigma-algebra on Ω. Each element $E \in \mathscr{B}$ is an event; its negation is denoted $\neg E = \Omega \backslash E$. An event about i is any $E = E_i \times \Omega_{-i}$, where $E_i \subseteq \Omega_i$ is a Borel set. $\mathscr{E}_i$ is the family of events about i. Events about other players are similarly defined; the collection of such events is denoted $\mathscr{E}_{-i}$.

As is standard in most epistemology, we disregard players' beliefs about themselves. A state $\omega = (s_i, \mu_i, \omega_{-i})$, player i would believe event $E = E_i \times \Omega_{-i} \in \mathscr{E}_{-i}$ conditional on history h_T in the set H_T with probability $f_{i,\boldsymbol{h}_T}(\mu_{i,T})(E_{-i})$. Thus $\{(s_i, \mu_i, \omega_{-i}) : f_{i,\boldsymbol{h}_T}(\mu_{i,T})(E_{-i}) = 1\}$ is the event 'i would know E conditional on $\boldsymbol{h}_T$.' E may concern the beliefs of the other players. Note that player i's induced beliefs are confined to this awareness level T.

- We define a belief operator as a mapping $\boldsymbol{B}_{i,h_T} : \mathscr{E}_{-i} \to \mathscr{E}_i$ defined as follows for all $h_T \in H_T$, $E = \Omega_i \times E_{-i} \in \mathscr{E}_{-i}$:

$$\boldsymbol{B}_{i,h_T}(E) = \{(s_i, \mu_i, \omega_{-i}) : f_{i,\boldsymbol{h}_T}(\mu_{i,T})(E_{-i}) = 1\}. \tag{3}$$

  That is, $\boldsymbol{B}_{i,h_T}$ contains events that player i, given his confined awareness T, would know to obtain.

- An awareness operator is an mapping $\boldsymbol{A}_{i,h_T} : \mathscr{E}_{-i} \to \mathscr{E}_i$ such that for all $h_T \in H_T$, $p \in [0,1]$, $E = \Omega_i \times E_{-i} \in \mathscr{E}_{-i}$:

$$\boldsymbol{A}_{i,h_T}(E) = \{(s_i, \mu_i, \omega_{-i}) : f_{i,\boldsymbol{h}_T}(\mu_{i,T})(E_{-i}) = p\}. \tag{4}$$

$\boldsymbol{A}_{i,h_T}$ is thus defined in the spirit of Monderer and Samet (1989)'s 'p-belief' operator. Events in $\boldsymbol{A}_{i,h_T}$ are those to which player i can assign some probability to.

- The unawareness operator is naturally defined as the negation of awareness:

$$\boldsymbol{U}_{i,h_T}(E) = \neg\boldsymbol{A}_{i,h_T}(E). \tag{5}$$

  $\boldsymbol{U}_{i,h_T}(E)$ contains events which the player can assign no probability to. For player i, confined to subtree T, these events $[E = \Omega_i \times E_{-i} \in \mathscr{E}_{-i}]$ are those for which $E_{-i} \subseteq \Omega_{-i} \backslash (S_{-i}^{\boldsymbol{H}_T} \times B_{-i,T})$. That is, a player need not to be aware of all tautologies. This is a violation of the 'axiom of wisdom' and our model is therefore not a standard state space model.

- With slight abuse of notation we write $\boldsymbol{B}_{i,h_T}(E_{-i})$, $\boldsymbol{A}_{i,h_T}(E_{-i})$ and $\boldsymbol{U}_{i,h_T}(E_{-i})$ for the events $\mathscr{E}_i$ which corresponds to events $\boldsymbol{B}_{i,h_T}(E)$, $\boldsymbol{A}_{i,h_T}(E)$ and $\boldsymbol{U}_{i,h_T}(E)$ in $\mathscr{E}$, respectively. For example, $\boldsymbol{B}_{i,h_T}(E) = \Omega_i \times \boldsymbol{B}_{i,h_T}(E_{-i})$ in $\mathscr{E}_i$.

By showing that the unawareness operator complies with the properties that any appealing concept, as suggested by Dekel *et al.* (1998), the following proposition proves that unawareness in our model is non-trivial.

Proposition 2. Let $E = \Omega_i \times E_{-i} \in \mathscr{E}_{-i}$ be an event. In stepwise thinking the following properties of unawareness obtains for all $h_T \in H_T$:

1. Plausibility: $\boldsymbol{U}_{i,h_T}(E) \subseteq \neg\boldsymbol{B}_{i,h_T}(E) \cap \neg\boldsymbol{B}_{i,h_T}\neg\boldsymbol{B}_{i,h_T}(E)$,
2. *BU* introspection: $\boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E) = \varnothing$,
3. *AU* introspection: $\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{U}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$.
4. Weak necessitation: $\neg\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{B}_{i,h_T}(\Omega)$.

Proof. Proof of each of the propositions follows:

1. *Plausibility*: This property is equivalent to $\boldsymbol{B}_{i,h_T}(E) \cup \boldsymbol{B}_{i,h_T}\neg\boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. By Equation 3 and 4 we have that $\boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. To see that $\boldsymbol{B}_{i,h_T}\neg\boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$, note that $\omega \in \boldsymbol{B}_{i,h_T}\neg\boldsymbol{B}_{i,h_T}(E)$ *iff* $f_{i,\boldsymbol{h}_T}(\mu_{i,T})(\neg\boldsymbol{B}_{i,h_T}(E_{-i})) = 1$. This implies that $\neg\boldsymbol{B}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. Hence $\omega \in \boldsymbol{A}_{i,h_T}(E)$.

2. *BU introspection*: $\boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E) = \varnothing$. To see that this is true consider that some $\omega \in \boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$ *iff* $f_{i,\boldsymbol{h}_T}(\mu_{i,T})(\boldsymbol{U}_{i,h_T}(E_{-i})) = 1$, which can only be true if $\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{A}_{i,h_T}(E)$. By Equation 5 this is impossible and $\omega \notin \boldsymbol{B}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$.

3. *AU introspection*: $\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{U}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$ is equivalent to $\boldsymbol{A}_{i,h_T}\boldsymbol{U}_{i,h_T}(E) = \boldsymbol{A}_{i,h_T}(E)$. Then $\omega \in \boldsymbol{A}_{i,h_T}\boldsymbol{U}_i(E)$ *iff* $f_{i,\boldsymbol{h}_T}(\mu_{i,T})(\boldsymbol{U}_{i,h_T}(E_{-i})) \geq \pi$. Hence $\omega \in \boldsymbol{A}_{i,h_T}\boldsymbol{U}_{i,h_T}(E)$ *iff* $\omega \in \boldsymbol{A}_{i,h_T}(E)$ by Equation 4.

4. *Weak necessitation*: $\neg\boldsymbol{U}_{i,h_T}(E) \subseteq \boldsymbol{B}_{i,h_T}(\Omega)$ is equivalent to $\boldsymbol{A}_{i,h_T}(E) \subseteq \boldsymbol{B}_{i,h_T}(\Omega)$. $\omega \in \boldsymbol{A}_{i,h_T}(E)$ *iff* $f_{i,\boldsymbol{h}_T}(\mu_{i,T})(E_{-i}) \geq p$ (Equation 4), and $\omega \in \boldsymbol{B}_{i,h_T}(\Omega)$ *iff* $f_{i,\boldsymbol{h}_T}(\mu_{i,T})(\Omega_{-i}) = 1$ (Equation 3). Since $E_{-i} \subseteq \Omega_{-i}$ and $p \leq 1$ (awareness if a weaker condition than belief) then it hold true that $\omega \in \boldsymbol{A}_{i,h_T}(E)$ *iff* $\omega \in \boldsymbol{B}_{i,h_T}(\Omega)$.

■

Plausibility implies that a player is unaware of E if he does not have any beliefs about E, and does not have any beliefs about not having any beliefs about E. *BU* introspection states that a player cannot have any beliefs about her own unawareness. *AU* introspection is the property that if a player is unaware of an event E, then she must be unaware of being unaware. Finally, weak necessitation says that if a player is not unaware of E, then he knows any tautology involving E. The four properties together preclude unawareness in any standard state space model.

B Appendix

B.1 Proof of Proposition 1

The Proof follows naturally from the following Lemma, which itself is essentially an adaptation of the dynamic programming approach due to Battigalli and Dufwenberg (2007a, Section 3). We want to relate the problem $\max_{s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'})} \mathbb{E}_{s_i^{T'}, \mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}]$ to a 'multi-dimentional' dynamic programming on a decision tree induced by $\mu_{i,T'}$. First we develop some notation needed for the Lemma:

Define the value functions $V_{\mu_{i,T'}} : \boldsymbol{H}_{T'} \to \mathbb{R}$ and $\overline{V}_{\mu_{i,T'}} : (\boldsymbol{H}_{T'} \backslash \boldsymbol{Z}_{T'}) \times C_{i,h_{T'}} \to \mathbb{R}$ as follows

- For terminal histories $\boldsymbol{z}_{T'} \subset \boldsymbol{Z}_{T'}$, let

$$V_{\mu_{i,T'}}(\boldsymbol{z}_{T'}) = u_{i,T'}(z_{T'}, \mu_{i,T'})$$

- For non-terminal histories $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_{T'} \backslash \boldsymbol{Z}_{T'}$ and any $s_{-i}^{T''} \in S_{-i}^{\boldsymbol{H}_{T'}}(\boldsymbol{h}_{T'})$, let

$$V_{\mu_{i,T'}}(\boldsymbol{h}_{T'}) = \mu_{i,T'}^1(S_{-i}^{\boldsymbol{H}_{T'}}(\boldsymbol{h}_{T'})|\boldsymbol{h}_{T'}) u_{i,T'}(\zeta(s_i^{T'}, s_{-i}^{T''}), \mu_{i,T'})$$

- Assume that the value function $V_{\mu_{i,T'}}(\boldsymbol{h}_{T'}, (c_i, c_{-i}))$ has been defined for all $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_T \backslash \boldsymbol{Z}_T$ and any $s_{-i}^{T''} \in S_{-i}^{\boldsymbol{H}_{T'}}(\boldsymbol{h}_{T'}, (c_i, c_{-i}))$ such that

$$V_{\mu_{i,T'}}(\boldsymbol{h}_{T'}, (c_i, c_{-i})) = \mu_{i,T'}^1(S_{-i}^{\boldsymbol{H}_{T'}}(\boldsymbol{h}_{T'}, (c_i, c_{-i}))|\boldsymbol{h}_{T'}, (c_i, c_{-i})) u_{i,T'}(\zeta(s_i^{T'}, s_{-i}^{T''}), \mu_{i,T'})$$

- Let

$$\overline{V}_{\mu_{i,T'}}(\boldsymbol{h}_{T'}, c_i) = \sum_{c_{-i} \in C_{-i,\boldsymbol{h}_{T'}}} \mu_{i,T'}^1(S_{-i}^{\boldsymbol{H}_{T'}}(\boldsymbol{h}_{T'}, c_{-i})|\boldsymbol{h}_{T'}) V_{\mu_{i,T'}}(\boldsymbol{h}_{T'}, (c_i, c_{-i})),$$

  where $C_{-i,\boldsymbol{h}_{T'}}$ is the set of other players' choices at each copy $h_{T''} \in \boldsymbol{h}_{T'}$.

- Then $V_{\mu_{i,T}}(\boldsymbol{h}_{T'})$ can be defined as

$$V_{\mu_{i,T}}(\boldsymbol{h}_{T'}) = \max_{c_i \in C_{i,h_{T'}}} \overline{V}_{\mu_{i,T'}}(\boldsymbol{h}_{T'}, c_i).$$

For any given strategy $s_i^{T'}$ and history $h_{T'} \in H_{T'} \backslash Z_{T'}$ for $H_{T'} \backslash Z_{T'} \subset \boldsymbol{H}_{T'} \backslash \boldsymbol{Z}_{T'}$, we use the following notation:

- For each k with $0 \leq k \leq l(h_{T'})$ (recall that $l(h_{T'})$ denotes the length of history $h_{T'}$). Let c_i^k be the choice made by i in $h_{T'}$ at the prefix of $h_{T'}$ of length k. Thus, by definition $h_{T'} = (c^0, c^1, \ldots, c^{l(h_{T'})-1})$ where $c^k = (c_1^k, \ldots, c_n^k)$.

- $(s_i^{T'}|h_{T'})$ denotes the strategy that takes all the choices of player i in history $h_{T'}$ and behaves as $s_i^{T'}$ otherwise:

$$(s_i^{T'}|h_{T'})_{\tilde{h}_{T'}} = \begin{cases} s^{T'}_{i,\tilde{h}_{T'}} & \text{if } \tilde{h}_{T'} \nprec h_{T'}, \\ c_i^{l(\tilde{h}_{T'})} & \text{if } \tilde{h}_{T'} \prec h_{T'}. \end{cases}$$

  Intuitively, $(s_i^{T'}|h_{T'})$ is a strategy that takes on the observed choices made prior to the history $h_{T'}$, and then agrees with strategy $s_i^{T'}$ at $h_{T'}$ and in what follows.

- Now change $(s_i^{T'}|h_{T'})$ at $h_{T'}$ so that it is the strategy obtained from $(s_i^{T'}|h_{T'})$ by replacing $s^{T'}_{i,\tilde{h}_{T'}}$ with $c_i \in C_{i,h_{T'}}$. The resulting strategy is denoted $(s_i^{T'}|h_{T'}, c_i)$. That is,

$$(s_i^{T'}|h_{T'}, c_i)_{\tilde{h}_{T'}} = \begin{cases} (s_i^{T'}|h_{T'})_{\tilde{h}_{T'}} & \text{if } \tilde{h}_{T'} \neq h_{T'}, \\ c_i & \text{if } \tilde{h}_{T'} = h_{T'}. \end{cases}$$

  In words, $(s_i^{T'}|h_{T'}, c_i)$ is the strategy consistent with $h_{T'}$ that chooses c_i at $h_{T'}$ and behaves as $(s_i^{T'}|h_{T'})$ in all other histories $\tilde{h}_{T'}$. If $s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'}, c_i)$, then $(s_i^{T'}|h_{T'}, c_i) = s_i^{T'}$. That is, $(s_i^{T'}|h_{T'})$ takes an ex ante (before player i makes his choice at $h_{T'}$) point of view of the strategy $s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'})$ which is consistent with $h_{T'}$, while $(s_i^{T'}|h_{T'}, c_i)$ takes on an ex post (after player i makes his choice at $h_{T'}$) view of the strategy $s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'}, c_i)$ which is consistent with $h_{T'}$ and the choice c_i he is about to make.

- Finally, let $d(h_{T'}) = \max_{h_{T'} \leq z_{T'}}[l(z_{T'}) - l(h_{T'})]$ denote the depth of the subtree with root $h_{T'}$.

Lemma 2 (Dynamic Programming)**.** Suppose that for all $h_{T'} \in H_{T'} \backslash Z_{T'}$ where $H_{T'} \backslash Z_{T'} \subset \boldsymbol{H}_T \backslash \boldsymbol{Z}_T$,

$$s^{T',*}_{i,h_{T'}} \in \arg \max_{c_i \in C_{i,h_{T'}}} \overline{V}_{\mu_{i,T}}(\boldsymbol{h}_{T'}, c_i).$$

Then for all $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_T \backslash \boldsymbol{Z}_T$,

$$\mathbb{E}_{(s_i^{T',*}|h_{T'}),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}] = V_{\mu_{i,T'}}(\boldsymbol{h}_{T'}) = \max_{s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'})} \mathbb{E}_{s_i^{T'},\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}]. \tag{DP}$$

Proof. The proof is by induction on $d(h_T)$.

Basic step: We start from the last stage of any T'-partial game for which $T' \subseteq T$: $h_{T'}$ is such that all feasible choices at $h_{T'}$ terminate the game, i.e. $d(h_{T'}) = 1$. Clearly DP holds for all $h_{T'}$ for which $d(h_{T'}) = 1$ because strategies and choices coincides.

Inductive step: We now fix some stage $k \geq 1$, which is not the last stage, and look at the stage just preceding it. Suppose DP holds for all $h_{T'}$ such that $1 \leq d(h_{T'}) \leq k$. Let $d(h_{T'}) = k+1$. By the law of iterated expectations for all $c_i \in C_{i,h_{T'}}$

$$\mathbb{E}_{(s_i^{T',*}|h_{T'},c_i),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}] = \sum_{c_{-i}\in C_{-i,\boldsymbol{h}_{T'}}} \mu^1_{i,T'}(S_{-i}^{\boldsymbol{H}_{T'}}(\boldsymbol{h}_{T'},c_{-i})|\boldsymbol{h}_{T'})\times \\ \mathbb{E}_{(s_i^{T',*}|h_{T'},c_i),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'},(c_i,c_{-i})]$$

By the inductive hypothesis, for all $c_i \in C_{i,h_{T'}}$ and $c_{-i} \in C_{-i,\boldsymbol{h}_{T'}}$ (that is, all the choices in the T'-partial game i believes others can be aware of),

$$\begin{aligned}\mathbb{E}_{(s_i^{T',*}|h_{T'},c_i),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'},(c_i,c_{-i})] =& V_{\mu_{i,T'}}(\boldsymbol{h}_{T'},(c_i,c_{-i})) \\ =& \max_{s_i^{T'}\in S_i^{\boldsymbol{H}_{T'}}(h_{T'},(c_i,c_{-i}))} \mathbb{E}_{s_i^{T'},\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'},(c_i,c_{-i})]\end{aligned}$$

Taking expectations w.r.t. $c_{-i} \in C_{-i,\boldsymbol{h}_{T'}}$:

$$\mathbb{E}_{(s_i^{T',*}|\boldsymbol{h}_{T'},c_i),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}] = \overline{V}_{\mu_{i,T'}}(\boldsymbol{h}_{T'},c_i).$$

Therefore

$$\mathbb{E}_{(s_i^{T',*}|h_{T'}),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}] = V_{\mu_{i,T'}}(\boldsymbol{h}_{T'}) = \max_{s_i^{T'}\in S_i^{\boldsymbol{H}_{T'}}(h_{T'})} \mathbb{E}_{s_i^{T'},\mu_{i,T'}}[u_{i,T}|\underline{,}h_{T'}]$$

$$\textit{if and only if}$$

$$s^{T',*}_{i,h_{T'}} \in \arg\max_{c_i\in C_{i,h_{T'}}} \mathbb{E}_{(s_i^{T',*}|h_{T'},c_i),\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_{T'}]$$

$$\textit{if and only if}$$

$$s^{T',*}_{i,h_{T'}} \in \arg\max_{c_i\in C_{i,h_{T'}}} \overline{V}_{\mu_{i,T'}}(\boldsymbol{h}_{T'},c_i).$$

The latter condition holds by assumption and the inductive step is hereby proven. ■

Since $\overline{V}_{\mu_{i,T'}} = \mathbb{E}_{\sigma_{T'},\mu_{T'}}[u_{i,T'}|\boldsymbol{h}_{T'}, c_i]$ it follows from Lemma 2 that:

$$V_{\mu_{i,T'}} = \max_{c_i \in C_{i,h_{T'}}} \mathbb{E}_{\sigma_{T'},\mu_{T'}}[u_{i,T'}|\boldsymbol{h}_{T'}, c_i] = \max_{s_i^{T'} \in S_i^{\boldsymbol{H}_{T'}}(h_{T'})} \mathbb{E}_{s_i^{T'},\mu_{i,T'}}[u_{i,T'}|\boldsymbol{h}_T].$$

■

B.2 Proof of Theorem 1

First let $\beta^1(\sigma_{T'}) = (\beta^1(\sigma_{T'}))_{i\in N}$ denote the profile of first-order beliefs derived from $\sigma_{T'}$ according to condition (i) in Definition 10. The profile of infinite belief hierarchies $\mu_{T'} = \beta(\sigma_{T'})$ is obtained by condition (ii) in Definition 10. By construction, the assessment $(\sigma_{T'}, \beta(\sigma_{T'}))$ is consistent. It follows that $\beta(\cdot)$ is a continuous function.

Suppose that each player i is subject to a slight imperfection of rationality (tremble) of the following kind. At every history $h_{T'}$ there is a small positive probability $\epsilon_{i,h_{T'}}$ for the breakdown of rationality. Whenever rationality breaks down, every choice c_i will be selected with some positive probability $\sigma_{i,T'}(c_i|h_{T'}) = \epsilon_{i,h_{T'}}(c_i)$. Formally, fix a strictly positive vector $\epsilon = (\epsilon_{i,h_{T'}}(c_i)_{c_i\in C_{i,h_{T'}}})_{i\in N, h_{T'}\in \boldsymbol{H}_{T'}\setminus \boldsymbol{Z}_{T'}}$ s.t. for all $h_{T'} \in \boldsymbol{H}_{T'}\setminus\boldsymbol{Z}_{T'}$, $\sum_{c_i\in C_{i,h_{T'}}} \epsilon_{i,h_{T'}}(c_i) < 1$. Now define an (agent-form, psychological) ϵ-constrained equilibrium in the T-partial game:

Definition 13 (ϵ-constrained equilibrium)**.** An ϵ-constrained equilibrium in the T-partial game is a set of behavioral strategies profiles $\sigma_{T'}$ (for each $T' \subseteq T$) s.t. for all $i \in N$, $T' \subseteq T$, $\boldsymbol{h}_{T'} \subset \boldsymbol{H}_{T'}$, $c_i \in C_{i,h_{T'}}$,

- **(i)** $\sigma_{i,T'}(c_i|h_{T'}) \geq \epsilon_{i,h_{T'}}(c_i)$,
- **(ii)** $c_i \notin \arg\max_{c_i\in C_{i,h_{T'}}} \mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'}, c_i] \Rightarrow \sigma_{i,T'}(c_i|h_{T'}) = \epsilon_{i,h_{T'}}(c_i)$.

Let $\Sigma_\epsilon = \prod_{i\in N} \Sigma_{\epsilon,i}$ be the set of behavioral strategy profiles in $T' \subseteq T$ satisfying condition (i) in Definition 13, and let $\mathrm{BR}_\epsilon : \Sigma_\epsilon \to \Sigma_\epsilon$ be the ϵ-best response correspondence that assigns to each profile $\sigma_{T'}$ the subset of profiles in Σ_ϵ satisfying condition (ii) of the definition,

$$\begin{aligned}\mathrm{BR}_{\epsilon,i}(\sigma_{T'}) = \{\sigma_{i,T'} \in \Sigma_{\epsilon,i} : & c_i \notin \arg \max_{\tilde{c}_i\in C_{i,h_{T'}}} \mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'}, \tilde{c}_i] \\ & \Rightarrow \sigma_{i,T'}(c_i|h_{T'}) = \epsilon_{i,h_{T'}}(c_i), \forall \boldsymbol{h}_{T'} \subset \boldsymbol{H}_T, \forall h_{T'} \in H_T, \forall c_i \in C_{i,h_{T'}}\},\end{aligned}$$

$$\mathrm{BR}_\epsilon(\sigma_{T'}) = \prod_{i\in N} \mathrm{BR}_{\epsilon,i}(\sigma_{T'}).$$

$\mathrm{BR}_{\epsilon,i}(\sigma_{T'})$ is a nonempty convex subset of Euclidean space $\Delta(C_{i,h_{T'}})$. Since $\mathbb{E}_{\sigma_{T'},\mu_{T'}}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]$ is continuous in $(\sigma_{T'},\mu_{T'})$ and $\mu_{T'}=\beta(\sigma_{T'})$ is a continuous function, $\mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]$ is continuous in $\sigma_{T'}$.

We now have enough structure to apply Kakutani's fixed point theorem to the best response correspondence. $\mathrm{BR}_{\epsilon}(\sigma_{T'})$ is upper hemicontinuous because $\mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]$ is continuous for each (finite) $\boldsymbol{h}_{T'}\subset\boldsymbol{H}_T$ and $c_i\in C_{i,h_{T'}}$, nonempty since each $\mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]$ is continuous and Σ_ϵ is compact, and convex valued because each $\mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]$ is quasi-concave on Σ_ϵ. Therefore $\mathrm{BR}_{\epsilon}(\sigma_{T'})$ has a fixed point, which is an ϵ-constrained equilibrium.

Fix a sequence $\epsilon^k\to 0$ and a corresponding sequence of ϵ^k-constraint equilibrium strategies $\sigma^k_{T'}$. By compactness, the sequence $(\sigma^k_{T'})$ has a limit point $\sigma^*_{T'}$. A trembling-hand perfect equilibrium is any limit of ϵ-constraint equilibria as $\epsilon^k\to 0$. We will now prove that the trembling-hand perfect equilibrium $(\sigma^*_{T'},\beta(\sigma^*_{T'}))$ is a sequential equilibrium.

Assessment $(\sigma^*_{T'},\beta(\sigma^*_{T'}))$ is continuous: to see this note that, by continuity, $\beta(\sigma^*_{T'})$ is a limit point of $\beta(\sigma^k_{T'})$, and that the set of consistent assessment is closed. By continuity of $\mathbb{E}_{\sigma_{T'},\beta(\sigma_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]$ in $\sigma_{T'}$ (and fitness of $C_{i,h_{T'}}$), for k sufficiently large

$$\arg\max_{c_i\in C_{i,h_{T'}}}\mathbb{E}_{\sigma^*_{T'},\beta(\sigma^*_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i]=\arg\max_{c_i\in C_{i,h_{T'}}}\mathbb{E}_{\sigma^k_{T'},\beta(\sigma^k_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i].$$

This implies that

$$\mathrm{supp}(\sigma^*_{i,T'}(\cdot|h_{T'})\subseteq\arg\max_{c_i\in C_{i,h_{T'}}}\mathbb{E}_{\sigma^*_{T'},\beta(\sigma^*_{T'})}[u_{i,T'}|\boldsymbol{h}_{T'},c_i].$$

By Definition 12 each $(\sigma^*_{T'},\beta(\sigma^*_{T'}))$ is a sequential equilibrium. ■

B.3 Proof of Corollary 1

First note that the existence of an equilibrium assessment in each partial game is ensured by Theorem 1.

Now imagine an equilibrium assessment $(\sigma^*_T,\beta(\sigma^*_T))$ for some players confined to the T-partial game G_T, and assume that some other players confined to the T'-partial game $G_{T'}$ play their component in the equilibrium assessment $(\sigma^*_{T'},\beta(\sigma^*_{T'}))$.

We need to show that $(\sigma^*_T, \beta(\sigma^*_T))$ is an equilibrium assessment in G_T. Suppose not, then there would be a profitable deviation

$$\mathbb{E}_{((\tilde{\sigma}_{i,X}, \sigma^*_{-i,X}), \beta(\tilde{\sigma}_{i,X}, \sigma^*_{-i,X}))}[u_{i,X} | \boldsymbol{h}_X, c_i] > \mathbb{E}_{(\sigma^*_X, \beta(\sigma^*_X))}[u_{i,X} | \boldsymbol{h}_X, c_i]$$

for some $i \in N$, $\boldsymbol{h}_X \subset \boldsymbol{H}_X$, $c_i \in C_{i,h_X}$ and $X = \{T, T'\}$.

1. For players confined to the T-partial game G_T $(X = T)$: a player's assessment $((\tilde{\sigma}_{i,T}, \sigma^*_{-i,T}), \beta(\tilde{\sigma}_{i,T}, \sigma^*_{-i,T}))$ is not an equilibrium assessment in G_T by Definition 12–a contradiction.

2. For players confined to the T'-partial game $G_{T'}(X = T')$: a player's expected payoff is (due to unawareness) identical in $G_{T'}$ and G_T, thus his assessment $((\tilde{\sigma}_{i,T'}, \sigma^*_{-i,T'}), \beta(\tilde{\sigma}_{i,T'}, \sigma^*_{-i,T'}))$ is not an equilibrium strategy in $G_{T'}$ by Definition 12–a contradiction.

Hence $(\sigma^*_T, \beta(\sigma^*_T))$ must be an equilibrium assessment in G_T. ■

C Appendix

C.1 Proof of Result 3

Remember in scenario 2 *Bob* is aware of everything. Hence, if *Ann* chooses *Defect*, *Bob* evaluates *Ann*'s kindness on the basis of $T_{15} = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$ in the history that he finds himself in. In result 1 we have shown that full awareness would imply that *Bob* chooses *defect* out of monetary and reciprocity reasons. Although *Bob* is aware of everything and observes *Ann*'s choice *Defect*, he knows that *Ann* is unaware of her action *Cooperate* and his subsequent actions. *Bob*, hence, forms an equilibrium belief about what *Ann* would have done had she been of the same awareness level as he is. From scenario 1 we know that the only sequential equilibrium given full awareness and $Y \geq 1$ involves *Ann* playing *Cooperate* and *Bob* playing *cooperate*. This means, *Bob* holds the equilibrium belief given his awareness level that $(Cooperate, (cooperate, defect))$ would have been the actions in the joint equilibrium strategy, if *Ann* had been of the same awareness level as he is. Given this, *Bob*'s evaluation of *Ann*'s kindness even following *Ann*'s choice *Defect* is:

$$\lambda_{BAB} = 1 - \frac{1}{2}[1 + 0] = 0.5.$$

Note $\lambda_{BAB} = 0.5$ is Bob's perception about *Ann*'s kindness after *Ann*'s action *Cooperate* in the equilibrium they would have played had both been aware of everything. As *Bob* does not hold her responsible for being unaware, this is also his perception concerning *Ann*'s kindness following her choice *Defect* and awareness levels $T_{15} = \{n^0, n^2, n^5, n^6\}$. In other words, this is *Bob*'s equilibrium belief about *Ann*'s kindness given her awareness level $T = \{n^0, n^2, n^5, n^6\}$ and following her choice of action *Defect*. On the other hand, the kindness that *Bob* can show to *Ann* is given by

$$\kappa_{BA} = 2 - \frac{1}{2}(2 + 0) = 1$$

by choosing *cooperate* and

$$\kappa_{21} = 0 - \frac{1}{2}(2 + 0) = -1$$

by choosing *defect*. Bringing things together, *Bob* chooses *cooperate* if the utility from choosing *cooperate*, i.e. $-1 + Y \cdot (0.5) \cdot (1)$, is higher than the utility from choosing *defect*, i.e. $0 + Y \cdot (0.5) \cdot (-1)$. This is the case when $Y \geq 1$. In other words, *Bob* chooses to accept -1 in order not to be unkind to *Ann* who he believes would have been kind to him if she had been aware of everything that he is aware of. ■

C.2 Proof of Result 5

To understand result 5 it is important to see that whatever *Ann* beliefs about *Bob*'s strategy following her choice $Defect$, *Bob* is worse of than if she would have chosen $Cooperate$ (see also result 1 and the proof to observation 1 in Dufwenberg and Kirchsteiger (2004)). This means it is sure that *Bob* who becomes aware of everything when *Ann* chooses $Defect$ considers $Defect$ as an unkind choice. Given this, his belief-dependent reciprocity preferences plus his own monetary payoff makes him to choose his action $defect$. Furthermore, as *Bob* correctly believes that *Ann* is also aware of everything, messages do not play any strategic role for him and, hence, he chooses any message. ■

C.3 Proof of Result 6

Consider first part (i): By sending a message which does not contain any new information *Bob* does not become aware of any new feasible path of play. This implies that *Bob* will continue to evaluate *Ann*'s kindness on the basis of $T_5 = \{n^0, n^1, n^3, n^4\}$. As $T_5 = \{n^0, n^1, n^3, n^4\}$ only entails one action for *Ann*, *Bob*'s belief about the intentions of *Ann* towards him as well as *Bob*'s belief about the maximum and minimum that *Ann* could have given to him coincide. Hence, $\lambda_{BAB} = 0$. Given this, *Bob*'s psychological utility from reciprocity is $Y \cdot \kappa_{BA} \cdot \lambda_{BAB} = 0$ and he consequently maximizes his own monetary payoff, i.e. *Bob* chooses action $defect$. Consider now part (ii) and (iii): if *Ann* chooses $Cooperate$ and a message that contains at least $T_2 = \{n^0, n^2, n^6\}$ as new information, then *Bob* evaluates *Ann*'s kindness either on $T_{15} = \{n^0, n^1, n^2, n^3, n^4, n^5, n^6\}$ or $T_{13} = \{n^0, n^1, n^2, n^3, n^4, n^6\}$ depending on *Ann*'s message. To evaluate *Bob*'s perception concerning *Ann*'s kindness in this case we have to specify his belief concerning *Ann*'s belief regarding his choice following *Ann*'s action $Cooperate$. Denote *Bob*'s belief concerning *Ann*'s belief concerning the likelihood with which he plays *cooperate* following her action $Cooperate$ by β. This implies that he believes that *Ann* believes that he plays $defect$ following her choice of $Cooperate$ with probability $(1-\beta)$. Furthermore, note that in this situation *Bob* believes that in equilibrium he would have chosen $defect$ following *Ann*'s choice $Defect$ giving him a payoff of 0. Given this, *Bob* perceives *Ann*'s choice $Cooperate$ and the message which contains *at least* $T_2 = \{n^0, n^2, n^6\}$ as

$$\lambda_{BAB} = \beta + (1-\beta)2 - \frac{1}{2}[\beta + (1-\beta)2 + 0]$$

where $\frac{1}{2}[\beta + (1-\beta)2 + 0]$ is *Bob*'s perception given his awareness level concerning the average that *Ann* could have given him. λ_{BAB} reduces to $1 - \frac{1}{2}\beta$. In equilibrium beliefs have to be correct! Hence, *Bob*'s perception of *Ann*'s kindness in an equilibrium involving his action *cooperate* following *Ann*'s action *Cooperate* ($\beta = 1$) is $\frac{1}{2}$. On the other hand, in this situation *Bob*'s kindness towards *Ann* by choosing *cooperate* and any message is $\kappa_{BA} = 1 - \frac{1}{2}[1 + (-1)] = 1$ and his kindness from choosing *defect* and any message is $\kappa_{BA} = -1 - \frac{1}{2}[1 + (-1)] = -1$. This means he chooses *cooperate* in equilibrium if:

$$1 + Y(\frac{1}{2})(1) \geq 2 + Y(\frac{1}{2})(-1)$$

which holds if $Y \geq 1$. Consider now part (iv). We follow the same reasoning as before: if *Ann* chooses *Cooperate* and a message that contains only $T_1 = \{n^0, n^2, n^5\}$ as new information, then *Bob* evaluates *Ann*'s kindness on $T_{12} = \{n^0, n^1, n^2, n^3, n^4, n^5\}$. In this case *Bob* believes that he would have chosen *cooperate* following *Ann*'s choice *Defect* as this is the only of his actions following *Ann*'s choice *Defect* that he has become aware of by *Ann*'s message. Again, denote *Bob*'s belief concerning *Ann*'s belief concerning the likelihood with which he plays *cooperate* following *Ann*'s action *Cooperate* by β. This means that *Bob* perceives *Ann*'s choice *Cooperate* and the message which contains only $T_1 = \{n^0, n^2, n^5\}$ as new information as:

$$\lambda_{BAB} = \beta + (1 - \beta)2 - \frac{1}{2}[\beta + (1 - \beta)2 + (-1)]$$

which reduces to $1\frac{1}{2} - \frac{1}{2}\beta$. As before, in equilibrium beliefs have to be correct. Hence, *Bob*'s perception of *Ann*'s kindness in an equilibrium involving his action *cooperate* following *Ann*'s choice *Cooperate* ($\beta = 1$) is 1. As in the cases (ii) and (iii), in this situation *Bob*'s kindness towards *Ann* by choosing *cooperate* and any message is $\kappa_{BA} = 1 - \frac{1}{2}[1 + (-1)] = 1$ and his kindness from choosing *defect* and any message is $\kappa_{BA} = -1 - \frac{1}{2}[1 + (-1)] = -1$. This means he chooses *cooperate* in equilibrium if:

$$1 + Y(1)(1) \geq 2 + Y(1)(-1)$$

which holds if $Y \geq \frac{1}{2}$. ■

C.4 Proof of Result 7

Case (i): If *Bob*'s sensitivity to reciprocity is $Y < \frac{1}{2}$, *Ann* knows that *Bob* will defect no matter what she does and which messages she sends. Hence, she chooses *Defect* to get in equilibrium 0, rather than −1 which she would get by choosing *Cooperate*. Case (ii): If *Bob*'s sensitivity to reciprocity is $\frac{1}{2} \leq Y \leq 1$, *Ann* knows that *Bob* will *cooperate* when she chooses *Cooperate* and a message which contains only $T_1 = \{n^0, n^2, n^5\}$ as new information. As this gives her 1 in monetary payoffs which is more than with any of her other actions and messages, she chooses to cooperate and send a message which contains only $T_1 = \{n^0, n^2, n^5\}$

as new information. Case (iii): In case (iii) we can apply the same reasoning as in case (ii). But, as a message that contains either $T_3 = \{n^0, n^2, n^5, n^6\}$ or $T_2 = \{n^0, n^2, n^6\}$ implies a lower kindness perception in *Bob*'s eyes about *Ann*'s action *Cooperate*, *Bob* chooses to *cooperate* in equilibrium only if $Y \geq 1$. Hence, *Ann* chooses this action and message only if $Y \geq 1$.

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