Price setting with menu cost for multi-product firms*

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Abstract

We model the decisions of a multi-product firm that can revise prices after paying a fixed "menu" cost. The key assumption, introduced by Lach and Tsiddon (1996, 2007) and Midrigan (2007, 2009), is that once the menu cost is paid the firm can adjust the price of *all* its products. We completely characterize the decision rule of a simple symmetric problem in terms of the structural parameters: the variability of the flexible prices, the curvature of the profit function, the size of the menu cost, and the number of products sold by the firm. We provide analytical expressions for the frequency of adjustment, the hazard rate of price adjustments, and the distribution of price changes in terms of the structural parameters. We show analytically that economies with firms that sell more goods are more sticky: the impact effect of a monetary shock on aggregate prices is decreasing in the number of products sold by each firm.

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1 Introduction

Several papers document how price setting behavior, as summarized by the size-distribution and by the timing of price changes, varies systematically with the number of products sold. The recent empirical work of Bhattarai and Schoenle (2010) documents that firms selling more goods display a higher frequency of price adjustment as well as smaller adjustments. Lach and Tsiddon (1996, 2007) show that price changes are synchronized within stores, but staggered across stores. Cavallo (2010) studies online large supermarket chains, and finds that price changes of similar goods are synchronized within a store.[1]

Despite the rich and growing evidence on this phenomenon, there is scant theoretical work on this problem. Midrigan (2007, 2009) begun to study this problem by explicitly writing down and solving numerically a model where a firm is selling 2 goods, subject to a common menu cost.[2] Compared to the classic case of one good, his model generates a distribution of price changes with "small" price adjustments. Indeed the main motivation on the seminal paper by Lach and Tsiddon (1996) was to argue that, due to the synchronization of adjustments, the presence of small price changes does not imply that menu cost are not important. But some important questions remain to be answered: what forces shape the optimal pricing decisions as the number of goods n sold by the firm changes? Going beyond the $n = 2$ case is important, as the number of goods sold by the retail stores, where much of the micro data are measured, is much larger.[3]

We analytically solve for the firm optimal decision rules and compute the invariant distribution of price changes and hazard functions implied by these rules in a tractable set up. By modeling how the price setting decision of firms depend on n the paper addresses two types of

[1] An incomplete list of additional contributions documenting these type of behaviour includes Lach and Tsiddon (1992), Baudry et al. (2007), Dhyne and Konieczny (2007), Dutta et al. (1999), Midrigan (2007, 2009), and Neiman (2010).

[2] Importantly, he also solved for a general equilibrium and analyzed the effect of a monetary shock. Bhattarai and Schoenle (2010) also solve numerically the problem of a firm selling three goods.

[3] Bhattarai and Schoenle (2010) analyze the BLS data on US producer's prices. The median number of goods *sampled* by the BLS for each producer is between 3 to 5. Obviously this is a lower bound of the median number of goods that they actually sell.

questions that are hard to tackle without a formal frame. First, the model reproduces several cross-sectional empirical regularities that have become well-known after the contributions of Bils and Klenow (2004) and Nakamura and Steinsson (2008). Examples include the average size of price changes, the fraction of "small" price changes, the frequency of price changes. Naturally the model allows us to understand what is the role of the various fundamental variables, such as n, the volatility of the underlying shocks, the size of the menu cost, and the benefit of adjusting prices, in affecting the cross-section statistics on price setting behavior.

Second, the model is useful to advance our understanding of the impact effect of an aggregate monetary shock. This question has been tacked numerically by Midrigan (2007) for the case of $n = 2$, and compared with the case of $n = 1$, analyzed by Golosov and Lucas (2007), but as we explain below we provide tools to analyze the case of $n \geq 2$, and discuss why this is important. We consider a permanent unexpected increase in money supply in an economy that starts at the cross sectional stationary distribution corresponding to zero inflation. We characterize which firms will change prices, and develop simple analytical expressions for the fraction of adjusters and the response of the aggregate price level. In the language of Golosov and Lucas (2007), economies with different values of n have a different amount of "selection". Indeed Caballero and Engel (2007) argue that the cross section distribution of the "desired adjustments", or price gaps in our set up, is one of the key ingredients to understand the aggregate effect in a model with sS policies. As n increases we show that there are more firms close to the point where they want to adjust, thus increasing the flexibility of prices after a shock. But there is also an additional effect from the number of products n, namely how large are the changes in the prices of those firms that will adjust. The net effect of these two forces depend on both the size of the monetary shock and on the number of product n. We show that the impact effect of a monetary shock indeed depends both on the size of the shock and on the number of products that the firms in the economy sell. For a given n the larger the shock the larger the impact on the aggregate price level, and hence the smaller the real effect. Interestingly, whether an economy with more goods is more flexible depends on the size of the shock. For small shocks, the flexibility of the aggregate price level decreases with n, the opposite is true for large shocks.

To our knowledge this is the first fixed cost adjustment problem in n-dimensions whose solution is analytically characterized. We believe that this is because of the difficulty of finding a tractable boundary condition and a candidate solution that is smooth enough on the boundary of the inaction region. Baccarin (2009) gives a recent statement of the general problem, and an existence results of a viscosity solution. Instead we look for a strong solution, i.e. a smooth one. In our case we can reduce the dimension of the problem to one, by keeping track of y, the square of the radius of the vector of the price gaps. This reduction is possible because of the quadratic nature of the objective function, and the lack of drift of the uncontrolled price gap. Thus, we trade off high dimension for a non-linearity on the evolution of the system. In Section 5 we study the sensitivity of our results to the introduction of an inflation drift. We show that, up to a first order, inflation has no effect on the statistics that we focus, so that the analysis should be accurate for countries with low inflation rates. Our proof strategy is to convert back the one-dimensional problem into the original n-dimensional problem and to check that the candidate solution satisfies the solution of the n-dimensional variational inequality verification theorem for stopping time problems by Øksendal (2000).

The setup and summary of main results

We study a stylized version of the problem of a multi-product firm that can revise prices only after paying a fixed cost. The key assumption, introduced by Lach and Tsiddon (1996, 2007) and Midrigan (2007, 2009), is that once the fixed menu cost is paid the firm can adjust the price of *all* its products. The problem is set up as to minimize the deviations of the profits incurred relative to the flexible price case, i.e. the case with no menu cost. We assume that the static profit maximizing prices for each of the n products, which coincide with the price that would be charged without menu cost, follow n independent random walks without drift and with volatility σ per unit of time. We refer to the vector of the difference between the frictionless prices and the actual prices charged as the vector of the price gaps. The period return function is assumed to be proportional to the sum of the squares of the price gaps. The proportionality constant B measures the second order per period losses associated with charging a price different from the optimum, i.e. it is a measure of the curvature of the profit function.[4] We assume that if a fixed cost ψ is paid the firm can simultaneously change all the prices. The firm minimizes the expected discounted cost, which include the stream of lost profit from charging prices different from the frictionless as well as the fixed cost at the time of adjustments. We completely characterize the solution of problem in terms of the structural parameters: the variability of the flexible prices σ, the curvature of the profit function B, the size of the menu cost ψ, the discount rate r, and the number of products n. We also provide analytical expressions for the invariant distribution of the price gaps, the frequency of adjustment, the hazard rate of price adjustments, and the marginal distribution of price changes in terms of the fundamental parameters.

The solution of the firm's problem involves finding the set over which prices are adjusted, and the set where they are not, i.e. the inaction set. Due to the lack of drift, when prices are adjusted they are set equal to the frictionless prices, i.e. the price gaps are set to zero in all dimensions. We show that the optimal decision is to control the price gap as to remain in the interior of the n-dimensional ball centered at the origin. The economics of this is clear: the firm will adjust either if all the prices of its product have a medium size deviation, or if only one has a large deviation, since in the margin a larger deviation hurst profits more. The size of this ball, whose square radius we denote by $\bar{y}$, is chosen optimally. We solve for the value function and completely characterize the size of the inaction set $\bar{y}$ as a function of the parameters of the problem. As we let $r \downarrow 0$ the ratio $\bar{y}/\sigma^2$ can be written as an increasing function of two arguments: $\sigma^2 B/\psi$ and n. We also obtain a very accurate approximation for small cost ψ, where we show that $\bar{y}$ takes the form of a square

[4]The first order losses are zero, since the maximum per period profits are obtained at the frictionless price.

root function, $\bar{y} \approx [2(n+2)\sigma^2 B/\psi]^{1/2}$. To compare the model with tabulation for the US economy as functions of n, we consider two extreme cases of how the technology to adjust prices. In one case we assume that the fixed cost increases proportionally with the number of products, i.e. $\psi = \psi_1\ n$ for some $\psi_1 > 0$, a case that we referred to as *constant returns to scale*. In the other extreme the fixed cost remains constant as n changes, an assumption that we referred to as *constant fixed cost*, so $\psi = \psi_1$ for all n. Thus, when a prediction of the model depends on how the technology varies across n we present both cases.

We characterize the implications for the timing of price changes given $\bar{y}$, σ^2 and n. We show that the expected number of price adjustments per unit of time is given by $n\sigma^2/\bar{y}$, which together with our result for $\bar{y}$ gives a complete characterization of the frequency of price adjustments. This characterization can be used to disentangle the effects on the frequency of adjustments while comparing firms with different number of products, since it points out to all its determinants. Moreover, when used together with other information described below, it can be used to identify the parameters of the model and test its implications. For instance we compare the elasticity of the formulas implied in our paper with the ones implied by tabulations on Bhattarai and Schoenle (2010). We find that the elasticities predicted by the theory are closer to the ones in US data in the case of *constant returns to scale cost* case.

We solve in closed form for the hazard rate of the price changes as a function of the time elapsed since the last change. The shape of this function, except for its scale, depends *exclusively* on the number of products n. The scale of the function is completely determined by the expected number of adjustment per unit of time, which we have already solve for. For a given n, the hazard rates are increasing in duration, have an elongated S shape, with a finite asymptote. Comparing across different values of n, while keeping the expected number of adjustment constant, we show that the asymptote of the hazard rate is increasing in n. As we let n increase without bound, the asymptote diverges to $+\infty$ and the hazard rate function converges to the one with deterministic adjustments, i.e. towards one with an inverted L shape. In words, as n increases, adjustment is less likely early on, and more likely later on, converging to the extreme case of deterministic adjustment as $n \to \infty$.

We characterize the shape of the distribution of price changes. While price changes occur simultaneously for n products, we characterize the marginal distribution of prices, because this is the object that is usually computed in actual data sets. We give a closed form expression for the density of the marginal distribution of price changes as a function of $\bar{y}$ and n. Based on these results we compute several statistics that measure the size of the price changes, such as $\mathbb{E}[|\Delta p|]$, the expected value of the *absolute value* of price changes. We show that, regardless of how the fixed cost changes with n in our two extreme cases, as the number of products increases, the size of the adjustments decreases for all n. Thus, the insight of $n = 2$ generalizes, i.e. with more products the typical adjustment is smaller in each product. We use this statistics, as well as our solution for $\bar{y}$ for different n to compare it with the tabulations in the data from Bhattarai and Schoenle (2010). We find that the elasticities predicted by the theory are close to the ones in the US data for the *constant returns to scale case.*

We show that once the size of the changes is controlled for, the shape of the price change distribution is *exclusively* a function of the number of products n. We obtain then several statistics that have been computed in the data, such as the coefficient of variation of the absolute value of price changes, or the excess kurtosis, as purely functions of the n. We compare this statistics with the tabulations in US data by Bhattarai and Schoenle (2010) and find the same pattern: higher values of n imply higher dispersion and fatter tails. Indeed the shape of the distribution of price changes is as follows: for $n = 2$ it is bimodal, with modes at the absolute value of $\sqrt{\bar{y}}$, for $n = 3$ is uniform, for $n = 4$ peaks at zero and it is concave, and for larger n it is bell shape. Indeed, as $n \to \infty$, once normalized, the distribution converged to a standard normal. We find the sensitivity of the shape of price changes with respect to n an interesting result to identify different type of models of price adjustments. In particular, bimodality is only predicted for $n = 1$ or $n = 2$. This helps to discriminate with respect to other theories of price adjustments, as the ones bases on a mixture of information and menu cost, worked out in Alvarez, Lippi, and Paciello (2011). Additionally, bimodality receives some support in the data in studies by Cavallo (2010) and Cavallo and Rigobon (2010) which use data from stores that sell large number of products.

As an illustration of the advantage of the analytical characterization of the firm's problem in Section 6 we study the impact effect of a monetary shock shocks on the aggregate price level, i.e. the first point of an impulse response function. We derive an analytical representation of the impact effect on aggregate prices as a function of the normalized size of the monetary shock for economies with different number of products. We show that, for an economy where firms sell n products, the impact effect of a monetary shock on the aggregate price level depends, exclusively on the steady state size of price changes as measured by its standard deviation -and it is independent of the steady state frequency of price changes. For a given steady state size of price changes, economies with firms that sell more goods, i.e. larger values of n, have stickier aggregate price level, and thus have larger effect of monetary shock on output.

2 A stylized multiproduct menu cost model

Let n be the number of goods sold by the firm. Each price p_i evolves according to a random walk without drift, so that $\mathrm{d}p_i = \sigma \, \mathrm{d}W_i$ where $\mathrm{d}W_i$ is a standard Brownian Motion. The n Brownian Motions (BM henceforth) are independent, so $\mathbb{E}\left[W_i(t)W_j(t')\right] = 0$ for all $t, t' \geq 0$ and $i, j = 1, ..., n$. The problem is:

$$V(p) = \min_{\{\tau_j, \Delta p_i(\tau_j)\}_{j=1}^{\infty}} \mathbb{E}\left[\sum_{j=1}^{\infty} e^{-r\tau_j}\psi + \int_0^{\infty} e^{-rt} B\left(\sum_{i=1}^{n} p_i^2(t)\right) dt \,\middle|\, p(0) = p\right] \tag{1}$$

where

$$p_i(t) = \sigma W_i(t) + \sum_{j:\tau_j < t} \Delta p_i(\tau_j) \text{ for all } t \geq 0 \text{ and } i = 1, 2, ..., n, \tag{2}$$

and $p(0) = p$.

So that τ_j are the (stopping) times at which control is exercised. At these times, after paying the cost ψ, the state can be changed to any value in $\mathbb{R}^n$. We denote the vector of price changes as $\Delta p(\tau_j) \in \mathbb{R}^n$. This is a standard adjustment cost problem subject to a fixed cost, with the exception that after paying the adjustment cost ψ the decision maker can adjust the state in the n dimension.

At an abstract level equation (1) and equation (2) can be used to solve a symmetric quadratic loss tracking problem in n dimensions, subject to a fixed adjustment cost. To map it into a tracking problem, let the state of the system be two n dimensional vectors $\hat{p}(t)$, and $p^*(t)$. The interpretation of $\hat{p}(t)$ is the location of the system, and $p^*(t)$ a bliss point, the location that the decision maker is tracking. The instantaneous cost of the decision maker is proportional to the distance between the location of the system and the bliss point, $B||p^*(t) - \hat{p}(t)||^2$, where $B > 0$. Each component of the bliss points evolve as an independent random walk without drift, with variance σ^2 per unit of time. If the decision maker pays a fixed cost ψ/n she can change the location of the system anywhere that she desires. For the purpose of finding the times at which the decision maker chooses to change the state, and to find the value of the changes of the state, we can simplify the problem and consider the distance between the location of the state and the bliss point the of the system, and simple let the state be $p(t) = \hat{p}(t) - p^*(t)$. We have written equation (1) and equation (2) using this "gap" notation. For future reference we note if B and ψ are multiplied by a constant $\lambda > 0$ the value function is scaled by λ with no change on the decisions. This explains why all the decisions are functions of B/ψ. We will use this property to interpret different assumptions about how the B, ψ parameters vary across firms with different number of products n.

Nest we discuss an economic interpretation of the problem which can be summarized to say that the firm "tracks" the prices that maximize instantaneous profits from the n products. Consider a system of n independent demands, with constant elasticity η for each product, and a time varying constant marginal cost $C_i(t)$. In the context of the price setting models, our model is a stylized version of the problem introduced by Midrigan (2007, 2009) where the elasticity of substitution between the goods produced *within* the firm is the same as the one of the bundle of goods produced *across* firms. The instantaneous profit maximizing price is proportional to the marginal cost, or in logs $p_i^*(t) = \log C_i(t) + \log(\eta/(\eta-1))$. In this case we assume that the log of the marginal cost evolves as a random walk with drift so that $p_i^*(t)$ inherits this property. We can interpret the period cost as a second order expansion of the (log) of the profit function with respect to the vector of the log of prices, around the log of the profit maximized price vector (see Appendix A) The first order term are zero because we are expanding around $p^*(t)$. The *log* expansion is equivalent to measuring the profits relative to the value of the maximized profit for the n goods. There are no second order cross terms due to the separability of the demand. Thus we can write the problem in terms of the gap between the actual price and the profit maximizing price: $p(t) = \hat{p}(t) - p^*(t)$. The constant B is given by $B = (1/2)\eta(\eta-1)/n$, where the term $1/2$ is due to a second order expansion, the terms with η are due to the fact that the curvature of the profits depend on the elasticity of demand, and the term $(1/n)$ is the share of profits from each product relative to the profits across the n goods. In this interpretation the value of the fixed cost is measured relative to the profit of the n goods, thus it costs ψ/n in units of the numeraire good. Since all that matter for the decision is the ratio of B to ψ this normalization only scales the units of the vale function.

Below we consider two cases for scaling of the cost of adjustment with respect to the number of goods. In the first case, which we refer to as constant returns to scale (CRTS) technology for adjustment cost, when we compare firms with different values of n, the adjustment cost scales linearly with it, so that $\psi = n\ \psi_1$. In this case a firm with twice as many products pays twice as much in terms of the numeraire good to adjust all the prices simultaneously. We refer to the case of a *constant fixed cost*, if $\psi = \psi_1$, so that a firm with twice as many products pays the same cost in terms of numeraire to adjust the twice as many prices. We think that these two extreme simple cases bracket all of the interesting setups.

We note the following basic properties of the value function and the optimal policy.

1. Given the symmetry of the BM and of the objective function around zero, and the independence of the BM's, one can use reflection around zero to show that the value function only depends on the absolute values of p_i, i.e. $V(p) = V(|p_1|, |p_2|, ..., |p_n|)$ for all $p \in \mathbb{R}^n$.

2. Due to the symmetry of the return function, in the law of motion the target prices and the lack of drift, it is easy to see that after an adjustment the state is reset at the origin, i.e. $p(\tau_j^+) = 0$, or $\Delta p(\tau_j) = -p(\tau_j^-)$. See Proposition 13 in Appendix B for a formal argument.

3. The state space $\mathbb{R}^n$ can be divided in two regions, an inaction region $\mathcal{I} \subset \mathbb{R}^n$ and control region $\mathcal{C} \subset \mathbb{R}^n$. We use $\text{Int}(\mathcal{C})$ for the interior of the control region and $\partial\mathcal{I}$ for the boundary of the inaction region. We have that $\mathcal{C} \cap \mathcal{I} = \emptyset$, that inaction is strictly preferred in $\mathcal{I}$, that control is strictly preferred in $\text{Int}(\mathcal{C})$, and that in $\partial\mathcal{I}$ the agent is indifferent between control and inaction.

We write down the conditions for the solution of the problem, provided that a value function is smooth enough, i.e. we look for a solution of the "strong" formulation of the problem with: $V \in C^1(\mathbb{R}^n)$ and $V \in C^2(\mathbb{R}^n \backslash \partial\mathcal{I})$, so the function is once differentiable in the whole domain, and twice differentiable everywhere, but in the boundary of the inaction set. In the range of inaction the cost for the firm is given by the following Bellman equation:

$$r\ V(p_1, p_2, ..., p_n) = B\ \sum_{i=1}^{n} p_i^2 + \frac{\sigma^2}{2} \sum_{i=1}^{n} V_{ii}(p_1, p_2, ..., p_n) \tag{3}$$

for all $p \in \mathcal{I}$. In the control region we have:

$$V(p_1, p_2, ..., p_n) = V(0) + \psi \tag{4}$$

for all $p \in \mathcal{C}$. The optimality of returning to the origin implies that,

$$V_i(0, 0, ..., 0) = 0 \text{ for all } i = 1, 2, ..., n. \quad (5)$$

Finally, differentiability in the boundary of the inaction region gives

$$V_i(p_1, p_2, ..., p_n) = 0 \text{ for } i = 1, 2, ..., n \text{ and for all } p \in \partial\mathcal{I}, \quad (6)$$

We refer to this condition as smooth pasting.

We briefly comments on the results on control theory that apply to our problem. Theorem 1 in Baccarin (2009) shows the existence of a continuous value function V and a policy described by a continuation and control region for a class of problem that include ours. The set-up in Baccarin (2009) includes a more general form of adjustment cost, more general period return function, and more general law of motion for the state, as well as weaker differentiability assumption on these function.[5] Øksendal (2000) and Aliev (2007) analyze a general class of slightly simpler stopping time problem in n dimensions. Their consider a problem with a one time decision of when to collect a given reward function of the state, denoted by g. Before that time the decision maker has either zero flow returns, or in the case of Øksendal (2000) she receives a flow return f, as function the state. The decision maker maximized the expected discounted value of the reward.[6] Their problem maps into our by making the reward $g(p) = V(0) + \psi$ and the flow return $f(p) = B||p||^2$. Aliev (2007) shows that equation (6) is necessary for optimality, provided that $p \in \partial\mathcal{I}$ is a regular point for the stopping set $\mathcal{C}$ with respect to the process $\{\sigma W(t)\}$ and that the derivatives of the value function in a neighborhood of $\partial\mathcal{I}$ are bounded. Theorem 10.4.1 in Øksendal (2000) is a verification theorem in term of variational inequalities which, when adapted to our set-up,

[5] Strictly speaking, our problem does not fit one of the assumptions for Theorem 1 in Baccarin (2009). In particular, Assumption (2.4) requires that the cost diverges to infinity as the norm of the adjustment diverges. Nevertheless, we can artificially modify our problem by incorporating a proportional adjustment cost that applies only when $||p||$ is very large, without altering our solution.

[6] While in their analysis discounting is not explicitly included, it is easy to introduce it by taking time as one of the n states.

says that if a function V that satisfies conditions equations (3)-(6) and several additional conditions -which we state and check in our proof- the value function solves the stopping time problem.

3 Characterization of the solution

Before presenting the solution of this problem we change the state space, which we summarize using a single variable. Let

$$y = \sum_{i=1}^{n} p_i^2 \tag{7}$$

measure the deviation of prices from their optimal value across the n goods. We consider policies summarized by a single number $\bar{y}$. In this class of policies the firm controls the state so that if $y < \bar{y}$, there is inaction. The first time that y reaches $\bar{y}$, all prices are adjusted to the origin, so that $y = 0$. We will find the optimal policy in this class. Then we will show that the optimal policy of the original problem is of this form.

The variable y measures the square of the ray of a sphere centered on the origin. Since each of the prices follows identical independent standard BM in the inaction region, then y follows a simple diffusion in the inaction. Using Ito's Lemma on equation (7) the evolution of y is

$$\mathrm{d}y = n\sigma^2 \mathrm{d}t + 2\sigma \sum_{i=1}^{n} p_i(t) \mathrm{d}W_i$$

This implies that the quadratic variation of y is:

$$\mathbb{E}(\mathrm{d}y)^2 = 4\sigma^2 \left(\sum_{i=1}^{n} p_i^2(t) \right) \mathrm{d}t$$

Thus we can define a stochastic differential equation for y with a new standard BM $\{W(t)\}$ that solves:

$$\mathrm{d}y = n\sigma^2 \mathrm{d}t + 2\sigma\sqrt{y}\ \mathrm{d}W \text{ for } y \in [0, \bar{y}]. \tag{8}$$

We note that for the unregulated process, i.e. when $\bar{y} = \infty$, if $y(0) > 0$ then $y(t) > 0$ for $t > 0$ with probability one provided that $n \geq 2$, see Karatzas and Shreve (1991) Proposition 3.22.[7]

Note that the drift and diffusion terms in equation (8) are only functions of y. We also note that the instantaneous return is a function of y, so we can write the following

$$v(y) = \min_{\bar{y}} \mathbb{E} \left[\sum_{j=1}^{\infty} e^{-r\tau_j} \psi + \int_0^{\infty} e^{-rt} B \; y(t) \; dt \;\middle|\; y(0) = y \right] \tag{9}$$

subject to equation (8) when $y \in [0, \bar{y}]$, where τ_j are the first time that $y(t)$ hits $\bar{y}$. The function v solves:

$$r \; v(y) = By + n\sigma^2 \; v'(y) + 2\sigma^2 y \; v''(y), \quad \text{for } y \in (0, \bar{y}) \; . \tag{10}$$

Since policy calls for adjustment at values higher than $\bar{y}$ we have:

$$v(y) = v(0) + \psi, \quad \text{for all } y \geq \bar{y} \; . \tag{11}$$

If v is differentiable at $\bar{y}$ we can write the two boundary conditions:

$$v(\bar{y}) = v(0) + \psi \quad \text{and} \quad v'(\bar{y}) = 0 \; . \tag{12}$$

These conditions are typically referred to as value matching and smooth pasting. For $y = 0$ to be the optimal return point, it must be a global minimum, and thus we require that:

$$v'(0) \geq 0 \; . \tag{13}$$

[7] This result was obtained for Bessel processes, which are the square root of $y(t)$. Additionally, Karatzas and Shreve (1991) have shown in Problem 3.23 and 3.24 that for if $y(0) > 0$, then for $n = 2$ the unregulated process can become arbitrarily close to zero but for $n \geq 3$ almost every path remains bounded away from zero. Furthermore, for the regulated process the classification for the boundaries of a diffusion gives that for $n \geq 2$ the point $y = 0$ is an entrance boundary, as verified in Karlin and Taylor (1999) Example 6, Chapter 12.6.

Note the weak inequality, since y is non-negative.

The next proposition finds an analytical solution for v in the range of inaction.

Proposition 1. Let $\sigma > 0$. The ODE given by equation (10) is solved by the following analytical function:

$$v(y) = \sum_{i=0}^{\infty} \beta_i \, y^i \ , \quad \text{for } y \in [0, \bar{y}] \ . \tag{14}$$

where the coefficients $\{\beta_i\}$ solve:

$$\beta_0 = \frac{n\sigma^2}{r}\beta_1 \ \ , \ \ \beta_2 = \frac{r\beta_1 - B}{2\sigma^2(n+2)} \ , \ \ \beta_{i+1} = \frac{r}{(i+1)\ \sigma^2\ (n+2i)}\ \beta_i \ , \quad \text{for } i \geq 2 \ . \tag{15}$$

for any β_0.

The proof follows by replacing the function given in equation (14) into the ODE (10) and matching the coefficients for the powers of y^i. By the Cauchy-Hadamard theorem, the power series converges absolutely for all $y > 0$ since $\lim_{i\to\infty} \beta_{i+1}/\beta_i = 0$. The next proposition shows that there exist a unique solution of the ODE (10) satisfying the relevant boundary conditions (see Appendix B for the proof).

Proposition 2. Assume $r > 0, \sigma > 0, n \geq 1$. There exists $\bar{y}$ and a unique solution of the ODE (10) satisfying the two boundary conditions described in equations (12) for which $v(\cdot)$ satisfies: i) it is minimized at $y = 0$, ii) it is strictly increasing in $(0, \bar{y})$, and iii) $\bar{y}$ is a local maximum, i.e. $\lim_{y \uparrow \bar{y}} v''(y) < 0$.

The next proposition uses a slightly modified version of the verification theorem in Øksendal (2000) to show that value function v and threshold policy $\bar{y}$ that we found in Proposition 2 for the one-dimensional representation indeed characterize the inaction $\mathcal{I} = \{p : ||p||^2 < \bar{y}\}$ and control sets $\mathcal{C}$ as well as the value function V for the original n-dimensional problem (see Appendix B for the proof).

Proposition 3. Let v be the solution of the restricted problem equation (9) and equation (8). Let $V(p) = v\left(\sum_{i=1}^{n} p_i^2\right)$. This is the solution of the problem described equation (1) and equation (2).

For completeness we comment on how the $n = 1$ products and the case of $n > 1$ perfectly correlated target prices look like. In the case of one product, i.e. $n = 1$, the solution to V is easily seen to be the sum of a quadratic and of two exponentials

$$V(p) = \frac{B}{r} p^2 + \beta \left(\exp\left(\zeta p\right) + \exp\left(-\zeta p\right)\right) + B \left(\frac{\sigma}{r}\right)^2$$

where $\zeta = \sqrt{2r}/\sigma$ and the constant β is chosen to enforce smooth pasting and value matching. Moreover, it is easy to see that in that case $v(y) = V(\sqrt{y})$ solves the ODE in (10) and its boundary conditions. We note that the solution for the $n = 1$ case and the expression for the approximation for $\bar{y}$ are the same ones derived in Dixit (1991), which we explore in the price setting context in Alvarez, Lippi, and Paciello (2011). In the case of n *perfectly correlated target prices* the problem has a single state variable after the first adjustment. In this case, in terms of the threshold policy and value function, the problem is identical to the one with only one price. The static return is thus $nBp(t)^2$ where $p(t)$ is, when uncontrolled, a one dimensional brownian motion. The only difference with the problem with only one price is that the value of B is multiplied by n, or more importantly, the ratio B/ψ is proportional to n. This is quite natural, since the adjustment has the same effectiveness for all products, and hence it is as if it were cheaper. Note that, in the case of the CRTS assumption, the value of the adjustment threshold, and hence the frequency of adjustment, is independent of n. Instead, in terms of the implication for price changes, the problem with perfectly correlated shocks is quite differently, since there are *no* small price changes. When adjustment takes place, all products have the same price gap. We return to this simple case later on to speculate on the case of positive, but less than one correlation between the innovations.

We finish this section by characterizing the optimal policy $\bar{y}$ in terms of the structural parameters of the model $(\frac{\psi}{B}, \sigma^2, n, r)$.

Proposition 4. The optimal threshold is given by a function $\bar{y} = \frac{\sigma^2}{r} Q(\frac{\psi}{B}\frac{r^2}{\sigma^2}, n)$ so that

(i) $\bar{y}$ is strictly increasing in $\frac{\psi}{B}$ with $\bar{y} = 0$ if $\frac{\psi}{B} = 0$ and $\bar{y} \to \infty$ as $\frac{\psi}{B} \to \infty$,

(ii) $\bar{y}$ is strictly increasing in n and $\bar{y} \to \infty$ as $n \to \infty$,

(iii) $\bar{y}$ is bounded below by $\sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}$ and as $\frac{\psi}{B}\frac{r^2}{\sigma^2} \to 0$ then $\frac{\bar{y}}{\sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}} \to 1$,

(iv) as $\frac{\psi}{B}\frac{r^2}{\sigma^2} \to \infty$ then $\frac{\bar{y}}{\sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}} \to \infty$, and $\frac{\psi/B}{\bar{y}} \frac{\partial \bar{y}}{\partial \psi/B} > \frac{1}{2}$

(v) the elasticity of $\bar{y}$ with respect to r and σ^2 satisfy:

$$\frac{r}{\bar{y}}\frac{\partial \bar{y}}{\partial r} = 2\frac{(\psi/B)}{\bar{y}}\frac{\partial \bar{y}}{\partial(\psi/B)} - 1 \quad \text{and} \quad \frac{\sigma^2}{\bar{y}}\frac{\partial \bar{y}}{\partial \sigma^2} = 1 - \frac{(\psi/B)}{\bar{y}}\frac{\partial \bar{y}}{\partial(\psi/B)} .$$

See Appendix B for the proof. That $\bar{y}$ is only a function of the ratio ψ/B is apparent from the definition of the sequence problem. That, as stated in part (i), $\bar{y}$ is strictly increasing in the ratio of the fixed cost to the benefit of adjustment ψ/B is quite intuitive. Item (ii) says that threshold is increasing in the number of products n. This is because as n increases, equation (8) shows that the drift of $y = ||p||^2$ increases, thus if $\bar{y}$ would stay constant there will be more adjustments per unit of time, and hence higher menu cost will be paid. Additionally, if $\bar{y}$ remains unchanged, the average cost per unit of time also increases. One can show that the second effect is smaller, and hence an increase in n makes it optimal to increase $\bar{y}$. Part (iii) gives an expression for a lower bound for $\bar{y}$, which becomes arbitrary accurate for either a small value of the cost ψ/B, so that the range of inaction is small, or a small value of the interest rate r, so that the problem is equivalent to minimize the steady state average net cost. We note that in the approximation:

$$\bar{y} = \sqrt{\frac{\psi \ \sigma^2 \ 2 \ (n+2)}{B}} , \tag{16}$$

the effect of $\psi\sigma^2/B$ is exactly the same as in the case of one product. Indeed the quartic root (implied for the optimal threshold $\bar{p} = \sqrt{\bar{y}}$) is the one obtained by Dixit (1991) in a model with $n = 1$. Part (iv) shows that the approximation worsens, and that the elasticity of $\bar{y}$ with respect to ψ/B increases above $1/2$ as $\frac{\psi}{B}\frac{r^2}{\sigma^2}$ becomes large. Note that the approximation in part (iii) implies that the elasticity of $\bar{y}$ with respect to ψ/B is $1/2$ for small values of the ψ/B ratio. Then, using part (v), we obtain that $\bar{y}$ has elasticity $1/2$ with respect to σ^2 and also that it is independent of r. Despite the result in Part (iv), we found that the quadratic approximation to $v(\cdot)$, which amounts to a quartic approximation to $V(\cdot)$, gives very accurate values for $\bar{y}$ across a very large range of parameters, as documented in Appendix C. What happens is that for any realistic application the values of r and ψ are small relative to $B\sigma^2$, hence the approximation given in part (iii) applies.

4 Implications for timing and size of price changes

In this section we explore the implications for the frequency and distribution of price changes.

We let the expected time for $y(t)$ to hit the barrier $\bar{y}$ starting at y by the function $\mathcal{T}(y)$. This function satisfies:

$$0 = 1 + \ n\, \sigma^2\, \mathcal{T}'(y) + \ 2\, y\, \sigma^2\, \mathcal{T}''(y) \ \text{ for } y \in (0, \bar{y}) \quad \text{and} \quad \mathcal{T}(\bar{y}) = 0\ ,$$

where the first condition gives the law of motion inside the range of inaction and the second one imposes the terminal condition on the boundary of the range of inaction. The unique solution of this ODE that satisfies the relevant boundary condition is:

$$\mathcal{T}(y) = \frac{\bar{y} - y}{n\, \sigma^2} \ \text{ for } y \in [0, \bar{y}]\ . \tag{17}$$

We use $\mathcal{T}(0)$ as the expected time between successive price adjustments, and thus the average number of adjustment, denoted by N_a is given by $\frac{1}{\mathcal{T}(0)}$. We summarize this result in the following proposition:

Proposition 5. Let N_a be the expected number of price changes for a multi-product firm with n goods. It is given by

$$N_a = \frac{n\sigma^2}{\bar{y}} = \frac{nr}{Q\left(\frac{\psi r^2}{B\ \sigma^2},\ n\right)} \cong \sqrt{\frac{B\ \sigma^2}{2\ \psi}\frac{n^2}{(n+2)}}\ . \tag{18}$$

The second equality in equation (18) we use the function $Q(\cdot)$ derived in Proposition 4, while in the last equality we use the approximation of $\bar{y}$ for small $\psi r^2/(B\sigma^2)$ (see Appendix C for more documentation on the accuracy of the approximation). It is interesting that this expression extends the well known expression for the case of $n = 1$, simply by adjusting the value of the variance from σ^2 to $n\sigma^2$. The number of products n affects N_a through two opposing forces. One is that with more products, the variance of the deviations of the price gaps increases, and thus a given value of $\bar{y}$ is hit sooner in expected value, which we refer to as the direct effect. On the other hand, with more products, the optimal value of $\bar{y}$ is higher. Expression equation (18) shows that, as often happens in these models, the direct effect dominates, and the frequency of adjustment increases with n.

We use this expression to study how the bundling of menu costs, i.e. the fact that a single menu cost relates to several products, affects the frequency of adjustment of individual prices. This is interesting because recent evidence in Bhattarai and Schoenle (2010) shows that the frequency of price adjustment appears higher for firms that sell a larger number of goods.[8] They find that the average frequency of price adjustment increases.

This pattern is qualitatively consistent with the formula in equation (18), which shows that N_a is increasing in n. Notice however that in this comparison we are keeping ψ constant, so that as n increases the menu cost per good is decreasing. One may wonder whether the increased activity by the firms follows from the fact that the menu cost is smaller (per good) or because of the bundling of the goods prices. To separate the effects of the economies of

[8]See Figures 1 and 2 in their paper. These authors group firms into 4 bins, according to the number of items sold (and recorded by the BLS), from 1 to 3 goods in the first bin to more than 7 goods in the fourth bin. They first measure the frequency of price changes at the good level, then compute the median frequency across the goods produced in the firm. Finally, they average these medians inside each of the 4 bins.

Table 1: Frequency of Price changes N_a

	number of products n						implied
	1	2	4	6	10	50	ψ_1
N_a for C.R.T.S. model $\psi = n\ \psi_1$:	2	2.4	2.8	2.9	3.1	3.3	0.02
N_a for C.F.C. model: $\psi = \psi_1$:	1.4	2.4	3.9	5.1	6.9	17	0.04
N_a for US Data:	-	2.4	2.3	2.8	3.5	-	-

Value of $\frac{\psi_1}{B\sigma^2}$ chosen to match the size of price changes at $n = 2$. US data from Bhattarai and Schoenle (2010) Figure 1. Implied ψ_1 using $B = 20$ and $\sigma = 0.15$.

scale in the menu cost from the bundling of the goods, consider the case where the cost ψ grows linearly with the number of goods n, i.e. : $\psi = \psi_1\ n$. This gives

$$N_a \cong \sqrt{\frac{B\ \sigma^2}{2\ \psi_1}\frac{n}{(n+2)}}\ . \tag{19}$$

which is also increasing in n, although at a lower rate. Thus, even under "CRTS" for the menu cost, the bundling of the goods pricing induces more frequent adjustments than in the case where the menu costs are dissociated, i.e. when $n = 1$. We explain the economics behind this result for the CRTS case. Define N_1 as the optimal number of adjustments per year for a firm selling only 1 good, i.e. with $n = 1$. Consider a multi-product firm with $n > 1$ that follows a policy of doing N_1 adjustments per year. Since we are considering the CRTS case, the expected amount spent in adjustment per year is the same for both firms: $N_1\ \psi_1$. But the one good firm tailors all adjustments to those instances where the deviations of the optimal price are large. Instead, due to the fact that all prices are adjusted at the same time, the goods sold by the multiproduct firm will be adjusted sometimes when the price has small deviations, and other times when they have very large deviations. Since the profit function is concave in price, it is profitable for the multiproduct firm to increase the number of adjustment to decrease the per good expected price deviation from its optimal price relative to the firm with only one good. This is exactly what the expression in equation (19) shows for the optimal policy, since this expression is increasing in n. In the case with constant fixed cost the effect is even stronger, since in addition mechanically the cost of adjustment per good decreases. Table 1 uses equation (19) for the case of constant returns to scale (CRTS) and equation (18) for the case of constant fixed cost (CFC) to calculate hypothetical values of N_a for different values of n. For both cases we have selected the values of $B\,\sigma^2/\psi_1$ so that its value is 2.4 adjustments per year, the value estimated by Bhattarai and Schoenle (2010) for the US for firms with $n = 2$. Table 1 also includes a row with US data. Comparing the case of CRTS with the one with CFC, the former displays a pattern much closer to the one in the US data.

We now move to the study of the hazard rate of price adjustments.

Proposition 6. Let t denote the time elapsed since the last price change. Let $J_\nu(\cdot)$ be the Bessel function of the first kind. The hazard rate for price changes is given by

$$h(t) = \sum_{k=1}^{\infty} \frac{\xi_{n,k}}{\sum_{s=1}^{\infty} \xi_{n,s} \exp\left(-\frac{q_{n,s}^2\sigma^2}{2\bar{y}}\, t\right)} \frac{q_{n,k}^2\sigma^2}{2\bar{y}} \exp\left(-\frac{q_{n,k}^2\sigma^2}{2\bar{y}}\, t\right) \text{ , where } \nu = \frac{n}{2} - 1 \text{ ,}$$

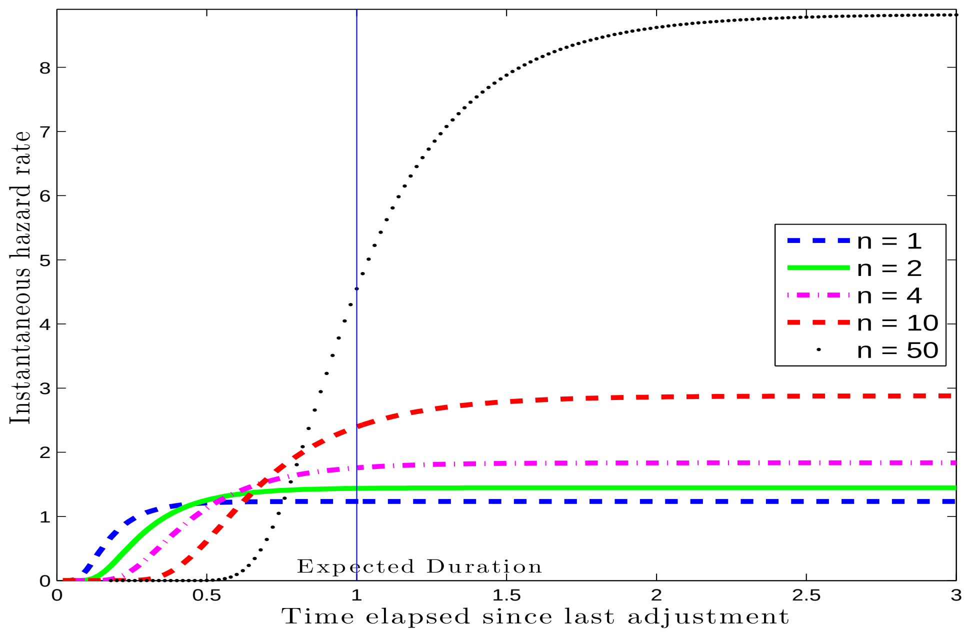
$$\xi_{n,k} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{n,k}^{\nu-1}}{J_{\nu+1}(q_{n,k})} \text{ , and } q_{n,k} \text{ are the positive zeros of } J_\nu(\cdot),$$

$$\text{which asymptotes to } \lim_{t\to\infty} \frac{h_n(t)}{\mathcal{T}(0)} = \frac{q_{n,1}^2}{2n} > \frac{(n-1)^2}{2n} \text{ .} \tag{20}$$

See Appendix B for the proof. In the proof we use results from probability theory on the first passage of time of a n brownian motion in a sphere center to the origin by Ciesielski and Taylor (1962) as well as characterization from the zeros of the Bessel function from Qu and Wong (1999) and Hethcote (1970).

Proposition 6 compares the asymptote of the hazard rate with the expected time until adjustment, which equals $\mathcal{T}(0) = \bar{y}/(n\sigma^2)$, as derived above. Notice that for a model with constant hazard rate these two quantities are the reciprocal of each other, i.e. the expected duration is the reciprocal of the hazard rate. We use this ratio, as a function of n as a measure of how close the model is to have constant hazard rates. We note that this ratio is exclusively a function of n. Indeed from the expression Proposition 6, it is immediate that the shape of the hazard rate function depends only on the number of products n. Changes in σ^2, B, ψ only stretch linearly the horizontal axis. More precisely, once keeping the expected time until adjustment $\mathcal{T}(0)$ fixed, the hazard rate is only a function of n.

Figure 1: Hazard rate of Price Adjustments for various choices of n



For each n the value of $\sigma^2/\bar{y}$ is chosen so that the expected time elapsed between adjustments is one.

Figure 1 plots the hazard rate function h for different choices of n keeping the expected time between price adjustment fixed at one. As Proposition 6 shows the function h has an asymptote, which is increasing in the number of products n. Moreover, since the asymptote diverges to ∞ as n increases with no bound, the hazard rate converges to a an inverted L shape, as the one for a model where adjustment are done exactly every $\mathcal{T}(0) = 1$ periods. To see this note that, defining $\tilde{y} \equiv y/\bar{y}$ and fixing the ratio $\sigma^2/\bar{y} = \mathcal{T}(0)/n$ so that for any n the expected time elapsed between price changes is $\mathcal{T}(0)$, we have:

$$\mathrm{d}\tilde{y} = \mathcal{T}(0)\mathrm{d}t + 2\sqrt{\tilde{y}\ \frac{\mathcal{T}(0)}{n}}\ \mathrm{d}W \text{ for } \tilde{y} \in [0,1]. \tag{21}$$

As $n \to \infty$ the process for the normalized size of the price gap $\tilde{y}$ described in equation (21) converges to the deterministic one, in which case the hazard rate is zero between times 0 and below $\mathcal{T}(0)$ and ∞ precisely at $\mathcal{T}(0)$. For completeness, Table 2 computes the first zero for the relevant Bessel functions and the asymptotic hazard rate for several value of n.

Table 2: Limit hazard rates for various values of n

	number of products n									
	1	2	3	4	6	8	10	20	50	100
zeroes of $J_{\frac{n}{2}-1}(\cdot) : q_{n1}$	1.6.	2.4	3.1	3.8	5.1	6.4	7.6	13	30	56
$\frac{\text{Limit Hazard rate}}{\text{Expected duration}} : \lim_{t\to\infty} \frac{h_n(t,\bar{y})}{\mathcal{T}(0)}$	1.2	1.4	1.6	1.8	2.2	2.5	2.9	4.5	8.8	16

Note: for $n = 1$ and $n = 3$ the zeros are multiples of π, i.e. $q_{1,k} = (2(k-1)+1)\pi/2$ and $q_{3,k} = k\ \pi$.

The shape of estimated hazard rates varies across studies, but many have found flat or decreasing ones, and some have found hump-shape ones. As can be seen from Figure 1 the hazard rate for the case of $n = 1$ is increasing but rapidly reaches its asymptote. As n is increased, the shape of the hazard rate becomes closer to the inverted L shape of its limit as $n \to \infty$. For instance, when $n = 10$ the level of the hazard rate evaluated at the expected duration is about twice as large as the one for $n = 2$. This is a prediction that can be tested using the data set in Bhattarai and Schoenle (2010).

Finally we discuss the distribution of price changes. This distribution is characterized by two parameters: the number of goods n, and the optimal boundary of the inaction set $\bar{y}$. The value of $\bar{y}$, as discussed above, depends on all the parameters. Since after an adjustment price gaps are reset to zero, price changes coincide with the value of $p(\tau) \in \partial\mathcal{I} \subset \mathbb{R}^n$, the surface of an n-dimensional sphere of radius $\sqrt{\bar{y}}$. Let τ, be a time where y hits the boundary of the range of inaction: then given that each of the (uncontrolled) $p_i(t)$ is independently and identically normally distributed, price changes $\Delta p(\tau) = -p(\tau)$ are uniformly distributed in the n-dimensional surface of the sphere of radius $\sqrt{\bar{y}}$.[9] The next proposition characterizes the marginal distribution of price changes.

Proposition 7. Let $\Delta p \in \partial \mathcal{I} \subset \mathbb{R}^n$ denote a price change for the n goods. The distribution of the price change of an individual good, i.e. the marginal distribution of $\Delta p_i \in [0, \ \sqrt{\bar{y}}\ ]$, has density:

$$w(\Delta p_i) \quad = \quad \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\ \sqrt{\bar{y}}}\ \left(1-\left(\frac{\Delta p_i}{\sqrt{\bar{y}}}\right)^2\right)^{(n-3)/2} \tag{22}$$

where $Beta(\cdot,\cdot)$ denotes the Beta function. The standard deviation and kurtosis of the price changes, and expected value of the absolute value of price changes and its coefficient of variations are given by:

$$\begin{aligned} Std\,(\ \Delta p_i\ ) \quad &= \quad \sqrt{\bar{y}/n}\ , \qquad Kurt\,(\ \Delta p_i\ ) = \frac{3\ n}{n+2}\ , \\ \mathbb{E}\,[\ |\Delta p_i|\ ] \quad &= \quad \frac{\sqrt{\bar{y}}}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\ ,\ \frac{Std\,(\ |\Delta p_i|\ )}{\mathbb{E}\,(\ |\Delta p_i|\ )} = \sqrt{\left[\frac{n-1}{2}Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\right]^2\frac{1}{n}-1}\ . \end{aligned}$$

Moreover, as $n \to \infty$, the distribution of $\Delta p_i/Std(\Delta p_i)$ converges point-wise to a standard normal.

The proof uses results from the characterization of spherical distributions by Song and Gupta (1997) (see Appendix B for the details). Using the previous proposition and the approximation for $\bar{y}$ we obtain the following expression for the standard deviation of price changes:

$$Std\,(\ \Delta p_i\ ) = \left(\frac{\sigma^2\psi}{B}\frac{2(n+2)}{n^2}\right)^{1/4} \quad \text{and in the CRTS case} \quad Std\,(\ \Delta p_i\ ) = \left(\frac{\sigma^2\psi_1}{B}\frac{2(n+2)}{n}\right)^{1/4},$$

where both expressions are decreasing in n. The expression for the kurtosis of the price

[9] The distribution of $\Delta p(\tau)$ is uniform in the surface of the sphere. To see this notice that the p.d.f. of a jointly normally distributed vector of n identical and independent normals is given by a constant times the exponential of the *square radius of the sphere*, divided by half of the common variance.

changes shows that this statistic is an increasing function of n.

We can approximate some of the expressions in Proposition 7 for statistics for $|\Delta p_i|$ involving the *Beta* function to obtain the following simpler expressions:[10]

$$\begin{aligned}
\mathbb{E}\left[\ |\Delta p_i|\ \right] &\approx \sqrt{\bar{y}/n}\ \sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{1.1}{2\ n}} = Std\left(\ \Delta p_i\ \right)\ \sqrt{\frac{2}{\pi}}\ \sqrt{1+\frac{1.1}{2\ n}} \quad \text{and} \\
\frac{Std\left(\ |\Delta p_i|\ \right)}{\mathbb{E}\left(\ |\Delta p_i|\ \right)} &\approx \sqrt{\frac{\pi}{2}\left(\frac{2n}{1.1+2n}\right)-1}\ .
\end{aligned}$$

The expression for the approximate value of $\mathbb{E}\left[|\Delta p_i|\right]$ is given by $Std\left(\Delta p_i\right)$ times a decreasing function of n. The expression for the approximate value of $Std\left(|\Delta p_i|\right)/\mathbb{E}\left(|\Delta p_i|\right)$ show that this statistic is an increasing function of n.

We note that the shape of the distribution h for price changes differs substantially for small values of n. For $n = 2$ is U-shaped, for $n = 3$ is uniform, for $n = 4$ it has the shape of a half circle, and for $n \geq 6$ it has bell shape.[11] Proposition 7 establishes that when $n \to \infty$ the distribution converges to a normal: this can be seen in Figure 2 by the comparison of the distribution for $n = 50$ and the p.d.f of a normal distribution with standard deviation equal to $Std(\Delta p_i)$ for $n = 50$.

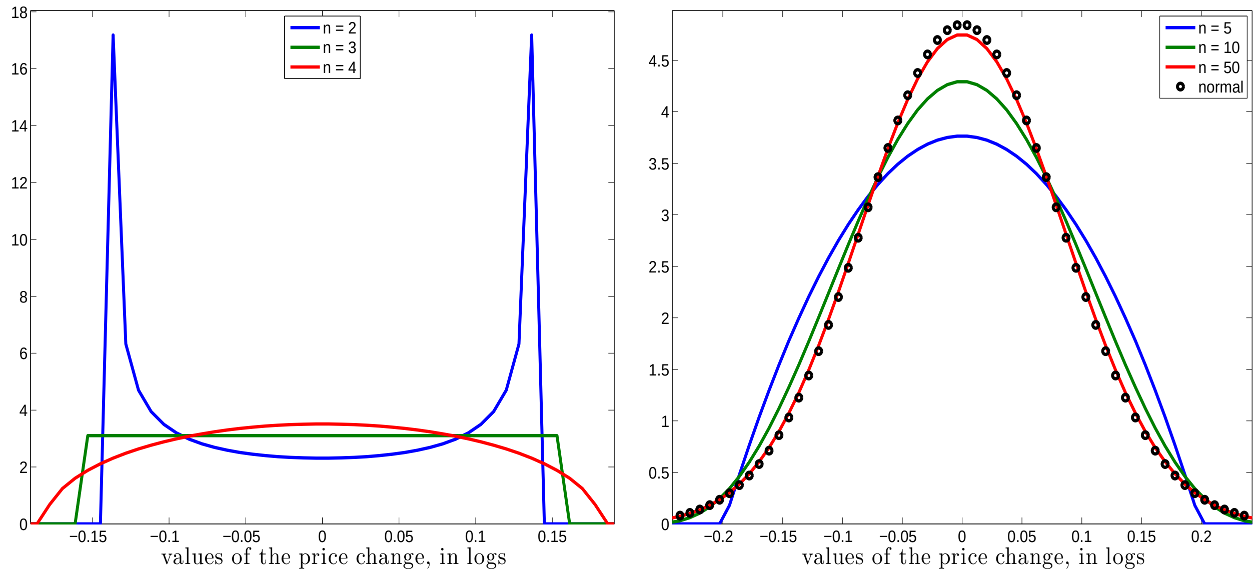
Table 3 computes the size of the price adjustments, measured as $E[|\Delta p|]$, as a function of n. We do so for the two extreme technologies, the constant returns to scale (CRTS) and the constant fixed cost (CFC) case. In each case we fix the value of the parameter $B\sigma^2/\psi_1$ so that this statistic is 0.085, the value estimated by Bhattarai and Schoenle (2010) in US data. We also report the values estimated for the US for other values of n. Comparing both assumptions, it seems that the US data is somewhere in the middle, but closer to the case of CRTS. The better fit of the CRTS case for the size of price changes is consistent with the better bit obtained for the frequency of price changes as reported in Table 1.

Furthermore, from the expressions in Proposition 7 the distribution of price changes Δp,

[10] We note that error on the approximation error for $\mathbb{E}\left[\ |\Delta p_i|\ \right]$ and $Std\left(\ |\Delta p_i|\ \right)/\mathbb{E}\left(\ |\Delta p_i|\ \right)$ are smaller than 0.26% and 0.91%.

[11] For n equal to 2, 3 and 4 one can grasp the shape of the distribution h from geometrical considerations, together with the fact the maximum of a density of a univariate normal is at one.

Figure 2: Density $w(\cdot)$ of the price changes for various choices of n



Parameter values: $B = 20$, $\sigma = 0.15$, $\psi_1 = 0.02$. Menu cost proportional to n. Solid lines are the p.d.f for w for different n. Circles denote the p.d.f. of a normal with standard deviation equal to that of Δp_i for $n = 50$.

Table 3: Size of Price changes $E[|\Delta p|]$

	number of products n						implied
	1	2	4	6	10	50	ψ_1
$E[\|\Delta p\|]$ for C.R.T.S. model $\psi = \ n\ \psi_1$:	10%	8.5%	7.5%	7.1%	6.8%	6.4%	0.03
$E[\|\Delta p\|]$ for C.F.C. model: $\psi = \psi_1$:	12%	8.5%	6.3%	5.4%	4.6%	2.9%	0.02
$E[\|\Delta p\|]$ for US Data:	-	8.5%	7.75%	6.75%	6.5%	-	-

Value of $\frac{\psi_1}{B\sigma^2}$ chosen to match the size of price changes at $n = 2$. US data from Bhattarai and Schoenle (2010), Figure 4. Implied ψ_1 using $B = 20$ and $\sigma = 0.15$.

and of their absolute value $|\Delta p|$ depend only on n *and* $\bar{y}$. Thus, any normalized statistics such as ratio of moments (kurtosis, skewness, etc) or a ratio of points in the c.d.f. depends exclusively on n. Indeed the kurtosis is given in Proposition 7, as $Kurtosis(\Delta p_i) = 3n/(2+n)$, which is an increasing concave function, starting at 1 and converging to 3. Table 4 uses the expressions of the model to compute several moments of interest. These moments have been estimated using two scanner data sets by Midrigan (2009) and also using BLS producer data by Bhattarai and Schoenle (2010). A summary of the selected statistics from these papers is reproduced in Table 5.

Table 4: Statistics for price changes as function of number of products, Model economy

statistics \ number of products n	1	2	3	4	5	6	10	20	50
$Std(\lvert\Delta p_i\rvert)$ / $E(\lvert\Delta p_i\rvert)$	0	0.48	0.58	0.62	0.65	0.67	0.70	0.74	0.75
Kurtosis(Δp_i)	1.0	1.7	2.0	1.9	2.1	2.3	2.5	2.8	2.9
Fraction: $\lvert\Delta p_i\rvert < \frac{1}{2}E(\lvert\Delta p_i\rvert)$	0	0.21	0.25	0.27	0.28	0.28	0.30	0.31	0.31
Fraction: $\lvert\Delta p_i\rvert < \frac{1}{4}E(\lvert\Delta p_i\rvert)$	0	0.10	0.12	0.13	0.14	0.14	0.15	0.16	0.16

Δp_i denotes the log of the price change, and $|\Delta p_i|$ the absolute value of the log of price changes. They are computed using the results in Proposition 7. All statistics in the table depend exclusively on n. Kurtosis defined as the fourth moment relative to the square of the second.

We briefly comment on the reasons why the statistics chosen in Table 4 with Table 5 are of interest. Note that the case of $n = 1$, price changes are binomial, either $-\sqrt{\bar{y}}$ or $+\sqrt{\bar{y}}$ with the same probability, so its absolute value has a degenerate distribution. As the number of goods increases the dispersion of the absolute value increases. The distribution includes larger price changes, so that its kurtosis also increases with n. As there are more goods, some goods will be adjusted even if their price is almost optimal, and hence the fraction of small price changes increases with n. We draw two conclusions from the comparison of Table 4 with Table 5. First, for the four moments computed our model falls short from the data. In particular, as shown in Proposition 7 the distribution in the model converged to a normal as n goes to ∞. Yet the data displays values for the four moments even larger than the ones corresponding to a standard normal. Second, our model reproduces the pattern of the four moments in terms of their variation with respect to the number of products n.

Table 5: Statistics for price changes as function of the number of products, US data

Statistics	*Bhattarai and Schoenle* Number of products n 2	4	6	10	*Midrigan* AC Nielsen All	No Sales	Dominick's All	No Sales
$Std(\lvert\Delta p_i\rvert)\ /\ E(\lvert\Delta p_i\rvert)$	1.02	1.15	1.30	1.55	0.68	0.72	0.84	0.81
Kurtosis(Δp_i)	5.5	7.0	11	17	3.0	3.6	4.1	4.5
Fraction: $\lvert\Delta p_i\rvert < \frac{1}{2}E(\lvert\Delta p_i\rvert)$	0.39	0.45	0.47	0.50	0.24	0.25	0.34	0.31
Fraction: $\lvert\Delta p_i\rvert < \frac{1}{4}E(\lvert\Delta p_i\rvert)$	0.27	0.32	0.35	0.38	0.10	0.10	0.17	0.14

Sources: For the Bhattarai and Schoenle (2010) data: the number of product n is the mean of the categories considered based on the information in Table 1, the ratio $Std(|\Delta p_i|)\ /\ E(|\Delta p_i|)$ is from Table 2 (Firm-Based), the fraction of $|\Delta p_i|$ which are small is from Table 14, the Kurtosis is from Figure 7. The data from Midrigan (2007) are taken from distribution of standardized prices in Table 2a.

5 Sensitivity to inflation

In this section we analyze the effect of inflation on the frequency of price adjustments and the size distribution of price changes under the assumption that the inflation rate, which we denote by μ, is small.

We model inflation as introducing a constant common drift on each of the n target prices $\{p_i^*(t)\}$. Equivalently, this means that each of the price gaps $\{p_i(t)\}$ has a *negative* drift μ, so equation (2) becomes

$$p_i(t) = -\mu t + \sigma W_i(t) + \sum_{j:\tau_j<t} \Delta p_i(\tau_j) \text{ for all } t \geq 0 \text{ and } i = 1, 2, ..., n. \tag{23}$$

We were not able to characterize the solution of the problem for arbitrary values of μ. Recall that in the case of $\mu = 0$ the time t conditional distribution of $||p(t + \Delta t)||^2$ at time $t + \Delta t$ depends only on $||p(t)||^2$. Thus, since in the objective function is proportional to $y(t) \equiv ||p(t)||^2$, the state of the problem can be taken to be the scalar $y(t)$, and hence the shape of the control and inaction regions are all functions of $y(t)$. In the case of $\mu \neq 0$ the time t conditional distribution of $||p(t+\Delta t)||^2$ at time $t+\Delta t$ depends on $||p(t)||^2$ as well as on $\mu\left[\sum_{i=1}^{n} p_i(t)\right]$. Thus the state of the problem will not be solely y, and hence the control and inaction sets will not be functions exclusively of $y = ||p||^2$. Also in the case of $\mu \neq 0$ it will not be the case that at a time τ where the firm adjust prices: $\Delta p_i(\tau) = -p_i(\tau)$. In other words, conditional on an adjustments, firms will not set the price gap equal to the static optimal value, since the state has a drift. Yet, even though we have not solved the model for positive inflation, the next proposition shows that for many statistics inflation has a second order effect.

For the next proposition we explicitly write μ as an argument of the value function $V(p,\mu)$, and of the statistics such as the frequency of price changes $N_a(\mu)$, the hazard rate of price changes $h(t,\mu)$, the moments of the distribution of price changes $\mathbb{E}[\Delta p_i, \mu]$, etc. We also define the density of the marginal distribution of the *absolute* value of price changes $\ell(|\Delta p_{-}i|, \mu)$ and the average value function: $\mathbb{E}[V](\mu)$, i.e. the expected value of the value function under the invariant distribution of the price gaps $g(p)$ as $\mathbb{E}[V](\mu) \equiv \int_{\mathbb{R}^n} V(p,\mu) g(p,\mu) dp$. We have:

Proposition 8. Assume that all the functions below are differentiable. Then

(i) $\frac{\partial}{\partial \mu} N_a(\mu)\big|_{\mu=0} = 0$, and $\frac{\partial}{\partial \mu} h(t,\mu)\big|_{\mu=0} = 0$ for all $t \geq 0$,

(ii) $\frac{\partial}{\partial \mu} \mathbb{E}\left[\Delta p_i, \mu\right]\big|_{\mu=0} = \frac{1}{N_a(0)} > 0$ and $\frac{\partial^2}{\partial \mu^2} \mathbb{E}\left[\Delta p_i, \mu\right]\big|_{\mu=0} = 0$,

(iii) $\frac{\partial}{\partial \mu} \mathbb{E}\left[\left(\Delta p_i - \mathbb{E}[\Delta p_i]\right)^{2k}, \mu\right]\Big|_{\mu=0} = 0$, for $k = 1, 2, ...,$

(iv) $\frac{\partial}{\partial \mu} \ell\left(|\Delta p_i|, \mu\right)\big|_{\mu=0} = 0$ for all $|\Delta p_i| < \sqrt{\bar{y}}$ and

(v) $\frac{\partial}{\partial \mu} \mathbb{E}[V](\mu)\big|_{\mu=0} = 0$.

Part (i) shows that the average number of adjustments per unit of time, $N_a(\mu)$, is insensitive to inflation at $\mu = 0$. Indeed, the whole hazard rate function of price adjustment, $h(t,\mu)$ is insensitive to inflation at $\mu = 0$. Part (ii) states that the expected value of price changes increases linearly with μ with slope $1/N_a(0)$, at least for small values of $\mu = 0$. This follows from (i) and from the identity: $\mu = n_a(\mu)\ \mathbb{E}[\Delta p_i, \mu]$, i.e. that the product of the average price change times the number of adjustments equals the inflation rate.

The result that the "intensive" margin of price adjustment is insensitive to inflation at $\mu = 0$ applies to the special case of models with only one product, i.e. $n = 1$, as it is illustrated in the numerical results reported in Figure 3 of Golosov and Lucas (2007), when $\sigma > 0$. The proof of each of these results, as well as of the other parts of this proposition, is based on the symmetry of the problem. For instance it is easy to see that given the symmetry of the objective function and the distribution of the BMs $\{W_i(t)\}$, then for all $p \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ we have: $V(p, \mu) = V(-p, -\mu)$ and that $p \in \mathcal{I}(\mu)$ if and only if $-p \in \mathcal{I}(-\mu)$, where $\mathcal{I}(\mu)$ is the control set, viewed as a correspondence of inflation. This implies that $N_a(\mu) = N_a(-\mu)$, as well as $h(t, \mu) = h(t, -\mu)$. Thus, if $N_a(\mu)$ is differentiable at $\mu = 0$, then it must be flat. We skip a proof of the symmetry and of the proposition, since it follows the same lines than the proof for the analogous results in the model with $n = 1$ but with observation and menu cost in Alvarez, Lippi, and Paciello (2011).[12]

The theoretical result about the insensitivity of N_a -and the associated linearity of $\mathbb{E}[\Delta p]$- is supported by the evidence in Gagnon (2009) who, among others, finds that when inflation is low (say below 10-15%), the frequency of price changes is almost unrelated to inflation, and that the average magnitude of price changes has a tight linear relationship with inflation.

To understand (iii) and (iv) it is useful to realize that for $\mu = 0$ the marginal distribution of price changes is symmetric around zero, a consequence of the symmetry of the loss function and of the distribution of the shocks. Part (iii) shows that all the even centered moments are approximately the same for zero and low inflation. Importantly, this includes the variance and the kurtosis which is one of the moments that researchers have focused in the analysis of the effect of multi-products firms. Yet we are pretty sure that inflation will have a first order effect on other aspects of the distribution of price changes such as skewness. Part (iv)

[12] The case of $\sigma = 0$, which corresponds to the model in Sheshinski and Weiss (1977), the "insensitivity result" does not hold, since while the function n_a is symmetric, it has a kink at $\mu = 0$. We conjecture, but have not proven, that as long as $\sigma > 0$, all these functions are differentiable at $\mu = 0$. The economics are clear: the effect of small inflation is swamped by idiosyncratic shocks when $\sigma > 0$.

shows that the whole distribution of the *absolute value* of price changes is approximately the same for low and zero inflation. Finally part (v) shows that inflation has only a second order effect on the expected value function. Equivalently, inflation causes a second order increase in the unconditional expectation of losses for the firm.

These results show that the expected losses of the firm as well as the frequency and several moments of the size distribution of price changes are insensitive to inflation at $\mu = 0$. Thus, the analysis of the problem in a low inflation environment is well approximated by studying the case of zero inflation.

6 On the impact effect of an aggregate monetary shock

In this section we study the aggregate impact effect on prices of an unexpected permanent monetary shock. Understanding this response is useful to quantify the real effects of monetary policy shocks in the presence of menu costs, and how this effect varies with the number of products sold by the firm.

We start with a unit mass of multi-products firms receiving independent shocks, distributed according to the invariant distribution of price gaps. We study the immediate effect of an aggregate shock on the fraction of firms that change their prices, denoted by Φ, as well as the average price change across the n products across *all* the firms in the economy, which we denote by Θ. The aggregate shock consists of decreasing each of the coordinates of the vector of price gaps by a constant $\delta > 0$, which we interpret as an increase in the marginal cost faced by all firms. Alternatively, the experiment consists of an unexpected aggregate shock that, permanently and uniformly across all firms and all products, increases the target prices by δ. We assume that firms use same the decision rules before and after the *shock*, i.e. the value of $\bar{y}$ that generates the invariant distribution is the same one that is used to compute the fraction of firms that adjust their prices, as well as the average price change conditional on adjustment.[13] In the background, we have in mind the mechanism of

[13]In this sense our experiment imitates the analysis of Caballero and Engel (2007), and ignores the general

a general equilibrium sticky price model, such as Danziger's (1999) or Golosov and Lucas's (2007), where a permanent increase in the (log of) money of size δ, increases permanently (the log of) nominal wages, and hence marginal cost, by δ, and where the impact effect on employment depends on how much the aggregate price level rises on impact. Our objective is to understand how the number of goods n affects the impact effect on the aggregate price level.

It may help to understand the problem to have a graphical depiction of the case where $n = 1$ which, abusing a bit the analogy, we refer to as the Golosov and Lucas (GL) case, and the multiproduct case with $n = 2$, which we refer to as the Midrigan-Lach-Tsiddon (MLT) case. In the GL case, the firms controls the price gap between two symmetric thresholds, $\pm\bar{p}$, and when the price gap hits either of them it returns it to zero. Hence, in the GL case the invariant distribution of price gaps is triangular: the density function has a maximum at the price gap $p = 0$ and decreases linearly on both sides to reach a value of zero at the thresholds $\bar{p}$ and $-\bar{p}$, since firms that reach the thresholds will adjust upon a further shock. An example of such a distribution is depicted by the solid line in the left panel Figure 3. A straightforward computation gives that the slope of this density is $\pm(1/\bar{p})^2$. Consider an aggregate shock that displaces the distribution by reducing all price gaps by δ. If the value of $\delta > 2\bar{p}$ then all the firms will adjust their price, so that $\Phi_1 = 1$, and after a simple calculation one can see that the aggregate price level is increased by δ. Instead, if the value of δ is smaller than $2\bar{p}$, only the firms with a sufficiently small price gap will adjust, these are the firms that end up with $p < -\bar{p}$. For a shock of size δ the mass of such firms is $\Phi_1 = (1/2)(\delta/\bar{p})^2$, which uses the slope of the density given above. Note that the magnitude of this fraction is proportional to the square of the shock, a feature that is due to the fact that there are a few firms close to the boundary of the inaction set. This case is depicted by the dotted line in the left panel of Figure 3. Firms that change prices "close the price gap" completely, so that price increase

equilibrium feedback on the decision rules during the transition. Alternatively, we are concentrating on the aggregation effects. In the Golosov and Lucas's (2007) setup this result is exact when the intertemporal elasticity of substitution of the composite aggregate consumption across time equals the elasticity of substitution between goods of different variety.

will be $\delta + \bar{p}$ for the firm that prior the shock had price gap $-\bar{p}$ and it will be equal to $\bar{p}$ for the firm with pre-shock price gap equal to $-\bar{p} + \delta$. Using the triangular distribution of price gaps we have that the average price increase among those that adjust prices equals $\bar{p} + \delta/3$. Let's denote by Θ_1 the impact effect on aggregate prices of a monetary shock of size δ, the product of the number of firms that adjust times the average adjustment among them. Note that in steady state the average size of price changes, as measured by the standard deviation of price changes $Std[\Delta p]$, is given by $\bar{p}$. Thus we can write

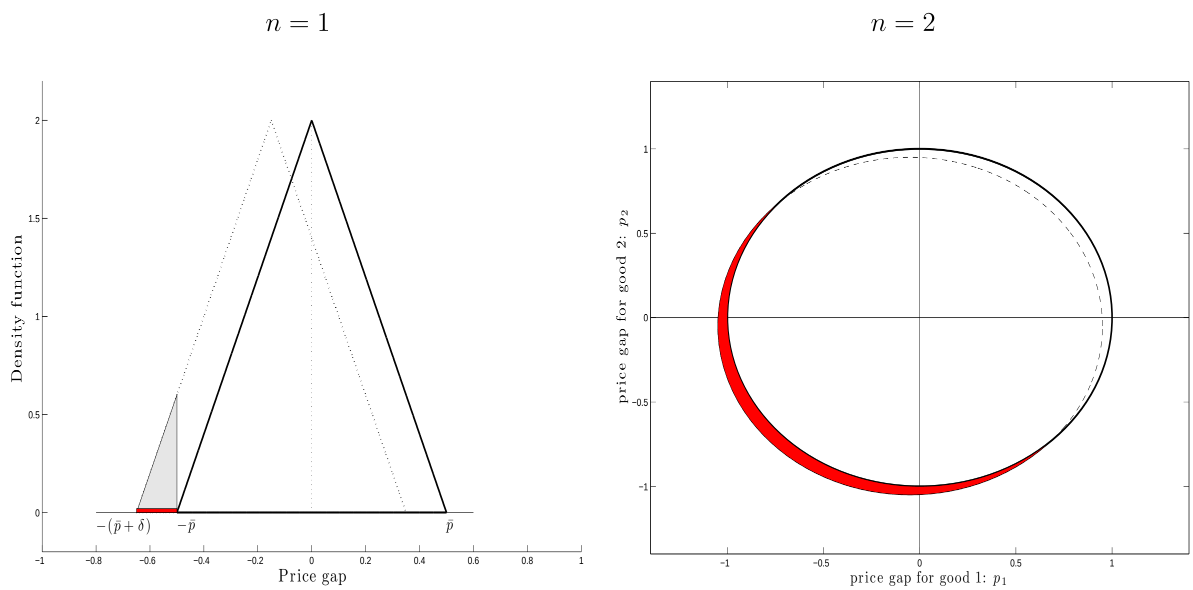
$$\frac{\Theta_1}{Std[\Delta p]} = \frac{1}{2}\left(\frac{\delta}{Std[\Delta p]}\right)^2 \left(1 + \frac{1}{3}\frac{\delta}{Std[\Delta p]}\right) \tag{24}$$

so that for an economy with one good, the impact effect on prices, normalized by the steady state average price change, depends on the normalized monetary shock, and it is locally quadratic, at least for a small enough shock. Note that the degree of aggregate stickyness is independent of the steady state fraction of price changes. In the next four propositions we will extend this analysis to the case of n products, and compare both the mass of firms that adjust, and the average price adjustment with the one for only one firm. We find that, when shocks are not too large, economies with firms that sell more goods and that are calibrated to the same standard deviation of price changes, have smaller impact effect of prices, i.e. they are stickier.

As a first step, and because of its independent interest, we study the invariant distribution of the sum of the squares of the price gaps $||p||^2 = \sum_{i=1}^{n} p_i^2(t)$ under the optimal policy. We will denote the density of the invariant distribution by $f(y)$ for $y \in [0, \bar{y}]$. This is interesting to study the response of firms that are in the steady state to an unexpected shock to their target that displaces the price gaps uniformly. The density of the invariant distribution for y is found by solving the corresponding forward Kolmogorov equation, and the relevant boundary conditions (see Appendix B for the proof).

Proposition 9. The density $f(\cdot)$ of the invariant distribution of the sum of the squares

Figure 3: The selection effect for the $n = 1$ and $n = 2$ case



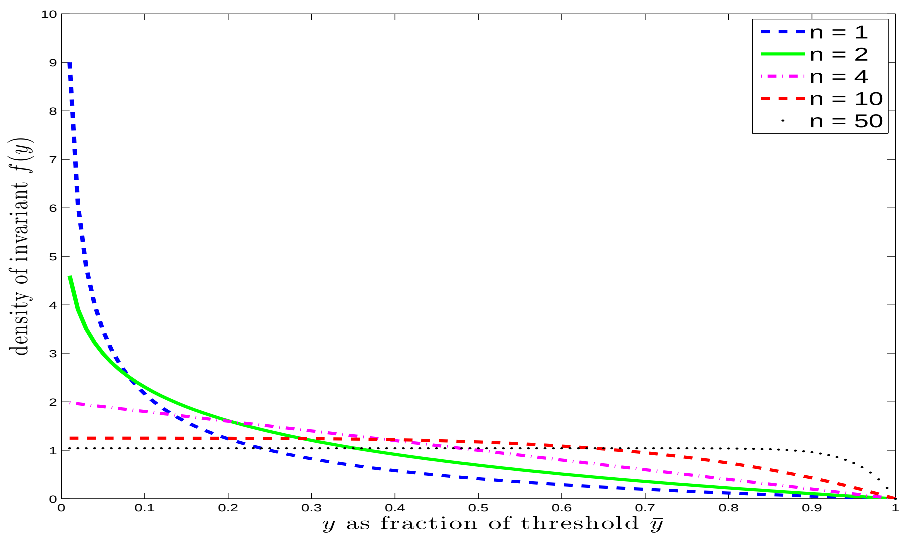
of the price gaps y, for a given thresholds $\bar{y}$ in the case of $n \geq 1$ products is for all $y \in [0, \bar{y}]$

$$
\begin{aligned}
f(y) &= \frac{1}{\bar{y}} \left[\log(\bar{y}) - \log(y)\right] \text{ if } \; n = 2, \text{ and} \\
f(y) &= (\bar{y})^{-\frac{n}{2}} \left(\frac{n}{n-2}\right) \left[(\bar{y})^{\frac{n}{2}-1} - (y)^{\frac{n}{2}-1}\right] \text{ otherwise.}
\end{aligned} \tag{25}
$$

The density has a peak at $y = 0$, decreases in y, and reaches zero at $\bar{y}$. The shape depends on n. The density is convex in y for $n = 1, 2, 3$, linear for $n = 4$, and concave for $n \geq 5$. This is intuitive, since the drift of the process for y increases linearly with n, hence the mass accumulates closer to the upper bound $\bar{y}$ as n increases. Indeed as $n \to \infty$ the distribution converges to a uniform in $[0, \bar{y}]$. Proposition 9 makes clear also that the shape of the invariant density depends exclusively on n, the value of the other parameters, ψ, B, σ^2 only enters in determining $\bar{y}$, which only stretches the horizontal axis proportionally.

Now we turn to studying the economy-wide effect of the aggregate shock. To find out what is the fraction of firms that will adjust prices under the invariant we need to characterize some features of the invariant distribution of $p \in \mathbb{R}^n$. We assume that the aggregate shock

Figure 4: $f(\cdot)$ density of invariant distribution of y, for various choices of n



happens once and for all, so that the price gap process remains the same and the firms solve the problem stated above. First we find out which firms will choose to change prices and, averaging among their n products, by how much. A firm with price gap $p \in \mathbb{R}^n$ and state $||p||^2 = y \leq \bar{y}$ before the shock, will have its price gaps displaced down by δ in each of its n goods, i.e. its state immediately after the shock is $||p - 1_n \delta||$, where 1_n is a vector of ones. This firm will change its prices if and only if the state will fall outside the range of inaction, i.e. $||p - 1_n\delta|| \geq \bar{y}$, or equivalently if and only if:

$$||p||^2 - 2\delta \left(\sum_{i=1}^{n} p_i \right) + n\delta^2 \geq \bar{y} \quad \text{or} \quad \frac{\sum_{i=1}^{n} p_i}{\sqrt{y}} \leq \nu(y, \delta) \equiv \frac{y - \bar{y}}{2\delta\sqrt{y}} + n\frac{\delta}{2\sqrt{y}} \; . \tag{26}$$

Thus $\nu(y, \delta)$ gives the highest value for the sum of the n price gaps for which a firm with state y will adjust the price. The normalized sum of price gaps $\sum_{i=1}^{n} p_i/\sqrt{y}$ takes values on $[-\sqrt{n}, \sqrt{n}]$. The right panel of Figure 3 shows the case of two goods by plotting a sphere centered at zero that contains all the pre-shock price gap, and showing the "displaced" price gaps right after the δ shock, which are given by a sphere centered at $(-\delta, \delta)$. The (red) shaded area contains all the price gaps of the firms that, after the shock, will find it optimal to adjust their prices, i.e. firms for which equation (26) holds.

A firm whose price gap p satisfies equation (26), i.e. one with $(1/\sqrt{y})\sum_{i=1}^{n} p_i \leq \nu(y,\delta)$, will change all its prices. The mean price change, averaging across its n products, is $\delta - (1/n)\sum_{i=1}^{n} p_i$.[14] Thus we can determine the fraction of firms that change its prices, and the amount by which they change them, analyzing the invariant distribution of the squared price gaps, $f(y)$. Let $S(z)$ denote the cumulative distribution function of the sum of the coordinates of the vectors distributed uniformly in the n dimensional unit sphere. Formally we define $S : \mathbb{R} \to [0,1]$ as

$$S(z) = \frac{1}{L(\mathbb{S}^n)} \int_{x \in \mathbb{R}^n, ||x||=1} \mathbf{I}\left\{x_1 + x_2 + ... + x_n \leq z\right\} L(dx) \ .$$

where $\mathbb{S}^n$ is the $n-$dimensional sphere and where L denotes its $n-1$ Lebesgue measure. Note that $S(\cdot)$ is weakly increasing, that $0 = S\left(-\sqrt{n}\right)$, $S(0) = 1/2$, $S\left(\sqrt{n}\right) = 1$ and that it is strictly increasing for $z \in (-\sqrt{n}, \sqrt{n})$. Remarkably, the distribution of the sum of the coordinates of a uniform random variable in the unit n-dimensional sphere is the same, up to a scale, than the marginal distribution of any of the coordinates of a uniform random variable in the unit n-dimensional sphere (which we discussed in Proposition 7), i.e.:

$$S'(z) \equiv s(z) = \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)\ \sqrt{n}} \ \left(1 - \left(\frac{z}{\sqrt{n}}\right)^2\right)^{(n-3)/2} \quad \text{for} \ \ z \in \left(-\sqrt{n}, \sqrt{n}\right) \qquad (27)$$

for $n \geq 2$, and for $n = 1$ the c.d.f. S has two points with mass $1/2$ at -1 and at $+1$. This result is shown in the proof of Proposition 10. Now we are ready to give expressions for the effect of an aggregate shock δ. First consider Φ_n, the fraction of firms that adjust prices. There are $f(y)dy$ firms with state y in the invariant distribution; among them the fraction $S(\nu(y,\delta))$ adjust. Integrating across all the values of y we obtain the desired expression. Second,

[14]Recall that p_i are the price gaps, thus in order to set them to zero the price changes must take the opposite sign. Moreover, since δ has the interpretation of a cost increase, it decreases the price gap, and hence it correction requires a price increase.

consider Θ_n, the change in the price level across all firms. There are $f(y)dy$ firms with state y in the invariant distribution; among them we consider all the firms with normalized sum of price gaps less than $\nu(y,\delta)$, for which the fraction $s(z)dz$ adjust prices by $\delta - \sqrt{y}\ z/n$. Considering all the values of y we obtain the relevant expression. This gives:

PROPOSITION 10. Consider an aggregate shock of size δ. The fraction of price changes on impact, Φ_n, and the average price change across the n goods among all the firms in the economy, Θ_n, are given by:

$$\Phi_n(\delta,\bar{y}) = \int_0^{\bar{y}} f(y)\ S\left(\nu(y,\delta)\right)\ dy \tag{28}$$

$$\Theta_n(\delta,\bar{y}) = \delta\Phi_n(\delta,\bar{y}) - \int_0^{\bar{y}} f(y)\frac{\sqrt{y}}{n}\left[\int_{-\sqrt{n}}^{\nu(y,\delta)} z\ s(z)dz\right] dy \tag{29}$$

where $s(\cdot)$ is given by equation (27) which depends on n, and where $f(\cdot)$ and $\nu(\cdot)$, which are also functions of $\bar{y}$ and n, are given in equation (25) and equation (26) respectively.

See Appendix B for the proof. The expression in equation (28) is readily evaluated by either numerical integration, or using that $S(z)$ is proportional to the hypergeometric function ${}_2F_1(\cdot)$. Likewise, equation (29) is easy to evaluate, since it has the following closed form solution:

$$S(z) = \frac{{}_2F_1\left(\frac{1}{2},\frac{3-n}{2},\frac{3}{2},\frac{z^2}{n}\right)}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\ \sqrt{n}} \quad \text{and} \quad \int_{-\sqrt{n}}^{\nu} z\ s(z)dz = -\frac{n\left(1-\frac{\nu^2}{n}\right)^{(n-1)/2}}{(n-1)Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\ \sqrt{n}}\ .$$

The next proposition defines some properties of the functions Θ_n and Φ_n that will allow us to characterize the impact effect of an aggregate shock (see Appendix B for the proof).

PROPOSITION 11. Let Φ_n and Θ_n be the fraction of firms that change prices and the average price change across the n goods after a monetary shock of size δ. The functions Φ_n and Θ_n are both weakly increasing in δ; the function Φ_n is decreasing in $\bar{y}$ and, for small δ, Θ_n is decreasing in $\bar{y}$. The function Θ_n is homogenous of degree one in $(\delta,\sqrt{\bar{y}})$. For large shocks we have that prices are flexible, i.e. for all n:

$$\Phi_n(\delta, \bar{y}) \;=\; 1 \text{ and } \Theta_n(\delta,\bar{y}) = \delta \text{ if } \delta \geq 2\sqrt{\bar{y}/n} \text{ and } \bar{y} > 0 \, . \tag{30}$$

Let $\bar{y}_n$ be the (approximate) expression for the optimal threshold of a firm with n goods described in equation (16) for the parameters $(\psi_n, B_n, \sigma_n^2)$. For small shocks we have that:

$$\frac{\partial \Phi_n(0,\bar{y}_n)}{\partial \delta} \;=\; \frac{\partial \Theta_n(0,\bar{y}_n)}{\partial \delta} = 0 \, ,$$

$$\lim_{\delta \downarrow 0} \frac{\Phi_n(\delta,\bar{y}_n)}{\Phi_1(\delta,\bar{y}_1)} \;=\; \mathcal{F}_n \cdot \left[\left(\frac{\psi_1 \sigma_1^2}{B_1} \right) / \left(\frac{\psi_n \sigma_n^2}{B_n} \right) \right]^{1/2} \text{ and} \tag{31}$$

$$\lim_{\delta \downarrow 0} \frac{\Theta_n(\delta,\bar{y}_n)}{\Theta_1(\delta,\bar{y}_1)} \;=\; \mathcal{Q}_n \cdot \left[\left(\frac{\psi_1 \sigma_1^2}{B_1} \right) / \left(\frac{\psi_n \sigma_n^2}{B_n} \right) \right]^{1/4} \, , \tag{32}$$

where the functions $\mathcal{F}_n$ and $\mathcal{Q}_n$ depend *only* on n.

As a benchmark, note that in a flexible price economy all firms change prices, i.e. $\Phi(\delta) = 1$ and hence the average price change equals the monetary impulse, i.e. $\Theta(\delta) = \delta$ for all δ. Instead, this proposition say that the effect of a small monetary shock are second order, i.e. $\phi(\delta) = \phi(0) + \phi'(0)\delta + o(\delta) = o(\delta)$. This is because the invariant distribution has no mass of firms at the threshold value for $\bar{y}$, the value at which they adjust, for any number of products $n \geq 1$. On the other hand, for very large shocks, i.e. for $\delta \geq 2\sqrt{\bar{y}/n}$ all the firms adjust and all firms prices change, in average by δ. For a small shock, we are interested in analyzing the fraction of firms that change prices as well as the total change prices relative to the case of $n = 1$.

The next proposition analysis the determinants of the impulse response. To aid in its measurement, it seeks to relate them to steady state statistics of the same economy. It uses the homogeneity shown in Proposition 11 to scale the effect on aggregate prices, as well as the size of the monetary shock δ, by the steady state standard deviation of price changes $Std[\Delta p]$.

Proposition 12. Let $n \geq 1$ be the number of products and $Std[\Delta p]$ be the steady state standard deviation of price changes. The normalized impact effect on aggregate prices of a normalized monetary shock, $\Theta_n\left(\frac{\delta}{Std[\Delta p]}\right) \; / \; Std[\Delta p]$ depends only on n. Keeping fixed the average size of price changes, the remaining part of the impulse response stretches out inversely proportionally to the steady state number of price adjustments N_a

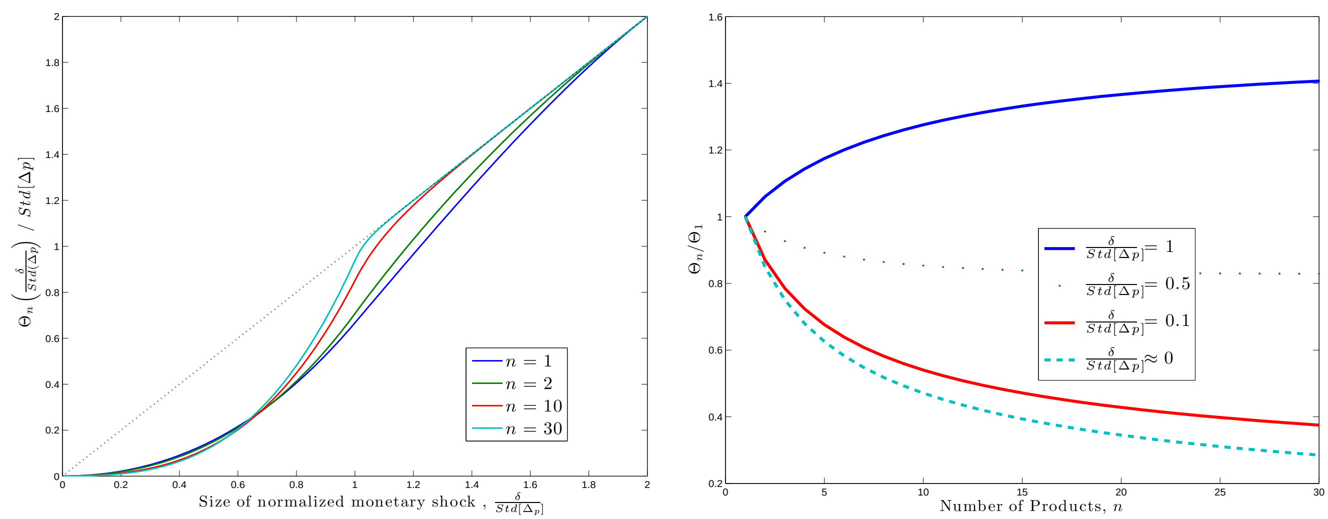
Proposition 12 implies that the impact effect of a monetary shock on the aggregate price level, as measured by Θ_n, is the same for any two economies with the same steady state average size of price changes, measured by either $Std[\Delta p]$ or $\mathbb{E}|\Delta p|$. Interestingly, this means that economies with the same value of $Std[\Delta p]$ but different steady state frequency of price changes, as measured by N_a, have the same impact effect of monetary shocks in the aggregate price level. Yet this proposition also implies that, keeping fixed $Std[\Delta p]$, the half life of the impulse response of the aggregate price level is proportional to $1/N_a$. The proposition makes explicit that no other parameters, different from n, enters into the computation of this function, so plotting it answer the question of whether economies with multi-product firms with higher n are more sticky or not.

Figure 5 has two panels aim to understand the impact effect on prices of monetary shocks of different size for economies with different values of n, illustrating the results of Proposition 11 and Proposition 12. The left panel of Figure 5 shows in the vertical axis the normalized impact effect on the aggregate price level, $\Theta_n/Std[\Delta p]$, of a normalized monetary shock, i.e. a shock $\delta/Std[\Delta p]$, in the horizontal axis. Each line plotted in this panel corresponds to a different choice of the number of products n. Recall that, if $\Theta_n(\delta) = \delta$ the shock is neutral, and that instead when $\Theta_n(\delta) < \delta$ the shock implies an increase in real output. As Proposition 11 states, this figure shows that, if $\delta \geq 2\sqrt{\bar{y}/n} = 2Std[\Delta p]$, then all firms adjust prices, and hence the shock is neutral. This last fact also explain the range of the normalized shock, between 0 and 2. For the quantification of this figure it is helpful to notice that a typical estimate of the standard deviation of price changes for US or European countries is 10% or higher, i.e. $Std[\Delta p] \approx 0.1$. This figure also shows that for small δ, as anticipated in

Figure 5: The impact effect of an aggregate shock on the price level

Effect on the price level

Effect relative to the $n = 1$ case



Normalized impact response of the aggregate price level to a permanent shock in the level of money of size $\delta/Std[\Delta p]$. The normalization in the left panel consists on dividing the change in the aggregate price level by $Std[\Delta p]$, the steady state standard deviation of price changes. Note that for the US, $Std[\Delta p] \approx 0.1$. See the text for more details).

Proposition 11 and in our analysis of the $n = 1$ case, the aggregate price effects are of order δ^2, i.e. the impact responses are approximately quadratic. Interestingly, the impact response of a monetary shock change order with respect to n as the value of δ increases, as can be seen for shocks smaller or larger than $\delta/Std[\Delta p] \approx 0.7$. Note that using $Std[\Delta p] = 01$ this means that shock for which they reverse order larger than 7%, a very large value. The right panel of Figure 5 displays four lines, each line is drawn for a different value of δ, and plots the ratio of the effects on the price level of an economy with n goods relative to one with one good. We include the limit case of $\delta \downarrow 0$, where, as shown in the case above, the results are independent of the value of $\bar{y}$. Each of these lines has the aggregate effect on prices of the corresponding shock for the number of products indicated in the horizontal axis, relative to the case of $n = 1$. From these two panels it can be seen that, as long as the monetary shocks are not very large, i.e. for increases in money $\delta/Std[\Delta p]$ smaller than a 0.5 (or for the benchmark value, for δ smaller than 5%), economies with more products are more sticky than those with fewer. Instead for very large shocks, economies with fewer products are more sticky than those with many products.

References

Aliev, A.F. 2007. "On the smooth-fit principle in $\mathbb{R}^n$." *Russian Mathematical Surveys* 62 (4):781–783.

Alvarez, Fernando E., Francesco Lippi, and Luigi Paciello. 2011. "Optimal price setting with observation and menu costs." *The Quarterly Journal of Economics, forthcoming* .

Baccarin, Stefano. 2009. "Optimal impulse control for a multidimensional cash management system with generalized cost functions." *European Journal of Operational Research* 196 (1):198–206.

Baudry, L., H. Le Bihan, P. Sevestre, and S. Tarrieu. 2007. "What do Thirteen Million Price Records have to Say about Consumer Price Rigidity?" *Oxford Bulletin of Economics and Statistics* 69 (2):139–183.

Bhattarai, Saroj and Raphael Schoenle. 2010. "Multiproduct Firms and Price-Setting: Theory and Evidence from U.S. Producer Prices." Working papers, Princeton University.

Bils, Mark and Peter J. Klenow. 2004. "Some Evidence on the Importance of Sticky Prices." *Journal of Political Economy* 112 (5):947–985.

Caballero, Ricardo J. and Eduardo M.R.A. Engel. 2007. "Price stickiness in Ss models: New interpretations of old results." *Journal of Monetary Economics* 54 (Supplement):100–121.

Cavallo, Alberto. 2010. "Scraped Data and Sticky Prices." Tech. rep., MIT Sloan.

Cavallo, Alberto and Roberto Rigobon. 2010. "The distribution of the Size of Price Changes." Working papers, MIT.

Ciesielski, Z. and SJ Taylor. 1962. "First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path." *Transactions of the American Mathematical Society* 103 (3):434–450.

Danziger, Leif. 1999. "A Dynamic Economy with Costly Price Adjustments." *American Economic Review* 89 (4):878–901.

Dhyne, Emmanuel and Jerzy Konieczny. 2007. "Temporal Distribution of Price Changes: Staggering in the Large and Synchronization in the Small." Working Paper Series 01-07, Rimini Centre for Economic Analysis.

Dixit, Avinash. 1991. "Analytical Approximations in Models of Hysteresis." *Review of Economic Studies* 58 (1):141–51.

Dutta, Shantanu, Mark Bergen, Daniel Levy, and Robert Venable. 1999. "Menu Costs, posted prices and multiproduct retailers." *The Journal of Money, Credit and Banking* 31 (4):683–703.

Gagnon, E. 2009. "Price Setting During Low and High Inflation: Evidence from Mexico." *Quarterly Journal of Economics* 124 (3):1221–1263.

Golosov, Mikhail and Robert E. Jr. Lucas. 2007. “Menu Costs and Phillips Curves.” *Journal of Political Economy* 115:171–199.

Hethcote, H.W. 1970. “Bounds for zeros of some special functions.” *Proceedings of the American Mathematical Society* :72–74.

Karatzas, I. and S.E. Shreve. 1991. *Brownian motion and stochastic calculus*. Springer-Verlag.

Karlin, S. and H.M. Taylor. 1999. *A second course in stochastic processes*. Academic press.

Khokhlov, VI. 2006. “The Uniform Distribution on a Sphere in R^S. Properties of Projections. I.” *Theory of Probability and its Applications* 50:386.

Lach, S. and D. Tsiddon. 2007. “Small price changes and menu costs.” *Managerial and Decision Economics* 28 (7):649–656.

Lach, Saul and Daniel Tsiddon. 1992. “The Behavior of Prices and Inflation: An Empirical Analysis of Disaggregated Price Data.” *The Journal of Political Economy* 100 (2):349–389.

———. 1996. “Staggering and Synchronization in Price-Setting: Evidence from Multiproduct Firms.” *The American Economic Review* 86 (5):1175–1196.

Midrigan, Virgiliu. 2007. “Menu Costs, Multi-Product Firms, and Aggregate Fluctuations.” CFS Working Paper Series 2007/13, Center for Financial Studies.

———. 2009. “Menu Costs, Multi-Product Firms, and Aggregate Fluctuations.” Working paper, NYU.

Nakamura, Emi and Jon Steinsson. 2008. “Five Facts about Prices: A Reevaluation of Menu Cost Models.” *The Quarterly Journal of Economics* 123 (4):1415–1464.

Neiman, Brent. 2010. “Stickiness, synchronization, and passthrough in intrafirm trade prices.” *Journal of Monetary Economics* 57 (3):295–308.

Øksendal, B.K. 2000. *Stochastic differential equations: an introduction with applications*. Springer Verlag.

Qu, CK and R. Wong. 1999. “Best Possible Upper and Lower Bounds for the Zeros of the Bessel Function J ν (x).” *Transactions of the American Mathematical Society* 351 (7):2833–2859.

Sheshinski, Eytan and Yoram Weiss. 1977. “Inflation and Costs of Price Adjustment.” *Review of Economic Studies* 44 (2):287–303.

Song, D. and A. K. Gupta. 1997. “Lp-Norm Uniform Distribution.” *Proceedings of the American Mathematical Society* 125 (2):595–601.

A Approximating the Profit Function

Consider the expression for the rate of profits of a multiproduct firm. The marginal cost for each of the products is C_i. The demand system is given by the sum of n independent demands, with own price elasticity given by η. The parameter A_i is the intercept, in logs, of the demand for the $i-th$ product. Given the constant elasticity of demand and constant marginal cost, the frictionless optimal price for the monopolist is a multiple of the marginal cost, and independent of A_i. To keep the n goods symmetric we assume that C_i and A_i are perfectly correlated, so that when the cost is high, and hence the frictionless prices is high, demand is also high. In this way we can keep the share of profits coming from each of the n goods comparable, even if the costs differ. We write the total profits per product

$$\Theta(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n) \equiv \sum_{i=1}^{n} \Theta(P_i,C_i,A_i) = \sum_{i=1}^{n} A_i \; P_i^{-\eta} \left(P_i - C_i\right)$$

Let $P_i^* = \arg\max_P \Theta(P, C_i, A_i)$. Assuming that

$$A_i = A \; (C_i)^{\eta-1} \; ,$$

we obtain that profits, relative to the maximized profits, can be written as

$$\begin{aligned}
&\frac{\Theta(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n) - \Theta(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)}{\Theta(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \\
&= B \sum_{i=1}^{n} \left(\frac{P_i - P_i^*}{P_i^*}\right)^2 + o\left(\sum_{i=1}^{n}\left(\frac{P_i - P_i^*}{P_i^*}\right)^2\right)
\end{aligned}$$

where $B = \frac{(\eta-1)\eta}{2\;n}$.

To obtain the quadratic expression above we write a second order expansion of the profits, divide both sides by the maximized total profits, and complete elasticities:

$$\begin{aligned}
&\frac{\Theta(P_1,..,P_n,C_1,...,C_n,A_1,...,A_n)}{\Theta(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \\
&= 1 + \sum_{i=1}^{n} \frac{1}{\Theta(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \left.\frac{\partial}{\partial P_i}\Theta\left(P_i,C_i,A_i\right)\right|_{P_i^*} P_i^* \left(\frac{P_i - P_i^*}{P_i^*}\right) \\
&+ \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\Theta(P_1^*,..,P_n^*,C_1,...,C_n,A_1,...,A_n)} \left.\frac{\partial^2}{\partial P_i^2}\Theta\left(P_i,C_i\right)\right|_{P_i^*} \left(P_i^*\right)^2 \left(\frac{P_i - P_i^*}{P_i^*}\right)^2
\end{aligned}$$

Computing the derivatives for our functional forms:

$$\begin{aligned}
\frac{\partial}{\partial P_i}\Theta\left(P_i,C_i,A_i\right) &= A_i P^{-\eta}\left(-\eta\left(\frac{P_i - C_i}{P_i}\right) + 1\right) \\
\frac{\partial^2}{\partial P_i^2}\Theta\left(P_i,C_i\right) &= -A_i P^{-\eta}\eta\frac{1}{P_i}\left(-\eta\left(\frac{P_i - C_i}{P_i}\right) + 1\right) - A_i P^{-\eta}\eta\left(\frac{C_i}{P_i^2}\right)
\end{aligned}$$

We have the standard result of a constant mark-up $P_i^* = \frac{\eta}{\eta-1} C_i$ and the maximized value of profits given by

$$\Theta(P_i^*, C_i, A_i) = A_i C_i^{-\eta} \left(\frac{\eta}{\eta-1}\right)^{-\eta} C_i \left(\frac{1}{\eta-1}\right) = A_i \; C_i^{1-\eta} \left(\frac{\eta}{\eta-1}\right)^{-\eta} \left(\frac{1}{\eta-1}\right) \; .$$

Hence the first and second derivatives, evaluated a the optimal prices are:

$$\begin{aligned}
\left.\frac{\partial}{\partial P_i} \Theta\left(P_i, C_i, A_i\right)\right|_{P^*} &= 0 \\
\left.\frac{\partial^2}{\partial P_i^2} \Theta\left(P_i, C_i, A_i\right)\right|_{P^*} &= -A_i P^{*-\eta} \eta \, \frac{C_i}{P_i^{*2}} = -A_i \left(C_i \frac{\eta}{\eta-1}\right)^{-\eta} \frac{\eta C_i}{P_i^{*2}}
\end{aligned}$$

and

$$\frac{1}{\Theta\left(P_i^*, C_i, A_i\right)} \left.\frac{\partial^2}{\partial P_i^2} \Theta\left(P_i, C_i, A_i\right)\right|_{P^*} \left(P_i^*\right)^2 = -\frac{A_i \left(C_i \frac{\eta}{\eta-1}\right)^{-\eta} \; \eta C_i}{A_i \; C_i^{1-\eta} \left(\frac{\eta}{\eta-1}\right)^{-\eta} \left(\frac{1}{\eta-1}\right)} = -\left(\eta-1\right)\eta$$

Thus the expansion can be written as:

$$\begin{aligned}
&\frac{\Theta(P_1, .., P_n, C_1, ..., C_n, A_1, ..., A_n)}{\Theta(P_1^*, .., P_n^*, C_1, ..., C_n, A_1, ..., A_n)} \\
&= 1 + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\Theta(P_1^*, .., P_n^*, C_1, ..., C_n, A_1, ..., A_n)} \left.\frac{\partial^2}{\partial P_i^2} \Theta\left(P_i, C_i\right)\right|_{P_i^*} \left(P_i^*\right)^2 \left(\frac{P_i - P_i^*}{P_i^*}\right)^2 \\
&= 1 - \frac{1}{2} \sum_{i=1}^{n} \frac{\Theta(P_i^*, C_i, A_i)}{\Theta(P_1^*, .., P_n^*, C_1, ..., C_n, A_1, ..., A_n)} \left(\eta-1\right)\eta \left(\frac{P_i - P_i^*}{P_i^*}\right)^2
\end{aligned}$$

Using the assumption that $A_i = A \; (C_i)^{\eta-1}$ we have that

$$\frac{\Theta(P_i^*, C_i, A_i)}{\Theta(P_1^*, .., P_n^*, C_1, ..., C_n, A_1, ..., A_n)} = \frac{A_i C_i^{1-\eta}}{\sum_{j=1}^{n} A_j C_j^{1-\eta}} = \frac{1}{n} \; ,$$

and hence the expression for B is:

$$B = \frac{(\eta-1)\eta}{2 \;\; n} \; .$$

B Proofs

Proposition 13. The origin is the optimal return point.

Proof. (of Proposition 13) By contradiction, suppose that it is not and assume without loss of generality that $t = 0$ is a period where an adjustment takes place and that $\hat{p}_i > 0$. Then, consider an alternative plan where $\hat{p}_i' = 0$ and where $\mathcal{I}' = \mathcal{I} + \{(0, 0, ..., -\hat{p}_i, ..., 0)\}$ so that the next adjustment happens exactly with the same probabilities. Letting τ be the next stopping time, then for $0 \le t \le \tau$ we have $\mathbb{E}_0[||p(t)||^2] = \mathbb{E}_0[||p(t)'||^2] + \hat{p}_i^2$, and thus setting $\hat{p}_i > 0$ is not optimal.

Proof. (of Proposition 2) Notice that $v'(0) = \beta_1$ and that $v(0) = \beta_0$, so that we require $\beta_1 > 0$, which implies $\beta_0 > 0$. Moreover, if $\beta_1 > B/r$ then v is strictly increasing and strictly convex. If $\beta_1 = B/r$ then v is linear in y. If $0 < \beta_1 < B/r$, then v is strictly increasing at the origin, strictly concave, and it reaches its unique maximum at a finite value of y. Thus, a solution that satisfies smooth pasting requires that $0 < \beta_1 < B/r$, and the maximizer is $\bar{y}$. In this case, $y = 0$ achieves the minimum in the range $[0, \bar{y}]$. Thus we have verified i), ii) and iii). Finally, we require value matching at $\bar{y}$, i.e. $v(\bar{y}) = v(0) + \psi$. Let $\beta_i(\beta_1)$ be the solution of equation (15), as a function of β_1. Note that for $0 < \beta_1 < B/r$, all the $\beta_i(\beta_1) < 0$ for $i \ge 2$ and are increasing in β_1, converging to zero as β_1 goes to B/r. Smooth pasting can be written as

$$0 \quad = \quad v'(\bar{y}; \beta_1) \equiv \sum_{i=1}^{\infty} \; i \; \beta_i(\beta_1) \; \bar{y}^{i-1} \; ,$$

where we emphasize that all the β_i can be written as a function of β_1. From the properties of the $\beta_i(\cdot)$ discussed above, it follows that we can write the unique solution of $0 = v'(\bar{\rho}(\beta_1); \beta_1)$ as an strictly increasing function of β_1, i.e. $\bar{\rho}'(\beta_1) > 0$. Now we write value matching at $\bar{y}$ which gives:

$$\psi \quad = \quad v(\bar{y}, \beta_1) - v(0, \beta_1) = v(\bar{y}, \beta_1) - \beta_0(\beta_1) = \sum_{i=1}^{\infty} \; \beta_i(\beta_1) \; \bar{y}^i \; .$$

We note that, given the properties of $\beta_i(\cdot)$ discussed above, for *any given* $y > 0$ we have: $v(y, \beta_1) - \beta_0(\beta_1)$ is strictly increasing in β_1, as long as $0 < \beta_1 < B/r$. Thus, define

$$\Psi(\beta_1) \quad = \quad v(\bar{\rho}(\beta_1), \beta_1) - v(0, \beta_1) = \sum_{i=1}^{\infty} \; \beta_i(\beta_1) \; \bar{\rho}(\beta_1)^i \; .$$

From the properties discussed above we have that $\Psi(\beta_1)$ is strictly increasing in β_1 and that it ranges from 0 to ∞ as β_1 ranges from 0 to B/r. Thus Ψ is invertible. The solution of the problem is given by setting:

$$\beta_1(\psi) = \Psi^{-1}(\psi) \qquad \text{and} \qquad \bar{y}(\psi) = \bar{\rho}(\beta_1(\psi)) \; .$$

Proof. (of Proposition 3) We first recall a useful theorem by Øksendal (2000)

Theorem 1. Øksendal (2000) Theorem 10.4.1 adds the following to equations (3)-(6) to show that a function verifying these conditions is the solution of the problem.

1. $0 \le V(p) \le A(p)$ for all $p \in \mathbb{R}^n$ where $A(p) = B||p||^2 \; n \; (\sigma/r)^2$ is the expected discounted value of never-adjusting,

2. $rV(p) \leq B\ ||p||^2 + \frac{\sigma^2}{2} \sum_{i=1}^n V_{ii}(p)$ for all $p \in \mathbb{R}^n \backslash \bar{\mathcal{I}}$,

3. $V(p) \leq \min_{\hat{p}} V(\hat{p}) + \psi$ for all $p \in \mathbb{R}^n$,

4. $\partial \mathcal{I}$ is a Lipschitz surface: i.e. it is locally the graph of an Lipschitz function,

5. the process $\{p(t)\}$ spends no time in the boundary of the inaction region:

$$\mathbb{E}\left[\int_0^\infty \chi_{\{\partial \mathcal{I}\}}(p(t))\ dt\ |p(0) = p\right] = 0, \text{ for all } p \in \mathbb{R}^n\ .$$

6. The second derivatives of V are bounded in a neighborhood of $\partial \mathcal{I}$,

7. the stopping times τ_i^* that achieve the solution are finite,

8. Let τ^* be the optimal stopping time starting from $p(0)$, the family $\{e^{-r\tau} V(p(\tau)); \tau \leq \tau^*\}$ is uniformly integrable for all $p(0)$.

For completeness we state the definition of a Lipschitz surface.

Definition 1. The boundary of a bounded set $\mathcal{I} \subset \mathbb{R}^n$ denoted by $\partial \mathcal{I}$ has Lipschitz domain (or it is a Lipschitz surface) if there is constant $K > 0$ such that for all $p \in \partial I$ there is a neighborhood $B_\epsilon(p) \cap \mathcal{I}$ and a system of coordinates $x = (p_1, p_2, ..., p_{n-1}), y = p_n$ and a function h_p such that for all:

1. $|h(x_1) - h(x_2)| < K|x_1 - x_2|$ for all x_1, x_2,

2. $B_\epsilon(p) \cap \mathcal{I} = B_\epsilon(p) \cap \{(x, y) : y > h_p(x)\}$, and

3. $B_\epsilon(p) \cap \partial \mathcal{I} = B_\epsilon(p) \cap \{(x, y) : y = h_p(x)\}$.

We now show that V so constructed has the following properties:

1. it only depends on the absolute value of the prices, since for all $p \in \mathbb{R}^n$:

$$v\left(\sum_{i=1}^n p_i^2\right) = v\left(\sum_{i=1}^n |p_i|^2\right)\ .$$

   for all $p \in \mathbb{R}^n$,

2. The range of inaction is given by $\mathcal{I} = \{p \in \mathbb{R}^n \big| \sum_{i=1}^n p_i^2 \leq \bar{y}\}$.

3. It solves the ODE given by equation (3). This can be seen by computing:

$$V_i(p) = v'\left(\sum_{i=1}^n p_i^2\right) 2p_i \ \text{ and } \ V_{ii}(p) = v''\left(\sum_{i=1}^n p_i^2\right)(2p_i)^2 + v'\left(\sum_{i=1}^n p_i^2\right) 2\ ,$$

   replacing this into the ODE equation (3) we obtain the ODE equation (10), which v solves by hypothesis.

4. It satisfies value matching equation (4), which is immediate since it satisfied the value matching condition for v given in equation (11).

5. it satisfies smooth pasting equation (6). Using the form of the solution for v, namely:

$$V_i(p) = v' \left(\sum_{i=1}^{n} p_i^2 \right) 2\ p_i = \sum_{j=1}^{n}\ \beta_j\ j\ \left(\sum_{k=1}^{n} p_k^2 \right)^{j-1}\ 2\ p_i$$

   Using that v satisfies smooth pasting we have:

$$0 = \sum_{j=1}^{n}\ \beta_j\ j\ \left(\sum_{k=1}^{n} p_k^2 \right)^{j-1}$$

   for any p with $\sum_{k=1}^{n} p_k^2 = \bar{y}$, which establishes that $V_i(p) = 0$ for all $i = 1, .., n$ and for any $p \in \partial \mathcal{I}$.

6. It satisfies optimality of the origin as return point, as given by equation (5). Direct computation gives:

$$V_i(p) = v' \left(\sum_{i=1}^{n} p_i^2 \right) 2\ p_i = \sum_{j=1}^{n}\ \beta_j\ j\ \left(\sum_{k=1}^{n} p_k^2 \right)^{j-1}\ 2\ p_i$$

   which equals zero when evaluated at $p = 0$. Notice also that

$$V_{ii}(0) = 2\ \beta_1 > 0 \ \text{ for all } i = 1, ..., n \ \text{ and } \ V_{ij}(0) = 0\ ,$$

   thus, $p = 0$ is a local minimum.

Finally we show that a function V with these properties is a strong solution to the variational inequality of the problem, and hence it is the value function by checking the extra conditions of Theorem 1.
Item (1) holds by construction of v as in Proposition 2, were we have for all $y > 0$ or $p \neq 0$:

$$V(p) = v(y) > v(0) = \beta_0 = \frac{n\sigma^2}{r}\beta_1 < \frac{Bn\sigma^2}{r^2} = A(p)\ .$$

Item (2) holds with equality in $\mathcal{I}$ by construction. It holds as inequality in $Int(\mathcal{C})$. To see why

$$V_i(p) = v'(||p||^2)2p_i \ \text{ and } \ V_{ii}(p) = v''(||p||^2)4(p_i)^2 + v'(||p||^2)2\ ,$$

but using Proposition 2 at $p \in \partial\mathcal{I}$ we have $v'(||p||^2) = 0$ and $v''(||p||^2) < 0$. Additionally, $||p||^2 > \bar{y} \in Int(\mathcal{C})$, thus

$$rV(\bar{y}) = B\bar{y} + \frac{\sigma^2}{2}v''(\bar{y})n4\bar{y} < rV(p) = B||p||^2 \text{ for all } p \in Int(\mathcal{C})\ .$$

Item (3) holds since by Proposition 2 we have $v(y)$ is strictly increasing in $(0,\bar{y})$ and $v(\bar{y}) - v(0) = \psi$, thus $V(p) > V(0) + \psi$ for all $p \neq 0$.
Item (4) holds by taking $h(p_1, p_2, ..., p_{n-1}) = \sqrt{\bar{y} - \sum_{i=1}^{n-1} p_i^2}$ for $p_n^2 > 0$, otherwise take a different coordinate system, i.e. solve for the ith coordinate for which $p_i^2 > 0$. Clearly h is Lipschitz.
Item (5) holds by considering the uncontrolled process $\mathrm{d}y = n\sigma^2 \mathrm{d}t + 2\sigma\sqrt{y}\ \mathrm{d}W$ and thus $\mathbb{E}_0[y(t)] = n\sigma^2 t + y(0)$.
Item (6) holds since, as shown above, $V_{ii}(p) = v''(\bar{y})\ 4\ p_i^2$ and

$$v''(\bar{y}) = \sum_{i=2}^{\infty} \beta_i\ i\ (i-1)\ (\bar{y})^{i-2}$$

and since, as shown in Proposition 1, $\lim_{i\to\infty} \beta_{i+1}/\beta_i = 0$ and thus the function v is analytical for all $y > 0$.
Item (7) holds, since $y(t)$ has a strictly positive drift $n\sigma^2$.
Item (8) holds since $e^{-r\tau} V(p(\tau)) \leq e^{-r\tau^*}(\psi + V(0))$.

Proof. (of Proposition 4) Using the expression for $\{\beta_i\}$ obtained in Proposition 1 value matching and smooth pasting can be written as two equations in β_2 and $\bar{y}$:

$$\begin{aligned}
\psi &= \frac{B}{r}\bar{y} + \beta_2 \left[\frac{2\sigma^2(n+2)}{r}\bar{y} + \bar{y}^2 + \bar{y}^2 \sum_{i=1}^{\infty} \kappa_i\ r^i\ \bar{y}^i \right] \\
0 &= \frac{B}{r}\bar{y} + \beta_2 \left[\frac{2\sigma^2(n+2)}{r}\bar{y} + 2\bar{y}^2 + \bar{y}^2 \sum_{i=1}^{\infty} \kappa_i\ (i+2)\ r^i\ \bar{y}^i \right]
\end{aligned}$$

where $\kappa_i = r^{-i}\frac{\beta_{2+i}}{\beta_2} = \prod_{s=1}^{i} \frac{1}{\sigma^2(s+2)(n+2s+2)}$. This gives an implicit equation for $\bar{y}$:

$$\psi = \frac{B}{r}\bar{y} \left[1 - \frac{\frac{2\sigma^2(n+2)}{r}\bar{y} + \bar{y}^2 + \bar{y}^2 \sum_{i=1}^{\infty} \kappa_i\ r^i\ \bar{y}^i}{\frac{2\sigma^2(n+2)}{r}\bar{y} + 2\bar{y}^2 + \bar{y}^2 \sum_{i=1}^{\infty} \kappa_i\ (i+2)\ r^i\ \bar{y}^i} \right] \tag{33}$$

Since the right hand side of equation (33) is strictly increasing in $\bar{y}$, and goes from zero to infinity, then we obtain Part (i). Since the right hand side of equation (33) is strictly decreasing in n, and goes to zero as $n \to \infty$, then we obtain Part (ii).

Rearranging this equation and defining $z = \bar{y}\ r/\sigma^2$

$$\frac{\psi\ 2(n+2)}{B\sigma^2}\ r^2 = z^2 + z^3 \left[\frac{2(n+2)\sum_{i=1}^{\infty} \omega_i\ (i+1)\ z^{i-1} - 2 - \sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i}{2(n+2) + 2z + z\sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i} \right] \tag{34}$$

where $\omega_i = \prod_{s=1}^{i} \frac{1}{(s+2)(n+2s+2)}$. Using the expression for ω_i and collecting terms on z^i one can show that the square bracket of equation (34) that multiplies z^3 is negative, and hence $\bar{y} > \sqrt{\psi\ 2(n+2)\sigma^2/B}$. Letting $b = \psi r^2 2(n+2)/(B\sigma^2)$ we can write equation (34) as:

$$1 = \frac{z^2}{b} \left(1 + z \left[\frac{2(n+2)\sum_{i=1}^{\infty} \omega_i\ (i+1)\ z^{i-1} - 2 - \sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i}{2(n+2) + 2z + z\sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i} \right] \right) \tag{35}$$

Since $z \downarrow 0$ as $b \downarrow 0$, then $z^2/b \downarrow 1$ as $b \downarrow 0$, establishing Part (iii). Inspection of equation (35) reveals that as $b \uparrow \infty$, then $z^2/b \uparrow \infty$ since the term in the round parenthesis that multiplies z^2/b on the right hand side goes to zero. Let $\Omega(z)$ denote this term. Taking logs on both sides of equation (35) and differentiating with respect to b gives

$$\frac{\partial \log z}{\partial \log b} = \left[2 + \frac{\partial \log \Omega(z)}{\partial \log z}\right]^{-1}$$

Noting that $\Omega(z) > 0$ and that it is decreasing in z implies that the elasticity is $\frac{\partial \log z}{\partial \log b}$ is increasing in b. This establishes Part (iv). From equation (34) it is clear that the optimal threshold satisfies $\bar{y} = \frac{\sigma^2}{r} Q\left(\frac{\psi}{B\sigma^2}\, r^2, n\right)$. Differentiating this expression we obtain Part (v).

Proof. (of Proposition 6) Let τ be the stopping time defined by the first time where the sum of the square of the price gaps vector $||p(\tau)||^2$ reaches the critical value $\bar{y}$, starting at the origin at time zero, i.e. starting at $||p(0)|| = 0$. Let $S_n(t, \bar{y})$ be the probability distribution for stopping times $\tau \geq t$, alternatively let $S_n(\cdot, \bar{y})$ be the survival function. Theorem 2 Ciesielski and Taylor (1962) shows that for $n \geq 1$:

$$S_n(t, \bar{y}) = \sum_{k=1}^{\infty} \xi_{n,k} \exp\left(-\frac{q_{n,k}^2}{2\bar{y}}\, \sigma^2\, t\right), \text{ where } \xi_{n,k} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{n,k}^{\nu-1}}{J_{\nu+1}(q_{n,k})}. \tag{36}$$

where $J_\nu(z)$ is the Bessel function of the first kind, where $\nu = (n-2)/2$, where $q_{n,k}$ are the positive zeros of the Bessel function $J_\nu(z)$, index in ascending order according to k, and where Γ is the gamma function. The hazard rate is then given by:

$$h_n(t, \bar{y}) = -\frac{1}{S_n(t, \bar{y})} \frac{\partial S_n(t, \bar{y})}{\partial t}, \text{ with asymptote } \lim_{t \to \infty} h_n(t, \bar{y}) = \frac{q_{n,1}^2\, \sigma^2}{2\, \bar{y}}. \tag{37}$$

As shown by Qu and Wong (1999), the zeroes of the Bessel function $q_{n,k}$ satisfy for $n > 2$ the following inequalities:

$$\left(\frac{n}{2}-1\right) - \frac{a_k}{2^{1/3}}\left(\frac{n}{2}-1\right)^{1/3} < q_{n,k} < \left(\frac{n}{2}-1\right) - \frac{a_k}{2^{1/3}}\left(\frac{n}{2}-1\right)^{1/3} + \frac{3}{10} a_k^2 \frac{2^{1/3}}{\left(\frac{n}{2}-1\right)^{1/3}} \tag{38}$$

where a_k are the first negative zero of the Airy function. For instance $a_1 \approx -2.33811$, giving a tight bound for the first zero $q_{n,1}$, which determines the asymptote of the hazard rate. A related simpler lower bound given by Hethcote (1970) for $n \geq 2$ is

$$q_{n,k}^2 > \left(k - \frac{1}{4}\right)^2 \pi^2 + \left(\frac{n}{2}-1\right)^2. \tag{39}$$

Proof. (of Proposition 7) We first establish the following Lemma.

Lemma 1. Let z be distributed uniformly on the surface in the surface of the n-dimensional sphere of radius one. We use x for the projection of z in any of the dimension, so $z_i = x \in [-1,1]$. The marginal distribution of $x = z_i$ has density:

$$\begin{aligned} f_n(x) &= \int_0^\infty \frac{s^{(n-3)/2} e^{-s/2}}{2^{(n-1)/2}\,\Gamma[(n-1)/2]} \frac{e^{-sx^2/[2(1-x^2)]}}{\sqrt{2\pi}} \frac{s^{1/2}}{(1-x^2)^{3/2}}\, ds \\ &= \frac{\Gamma(n/2)}{\Gamma(1/2)\,\Gamma[(n-1)/2]} (1-x^2)^{(n-3)/2} \end{aligned} \tag{40}$$

where the Γ function makes the density integrate to one.

This lemma is an application of Theorem 2.1, part 1 in Song and Gupta (1997), setting $p = 2$, so it is euclidian norm, and $k = 1$ so it is the marginal of one dimension. We give a simpler proof below. Now we consider the case where the sphere has radius different from one. Let $p \in \partial\mathcal{I}$, then

$$p = \frac{p}{\sum_{i=1}^n p_i^2}\bar{y} = \frac{p}{\sqrt{\sum_{i=1}^n p_i^2}}\ \sqrt{\bar{y}} = z\ \sqrt{\bar{y}}$$

where z is uniformly distributed in the n dimensional sphere of radius one. Thus each p_i has the same distribution than $x\ \sqrt{\bar{y}}$. Using the change of variable formula we obtain the required result.

Part 2 of Theorem 2.1 in in Song and Gupta (1997) shows that if x the marginal of a uniform distributed vector in the surface of the n-dimensional sphere, then x^2 is distributed as a $Beta(\frac{1}{2}, n-12)$. If y is distributed as a $Beta(\alpha, \beta)$ then it has $\mathbb{E}(y) = \alpha/(\alpha+\beta)$ and $\mathbb{E}(y^2) = (\alpha+1)/(\alpha+\beta+1)\mathbb{E}(y)$. Using these expressions for $\alpha = 1/2$ and $\beta = n/2$ we obtain the results for the standard deviation of Δp_i and its kurtosis. For the expected value of the absolute value of price changes we note that

$$\begin{aligned} \mathbb{E}[|\Delta p_i|] &= 2\int_0^{\sqrt{\bar{y}}} \Delta p_i\ w(\Delta p_i)\ d\Delta p_i \\ &= \frac{2}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \int_0^{\sqrt{\bar{y}}} \Delta p_i \left(1 - \left(\frac{\Delta p_i}{\sqrt{\bar{y}}}\right)^2\right)^{(n-3)/2} d\Delta p_i \\ &= \frac{\sqrt{\bar{y}}}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \end{aligned}$$

where the second line uses the form of h and the last line uses that the following result:

$$\int_a^b x\left(1-x^2\right)^{(n-3)/2} dx = \left.\frac{\left(1-x^2\right)^{(n-1)/2}}{1-n}\right|_a^b .$$

Then we have, using the fundamental property of the Gamma function

$$\frac{1}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} = \frac{\Gamma\left(\frac{n}{2}\right)}{\frac{n-1}{2}\ \Gamma\left(\frac{n-1}{2}\right)\ \Gamma(1/2)} = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\ \Gamma(1/2)}$$

Thus

$$\mathbb{E}[|\Delta p_i|] = \frac{\sqrt{\bar{y}}}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} = \sqrt{\bar{y}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\ \Gamma(1/2)}$$

We can approximate these ratio of Gamma functions as

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\ \Gamma(1/2)} \approx \sqrt{\frac{2}{\pi}} \frac{\sqrt{n+1/2}}{n}$$

from which we obtain our expression.

For $Std\,(\ |\Delta p_i|\ )\ \ /\ \mathbb{E}\,(\ |\Delta p_i|\ )$ we use that, given the symmetry around zero we have:

$$\begin{aligned} Std\,(\ |\Delta p_i|\ )\ \ /\ \mathbb{E}\,(\ |\Delta p_i|\ ) &= \sqrt{\frac{\mathbb{E}[\Delta p_i^2]}{\mathbb{E}[|\Delta p_i|]^2} - 1} = \sqrt{\left(\frac{Std(\Delta p_i)}{\mathbb{E}[|\Delta p_i|]}\right)^2 - 1} \\ &= \sqrt{\left(\frac{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)}{\sqrt{n}}\right)^2 - 1} \approx \sqrt{\frac{\pi}{2}\left(\frac{2n}{1+2n}\right) - 1} \end{aligned}$$

For the convergence of $\Delta p_i / Std(\Delta p_i)$ to a normal, we show that $y = x^2\ n$ converges to a chi-square distribution with 1 d.o.f., where x is the marginal of a uniform distribution in the surface of the n-dimensional sphere. The p.d.f of $y \in [0, n]$, the square of the standardized x, is

$$\frac{\Gamma\left(\frac{n}{2}\right)}{n\ \Gamma\left(\frac{n-1}{2}\right)\ \Gamma\left(\frac{1}{2}\right)} \left(1 - \left(\frac{y}{n}\right)\right)^{(n-3)/2} \left(\frac{y}{n}\right)^{-1/2},$$

and the p.d.f. of a chi-square with 1 d.o.f. is

$$\frac{\exp(-y/2)\ y^{-1/2}}{\sqrt{2}\,\Gamma\left(\frac{1}{2}\right)}\ .$$

Then, fixing y, taking logs in the ratio of the two p.d.f.'s, and taking the limit as $n \to \infty$, using that

$$\frac{\Gamma\left(\frac{n}{2}\right)\ \sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)\ \sqrt{n}} \to 1 \text{ as } n \to \infty\ .$$

we obtain that the ratio of the two p.d.f.'s converges to one.

Proof. (of Proposition 9) The forward Kolmogorov equation is:

$$0 = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left([2\sigma\sqrt{y}]^2\, f(y)\right) - \frac{\partial}{\partial y}\left(n\sigma^2 f(y)\right) \quad \text{for } y \in (0, \bar{y})\ , \tag{41}$$

with boundary conditions:

$$1 = \int_0^{\bar{y}} f(y)\ dy \ \text{ and } \ f(\bar{y}) = 0\ . \tag{42}$$

The first boundary conditions ensures that f is a density. The second is implied by the fact that when the process reaches $\bar{y}$ it is return to the origin, so the mass escape from these points. Equation (42) implies the second order ODE: $f'(y)(\frac{n}{2}-2) = yf''(y)$. The solution of this ODE for $n \neq 2$ is $f(y) = A_1 y^{n/2-1} + A_0$ for two constants A_0, A_1 to be determine using the boundary conditions equation (42):

$$\begin{aligned} 0 &= A_1(\bar{y})^{n/2-1} + A_0 \ , \\ 1 &= \frac{A_1}{n/2}(\bar{y})^{n/2} + A_0\bar{y} \ . \end{aligned}$$

For $n = 2$ the solution is $f(y) = -A_1 \log(y) + A_0$ subject to the analogous conditions. Solving for the coefficients A_0, A_1 gives the desired expressions.

Proof. (of Proposition 10) The only result to be established is that the distribution of the sum of the coordinates of a vector uniformly distributed in the $n-$dimensional sphere has density given by equation (27). Using the result in page 387 of Khokhlov (2006), let $c : \mathbb{R} \to \mathbb{R}$ be measurable, and let L be the Lebesgue measure in n dimensional sphere, then

$$\begin{aligned} \int_{x\in\mathbb{R}^n, ||x||=1} c(x_1 + ... + x_n)dL(x) &= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} c\left(\sqrt{n}u\right)(1-u^2)^{(n-3)/2} du \\ = \ \frac{2\pi^{n/2}}{\sqrt{n}\,\Gamma\left(\frac{1}{2}\right)\Gamma(\frac{n-1}{2})} & \int_{-\sqrt{n}}^{\sqrt{n}} c\left(\sqrt{n}u\right)\left(1-\left(\frac{\sqrt{n}u}{\sqrt{n}}\right)^2\right)^{(n-3)/2} d\left(\sqrt{n}u\right) \end{aligned}$$

Consider a function $c(x_1 + \cdots + x_n) = 1$ if $\alpha \leq x_1 + \cdots + x_n \leq \beta$ and dividing by the surface area of the $n-$dimensional sphere we obtain the desired result in equation (27). □

Proof. (of Proposition 11) The monotonicity of Φ_n and Θ_n with respect to δ follows, after manipulating the derivatives, from the monotonicity of $\nu(\cdot)$ and $S(\cdot)$. The zero derivative of Φ_n and Θ_n with respect to δ at zero can be obtained by considering two related functions, $\bar{\Phi}_n$ and $\bar{\Theta}_n$ which are obtained by replacing ν with $\bar{\nu} = \delta/(2\sqrt{y})$ and, without loss of generality, by replacing the lower extreme of integration w.r.t. y by the $\underline{y}(\delta)$, which solves $\nu(\underline{y}(\delta), \delta) = -\sqrt{n}$. Note that $\underline{y}(\delta)$ goes to $\bar{y}$ as $\delta \downarrow 0$, and that, by monotonicity, these functions are upper bounds for Φ_n and Θ_n. The result follows now by differentiating $\bar{\Phi}_n$ and $\bar{\Theta}_n$ w.r.t. δ and letting δ go to zero. For the last result we first state a useful homogeneity property, and define two related functions ϕ_n and θ_n:

$$\phi_n\left(\frac{\delta}{\sqrt{\bar{y}}}\right) \ \equiv \ \Phi_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right) = \Phi_n(\delta, \bar{y}) \ \text{ and } \ \theta_n\left(\frac{\delta}{\sqrt{\bar{y}}}\right) \equiv \ \Theta_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right) = \frac{\Theta_n(\delta, \bar{y})}{\sqrt{\bar{y}}} \ .$$

The homogeneity follows from a change of variables, and by noticing that ν and f are also functions of $\bar{y}$. To show that for small δ the function Θ_n is decreasing on $\bar{y}$, differentiate totally with respect to $\bar{y}$ the expression stating the homogeneity of this function obtaining

$$\Theta_{n,2}(\delta, \bar{y}) = \frac{1}{2}\frac{\delta}{\sqrt{\bar{y}}}\left[\frac{\Theta_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right)}{\delta} - \Theta_{n,1}\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right)\right] \ ,$$

where $\Theta_{n,i}$ denote the derivative with respect to the i^{th} argument of this function. Use that, for small values of δ, the function $\Theta_n(\delta/\sqrt{\bar{y}}, 1)$ is increasing and convex on θ, so that the expression in squares brackets is negative. That this function is increasing and convex follows since its first derivative is zero at $\delta = 0$ and the function is increasing. For Φ_n it follows from differentiating with respect to $\bar{y}$ the definition of homogeneity, obtaining:

$$\Phi_{n,2}(\delta, \bar{y}) = -\frac{1}{2}\frac{\delta}{\sqrt{\bar{y}}}\Phi_{n,1}\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right) \ ,$$

hence this function is decreasing in $\bar{y}$ in its domain. To compute the limit of the ratio in equation (31) we use the homogeneity to compute the first and second derivatives, and apply L'Hopital twice obtaining:

$$\begin{aligned}
\lim_{\delta \downarrow 0} \frac{\Phi_n(\delta, \bar{y}_n)}{\Phi_1(\delta, \bar{y}_1)} &= \frac{\lim_{\delta\downarrow 0} \phi_n''\left(\delta/\sqrt{\bar{y}_n}\right)/\bar{y}_n}{\lim_{\delta\downarrow 0} \phi_1''\left(\delta/\sqrt{\bar{y}_1}\right)/\bar{y}_1} = \frac{\phi_n''(0)}{\phi_1''(0)}\frac{\bar{y}_1}{\bar{y}_n} \\
&= \frac{\phi_n''(0)}{\phi_1''(0)}\left[\frac{3}{n+2}\right]^{1/2}\left[\left(\frac{\psi_1\sigma_1^2}{B_1}\right)/\left(\frac{\psi_n\sigma_n^2}{B_n}\right)\right]^{1/2}
\end{aligned}$$

where the last lines use the form of equation (16). So that $\mathcal{F}_n = \frac{\phi_n''(0)}{\phi_1''(0)}\left[\frac{3}{n+2}\right]^{1/2}$, which depends only on n. In the text outside the proof we show that $\phi_1''(0) > 0$. By a similar argument we obtain: $\mathcal{Q}_n = \frac{\theta_n''(0)}{\theta_1''(0)}\left[\frac{3}{n+2}\right]^{1/4}$. $\square$

Proof. (of Proposition 11)

The homogeneity with respect of $(\delta, \sqrt{\bar{y}})$ follows by a change in variable in expressions equation (28) and equation (29), taking into account that $f(y)$ is homogenous of degree -1 in $(\bar{y}, y)$ as displayed in equation (25), and that $\nu(y, \delta)$ is homogenous of degree zero in $(\sqrt{y}, \sqrt{\bar{y}}, \delta)$, as displayed in equation (26). In contrast, $S(z)$ and $s(z)$ once evaluated in a given z, do not depend on δ or $\bar{y}$.

To show that $\phi'(0) = 0$ we use the following. i) equation (26) implies that

$$S(\nu(y, \delta)) = 0 \ \text{ for } 0 \le y \le \underline{y}(\delta) \equiv \left(\max\left\{\sqrt{\bar{y}} - \delta\sqrt{n}\ ,\ 0\right\}\right)^2 \ , \tag{43}$$

hence the integration with respect to y in equation (28) and equation (29) can be done between $\underline{y}(\delta)$ and $\bar{y}$. ii) note that $\underline{y}(\delta) \to \bar{y}$ as $\delta \downarrow 0$, iii) once we have defined the integral w.r.t y in Φ in the interval $(\underline{y}, \bar{y})$, we can replace the function ν by an upper bound $\bar{\nu}(y, \delta) \equiv n\delta/(2\sqrt{y})$ and define upper bounds for Φ as

$$\bar{\Phi} \equiv \int_{\underline{y}(\delta)}^{\bar{y}} f(y)\ S\left(\bar{\nu}(y, \delta)\right) dy \le \Phi = \int_{\underline{y}(\delta)}^{\bar{y}} f(y)\ S\left(\nu(y, \delta)\right) dy \tag{44}$$

iv) the density $f(\bar{y}) = 0$. Then differentiating the expression for $\bar{\Phi}$ w.r.t. δ, evaluating the derivative at $\delta = 0$ we obtain that $\bar{\Phi}'(0, \sqrt{\bar{y}}) = 0$. v) since $\bar{\Phi} \ge \Phi \ge 0$ this establishes the desired result.

To show that $\theta'(0) = 0$ notice that $\Theta = \Phi \times \mathbb{E}_\delta[\Delta p | \Delta p \neq 0]$, i.e. the change in prices is equal to the fraction of firms that change prices times the expectation that of a price change, conditional on having a price change. Using that $\Phi(0, \sqrt{\bar{y}}) = \Phi'(0, \sqrt{\bar{y}}) = 0$ we obtained that $\Theta'(0, \sqrt{\bar{y}}) = 0$.

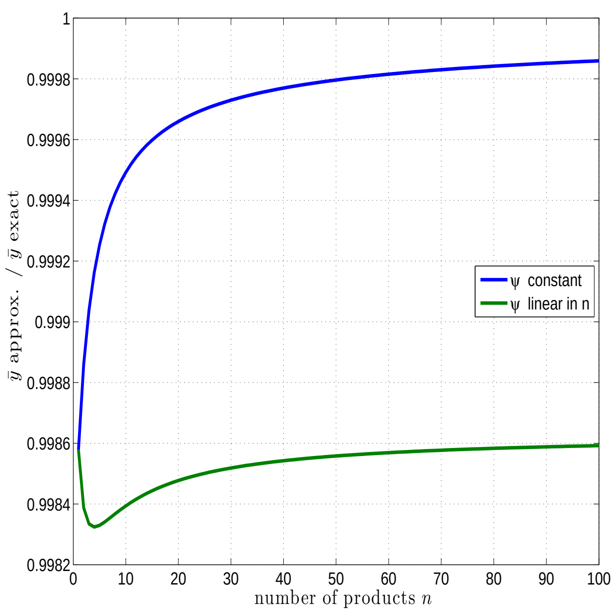
That Φ is strictly increasing in δ follows from the monotonicity of S and ν. That $\phi(2\sqrt{y/n}) = 1$ follows since values of δ so high $\nu(y, \delta) \geq \sqrt{n}$ and hence $S(\nu(y, \delta)) = 1$ for all $y \in (0, \bar{y})$. That $\theta(\delta) = \delta$ for $\delta \geq 2\sqrt{y/n}$ follow from a similar argument, since in this case the inside integral in equation (29) equals zero for all y.

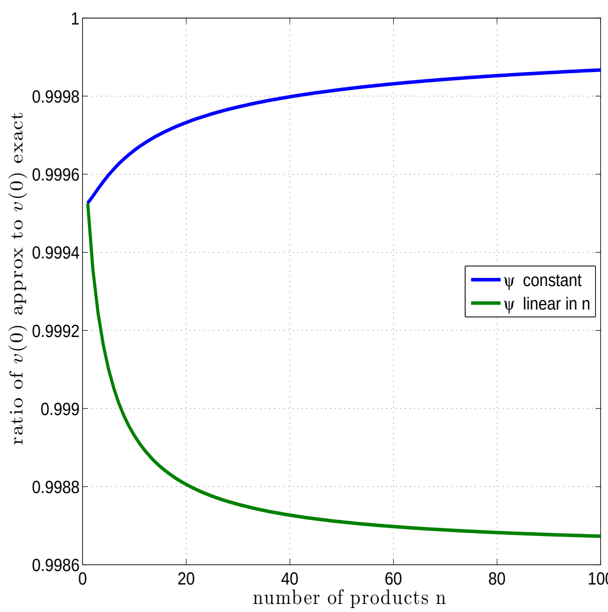
□

C Numerical accuracy of the approximation

In this section we present some evidence on the numerical accuracy of the approximation. We compare the value of $\bar{y}$ obtained from the quadratic approximation to v described above, with what we call the "exact" solution, which is the numerical solution using up to 30 terms for β_i in its the expansion. The approximation are closer for smaller values of σ and ψ, which we regard as more realistic.

Figure 6: Ratio of $\bar{y}'s$ and of $v(0)'s$ for the approximation relative to the "exact" solution



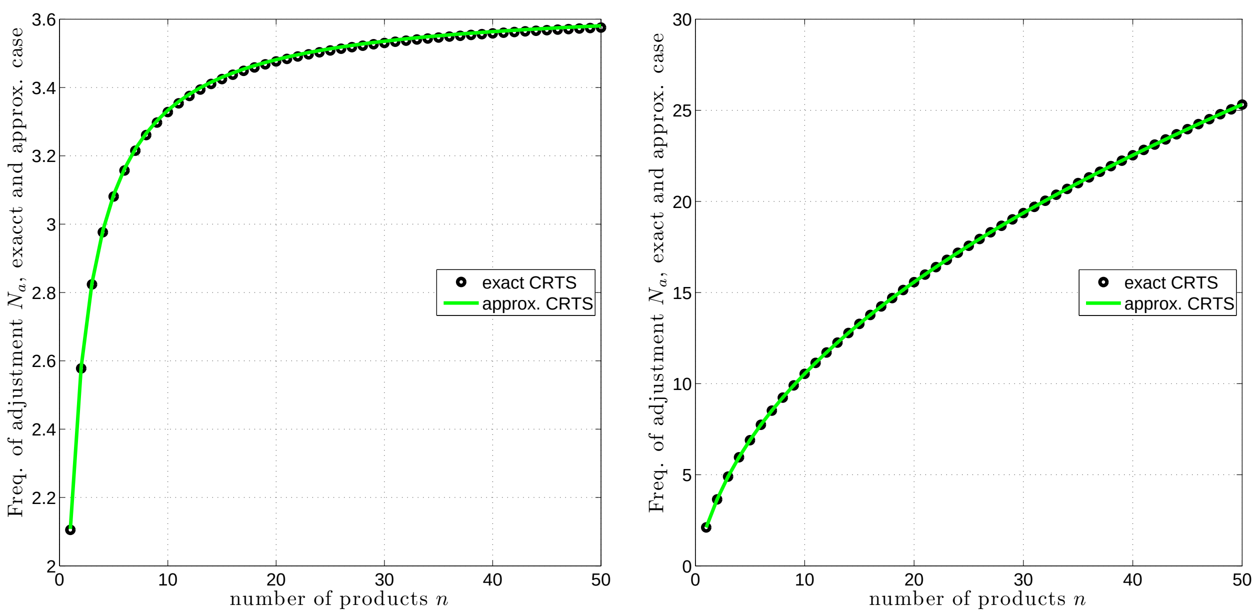


Note: parameter values are $B = 20$, $\sigma = 0.25$, $\psi_1 = 0.03$ and $r = 0.03$.

The next figure shows the value of $N_a(n)$ for various n when the menu cost are constant returns to scale, so $\psi = \psi_1 n$ using the approximation and using the "exact" expression.

Figure 7: Frequency of adjustment N_a for the CRTS $\psi_1 n$ and constant ψ.

CRTS: ψ proportional to n | Constant ψ



Note: parameter values are $B = 20$, $\sigma = 0.20$, $\psi_1 = 0.03$ and $r = 0.03$.