The general dynamic factor model: One-sided representation results

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A B S T R A C T
Recent dynamic factor models have been almost exclusively developed under the assumption that the common components span a finite-dimensional vector space. However, this finite-dimension assumption rules out very simple factor-loading patterns and is therefore severely restrictive. The general case has been studied, using a frequency domain approach, in Forni et al. (2000). That paper produces an estimator of the common components that is consistent but is based on filters that are two-sided and therefore unsuitable for prediction. The present paper, assuming a rational spectral density for the common components, obtains a one-sided estimator without the finite-dimension assumption.

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1. Introduction

The dynamic factor model

\[ x_t = \chi_t + \xi_t = b_1(L)u_{1t} + b_2(L)u_{2t} + \cdots + b_q(L)u_{qt} + \xi_t, \tag{1.1} \]

where \( i \in \mathbb{N}, t \in \mathbb{Z} \), has been studied in a vast literature starting with Stock and Watson (2002a, b), Forni et al. (2000) and Forni and Lipi (2001).

The components \( \xi_t \), called idiosyncratic, are assumed to be orthogonal to the common components \( \chi_t \) and cross-sectionally weakly correlated (see Section 2), so the comovement of the series is mainly accounted for by the common shocks \( u_{jt} \). Usually, the assumptions include that the Hilbert space spanned by the common components \( \chi_t \), for a given \( i \) and \( j \in \mathbb{N} \), is finite dimensional. Under this assumption, the components \( \chi_t \) and \( \xi_t \) can be consistently estimated, as \( n \) and \( T \) (the number of series and the number of observations for each series, respectively) tend to infinity, using principal components (standard or generalized) of the observable series \( x_t \) (see Stock and Watson, 2002a, b; Bai and Ng, 2002; Forni et al., 2005, 2009). Moreover, these estimators only involve present and past values of the variables \( x_t \).

Dynamic principal components, based on the spectral density of the \( x_t \)'s, have been used in Forni et al. (2000), where the above mentioned finite-dimension assumption is not required. However, dynamic principal components result in two-sided filters, involving present and past but also future values of the variables \( x_t \), with the consequence that the estimates are unreliable at the end of the sample and therefore useless for prediction.

The present paper starts with the observation that the finite-dimension assumption is very strict, as it does not include a model as simple as

\[ x_t = \frac{1}{1 - \alpha_i L} u_t + \xi_t, \tag{1.2} \]

with the coefficients \( \alpha_i \) independently drawn, for example, from the uniform distribution between \(-0.9\) and \(0.9\).

This seems sufficient motivation to go back to model (1.1) without the finite-dimension assumption. Combining the approach taken in Forni et al. (2000) with recent results obtained by Anderson, Deistler and coauthors (see Section 3), we show that under the assumption that the filters \( b_j(L) \) are rational, plus reasonable technical assumptions, model (1.1) can be rewritten as

\[ H_n(L)x_{nt} = R_u u_t + H_n(L)\xi_{nt}, \tag{1.3} \]

where \( x_{nt} \) and \( \xi_{nt} \) stack the first \( n \) series \( x_t \) and \( \xi_t \), respectively, \( u_t = (u_{1t}, u_{2t}, \cdots, u_{qt}) \), \( H_n(L) \) is a finite matrix polynomial. Moreover:

(i) \( H_n(L) \), which is \( n \times n \), and \( R_u \), which is \( n \times q \), can be obtained from the spectral density of \( X_{nt} \).
(ii) \( H_n(L)\xi_{nt} \) is idiosyncratic (this is not obvious; see Section 4).

Though the paper is limited to representation results, Eq. (1.3), combined with the estimate of the spectral density of \( X_{nt} \) proposed in Forni et al. (2000), can be seen as a basis for estimating the common components \( \chi_{nt} \) without the finite-dimension assumption and using only contemporaneous and past values of the series \( x_{nt} \).
Section 2 reviews previous results on model (1.1). Section 3 introduces and discusses the main assumptions. Section 4 derives representation (1.3). Section 5 discusses estimation based on (1.3). Section 6 concludes.

2. Previous results

2.1. The general model

Let us rewrite model (1.1) in vector form:

\[ \mathbf{x}_n = \mathbf{\xi}_n + \mathbf{\eta}_n \]

(2.4)

with \( \mathbf{b}_n(L) \) being the \((i,j)\) entry of \( \mathbf{B}_n(L) \) for all \( n \geq i \) (the matrices \( \mathbf{B}_n(L) \) are nested). We assume that:

A1. (Common components) \( \mathbf{u}_i \) is an orthonormal \( q \)-dimensional white noise. The filters \( \mathbf{b}_n(L) \) are square summable.

A2. (Idiosyncratic components) \( \mathbf{\xi}_n \) is weakly stationary.

A3. (Orthogonality of common and idiosyncratic components) \( \mathbf{\xi}_n \perp \mathbf{u}_i \), for all \( n, i, t \).

A4. (Eigenvalues of the idiosyncratic components) Let \( \Sigma_\chi^2(\theta) \) be the spectral density matrix of \( \mathbf{\xi}_n \) and \( \lambda_\chi^2(\theta) \) its first eigenvalue (in descending order). We assume that there exists a positive real number \( \lambda \) such that \( \lambda_\chi^2(\theta) \leq \lambda \) for all \( n \).

A5. (Eigenvalues of the common components) Let \( \Sigma_\mathbf{u}^2(\theta) \) be the spectral density matrix of \( \mathbf{u}_i \) and \( \lambda_\mathbf{u}(\theta) \) its qth eigenvalue.

We assume that \( \lambda_\mathbf{u}(\theta) \to \infty \), for all \( \theta \), as \( n \to \infty \).

Forni and Lippi (2001) prove that (2.4) and Assumptions A1 through A5 impose little structure on the \( x_t \)'s. They show that the following two assumptions: (1) \( \mathbf{x}_n \) is stationary for all \( n \), (2) there exists an integer \( q \) such that, for \( n \), the \( q \)-th eigenvalue of the spectral density matrix of \( \mathbf{x}_n \) diverges for all frequencies while the \((q+1)\)-th is uniformly bounded, imply that the \( x_t \)'s can be represented as in (2.4) with A1 through A5 holding.

Under Assumptions A1 through A5, the decomposition of the \( x_t \)'s into common and idiosyncratic components is unique. To be precise, if

\[ x_{it} = \chi_{it} + \xi_{it} = b_{i1}(L)u_{it} + b_{i2}(L)u_{it} + \cdots + b_{iq}(L)u_{it} + \xi_{it} \]

(1.1')

for all \( i \in \mathbb{N} \) and \( t \in \mathbb{Z} \), and Assumptions A1 through A5 are fulfilled for (1.1'), then

\[ q' = q, \quad \chi_{it} = \chi_{it}, \quad \xi_{it} = \xi_{it} \]

for all \( i \in \mathbb{N} \) and \( t \in \mathbb{Z} \) (see Forni and Lippi, 2001).

Note that the asymptotic condition in Assumption A4 does not require mutual orthogonality of the idiosyncratic components, a standard identification condition in finite-\( n \) factor models. For example, a non-zero correlation of \( \xi_{it} \) with \( \xi_{i+1,t} \) does not conflict with A4. As a consequence, the decomposition of the \( x_t \)'s into common and idiosyncratic components is identified only under the condition \( x_{it} = \chi_{it} + \xi_{it} = \chi_{it} + \xi_{it} \), for all \( i \in \mathbb{N} \) and \( t \in \mathbb{Z} \). Note also that uniqueness does not extend to \( \mathbf{B}_n(L) \) or the common shocks \( \mathbf{u}_i \). For, if \( \mathbf{B}(L) \) is a \( q \times q \) filter such that \( \mathbf{B}(z)B'(z^{-1}) = I_q \) for \( |z| = 1 \), then defining

\[ \mathbf{B}_n(L) = \mathbf{B}_n(L) \mathbf{B}(L), \quad \mathbf{u}_i = \mathbf{B}'(L^{-1}) \mathbf{u}_i, \]

(2.5)

we have \( \mathbf{x}_n = \tilde{\mathbf{B}}_n(L) \mathbf{u}_i \), which can replace the second equation in (2.4).

Now consider \( \Sigma_\mathbf{u}^2(\theta) \), its first \( q \) eigenvalues and corresponding eigenvectors:

\[ \lambda_1^2(\theta), \lambda_2^2(\theta), \ldots, \lambda_q^2(\theta), \]

\[ p_{n1}(\theta), p_{n2}(\theta), \ldots, p_{nq}(\theta), \]

where \( |p_{n1}^2(\theta)| + \cdots + |p_{nj}^2(\theta)| = 1 \) for all \( \theta \in [-\pi, \pi] \).

Define \( P_n(L) \) as the inverse Fourier transform of

\[ \frac{1}{\sqrt{\lambda_n^2(\theta)}}p_n^2(\theta). \]

The vector

\[ \mathbf{u}_{i,n} = (P_{n1}(L)p_{n1}(\theta) \cdots p_{nq}(\theta)) \mathbf{x}_n \]

is a \( q \)-dimensional orthonormal white noise. Moreover, define

\[ \chi_{i,n} = \text{Proj}(\mathbf{x}_n | \text{span}(\mathbf{u}_{i,n}, s \in \mathbb{Z})). \]

Then as \( n \to \infty \) we have \( \chi_{i,n} \to \chi_{i} \) in quadratic mean.

The above matrices and vectors have sample counterparts:

\[ \Sigma_{\chi}^2(\theta), \quad P_{n1}(L), \quad \mathbf{u}_{i,n}, \quad \chi_{i,n}, \]

and the result is that

\[ \chi_{i,n} \to \chi_{i} \]

in probability as \( n, T \to \infty \) (see Forni et al., 2000).

The following elementary example shows how the dynamic principal components work and their main drawback:

\[ \chi_{i} = \begin{cases} \mathbf{u}_{i-1} & \text{if } i \text{ is odd} \\ \mathbf{u}_{i} & \text{if } i \text{ is even} \end{cases}. \]

(2.6)

Moreover, assume that \( \Sigma_\mathbf{u}^2(\theta) = \frac{1}{2\pi}I_q \) (the idiosyncratic components are orthogonal to one another and have unit variance). Then

\[ \Sigma_{\chi}^2(\theta) = \frac{1}{2\pi} \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} \begin{pmatrix} e^{i0} & 1 \cdots e^{i0} & 1 \\ e^{i\theta} & \cdots & 1 \end{pmatrix} + \frac{1}{2\pi}I_q. \]

The first eigenvalue is \( 1 + n \), with eigenvector \( \frac{1}{\sqrt{n}}(e^{i0} 1 \cdots e^{i0}) \), so

\[ P_{n1}(L) = \frac{1}{\sqrt{n}(1 + n)}(L^{-1} 1 \cdots L^{-1} 1), \]

where \( L^{-1} \) is the forward shift operator: \( L^{-1}x_{it} = x_{i,t+1} \). As a consequence this estimator can be used only for \( t < T - 1 \).

2.2. The restricted model

An important simplification is obtained with the following assumption, which is used in Stock and Watson (2002a,b), Bai and Ng (2002), Forni et al. (2005) and Forni et al. (2009). For a given \( t \), we will denote by \( S_{\chi}^t \) the Hilbert space \( \text{span}(\chi_{i,n}, s \in \mathbb{Z}) \), i.e., the closure of the set of all linear combinations of the variables \( \chi_{i,n} \). Note that stationarity of the vectors \( \chi_{i,n} \) implies that the dimension of \( S_{\chi}^t \) is independent of \( t \).

AF. The space \( S_{\chi}^t \) is finite dimensional.

Under A1 through A5 plus AF, denote by \( r \) the dimension of \( S_{\chi}^t \). There exist:

(i) an \( r \)-dimensional stationary process \( F_t \), which has the representation

\[ F_t = N(L)\mathbf{u}_t, \]

(2.7)

\( N(L) \) being a square-summable \( r \times q \) filter;

(ii) nested \( n \times r \) matrices \( C_s \), such that

\[ \chi_{i,n} = \mathbf{B}_n(L)\mathbf{u}_t = C_sF_t \]

(2.8)
(for a proof of this trivial fact, see Forni et al. (2009). The processes $F_0$ are called the static factors. Note that the static factors evolve according to a dynamic equation; see (2.7). “Static” only refers to the loading of $F_i$ by the $\chi$’s; see (2.8).

Summing up, in general, the stochastic variables $\{x_{it}, \ i \in N, t \in \mathbb{Z}\}$, span an infinite-dimensional Hilbert space $X$, which is contained in the Hilbert space spanned by $\{u_{ij}, \ j = 1, \ldots, q, t \in \mathbb{Z}\}$. Under AF the Hilbert space spanned by $\{x_{it}, \ i \in \mathbb{Z}\}$, for any given $t$, is finite dimensional with stationary basis $F_i$. Of course in this case $X$ is also contained in the Hilbert space spanned by $\{F_i, j = 1, \ldots, r, t \in \mathbb{Z}\}$.

Let $\Gamma_n$ be the covariance matrix of $x_{it}$. Under AF, estimation of the common components can be achieved using the first $r$ eigenvalues and corresponding eigenvectors of $\Gamma_n$ to obtain $F_in$, then projecting $x_{it}$ on $F_in$. In this case only contemporaneous values of the $x$’s are involved, so no two-sidedness problem arises.

3. Back to the general model

As we have observed in the Introduction, taking the simple case (1.2), rewritten here:

$$x_{it} = \frac{1}{1 - \alpha(L)}u_t + \xi_{it},$$

where $\alpha$ is drawn from the uniform distribution on the interval $[-.9, .9]$, we see that $S$ is not finite dimensional; thus, so to speak, we have an infinite number of static factors.

Criteria for determining $r$, the number of static factors, when applied to models like (1.2), will produce wrong results, with the estimated $r$ growing to infinity with $n$. Moreover, all criteria for determining $q$ that are based on firstly estimating $F_i$, then estimating a VAR for $F_i$, are misspecified. To our knowledge, the only criterion for determining $q$, which does not depend on the assumption of a finite $r$ and has therefore general applicability, is that of Hallin and Liška (2007).

3.1. Fundamental and zeroless representations

We believe that model (1.2) provides a strong motivation for not assuming AF. Instead, we assume here that:

A6. The spectral density of $x_{it}$ is rational.

Assumptions A6 and A5 imply that there exists $n_0 > 0$ such that for $n \geq n_0$, rank $(\Sigma^{(0)}(i)) = q$ for all $\xi$. As a consequence, for $n \geq n_0$ the vector $x_{it}$ has a fundamental rational representation of rank $q$, i.e.,

$$X_{nt} = C_n(L)\psi^{(n)}_t,$$  
(3.9)

where: (1) the entries of $C_n(L)$, denoted by $c_{ij}(L)$, are rational functions

$$c_{ij}(L) = \frac{d_{ij}(L)}{e_{ij}(L)},$$

where $d_{ij}$ and $e_{ij}$ have no common roots and $e_{ij}(0) = 1$; (2) $\psi^{(n)}_t$ is a $q$-dimensional orthonormal white noise; (3) $C_n(L)$ has no zeros for $|z| < 1$, a zero of $C_n(z)$ being defined as a complex number $\xi$ such that the rank of $C_n(\xi)$ is lower than the maximum rank of $C_n(0)$, and no poles for $|z| \leq 1$, the poles of $C_n(z)$ being defined as the poles of the polynomials $e_{ij}(z)$. This implies that $\psi^{(n)}_t$ belongs to the space spanned by $X_{nt-k}$, for $k \geq 0$. As (3.9) implies that $x_{it}$ belongs to the space spanned by $\psi^{(n)}_{n-k}$, for $k \geq 0$, the two spaces coincide.

Fundamental representations are unique up to an orthogonal matrix. To be precise, $X_{nt} = \tilde{C}_n(L)\psi^{(n)}_t$, is fundamental if and only if there exists an orthogonal matrix $K_n$ such that

$$\tilde{C}_n(L) = C_n(L)K_n,$$

$$\tilde{\psi}^{(n)}_t = K_n\psi^{(n)}_t.$$  

To understand the relationship between representations (3.9) and (2.4), consider again example (2.6):

$$x_{it} = \begin{cases} u_{t-1} & \text{for } i \text{ odd} \\ u_t & \text{for } i \text{ even}. \end{cases}$$

In this case a fundamental white noise for $x_{nt}$ is $u_{t-1}$ for $n = 1, u_t$ for $n > 1$. Note also that the $(1,1)$ entry of $C_n(L)$ is for $n = 1$, $L$ for $n > 1$. The example shows that, firstly, reference to $n$ in $\psi^{(n)}_t$ is necessary and, secondly, that the matrices $C_n(L)$, unlike the matrices $B_n(L)$, are not necessarily nested.

In the following example, though $C_n(L) \neq B_n(L)$ for all $n$, the matrices $C_n(L)$ are nested. Let $q = 1$ and let representation (1.1) be

$$x_{it} = b(L)u_t, \quad b(L) = \frac{1 - \alpha^{-1}L}{1 - \alpha L},$$

with $|\alpha| < 1$. As the polynomial $1 - \alpha^{-1}L$ is not invertible, the white noise $u_t$ does not belong to the space spanned by present and past values of the $x$’s. However, elementary calculations show that

$$1 - \alpha^{-1}L u_t = -\alpha^{-1}\left(1 - \alpha^{-1}L(Lu_t)\right) = -\alpha^{-1}u_t,$$

and that the spectral density of $u_t$ is equal to unity at all frequencies. Thus $u_t$ is a unit-variance white noise. Representation (3.9) is immediately obtained:

$$x_{it} = \psi^{(n)}_t, \quad \psi^{(n)}_t = u_t, \text{ independent of } n.$$  

Thus the matrices $C_n(L)$ are nested and $\psi^{(n)}_t = u_t$ is independent of $n$.

More generally, under Assumption A7’, to be introduced below, we can choose the fundamental representations (3.9) in such a way that $\psi^{(n)}_t$ is independent of $n$ and the matrices $C_n(L)$ are nested.

Now consider the set of all $n \times q$ matrices $D(L)$, with rational entries

$$d_{ij}(K) = \frac{f_{ij}(L)}{g_{ij}(L)},$$

with $g_{ij}(0) = 1$, such that

$$\text{degree}(f_{ij}) \leq p_1, \quad \text{degree}(g_{ij}) \leq p_2.$$  

The parameter space for $D(L)$ has dimension $pq(p_1 + p_2 + 1)$. If the matrix $D(L)$ is tall, i.e. if $n > q$, then, for generic values of the parameters, $D(L)$ is zeroless, i.e. the rank of $D(z)$ is $q$ for all complex numbers $z$.

To see why this result holds, consider firstly the following example, in which $q = 1$:

$$x_{it} = (\alpha_1 + \beta_1 L)u_t,$$  
(3.10)

for $i = 1, \ldots, n$, with $n > 1$. Obviously in this case $D(z)$ is zeroless unless $\alpha_1/\beta_1 = \gamma$ for all $i$. In general, existence of a zero of $D(z)$ means that the determinants of all the $q \times q$ submatrices of $D(z)$ vanish for the same complex number. This implies algebraic restrictions on the coefficients of $D(L)$, as argued in Forni et al. (2009) and Zinner (2008). For a formal proof see Anderson and Deistler (2008a) and Deistler et al. (2010).

This motivates the following assumption, which will be enhanced in the next section:

A7. For $n \geq q + 1$, the matrix $C_n(z)$, corresponding to the fundamental representation $X_{nt} = C_n(L)\psi^{(n)}_t$, is zeroless.
3.2. Autoregressive representations for n > q

Tall, zeroless moving average rational matrices possess a finite inverse: 

\((F)\) Let \(n > q\). Consider the rational representation \(y_t = D(L)z_t\), where \(y_t\) is \(n\)-dimensional and \(z_t\) is an orthonormal \(q\)-dimensional white noise. If \(D(L)\) is zeroless then \(y_t\) has a finite autoregressive representation 
\[ A(L)y_t = D(0)z_t. \]

For a formal proof see Anderson and Deistler (2008b) and Deistler et al. (2010). Example (3.10) for \(n = 2\) provides an intuition:

\[ \begin{align*}
\chi_{1t} & = \alpha_1 u_t + \beta_1 u_{t-1}, \\
\chi_{2t} & = \alpha_2 u_t + \beta_2 u_{t-1}.
\end{align*} \tag{3.11} \]

We see that 
\[ u_t = \frac{1}{\alpha_1 \beta_2 - \alpha_2 \beta_1} (\beta_2 \chi_{1t} - \beta_1 \chi_{2t}), \]
and so 
\[ \begin{pmatrix} 1 - \delta \beta_1 \beta_2 L - \delta \beta_1^2 L \alpha_1 \beta_2 & \delta \beta_1 L \alpha_2 \\
-\delta \beta_2 L \alpha_1 & 1 + \delta \beta_1 \beta_2 L \end{pmatrix} \chi_{1t} \chi_{2t} = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) u_t, \]
where \(\delta = 1/(\alpha_1 \beta_2 - \alpha_2 \beta_1)\). Note that the autoregressive representation exists if and only if \(\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0\), that is when \(D(z)\) is zeroless. Moreover, \(\chi_{1t-1} = \chi_{2t-1}\) and \(\chi_{1t+1}\) are linearly independent. Therefore the autoregressive representation of order 1 is unique.

But as soon as \(n = 3\), 
\[ \begin{align*}
\chi_{1t} & = \alpha_1 u_t + \beta_1 u_{t-1}, \\
\chi_{2t} & = \alpha_2 u_t + \beta_2 u_{t-1}, \\
\chi_{3t} & = \alpha_3 u_t + \beta_3 u_{t-1},
\end{align*} \tag{3.12} \]
we see that infinitely many autoregressive representations of order 1 are possible. For, \(c = (\alpha_1 \alpha_2 \alpha_3)\), we have 
\[ u_t = \frac{1}{\alpha_1 \alpha_2} (c_1 \chi_{1t} + c_2 \chi_{2t} + c_3 \chi_{3t}), \tag{3.13} \]
where \(c = (c_1 c_2 c_3)\) is any vector orthogonal to \((\beta_1 \beta_2 \beta_3)\) and such that \(c \cdot \alpha = 0\). Using (3.13) to replace \(u_{t-1}\) in (3.12), we obtain an autoregressive representation of order one depending on \(c\).

Consider now \(q + 1\) integers \(i_1, i_2, \ldots, i_{q+1}\), with \(1 \leq i_k < i_{k+1} \leq n\), and let 
\[ \chi_{i_1,\ldots,i_{q+1},t} = (\chi_{i_1t} \chi_{i_2t} \cdots \chi_{i_{q+1}t}) = C_{i_1,\ldots,i_{q+1}}(L) \psi_t^{(n)} \tag{3.14} \]
be obtained from (3.9) by selecting the rows \(i_1, i_2, \ldots, i_{q+1}\). The vector (3.14) is tall (it has dimension \(q + 1\) and rank \(q\)), so for generic values of the parameters the matrix \(C_{i_1,\ldots,i_{q+1}}(L)\) is zeroless. As a consequence, by Proposition (F), for generic values of the parameters the vector \(\chi_{i_1,\ldots,i_{q+1},t}\) has a finite autoregressive representation. This motivates almost all of Assumption A7 below, which enhances Assumption A7. The uniqueness in part (ii) is motivated by the discussion of examples (3.11) and (3.12).

A7'. For all \(n\) and all choices of \(i_1, i_2, \ldots, i_{q+1}\), we assume that (i) 
\[ C_{i_1,\ldots,i_{q+1}}(z) \text{ is zeroless}, \]
and that (ii) 
\[ \chi_{i_1,\ldots,i_{q+1},t} \text{ has a unique minimum-lag autoregressive representation.} \]

As the vector (3.14) is tall, being of dimension \((q + 1)\) but of rank \(q\), part (i) of A7' can be motivated by the genericity argument. Part (ii) has a motivation in the discussion of examples (3.11) and (3.12).

A consequence of A7'(i) is that the space spanned by present and past values of \(\chi_{i_t}, i \in \mathbb{Z}\) is equal to that spanned by present and past values of any \(q + 1\) among the variables \(\chi_{it}\). For, present and past values of \(\chi_{i_1,\ldots,i_{q+1},t}\) span the same space as is spanned by present and past values of \(\psi_t^{(n)}\), and therefore by present and past values of \(\chi_{it}\), for any \(n\).

Assumption A7' rules out examples like (2.6), which fulfills A7. Note however that (2.6) is a special case of (3.10), in which Assumption A7' is fulfilled for generic values of \(\alpha_i\) and \(\beta_i\).

Lastly, consider a fundamental representation for \(\chi_{q+1,t}\): 
\[ \chi_{q+1,t} = F(L)w_t. \]

By A7', for \(i > q + 1\), \(\chi_{it}\) belongs to the space spanned by present and past values of \(\chi_{q+1,t}\) and therefore of \(w_t\), so 
\[ \chi_{it} = f_{1i}(L)v_{it} + f_{2i}(L)v_{2t} + \cdots + f_{qi}(L)v_{qt}, \]
for all \(i \in \mathbb{N}\). Thus under A7' representation (3.9) can be written with a white noise \(w_t\), which is independent of \(n\), and nested matrices \(C_n(L)\):
\[ \chi_{it} = c_n(L)w_t. \tag{3.15} \]
and common components has been obtained in Bai and Ng (2004), but only for the restricted model. Methods allowing estimation of the components $\delta_1$ and testing for their cointegration in the general model are not available. On the other hand, we do not really need as much as Assumption A7. In the next section we show that what is needed to obtain a finite autoregressive representation for $X_{nt}$ is the existence of a partition of $X_{nt}$ into $(q + 1)$-dimensional subvectors each fulfilling $A7'$. In empirical situations, careful grouping of the variables, based for example on their economic relationships, should help with avoiding "dangerous" $(q + 1)$-dimensional vectors.

4. Transforming the dynamic model into a static model with $q$ factors

We assume for convenience that $n = (q + 1)m$ and partition $X_{nt}$ as

$$X_{nt} = (X_{n1t} X_{n2t} \cdots X_{ntm})',$$

where $X_{ntl} = (X_{n1-(l-1)(q+1)+1,t} X_{n1-(l-1)(q+1)+2,t} \cdots X_{n1-l (q+1),t})'.

We start with (3.15) and denote by

$$A_{l1}(L)X_{ntl} = R_{ntl}v_t,$$  \hspace{1cm} (4.16)

the minimum-lag autoregressive representation of the $(q + 1)$-dimensional vector $X_{ntl}$ (see Assumption A7'). Combining Eq. (4.16), $X_{nt}$ has the following autoregressive representation:

$$\begin{bmatrix}
A_{l1}(L) & 0 & \cdots & 0 \\
0 & A_{l2}(L) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{lm}(L)
\end{bmatrix}X_{nt} = R_{nt}v_t,$$

(4.17)

where $R_{nt} = (R_{nt1} \cdots R_{ntm})'$. Of course other representations like (4.17) can be obtained by reordering the components of $X_{nt}$. However, the component of $R_{nt}v_t$ which corresponds to a given component of $X_{nt}$ is independent of which ordering has been chosen.

A8. We assume that the $q$th eigenvalue of $R_{nt}R_{nt}'$, call it $\nu_n$, tends to infinity as $n \to \infty$.

Assumption A8 is not a consequence of A5. In example (3.10), A5 requires that $\sum |\alpha_l + \beta_l e^{-i \theta}|^2$ diverges for all $\theta$, while A8 requires that $\sum \alpha_l^2$ diverges. Note that A8 is not affected if $R_{nt}$ is multiplied on the right by an orthogonal matrix.

We denote by $G^*$ the complex conjugate of the matrix $G$.

A9. Let $A_{l1-\ldots-l_{q+1}}(L)$ be the minimum-lag autoregressive matrix of $X_{l1-\ldots-l_{q+1},t}$. Denote by $\mu_{l1-\ldots-l_{q+1},t}(\theta)$ the maximum eigenvalue of

$$A_{l1-\ldots-l_{q+1}}(e^{-i \theta})A_{l1-\ldots-l_{q+1}}(e^{-i \theta})^*.$$

We assume that $\mu_{l1-\ldots-l_{q+1},t}(\theta) \leq \mu$ for a positive real $\mu$, for all choices of $l_k, k = 1, \ldots, q + 1$, for all $\theta$.

Assumption A9 is reasonable but not trivial. Take

$$A(L) = \begin{pmatrix}
1 & \alpha L \\
\beta L & 1
\end{pmatrix}.$$  

The trace of $A(e^{-i \theta})A(e^{-i \theta})^*$ is $|1 + \alpha e^{-i \theta}|^2 + |1 + \beta e^{-i \theta}|^2$, which is not bounded under the stability condition $|\alpha| < 1$.

Defining $H_n(L)$ as the autoregressive matrix in (4.17), we have

$$H_n(L)X_{nt} = R_{nt}v_t + H_n(L)\xi_{nt}$$  \hspace{1cm} (4.18)

or, setting $\tilde{X}_{nt} = H_n(L)v_t$, $\tilde{X}_{nt} = R_{nt}v_t$ and $\tilde{\xi}_{nt} = H_n(L)\xi_{nt}$,

$$\tilde{X}_{nt} = R_{nt}v_t + \tilde{\xi}_{nt} = \tilde{X}_{nt} + \tilde{\xi}_{nt}.$$  \hspace{1cm} (4.19)

Let us prove that this is a static factor model with $q$ factors, i.e. that as $n \to \infty$ the first $q$ eigenvalues of the covariance matrix of $\tilde{X}_{nt}$ diverge and the first eigenvalue of the covariance matrix of $\tilde{\xi}_{nt}$ is bounded. The first statement is a consequence of A8. Moreover, using A4 and A9, we have

$$\begin{align*}
a^* \Sigma_{\tilde{X}_{nt}}(\theta) & = aH_n(e^{-i \theta})\Sigma_{\tilde{X}_{nt}}(\theta)H_n(e^{-i \theta})^*a^* \\
& \leq \lambda_{n1}(\theta)aH_n(e^{-i \theta})H_n(e^{-i \theta})^*a^* \leq \lambda_n(\theta)a^2.
\end{align*}

Thus the first eigenvalue of the spectral density $\Sigma_{\tilde{X}_{nt}}(\theta)$, call it $\lambda_{n1}(\theta)$, is bounded by $\lambda_n(\theta)$. On the other hand, the first eigenvalue of the covariance matrix of $\tilde{\xi}_{nt}$ is bounded by

$$\int_{-\pi}^{\pi} \lambda_{n1}(\theta)d\theta.$$

The result follows.

Other choices of the autoregressive representation of $X_{nt}$ may turn out into representations $\tilde{X}_{nt} = \tilde{X}_{nt} + \tilde{\xi}_{nt}$ with a non-idiotsyncratic $\tilde{\xi}_{nt}$. As an example, consider again model (3.10):

$$X_{nt} = \alpha_n u_t + \beta_n u_{t-1}.$$

If $c = (c_1 c_2 \cdots c_q)$ is orthogonal to $\beta_n$, then an autoregressive representation is

$$[I - (\delta \beta_n cL)]X_{nt} = \alpha_n u_t,$$

where $\delta = (c\alpha_n)^{-1}$ and therefore

$$[I - (\delta \beta_n cL)]X_{nt} = \alpha_n u_t + [I - (\delta \beta_n cL)]\tilde{\xi}_{nt} = \tilde{X}_{nt} + \tilde{\xi}_{nt}.$$  \hspace{1cm} (4.20)

We have

$$\tilde{\xi}_{nt} = \xi_{nt} + \delta \beta_n [c\xi_{nt-1}].$$

Thus the vector $\tilde{\xi}_{nt}$ is not idiotsyncratic.

5. Estimation: a sketch

In the previous section we have shown that Assumption A7' implies the existence of representation (4.18). We now provide a procedure for constructing $H_n(L)$, $R_{nt}$ and $\tilde{v}_t$. Starting with the spectral density of the common components $\Sigma_{\tilde{X}_{nt}}(\theta)$. As we assume that $\Sigma_{\tilde{X}_{nt}}(\theta)$ is known, this is to be considered only as a sketch of an estimation procedure. In practical situations $\Sigma_{\tilde{X}_{nt}}(\theta)$ is not known; we start with an estimate $\hat{\Sigma}_{\tilde{X}_{nt}}(\theta)$ and compute the corresponding sample-dependent $H_n(L)$, $R_{nt}$ and $\tilde{v}_t$. A proof of consistency of such estimates, for $n$ and $T$ tending to infinity, is beyond the scope of the present paper and left for future research. Let us only observe here that our assumptions, A1 through A9, must be enhanced with conditions ensuring consistency of a smoothed periodogram of $X_{nt}$ (see e.g. Brockwell and Davis, 1991, pp. 445–7).

Firstly we determine $H_n(L)$ and $R_{nt}$. We keep assuming that $n = (q + 1)m$. Using the $m$ diagonal $(q + 1) \times (q + 1)$ blocks of $\Sigma_{\tilde{X}_{nt}}(\theta)$ we can obtain the matrices

$$G_{ij}(L), \hspace{1cm} \Gamma_{ij}, \hspace{1cm} j = 1, 2, \ldots, m,$

corresponding to the Wold representation

$$X_{nt} = G_{ij}(L)w_{ntj}.$$  \hspace{1cm} (5.20)

Note that neither the $X_{ntj}$ nor the $w_{ntj}$ are observable. The matrix $G_{ij}(L)$ is $(q + 1) \times (q + 1)$ and has rational entries. Moreover, $G_{ij}(0) = I_{q+1}$. The matrix $\Gamma_{ij}$ is the covariance matrix of the $(q + 1) \times 1$ one-step-ahead prediction error vector $w_{ntj}$. The matrix
The whitenoise vectors \( \mathbf{w}_{j\ell} \) are different in general but, by Assumption A7', span the same space. Therefore, for \( j=2, \ldots, m \),\n
\[
\mathbf{w}_{j\ell} = K_j \mathbf{v}_{1\ell},
\]

where \( K_j \) is orthogonal. Using (5.21),

\[
K_j = \mathbb{E} \left( \mathbf{v}_{1\ell} \mathbf{v}_{1\ell}' \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( A_{1j}^{-1} P_{1j} \mathbf{w}_{1\ell} \right) \left( e^{it\theta} \Sigma_{11} \left( \theta \right) A_{1j} \left( e^{it\theta} \right)^{-1} P_{1j} A_{1j}^{-1} \right) d\theta,
\]

where \( \Sigma_{11} \left( \theta \right) \) is the \((q+1) \times (q+1)\) cross-spectrum of \( \mathbf{X}_{1\ell} \) and \( \mathbf{X}_{1\ell} \) (a submatrix of \( \mathbf{X} \)). In conclusion, setting \( \mathbf{v}_i = \mathbf{v}_{1\ell} \), we have

\[
\begin{pmatrix}
A_{1j}(L) & 0 & \cdots & 0 \\
0 & A_{2j}(L) & \cdots & 0 \\
0 & 0 & \cdots & A_{mj}(L)
\end{pmatrix}
\begin{pmatrix}
\mathbf{X}_{1\ell} \\
\mathbf{X}_{2\ell} \\
\vdots \\
\mathbf{X}_{m\ell}
\end{pmatrix}
=
\begin{pmatrix}
\Sigma_{1j} \\\n\Sigma_{2j} K_{1j} \\\n\vdots \\
\Sigma_{mj} K_{mj}
\end{pmatrix}
\mathbf{v}_i,
\]

and therefore

\[
H_n(L) \mathbf{x}_n = \mathbf{R}_n \mathbf{v}_i + H_n(L) \mathbf{\xi}_n,
\]

where \( H_n(L) \) and \( \mathbf{R}_n \) are defined in (5.22).

The next step determines \( \mathbf{v}_i \). Note that the matrix \( \mathbf{R}_n \) has mutually orthogonal columns. As a consequence, \( \mathbf{R}_n \mathbf{R}_n' \) has the eigenvalues of \( \mathbf{R}_n \mathbf{R}_n' \) on the main diagonal (this is easily seen) and zero elsewhere. Setting \( \mathbf{M}_n = \left( \mathbf{R}_n \mathbf{R}_n' \right)^{-1} \)

\[
\mathbf{M}_n \mathbf{R}_n H_n(L) \mathbf{x}_n = \mathbf{M}_n \mathbf{R}_n \mathbf{v}_i + \mathbf{M}_n \mathbf{R}_n H_n(L) \mathbf{\xi}_n = \mathbf{v}_i + \mathbf{M}_n \mathbf{R}_n H_n(L) \mathbf{\xi}_n.
\]

Denoting by \( R_k \) the entries of \( \mathbf{R}_n \), the sth row of \( \mathbf{M}_n \mathbf{R}_n' \) is

\[
\frac{1}{n} \sum_{k=1}^{R_k^2} R_{ks} \mathbf{x}_n \mathbf{v}_i.
\]

Thus the sum of its squares is

\[
\frac{1}{n} \sum_{k=1}^{R_k^2} R_{ks}^2 \mathbf{x}_n \mathbf{v}_i = \frac{1}{n} \sum_{k=1}^{R_k^2} R_{ks}^2
\]

i.e. the reciprocal of the sth eigenvalue of \( \mathbf{R}_n \mathbf{R}_n' \). By Assumption A8, this reciprocal tends zero as \( n \to \infty \). Because \( H_n(L) \mathbf{\xi}_n \) is idiosyncratic, the term \( \mathbf{M}_n \mathbf{R}_n H_n(L) \mathbf{\xi}_n \) tends to zero in mean square as \( n \to \infty \) (see e.g. Forni and Lippi, 2001). Thus

\[
\mathbf{M}_n \mathbf{R}_n H_n(L) \mathbf{x}_n \to \mathbf{v}_i
\]

in mean square as \( n \to \infty \). Lastly, \( \mathbf{X}_n \) results from inversion of \( H_n(L) \).

6. Conclusions

Forni et al. (2000) estimate \( \Sigma^2_n(\theta) \), the spectral density of the common components of model (1.1), by means of \( q \) dynamic principal components, and provide a factorization of \( \Sigma^2_n(\theta) \). However, the estimator of the common components based on such factorization, though consistent, applies two-sided filters to the observable variables \( x_t \).

In the present paper, under the assumption of rationality for \( \Sigma^2_n(\theta) \) and other mild requirements, we obtain a factorization of \( \Sigma^2_n(\theta) \) which only employs one-sided filters.

An important feature of our method is that the problem of factoring \( \Sigma^2_n(\theta) \), which is of dimension \( n \) and rank \( q \), is solved by separately factoring many spectral matrices of dimension \( q + 1 \).

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References


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