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Second-Order Approximation of Dynamic Models with Time-Varying Risk

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Abstract

This paper provides first and second-order approximation methods for the solution of non-linear dynamic stochastic models in which the exogenous state variables follow conditionally-linear stochastic processes displaying time-varying risk. The first-order approximation is consistent with a conditionally-linear model in which risk is still time-varying but has no distinct role – separated from the primitive stochastic disturbances – in influencing the endogenous variables. The second-order approximation of the solution, instead, is sufficient to get this role. Moreover, risk premia, evaluated using only a first-order approximation of the solution, will be also time varying.

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1 Introduction

In the last decade, there has been an increasing interest among researchers and policymakers in developing dynamic general equilibrium models to study business cycle properties of macroeconomic variables and to conduct policy analysis. This research agenda has been accompanied by parallel developments in solution methods and estimation techniques aimed at handling different challenges that richer models pose to economists. For example, second-order approximation techniques have been proposed by Schmitt-Grohé and Uribe (2004) and Benigno and Woodford (2008) to address welfare comparisons across policy regimes while Bayesian analysis has been developed for estimating dynamic general equilibrium models (An and Schorfeide, 2007).

In this work, we propose a solution method for non-linear dynamic stochastic models in which the exogenous stochastic processes display time-varying risk. While the use of models with time-varying risk is quite popular in finance, only recently there has been considerable attention on the role and the effects that risk or uncertainty and their variations over time have on macroeconomic variables.[1] Our solution method is based on appropriately-defined first and second-order approximations of the solution which can be effective in studying how time-variation in the exogenous risk influences the equilibrium allocation in standard macroeconomic models. This is in contrast with other solution methods, recently proposed, relying on third-order approximations as in Fernandez-Villaverde et al. (2009).[2]

We consider a class of non-linear dynamic stochastic models in which the exogenous state variables follow conditionally-linear stochastic processes where either variances or standard deviations of the primitive shocks are modelled through stochastic linear processes. We show that a first-order approximation of the solution can be consistent with a conditionally-linear model in which the process for the exogenous state variables is not approximated and still displays time-varying volatility. Indeed, whether the exogenous state process is approximated or not does not affect the other coefficients of the linear

[1]Bloom (2009) examines the effects of an increase in uncertainty on investment and hiring decisions by firms, Bloom, Floetotto and Jaimovich (2009) extend a canonical real business cycle model to study the impact of change in the variance to productivity innovation on economic activity while Fernandez-Villaverde, Guerron-Quintana, Rubio-Ramirez and Uribe (2009) show how changes in the volatility of the foreign real interest rate are an important mechanism in explaining the behavior of output, consumption and investment in emerging market economies.

[2]Bloom et al. (2009), following Krussell and Smith (1998), use instead a value function iteration approach which is more computationally demanding and difficult to implement even in small scale dynamic general equilibrium models.

approximation nor the dimension of the relevant endogenous state variables.[3]

There are three clear advantages of following a conditionally-linear approximation instead of a fully-linear approximation. First, the approximated linear solution would still display a role for time-varying risk in affecting the evolution of the endogenous variables of the model.[4] However, this is not a "distinct and direct" role, as risk and primitive shocks are not disjoint arguments: if shocks are zero, risk does not influence directly the endogenous variables. Second, the fact that stochastic volatility enters the first-order approximation, although not disjointly, has important implications also for higher-order approximations. In particular, we show that a second-order approximation of the policy rules is sufficient to imply a "distinct and direct" role for time-varying volatility in affecting the endogenous variables, whereas with other approaches a more computationally-demanding third-order approximation is needed. Third, a conditionally-linear approximation, where volatility is still time-varying, can be sufficient to characterize time variation in covariances and therefore in risk premia, whereas a standard linear approximation would only deliver constant risk premia.

Our paper is related to Justiniano and Primiceri (2008) since their *partially-nonlinear* approximation, as a first-order approximation of the solution, agrees with our proposed conditionally-linear approximation when the exogenous state variables follow conditionally-linear processes. We also provide a second-order approximation of the solution to characterize a distinct role for exogenous risk in affecting the endogenous variables. In particular we consider two models of time-varying volatility, one with a stochastic linear process for the standard deviation of the primitive shocks, as in Justiniano and Primiceri (2008), and another with a linear process for the variance.[5] The latter model is indeed also more parsimonious in the second-order approximation.

Our contribution can also be read as a generalization of the second-order approximation methods of Schmitt-Grohé and Uribe (2004), Kim et al. (2008) and Gomme and Klein (2008) to the case in which the exogenous state variables follow heteroskedastic processes. Recent works by Fernandez-Villaverde et al. (2009, 2010) have provided approximation methods for exactly the same model as ours in which the standard-deviation

[3]We follow here the insights of Justiniano and Primiceri (2008) which indeed define a *partially-nonlinear* approximation.

[4]This role has been particular relevant for Justiniano and Primiceri (2008) to deliver a model that can be estimated parsimoniously in order to investigate which sources of risk have contributed the most to the fall in macroeconomic volatility associated with the US Great Moderation.

[5]Justiniano and Primiceri (2008), however, model the log of the standard deviation as a stochastic linear process.

of the primitive shocks is time-varying. However, they consider a fully-linear approximation in which even the process for the exogenous state variable is linearized.[6] By doing this, time-varying volatility is lost in the first-order approximation and, to get a distinct role for risk in affecting the endogenous variables, a third-order approximation is needed. Amisano and Tristani (2009) analyze models where volatility is subject to discrete switching-regime changes and show that the time-varying volatility can affect the second-order approximation. Finally, there are other contributions which have been interested in characterizing how time-varying risk affects endogenous variables. But in these cases, as in Rudebush and Swanson (2008), exogenous state variables follow homoskedastic processes as in Schmitt-Grohé and Uribe (2004) and time-varying endogenous (not exogenous) risk affects the endogenous variables only in a third-order approximation.

The structure of this work is the following. Section 2 presents first and second-order approximations in a model in which the exogenous state variables have time-varying linear process for the conditional standard deviation. Section 3 considers the case in which time-varying risk is modelled using a linear process for the conditional variance of the primitive shocks. Section 4 applies our methods to the benchmark neoclassical growth model. Section 5 concludes.

2 A model with time-varying standard deviations

We consider the following general model which encompasses a wide variety of dynamic stochastic models:

$$E_t \left\{ f(y_{t+1}, x_{t+1}, y_t, x_t) \right\} = \mathbf{0}, \tag{1}$$

where $E_t\{\cdot\}$ denotes the mathematical expectations operator conditional on the information available at date t and $f(\cdot)$ is a vector, of size n, of functions. The vector y_t, of non-predetermined variables, is of size $n_y \times 1$ while the vector x_t of state variables is of size $n_x \times 1$, with $n_y + n_x = n$. In particular, the vector x_t can be partitioned into a vector of endogenous state variables k_t and a vector of exogenous predetermined variables z_t of size $n_z \times 1$, as follows:

$$x_t = \begin{bmatrix} k_t \\ z_t \end{bmatrix}.$$

[6]This is because they do not necessarily assume a conditionally-linear process for the exogenous state variables.

The vector z_t follows the exogenous stochastic process given by

$$z_{t+1} = \Lambda_z z_t + Z\xi_{t+1} \tag{2}$$

where Z and Λ_z are matrices of order $n_z \times n_z$. The vector ξ_{t+1} is also of dimension $n_z \times 1$ and is given by

$$\xi_{t+1} = U_t \varepsilon_{z,t+1} \tag{3}$$

where $\varepsilon_{z,t+1}$ is a $n_z \times 1$ vector of innovations, which are assumed to have a bounded support and to be independently and identically distributed with mean zero and variance/covariance matrix I_z, where I_z is an identity matrix of dimension $n_z \times n_z$; U_t is a diagonal matrix of dimension $n_z \times n_z$ whose elements on the diagonal are collected into vector u_t, of dimension $n_z \times 1$. In particular u_t follows the exogenous stochastic linear process given by

$$u_{t+1} = \sigma_z(I_z - \Lambda_u)\bar{u} + \Lambda_u u_t + \sigma_v V \varepsilon_{v,t+1} \tag{4}$$

where V and Λ_u are matrices of order $n_z \times n_z$, $\varepsilon_{v,t+1}$ is a $n_z \times 1$ vector of innovations which are assumed to have a bounded support and to be independently and identically distributed with mean zero and variance/covariance matrix I_z; $\bar{u}$ is a vector of dimension $n_z \times 1$ while σ_z and σ_v are scalars with σ_z,$\sigma_v \geq 0$.[7] We also assume that the initial condition on the process for u_t is such that $u_{t_0-1} = \sigma_z \bar{u}$.[8]

Given equations (3) and (4), the model generalizes the framework of Schmitt-Grohé and Uribe (2004) to a case in which the volatility is time varying and stochastic. In particular, the process for the exogenous state variable (2) is conditionally linear where each element of the vector u_t captures the conditional standard deviation of each element of the stochastic disturbance ξ_{t+1}; such standard deviations are allowed to vary over time in a stochastic way following the autoregressive process described by equation (4). The model boils down to the framework of Schmitt-Grohé and Uribe (2004) under the assumptions $\sigma_v = 0$ and $\bar{u}_i = 1$ for all $i = 1, ..., n_z$, since in this case

$$\xi_{t+1} = \sigma_z \varepsilon_{z,t+1}.$$

[7]Fernandez-Villaverde et al. (2009, 2010) and Justiniano and Primiceri (2008) model a linear process for the log of the standard deviations to assure that variances remain always positive. This is not necessary in our case since we are assuming a bounded support for the shock $\varepsilon_{v,t}$ which is needed anyway, for the goodness of the approximation.

[8]Notice that in (3), there is no need to add another scalar of the type σ_ξ as a perturbation parameter since the variables on the diagonal matrix U_t are already subject themselves to perturbation. Indeed, this is why we add the perturbation parameters σ_z and σ_v in (4) and assume an initial condition for u_t of the form $u_{t_0-1} = \sigma_z \bar{u}$.

We make three important remarks on the above structure which are important to define the class of models which we are interested in. First, equation (4) is not part of the equilibrium conditions (1). Second, the vector u_t is not a distinct argument of the set of equilibrium conditions with respect to what is already captured by the state vector x_t. Third, the vector of exogenous state variables z_t follows a conditionally-linear process given by (2). We are not interested in characterizing approximations of more general models in which the exogenous state variables follow instead non-linear models.

2.1 Solution

Given the above defined model and structure of the stochastic processes, a solution of (1) takes the form

$$y_t = g(x_t, u_t, \sigma_z, \sigma_v) \tag{5}$$

$$x_{t+1} = h(x_t, u_t, \sigma_z, \sigma_v) + \bar{h}_\xi \xi_{t+1} \tag{6}$$

for generic functions $g(\cdot)$ and $h(\cdot)$ where $\bar{h}_\xi$ is a known $n_x \times n_z$ matrix of the form

$$\bar{h}_\xi \equiv \begin{bmatrix} \mathbf{0} \\ Z \end{bmatrix}.$$

We are interested in a second-order approximation of (5) and (6) around a deterministic steady state in which $\sigma_z = \sigma_v = 0$ and $u_t = \sigma_z \bar{u} = \mathbf{0}$. In this deterministic steady state $x_t = \bar{x}$ and $y_t = \bar{y}$ satisfy

$$\begin{aligned} \bar{y} &= g(\bar{x}, \mathbf{0}, 0, 0) \\ \bar{x} &= h(\bar{x}, \mathbf{0}, 0, 0) \end{aligned}$$

or, equivalently

$$f(\bar{y}, \bar{x}, \bar{y}, \bar{x}) = \mathbf{0}.$$

2.2 First-order approximation

First, we characterize a first-order approximation of (5) and (6). We guess and verify that this approximation takes the form

$$\tilde{y}_t = \bar{g}_x \tilde{x}_t \tag{7}$$

$$\tilde{x}_{t+1} = \bar{h}_x \tilde{x}_t + \bar{h}_\xi \xi_{t+1} \tag{8}$$

where $\tilde{y}_t \equiv y_t - \bar{y}$, $\tilde{x}_t \equiv x_t - \bar{x}$ and $\bar{g}_x$ and $\bar{h}_x$ are the Jacobian matrices of the functions $g(\cdot)$ and $h(\cdot)$ with respect to x, of size $n_y \times n_x$ and $n_x \times n_x$, respectively, and evaluated at the steady state. To verify this guess, we take a first-order approximation of (1), obtaining

$$D\bar{f}_{\hat{y}} \cdot E_t\tilde{y}_{t+1} + D\bar{f}_{\hat{x}} \cdot E_t\tilde{x}_{t+1} + D\bar{f}_y \cdot \tilde{y}_t + D\bar{f}_x \cdot \tilde{x}_t = 0 \tag{9}$$

where $D\bar{f}_{\hat{y}}$, $D\bar{f}_{\hat{x}}$, $D\bar{f}_y$, and $D\bar{f}_x$ are matrices containing the respective gradients of the vector of functions $f(\cdot)$ taken with respect to the arguments of the function and evaluated at the above-defined steady state. In particular hats denote the gradient with respect to time $t+1$ vectors, $\hat{y}$ stands for y_{t+1} and $\hat{x}$ for x_{t+1}.

To verify our guess, we plug (7) and (8) into (9) noting that $E_t\xi_{t+1} = 0$. It follows that the matrices $\bar{g}_x$ and $\bar{h}_x$ have to satisfy the following set of $n \times n_x$ conditions.

$$D\bar{f}_{\hat{y}}\bar{g}_x\bar{h}_x + D\bar{f}_y\bar{g}_x + D\bar{f}_{\hat{x}}\bar{h}_x + D\bar{f}_x = \mathbf{0}. \tag{10}$$

The above set of conditions can be solved using standard algorithms. Indeed, it corresponds to that of Schmitt-Grohé and Uribe (2004) in the case in which the volatility is non stochastic: the matrices $\bar{g}_x$ and $\bar{h}_x$ are the same as in their framework. However, the overall solution given by (7) and (8) does not correspond to their solution since the driving stochastic disturbance is still a non-linear process, which is described by (3). In particular, (7), (8) together with (3) and (4) represent the best conditionally-linear solution of (9) given that the exogenous state variables follow (2)-(4) and given that the vector u_t does not enter the set of equations (1) nor their arguments. Notice first that (9) just imposes restrictions on the linear approximations of the functions $g(\cdot)$ and $h(\cdot)$ of (5) and (6). Since $E_t\xi_{t+1} = 0$, the approximations (7) and (8) are conditionally linear. Moreover since $\bar{h}_\xi$ is known, the best approximation of the term $\bar{h}_\xi\xi_{t+1}$, in equation (6), is just the term itself which is what appears in (8).

Fernandez-Villaverde et al. (2009, 2010) assume that

$$\xi_{t+1} = \Omega_{t+1}\sigma_z\varepsilon_{z,t+1} \tag{11}$$

where Ω_{t+1} is a diagonal matrix whose diagonal contains the vector of standard deviations ω_{t+1}; the log of the standard deviations follow

$$\log\omega_{t+1} = \log\bar{\omega} + \Lambda_\omega \log\omega_t + \sigma_{_\omega} V_\omega\varepsilon_{\omega,t+1} \tag{12}$$

given appropriately defined matrices Λ_ω and V_ω, given the vector $\log\bar{\omega}$ and stochastic disturbances $\varepsilon_{\omega,t+1}$ where $\sigma_{_\omega}$ and σ_z are scalars with $\sigma_{_\omega}, \sigma_z \geq 0$. If $\varepsilon_{\omega,t+1}$ and $\varepsilon_{z,t+1}$ are statistically independent, the process for the exogenous state variables is also conditionally linear.[9] In this framework, Fernandez-Villaverde et al. (2009, 2010) look for a fully linear approximation in which (11) is also linearized. However, an appropriate linear approximation of (11) – also with respect to the scalar σ_z – would be zero and therefore the overall first-order approximation of the solution would no longer be stochastic. However, an alternative linear approximation would be to approximate ξ_{t+1} as a linear function of $\sigma_z \varepsilon_{z,t+1}$ in a way that also (8) becomes linear in the stochastic disturbances $\sigma_z \varepsilon_{z,t+1}$. In their case a linear approximation of the exogenous state variables takes the form

$$\tilde{x}_{t+1} = \bar{h}_x \tilde{x}_t + \bar{h}_\xi \bar{\Omega} \sigma_z \varepsilon_{z,t+1}, \tag{13}$$

in which $\bar{\Omega}$ is the diagonal matrix containing the vector $\bar{\omega}$ on its diagonal. Applying this approximation to our context requires to set $\sigma_v = 0$ in (4) to obtain a linear approximation of the exogenous state variables of the form

$$\tilde{x}_{t+1} = \bar{h}_x \tilde{x}_t + \bar{h}_\xi \bar{U} \sigma_z \varepsilon_{z,t+1}, \tag{14}$$

in which $\bar{U}$ is the diagonal matrix containing the vector $\bar{u}$ on its diagonal. Solution (14) is now in the form of a linear multivariate autoregressive process, but it is not the best conditionally-linear approximation of (6). In our approximation (7) and (8) together with (3) and the linear process (4) are all that is needed to characterize the conditionally-linear approximation. In Fernandez-Villaverde et al. (2009, 2010), it suffices instead to consider (7), (8) and (14) where time-varying volatility ceases to play a role. However, there is no restriction in our and their approximation methods that should require to linearize also ξ_{t+1}. This is not even a requirement for analytical tractability since conditionally linear heteroskedastic models are commonly used and most recently in macro models.[10]

Indeed, we will show that there are actually several advantages of our conditionally-linear approximation. A first one is that in our case, first-order approximations will retain a role for stochastic volatility, as in Justiniano and Primiceri (2008), although not a distinct role, since risk enters only jointly with the structural shock. In Fernandez-Villaverde et al. (2009, 2010), on the contrary, first-order approximations will lose any role for time-varying risk. Such difference between our and their linear approximations will also be importantly reflected in the second-order approximation and especially in

[9] Fernandez-Villaverde et al. (2009, 2010) do not assume explicitly conditional linearity and, perhaps, they are looking at the broader class of non-linear processes for the exogenous state variables.

[10] See Justiniano and Primiceri (2008) for further arguments to justify what they call a "partially nonlinear" approximation in the same model of Fernandez-Villaverde et al. (2009, 2010). This would be a conditionally-linear approximation when $\varepsilon_{\omega,t+1}$ and $\varepsilon_{z,t+1}$ are statistically independent.

the role that time-varying volatility plays in it. A further advantage of our approach, indeed, is that time-varying volatility will play a "distinct and direct" role already in a second-order approximation whereas in Fernandez-Villaverde et al. (2009, 2010) at least a third-order approximation is needed. With "distinct and direct" role, we mean that the impulse response functions of the variables of interest with respect to the primitive volatility shock $\varepsilon_{v,t+1}$ can be in general different from zero.[11] As a consequence, a very appealing implication of our method is that risk premia evaluated using first-order approximations will be time-varying, in contrast to the constant risk premia implied by the framework of Fernandez-Villaverde et al. (2009, 2010). In their context, indeed, higher-order approximations will be needed to characterize time-varying risk premia.

We conclude this section by noting that a complete linear approximation to (5) and (6) can be represented as

$$\tilde{y}_t = \bar{g}_x \tilde{x}_t + \bar{g}_u u_t + \bar{g}_z \sigma_z + \bar{g}_v \sigma_v$$

$$\tilde{x}_{t+1} = \bar{h}_x \tilde{x}_t + \bar{h}_u u_t + \bar{h}_z \sigma_z + \bar{h}_v \sigma_v + \bar{h}_\xi \xi_{t+1}.$$

However, plugging the above equations into (9) shows that $\bar{g}_u$, $\bar{g}_z$, $\bar{g}_v$, $\bar{h}_u$, $\bar{h}_z$, $\bar{h}_v$ are all zero matrices.

2.3 Second-order approximation

In this section, we characterize a second-order approximation of the solutions (5) and (6). We guess and verify that it takes the form

$$\tilde{y}_t = \bar{g}_x \tilde{x}_t + \frac{1}{2}(I_y \otimes \tilde{x}_t')\bar{g}_{xx}\tilde{x}_t + \frac{1}{2}(I_y \otimes u_t')\bar{g}_{uu}u_t + \frac{1}{2}\bar{g}_{vv}\sigma_v^2 + \frac{1}{2}\bar{g}_{zz}\sigma_z^2 + \bar{g}_{zu}\sigma_z u_t \tag{15}$$

$$\tilde{x}_{t+1} = \bar{h}_x \tilde{x}_t + \frac{1}{2}(I_x \otimes \tilde{x}_t')\bar{h}_{xx}\tilde{x}_t + \frac{1}{2}(I_x \otimes u_t')\bar{h}_{uu}u_t + \frac{1}{2}\bar{h}_{vv}\sigma_v^2 + \frac{1}{2}\bar{h}_{zz}\sigma_z^2 + \bar{h}_{zu}\sigma_z u_t + \bar{h}_\xi \xi_{t+1} \tag{16}$$

where I_y and I_x are identity matrices of order $n_y \times n_y$ and $n_x \times n_x$, respectively, $\otimes$ denotes the Kronecker product and $\bar{g}_{xx}$, $\bar{g}_{uu}$, $\bar{g}_{zz}$, $\bar{g}_{vv}$, $\bar{g}_{zu}$, $\bar{h}_{xx}$, $\bar{h}_{uu}$, $\bar{h}_{zz}$, $\bar{h}_{vv}$, $\bar{h}_{zu}$ are conformable matrices, corresponding to the Magnus-Neudecker Hessian matrices of functions $\bar{g}$ and $\bar{h}$ with respect to the arguments in the indexes.[12] Specifically, $\bar{g}_{xx}$ is defined as

$$\bar{g}_{xx} = \frac{\partial^2 g(x, u, \sigma_z, \sigma_v)}{\partial x \partial x'} = D_x \text{vec}\Big[\big(D_x g(\bar{x}, \mathbf{0}, 0, 0)\big)'\Big],$$

[11] Accordingly, since in our first-order approximation there is no distinct role for volatility in affecting the endogenous variables, the impulse response of any variable with respect to a volatility shock is always zero.

[12] See Magnus and Neudecker (1999).

evaluated at the steady state, and consists of n_y vertically stacked symmetric $n_x \times n_x$ matrices ($\bar{g}_{xx}$ is therefore of size $n_y \cdot n_x \times n_x$). All remaining matrices are defined analogously.

To evaluate this guess, we take a second-order approximation of (1), to get

$$\begin{aligned} 0 = E_t \Big\{ & D\bar{f}^i_{\hat{y}} \cdot \tilde{y}_{t+1} + D\bar{f}^i_{\hat{x}} \cdot \tilde{x}_{t+1} + D\bar{f}^i_{y} \cdot \tilde{y}_t + D\bar{f}^i_{x} \cdot \tilde{x}_t \\ & + \frac{1}{2}\tilde{y}'_{t+1} \cdot D\bar{f}^i_{\hat{y}\hat{y}} \cdot \tilde{y}_{t+1} + \tilde{x}'_{t+1} \cdot D\bar{f}^i_{\hat{y}\hat{x}} \cdot \tilde{y}_{t+1} + \tilde{y}'_t \cdot D\bar{f}^i_{\hat{y}y} \cdot \tilde{y}_{t+1} \\ & + \tilde{x}'_t \cdot D\bar{f}^i_{\hat{y}x} \cdot \tilde{y}_{t+1} + \frac{1}{2}\tilde{x}'_{t+1} \cdot D\bar{f}^i_{\hat{x}\hat{x}} \cdot \tilde{x}_{t+1} + \tilde{y}'_t \cdot D\bar{f}^i_{\hat{x}y} \cdot \tilde{x}_{t+1} \\ & + \tilde{x}'_t \cdot D\bar{f}^i_{\hat{x}x} \cdot \tilde{x}_{t+1} + \frac{1}{2}\tilde{y}'_t \cdot D\bar{f}^i_{yy} \cdot \tilde{y}_t + \tilde{x}'_t \cdot D\bar{f}^i_{yx} \cdot \tilde{y}_t + \frac{1}{2}\tilde{x}'_t \cdot D\bar{f}^i_{xx} \cdot \tilde{x}_t \Big\}, \end{aligned} \tag{17}$$

for each $i = 1, ..., n$ and where f^i denotes the i-component of the vector f.

The second-order approximation of (1) can be cast in a more compact form as

$$\mathbf{0} = E_t \left\{ D\bar{f} \begin{bmatrix} \tilde{y}_{t+1} \\ \tilde{x}_{t+1} \\ \tilde{y}_t \\ \tilde{x}_t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} I_n \otimes \tilde{y}_{t+1} \\ I_n \otimes \tilde{x}_{t+1} \\ I_n \otimes \tilde{y}_t \\ I_n \otimes \tilde{x}_t \end{bmatrix}' H\bar{f} \begin{bmatrix} \tilde{y}_{t+1} \\ \tilde{x}_{t+1} \\ \tilde{y}_t \\ \tilde{x}_t \end{bmatrix} \right\}, \tag{18}$$

where $D\bar{f} \equiv \begin{bmatrix} D\bar{f}_{\hat{y}} & D\bar{f}_{\hat{x}} & D\bar{f}_{y} & D\bar{f}_{x} \end{bmatrix}$ denotes the $n \times 2n$ Jacobian matrix of function f, and $H\bar{f}$ the corresponding $2n^2 \times 2n$ Magnus–Neudecker Hessian matrix, evaluated at the steady state:

$$H\bar{f} = D\text{vec}\big[(D\bar{f})'\big].$$

We use equations (7) and (8) into (18) to evaluate the second-order terms and (15) and (16) to evaluate the first-order terms, taking into account the restrictions (10).

Making use of $E_t\xi_{t+1} = \mathbf{0}$, we obtain:

$$\begin{aligned} \mathbf{0} = & \frac{1}{2} E_t \Big\{ D\bar{f}_{\hat{y}} \Big[(\bar{g}_x \otimes \tilde{x}'_t)\, \bar{h}_{xx}\tilde{x}_t + (\bar{g}_x \otimes u'_t)\, \bar{h}_{uu}u_t + \bar{g}_x\bar{h}_{zz}\sigma_z^2 + \bar{g}_x\bar{h}_{vv}\sigma_v^2 + 2\bar{g}_x\bar{h}_{zu}\sigma_z u_t \\ & + \left(I_y \otimes u'_{t+1}\right) \bar{g}_{uu}u_{t+1} + \left[I_y \otimes (\bar{h}_x\tilde{x}_t + \bar{h}_\xi\xi_{t+1})'\right] \bar{g}_{xx}(\bar{h}_x\tilde{x}_t + \bar{h}_\xi\xi_{t+1}) + \bar{g}_{zz}\sigma_z^2 + \bar{g}_{vv}\sigma_v^2 \\ & + 2\bar{g}_{zu}\sigma_z u_{t+1} \Big] + D\bar{f}_{\hat{x}} \Big[(I_x \otimes \tilde{x}'_t)\, \bar{h}_{xx}x_t + (I_x \otimes u'_t)\, \bar{h}_{uu}u_t + \bar{h}_{zz}\sigma_z^2 + \bar{h}_{vv}\sigma_v^2 + 2\bar{h}_{zu}\sigma_z u_t \Big] \\ & + D\bar{f}_y \Big[(I_y \otimes \tilde{x}'_t)\, \bar{g}_{xx}x_t + (I_y \otimes u'_t)\, \bar{g}_{uu}u_t + \bar{g}_{zz}\sigma_z^2 + \bar{g}_{vv}\sigma_v^2 + 2\bar{g}_{zu}\sigma_z u_t \Big] \\ & + \left[I_n \otimes (\bar{g}_x\bar{h}_x\tilde{x}_t + \bar{g}_x\bar{h}_\xi\xi_{t+1})'\right] H\bar{f}_{\hat{y}} \cdot \tilde{w}_{t+1} + \left[I_n \otimes (\bar{h}_x\tilde{x}_t + \bar{h}_\xi\xi_{t+1})'\right] H\bar{f}_{\hat{x}} \cdot \tilde{w}_{t+1} \\ & + (I_n \otimes \tilde{x}'_t\bar{g}'_x)\, H\bar{f}_y \cdot \tilde{w}_{t+1} + (I_n \otimes \tilde{x}'_t)\, H\bar{f}_x \cdot \tilde{w}_{t+1} \Big\}, \end{aligned} \tag{19}$$

where $\tilde{w}_{t+1} \equiv [\tilde{y}'_{t+1} \quad \tilde{x}'_{t+1} \quad \tilde{y}'_t \quad \tilde{x}'_t]'$ is a $2n \times 1$ vector and $H\bar{f}_{\hat{y}}$, $H\bar{f}_{\hat{x}}$, $H\bar{f}_y$, and $H\bar{f}_x$ are the Magnus-Neudecker Hessian matrices of the vector of functions $f(\cdot)$ taken with respect to the arguments of the function and evaluated at the above-defined steady state, such that

$$H\bar{f} = \begin{bmatrix} H\bar{f}_{\hat{y}} \\ H\bar{f}_{\hat{x}} \\ H\bar{f}_y \\ H\bar{f}_x \end{bmatrix}.$$

Specifically, $H\bar{f}_{\hat{y}}$ is defined as

$$H\bar{f}_{\hat{y}} = D\text{vec}\big[(D\bar{f}_{\hat{y}})'\big],$$

and analogously for the other terms. Moreover, equations (7) and (8) imply

$$\tilde{w}_{t+1} = \bar{M}_x \tilde{x}_t + \bar{M}_\xi \xi_{t+1}, \tag{20}$$

where M_x and M_ξ are matrices of order $2n \times n_x$ and $2n \times n_z$, respectively, defined by

$$\bar{M}_x \equiv \begin{bmatrix} \bar{g}_x \bar{h}_x \\ \bar{h}_x \\ \bar{g}_x \\ I_x \end{bmatrix} \qquad \bar{M}_\xi \equiv \begin{bmatrix} \bar{g}_x \bar{h}_\xi \\ \bar{h}_\xi \\ \mathbf{0}_{(n_y \times n_z)} \\ \mathbf{0}_{(n_x \times n_z)} \end{bmatrix}. \tag{21}$$

From equation (19), and using (20), we can collect the quadratic terms in the vector $\tilde{x}_t$, to obtain

$$\begin{aligned} \mathbf{0} = \frac{1}{2} E_t \Big\{ & \left(D\bar{f}_{\hat{y}} \cdot \bar{g}_x \otimes \tilde{x}'_t\right) \bar{h}_{xx} \tilde{x}_t + \left(D\bar{f}_{\hat{y}} \otimes \tilde{x}'_t \bar{h}'_x\right) \bar{g}_{xx} \bar{h}_x \tilde{x}_t + \left(D\bar{f}_{\hat{x}} \otimes \tilde{x}'_t\right) \bar{h}_{xx} x_t \\ & + \left(D\bar{f}_y \otimes \tilde{x}'_t\right) \bar{g}_{xx} x_t + \left(I_n \otimes \tilde{x}'_t \bar{h}'_x \bar{g}'_x\right) H\bar{f}_{\hat{y}} \cdot \bar{M}_x \tilde{x}_t + \left(I_n \otimes \tilde{x}'_t \bar{h}'_x\right) H\bar{f}_{\hat{x}} \cdot \bar{M}_x \tilde{x}_t \\ & + \left(I_n \otimes \tilde{x}'_t \bar{g}'_x\right) H\bar{f}_y \cdot \bar{M}_x \tilde{x}_t + \left(I_n \otimes \tilde{x}'_t\right) H\bar{f}_x \cdot \bar{M}_x \tilde{x}_t \Big\}. \end{aligned} \tag{22}$$

Following Gomme and Klein (2008), given a generic $n \cdot m \times m$ matrix A consisting of n square matrices A_i stacked vertically, with $i = 1, ..., n$, we define $\text{trm}(A)$ as the $n \times 1$ vector of traces of the n matrices A_i:

$$\text{trm}(A) = [\text{tr}(A_1) \quad \text{tr}(A_2) \quad ... \quad \text{tr}(A_n)]'.$$

We can use the above operator to show that moment condition (22) implies the following set of $n \cdot n_x \times n_x$ equations:

$$\begin{aligned}\mathbf{0} = &\left(D\bar{f}_{\hat{y}} \cdot \bar{g}_x \otimes I_x\right)\bar{h}_{xx} + \left(D\bar{f}_{\hat{y}} \otimes \bar{h}'_x\right)\bar{g}_{xx}\bar{h}_x + \left(D\bar{f}_{\hat{x}} \otimes I_x\right)\bar{h}_{xx} \\ &+ \left(D\bar{f}_y \otimes I_x\right)\bar{g}_{xx} + \left(I_n \otimes \bar{h}'_x\bar{g}'_x\right)H\bar{f}_{\hat{y}} \cdot \bar{M}_x + \left(I_n \otimes \bar{h}'_x\right)H\bar{f}_{\hat{x}} \cdot \bar{M}_x \\ &+ \left(I_n \otimes \bar{g}'_x\right)H\bar{f}_y \cdot \bar{M}_x + H\bar{f}_x \cdot \bar{M}_x, \quad (23)\end{aligned}$$

which can be solved for the unknown matrices $\bar{g}_{xx}$ and $\bar{h}_{xx}$, given $\bar{h}_x$, $\bar{g}_x$, $D\bar{f}$ and $H\bar{f}$.

We can then collect the remaining terms and obtain

$$\begin{aligned}\mathbf{0} = \frac{1}{2}E_t\Bigg\{ &D\bar{f}_{\hat{y}}\Big[\left(\bar{g}_x \otimes u'_t\right)\bar{h}_{uu}u_t + \bar{g}_x\bar{h}_{zz}\sigma_z^2 + \bar{g}_x\bar{h}_{vv}\sigma_v^2 + 2\bar{g}_x\bar{h}_{zu}\sigma_z u_t + \left(I_y \otimes u'_{t+1}\right)\bar{g}_{uu}u_{t+1} \\ &+ \left(I_y \otimes \xi'_{t+1}\bar{h}'_\xi\right)\bar{g}_{xx}\bar{h}_\xi\xi_{t+1} + \bar{g}_{zz}\sigma_z^2 + \bar{g}_{vv}\sigma_v^2 + 2\bar{g}_{zu}\sigma_z u_{t+1}\Big] \\ &+ D\bar{f}_{\hat{x}}\Big[\left(I_x \otimes u'_t\right)\bar{h}_{uu}u_t + \bar{h}_{zz}\sigma_z^2 + \bar{h}_{vv}\sigma_v^2 + 2\bar{h}_{zu}\sigma_z u_t\Big] \\ &+ D\bar{f}_y\Big[\left(I_y \otimes u'_t\right)\bar{g}_{uu}u_t + \bar{g}_{zz}\sigma_z^2 + \bar{g}_{vv}\sigma_v^2 + 2\bar{g}_{zu}\sigma_z u_t\Big] \\ &+ \left(I_n \otimes \xi'_{t+1}\bar{h}'_\xi\bar{g}'_x\right)H\bar{f}_{\hat{y}} \cdot \bar{M}_\xi\xi_{t+1} + \left(I_n \otimes \xi'_{t+1}\bar{h}'_\xi\right)H\bar{f}_{\hat{x}} \cdot \bar{M}_\xi\xi_{t+1}\Bigg\}. \quad (24)\end{aligned}$$

Given a generic square matrix A, of order m, we define $\text{diagm}(A)$ as the diagonal matrix whose main diagonal is that of matrix A. Moreover, given a generic $n \cdot m \times m$ matrix B consisting of n square matrices B_i stacked vertically, with $i = 1, ..., n$, we define $\text{dgm}(B)$ as the $n \cdot m \times m$ matrix that stacks vertically the $m \times m$ diagonal matrices $\text{diagm}(B_i)$:

$$\text{dgm}(B) = [\text{diagm}(B_1) \quad \text{diagm}(B_2) \quad ... \quad \text{diagm}(B_n)]'.$$

We can use the above operator to show, for generic and conformable matrices A and B:

$$\begin{aligned}E_t\left\{\left(I \otimes \xi'_{t+1}A'\right)BA\xi_{t+1}\right\} &= E_t\left\{\text{trm}\left[\left(I \otimes \xi'_{t+1}A'\right)BA\xi_{t+1}\right]\right\} = \\ &\text{trm}\left[\left(I \otimes A'\right)BAE_t\left\{\xi_{t+1}\xi'_{t+1}\right\}\right] = \text{trm}\left[\left(I \otimes A'\right)BAU_tU'_t\right],\end{aligned}$$

where "trm" is the matrix trace operator defined earlier, and in the last equality we used $E_t(\xi_{t+1}\xi'_{t+1}) = U_tU'_t$, as implied by equation (3). Moreover, since U_t is a diagonal matrix whose vector on the main diagonal is u_t, the following also holds:

$$\text{trm}\left[\left(I \otimes A'\right)BAU_tU'_t\right] = \text{trm}\left\{\text{dgm}\left[\left(I \otimes A'\right)BA\right] \cdot u_tu'_t\right\},$$

from which we can conclude:

$$E_t\left\{\left(I\otimes\xi'_{t+1}A'\right)BA\xi_{t+1}\right\}=\mathrm{trm}\left\{\mathrm{dgm}\left[\left(I\otimes A'\right)BA\right]\cdot u_t u'_t\right\}. \tag{25}$$

Recall the definition of the process for the standard deviations:

$$u_{t+1}=\sigma_z(I_z-\Lambda_u)\bar{u}+\Lambda_u u_t+\sigma_v V\varepsilon_{v,t+1}.$$

We can use the above definition to write the quadratic term in u_{t+1} in equation (24) as:

$$\begin{aligned}E_t\Big\{\left(D\bar{f}_{\hat{y}}\otimes u'_{t+1}\right)\bar{g}_{uu}u_{t+1}\Big\}=E_t\Big\{&\sigma_z^2\left[D\bar{f}_{\hat{y}}\otimes\bar{u}'(I_z-\Lambda_u)'\right]\bar{g}_{uu}(I_z-\Lambda_u)\bar{u}\\&+\left(D\bar{f}_{\hat{y}}\otimes u'_t\Lambda'_u\right)\bar{g}_{uu}\Lambda_u u_t+\sigma_v^2\left(D\bar{f}_{\hat{y}}\otimes\varepsilon'_{v,t+1}V'\right)\bar{g}_{uu}V\varepsilon_{v,t+1}\\&+2\sigma_z\left[D\bar{f}_{\hat{y}}\otimes\bar{u}'(I_z-\Lambda_u)'\right]\bar{g}_{uu}\Lambda_u u_t\Big\}.\end{aligned} \tag{26}$$

Using the above to collect all second-order terms in u_t from equation (24), considering equation (25) and exploiting the operators “trm” and “dgm”, we obtain the following system of $n\cdot n_z\times n_z$ equations

$$\begin{aligned}\mathbf{0}=&\left(D\bar{f}_{\hat{y}}\cdot\bar{g}_x\otimes I_z\right)\bar{h}_{uu}+\left(D\bar{f}_{\hat{y}}\otimes\Lambda'_u\right)\bar{g}_{uu}\Lambda_u+\left(D\bar{f}_{\hat{x}}\otimes I_z\right)\bar{h}_{uu}+\left(D\bar{f}_y\otimes I_z\right)\bar{g}_{uu}\\&+\mathrm{dgm}\Big[\left(D\bar{f}_{\hat{y}}\otimes\bar{h}'_\xi\right)\bar{g}_{xx}\bar{h}_\xi+\left(I_n\otimes\bar{h}'_\xi\bar{g}'_x\right)H\bar{f}_{\hat{y}}\cdot\bar{M}_\xi+\left(I_n\otimes\bar{h}'_\xi\right)H\bar{f}_{\hat{x}}\cdot\bar{M}_\xi\Big],\end{aligned} \tag{27}$$

which can be solved for matrices $\bar{g}_{uu}$ and $\bar{h}_{uu}$, given $\bar{h}_x$, $\bar{g}_x$, $\bar{h}_{xx}$, $\bar{g}_{xx}$, $D\bar{f}$ and $H\bar{f}$. Notice that $\bar{h}_{uu}$ and $\bar{g}_{uu}$ will therefore consist of n_x and n_y, respectively, vertically stacked matrices of dimensions $n_z\times n_z$ which will be diagonal matrices.

We can further collect terms in $\sigma_z u_t$ from equation (24), considering equation (26) and using the trm operator, to obtain a set of $n\times n_z$ equations:

$$\mathbf{0}=(D\bar{f}_{\hat{y}}\cdot\bar{g}_x+D\bar{f}_{\hat{x}})\bar{h}_{zu}+D\bar{f}_{\hat{y}}\cdot\bar{g}_{zu}\Lambda_u+D\bar{f}_y\cdot\bar{g}_{zu}+\left[D\bar{f}_{\hat{y}}\otimes\bar{u}'(I_z-\Lambda_u)'\right]\bar{g}_{uu}\Lambda_u, \tag{28}$$

which can be solved for the unknown matrices $\bar{g}_{zu}$ and $\bar{h}_{zu}$, given $\bar{g}_x$, $\bar{g}_{uu}$, and $D\bar{f}$.

Similarly, we can collect the terms in σ_z^2 obtaining a set of $n\times 1$ equations

$$\begin{aligned}\mathbf{0}=&(D\bar{f}_{\hat{y}}\cdot\bar{g}_x+D\bar{f}_{\hat{x}})\bar{h}_{zz}+(D\bar{f}_{\hat{y}}+D\bar{f}_y)\bar{g}_{zz}\\&+2D\bar{f}_{\hat{y}}\cdot\bar{g}_{zu}(I_z-\Lambda_u)\bar{u}+\left[D\bar{f}_{\hat{y}}\otimes\bar{u}'(I_z-\Lambda_u)'\right]\bar{g}_{uu}(I_z-\Lambda_u)\bar{u},\end{aligned} \tag{29}$$

which can be solved for $\bar{g}_{zz}$ and $\bar{h}_{zz}$, given $\bar{g}_x$, $\bar{g}_{zu}$, $\bar{g}_{uu}$, and $D\bar{f}$.

Finally, we can collect the terms in σ_v^2 obtaining a set of $n\times 1$ equations

$$\mathbf{0}=(D\bar{f}_{\hat{y}}\cdot\bar{g}_x+D\bar{f}_{\hat{x}})\bar{h}_{vv}+(D\bar{f}_{\hat{y}}+D\bar{f}_y)\bar{g}_{vv}+\mathrm{trm}\Big[(D\bar{f}_{\hat{y}}\otimes V')\bar{g}_{uu}V\Big], \tag{30}$$

which deliver $\bar{g}_{vv}$ and $\bar{h}_{vv}$, given $\bar{g}_x$, $\bar{g}_{uu}$, and $D\bar{f}$.

3 A model with time-varying variances

In this section we discuss a more parsimonious model with time-varying volatility in which the volatility is modeled through an autoregressive linear process for conditional variances rather than for conditional standard deviations, as in the previous section. The only difference with respect to the previous model is in equation (4) which we now replace with

$$u_{t+1}^2 = \sigma_z^2(I_z - \Lambda_u \bar{u}^2) + \Lambda_u u_t^2 + \sigma_v^2 V \varepsilon_{v,t+1}. \tag{31}$$

Each element of u_t^2 is the corresponding squared value of each element of u_t, which still corresponds to the diagonal of matrix U_t as in (3); $\bar{u}$ is a vector of dimension $n_z \times 1$ with $\bar{u}^2$ being a vector of the same dimension whose elements are each the square of the respective element of $\bar{u}$; V and Λ_u are matrices of order $n_z \times n_z$, $\varepsilon_{v,t+1}$ is a vector of innovation of dimension $n_z \times 1$ which are assumed to have a bounded support and to be independently and identically distributed with mean zero and variance/covariance matrix I_z; σ_v and σ_z are scalars with $\sigma_v, \sigma_z \geq 0$.

Differently from the previous model, it is now the conditional variance of each element of the stochastic disturbances ξ_{t+1} which is modeled as a linear process, (31). As a consequence, the scale factors σ_z and σ_v have been appropriately squared.

It is straightforward to show that a first-order approximation of this alternative model is identical to that of the previous section except that now (4) replaces (31).

Instead, a second order approximation will be of the form

$$\tilde{y}_t = \bar{g}_x \tilde{x}_t + \frac{1}{2}(I_y \otimes \tilde{x}_t')\bar{g}_{xx}\tilde{x}_t + \frac{1}{2}\bar{g}_{uu}u_t^2 + \frac{1}{2}\bar{g}_{zz}\sigma_z^2, \tag{32}$$

$$\tilde{x}_{t+1} = \bar{h}_x \tilde{x}_t + \frac{1}{2}(I_x \otimes \tilde{x}_t')\bar{h}_{xx}\tilde{x}_t + \frac{1}{2}\bar{h}_{uu}u_t^2 + \frac{1}{2}\bar{h}_{zz}\sigma_z^2 + \bar{h}_\xi \xi_{t+1}, \tag{33}$$

where the sizes of matrices $\bar{g}_x$, $\bar{g}_{xx}$, $\bar{g}_{zz}$, and $\bar{h}_x$, $\bar{h}_{xx}$, $\bar{h}_{zz}$, are the same as in the previous section. Instead $\bar{g}_{uu}$ and $\bar{h}_{uu}$ are matrices of order $n_y \times n_z$ and $n_x \times n_z$, respectively.[13]

We now evaluate the second-order expansion (18), using equations (7) and (8) for the second-order terms, taking into account (31), and the second-order guess solutions (32) and (33) for the first-order terms, taking into account the restrictions implied by (10).

[13] Notice that the expansion with respect to σ_v^2 is zero up to second-order terms.

We obtain:

$$
\begin{aligned}
\mathbf{0} =& \frac{1}{2} E_t \Bigg\{ D\bar{f}_{\hat{y}} \Big[\left(\bar{g}_x \otimes \tilde{x}_t'\right) \bar{h}_{xx}\tilde{x}_t + \bar{g}_x \bar{h}_{uu} u_t^2 + \bar{g}_x \bar{h}_{zz}\sigma_z^2 + \bar{g}_{zz}\sigma_z^2 \\
&+ \left[I_y \otimes (\bar{h}_x \tilde{x}_t + \bar{h}_\xi \xi_{t+1})' \right] \bar{g}_{xx} (\bar{h}_x \tilde{x}_t + \bar{h}_\xi \xi_{t+1}) + \sigma_z^2 \bar{g}_{uu} (I_z - \Lambda_u) \bar{u}^2 + \bar{g}_{uu} \Lambda_u u_t^2 \Big] \\
&+ D\bar{f}_{\hat{x}} \Big[\left(I_x \otimes \tilde{x}_t'\right) \bar{h}_{xx} x_t + \bar{h}_{uu} u_t^2 + \bar{h}_{zz}\sigma_z^2 \Big] + D\bar{f}_y \Big[\left(I_y \otimes \tilde{x}_t'\right) \bar{g}_{xx} x_t + \bar{g}_{uu} u_t^2 + \bar{g}_{zz}\sigma_z^2 \Big] \\
&+ \left[I_n \otimes (\bar{g}_x \bar{h}_x \tilde{x}_t + \bar{g}_x \bar{h}_\xi \xi_{t+1})' \right] H\bar{f}_{\hat{y}} \cdot \tilde{w}_{t+1} + \left[I_n \otimes (\bar{h}_x \tilde{x}_t + \bar{h}_\xi \xi_{t+1})' \right] H\bar{f}_{\hat{x}} \cdot \tilde{w}_{t+1} \\
&+ \left(I_n \otimes \tilde{x}_t' \bar{g}_x'\right) H\bar{f}_y \cdot \tilde{w}_{t+1} + \left(I_n \otimes \tilde{x}_t'\right) H\bar{f}_x \cdot \tilde{w}_{t+1} \Bigg\},
\end{aligned} \tag{34}
$$

From the above equations it is clear that matrices $\bar{g}_{xx}$ and $\bar{h}_{xx}$ are equivalent to those of the previous model and satisfy equation (23).

We can collect the remaining terms:

$$
\begin{aligned}
\mathbf{0} = E_t \Bigg\{ & D\bar{f}_{\hat{y}} \Big[\bar{g}_x \bar{h}_{uu} u_t^2 + \bar{g}_x \bar{h}_{zz}\sigma_z^2 + \bar{g}_{zz}\sigma_z^2 + \left(I_y \otimes \xi_{t+1}' \bar{h}_\xi'\right) \bar{g}_{xx} \bar{h}_\xi \xi_{t+1} \\
&+ \sigma_z^2 \bar{g}_{uu} (I_z - \Lambda_u) \bar{u}^2 + \bar{g}_{uu} \Lambda_u u_t^2 \Big] + D\bar{f}_{\hat{x}} \Big[\bar{h}_{uu} u_t^2 + \bar{h}_{zz}\sigma_z^2 \Big] \\
&+ \left(I_n \otimes \xi_{t+1}' \bar{h}_\xi' \bar{g}_x'\right) H\bar{f}_{\hat{y}} \cdot \bar{M}_\xi \xi_{t+1} + \left(I_n \otimes \xi_{t+1}' \bar{h}_\xi'\right) H\bar{f}_{\hat{x}} \cdot \bar{M}_\xi \xi_{t+1} \Bigg\}.
\end{aligned} \tag{35}
$$

Given a generic $n \cdot m \times m$ matrix A consisting of n square matrices A_i stacked vertically, with $i = 1, ..., n$, we define $\mathrm{dgv}(A)$ as the $m \times n$ matrix that stacks horizontally the main diagonals of each of the $m \times m$ matrices A_i:

$$
\mathrm{dgv}(A) = [\mathrm{diagv}(A_1) \quad \mathrm{diagv}(A_2) \quad ... \quad \mathrm{diagv}(A_n)],
$$

where $\mathrm{diagv}(A_i)$ is an $m \times 1$ vector collecting the elements on the main diagonal of A_i. We can use the above operator, together with the matrix trace operator defined in the previous section, to show, for generic and conformable matrices A and B:

$$
\begin{aligned}
& E_t \left\{ \left(I \otimes \xi_{t+1}' A'\right) BA\xi_{t+1} \right\} = E_t \left\{ \mathrm{trm} \left[\left(I \otimes \xi_{t+1}' A'\right) BA\xi_{t+1} \right] \right\} = \\
& \qquad \mathrm{trm} \left[\left(I \otimes A'\right) BA E_t \left\{ \xi_{t+1} \xi_{t+1}' \right\} \right] = \mathrm{trm} \left[\left(I \otimes A'\right) BA U_t U_t' \right] = \mathrm{dgv} \left[\left(I \otimes A'\right) BA \right]' u_t^2.
\end{aligned}
$$

Using the above to express the quadratic terms in ξ_{t+1} in equation (35) in terms of u_t^2, we collect the latter to obtain the following system of $n \times n_z$ conditions:

$$
\begin{aligned}
\mathbf{0} =& \left(D\bar{f}_{\hat{y}} \cdot \bar{g}_x + D\bar{f}_{\hat{x}}\right) \bar{h}_{uu} + D\bar{f}_{\hat{y}} \cdot \bar{g}_{uu} \Lambda_u + D\bar{f}_y \cdot \bar{g}_{uu} \\
&+ \mathrm{dgv} \Big[\left(D\bar{f}_{\hat{y}} \otimes \bar{h}_\xi'\right) \bar{g}_{xx} \bar{h}_\xi + \left(I_n \otimes \bar{h}_\xi' \bar{g}_x'\right) H\bar{f}_{\hat{y}} \cdot \bar{M}_\xi + \left(I_n \otimes \bar{h}_\xi'\right) H\bar{f}_{\hat{x}} \cdot \bar{M}_\xi \Big]',
\end{aligned} \tag{36}
$$

which can be solved for matrices $\bar{h}_{uu}$ and $\bar{g}_{uu}$.

Finally, we can collect the terms in σ_z^2 from equation (35), to show that matrices $\bar{h}_{zz}$ and $\bar{g}_{zz}$ solve the following system of $n \times 1$ equations:

$$\mathbf{0} = \left(D\bar{f}_{\hat{y}}\bar{g}_x + D\bar{f}_{\hat{x}}\right)\bar{h}_{zz} + \left(D\bar{f}_{\hat{y}} + D\bar{f}_y\right)\bar{g}_{zz} + D\bar{f}_{\hat{y}}\bar{g}_{uu}\left(I_z - \Lambda_u\right)\bar{u}^2. \tag{37}$$

4 Application: the neoclassical growth model

To apply our method to a simple example, we consider the standard neoclassical growth model as it is also done in Schmitt-Grohé and Uribe (2004). We denote with C_t consumption and with K_t the capital stock at the beginning of period t. The parameters β, δ, γ and α represent (respectively) the subjective discount factor, the depreciation rate of capital, relative risk aversion and the return to scale of capital in the production function. The equilibrium conditions of the model are given by:

$$K_{t+1} - e^{a_t}K_t^{\alpha} - (1-\delta)K_t + C_t = 0 \tag{38}$$

$$E_t\left\{\beta\left[\alpha e^{a_{t+1}}K_{t+1}^{\alpha-1} + (1-\delta)\right]\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right\} - 1 = 0 \tag{39}$$

$$a_{t+1} = \rho a_t + u_t\varepsilon_{a,t+1} \tag{40}$$

$\forall t \geq 0$, given K_0 and $a_0 = 0$; where a_t denotes the log of the productivity shock. In particular, the innovation $\varepsilon_{a,t+1}$ to the log-productivity process (40) is identically and independently distributed process with mean zero and unitary variance; u_t captures the time-varying conditional standard deviation of a_{t+1} and ρ is a parameter, with $0 \leq \rho < 1$. We model the square of u_t, i.e. the conditional variance of a_{t+1}, as an exogenous stochastic linear process

$$u_{t+1}^2 = (1-\lambda)\sigma_a^2\bar{u}^2 + \lambda u_t^2 + \sigma_v^2\varepsilon_{v,t+1} \tag{41}$$

with initial condition $u_0^2 = \sigma_a^2\bar{u}^2$ where λ is a coefficient such that $0 \leq \lambda < 1$, while σ_a and σ_v are non-negative scalars; $\bar{u}$ is a positive parameter and the innovation $\varepsilon_{a,t+1}$ is identically and independently distributed process with mean zero and unitary variance. Notice that since $E_t\left(u_t\varepsilon_{a,t+1}\right) = 0$, the log-productivity process (40) is a conditionally linear stochastic process.

We can cast this model in the general notation of Section 2. Defining $c_t \equiv \ln C_t$,

$k_t \equiv \ln K_t$, we can write $y_t = [c_t]$ and $x_t = [k_t, a_t]$ and therefore

$$E_t \left\{ f(y_{t+1}, y_t, x_{t+1}, x_t) \right\} = E_t \begin{bmatrix} \beta \left[\alpha e^{a_{t+1}+(\alpha-1)k_{t+1}} + (1-\delta) \right] e^{-\gamma c_{t+1}} - e^{-\gamma c_t} \\ e^{k_{t+1}} - e^{a_t+\alpha k_t} - (1-\delta)e^{k_t} + e^{c_t} \\ a_{t+1} - \rho a_t \end{bmatrix} = 0. \quad (42)$$

According to (5) and (6), a solution to (42) takes the form

$$c_t = g(k_t, a_t, u_t, \sigma_a, \sigma_v) \quad (43)$$

$$k_{t+1} = h(k_t, a_t, u_{t,} \sigma_a, \sigma_v) \quad (44)$$

$$a_{t+1} = \rho a_t + \xi_{t+1}$$

with $\xi_{t+1} \equiv u_t \varepsilon_{a,t+1}$ and where the square of u_t follows (41).

In the non-stochastic steady-state, in which $\sigma_a = \sigma_v = 0$ and $f(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0$, the following system is used to solve for $\bar{K}$ and $\bar{C}$

$$\delta \bar{K} - \bar{K}^{\alpha} - \bar{C} = 0,$$

$$\beta \left[\alpha \bar{K}^{\alpha-1} + (1-\delta) \right] = 1.$$

Using the calibration of Schmitt-Grohé and Uribe (2004), i.e. $\beta = 0.95$, $\delta = 1$, $\alpha = 0.3$, $\rho = 0$, $\gamma = 2$, we obtain:

$$\bar{K} = 0.1664, \qquad \bar{C} = 0.4175.$$

According to (7) and (8), a first-order approximation of (43) and (44) takes the form

$$\tilde{c}_t = g_k \tilde{k}_t + g_a a_t \quad (45)$$

$$\tilde{k}_{t+1} = h_k \tilde{k}_t + h_a a_t \quad (46)$$

where we have defined $\tilde{c}_t \equiv \ln C_t - \ln \bar{C}$, $\tilde{k} \equiv \ln K_t - \ln \bar{K}$ and the coefficients g_k, g_a, h_k and h_a coincide with those of Schmitt-Grohé and Uribe (2004):

$$\begin{aligned} g_k &= 0.2525, \ g_a = 0.8417 \\ h_k &= 0.4191, \ h_a = 1.3970. \end{aligned}$$

However, there is an important difference between our approximation and that of Schmitt-Grohé and Uribe (2004). In our case, a_t follows the conditionally-linear and heteroskedastic process (40), in which the conditional variance is modelled as in (41). In their framework, instead, shocks are homoskedastic and a_t follows the following linear process:

$$a_{t+1} = \rho a_t + \sigma_a \bar{u} \varepsilon_{a,t+1}. \tag{47}$$

In Fernandez-Villaverde et al. (2009, 2010) the original stochastic process for the exogenous state variables is heteroskedastic as in (11) and (12), but a linear approximation of this process would be consistent with (47) in which risk is no longer time-varying. Instead, in our first-order approximation stochastic volatility still matters and will be particularly relevant when estimating the model, as it is done in Justiniano and Primiceri (2008).

However, as mentioned in section 2.2, in our first-order approximation risk does not play a "distinct and direct" role. To see this point, we discuss the impulse response functions. Defining the impulse response of a generic variable x_t at time $t+j$ with respect to the shock ε_t as

$$I(x_{t+j}|\varepsilon_t) = \frac{\partial (E_t x_{t+j} - E_{t-1} x_{t+j})}{\partial \varepsilon_t},$$

we obtain that the impulse response with respect to the shock $\varepsilon_{a,t}$ is given by

$$I(\tilde{c}_{t+j}|\varepsilon_{a,t}) = g_k I(\tilde{k}_{t+j}|\varepsilon_{a,t}) + g_a I(a_{t+j}|\varepsilon_{a,t})$$

$$I(\tilde{k}_{t+j+1}|\varepsilon_{a,t}) = h_k I(\tilde{k}_{t+j}|\varepsilon_{a,t}) + h_a I(a_{t+j}|\varepsilon_{a,t})$$

for each $j \geqslant 0$ with $I(\tilde{k}_t|\varepsilon_{a,t}) = 0$ where

$$I(a_{t+j+1}|\varepsilon_{a,t}) = \rho I(a_{t+j}|\varepsilon_{a,t})$$

for each $j \geqslant 0$ and

$$I(a_t|\varepsilon_{a,t}) = \sigma_a \bar{u}.$$

The impulse response with respect to the shock $\varepsilon_{a,t}$ will not be affected by the fact that shocks are heteroskedastic or not and therefore will coincide with those of Schmitt-Grohé and Uribe (2004). However, even if we compute the impulse response with respect to risk, i.e. with respect to the shock $\varepsilon_{v,t}$, this will be zero at all times: $I(\tilde{c}_{t+j}|\varepsilon_{v,t}) = 0$ and $I(\tilde{k}_{t+j}|\varepsilon_{v,t}) = 0$ for each $j \geqslant 0$. Therefore risk will not play a distinct and separate role in affecting the variables of interest even in our first-order approximation. To get this role, we need to go to a second-order approximation.

Following (32) and (33), the second-order approximation will be of the form

$$\tilde{c}_t = g_k \tilde{k}_t + g_a a_t + \frac{1}{2} g_{uu} u_t^2 + \frac{1}{2} g_{kk} \tilde{k}_t^2 + \frac{1}{2} g_{aa} a_t^2 + g_{ka} a_t \tilde{k}_t + \frac{1}{2} g_{\sigma\sigma} \sigma_a^2$$

$$\tilde{k}_{t+1} = h_k \tilde{k}_t + h_a a_t + \frac{1}{2} h_{uu} u_t^2 + \frac{1}{2} h_{kk} \tilde{k}_t^2 + \frac{1}{2} h_{aa} a_t^2 + h_{ka} a_t \tilde{k}_t + \frac{1}{2} h_{\sigma\sigma} \sigma_a^2$$

where again a_t follows (40) and u_t^2 follows (41). To compute the numerical values for the remaining coefficients, we consider the calibration adopted by Schmitt-Grohé and Uribe (2004) for the structural parameters, and $\sigma_a = \sigma_v = \bar{u} = 1$ and $\lambda = 0.5$ for the parameters entering equation (41) and governing the dynamics of stochastic volatility. This calibration implies:

$$g_{uu} = -0.1444, \quad g_{kk} = -0.0051, \quad g_{aa} = -0.0569, \quad g_{ka} = -0.0171, \quad g_{\sigma\sigma} = -0.0478,$$
$$h_{uu} = 0.3622, \quad h_{kk} = -0.0070, \quad h_{aa} = -0.0778, \quad h_{ka} = -0.0233, \quad h_{\sigma\sigma} = 0.1199.$$

It is also clear that second-order-approximation impulse response function with respect to the shock $\varepsilon_{a,t}$ will not be affected by the fact that shocks are heteroskedastic or not and therefore will correspond to those of Schmitt-Grohé and Uribe (2004). Instead, now there is a distinct role for risk to affect the variables of interest. Indeed, the impulse responses with respect to the shock $\varepsilon_{v,t}$ will be of the form

$$I(\tilde{c}_{t+j}|\varepsilon_{v,t}) = g_k I(\tilde{k}_{t+j}|\varepsilon_{v,t}) + g_{uu} I(u_{t+j}^2|\varepsilon_{v,t})$$

$$I(\tilde{k}_{t+j+1}|\varepsilon_{v,t}) = h_k I(\tilde{k}_{t+j}|\varepsilon_{v,t}) + h_{uu} I(u_{t+j}^2|\varepsilon_{v,t})$$

for each $j \geqslant 0$ with $I(\tilde{k}_t|\varepsilon_{v,t}) = 0$ where

$$I(u_{t+1+j}^2|\varepsilon_{v,t}) = \lambda I(u_{t+j}^2|\varepsilon_{v,t})$$

for each $j \geqslant 0$ and

$$I(u_t^2|\varepsilon_{v,t}) = \sigma_v^2.$$

Obviously, in Schmitt-Grohé and Uribe (2004) there is no role at all for time-varying volatility while in Fernandez-Villaverde et al. (2009, 2010) there will not be a distinct role and therefore impulse responses with respect to $\varepsilon_{v,t}$ will be zero. To get this role, they have to go to higher-order approximations.

In Figure 1 we show the impulse response of consumption and capital to 1% change in risk to productivity shock. The impact response of consumption and investment depends on the relative strength of two opposite forces. On the one hand, higher volatility tends to increase the supply of saving for future production and therefore for precautionary reasons.[14] On the other hand, higher volatility increases the expected excess return on capital reducing its appeal as an asset to accumulate. Under our parametrization, in particular with $\delta = 1$, the precautionary-saving effect dominates and on impact consumption

[14]This channel is stronger when the depreciation is larger, and is clearly dominant with full depreciation.

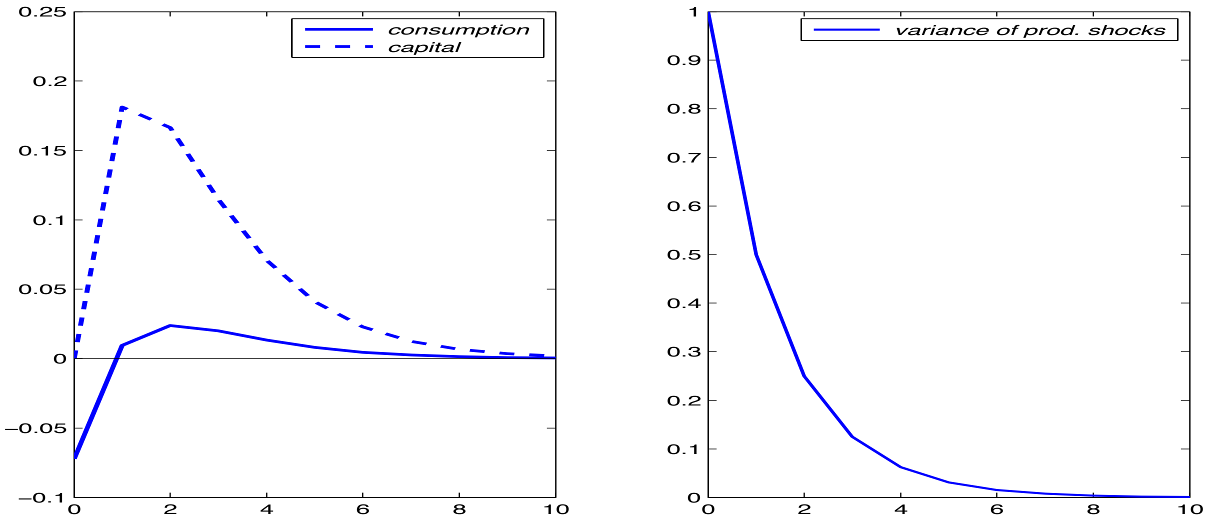

Figure 1: Dynamic response of consumption and capital to a 1% innovation to the variance of productivity shocks. Percentage points.

decreases while investment rises.[15] In the following periods because of capital accumulation, production and consumption increase above their steady state levels as long as agents still accumulate capital above steady state.

As we have already discussed, another important advantage of our approach with respect to Schmitt-Grohé and Uribe (2004) and Fernandez-Villaverde et al. (2009, 2010) is that risk-premia evaluated using first-order approximation will be time-varying. To see this, let r_{t+1} be the risk-free real rate, and define $r_{k,t+1}$ as the return on capital from period t to period $t+1$:

$$r_{k,t+1} = \alpha e^{a_{t+1}+(\alpha-1)k_{t+1}} + (1-\delta).$$

Using the above, we can show that in a second-order approximation the expected excess return of capital is given by

$$E_t(\tilde{r}_{k,t+1} - \tilde{r}_{t+1}) + \frac{1}{2} var_t(\tilde{r}_{k,t+1}) = \gamma cov_t(\tilde{r}_{k,t+1}, \Delta\tilde{c}_{t+1})$$

where $\tilde{r}_{k,t+1}$ and $\tilde{r}_{t+1}$ denote the log deviation from steady state of the real return on capital and the risk-free rate, respectively. The right-hand side measures the risk premium and

[15]When $\gamma = \delta = 1$, saving is always a constant fraction of income and therefore risk does not have a distinct role.

this will be time-varying because

$$
\begin{aligned}
cov_t(\tilde{r}_{k,t+1}, \Delta\tilde{c}_{t+1}) &= \phi cov_t(a_{t+1} + (\alpha-1)\tilde{k}_{t+1}, g_k\tilde{k}_{t+1} + g_a a_{t+1}) \\
&= \phi\Big\{ E_t\left[g_a a_{t+1}^2 + ((\alpha-1)g_a + g_k)\tilde{k}_{t+1}a_{t+1} + (\alpha-1)g_k\tilde{k}_{t+1}^2\right] \\
&\qquad\qquad -E_t\left[a_{t+1} + (\alpha-1)\tilde{k}_{t+1}\right] E_t\left[g_k\tilde{k}_{t+1} + g_a a_{t+1}\right]\Big\} \\
&= \phi g_a u_t^2
\end{aligned}
$$

which will indeed depend on the time-variation of the variance u_t^2, where $\phi \equiv 1-\beta(1-\delta)$. In Schmitt-Grohé and Uribe (2004) and Fernandez-Villaverde et al. (2009, 2010), this risk premium, computed using a first-order approximation, will be constant.

5 Conclusion

Recent models used in macroeconomics examine the role of stochastic volatility for the equilibrium allocation. To solve these models, researchers have appealed to global solutions or high-order approximation techniques. Global-solution techniques suffer from the 'curse of dimensionality', since the number of state variables limits their computational efficiency. Commonly used approximation techniques require third-order expansion of the equilibrium conditions in order to display a distinct role for stochastic volatility.

Here we propose a first and second-order approximation method to study the role of time-varying exogenous risk in discrete-time dynamic stochastic models which encompass standard dynamic general equilibrium models with rational expectations. In our framework, an important assumption is that the exogenous state variables follow a conditionally-linear stochastic process in which either the variance or the standard deviation of the primitive shocks are modelled through a stochastic linear process. In this way, we generalize the framework and the method developed by Schmitt-Grohé and Uribe (2004), Kim et al. (2008) and Gomme and Klein (2008) to the case in which the exogenous state variables follow an heteroskedastic process.

The main contribution of our paper is to show that first and second-order approximations of the solution are sufficient to capture most of the relevant elements needed to study the impact of uncertainty in standard macroeconomic models. There are three main advantages following our method. First, a first-order approximation falls in the broader class of conditionally-linear approximations displaying a role for time-varying volatility, although not a distinct one. Second, given that a first-order approximation retains a role for stochastic volatility, the second-order approximation of the solution implies that the time-varying volatility of primitive shocks can directly affect the endogenous variables. Third, it follows from the previous results that risk-premia evaluated using first-order approximations will be time-varying. All these advantages translate into a more parsimonious model, more easily tractable for estimation purposes.

In addition to characterizing the second-order approximation of the solution when shocks are conditionally linear, the paper offers a set of MATLAB codes designed to compute the coefficients of the first and second-order approximations and provides a simple example to illustrate the applicability of the method.[16] In general, indeed, our method can be applied easily to several macroeconomic models ranging from real business cycle models, to monetary models and also to asset-pricing or finance models. In Benigno et al. (2010), we employ this method to analyze how risk and monetary policy interact to determine prices, exchange rates and asset prices in an open-economy model.

[16]The set of codes is available under the webpage of the authors.

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