Pessimistic Optimal Choice for Risk-averse Agents*

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ABSTRACT

We propose a general framework for the analysis of dynamic optimization with risk-averse agents, extending Whittle's (Whittle, 1990) formulation of risk-sensitive optimal control problems to accommodate time-discounting. We show how, within a Markovian set-up, optimal risk-averse behavior is identified via a *pessimistic* choice mechanism and described by simple recursive formulae. We apply this methodology to two distinct problems formulated respectively in discrete- and continuous-time. In the former, we extend Svennson's (Svennson, 1997) analysis of optimal monetary policy, showing that with a risk-averse central bank the inflation forecast is not longer an explicit intermediate target, the monetary authorities do not expect the inflation rate to mean revert to its target level and apply a more aggressive Taylor rule than under risk-neutrality, while the inflation rate is less volatile. In the latter, we investigate the optimal production policy of a monopolist which faces a demand schedule subject to stochastic shocks, once again showing that risk-aversion induces her to act more aggressively.

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Introduction

Under risk-neutrality optimal control problems can be easily solved employing well-established results. Thus, in a linear quadratic set-up straightforward recursive formulae immediately yield the optimal policy, while applying the certainty equivalence principle unknown variables can be replaced by their maximum likelihood estimates. However, risk-aversion is an important aspect of agents' preferences, which heavily influences their actions, in particular when the economic environment is complex and uncertain, and agents need to consider the future implications of their decisions. Investigating the nexus between risk-aversion and agents' behavior is a challenge, in that optimal control problems are difficult to solve under risk-aversion.

Whittle (Whittle, 1990) introduces risk-aversion in the standard linear quadratic set-up and shows that: i) the optimal policy is identified via a *pessimistic* choice mechanism; ii) modified (or risk-sensitive) certainty equivalence and separation principles hold; and iii) recursive formulae describe the optimal policy. Despite its versatility, few researchers have employed Whittle's methodology in economics (exceptions are Mamaysky and Spiegel, 2002; van der Ploeg, 2009, 2010; Vitale, 1995, 2012; Zhang, 2004). This is because an important limitation of his methodology is that it does not consider time-discounting.

Hansen and Sargent (Hansen and Sargent, 1994, 1995, 2005) have introduced time-discounting into risk-sensitive optimal control problems by formulating a recursive optimization criterion *à la* Epstein and Zin. We modify their optimization criterion to accommodate time-discounting within Whittle's methodology. We are then able to reformulate Whittle's *pessimistic* choice mechanism, his risk-sensitive certainty equivalence and separation principles and the recursive formulae for the optimal policy he derives. Importantly, following Whittle's lead, we are also able to analyze risk-sensitive optimal control problems with time-discounting under imperfect state observation, a scenario that Hansen and Sargent do not explicitly consider and that their recursive optimization criterion cannot accommodate.

This paper is organized as follows. In Section 1 we extend the class of Markovian linear exponential quadratic Gaussian (LEQG) problems originally studied by Whittle to accommodate time-discounting by proposing a recursive optimization criterion which differs from that put forward by Hansen and Sargent. The suggested recursive criterion allows: i) to apply, with simple adjustments, Whittle's methodology and derive recursive solution formulae for the optimal policy; and ii) to solve LEQG problems with time-discounting (DLEQG problems henceforth) when only noisy signals on the state variables are observed, a scenario which cannot be investigated using Hansen and Sargent's recursive criterion. In this Section we also see how the risk-sensitive certainty equivalence principle (risk-sensitive CEP) originally derived by Whittle for the class of LEQG problems is maintained while the corresponding Riccati equation, which yields the optimal policy for the class of Markovian LEQG problems, is modified. In the following Section we study the class of DLEQG problems under imperfect state observation. Here, Whittle's risk-sensitive separation principle (risk-sensitive SP), which allows to separate control and estimation, is reformulated for the class of DLEQG problems.

DLEQG problems can be formulated to deal with several issues in economics and finance. Thus, Hansen and Sargent's lead has been followed to analyze consumption, precautionary savings and the real business cycle (see Hansen and Sargent, 2005; Hansen, Sargent, and Tallarini, 1999; Luo, 2004; Luo and Young, 2010; Tallarini, 2000, among others). In Section 3 we illustrate how the formulation of LEQG problems with time-discounting we propose can be applied to investigate output and inflation stabilization on the part of an independent central bank.

Specifically, we extend Svensson's (Svennson, 1997) analysis of optimal monetary policy to the case in which the central bank is risk-averse.[1] Because of risk-aversion the central bank selects its monetary policy via the revised pessimistic choice mechanism. Then, as the standard CEP cannot be applied: i) the inflation forecast is not longer an explicit intermediate target when inflation targeting is the exclusive mission of the central bank; and ii) the monetary authorities do not necessarily expect the inflation rate to mean revert to its target level when the monetary policy is also aimed at output stabilization. We actually see that even without output stabilization, a scenario in which the inflation forecast is always equal to its target level under risk-neutrality, if the central bank is risk-averse it may well be that the monetary authorities expect the inflation rate to wander away from the target level.

In addition, we find that the central bank follows a more aggressive Taylor rule under risk-aversion. This results in a smaller volatility for the inflation rate, while the unconditional variances of the output gap and the short-term interest rate are unaffected by risk-aversion. This is interesting, because it means that empirically the impact of risk-aversion only appears in a reduced variability for the inflation rate. Finally, we investigate the possibility that the central bank observes the state variables with a time lag and employ within this context the modified version of Whittle's risk-sensitive SP, confirming the empirical implications derived

[1] van der Ploeg (2009) studies a similar extension of Svensson's analysis. However, he introduces neither time-discounting nor a recursive optimization criterion.

under perfect state observation.

In the last two Sections of the paper we extend our analysis to consider non-homogeneous DLEQG problems and their continuous-time limit. Thus, in Section 4 we introduce predetermined disturbances in the law of motion which regulates the dynamics of the state vector, showing how the recursive formulae for the optimal policy derived in Section 2 must be modified. While, in Section 5 we consider the continuous-time limit of the Markovian DLEQG problem formulated in discrete-time. Here, the optimal policy is characterized by a modified version of the differential Riccati equation which applies to the LEQG formulation.

As an illustrative example of the Markovian DLEQG problem in continuous-time we consider the optimal production policy of a risk-averse monopolist which faces a demand schedule subject to stochastic shocks for the commodity she produces. Interestingly, risk-aversion makes the monopolist more aggressive, in that she finds it optimal to systematically produce a larger quantity than that selected by her risk-neutral counterpart. This is because, despite a larger supply of the commodity depresses its price and jeopardizes future profits, it also reduces their variability. Then, a risk-averse monopolist is willing to gain smaller profits on average to reduce their variability and hence willing to produce a larger quantity of the commodity.

In both economic problems we investigate risk-aversion induces the optimizing agent to act more aggressively. This may appear counter-intuitive as it contradicts results typically obtained within static formulations. However, such a feature of the impact of risk-aversion on the behavior of economic agents in dynamic optimization exercises appears elsewhere. Thus, Holden and Subrahmanyam (1994) and Vitale (1995, 2012) find that risk-aversion makes a privately informed strategic insider trade more aggressively in a sequential call auction market, while the same trader would be more cautions in a one-shot call auction market. In other words, a point which is worth emphasizing is that the Markovian DLEQG formulation allows to derive implications of risk-aversion which are both general and stark. Indeed, not only risk-averse agents are pessimistic, but also bold.

1 LEQG Problems with Time-Discounting

We define a specific class of optimal control problems, which are characterized by: i) a Markovian linear dynamic structure for a vector of state variables, $\mathbf{z}_t$; ii) a multi-normal distribution for an innovation vector $\boldsymbol{\epsilon}_t$; and iii) a recursive optimization criterion *à la* Epstein and Zin.

Definition 1 *An optimal control problem is said to be Markovian Linear Exponential Quadratic Gaussian with time-discounting and perfect state observation if the following recursive optimization*

$$\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t \quad = \quad \min_{\mathbf{u}_t}\left\{\frac{\rho}{2}c_t \;+\; \ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\,\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\}\,, \tag{1.1}$$

where ρ (with $\rho > 0$) is the coefficient of risk-aversion, δ (with $0 < \delta < 1$) is the time-discounting factor, c_t is the scalar-valued cost function and $\boldsymbol{\mathcal{V}}_t$ is the optimization criterion (with terminal condition $\boldsymbol{\mathcal{V}}_{T+1} = 0$), is solved over the periods $t = 1, 2, \ldots, T$ with respect to the free-valued control vector $\mathbf{u}_t$ under the conditions that:

(i) the cost function c_t is a quadratic form in the control vector, $\mathbf{u}_t$, and the state vector, $\mathbf{z}_t$,

$$c_t \;=\; \mathbf{u}_t'\,\mathbf{Q}\,\mathbf{u}_t \;+\; \mathbf{z}_t'\,\mathbf{R}\,\mathbf{z}_t \;+\; 2\,\mathbf{u}_t'\,\mathbf{S}\,\mathbf{z}_t\,,\ \textit{positive definite in } \mathbf{u}_t \textit{ and } \mathbf{z}_t\ ;$$

(ii) the vector of state variables, $\mathbf{z}_t$, is governed by the following linear plant equationf

$$\mathbf{z}_t \;=\; \mathbf{A}\,\mathbf{z}_{t-1} + \mathbf{B}\,\mathbf{u}_{t-1} + \boldsymbol{\epsilon}_t\,, \quad \textit{where}\ \ \boldsymbol{\epsilon}_t \sim \boldsymbol{\mathcal{N}}\,[\mathbf{0},\mathbf{N}]\ ,\ \textit{with}\ \boldsymbol{\epsilon}_t \perp \boldsymbol{\epsilon}_t'\,.$$

Imposing the condition that the recursive optimization is solved over a finite horizon T ensures that the optimization criterion $\boldsymbol{\mathcal{V}}_t$ is well defined. However, thanks to time-discounting an infinite horizon can be accommodated. Condition *(i)* that the cost function c_t is positive definite in the control and state vectors, while useful in finding a minimum in the recursive optimization, is not necessary. Similarly, the assumption that the control vector is free-valued, and hence not subject to any constraint, is also not strictly required for the existence of a minimum in the recursive optimization and it could be disposed of. Nevertheless, it is extremely useful in characterizing the optimal control path, in that it allows to derive recursive solutions to the Markovian DLEQG problem.

In Definition 1 the Markovian DLEQG problem is time-homogeneous, in that neither the plant equation nor the cost function explicitly depends on time t. However, this Definition can be adjusted to accommodate a non-homogenous plant equation and/or cost function, by making any of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{N}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ time-dependent.

To confirm the validity of our Definition for the class of Markovian DLEQG problems we notice that in the limit for $\rho \downarrow 0$ the solution to the recursive optimization criterion (1.1) converges to that for the standard dynamic programming recursion of a Markovian optimal control problem with time-discounting. Indeed, the following result holds.

Lemma 1 *Under perfect state observation, for $\rho \downarrow 0$ the argmin in the recursive optimization criterion (1.1) corresponds to the argmin in* $\min_{\mathbf{u}_t} \{c_t + \delta E_t [\boldsymbol{\mathcal{V}}_{t+1}]\}$.

Proof. For $\rho > 0$, the recursion

$$\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t = \min_{\mathbf{u}_t} \left\{\frac{\rho}{2}c_t + \ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\} = \rho \min_{\mathbf{u}_t} \left\{\frac{1}{2}c_t + \frac{1}{\rho}\ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\}.$$

For $\rho > 0$, the argmin in the right hand side can be obtained by solving the following minimization

$$\min_{\mathbf{u}_t} \left\{\frac{1}{2}\,c_t + \frac{1}{\rho}\ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\}.$$ Hence, consider that

$$\lim_{\rho\downarrow 0}\frac{1}{\rho}\ln\left(E_t\left[\exp\left(\delta\,\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right) = \lim_{\rho\downarrow 0}\delta\,\frac{1}{2}\,\frac{E_t\left[\exp\left(\delta\,\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\cdot\boldsymbol{\mathcal{V}}_{t+1}\right]}{E_t\left[\exp\left(\delta\,\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]} = \frac{1}{2}\,\delta\,E_t\left[\boldsymbol{\mathcal{V}}_{t+1}\right],$$

where we have used the Hôpital's rule and moved the derivative operator inside the expectation operator. Thus, in the limit we can solve $\min_{\mathbf{u}_t} \{c_t + \delta\, E_t [\boldsymbol{\mathcal{V}}_{t+1}]\}$. $\square$

This result implies that if c_t is a quadratic form in $\mathbf{u}_t$ and $\mathbf{z}_t$, while the innovation vector $\boldsymbol{\epsilon}_t$ is normally distributed, the Markovian DLEQG problem converges to the corresponding Markovian Linear Quadratic Gaussian problem with time-discounting (DLQG problem henceforth) for $\rho \downarrow 0$.

With respect to the formulation of the recursive optimization criterion proposed by Hansen and Sargent (1994, 1995) in their analysis of Markovian DLEQG problems, in (1.1) we move the discount factor inside the exponential function (so that rather than using $\delta \ln(E[\exp(\boldsymbol{\mathcal{X}})])$ we employ $\ln(E[\exp(\delta\boldsymbol{\mathcal{X}})]))$.[2] By doing this we can easily transform the recursive optimization criterion (1.1) into a formulation which allows to exploit a number of useful results derived by

[2] The two formulations are equivalent as also mentioned by Hansen and Sargent (1994).

Whittle (1990) for the class of LEQG problems.

In Whittle's formulation of LEQG problems there is no time-discounting,[3] so that in any period t the criterion $\ln\left(E_t\left[\exp\left(\rho\frac{\mathcal{C}}{2}\right)\right]\right)$ with $\mathcal{C}$ a time-separable cost function, $\mathcal{C} = \sum_{t=1}^{T} c_t$, is minimized with respect to the control vector $\mathbf{u}_t$. To accommodate time-discounting a recursive optimization criterion *à la* Epstein and Zin is called for. Despite we rely on an optimization criterion which differs from that put forward by Whittle, we are able to preserve most of his results with some minor adjustments.[4]

To show this we first state the following result:

Lemma 2 *Under perfect state observation, the recursive optimization criterion (1.1) can be equivalently formulated as follows,*

$$\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t = \ln\left(\min_{\mathbf{u}_t}\left\{E_t\left[\exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right]\right\}\right). \tag{1.2}$$

Proof. Under perfect state observation we can write

$$\begin{aligned}
\exp\left(\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t\right) &= \exp\left(\min_{\mathbf{u}_t}\left\{\frac{\rho}{2}c_t + \ln\left(E_t\left[\exp\left(\delta\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right\}\right)\\
&= \min_{\mathbf{u}_t}\left\{\exp\left(\frac{\rho}{2}c_t + \ln\left(E_t\left[\exp\left(\delta\frac{\rho}{2}\boldsymbol{\mathcal{V}}_{t+1}\right)\right]\right)\right)\right\}\\
&= \min_{\mathbf{u}_t}\left\{E_t\left[\exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right]\right\}, \text{ so that}\\
\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t &= \ln\left(\min_{\mathbf{u}_t}\left\{E_t\left[\exp\left(\frac{\rho}{2}(c_t + \delta\,\boldsymbol{\mathcal{V}}_{t+1})\right)\right]\right\}\right). \square
\end{aligned}$$

Then, following Whittle we introduce the concept of *discounted total stress function*:

Definition 2 *Under perfect state observation, the discounted total stress function,* $\boldsymbol{\mathcal{S}}_t$*, is given by* $\boldsymbol{\mathcal{S}}_t \equiv c_t - \frac{1}{\rho}d_{t+1} + \delta\boldsymbol{\mathcal{V}}_{t+1}$*, where* d_t *is a per-period discrepancy function equal to* $\boldsymbol{\epsilon}_t'\mathbf{N}^{-1}\boldsymbol{\epsilon}_t$ *for* $t = 1, 2, \ldots, T$ *and* 0 *for* $t = T + 1$*.*

[3]Another important difference is that he allows for negative values of the coefficient ρ, also considering a *risk-seeking* formulation for the LEQG class. Differently from Whittle and Hansen and Sargent, we only consider risk-averse behavior and consequently use the terms risk-averse and risk-sensitive interchangeably.

[4]For $\delta \uparrow 1$ $\boldsymbol{\mathcal{V}}_t$ does not converge to Whittle's optimization criterion, since in (1.1) the minimization argument does not contain the past cost components, c_h with $h < t$. However, under perfect state observation, for $\delta \uparrow 1$ the optimal policy for the recursive optimization criterion (1.1) converges to that for Whittle's Markovian LEQG problem, in that at time t c_h, with $h < t$, is deterministic and constant with respect to $\mathbf{u}_t$.

This concept is similar to that of *total stress* function originally introduced by Whittle. Our notion differs in two respects: firstly, it covers only periods t and $t+1$; secondly, the period $t+1$ optimization criterion $\boldsymbol{\mathcal{V}}_{t+1}$ is pre-multiplied by the discount factor δ. The discounted total stress function is useful in that we can rely on the following Lemma, which adapts a result firstly outlined by Whittle for the class of LEQG problems.

Lemma 3 *In a Markovian DLEQG problem, under perfect state observation, if the optimization criterion in* $t+1$, $\boldsymbol{\mathcal{V}}_{t+1}$, *is a quadratic form in the state vector* $\mathbf{z}_{t+1}$ *and the discounted total stress function* $\boldsymbol{\mathcal{S}}_t$ *satisfies a saddle point condition with respect to* $\boldsymbol{\epsilon_{t+1}}$ *and* $\mathbf{u}_t$, *so that* $\min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t$ *exists, the following proportionality condition holds,*

$$\min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2}(c_t \, + \, \delta\, \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad \propto \quad \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t \right) \, ,$$

where the proportionality constant is independent of the state vector $\mathbf{z}_t$, *while the optimization criterion* $\boldsymbol{\mathcal{V}}_t$ *is a quadratic form in* $\mathbf{z}_t$ *equal to the extremized discounted total stress function* $\min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t$ *plus a constant independent of* $\mathbf{z}_t$.

Proof. Firstly, we observe that if $Q(\mathbf{u}, \boldsymbol{\epsilon})$ is a quadratic form which admits the saddle point $\max_{\mathbf{u}} \min_{\boldsymbol{\epsilon}} Q(\mathbf{u}, \boldsymbol{\epsilon})$, then the following holds

$$\min_{\mathbf{u}} \int \exp\left[-\frac{1}{2} Q(\mathbf{u}, \boldsymbol{\epsilon}) \right] d\boldsymbol{\epsilon} \; \propto \; \exp\left[-\frac{1}{2} \max_{\mathbf{u}} \min_{\boldsymbol{\epsilon}} Q(\mathbf{u}, \boldsymbol{\epsilon}) \right] .$$

Secondly, consider that under perfect state observation, if $\boldsymbol{\mathcal{V}}_{t+1}$ is a quadratic form in $\mathbf{z}_{t+1}$, as the latter is linearly dependent on $\boldsymbol{\epsilon}_{t+1}$,

$$\begin{aligned} \min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2}(c_t \, + \, \delta\, \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad & \propto \quad \min_{\mathbf{u}_t} \int \exp\left(\frac{\rho}{2}(c_t \, + \, \delta\, \boldsymbol{\mathcal{V}}_{t+1}) - \frac{1}{2}\, \boldsymbol{\epsilon}'_{t+1} \mathbf{N}^{-1} \boldsymbol{\epsilon}_{t+1} \right) d\boldsymbol{\epsilon}_{t+1} \\ & = \quad \min_{\mathbf{u}_t} \int \exp\left(\rho\, \frac{\boldsymbol{\mathcal{S}}_t}{2} \right) d\boldsymbol{\epsilon}_{t+1} \, . \end{aligned}$$

Now, since $\boldsymbol{\mathcal{V}}_{t+1}$ can be expressed as a quadratic form in $\boldsymbol{\epsilon}_{t+1}$ and $\mathbf{u}_t$, so is $\boldsymbol{\mathcal{S}}_t$. If the discounted total stress function in t admits the saddle point $\min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t$, then $-\boldsymbol{\mathcal{S}}_t$ admits the saddle point in the aforementioned property. Exploiting this property we find that

$$\min_{\mathbf{u}_t} \int \exp\left(\rho \frac{\boldsymbol{\mathcal{S}}_t}{2}\right) d\boldsymbol{\epsilon}_{t+1} = \min_{\mathbf{u}_t} \int \exp\left(-\frac{1}{2} \underbrace{(-\rho \boldsymbol{\mathcal{S}}_t)}_{Q(\mathbf{u}_t, \boldsymbol{\epsilon}_{t+1})}\right) d\boldsymbol{\epsilon}_{t+1}$$

$$\propto \exp\left(-\frac{1}{2} \max_{\mathbf{u}_t} \min_{\boldsymbol{\epsilon}_{t+1}} (-\rho \boldsymbol{\mathcal{S}}_t)\right) = \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t\right).$$

From the saddle point condition we establish that the extremized value of $\boldsymbol{\mathcal{S}}_t$ will be a quadratic form in $\mathbf{z}_t$, while $\boldsymbol{\mathcal{V}}_t$ will be equal to the *extremized* value of $\boldsymbol{\mathcal{S}}_t$ plus a constant independent of $\mathbf{z}_t$. $\square$

The requirement that the discounted total stress function satisfies a *saddle point* condition may actually not hold. As it will be clearer later, for a sufficiently large degree of risk-aversion the discounted total stress function at time t will not be negative definite in $\boldsymbol{\epsilon}_{t+1}$ indicating that the saddle point condition cannot be met and that the DLEQG problem does not present a optimizing solution, in that the value of $\boldsymbol{\mathcal{V}}_t$ becomes infinite. In other words, while suggesting a way to solving Markovian DLEQG problems, this Lemma also indicates that under extreme circumstances such problems are not well-behaved and their optimization is meaningless.

As corollary of Lemma 3 we establish a simplified version of Whittle's risk-sensitive certainty equivalence principle (RSCEP) which is particularly useful in addressing Markovian DLEQG problems.

Theorem 1 - *(Risk-sensitive Certainty Equivalence Principle). In a Markovian DLEQG problem, under perfect state observation, if the discounted total stress function $\boldsymbol{\mathcal{S}}_{t+j}$ respects a saddle point condition with respect to $\boldsymbol{\epsilon}_{\mathbf{t+j+1}}$ and $\mathbf{u}_{t+j}$ for $j = 0, 1, \ldots, T-t$, the optimal value of the vector $\mathbf{u}_t$ is determined at time t by simultaneously minimizing $\boldsymbol{\mathcal{S}}_t$ with respect to $\mathbf{u}_t$ and maximizing it with respect to $\boldsymbol{\epsilon}_{t+1}$. In other words, an optimal current decision is obtained by minimizing with respect to the decision currently unmade, $\mathbf{u}_t$, and maximizing with respect to the currently unobservable future innovation vector, $\boldsymbol{\epsilon}_{t+1}$. The extremized discounted stress function is proportional to the recursive optimization criterion, $\boldsymbol{\mathcal{V}}_t \propto \min_{\mathbf{u}_t} \max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t$.*

Proof. Notice that in T c_T is a positive definite quadratic form in $\mathbf{u}_T$, while $\boldsymbol{\mathcal{V}}_{T+1} = d_{T+1} = 0$. This implies that $\boldsymbol{\mathcal{S}}_T$ is a quadratic form in $\mathbf{u}_T$ and $\boldsymbol{\epsilon}_{T+1}$ and hence that the conditions to apply Lemma 3 are met, so that the saddle point condition for $\boldsymbol{\mathcal{S}}_T$ yields the optimal control $\mathbf{u}_T$, with the extremized stress function, $\boldsymbol{\mathcal{S}}_T$, and the minimization criterion, $\boldsymbol{\mathcal{V}}_T$, both quadratic forms in $\mathbf{z}_T$. By backward induction the statement is established. $\square$

Theorem 1 is particularly useful in that it suggests that, when the discounted total stress function is well-behaved and hence the DLEQG problem admits a meaningful solution, to find the optimal control vector in period t it is sufficient to impose a saddle point condition for $\boldsymbol{\mathcal{S}}_t$. Indeed, first the discounted total stress function is maximized with respect to the innovation vector $\boldsymbol{\epsilon}_{t+1}$ and then the resulting expression is minimized with respect to the control vector $\mathbf{u}_t$, a procedure that Whittle refers to as *extremization*.[5] An economic interpretation of such extremization is that a risk-averse agent whose preferences are represented by the recursive optimization criterion (1.1) attempts to hedge against the worst possible values for the vector $\boldsymbol{\epsilon}_{t+1}$, by following a *min-max* strategy. Thus, she selects that policy $\mathbf{u}_t$ which minimizes her welfare loss (i.e. the discounted total stress) against the most unfavorable innovation vector $\boldsymbol{\epsilon}_{t+1}$. Such an agent acts as though she were pessimistic, considering these adverse realizations very likely. Consequently she tunes her actions on their impact on her welfare, applying what we term a *pessimistic* choice mechanism.

Theorem 1 revises the certainty equivalence principle of the Markovian Linear Quadratic Gaussian (LQG) problem: the normally distributed unobservable variables are no longer replaced by their maximum likelihood (ML) estimates, but by those that maximize the discounted total stress function in order to compensate for risk-aversion. In other words, in the Markovian LQG problem the separation principle between optimization of the control vector and estimation of the unknown values applies, in that the control vector is chosen by minimizing the criterion as it would be in the perfect information case, with the unobservable values replaced by their ML estimates. On the contrary, in the risk-sensitive case the derivation of the optimal control vector and the optimal estimation of the unknown values are intertwined, as the optimal control and optimal estimates are chosen in order to *extremize* the discounted stress function. Indeed, differently from the Markovian LQG problem, uncertainty over the innovation vector $\boldsymbol{\epsilon}_{t+1}$ conditions the optimal choice of the control vector, $\mathbf{u}_t$. Specifically, the statistical characteristics of $\boldsymbol{\epsilon}_{t+1}$, and hence its covariance matrix $\mathbf{N}$, influence the optimal value of the vector $\mathbf{u}_t$. *Viceversa*, the cost function and the degree of risk-aversion affect the optimal estimate of $\boldsymbol{\epsilon}_{t+1}$, which no longer corresponds to the ML estimate but it is given by the *maximum discounted total stress* estimate (MDTSE).

[5] One should notice that in the perfect state observation $\boldsymbol{\epsilon}_{t+1}$ and $\mathbf{z}_{t+1}$ are interchangeable. This means that at time t the extremization of the discounted total stress function can be equivalently conducted with respect to period $t+1$ state vector, $\mathbf{z}_{t+1}$. In other words, Theorem 1 could be equivalently reformulated with respect to the future value of the state vector $\mathbf{z}_{t+1}$.

Theorem 1 indicates that because of the recursive structure of the Markovian DLEQG problem the extremization of the discounted total stress function can be undertaken recursively. In particular, starting from T one proceeds backward imposing the saddle point conditions for the stress function in periods T, $T-1$, ..., 1. In this respect, we adapt to the Markovian DLEQG problem a result originally derived by Whittle.

Lemma 4 *The saddle point conditions for the discounted total stress function $\boldsymbol{\mathcal{S}}_t$, with $t = 1, 2, \ldots, T$, can be satisfied by solving the following discounted future stress backward recursion,*

$$F_t(\mathbf{z}_t) \quad = \quad \min_{\mathbf{u}_t} \left\{ \max_{\boldsymbol{\epsilon}_{t+1}} \left[c_t \;-\; \frac{1}{\rho}\, d_{t+1} \;+\; \delta F_{t+1}(\mathbf{z}_{t+1}) \right] \right\}, \tag{1.3}$$

with $t = T, T-1, \ldots, 1$, where $F_t(\mathbf{z}_t)$, denoted as the extremized discounted future stress function, is a quadratic form in the state vector $\mathbf{z}_t$, $F_t(\mathbf{z}_t) \equiv \mathbf{z}_t'\boldsymbol{\Pi}_t\mathbf{z}_t$ with $\boldsymbol{\Pi}_{T+1} \equiv \mathbf{0}$. The optimization criterion in t is $\boldsymbol{\mathcal{V}}_t = \kappa_t + F_t(\mathbf{z}_t)$, where κ_t is independent of $\mathbf{z}_t$.

Proof. Let us start from $t = T$. By definition $\boldsymbol{\mathcal{V}}_{T+1} = 0$ and $d_{T+1} = 0$. Then, given that c_T is a quadratic form in $\mathbf{u}_T$ and $\mathbf{z}_T$, we immediately see that: i) imposing the saddle point condition for the discounted stress function in T, $\boldsymbol{\mathcal{S}}_T$, is equivalent to solving recursion (1.3); and ii) there exist a matrix $\boldsymbol{\Pi}_T$ such that the extremized discounted future stress is $F_T(\mathbf{z}_T) \equiv \mathbf{z}_T'\boldsymbol{\Pi}_T\mathbf{z}_T$ and a constant k_T independent of $\mathbf{z}_T$ such that $\exp(\rho\boldsymbol{\mathcal{V}}_T/2) = \exp(\frac{1}{2}\rho[k_T + F_T(\mathbf{z}_T)])$. Proceeding backward, the optimal control vector in period $T-1$ is obtained by imposing the following saddle point condition,

$$\min_{\mathbf{u}_{T-1}} \max_{\boldsymbol{\epsilon}_T} \boldsymbol{\mathcal{S}}_{T-1} \;=\; \min_{\mathbf{u}_{T-1}} \left\{ \max_{\boldsymbol{\epsilon}_T} \left[c_{T-1} - \frac{1}{\rho} d_T \;+\; \delta\boldsymbol{\mathcal{V}}_T \right] \right\}$$

Since $\boldsymbol{\mathcal{V}}_T = k_T + F_T(\mathbf{z}_T)$ and k_T is independent of $\mathbf{z}_T$, this is equivalent to the saddle point condition

$$\min_{\mathbf{u}_{T-1}} \left\{ \max_{\boldsymbol{\epsilon}_T} \left[c_{T-1} \;-\; \frac{1}{\rho}\, d_T \;+\; \delta F_T(\mathbf{z}_T) \right] \right\}.$$

Given that c_{T-1} is a quadratic function in $\mathbf{u}_{T-1}$ and $\mathbf{z}_{T-1}$, d_T is a quadratic form in $\boldsymbol{\epsilon}_T$ and $F_T(\mathbf{z}_T)$ is a quadratic form in $\mathbf{z}_T$ while this is linear in $\mathbf{u}_{T-1}$, $\mathbf{z}_{T-1}$ and $\boldsymbol{\epsilon}_T$, we find that the result of this extremization is given by a quadratic form of $\mathbf{z}_{T-1}$, so that there exists a matrix $\boldsymbol{\Pi}_{T-1}$ such that $F_{T-1}(\mathbf{z}_{T-1}) = \mathbf{z}_{T-1}'\boldsymbol{\Pi}_T\mathbf{z}_{T-1}$ and $\exp(\rho\boldsymbol{\mathcal{V}}_{T-1}/2) = \exp(\frac{1}{2}\rho[\kappa_{T-1} + F_t(\mathbf{z}_{T-1})])$. Then, by backward induction the statement is proved. □

In conclusion, a recursion similar to the Bellman equation for the value function of dynamic programming is obtained: given the extremized discounted stress function at time $t+1$, the optimal policy at time t is obtained by solving the discounted future stress recursion.[6] Similarly to the Markovian LQG problem, the extremized future stress function is a quadratic form in the state vector, $F_t(\mathbf{z}_t) = \mathbf{z}_t'\mathbf{\Pi}_t\mathbf{z}_t$, while it can be established that the optimal policy is linear in the state vector, $\mathbf{u}_t = \mathbf{K}_t\mathbf{z}_t$, where $\mathbf{\Pi}_t$ and $\mathbf{K}_t$ respect recursions which correspond to modified versions of the *Riccati* recursions for the Markovian LQG problem.

Indeed, applying Lemma 4 we can establish the following Theorem, which extends Whittle's risk-sensitive Riccati equation to the class of Markovian DLEQG problems and reveals the nexus with the common solutions to the standard Markovian LQG problem.

Theorem 2 - *(Risk-sensitive Riccati Equation). If the matrix $(\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$ is positive definite, at time t the extremized discounted future stress function exists and it is given by*

$$F_t(\mathbf{z}_t) \;=\; \mathbf{z}_t'\,\mathbf{\Pi}_t\,\mathbf{z}_t \quad \textit{for the optimal control} \quad \mathbf{u}_t \;=\; \mathbf{K}_t\,\mathbf{z}_t \quad \textit{where} \tag{1.4}$$

$$\mathbf{\Pi}_t \;=\; \mathbf{R} \,+\, \mathbf{A}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{A} \,-\, (\mathbf{S}' \,+\, \mathbf{A}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B})(\mathbf{Q} \,+\, \mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S} \,+\, \mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{A})\,, \tag{1.5}$$

$$\mathbf{K}_t \;=\; -\,(\mathbf{Q} \,+\, \mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S} \,+\, \mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{A}) \tag{1.6}$$

$$\textit{and} \qquad \widetilde{\mathbf{\Pi}}_{t+1} \;=\; ((\delta\mathbf{\Pi}_{t+1})^{-1} \;-\; \rho\,\mathbf{N})^{-1}\,. \tag{1.7}$$

Proof. In the Markovian DLEQG problem the future stress function $F_t(\mathbf{z}_t)$ respects the double recursion $F_t \;=\; \boldsymbol{\mathcal{L}}\,\tilde{\boldsymbol{\mathcal{L}}}\,F_{t+1}$, based on the following two operators

$$\boldsymbol{\mathcal{L}}\,\phi(\mathbf{z}) \;=\; \min_{\mathbf{u}}\,[c(\mathbf{z},\mathbf{u}) \,+\, \phi(A\mathbf{z} + B\mathbf{u})] \quad \text{and} \quad \tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[\phi(\mathbf{z}+\boldsymbol{\epsilon}) \,-\, \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]\,,$$

where $\phi(\mathbf{z}) = \delta\mathbf{z}'\mathbf{\Pi}\mathbf{z}$, so that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[(\mathbf{z}+\boldsymbol{\epsilon})'\delta\mathbf{\Pi}(\mathbf{z}+\boldsymbol{\epsilon}) \,-\, \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]$. Taking first derivatives, we find that

$$\tilde{\boldsymbol{\epsilon}} \;=\; -\,(\delta\mathbf{\Pi} \,-\, \frac{1}{\rho}\mathbf{N}^{-1})^{-1}\delta\mathbf{\Pi}\,\mathbf{z} \;=\; -\,\breve{\mathbf{\Pi}}^{-1}\delta\mathbf{\Pi}\,\boldsymbol{z}\,,$$

which pins down a maximum if $\breve{\mathbf{\Pi}}$ is negative definite, or equivalently if $(\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N}$ is positive definite. Replacing this expression we conclude that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \mathbf{z}'((\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N})^{-1}\mathbf{z} \;=\; \mathbf{z}'\,\widetilde{\mathbf{\Pi}}\,\mathbf{z}$. For $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) = \mathbf{z}'\widetilde{\mathbf{\Pi}}\,\mathbf{z}$, solution of the operator $\boldsymbol{\mathcal{L}}$ yields the standard recursive formulae for $\mathbf{\Pi}$ and

[6] For $\delta \uparrow 1$ the discounted future stress recursion (1.3) converges to the recursion originally derived by Whittle for the Markovian LEQG problem. This confirms that under perfect state observation, the optimal policy for our recursive optimization criterion converges to that for Whittle's Markovian LEQG problem for $\delta \uparrow 1$. In other words, under perfect state observation, the DLEQG problem encompasses the LEQG one.

$\mathbf{K}$ from the Markovian LQG problem where $\widetilde{\mathbf{\Pi}} = ((\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N})^{-1}$ replaces $\mathbf{\Pi}$. Thus, applying the two operators at time t we obtain the recursive formulae for $\mathbf{\Pi}_t$ and $\mathbf{K}_t$ presented in the statement. Importantly, as the cost function c_t is positive definite in $\mathbf{u}_t$ and $\mathbf{z}_t$, $\mathbf{Q} + \mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B}$ is positive definite, so that the second order condition for minimization in the operator $\boldsymbol{\mathcal{L}}$ holds, $\mathbf{\Pi}_t$ is positive semidefinite. $\square$

It is worth noticing that with respect to the standard Riccati equation which applies to the standard Markovian LQG problem with perfect state observation, the matrix $\mathbf{\Pi}_{t+1}$ is now replaced by the modified matrix $\widetilde{\mathbf{\Pi}}_{t+1}$.[7] In other words, in the risk-sensitive case the optimal policy retains a specification which is very similar to the one that would prevail in a risk-neutral environment (with $\rho = 0$), in that only a correction for the impact of uncertainty and risk-aversion must be inserted in the expressions for the recursions of $\mathbf{\Pi}_t$ and $\mathbf{K}_t$.

The requirement that the matrix $(\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$ being positive definite derives from a second order condition which must hold for the discounted total stress function $\boldsymbol{\mathcal{S}}_t$ to satisfy the saddle point condition imposed by Theorem 1. As noted by Whittle, whenever the cost function c_t is non-negative such condition fails for ρ large enough, indicating that the minimization criterion $\boldsymbol{\mathcal{V}}_t$ is infinite and confirming our earlier claim that for a sufficiently large degree of risk-aversion the DLEQG problem is not well-behaved and does not admit an optimizing solution. An economic interpretation of the failure of the optimization problem is that in these extreme circumstances the optimizing agent becomes so *pessimistic* as to consider her control ineffective and hence useless.

Because of time-discounting it is possible to consider the limit case for $T \uparrow \infty$, that is a DLEQG problem with infinite horizon. As indicated by Hansen and Sargent (1994) there is no certainty that for $T \uparrow \infty$ the criterion $\boldsymbol{\mathcal{V}}_t$ is finite and hence the DLEQG may be not well-defined. However, when a minimum is reached we can identify a stationary solution, in that in the limit $\mathbf{\Pi}_t \to \mathbf{\Pi}$ and $\mathbf{K}_t \to \mathbf{K}$, where the limit matrices are determined by the fixed point in the risk-sensitive Riccati equation,

$$\mathbf{\Pi} \quad = \quad \mathbf{R} + \mathbf{A}'\widetilde{\mathbf{\Pi}}\mathbf{A} - (\mathbf{S}' + \mathbf{A}'\widetilde{\mathbf{\Pi}}\mathbf{B})(\mathbf{Q} + \mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}(\mathbf{S} + \mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{A}), \tag{1.8}$$

$$\text{with} \quad \widetilde{\mathbf{\Pi}} \equiv ((\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N})^{-1}. \tag{1.9}$$

[7] With respect to the original result by Whittle for the LEQG problem two adjustments are required as consequence of the introduction of time-discounting: i) the second order condition for the extremization of the discounted future stress function requires that $(\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$, rather than $\mathbf{\Pi}_{t+1}^{-1} - \rho\mathbf{N}$, is positive definite; and ii) in the modified Riccati equation $\widetilde{\mathbf{\Pi}}_{t+1} = ((\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N})^{-1}$ replaces $\widetilde{\mathbf{\Pi}}_{t+1} = (\mathbf{\Pi}_{t+1}^{-1} - \rho\mathbf{N})^{-1}$.

2 The Recursive Criterion Under Imperfect State Observation

Hansen and Sargent do not consider the scenario in which only a noisy signal on the state vector $\mathbf{z}_t$ is observed at time t.[8] In such a scenario the initial recursive criterion proposed in equation (1.1) is not well-defined, as the cost function c_t is no longer deterministic. However, we can employ the recursive criterion in equation (1.2). Under imperfect state observation, while no longer equivalent to the former one, this criterion is well-defined.

To allow for imperfect state observation we introduce the following modified Definition for the class of Markovian DLEQG problems.

Definition 3 *An optimal control problem is said to be Markovian Linear Exponential Quadratic Gaussian with time-discounting and imperfect state observation if the recursion (1.2) is solved over the periods $t = 1, 2, \ldots, T$ with respect to the free-valued control vector $\mathbf{u}_t$ under the conditions that:*

(i) for $t = 1,\ 2,\ \ldots T$, the cost function c_t is a quadratic form in the control vector, $\mathbf{u}_t$, and the state vector, $\mathbf{z}_t$, $c_t = \mathbf{u}_t'\mathbf{Q}\mathbf{u}_t + \mathbf{z}_t'\mathbf{R}\mathbf{z}_t + 2\mathbf{u}_t'\mathbf{S}\mathbf{z}_t$, positive definite in $\mathbf{u}_t$ and $\mathbf{z}_t$;
(ii) the vector of state variables, $\mathbf{z}_t$, follows a linear plant equation $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} + \boldsymbol{\epsilon}_t$;
(iii) the vector of observable variables is given by

$$\mathbf{w}_t \;=\; \mathbf{C}\,\mathbf{z}_{t-1} \;+\; \boldsymbol{\eta}_t\,,$$

with $\boldsymbol{\psi}_t \;=\; \begin{pmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{\eta}_t \end{pmatrix} \sim \boldsymbol{\mathcal{N}}\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{N} & \mathbf{L} \\ \mathbf{L}' & \mathbf{M} \end{pmatrix}\right]$ *and* $\boldsymbol{\psi}_t \perp \boldsymbol{\psi}_{t'}$.

As $\mathbf{z}_t$ is now unobservable, the discounted stress function takes a new formulation. In particular, let $\hat{\mathbf{z}}_{t-1}$ denote the expectation of the state vector $\mathbf{z}_{t-1}$ conditional on the information contained in observation history $\mathbf{H}_{t-1} = \{\mathbf{h}_0, \mathbf{U}_{t-2}, \mathbf{W}_{t-1}\}$, with $\mathbf{U}_{t-2} = (\mathbf{u}_1, \ldots, \mathbf{u}_{t-2})$ and $\mathbf{W}_{t-1} = (\mathbf{w}_1, \ldots, \mathbf{w}_{t-1})$, $\boldsymbol{\Omega}_{t-1}$ the corresponding conditional covariance matrix and $\mathbf{P}$ the covariance matrix for $\boldsymbol{\psi}_t$, with

$$\mathbf{P} = \begin{pmatrix} \mathbf{N} & \mathbf{L}' \\ \mathbf{L} & \mathbf{M} \end{pmatrix}.$$

We then introduce the following Definition.

[8]Hansen and Sargent, in deriving recursive linear control rules for their optimization criterion, rely on results developed by Jacobson (Jacobson, 1973, 1977) to analyze LEQG problems under perfect state observation.

Definition 4 *Under imperfect state observation, the discounted total stress function, $\boldsymbol{\mathcal{S}}_t$, is given by $\boldsymbol{\mathcal{S}}_t \equiv c_t - \frac{1}{\rho}\left(\boldsymbol{\mathcal{D}}_{t-1} + d_t + d_{t+1}\right) + \delta \boldsymbol{\mathcal{V}}_{t+1}$, where d_t is equal to $\boldsymbol{\psi}_t' \mathbf{P}^{-1} \boldsymbol{\psi}_t$ for $t = 1, 2, \ldots T$ and 0 for $t = T+1$, while $\boldsymbol{\mathcal{D}}_{t-1} = (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \, \boldsymbol{\Omega}_{t-1}^{-1} \, (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})$.*

We can then prove the following Lemma, which reformulates a result stated in Lemma 3 for the perfect state observation scenario.

Lemma 5 *In a Markovian DLEQG problem, under imperfect state observation, if the optimization criterion in $t+1$, $\boldsymbol{\mathcal{V}}_{t+1}$, is a quadratic form in the state vector $\mathbf{z}_{t+1}$ and the discounted total stress function $\boldsymbol{\mathcal{S}}_t$ satisfies a saddle point condition with respect to $\boldsymbol{\xi_t}$ (with $\boldsymbol{\xi}_t' \equiv (\mathbf{z}_{t-1}' - \hat{\mathbf{z}}_{t-1}' \ \boldsymbol{\psi}_t' \ \boldsymbol{\psi}_{t+1}')$) and $\mathbf{u}_t$, so that $\min_{\mathbf{u}_t} \max_{\boldsymbol{\xi}_t} \boldsymbol{\mathcal{S}}_t$ exists, the following proportionality condition holds,*

$$\min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2} (c_t \, + \, \delta \, \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad \propto \quad \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\xi}_t} \boldsymbol{\mathcal{S}}_t \right) ,$$

where the proportionality constant is independent of the state vector $\mathbf{z}_t$, while the optimization criterion $\boldsymbol{\mathcal{V}}_t$ is a quadratic form in $\mathbf{z}_t$ equal to the extremized discounted total stress function $\min_{\mathbf{u}_t} \max_{\boldsymbol{\xi}_t} \boldsymbol{\mathcal{S}}_t$ plus a constant independent of $\mathbf{z}_t$.

Proof. Consider that under imperfect state observation $\boldsymbol{\mathcal{V}}_{t+1}$ is still a function of $\mathbf{z}_{t+1}$, while c_t is function of $\mathbf{z}_t$. Since under imperfect state information $\mathbf{z}_t$ and $\mathbf{z}_{t+1}$ can be expressed in terms of the vector $\boldsymbol{\xi}_t$, we have

$$\min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2} (c_t \, + \, \delta \, \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad \propto \quad \min_{\mathbf{u}_t} \int \exp\left(\frac{\rho}{2} (c_t \, + \, \delta \, \boldsymbol{\mathcal{V}}_{t+1}) - \frac{1}{2} \, \boldsymbol{\xi}_t' \boldsymbol{\Upsilon}_{t-1}^{-1} \boldsymbol{\xi}_t \right) d\boldsymbol{\xi}_t \, ,$$

where $\boldsymbol{\Upsilon}_{t-1}$ denotes the covariance matrix of $\boldsymbol{\xi}_t$ conditional on observation history $\mathbf{H}_{t-1}$. In addition, since $(\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \perp \boldsymbol{\psi}_t' \perp \boldsymbol{\psi}_{t+1}'$, we can write

$$\begin{aligned} \min_{\mathbf{u}_t} E_t \left[\exp\left(\frac{\rho}{2} (c_t \, + \, \delta \, \boldsymbol{\mathcal{V}}_{t+1}) \right) \right] \quad \propto \quad & \min_{\mathbf{u}_t} \int \exp\Bigg(\frac{\rho}{2} (c_t \, + \, \delta \, \boldsymbol{\mathcal{V}}_{t+1}) - \frac{1}{2} \Big(\boldsymbol{\psi}_{t+1}' \mathbf{P}^{-1} \boldsymbol{\psi}_{t+1} \; + \\ & \boldsymbol{\psi}_t' \mathbf{P}^{-1} \boldsymbol{\psi}_t \; + \; (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1})' \, \boldsymbol{\Omega}_{t-1}^{-1} \, (\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1}) \Big) \Bigg) \, d\boldsymbol{\xi}_t \\ = \quad & \min_{\mathbf{u}_t} \int \exp\left(\rho \, \frac{\boldsymbol{\mathcal{S}}_t}{2} \right) \, d\boldsymbol{\xi}_t \, . \end{aligned}$$

Then, we can proceed as in the Proof of Lemma 3. Since $\boldsymbol{\mathcal{V}}_{t+1}$ is a quadratic function of $\mathbf{z}_{t+1}$, as the latter is linearly dependent on $\boldsymbol{\xi}_t$ and $\mathbf{u}_t$, $\boldsymbol{\mathcal{S}}_t$ is a quadratic form in $\boldsymbol{\xi}_t$ and $\mathbf{u}_t$. If the discounted total stress function respects the aforementioned saddle point condition, exploiting the property of quadratic forms outlined in the Proof of Lemma 3, we conclude that

$$\begin{aligned}\min_{\mathbf{u}_t} \int \exp\left(\rho \frac{\boldsymbol{\mathcal{S}}_t}{2}\right) d\boldsymbol{\xi}_t &= \min_{\mathbf{u}_t} \int \exp\left(-\frac{1}{2} \underbrace{(-\rho \boldsymbol{\mathcal{S}}_t)}_{Q(\mathbf{u}_t, \boldsymbol{\xi}_t)}\right) d\boldsymbol{\xi}_t \\ &\propto \exp\left(-\frac{1}{2} \max_{\mathbf{u}_t} \min_{\boldsymbol{\xi}_t} (-\rho \boldsymbol{\mathcal{S}}_t)\right) = \exp\left(\frac{\rho}{2} \min_{\mathbf{u}_t} \max_{\boldsymbol{\xi}_t} \boldsymbol{\mathcal{S}}_t\right),\end{aligned}$$

where once again we have made use of the fact that $-\boldsymbol{\mathcal{S}}_t$ admits a saddle point in $\mathbf{u}_t$ and in $\boldsymbol{\xi}_t$. As the discounted total stress function $\boldsymbol{\mathcal{S}}_t$ respects the saddle point condition, its extremized value will be a quadratic form in $\mathbf{z}_t$ and so will be $\boldsymbol{\mathcal{V}}_t$. $\square$

Lemma 5 suggests a revision of RSCEP outlined in Theorem 1.

Theorem 3 - *(Risk-sensitive Certainty Equivalence Principle). In a Markovian DLEQG problem, under imperfect state observation, if the discounted total stress function $\boldsymbol{\mathcal{S}}_{t+j}$ respects a saddle point condition with respect to $\boldsymbol{\xi_{t+j}}$ and $\mathbf{u}_{t+j}$ for $j = 0, 1, \ldots, T - t$, the optimal value of the vector $\mathbf{u}_t$ is determined at time t by simultaneously minimizing $\boldsymbol{\mathcal{S}}_t$ with respect to $\mathbf{u}_t$ and maximizing it with respect to $\boldsymbol{\xi}_t$. The extremized discounted stress function is proportional to the recursive optimization criterion, $\boldsymbol{\mathcal{V}}_t \propto \min_{\mathbf{u}_t} \max_{\boldsymbol{\xi}_t} \boldsymbol{\mathcal{S}}_t$.*

The vector $\boldsymbol{\xi}_t$ contains the vectors $\mathbf{z}_{t-1} - \hat{\mathbf{z}}_{t-1}$, $\boldsymbol{\epsilon}_t$, $\boldsymbol{\eta}_t$, $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\eta}_{t+1}$ which at time t are unknown. Following Whittle's suggestion they can be expressed as linear functions of the unobservable (at time t) state vectors $\mathbf{z}_{t-1}$, $\mathbf{z}_t$ and $\mathbf{z}_{t+1}$ and signal vector $\mathbf{w}_{t+1}$. Then the saddle-point condition for the discounted total stress function in t can be equivalently expressed as follows,

$$\min_{\mathbf{u}_t} \max_{\mathbf{z}_{t-1}, \mathbf{z}_t, \mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \boldsymbol{\mathcal{S}}_t .$$

By writing the saddle point condition in this way, we see that it can be satisfied proceeding in two stages: in stage i), conditionally on the current state vector $\mathbf{z}_t$, the discounted total stress function is extremized with respect to $\mathbf{u}_t$, $\mathbf{z}_{t-1}$, $\mathbf{z}_{t+1}$ and $\mathbf{w}_{t+1}$; in stage ii) the resulting function is extremized with respect to $\mathbf{z}_t$. In fact, we notice that

$$\min_{\mathbf{u}_t} \max_{\mathbf{z}_{t-1}, \mathbf{z}_t, \mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \boldsymbol{\mathcal{S}}_t \Leftrightarrow \max_{\mathbf{z}_t} \left\{ \min_{\mathbf{u}_t} \max_{\mathbf{z}_{t-1}, \mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \boldsymbol{\mathcal{S}}_t \right\} .$$

In stage i), conditionally on $\mathbf{z}_t$, the extremization of the discounted stress function will be achieved by isolating terms in $\boldsymbol{\mathcal{S}}_t$ which pertain to *past* and *future*. This allows a partial separation between estimation and control. Specifically, let $P_t(\mathbf{z}_t, \mathbf{H}_t)$ denote the *extremized discounted past stress function* defined as

$$P_t(\mathbf{z}_t, \mathbf{H}_t) \quad = \quad \max_{\mathbf{z}_{t-1}} \left\{ -\frac{1}{\rho}\left(d_t \; + \; \boldsymbol{\mathcal{D}}_{t-1} \right) \right\} \,, \tag{2.1}$$

where $\mathbf{H}_t$ is observation history at time t. Then, the following Lemma, which adapts to the Markovian DLEQG problem a result originally derived by Whittle, holds.

Lemma 6 *Under imperfect state observation, the extremization of the discounted stress function at time t, with $t = 1, 2, \dots, T$, can be obtained operating into two stages. Firstly, the extremized discounted past and future stress functions, $P_t(\mathbf{z}_t, \mathbf{H}_t)$ and $F_t(\mathbf{z}_t)$, are calculated conditionally on the current vector, $\mathbf{z}_t$. Secondly, the saddle point for the discounted total stress function $\boldsymbol{\mathcal{S}}_t$ is achieved by maximizing $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$ with respect to $\mathbf{z}_t$.*

Proof. The discounted total stress function can be divided into two parts which contain values respectively depending on past and future variables,

$$-\frac{1}{\rho}\left(d_t \; + \; \boldsymbol{\mathcal{D}}_{t-1} \right) \;\; \text{and} \;\; c_t \; - \; \frac{1}{\rho} d_{t+1} \; + \; \delta \boldsymbol{\mathcal{V}}_{t+1} \, .$$

Hence, in stage i), extremizing $\boldsymbol{\mathcal{S}}_t$ with respect to $\mathbf{u}_t$ and $\mathbf{z}_{t-1}$, $\mathbf{z}_{t+1}$, $\mathbf{w}_{t+1}$, conditionally on the current state vector $\mathbf{z}_t$, is equivalent to solving separately the two programs,

$$\max_{\mathbf{z}_{t-1}} \left\{ -\frac{1}{\rho}\left(d_t \; + \; \boldsymbol{\mathcal{D}}_{t-1} \right) \right\} \;\; \text{and} \;\; \min_{\mathbf{u}_t} \; \max_{\mathbf{z}_{t+1}, \mathbf{w}_{t+1}} \; \left\{ c_t \; - \; \frac{1}{\rho} d_{t+1} \; + \; \delta \boldsymbol{\mathcal{V}}_{t+1} \right\} \, .$$

As $\mathbf{z}_{t+1}$ and $\mathbf{w}_{t+1}$ are linearly dependent on $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\eta}_{t+1}$, the latter program can be re-written as follows

$$\min_{\mathbf{u}_t} \; \max_{\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\eta}_{t+1}} \; \left\{ c_t \; - \; \frac{1}{\rho} d_{t+1} \; + \; \delta \boldsymbol{\mathcal{V}}_{t+1} \right\} .$$

The maximization of $c_t - (1/\rho) d_{t+1} + \boldsymbol{\mathcal{V}}_{t+1}$ with respect to $\boldsymbol{\eta}_{t+1}$ reduces to $\max_{\boldsymbol{\eta}_{t+1}} \{-(1/\rho) d_{t+1}\}$ $= -\frac{1}{\rho} \boldsymbol{\epsilon}'_{t+1} \mathbf{N}^{-1} \boldsymbol{\epsilon}_{t+1}$. Importantly, this means that under imperfect state information the extremization, conditionally on $\mathbf{z}_t$, of the discounted total stress function, $\boldsymbol{\mathcal{S}}_t$, with respect to $\mathbf{u}_t$, $\mathbf{z}_{t+1}$ and $\mathbf{w}_{t+1}$ is equivalent to the extremization of the discounted total stress function under perfect state information with respect to $\mathbf{u}_t$ and $\boldsymbol{\epsilon}_{t+1}$. Lemma 4 shows that this corresponds to calculating the extremized discounted future stress function, $F_t(\mathbf{z}_t)$, which yields the optimal policy, $\mathbf{u}(\mathbf{z}_t)$, conditional on the state vector, $\mathbf{z}_t$.

The former program instead corresponds to calculating the extremized discounted past stress function in t, $P_t(\mathbf{z}_t, \mathbf{H}_t)$. This extremization identifies the optimal estimate for $\mathbf{z}_t$ conditionally on information up to time t. Estimation and control are then recoupled by maximizing the sum $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$ with respect to the current state vector $\mathbf{z}_t$. $\square$

Theorem 3 and Lemma 6 extend to the class of Markovian DLEQG problems Whittle's risk-sensitive separation principle (RSSP) which allows to find the optimal policy by separating estimation and control. In particular, if the conditions listed in Theorem 3 for the existence of a meaningful solution to the DLEQG problem under imperfect state observation are met, this is found through the following Theorem.

Theorem 4 - *(Risk-sensitive Separation Principle). The extremized discounted past and future stress functions P_t and F_t relate to estimation and control respectively, as the evaluation of the former identifies the optimal estimate for the state vector $\mathbf{z}_t$ conditional on past observations, while that of the latter pins down the control $\mathbf{u}_t(\mathbf{z}_t)$ which would be optimal if $\mathbf{z}_t$ were known. The calculations of P_t and F_t are recoupled by maximizing $P_t + F_t$ with respect to the state vector $\mathbf{z}_t$ so as to find the maximum discounted total stress estimate (MDTSE) $\breve{\mathbf{z}}_t$ for the state vector $\mathbf{z}_t$.*

- (Risk-sensitive Certainty Equivalence Principle). The optimal value of the control vector at time t is then given by $\mathbf{u}_t(\breve{\mathbf{z}}_t)$.

As mentioned, conditionally on $\mathbf{z}_t$ the extremized discounted future stress function, F_t, respects the recursion (1.3) in Lemma 4. Then, Theorem 2 provides the conditional optimal policy, $\mathbf{u}_t(\mathbf{z}_t)$. The extremization of the discounted past stress function, P_t, is obtained via the following Lemma.

Lemma 7 *The extremized discounted past stress function is equal to $P_t(\mathbf{z}_t, \mathbf{H}_t) = -(1/\rho)\,(\mathbf{z}_t - \hat{\mathbf{z}}_t)'\,\mathbf{\Omega}_t^{-1}\,(\mathbf{z}_t - \hat{\mathbf{z}}_t) + \cdots$, where $\cdots$ indicates terms independent of $\mathbf{z}_t$, while $\hat{\mathbf{z}}_t$ denotes the ML estimate of the state vector $\mathbf{z}_t$, given by its expectation conditional on observation history $\mathbf{H}_t$, and $\mathbf{\Omega}_t$ the corresponding conditional covariance matrix.*

Proof. $P_t(\mathbf{z}_t, \mathbf{H}_t) = \max_{\mathbf{z}_{t-1}} -\frac{1}{\rho}\left\{d_t + \boldsymbol{\mathcal{D}}_{t-1}\right\}$. Given the definition of $\boldsymbol{\mathcal{D}}_{t-1}$ this is equivalent to $\max_{\mathbf{z}_{t-1}} \frac{1}{\rho} \log f(\mathbf{z}_t \mid \mathbf{H}_{t-1})$ where $f(.)$ is the conditional density function of $\mathbf{z}_t$ given $\mathbf{H}_{t-1}$. The maximum corresponds to $P_t(\mathbf{z}_t, \mathbf{H}_t) = -\frac{1}{\rho}\boldsymbol{\mathcal{D}}_t + \cdots$, where now $\cdots$ indicates terms independent of $\mathbf{z}_t$. $\square$

The extremized discounted past stress function depends on the maximum likelihood estimate (MLE) of the state vector $\mathbf{z}_t$. Given the noisy linear signal $\mathbf{w}_t$, from the *Kalman filter* we see that this value respects the following recursion

$$\hat{\mathbf{z}}_t \;=\; \mathbf{A}\,\hat{\mathbf{z}}_{t-1} + \mathbf{B}\,\mathbf{u}_{t-1} + (\mathbf{L} + \mathbf{A}\boldsymbol{\Omega}_{t-1}\,\mathbf{C}')(\mathbf{M} + \mathbf{C}\,\boldsymbol{\Omega}_{t-1}\,\mathbf{C}')^{-1}\,(\mathbf{w}_t - \mathbf{C}\,\hat{\mathbf{z}}_{t-1}), \quad (2.2)$$

where the conditional covariance matrix $\boldsymbol{\Omega}_t$ respects the Riccati recursion,

$$\boldsymbol{\Omega}_t \;=\; \mathbf{N} + \mathbf{A}\,\boldsymbol{\Omega}_{t-1}\,\mathbf{A}' - (\mathbf{L} + \mathbf{A}\boldsymbol{\Omega}_{t-1}\,\mathbf{C}')(\mathbf{M} + \mathbf{C}\,\boldsymbol{\Omega}_{t-1}\,\mathbf{C}')^{-1}\,(\mathbf{L}' + \mathbf{C}\,\boldsymbol{\Omega}_{t-1}\,\mathbf{A}')\,. \quad (2.3)$$

Importantly, Lemma 7 indicates that differently from what applies to the Markovian LEQG problem studied by Whittle, in the extremization of the discounted past stress function no adjustment is made to the MLE of $\mathbf{z}_t$ to correct for the impact of risk-aversion. This is because differently from Whittle's formulation the extremized discounted past stress function does not depend on the cost function c_t.[9] However, estimation and control are still intertwined as suggested in the discussion of Theorem 1. In fact, to re-couple the extremization of past and future stress functions we apply Theorem 4 and calculate the sum $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$. The resulting function is then maximized with respect to the state vector $\mathbf{z}_t$ to obtain the maximum discounted total stress estimate (MDTSE), $\breve{\mathbf{z}}_t$. Given that $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t) = -(1/\rho)(\mathbf{z}_t - \hat{\mathbf{z}}_t)'\boldsymbol{\Omega}_t^{-1}(\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}_t'\boldsymbol{\Pi}_t\mathbf{z}_t'$ plus terms independent of $\mathbf{z}_t$, from the first derivative of this sum with respect to $\mathbf{z}_t$ it is immediate to see that, for $\boldsymbol{\Omega}_t^{-1} - \rho\,\boldsymbol{\Pi}_t$ positive definite, $\breve{\mathbf{z}}_t$ is given by the following expression

$$\breve{\mathbf{z}}_t \;=\; (\mathbf{I} - \rho\,\boldsymbol{\Omega}_t\,\boldsymbol{\Pi}_t)^{-1}\,\hat{\mathbf{z}}_t, \quad (2.4)$$

which is function of ρ and the matrix $\boldsymbol{\Pi}_t$. As the latter depends on the components of c_t, we see that the MDTSE $\breve{\mathbf{z}}_t$ is clearly affected by the shape of the cost function alongside the degree of risk-aversion, confirming the close nexus between control and estimation for the class of DLEQG problems. Finally, the optimal control vector under imperfect state observation is given by Theorem 2 where $\breve{\mathbf{z}}_t$ replaces $\mathbf{z}_t$, i.e. $\mathbf{u}_t = \mathbf{K}\breve{\mathbf{z}}_t$, where $\mathbf{K}$ is the matrix of optimal coefficients presented in Theorem 2.

[9] Whittle shows that a maximum past stress estimate (MPSE) replaces the standard MLE in the expression for P_t. Such MPSE is obtained by introducing an adjustment to the Kalman filter which accounts for the impact of risk-aversion on the optimal estimation of the state vector $\mathbf{z}_t$.

3 Monetary Policy for a Risk-averse Central Bank

We now apply this new formulation of the DLEQG problem to the issue of output and inflation stabilization on the part of an independent central bank. In particular, we refer to a simple analytical framework developed by Svensson (Svennson, 1997) which describes the optimal monetary policy of a central bank with an infinite-horizon, time-separable quadratic cost function of inflation and output gap. We investigate an extension of Svensson's analysis to the scenario where the monetary authorities: i) face a risk-sensitive optimization criterion; and ii) observe imperfectly inflation and output. This allows to see what happens when the CEP cannot be applied as in the LQG problem investigated by Svensson and the actual values of the state variables cannot be replaced by their expectations when they are imperfectly observed.

Svensson studies the optimal policy of a risk-neutral central bank which controls the short-term (real) interest rate to minimize the expected value of the loss function $\boldsymbol{\mathcal{L}}_t \equiv \sum_{i=0}^{\infty} \delta^i c_{t+i}$, where the c_t is a quadratic cost function in the inflation rate, π_t, and the output gap y_t, $c_t \equiv \pi_t^2 + \lambda y_t^2$, with $0 \leq \lambda$. The cost c_t captures the loss in welfare the economy incurs in period t when inflation and output deviates from first-best values, so that $\boldsymbol{\mathcal{L}}_t$ represents a social welfare loss function.[10,11]

The dynamics of the state variables, π_t and y_t, is given by the following system of equations

$$\pi_t = \pi_{t-1} + \alpha y_{t-1} + \epsilon_t^{\pi}, \tag{3.1}$$

$$y_t = \beta y_{t-1} - \gamma r_{t-1} + \epsilon_t^{y}, \tag{3.2}$$

where r_t is a short-term (real) interest rate and the coefficients α, β and γ are non-negative constants. The variation in the inflation rate is increasing in lagged output, while the latter is serially correlated and decreasing in the lagged (real) interest rate. As noted by Svensson the short-term interest rate affects output with one lag and the inflation rate with two lags, this discrepancy being an important feature of this model which is however consistent with ample empirical evidence.

As in the plant equation the innovation terms ϵ_t^{π} and ϵ_t^{y} follow independent white noise processes, Svensson investigates a standard Markovian DLQG problem. However, while he

[10] The long-run natural output level is normalized to zero so that y_t corresponds to output gap.

[11] The cost function should depend on the deviation of the inflation rate from a target level π^*. We postpone the discussion of this more involving formulation to Section 4, where we show how the normalization introduced here is inconsequential.

envisages a risk-neutral central bank, it can be argued that monetary authorities are risk-averse and mostly concerned with adverse shocks, such as those associated with a strong deflation, which possess a large negative impact on social welfare. Thence, within Svensson's formulation it is important to determine the optimal monetary policy of a risk-averse central bank which seeks to hedge against the most adverse economic conditions. We achieve this by recasting Svensson's formulation into the DLEQG framework we described in Section 1, in that according to Theorem 1 within such framework the monetary authorities will act as though they were pessimistic, choosing their monetary policy in order to minimize the social welfare loss in the face the worst possible economic shocks.[12]

Then, we introduce the criterion $\boldsymbol{\mathcal{V}}_t$ and define the vector of state variables $\mathbf{z}_t \equiv (\pi_t \ y_t)'$, the vector of innovation terms $\boldsymbol{\epsilon}_t \equiv (\epsilon_t^\pi \ \epsilon_t^y)'$ and the scalar control variable $u_t \equiv r_t$. We have

$$\mathbf{A} \equiv \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \mathbf{B} \equiv \begin{pmatrix} 0 \\ -\gamma \end{pmatrix}, \mathbf{R} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \mathbf{Q} \equiv \mathbf{S} \equiv \mathbf{0}, \mathbf{N} \equiv \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}.$$

As the optimization horizon is infinite we concentrate on the steady-state solution: by solving the fixed point for the modified Riccati equation (1.8) we find that $\exp(\rho\boldsymbol{\mathcal{V}}_t/2) = \exp(\frac{1}{2}\rho[k + \mathbf{z}_t'\boldsymbol{\Pi}\mathbf{z}_t])$ and the optimal control is $u_t = \mathbf{K}\mathbf{z}_t$, where k is a constant independent of $\mathbf{z}_t$,

$$\boldsymbol{\Pi} = \begin{pmatrix} 1+\delta W & \alpha\delta W \\ \alpha\delta W & \lambda+\alpha^2\delta W \end{pmatrix},$$

W is a positive root of the following quadratic equation

$$\delta\left(\alpha^2 - \delta(\alpha^2+\lambda)\rho\sigma_\pi^2\right)W^2 - \left(\delta(\alpha^2+\lambda) - \lambda(1-\delta\rho\sigma_\pi^2)\right)W - \lambda = 0$$

and

$$\mathbf{K} = \frac{1}{\gamma}\left(\frac{\alpha\delta W}{\alpha^2\delta W+\lambda-\theta\rho\sigma_\pi^2} \quad \beta + \frac{\alpha^2\delta W}{\alpha^2\delta W+\lambda-\theta\rho\sigma_\pi^2}\right),$$

where $\theta = \delta(\lambda + \delta(\alpha^2+\lambda)W)$. This implies that the optimal monetary policy is reached by

[12] van der Ploeg (2009) introduces risk-aversion in an monetary policy model where the central bank minimizes the expected value of an infinite-horizon, time-separable quadratic cost function of inflation and output gap, and these state variables follow a dynamics similar to that described in equations (3.1) and (3.2). However, since he does not allow for time-discounting in the time-separable cost function, his formulation cannot be considered a direct extension of Svensson's model. In his analysis van der Ploeg does not rely on a recursive criterion as the one presented in equation (1.2), but he recasts Svensson's model within Whittle's LEQG framework.

setting the short-term (real) interest rate equal to

$$r_t \quad = \quad \frac{1}{\gamma}\,\frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\,\pi_t \;+\; \frac{1}{\gamma}\left(\beta + \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\right) y_t\,, \tag{3.3}$$

a Taylor's rule which clearly subsumes that derived by Svensson for $\rho = 0$, in that under risk-neutrality,

$$W \;=\; \frac{1}{2}\left(1 \;-\; \frac{(1-\delta)\lambda}{\alpha^2\delta} \;+\; \sqrt{\left(1 \;+\; \frac{(1-\delta)}{\alpha^2\delta}\right)^2 \;+\; \frac{4\,\lambda}{\alpha^2}}\right)$$

and

$$r_t \quad = \quad \frac{1}{\gamma}\,\frac{\alpha\delta W}{\alpha^2\delta W + \lambda}\,\pi_t \;+\; \frac{1}{\gamma}\left(\beta + \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda}\right) y_t\,.$$

It is interesting to emphasize that similarities with Svensson's solution appear. In particular, denoting with $\pi_{t+1|t}$ the expectation at time t of the inflation rate in period $t+1$, we have that $\pi_{t+1|t} = \pi_t + \alpha y_t$. It is immediate to verify that

$$r_t \quad = \quad \frac{1}{\gamma}\left(\beta\,y_t \;+\; \frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\;\pi_{t+1|t}\right)$$

and that

$$\exp\left(\frac{\rho\boldsymbol{\mathcal{V}}_t}{2}\right) \;=\; \exp\!\left(\frac{\rho}{2}\left[k \;+\; \pi_t^2 \;+\; \lambda\,y_t^2 \;+\; \delta\,W\pi_{t+1|t}^2\right]\right),$$

so that the control path and the optimization criterion can be defined in terms of the inflation forecast. In addition, denoting with $\pi_{t+2|t}$ the expectation at time t of the inflation rate in period $t+2$, we find that at the optimum

$$\pi_{t+2|t} \;=\; -\,\frac{1}{\alpha\delta W}\,\left(\lambda \;-\; \theta\,\rho\,\sigma_\pi^2\right)\,y_{t+1|t}\,,$$

where $y_{t+1|t}$ denotes the expectation at time t of period $t+1$ output gap. This condition implies that the two-period ahead inflation forecast is equal to its target level insofar the one-period ahead expected output gap is null.

However, significant differences also emerge between the risk-neutral and risk-averse scenarios. When $\lambda = 0$, and hence only inflation targeting motivates the monetary authorities, the expectation of the inflation rate π_{t+2} in period t is always null for $\rho = 0$. As suggested by Svensson, in the risk-neutral scenario, for $\lambda = 0$ the inflation forecast becomes an explicit intermediate target, in that the monetary policy is optimal insofar $\pi_{t+2|t} = 0$. On the other hand, in the risk-averse scenario, for $\lambda = 0$ at the optimum $\pi_{t+2|t} = \alpha\delta\rho\sigma_\pi^2 y_{t+1|t} \neq 0$ (since $\theta = \alpha^2\delta^2 W$ for $\lambda = 0$). Because of risk-aversion, the standard CEP cannot be applied as in the LQG problem investigated by Svensson and hence the inflation forecast is not longer an explicit intermediate target when inflation targeting is the only mission of the central bank.

In addition, even when the monetary policy is also aimed at output stabilization ($\lambda > 0$), in the risk-neutral scenario we see that inflation forecasts dampen out, in that for $\rho = 0$ $\pi_{t+2|t} = \frac{\lambda}{\alpha^2\delta W+\lambda} \pi_{t+1|t}$. This indicates that under risk-neutrality with output stabilization, as the inflation forecast slowly converges to zero, the central bank expects the inflation rate to reach the target level in the long-run. This is not necessarily the case for $\rho > 0$, as now $\pi_{t+2|t} = (\frac{\lambda-\theta\rho\sigma_\pi^2}{\alpha^2\delta W+\lambda-\theta\rho\sigma_\pi^2})\pi_{t+1|t}$. Strikingly, the central bank may actually expect the inflation rate to wander away from the target level even if λ is small or null, that is even if output stabilization is not the main objective of its monetary policy. In fact, for $\lambda = 0$ we see that for $\rho > 0$, $\pi_{t+2|t} = \frac{-\delta\rho\sigma_\pi^2}{1-\delta\rho\sigma_\pi^2} \pi_{t+1|t}$, and hence for $1/2 < \delta\rho\sigma_\pi^2 < 1$ $\text{abs}(\pi_{t+2|t}) > \text{abs}(\pi_{t+1|t})$. This implies that even for $\lambda = 0$, a situation in which a risk-neutral central bank would employ $\pi_{t+2|t}$ as an intermediate target and would set its value equal to zero, ie. equal to the optimal level, a risk-averse central bank may expect the inflation forecast to wander away from zero.

Finally, risk-aversion conditions deeply the Taylor rule selected by the monetary authorities. Figure 1 plots the inflation, k_π, and output gap, k_y, coefficients in the optimal Taylor rule described in equation (3.3), against the risk-aversion coefficient for values of ρ ranging from 0 to 3.[13] As $\rho = 0$ corresponds to risk-neutrality we see that a risk-averse central bank follows a more aggressive Taylor rule, in that the short-term (real) interest rate is more sensitive to: i) departures of the inflation rate from its target level; and ii) swings in output from full employment. While Figure 1 is obtained for a specific choice of parameters, numerical analysis shows that the same conclusion is reached for all parametric constellations for which an optimal monetary policy exists.

This result may appear counter-intuitive and hence surprising, in that one may conjecture that a pessimistic agent will necessarily act more cautiously, selecting a more conservative policy rule. However, a risk-averse central bank cares for the variability of the inflation rate and the output gap and hence attempts to reduce it by reacting more aggressively to monetary and

[13]These coefficients are determined by: solving for the positive root of the quadratic equation which pins down W; and ii) inserting the resulting value in the vector $\mathbf{K}$.

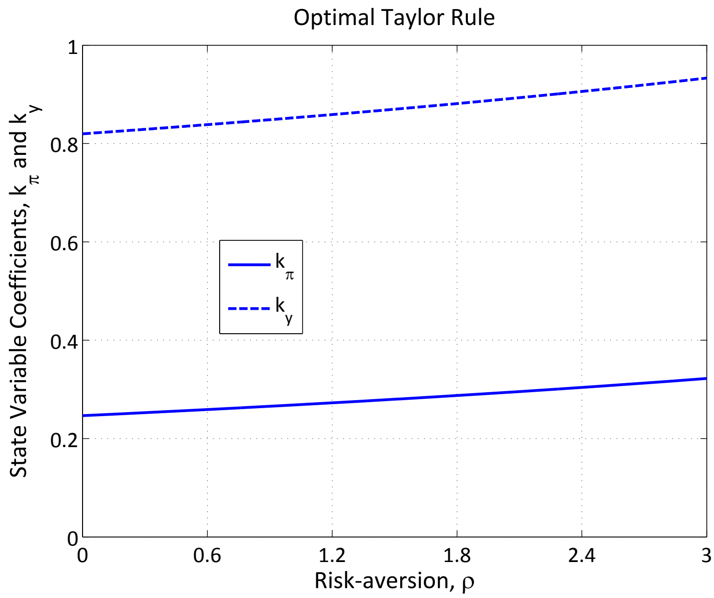

Figure 1: The values of the state variable coefficients k_π and k_y for $\alpha = 1.5$, $\beta = 0.9$, $\delta = 0.95$, $\gamma = 2$, $\lambda = 1$ and $\sigma_\pi^2 = \sigma_\pi^2 = 0.05$ are plotted against the risk-aversion coefficient ρ.

real shocks. In addition, it is worth noticing that this feature of the impact of risk-aversion is not confined to the specific model we investigate. Indeed, we can list other multi-period models with optimizing agents where risk-aversion makes them more rather than less aggressive (See, for instance, Holden and Subrahmanyam, 1994; Vitale, 2012).

Despite risk-averse monetary authorities select a more aggressive Taylor rule, the short-term (real) interest rate presents the same level of volatility which is observed under risk neutrality. This suggests that it may be difficult to detect empirically the impact of risk-aversion on the optimal monetary policy. In fact, if the monetary policy is analyzed on the basis of the moments of the short-term interest rate that of a risk-averse central bank is observationally equivalent to that of a risk-neutral one. To establish this result consider that in steady state, $\mathbf{z}_t = \mathbf{\Gamma}\mathbf{z}_{t-1} + \boldsymbol{\epsilon}_t$, where $\mathbf{\Gamma} = \mathbf{A} + \mathbf{BK}$. Therefore, $\mathbf{z}_t = \mathbf{\Lambda}\boldsymbol{\epsilon}_t$, with $\mathbf{\Lambda} = (\mathbf{I}_2 - \mathbf{\Gamma})^{-1}$, and hence $\mathrm{Var}[\mathbf{z}_t] = \mathbf{\Lambda}\mathbf{N}\mathbf{\Lambda}'$, while $\mathrm{Var}[r_t] = \mathbf{K}\mathbf{\Lambda}\mathbf{N}\mathbf{\Lambda}'\mathbf{K}'$, in that $r_t = \mathbf{K}\mathbf{z}_t$. Some tedious but straightforward algebra shows that

$$\mathrm{Var}[\mathbf{z}_t] \;=\; \begin{pmatrix} \frac{(1+\alpha\gamma\kappa_\pi)^2}{\alpha^2\gamma^2\kappa_\pi^2}\,\sigma_\pi^2 \;+\; \frac{1}{\gamma^2\kappa_\pi^2}\,\sigma_y^2 & -\,\frac{\gamma\,(1+\alpha\gamma\kappa_\pi)\,\kappa_\pi}{\alpha^2\gamma^2\kappa_\pi^2}\,\sigma_\pi^2 \\ -\,\frac{\gamma\,(1+\alpha\gamma\kappa_\pi)\,\kappa_\pi}{\alpha^2\gamma^2\kappa_\pi^2}\,\sigma_\pi^2 & \frac{1}{\alpha^2}\,\sigma_\pi^2 \end{pmatrix} \tag{3.4}$$

and hence that

$$\mathrm{Var}[r_t] \;=\; \frac{1}{\gamma^2}\left[\left(\frac{1-\beta}{\alpha}\right)^2 \sigma_\pi^2 \;+\; \sigma_y^2\right]. \tag{3.5}$$

This indicates that the unconditional variance of the short-term (real) interest rate is independent of ρ, the risk-aversion coefficient, and that the monetary policy shows the same level of volatility which prevails under risk-neutrality. The unconditional variances for the inflation rate, π_t, and the output gap, y_t, explains how this is possible. In fact, the variance of the latter is also independent of the risk-aversion coefficient, while that of the former is *decreasing* in ρ.[14] Then, in steady state, the smaller variability of the inflation rate under risk-aversion ($\mathrm{Var}[\pi_t]$ is smaller for $\rho > 0$) exactly compensates the augmented aggressiveness of the monetary authorities, so that, even if κ_π and κ_y are larger for ρ positive, the unconditional variance of the short-term (real) interest rate remains the same.

Indeed, the values of the unconditional variances $\mathrm{Var}[\pi_t]$ and $\mathrm{Var}[y_t]$ indicate that empirically the risk-neutral and risk-averse scenarios only differ in the variability of the inflation rate. While the unconditional variance of the output gap is unaffected by risk-aversion, that of the inflation rate is smaller for $\rho > 0$, suggesting that a risk-averse central bank will appear to be particularly concerned with the inflation rate volatility. This is because, given the specific lag structure in the law of motion for the vector of state variables, $\mathbf{z}_t$, the variability of the inflation rate represents the key factor in determining the welfare loss for the monetary authorities and it is therefore the main driver of the central bank's monetary policy.

Finally, before turning to the analysis of the imperfect information scenario, we should recall that for ρ large enough the second order condition, reported in Theorem 2, that $(\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N}$ being positive definite is violated, indicating that no optimal monetary policy exists for an extremely risk-averse central bank. In other words, an important non-linearity emerges in the relation between risk-aversion and monetary policy: as ρ augments the monetary authorities become more aggressive, but eventually their attempt to minimize their welfare loss completely fails and no optimal monetary policy can be undertaken.

[14] In fact, κ_π is increasing in ρ while the coefficients $(1+\alpha\gamma\kappa_\pi)^2/(\alpha^2\gamma^2\kappa_\pi^2)$ and $1/\gamma^2\kappa_\pi^2$ are both decreasing in κ_π.

3.1 Imperfect State Observation

It is interesting to see what happens in the case the central bank observes imperfectly the state variables. In the LQG case we know that thanks to the CEP it is sufficient to replace the state vector with its ML estimate. This is not the case when the central bank is risk-averse as the unobservable variables are replaced by those values which maximize the discounted total stress function.

With respect to the monetary policy of the central bank, a realistic scenario is that in which the monetary authorities observe the state variables with one lag. In this scenario the formulation of the discounted total stress function is simplified, in that $d_t = \boldsymbol{\epsilon}_t'\mathbf{N}^{-1}\boldsymbol{\epsilon}_t$ for $t = 1, 2, \ldots, T$. Then, the extremization of the discounted past stress is reached for $\mathbf{z}_{t-1} = \hat{\mathbf{z}}_{t-1}$ and is given by $P_t(\mathbf{z}_t, \mathbf{H}_t) = -\frac{1}{\rho}\,\boldsymbol{\epsilon}_t'\mathbf{N}^{-1}\boldsymbol{\epsilon}_t\, +\, \cdots$, where once again $+\cdots$ denotes terms independent of $\mathbf{z}_t$. Since in steady state $F_t(\mathbf{z}_t) = \mathbf{z}_t'\boldsymbol{\Pi}\mathbf{z}_t$, in re-coupling past and future extremization we solve

$$\max_{\mathbf{z}_t}\left\{-\frac{1}{\rho}\,\boldsymbol{\epsilon}_t'\mathbf{N}^{-1}\boldsymbol{\epsilon}_t + \mathbf{z}_t'\boldsymbol{\Pi}\mathbf{z}_t\right\}.$$

Given that at time t the observable vector is $\mathbf{w}_t = \mathbf{z}_{t-1}$, the conditional expectation of the state vector, $\mathbf{z}_t$, is $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}u_{t-1}$. Therefore, since we can write $\boldsymbol{\epsilon}_t = \mathbf{z}_t - \hat{\mathbf{z}}_t$, we need to solve

$$\max_{\mathbf{z}_t}\left\{-\frac{1}{\rho}\,(\mathbf{z}_t - \hat{\mathbf{z}}_t)'\mathbf{N}^{-1}(\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}_t'\boldsymbol{\Pi}\mathbf{z}_t\right\}.$$

We immediately conclude that, for $\mathbf{N}^{-1} - \rho\boldsymbol{\Pi}$ positive definite, the maximum discounted total stress estimate (MDTSE) $\breve{\mathbf{z}}_t$ is given by

$$\breve{\mathbf{z}}_t \;=\; (\mathbf{I}\,-\,\rho\mathbf{N}\,\boldsymbol{\Pi})^{-1}\hat{\mathbf{z}}_t\,.$$

As indicated in Theorem 4, the optimal control is obtained by inserting the MTSE, $\breve{\mathbf{z}}_t$, into the control rule which would prevail under perfect state observation. Within the monetary policy example we find that in equation (3.3) the actual values of the inflation rate and output gap are not replaced by their ML estimates, $\hat{\pi}_t$ and $\hat{y}_t$, but by the following values which correct for the impact of risk-aversion

$$\begin{pmatrix}\breve{\pi}_t\\ \breve{y}_t\end{pmatrix} = \begin{pmatrix}\hat{\pi}_t\\ \hat{y}_t\end{pmatrix} + \rho\,\mathbf{G}\begin{pmatrix}\hat{\pi}_t\\ \hat{y}_t\end{pmatrix},\ \text{where}\ \ \mathbf{G} = \begin{pmatrix}\frac{\boldsymbol{\pi}_1 - \det(\boldsymbol{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})}\,\sigma_\pi^2 & \frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})}\,\sigma_\pi^2\\ \frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})}\,\sigma_y^2 & \frac{\boldsymbol{\pi}_2 - \det(\boldsymbol{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})}\,\sigma_y^2\end{pmatrix},$$

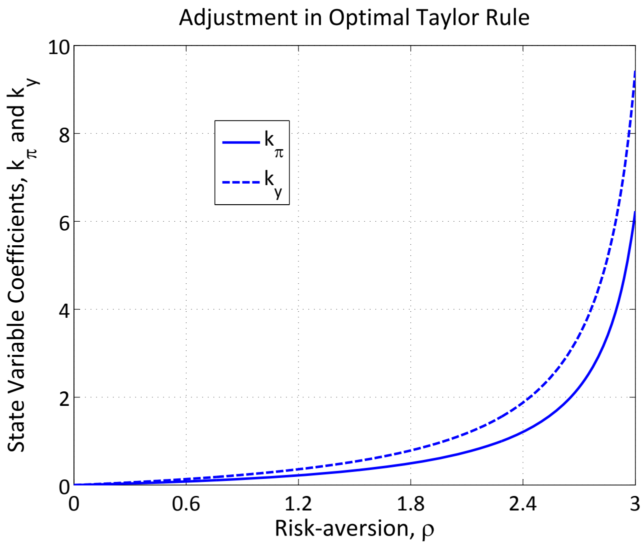

Figure 2: The adjustments to the Taylor rule coefficients k_π and k_y (i.e. the differences $\kappa_\pi^I - \kappa_\pi$ and $\kappa_y^I - \kappa_y$) are plotted against ρ for $\alpha = 1.5$, $\beta = 0.9$, $\delta = 0.95$, $\gamma = 2$, $\lambda = 1$ and $\sigma_\pi^2 = \sigma_\pi^2 = 0.05$.

while $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_{1,2}$ and $\boldsymbol{\pi}_2$ are the elements of matrix $\mathbf{\Pi}$. This implies that the optimal Taylor rule is given by a modified expression,

$$r_t \quad = \quad \mathbf{K}(\mathbf{I} + \rho\,\mathbf{G})\,\hat{\mathbf{z}}_t \;\; = \;\; \kappa_\pi^I\,\hat{\pi}_t \; + \; \kappa_y^I\,\hat{y}_t\,, \tag{3.6}$$

where the vector $\rho\mathbf{KG}$ contains adjustments to the Taylor rule coefficients induced by the risk-aversion correction to the ML estimate of the vector $\mathbf{z}_t$. According to the values in the matrix $\mathbf{G}$, imperfect state observation may entail a more (or less) aggressive Taylor rule, in so far the adjusted coefficients for inflation and output gap, κ_π^I and κ_y^I, are larger (smaller) than those which prevail under perfect state observation, κ_π and κ_y.

In Figure 2 we plot the differences between the adjusted coefficients, κ_π^I and κ_y^I, and the unadjusted ones, κ_π and κ_y, against the risk-aversion coefficient ρ. This plot proposes an apparently counter-intuitive result. In fact, we see that, as the difference is positive for both coefficients, the monetary authorities become even more aggressive when they observe with a time lag inflation and output. That is, when facing a more uncertain environment, the activism of the monetary authorities increases. As the adjustment to the Taylor rule coefficients increase with ρ we also observe that such activism augments with the degree of risk-aversion of the central bank. However, this increased activism is only apparent as the analysis of the unconditional variance of the inflation rate, π_t, output gap y_t, and the short-term (real) interest

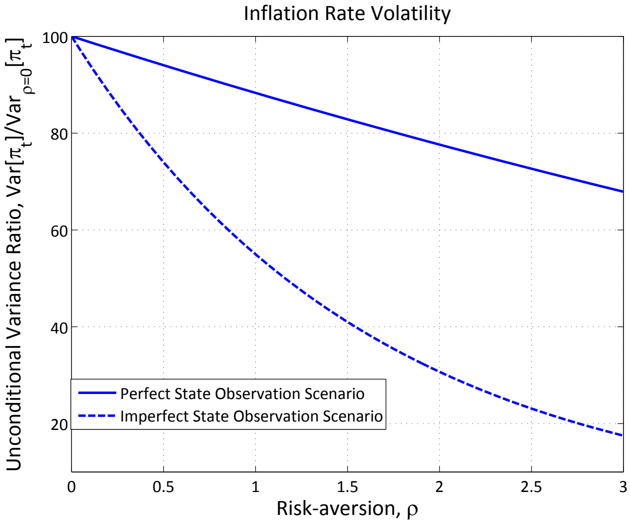

Figure 3: The ratio (in percentage terms) between the unconditional variance of the inflation ratio $\mathrm{Var}[\pi_t]$ and its base value for $\rho = 0$ is plotted against ρ, under perfect and imperfect state observation, for $\alpha = 1.5$, $\beta = 0.9$, $\delta = 0.95$, $\gamma = 2$, $\lambda = 1$ and $\sigma_\pi^2 = \sigma_\pi^2 = 0.05$.

rate, r_t, reveals.

In fact, even under imperfect state observation the unconditional variance of the short-term interest rate is independent of the risk-aversion coefficient, confirming that empirically it may be difficult to appreciate the impact of risk-aversion on the monetary policy. To show this result consider that under imperfect state observation $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\Psi}\hat{\mathbf{z}}_{t-1} + \boldsymbol{\epsilon}_t$, where $\boldsymbol{\Psi} = \mathbf{B}\mathbf{K}_I$ (with $\mathbf{K}_I$ corresponding to the vector of Taylor's rule coefficients under imperfect state observation, $\mathbf{K}_I = \mathbf{K}(\mathbf{I} + \rho\mathbf{G})$) and, as the state vector is observed with a lag, $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\Psi}\hat{\mathbf{z}}_{t-1}$. Then, $\hat{\mathbf{z}}_t = \boldsymbol{\Phi}\mathbf{z}_{t-1}$, where $\boldsymbol{\Phi} = (\mathbf{I}_2 - \boldsymbol{\Psi})^{-1}\mathbf{A}$. Replacing this expression in that for $\mathbf{z}_t$ we find that $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\Psi}\boldsymbol{\Phi}\,\mathbf{z}_{t-2} + \boldsymbol{\epsilon}_t$ which we can also write as $\mathbf{z}_t = \boldsymbol{\Lambda}_I\boldsymbol{\epsilon}_t$, where $\boldsymbol{\Lambda}_I = (\mathbf{I}_2 - \mathbf{A} - \boldsymbol{\Psi}\boldsymbol{\Phi})^{-1}$. We conclude that $\mathrm{Var}[\mathbf{z}_t] = \boldsymbol{\Lambda}_I\mathbf{N}\boldsymbol{\Lambda}_I'$, while $\mathrm{Var}[\hat{\mathbf{z}}_t] = \boldsymbol{\Phi}\boldsymbol{\Lambda}_I\mathbf{N}\boldsymbol{\Lambda}_I'\boldsymbol{\Phi}'$, and that hence, since under imperfect state observation $u_t = \mathbf{K}_I\hat{\mathbf{z}}_t$, $\mathrm{Var}[r_t] = \mathbf{K}_I\boldsymbol{\Phi}\boldsymbol{\Lambda}_I\mathbf{N}\boldsymbol{\Lambda}_I'\boldsymbol{\Phi}'\mathbf{K}_I'$.

Once again some long but straightforward algebra shows that

$$\mathrm{Var}^I[\mathbf{z}_t] = \begin{pmatrix} \left(1 + \frac{1+\gamma\kappa_y^I - \beta}{\alpha\gamma\kappa_\pi^I}\right)^2 \sigma_\pi^2 + \left(\frac{1-\gamma\kappa_y^I}{\gamma\kappa_\pi^I}\right)^2 \sigma_y^2 & -\frac{1}{\alpha}\left(1 + \frac{1+\gamma\kappa_y^I - \beta}{\alpha\gamma\kappa_\pi^I}\right)\sigma_\pi^2 \\ -\frac{1}{\alpha}\left(1 + \frac{1+\gamma\kappa_y^I - \beta}{\alpha\gamma\kappa_\pi^I}\right)\sigma_\pi^2 & \frac{1}{\alpha^2}\sigma_\pi^2 \end{pmatrix} \quad (3.7)$$

and that

$$\text{Var}^I[r_t] \;=\; \frac{1}{\gamma^2}\left[\left(\frac{1-\beta}{\alpha}\right)^2 \sigma_\pi^2 \;+\; \sigma_y^2\right]. \tag{3.8}$$

This shows that the unconditional variances of the short-term interest rate and the output gap are equal to those which prevail under risk-neutrality and perfect state observation ($\text{Var}^I[r_t] = \text{Var}[r_t]$ and $\text{Var}^I[y_t] = \text{Var}[y_t]$), while the unconditional variance of the inflation rate is function of the Taylor's rule coefficients κ_π^I and κ_y^I. As these coefficients depend on the risk-aversion coefficient, $\text{Var}^I[\pi_t]$ varies with ρ and indeed numerical analysis shows that such value decreases with ρ, explaining why it is possible that for a larger ρ (and hence with a more aggressive Taylor-rule) the variability of the short-term interest remains unchanged.

Thus, Figure 3 plots the ratio (in percentage terms) between the unconditional variance of the inflation rate, $\text{Var}[\pi_t]$, and its base value for $\rho = 0$ in both the perfect state and imperfect state scenarios.[15] The plot clearly illustrates the reduction in the volatility of the inflation rate in the presence of a risk-averse central bank in both scenarios. As the volatility of the inflation rate is smaller under risk-aversion, and decreasing in ρ, a more aggressive Taylor rule will not result in a more volatile short-term interest rate. We therefore conclude that in both scenarios the impact of the central bank's risk-aversion on its optimal monetary policy only manifests via a reduced volatility in the inflation rate, as the variability of both output gap and short-term interest rate is unaffected by ρ.

4 The Time-heterogeneous Formulation

In presenting the Markovian DLEQG problem we suggested that its formulation could be made time-heterogeneous by introducing time-dependent matrices $\mathbf{A}_t$, $\mathbf{B}_t$, $\mathbf{N}_t$, $\mathbf{Q}_t$, $\mathbf{R}_t$ and $\mathbf{S}_t$ in the specification of the plant equation and the cost function. Treatment of this generalization is straightforward. It is more involving the analysis of the formulation with deterministic disturbance terms into the plant equation. In particular, assume the state vector respects the following law of motion

$$\mathbf{z}_t \;=\; \mathbf{A}\mathbf{z}_{t-1} \;+\; \mathbf{B}\mathbf{u}_t \;+\; \boldsymbol{\mu}_t \;+\; \boldsymbol{\epsilon}_t\,,$$

[15] It should be noted that the base values for this unconditional variance differ between the two scenarios. In correspondence with the parametric choice of Figure 1, the value of the unconditional variance of the inflation rate under risk neutrality in the imperfect state scenario is about 4 times bigger than the corresponding value for the perfect state scenario, $\text{Var}^I_{\rho=0}[\pi_t] \approx 4\text{Var}_{\rho=0}[\pi_t]$.

where the vector $\boldsymbol{\mu}_t$ contains pre-determined values. These values are known in advance and represent unexpected disturbances which modify the original plant equation introduced in Definition 1.

Under perfect state observation, the pre-determined disturbance term $\boldsymbol{\mu}_t$ implies that the discounted future stress function is a non-homogenous quadratic form, $F_t(\mathbf{z}_t) = \mathbf{z}_t'\boldsymbol{\Pi}_t\mathbf{z}_t - 2\boldsymbol{\vartheta}_t'\mathbf{z}_t$ (with $\boldsymbol{\vartheta}_t$ a vector of coefficients), so that Theorem 2 must be amended as follows.

Theorem 5 - *(Risk-sensitive Riccati Equation). Under perfect state observation, if the matrix* $(\delta\boldsymbol{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$ *is positive definite and the state vector respects the linear plant equation with pre-determined disturbances, at time t the extremized discounted future stress function is given by*

$$F_t(\mathbf{z}_t) = \mathbf{z}_t'\,\boldsymbol{\Pi}_t\,\mathbf{z}_t \;-\; 2\boldsymbol{\vartheta}_t'\,\mathbf{z}_t \;\textit{for}\; \mathbf{u}_t \;=\; \mathbf{K}_t\,\mathbf{z}_t \,+\, (\mathbf{Q} \,+\, \mathbf{B}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{B})^{-1}\,\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\,(\boldsymbol{\Pi}_{t+1}^{-1}\,\boldsymbol{\vartheta}_{t+1} \,-\, \boldsymbol{\mu}_{t+1})\,, \tag{4.1}$$

where

$$\boldsymbol{\Pi}_t \;=\; \mathbf{R} \,+\, \mathbf{A}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{A} \,-\, (\mathbf{S}' \,+\, \mathbf{A}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{B})(\mathbf{Q} \,+\, \mathbf{B}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S} \,+\, \mathbf{B}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{A})\,, \tag{4.2}$$

$$\mathbf{K}_t \;=\; -\,(\mathbf{Q} \,+\, \mathbf{B}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S} \,+\, \mathbf{B}'\widetilde{\boldsymbol{\Pi}}_{t+1}\mathbf{A})\,, \tag{4.3}$$

$$\boldsymbol{\vartheta}_t \;=\; \boldsymbol{\Gamma}_t'\,\widetilde{\boldsymbol{\Pi}}_{t+1}(\boldsymbol{\Pi}_{t+1}^{-1}\,\boldsymbol{\vartheta}_{t+1} \,-\, \boldsymbol{\mu}_{t+1})\,, \tag{4.4}$$

$$\textit{with}\qquad \boldsymbol{\Gamma}_t \;=\; \mathbf{A} \,+\, \mathbf{B}\mathbf{K}_t \quad \textit{and} \quad \widetilde{\boldsymbol{\Pi}}_{t+1} \;=\; ((\delta\boldsymbol{\Pi}_{t+1})^{-1} \;-\; \rho\,\mathbf{N})^{-1}\,. \tag{4.5}$$

Proof. We just repeat the steps followed in the proof of Theorem 2. Recall that the discounted future stress respects the double recursion $F_t \;=\; \boldsymbol{\mathcal{L}}\,\tilde{\boldsymbol{\mathcal{L}}}\,F_{t+1}$ based on the two operators

$$\boldsymbol{\mathcal{L}}\,\phi(\mathbf{z}) \;=\; \min_{\mathbf{u}}\,[c(\mathbf{z},\mathbf{u}) \,+\, \phi(A\mathbf{z} + B\mathbf{u} + \boldsymbol{\mu})] \quad \text{and} \quad \tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[\phi(\mathbf{z}+\boldsymbol{\epsilon}) \,-\, \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]\,,$$

Thus, assume that $\phi(\mathbf{z}) \;=\; \delta\mathbf{z}'\,\boldsymbol{\Pi}\,\mathbf{z} \;-\; 2\delta\boldsymbol{\vartheta}'\,\mathbf{z}$, so that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[(\mathbf{z}+\boldsymbol{\epsilon})'\delta\boldsymbol{\Pi}(\mathbf{z}+\boldsymbol{\epsilon}) \,-\, 2\delta\boldsymbol{\vartheta}'\,(\mathbf{z}+\boldsymbol{\epsilon}) \,-\, \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]$. Taking first derivatives, we find that

$$\tilde{\boldsymbol{\epsilon}} \;=\; -\,(\delta\boldsymbol{\Pi} \,-\, \frac{1}{\rho}\mathbf{N}^{-1})^{-1}\delta\boldsymbol{\Pi}\,\mathbf{z} \;\;+\;\; (\delta\boldsymbol{\Pi} \,-\, \frac{1}{\rho}\mathbf{N}^{-1})^{-1}\delta\boldsymbol{\vartheta}\;,$$

which pins down a maximum if $(\delta\boldsymbol{\Pi})^{-1} - \rho\mathbf{N}$ is positive definite. Replacing this expression we conclude that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) \;=\; \mathbf{z}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{z} \;-\; 2\widetilde{\boldsymbol{\vartheta}}'\mathbf{z} + \cdots$, where $\widetilde{\boldsymbol{\Pi}} = ((\delta\boldsymbol{\Pi})^{-1} \,-\, \rho\mathbf{N})^{-1}$, $\widetilde{\boldsymbol{\vartheta}} = \widetilde{\boldsymbol{\Pi}}\boldsymbol{\Pi}^{-1}\boldsymbol{\vartheta}$ and $\cdots$ denotes terms independent of $\mathbf{z}$. For $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) = \mathbf{z}'\widetilde{\boldsymbol{\Pi}}\,\mathbf{z} - 2\widetilde{\boldsymbol{\vartheta}}'\mathbf{z}$, solution of the operator

$\boldsymbol{\mathcal{L}}$ yields the standard recursive formulae for $\mathbf{\Pi}$, $\mathbf{K}$ and $\boldsymbol{\vartheta}$ from the Markovian LQG problem with pre-determined disturbances where $\widetilde{\mathbf{\Pi}} = ((\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N})^{-1}$ and $\widetilde{\boldsymbol{\vartheta}}$ replace respectively $\mathbf{\Pi}$ and $\boldsymbol{\vartheta}$. Specifically applying the double recursion $F_t = \boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\mathcal{L}}} F_{t+1}$, we find that $F(\mathbf{z}_t) = \mathbf{z}_t'\mathbf{\Pi}_t\mathbf{z}_t - 2\widetilde{\boldsymbol{\vartheta}}_{t+1}'\mathbf{z}_t + \cdots$ for $\mathbf{u}_t = \mathbf{K}_t\mathbf{z}_t + (\mathbf{Q}+\mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B})^{-1}\mathbf{B}'(\widetilde{\boldsymbol{\vartheta}}_{t+1} - \widetilde{\mathbf{\Pi}}_{t+1}\boldsymbol{\mu}_{t+1})$, where $\mathbf{K}_t = -(\mathbf{Q}+\mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B})^{-1}(\mathbf{S}+\mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{A})$. Replacing $\widetilde{\boldsymbol{\vartheta}}_{t+1}$ with $\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{\Pi}_{t+1}^{-1}\boldsymbol{\vartheta}_{t+1}$ we find the recursive formulae presented in the statement. $\square$

Theorem 5 indicates that in the presence of pre-determined disturbances the optimal policy contains a risk-adjusted correction which takes into account their anticipated values. This also implies that the optimal control vector is not longer a simple linear function of the state vector, as an extra term enters into the optimal policy.

A second adjustment must be introduced under imperfect state observation, when pre-determined disturbances enter into the plant equation for the state vector. In fact, as Theorem 4 and Lemma 7 still apply, in recoupling the extremized discounted past and future stress function, the sum $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t)$ is maximized with respect to $\mathbf{z}_t$ to obtain the MDTSE, $\breve{\mathbf{z}}_t$. Given that $P_t(\mathbf{z}_t, \mathbf{H}_t) + F_t(\mathbf{z}_t) = -(1/\rho)(\mathbf{z}_t - \hat{\mathbf{z}}_t)'\mathbf{\Omega}_t^{-1}(\mathbf{z}_t - \hat{\mathbf{z}}_t) + \mathbf{z}_t'\mathbf{\Pi}_t\mathbf{z}_t' - 2\boldsymbol{\vartheta}_t'\mathbf{z}_t$ plus terms independent of $\mathbf{z}_t$, taking the first derivative of this sum with respect to $\mathbf{z}_t$ we see that, for $\mathbf{\Omega}_t^{-1} - \rho\,\mathbf{\Pi}_t$ positive definite, $\breve{\mathbf{z}}_t$ is

$$\breve{\mathbf{z}}_t = (\mathbf{I} - \rho\,\mathbf{\Omega}_t\,\mathbf{\Pi}_t)^{-1}(\hat{\mathbf{z}}_t - \rho\,\mathbf{\Omega}_t\,\boldsymbol{\vartheta}_t), \tag{4.6}$$

where $\hat{\mathbf{z}}_t$ is still the MLE of $\mathbf{z}_t$, which now, thanks to the presence of the pre-determined disturbances, is given by

$$\hat{\mathbf{z}}_t = \mathbf{A}\,\hat{\mathbf{z}}_{t-1} + \mathbf{B}\,\mathbf{u}_{t-1} + \boldsymbol{\mu}_t + (\mathbf{L} + \mathbf{A}\mathbf{\Omega}_{t-1}\,\mathbf{C}')(\mathbf{M} + \mathbf{C}\,\mathbf{\Omega}_{t-1}\,\mathbf{C}')^{-1}(\mathbf{w}_t - \mathbf{C}\,\hat{\mathbf{z}}_{t-1}) \tag{4.7}$$

4.1 Optimal Monetary Policy with a Positive Inflation Target

Our analysis of Svensson's model for a risk-averse central bank offers an opportunity to apply these results if we introduce the realistic assumption that the first-best value for the inflation rate is some positive constant π^*. Assuming, as in Svensson's original formulation, that at time t $c_t = (\pi_t - \pi^*)^2 + \lambda y_t^2$ implies that we should modify the DLEQG problem we investigated in Section 3. In particular define $\varsigma_t = \pi_t - \pi^*$. Then, we can rewrite the linear equations (3.1) and (3.2) governing the dynamics of the inflation rate and output gap as follows

$$\varsigma_t \;=\; \varsigma_{t-1} \;+\; \alpha y_{t-1} \;+\; \epsilon_t^{\pi}\,, \tag{4.8}$$

$$y_t \;=\; \beta y_{t-1} \;-\; \gamma(\iota_{t-1} \;-\; \pi^*) \;+\; \epsilon_t^{y}\,, \tag{4.9}$$

where now the control variable is the adjusted short-term real interest rate $\iota_t \equiv r_t + \pi^* = i_t - \varsigma_t$. For $\mathbf{z}_t' \equiv (\varsigma_t \; y_t)$, we can rewrite the plant equation as

$$\mathbf{z}_t \;=\; \mathbf{A}\mathbf{z}_{t-1} \;+\; \mathbf{B}\mathbf{u}_t \;+\; \boldsymbol{\mu}_t \;+\; \boldsymbol{\epsilon}_t\,,$$

with $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\epsilon}_t$ as in Section 3 and $\boldsymbol{\mu}_t \equiv \begin{pmatrix} 0 \\ \gamma\pi^* \end{pmatrix}$. As before we concentrate on a steady state solution. This is possible because the pre-determined disturbance terms are time-invariant. To pin down the steady state solution we just consider that the recursive expression for the vector $\boldsymbol{\vartheta}_t$ must yield a fixed point, $\boldsymbol{\vartheta} = \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}(\boldsymbol{\Pi}^{-1}\boldsymbol{\vartheta} - \boldsymbol{\mu})$, which implies that

$$\boldsymbol{\vartheta} \;=\; -\boldsymbol{\Pi}\,(\boldsymbol{\Pi}^{-1} \;-\; \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}})^{-1}\,\boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\mu}\,.$$

Given the expressions for $\boldsymbol{\Gamma}$ $\widetilde{\boldsymbol{\Pi}}$ and $\boldsymbol{\mu}$ it can be proved that $\boldsymbol{\vartheta} = \mathbf{0}$. Inserting this vector in the expression for the optimal control in Theorem 5 we find after some manipulation that the optimal adjusted short-term interest rate is $\iota_t = \kappa_\pi \varsigma_t + \kappa_y y_t + \pi^*$, where κ_π and κ_y respect the expressions given in Section 3. Given the definitions of ι_t and ς_t we conclude that the short-term real interest rate is equal to

$$r_t \;=\; \kappa_\pi(\pi_t - \pi^*) \;+\; \kappa_y y_t\,, \tag{4.10}$$

which corresponds to the Taylor rule derived in Section 3 for the inflation rate, π_t, replaced by its deviation from the optimal level, $\pi_t - \pi^*$. Bar this adjustment, the optimal policy is identical to that derived in Section 3. This is also true under imperfect state observation. In fact, for $\boldsymbol{\vartheta} = \mathbf{0}$ we still have that if the monetary authorities observe the inflation rate and the output gap with a lag, the MDTSE of the state vector, $\check{\mathbf{z}}_t$, respects the following expression

$$\check{\mathbf{z}}_t \;=\; (\mathbf{I} \;-\; \rho\mathbf{N}\,\boldsymbol{\Pi})^{-1}\hat{\mathbf{z}}_t\,,$$

where now the MLE for $\mathbf{z}_t$ is $\hat{\mathbf{z}}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}u_{t-1} + \boldsymbol{\mu}$. Given the expressions for $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\mu}$, and the definitions of $\mathbf{z}_t$ and ι_t, we have that this MLE can be written as in Section 3,

$$\begin{aligned} \hat{\pi}_t &= \pi_{t-1} + \alpha\, y_t\,, \\ \hat{y}_t &= \beta\, y_{t-1} - \gamma\, r_{t-1}\,. \end{aligned}$$

5 The Continuous-time Limit

To derive the continuous-time limit of the discrete-time formulation of the Markovian DLEQG problem we have discussed so far, assume that the interval of time $[0, T]$ is divided in n sub-periods of length $\Delta = T/n$. Then, let us adjust the plant equation in the discrete-time formulation as follows,

$$\mathbf{z}_t \;=\; (\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_{t-1} \;+\; \mathbf{B}\,\Delta\,\mathbf{u}_{t-1} \;+\; \boldsymbol{\epsilon}_t\,,$$

where $\boldsymbol{\epsilon}_t \sim \boldsymbol{\mathcal{N}}(\mathbf{0}, \Delta\mathbf{N})$, while in the recursive optimization criterion the per-period cost function is Δc_t and the discount factor is δ^{Δ}. Then, for $\Delta \downarrow 0$ the Markovian DLEQG problem converges to its continuous-time analogue. The following result holds.

Theorem 6 *In the continuous-time limit of the Markovian DLEQG problem, under perfect state observation, the optimal policy is*

$$\mathbf{u}(t) \;=\; \mathbf{K}(t)\,\mathbf{z}(t)\,, \textit{with} \tag{5.1}$$

$$\mathbf{K}(t) \;=\; -\,\mathbf{Q}^{-1}\,(\mathbf{S}' + \mathbf{B}'\,\boldsymbol{\Pi}(t))\,, \tag{5.2}$$

where the matrix $\boldsymbol{\Pi}(t)$ respects the following continuous-time risk-sensitive Riccati equation

$$\begin{aligned} \frac{d\,\boldsymbol{\Pi}(t)}{d\,t} \;+\; \mathbf{R} \;+\; \mathbf{A}'\,\boldsymbol{\Pi}(t) \;+\; \boldsymbol{\Pi}(t)\,\mathbf{A} \;-\; (\mathbf{S}' + \boldsymbol{\Pi}(t)\,\mathbf{B})\,\mathbf{Q}^{-1}\,(\mathbf{S} + \mathbf{B}'\,\boldsymbol{\Pi}(t)) \\ +\,\rho\,\boldsymbol{\Pi}(t)\,\mathbf{N}\,\boldsymbol{\Pi}(t) \;+\; \log\delta\,\boldsymbol{\Pi}(t) \;=\; \mathbf{0}\,. \end{aligned} \tag{5.3}$$

Proof. Theorem 2 still applies when the time interval $[0, T]$ is divided in n sub-periods. The extremized discounted future stress function is now $F_t(\mathbf{z}_t) = \mathbf{z}_t'\boldsymbol{\Pi}_t\mathbf{z}_t$ for $\mathbf{u}_t = \mathbf{K}_t\mathbf{z}_t$, with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ in the recursive equations for $\boldsymbol{\Pi}_t$ and $\mathbf{K}_t$ replaced respectively by $\mathbf{I} + \mathbf{A}\Delta$, $\mathbf{B}\Delta$,

$\mathbf{Q}\Delta$, $\mathbf{R}\Delta$ and $\mathbf{S}\Delta$, and $\widetilde{\mathbf{\Pi}}_{t+1}$ given by $((\delta^{\Delta}\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}\Delta)^{-1}$. In particular,

$$\begin{aligned}
\mathbf{K}_t &= -(\mathbf{Q} + \Delta\,\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\left(\mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}\,(\mathbf{I} + \mathbf{A}\,\Delta)\right), \\
\mathbf{\Pi}_t &= \widetilde{\mathbf{\Pi}}_{t+1} + \Delta\left(\mathbf{R} + \mathbf{K}_t'\,\mathbf{Q}\,\mathbf{K}_t + 2\mathbf{S}'\,\mathbf{K}_t + \mathbf{A}'\,\widetilde{\mathbf{\Pi}}_{t+1} + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{A} + 2\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\,\mathbf{K}_t\right) + o(\Delta^2)\,.
\end{aligned}$$

Given the expressions for $\mathbf{K}_t$ and $\widetilde{\mathbf{\Pi}}_{t+1}$ and that $\lim_{\Delta\downarrow 0}\widetilde{\mathbf{\Pi}}_{t+1} = \lim_{\Delta\downarrow 0}\mathbf{\Pi}_t = \mathbf{\Pi}(t)$, for $\Delta \downarrow 0$ the Riccati difference equation for $\mathbf{\Pi}_t$ converges to its continuous-time counterpart given in (5.3). Then, the discounted future stress function in the continuous-time limit is $F(\mathbf{z}(t), t) = \mathbf{z}(t)'\mathbf{\Pi}(t)\mathbf{z}(t)$, while the optimal policy is $\mathbf{u}(t) = \mathbf{K}(t)\mathbf{z}(t)$ with $\mathbf{K}(t) = -\mathbf{Q}^{-1}(\mathbf{S}' + \mathbf{B}'\mathbf{\Pi}(t))$. $\square$

It is worth noticing that in the continuous-time limit the solution to the Markovian DLEQG problem is simpler, in that for $\Delta \downarrow 0$ the modified matrix $\widetilde{\mathbf{\Pi}}_{t+1}$ converges to $\mathbf{\Pi}_t$. Indeed, this is consequence of the fact that for $\Delta \downarrow 0$ the stress function $\boldsymbol{\mathcal{S}}_t$ always possesses a maximum in $\boldsymbol{\epsilon}_{t+1}$ for $\boldsymbol{\epsilon}_{t+1} = \mathbf{0}$. This implies that the positive-definiteness condition for the extremization of the discounted total stress function in Theorem 2 is redundant. Furthermore, in the limit $(\mathbf{Q}+\Delta\mathbf{B}\widetilde{\mathbf{\Pi}}_{t+1}\mathbf{B}')$ converges to $\mathbf{Q}$, so that a sufficient condition for the existence of a solution to the Markovian DLEQG problem is that such matrix is positive definite.[16] This means that as long as the cost function, c_t, is positive definite in $\mathbf{u}_t$, in the continuous-time limit an optimal policy for the Markovian DLEQG always exists, since the stress function $\boldsymbol{\mathcal{S}}_t$ surely possesses a saddle point. Finally, notice that with respect to the standard Riccati equation which applies to the LQG problem in continuous-time, two extra terms appear in the modified (risk-averse) version. The term $\rho\mathbf{\Pi}(t)\mathbf{N}\mathbf{\Pi}(t)$ modifies the standard Riccati equation as in the Markovian LEQG problem discussed by Whittle and captures the impact in continuous-time of risk-aversion on the dynamics of the optimal policy. Similarly, the extra term $\log\delta\,\mathbf{\Pi}(t)$ captures the impact of time-discounting. Interestingly the two terms enter separately into the modified Riccati equation indicating that in the continuous-time limit the impact of risk-aversion and time-discounting is clearly disjointed.

Under imperfect state information one should see how optimal control and estimation behaves for $\Delta \downarrow 0$. It is immediate to see that Theorem 3 and Lemma 7 still apply, with the qualification that the MLE of $\mathbf{z}_t$ will be replaced by its continuous-time counter-part. The continuous-time version of Kalman filter indicates that for $\mathbf{w}(t) = \mathbf{C}\mathbf{z}(t) + \boldsymbol{\eta}(t)$, the MLE of

[16] In the definition of the DLEQG problem we have introduced the assumption that c_t is positive definite in $\mathbf{u}_t$ and $\mathbf{z}_t$. In the discrete formulation, when $(\delta\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}$ is positive definite, this is a *sufficient* condition for the existence a recursive solution of the DLEQG problem. In the continuous-time limit such condition is relaxed to c_t being positive definite in $\mathbf{u}_t$.

the state vector respects the following expression,

$$\frac{d\,\hat{\mathbf{z}}(t)}{d\,t} \;=\; \mathbf{A}\,\hat{\mathbf{z}}(t) \,+\, \mathbf{B}\,\mathbf{u}(t) \,+\, (\mathbf{L} \,+\, \boldsymbol{\Omega}(t)\,\mathbf{C}')\mathbf{M}^{-1}\,(\mathbf{w}(t) \,-\, \mathbf{C}\,\hat{\mathbf{z}}(t))\,, \tag{5.4}$$

where the conditional covariance matrix $\boldsymbol{\Omega}(t)$ respects the new Riccati differential equation,

$$\frac{d\,\boldsymbol{\Omega}(t)}{d\,t} \;=\; \mathbf{N} \,+\, \mathbf{A}\,\boldsymbol{\Omega}(t) \,+\, \boldsymbol{\Omega}(t)\,\mathbf{A}' \,-\, (\mathbf{L} \,+\, \boldsymbol{\Omega}(t)\,\mathbf{C}')\mathbf{M}^{-1}\,(\mathbf{L}' \,+\, \mathbf{C}\,\boldsymbol{\Omega}(t)\,)\,. \tag{5.5}$$

Finally, re-coupling the extremization of the discounted past and future stress functions still yields the MDTSE for the state vector. In the continuous-time limit this will still be given by the usual expression

$$\check{\mathbf{z}}(t) \;=\; (\mathbf{I} \,-\, \rho\,\boldsymbol{\Omega}(t)\,\boldsymbol{\Pi}(t))^{-1}\,\hat{\mathbf{z}}(t)\,. \tag{5.6}$$

As in the discrete-time formulation we may wonder what happens when we consider the continuous-time limit of the heterogeneous-time formulation discussed in Section 4. In this case, the discrete-time plant equation assumes the following formulation

$$\mathbf{z}_t \;=\; (\mathbf{I} \,+\, \mathbf{A}\,\Delta)\mathbf{z}_{t-1} \,+\, \mathbf{B}\,\Delta\,\mathbf{u}_{t-1} \,+\, \boldsymbol{\mu}_t\,\Delta \,+\, \boldsymbol{\epsilon}_t\,.$$

For the continuous-time limit of the Markovian DLEQG problem the following result holds.[17]

Theorem 7 *In the continuous-time limit of the Markovian DLEQG problem with pre-determined disturbances, under perfect state observation, the optimal policy is*

$$\mathbf{u}(t) \;=\; \mathbf{K}(t)\mathbf{z}(t) \,+\, \mathbf{Q}^{-1}\mathbf{B}'\boldsymbol{\vartheta}(t)\,, \tag{5.7}$$

$$\textit{with} \qquad \mathbf{K}(t) \;=\, -\,\mathbf{Q}^{-1}\,(\mathbf{S}' \,+\, \mathbf{B}'\,\boldsymbol{\Pi}(t))\,, \tag{5.8}$$

where the matrix $\boldsymbol{\Pi}(t)$ respects equation (5.3) and the vector $\boldsymbol{\vartheta}(t)$ the following one

$$\frac{d\,\boldsymbol{\vartheta}(t)}{d\,t} \,+\, \left(\log\delta \,+\, \widetilde{\boldsymbol{\Gamma}}(t)'\right)\boldsymbol{\vartheta}(t) \,-\, \boldsymbol{\Pi}(t)\,\boldsymbol{\mu}(t) \;=\; \mathbf{0}\,, \tag{5.9}$$

$$\textit{with} \;\; \widetilde{\boldsymbol{\Gamma}}(t) \;=\; \mathbf{A} \,-\, \mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{S} \,-\, \left(\mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{B}' \,-\, \rho\,\mathbf{N}\right)\boldsymbol{\Pi}(t)\,.$$

[17] The presence of the pre-determined disturbances will only add a pre-determined value in the expression for the MLE of the state vector under imperfect state observation in equation (5.4).

Proof. Theorem 5 still applies when the time interval $[0, T]$ is divided in n sub-periods. The extremized discounted future stress function is now $F(\mathbf{z}_t) = \mathbf{z}_t' \,\mathbf{\Pi}_t \,\mathbf{z}_t - 2\boldsymbol{\vartheta}_t' \,\mathbf{z}_t + \cdots$ for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, $\boldsymbol{\mu}_{t+1}$, $\mathbf{R}$ and $\mathbf{S}$ replaced respectively by $\mathbf{I} + \mathbf{A}\Delta$, $\mathbf{B}\Delta$, $\mathbf{Q}\Delta$, $\boldsymbol{\mu}_{t+1}\Delta$, $\mathbf{R}\Delta$ and $\mathbf{S}\Delta$ and $\widetilde{\mathbf{\Pi}}_{t+1}$ given by $((\delta^{\Delta}\mathbf{\Pi}_{t+1})^{-1} - \rho\mathbf{N}\Delta)^{-1}$. In particular, the optimal control vector is

$$
\begin{aligned}
\mathbf{u}_t &= -(\mathbf{Q} + \Delta\,\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\left(\mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}\,(\mathbf{I} + \mathbf{A}\,\Delta)\right)\mathbf{z}_t \\
&\quad + (\mathbf{Q} + \Delta\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\Delta\mathbf{B})^{-1}\mathbf{B}'(\widetilde{\boldsymbol{\vartheta}}_{t+1} - \widetilde{\mathbf{\Pi}}_{t+1}\boldsymbol{\mu}_{t+1}\,\Delta)\,, \quad \text{where} \\
\widetilde{\mathbf{\Pi}}_{t+1} &= ((\delta^{\Delta}\mathbf{\Pi}_{t+1})^{-1} - \rho\,\Delta\,\mathbf{N})^{-1} \quad \text{and} \quad \widetilde{\boldsymbol{\vartheta}}_{t+1} = \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{\Pi}_{t+1}^{-1}\,\boldsymbol{\vartheta}_{t+1}\,.
\end{aligned}
$$

While the matrix $\mathbf{\Pi}_t$ respects the Riccati difference equation derived in the proof of Theorem 6, the vector $\boldsymbol{\vartheta}_t$ respects the following difference equation

$$
\begin{aligned}
\boldsymbol{\vartheta}_t' &= \widetilde{\boldsymbol{\vartheta}}_{t+1}'\,[\mathbf{I} + \Delta\,(\mathbf{A} + \mathbf{B}\,\mathbf{K}_t)] - \Delta\boldsymbol{\mu}_{t+1}'\,\widetilde{\mathbf{\Pi}}_{t+1} \\
&\quad - \Delta\,(\widetilde{\boldsymbol{\vartheta}}_{t+1} - \widetilde{\mathbf{\Pi}}_{t+1}\boldsymbol{\mu}\Delta)'\,\mathbf{B}\,(\mathbf{Q} + \Delta\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\Delta\mathbf{B})^{-1}[\,\mathbf{Q}\,\mathbf{K}_t + \mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,(\mathbf{I} + \mathbf{A}\,\Delta)]\,.
\end{aligned}
$$

Given the expressions for $\mathbf{K}_t$, $\widetilde{\boldsymbol{\vartheta}}_{t+1}$ and $\widetilde{\mathbf{\Pi}}_{t+1}$ and that $\lim_{\Delta\downarrow 0}\widetilde{\mathbf{\Pi}}_{t+1} = \lim_{\Delta\downarrow 0}\mathbf{\Pi}_t = \mathbf{\Pi}(t)$ and $\lim_{\Delta\downarrow 0}\widetilde{\boldsymbol{\vartheta}}_{t+1} = \lim_{\Delta\downarrow 0}\boldsymbol{\vartheta}_t = \boldsymbol{\vartheta}(t)$, for $\Delta \downarrow 0$ the difference equation for $\boldsymbol{\vartheta}_t$ converges to its continuous-time counterpart given in (5.9). Then, the extremized discounted future stress function in the continuous-time limit is $F(\mathbf{z}(t), t) = \mathbf{z}(t)'\,\mathbf{\Pi}(t)\,\mathbf{z}(t) - 2\boldsymbol{\vartheta}(t)'\,\mathbf{z}(t) + \cdots$, while the optimal policy is $\mathbf{u}(t) = \mathbf{K}(t)\mathbf{z}(t) + \mathbf{Q}^{-1}\mathbf{B}'\boldsymbol{\vartheta}(t)$ with $\mathbf{K}(t) = -\mathbf{Q}^{-1}(\mathbf{S}' + \mathbf{B}'\mathbf{\Pi}(t))$. $\square$

5.1 Optimal Production Policy for a Risk-averse Monopolist

To illustrate how the Markovian DLEQG formulation operates in continuous time we consider a monopolist who produces a perishable commodity. Her profits at time t are equal to $p(t)x(t) - qx(t)^2$, where $p(t)$ is the commodity price and $x(t)$ the quantity produced. As q is a positive constant production presents increasing marginal costs. We assume the demand schedule for this commodity is linear in its price and its price variation,

$$
x_d(t) = a_d - b_d\,p(t) - c_d\,\frac{d\,p(t)}{d\,t} + \epsilon_d(t)\,,
$$

where all coefficients are positive. This means that the demand for the commodity is decreasing in both its price level and its price variation. Since the commodity is perishable, the monopolist will choose the supplied quantity and accept any price which will clear the market. Resorting to the inverted demand function we obtain the following plant equation

$$\frac{d\,p(t)}{d\,t} \quad = \quad a\,p(t) \;+\; b\,x(t) \;+\; \mu \;+\; \epsilon(t)\,, \tag{5.10}$$

where $a = -b_d/c_d$, $b = -1/c_d$ (with a and b negative), while $\mu = a_d/c_d$ and $\epsilon(t) = -\epsilon_d(t)/c_d$. Because the demand for the commodity is subject to stochastic shocks the monopolist faces an uncertain environment and will not be able to anticipate the profits her production decisions will bring about in the future. Then, assuming she is risk-averse, her optimal production problem can be casted within our Markovian DLEQG framework. For simplicity suppose $\mu = 0$, so that we restrict our analysis to a homogeneous formulation in which $\mathbf{Q} = q$, $\mathbf{R} = 0$, $\mathbf{S} = -1/2$, $\mathbf{A} = a$, $\mathbf{B} = b$, $\mathbf{z} = p$ and $\mathbf{u} = x$.

Then, applying Theorem 6, we find that the optimal production policy respects the following formulation,

$$\begin{aligned} x(t) \;&=\; \kappa(t)\,p(t)\,, \quad \text{where} \\ \kappa(t) \;&=\; \frac{1}{q}\left(\frac{1}{2} \,-\, b\,\pi(t)\right) \quad \text{and} \quad \pi(t) \;=\; -\,\frac{\zeta_1}{\frac{b^2}{q} \,-\, \rho\,\sigma_\epsilon^2}\left(\frac{e^{\sqrt{\mathcal{D}}(T-t)} \,-\, 1}{\frac{\zeta_1}{\zeta_2}\,e^{\sqrt{\mathcal{D}}(T-t)} \,-\, 1}\right), \end{aligned} \tag{5.11}$$

with σ_ϵ^2 the variance of ϵ_t, T the terminal date, $\mathcal{D} = (2a + b/q + \log\delta)^2 - (1/q)(\frac{b^2}{q} - \rho\sigma_\epsilon^2)$ and $\zeta_1, \zeta_2 = -\frac{1}{2}(2a + b/q + \log\delta) \pm \frac{1}{2}\sqrt{\mathcal{D}}$. Inspection of this solution indicates that $\pi(t)$ is never positive, as the monopolist would stop her activity if she did not earn any profits from her production. In addition, since for $t \uparrow T$ $\pi(t) \uparrow 0$ and $\kappa(t) \uparrow 1/(2q)$, as the monopolist approaches the final horizon of her optimization problem, her optimal policy converges to the static solution, in which the monopolist chooses her production to maximizes her expected per-period profits. On the contrary, for $t \downarrow -\infty$, $\pi(t) \downarrow -\zeta_2/(\frac{b^2}{q} - \rho\sigma_\epsilon^2)$, while $k(t) \downarrow \frac{1}{q}(\frac{1}{2} + \frac{b\zeta_2}{b^2/q - \rho\sigma_\epsilon^2})$, indicating that the optimal policy converges to the stationary solution for the formulation with infinite horizon.

In Figure 3 we plot the dynamics of the coefficients $\pi(t)$ and $\kappa(t)$ under a specific parametric choice for $\rho = 0$ and 5 and $\delta = 1$ and 0.5. This graphical representation clearly confirms that in the dynamic optimization exercise the monopolist will be more cautious than in the static formulation, as the slope of her production policy is smaller, $\kappa(t) < (1/2q)$. Indeed, the monopolist realizes that a larger quantity of the commodity brought to the market at time t will jeopardize future profits. In fact, a larger quantity $x(t)$ in t lowers the commodity price $p(t)$. Because of the inertia in the demand schedule, such reduction propagates through time,

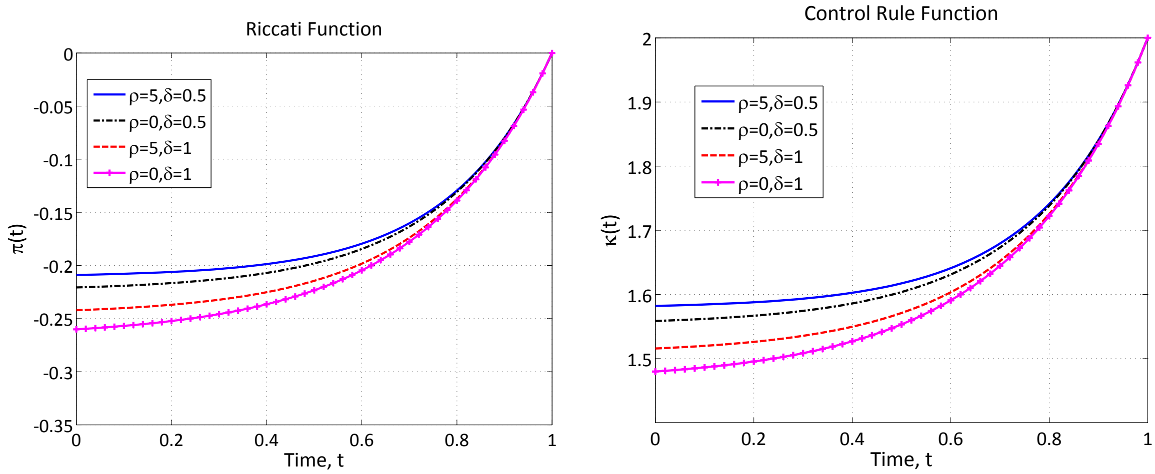

Figure 3: The dynamics $\pi(t)$ and $\kappa(t)$ for $T = 1$, $\sigma_\epsilon^2 = 0.25$, $a = -1$, $b = -0.5$ and $q = 0.25$.

so that a larger production today induces smaller prices and impaired profit opportunities tomorrow. As the terminal date T approaches, future profits have a smaller impact on her optimal policy and hence $\kappa(t)$ converges to the static value $1/(2q)$.

Both time-discounting and risk-aversion condition the production policy.[18] Unsurprisingly, for δ smaller the monopolist becomes more aggressive, as she gives less weight to future profits in determining her current production policy. This implies that the slope of the production policy, $\kappa(t)$, is larger for any t. Similarly, risk-aversion makes the monopolist more aggressive, as $\kappa(t)$ is larger for $\rho = 5$ than $\rho = 0$ throughout time.

Interestingly, the impact of risk-aversion is analogous to that described within Svennson's optimal policy model in Section 3. In the present context such impact can be explained considering that a larger quantity $x(t)$ in t will reduce both the expected value and the variance of future profits. Clearly, differently from a risk-neutral agent, a risk-averse monopolist will then be willing to accept smaller expected future profits to reduce their variance and will consequently choose a more aggressive production policy.

[18]While the plots represented in Figure 3 are specific to the parametric constellation we have chosen, qualitatively similar results are obtained for other choices of such parameters. This suggests that Figure 3 represents the general features of the impact of time-discounting and risk-aversion on the optimal policy of the monopolist.

6 Concluding Remarks

The impact of risk-aversion on the behavior of economic agents is a crucial issue in economics, in particular within dynamic problems where such agents are required to solve complicated optimization exercises. We have proposed a comprehensive analysis of a specific class of optimal control problems, termed the Discounted Linear Exponential Quadratic Gaussian (DLEQG), where risk-averse agents optimize a recursive criterion *à la* Epstein and Zin defined over a quadratic cost function in state and control vectors and a Markovian linear plant equation for the state vector dictates the dynamics of the economic environment. The DLEQG class is a generalization of Whittle's (Whittle, 1990) Linear Exponential Quadratic Gaussian (LEQG) class, which allows to accommodate time-discounting while preserving most of the results he derived. Thus, his risk-sensitive certainty-equivalence (RSCEP) and separation (SP) principles are reformulated, while his recursive formulae for the optimal control are modified to adapt them to the DLEQG class.

Our analysis of the DLEQG class is also an improvement over the contribution of Hansen and Sargent (Hansen and Sargent, 1994, 1995, 2005), as we are able to investigate DLEQG problems where agents only observe noisy signals of the state variables. In addition, our revised RSCEP confirms within the DLEQG class Whittle's result that the optimal behavior of risk-averse agents can be identified via a *pessimistic* choice mechanism, according to which the control variables are chosen by applying a *min-max* strategy, so that agents minimize their welfare loss against the most adverse shocks.

A possible conjecture is that amid an uncertain environment a *pessimistic* agent acts *cautiously*, choosing conservative control rules. In effect, it is immediate to verify that within static optimization problems under uncertainty risk-averse agents are less aggressive than their risk-neutral counter-parts. Common intuition would then suggest that a similar result also holds within fully dynamic optimization problems. The analysis of the two applications of the DLEQG class we propose suggests the contrary. In fact, we see that within these fully dynamic optimization problems risk-averse agents are bolder than their risk-neutral counter-parts, a conclusion that clearly contradicts common intuition. In other words, our analysis indicates that a *pessimistic* agent should not be confused with a *cautious* one, as we see that *pessimism* induces such agent to act *boldly*.

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Appendix: Detailed Calculations

A Property of Quadratic Forms. Consider the quadratic form $Q(\mathbf{u}, \boldsymbol{\epsilon})$, where

$$Q(\mathbf{u}, \boldsymbol{\epsilon}) \;=\; \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\epsilon} \end{pmatrix}' \begin{pmatrix} \mathbf{Q}_{\mathbf{u}\,\mathbf{u}} & \mathbf{Q}_{\mathbf{u}\,\boldsymbol{\epsilon}} \\ \mathbf{Q}_{\boldsymbol{\epsilon}\,\mathbf{u}} & \mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\epsilon} \end{pmatrix}.$$

Assume Q admits a minimum in $\boldsymbol{\epsilon}$ in that $\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}$ is positive definite. Then, the following holds,

$$\int \exp\left[-\frac{1}{2}\, Q(\mathbf{u}, \boldsymbol{\epsilon})\right] d\,\boldsymbol{\epsilon} \quad \propto \quad \exp\left[-\frac{1}{2}\, \min_{\boldsymbol{\epsilon}}\, Q(\mathbf{u}, \boldsymbol{\epsilon})\right].$$

This is because, for $\hat{\boldsymbol{\epsilon}}$ the vector $\boldsymbol{\epsilon}$ minimizing Q, we can write $Q(\mathbf{u}, \boldsymbol{\epsilon}) = Q(\mathbf{u}, \hat{\boldsymbol{\epsilon}}) + (\boldsymbol{\epsilon} - \hat{\boldsymbol{\epsilon}})'\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}(\boldsymbol{\epsilon} - \hat{\boldsymbol{\epsilon}})$. In fact, consider that as $\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}$ is positive definite and invertible, the minimum of Q with respect to $\boldsymbol{\epsilon}$ is obtained for $\hat{\boldsymbol{\epsilon}} = -\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}^{-1}\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\mathbf{u}}\,\mathbf{u}$ and is equal to $Q(\mathbf{u}, \hat{\boldsymbol{\epsilon}}) = \mathbf{u}'[\mathbf{Q}_{\mathbf{u}\,\mathbf{u}} \,-\, \mathbf{Q}_{\mathbf{u}\,\boldsymbol{\epsilon}}\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}^{-1}\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\mathbf{u}}]\mathbf{u}$. Then,

$$\begin{aligned} Q(\mathbf{u}, \boldsymbol{\epsilon}) - Q(\mathbf{u}, \hat{\boldsymbol{\epsilon}}) \quad &= \quad \boldsymbol{\epsilon}'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}\,\boldsymbol{\epsilon} \;+\; \boldsymbol{\epsilon}'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\mathbf{u}}\,\mathbf{u} \;+\; \mathbf{u}'\,\mathbf{Q}_{\mathbf{u}\,\boldsymbol{\epsilon}}\,\boldsymbol{\epsilon} \;+\; \mathbf{u}'\,\mathbf{Q}_{\mathbf{u}\,\boldsymbol{\epsilon}}\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}^{-1}\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\mathbf{u}}\mathbf{u} \\ &= \quad \boldsymbol{\epsilon}'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}\,\boldsymbol{\epsilon} \;-\; \boldsymbol{\epsilon}'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}\,\hat{\boldsymbol{\epsilon}} \;-\; \hat{\boldsymbol{\epsilon}}'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}\,\boldsymbol{\epsilon} \;+\; \hat{\boldsymbol{\epsilon}}'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}\,\hat{\boldsymbol{\epsilon}} \\ &= \quad (\boldsymbol{\epsilon} \,-\, \hat{\boldsymbol{\epsilon}})'\,\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}\,(\boldsymbol{\epsilon} \,-\, \hat{\boldsymbol{\epsilon}})\,. \end{aligned}$$

As $Q(\mathbf{u}, \hat{\boldsymbol{\epsilon}}) = \min_{\boldsymbol{\epsilon}}\, Q(\mathbf{u}, \boldsymbol{\epsilon})$ is a constant in the integration,

$$\int \exp\left[-\frac{1}{2}\, Q(\mathbf{u}, \boldsymbol{\epsilon})\right] d\,\boldsymbol{\epsilon} \quad = \quad \exp\left[-\frac{1}{2}\, \min_{\boldsymbol{\epsilon}}\, Q(\mathbf{u}, \boldsymbol{\epsilon})\right] \times \int \exp[-\frac{1}{2}\,(\boldsymbol{\epsilon} - \hat{\boldsymbol{\epsilon}})'\mathbf{Q}_{\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}}(\boldsymbol{\epsilon} - \hat{\boldsymbol{\epsilon}})]\, d\,\boldsymbol{\epsilon}\,.$$

Therefore, the constant of proportionality in the aforementioned equality is $\int \exp(-\frac{1}{2}\boldsymbol{\Delta}'\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}\boldsymbol{\Delta})\, d\boldsymbol{\Delta}$. Because $\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}$ is positive definite, for $\boldsymbol{\Delta}$ integrated over $\mathbb{R}^n$, this constant is given by $(2\pi)^{n/2}\det(\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}})^{-1/2}$, where n is the dimension of $\boldsymbol{\epsilon}$, and hence it is independent of $\mathbf{u}$.

Then, suppose that we solve the program $\min_{\mathbf{u}} \int \exp\left[-\frac{1}{2}Q(\mathbf{u},\boldsymbol{\epsilon})\right]$. Assume that Q admits a saddle point with respect to $\boldsymbol{\epsilon}$ and $\mathbf{u}$, so that $\max_{\mathbf{u}} \min_{\boldsymbol{\epsilon}} Q(\mathbf{u},\boldsymbol{\epsilon})$ exists. This is the case if the two conditions $\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} > 0$ and $\mathbf{Q}_{\mathbf{u}\mathbf{u}} - \mathbf{Q}_{\mathbf{u}\boldsymbol{\epsilon}}\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}^{-1}\mathbf{Q}_{\boldsymbol{\epsilon}\mathbf{u}} < 0$ hold Q. As a corollary of the former result we have

$$\begin{aligned} \min_{\mathbf{u}} \int \exp\left[-\frac{1}{2}Q(\mathbf{u},\boldsymbol{\epsilon})\right] d\boldsymbol{\epsilon} \quad &\propto \quad \min_{\mathbf{u}} \exp\left[-\frac{1}{2}\min_{\boldsymbol{\epsilon}} Q(\mathbf{u},\boldsymbol{\epsilon})\right] \\ &= \quad \exp\left[-\frac{1}{2}\max_{\mathbf{u}}\min_{\boldsymbol{\epsilon}} Q(\mathbf{u},\boldsymbol{\epsilon})\right]. \end{aligned}$$

It is worth noticing that the order of the extremizing operations is irrelevant in the determination of the saddle point if the two conditions $\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} > 0$ and $\mathbf{Q}_{\mathbf{u}\mathbf{u}} - \mathbf{Q}_{\mathbf{u}\boldsymbol{\epsilon}}\mathbf{Q}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}^{-1}\mathbf{Q}_{\boldsymbol{\epsilon}\mathbf{u}} < 0$ hold. More importantly, analogous results hold when Q is a non-homogeneous quadratic form, which depends on $\mathbf{u}$ and $\boldsymbol{\epsilon}$ alongside a third vector $\mathbf{z}$, insofar it admits a saddle point $\max_{\mathbf{u}} \min_{\boldsymbol{\epsilon}} Q(\mathbf{u},\boldsymbol{\epsilon},\mathbf{z})$.

The Discounted Future Stress Recursion. From Lemma 3 we know that if $\boldsymbol{\mathcal{V}}_{t+1}$ is a quadratic form in $\mathbf{z}_{t+1}$,

$$\exp\left(\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t\right) \quad = \quad \text{constant} \times \exp\left(\frac{\rho}{2}\min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t\right) \;=\; \exp\left(\frac{\rho}{2}\left[\gamma_t + \min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t\right]\right),$$

for γ_t a constant independent of $\mathbf{z}_t$. This implies that $\boldsymbol{\mathcal{V}}_t = \gamma_t + \min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t$. Then, assume that $\boldsymbol{\mathcal{V}}_{t+1} = \kappa_{t+1} + \mathbf{z}'_{t+1}\boldsymbol{\Pi}_{t+1}\mathbf{z}_{t+1}$, where κ_{t+1} is independent of $\mathbf{z}_{t+1}$. Considering that $\boldsymbol{\mathcal{S}}_t = c_t - \frac{1}{\rho}d_{t+1} + \delta\boldsymbol{\mathcal{V}}_{t+1}$, it follows that

$$\begin{aligned} \min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t \quad &= \quad \min_{\mathbf{u}_t}\left\{\max_{\boldsymbol{\epsilon}_{t+1}}\left[c_t - \frac{1}{\rho}d_{t+1} + \delta\kappa_{t+1} + \delta\,\mathbf{z}'_{t+1}\boldsymbol{\Pi}_{t+1}\mathbf{z}_{t+1}\right]\right\} \\ &= \quad \delta\kappa_{t+1} + \min_{\mathbf{u}_t}\left\{\max_{\boldsymbol{\epsilon}_{t+1}}\left[c_t - \frac{1}{\rho}d_{t+1} + \delta\,\mathbf{z}'_{t+1}\boldsymbol{\Pi}_{t+1}\mathbf{z}_{t+1}\right]\right\} \\ &= \quad \delta\kappa_{t+1} + \mathbf{z}'_t\boldsymbol{\Pi}_t\mathbf{z}_t \;=\; \delta\kappa_{t+1} + F_t(\mathbf{z}_t), \end{aligned}$$

so that $\boldsymbol{\mathcal{V}}_t = \gamma_t + \min_{\mathbf{u}_t}\max_{\boldsymbol{\epsilon}_{t+1}} \boldsymbol{\mathcal{S}}_t = \kappa_t + F_t(\mathbf{z}_t)$, with $\kappa_t = \gamma_t + \delta\kappa_{t+1}$ and $F_t(\mathbf{z}_t) = \mathbf{z}'_t\boldsymbol{\Pi}_t\mathbf{z}_t$.

The $\tilde{\boldsymbol{\mathcal{L}}}$-Recursion. Suppose $\phi(\mathbf{z}) = \delta\mathbf{z}'\boldsymbol{\Pi}\mathbf{z}$, so that $\tilde{\boldsymbol{\mathcal{L}}}\,\phi(\mathbf{z}) = \max_{\boldsymbol{\epsilon}}\,[(\mathbf{z}+\boldsymbol{\epsilon})'\delta\boldsymbol{\Pi}(\mathbf{z}+\boldsymbol{\epsilon}) - \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]$. Taking first derivatives, we find that

$$2\left(\delta\boldsymbol{\Pi} - \frac{1}{\rho}\mathbf{N}^{-1}\right)\boldsymbol{\epsilon} + 2\delta\,\boldsymbol{\Pi}\,\mathbf{z} \;=\; 0 \quad\Leftrightarrow\quad \tilde{\boldsymbol{\epsilon}} \;=\; -\left(\delta\boldsymbol{\Pi} - \frac{1}{\rho}\mathbf{N}^{-1}\right)^{-1}\delta\boldsymbol{\Pi}\,\mathbf{z} \;=\; -\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi}\,\mathbf{z},$$

which identifies a maximum for $\check{\boldsymbol{\Pi}}$ negative definite. Plugging this formula in the expression for $\tilde{\boldsymbol{\mathcal{L}}}\phi(\mathbf{z})$, we find that

$$\begin{aligned}\tilde{\boldsymbol{\mathcal{L}}}\phi(\mathbf{z}) &= -\frac{1}{\rho}\,\tilde{\boldsymbol{\epsilon}}'\,\mathbf{N}^{-1}\,\tilde{\boldsymbol{\epsilon}} \;+\; (\mathbf{z}+\tilde{\boldsymbol{\epsilon}})'\,\delta\,\boldsymbol{\Pi}\,(\mathbf{z}+\tilde{\boldsymbol{\epsilon}}) \\ &= -\frac{1}{\rho}\,\mathbf{z}'\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\mathbf{N}^{-1}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi}\,\mathbf{z} \;+\; \mathbf{z}'\,(\mathbf{I}-\check{\boldsymbol{\Pi}}^{-1}\delta\,\boldsymbol{\Pi})'\,\delta\boldsymbol{\Pi}\,(\mathbf{I}-\check{\boldsymbol{\Pi}}^{-1}\delta\,\boldsymbol{\Pi})\,\mathbf{z}\,.\end{aligned}$$

Now,

$$\begin{aligned}&-\frac{1}{\rho}\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\mathbf{N}^{-1}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;+\; (\mathbf{I}-\check{\boldsymbol{\Pi}}^{-1}\delta\,\boldsymbol{\Pi})'\,\delta\boldsymbol{\Pi}\,(\mathbf{I}-\check{\boldsymbol{\Pi}}^{-1}\delta\,\boldsymbol{\Pi}) \;= \\ &-\frac{1}{\rho}\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\mathbf{N}^{-1}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;+\; \delta\,\boldsymbol{\Pi} \;-\; 2\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;+\; \delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;= \\ &\delta\,\boldsymbol{\Pi} \;-\; 2\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;+\; \delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\left[\boldsymbol{\Pi}-\frac{1}{\rho}\mathbf{N}^{-1}\right]\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;= \\ &\delta\,\boldsymbol{\Pi} \;-\; 2\,\delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;+\; \delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;=\; \delta\,\boldsymbol{\Pi} \;-\; \delta\,\boldsymbol{\Pi}\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;=\; \delta\,\boldsymbol{\Pi}\,[\mathbf{I}-\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi}]\,.\end{aligned}$$

Notice that $\check{\boldsymbol{\Pi}} = \delta\,\boldsymbol{\Pi}\,[\mathbf{I}-\frac{1}{\rho}\,(\delta\,\boldsymbol{\Pi}\,)^{-1}\,\mathbf{N}^{-1}] = \delta\,\boldsymbol{\Pi}\,[\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1}]$, so that $\check{\boldsymbol{\Pi}}^{-1} = [\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1}]^{-1}\,(\delta\boldsymbol{\Pi})^{-1}$ and

$$\tilde{\boldsymbol{\mathcal{L}}}\phi(\mathbf{z}) \;=\; \mathbf{z}'\delta\,\boldsymbol{\Pi}\,[\mathbf{I}-(\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1})^{-1}]\,\mathbf{z}\,.$$

Since for $\mathbf{A}$ invertible $(\mathbf{I}+\mathbf{A})^{-1} = \mathbf{I}-(\mathbf{I}+\mathbf{A})^{-1}\mathbf{A}$, we find that

$$\mathbf{I}-(\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1})^{-1} \;=\; (\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1})^{-1}\,(-\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1}\,.$$

In addition since $(\mathbf{I}+\mathbf{A})^{-1} = (\mathbf{I}+\mathbf{A}^{-1})^{-1}\mathbf{A}^{-1}$, we also find that

$$(\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1})^{-1} = (\mathbf{I}-\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1}(-\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})\,, \quad \text{so that}$$

$$\begin{aligned}\delta\,\boldsymbol{\Pi}\,[\mathbf{I}-(\mathbf{I}-(\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1})^{-1}] &= \delta\,\boldsymbol{\Pi}\,(\mathbf{I}-\rho\,\mathbf{N}\delta\,\boldsymbol{\Pi})^{-1} \\ &= \delta\,\boldsymbol{\Pi}\,[((\delta\,\boldsymbol{\Pi})^{-1}-\rho\,\mathbf{N})\,\delta\,\boldsymbol{\Pi}]^{-1} \\ &= ((\delta\,\boldsymbol{\Pi})^{-1}-\rho\,\mathbf{N})^{-1} \;=\; \widetilde{\boldsymbol{\Pi}}\,.\end{aligned}$$

We conclude that $\tilde{\boldsymbol{\mathcal{L}}}\phi(\mathbf{z}) = \mathbf{z}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{z}$.

Second Order Conditions for the $\tilde{\boldsymbol{\mathcal{L}}}$-Recursion. Consider that the second order condition for the maximization in the $\tilde{\boldsymbol{\mathcal{L}}}$-recursion is that $\delta\boldsymbol{\Pi}-\frac{1}{\rho}\mathbf{N}^{-1}$ being negative definite. Now, as this is a symmetric matrix, there exists a coordinate transformation which diagonalizes it. This matrix will be negative definite iff all its eigenvalues are negative, or equivalently iff its elements on the main diagonal are negative, suggesting that is possible to proceed as in the scalar case. Hence, $\delta\boldsymbol{\Pi}-\frac{1}{\rho}\mathbf{N}^{-1} < 0$ is equivalent to $(\delta\boldsymbol{\Pi})^{-1}-\rho\mathbf{N} > 0$, as the elements on the main diagonal of the former matrix will be negative iff those on the latter are positive, or equivalently the former matrix is negative definite iff the latter is positive definite. We then establish that a solution to the $\tilde{\mathcal{L}}$-recursion exists if an only if $\widetilde{\mathbf{\Pi}}$, the inverse of $(\delta\mathbf{\Pi})^{-1} - \rho\mathbf{N}$, is positive definite.

Second Order Conditions for the $\mathcal{L}$-Recursion. Suppose that $\mathbf{\Pi}$ is positive semi-definite and $\mathbf{Q}$ and $\mathbf{R}$ are positive definite. This will be true if the cost function c is positive definite in $\mathbf{u}$ and $\mathbf{z}$ (that $\mathbf{Q}$ and $\mathbf{R}$ are positive definite when the cost function is a positive definite quadratic form is obvious; that in this case $\mathbf{\Pi}$ is also positive semi-definite will be shown below). Assume also that the condition in Theorem 2 for the DLEQG problem to have a proper solution holds, so that $\widetilde{\mathbf{\Pi}}$ is positive definite.

In solving the $\mathcal{L}$-recursion, standard result shows that the control is $\mathbf{u} = (\mathbf{Q} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}\,\mathbf{B})^{-1}\,(\mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}\,\mathbf{A}\,)$. As $\mathbf{Q}$ and $\widetilde{\mathbf{\Pi}}$ are positive definite a minimum is certainly reached since the denominator in the expression for the optimal control is also positive definite.

Then, suppose that in $t+1$ $\mathbf{\Pi}_{t+1}$ is positive semi-definite, while $\widetilde{\mathbf{\Pi}}_{t+1}$ is positive definite. At time t, plugging the optimal control vector into the $\mathcal{L}$-recursion, standard algebra shows that $\mathcal{L}\phi(\mathbf{z}_t) = \mathbf{z}_t'\mathbf{\Pi}_t\mathbf{z}_t$, where

$$\mathbf{\Pi}_t \;=\; \mathbf{R} + \mathbf{A}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{A} - \left(\mathbf{S}' + \mathbf{A}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\right)\left(\mathbf{Q} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\right)^{-1}\left(\mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{A}\right).$$

Now, consider that

$$\mathcal{L}\phi(\mathbf{z}_t) \;=\; \min \mathbf{u}_t\left[c(\mathbf{z}_t,\mathbf{u}_t) + (\mathbf{A}\,\mathbf{z}_t + \mathbf{B}\,\mathbf{u}_t)'\,\tilde{\mathbf{\Pi}}_{t+1}\,(\mathbf{A}\,\mathbf{z}_t + \mathbf{B}\,\mathbf{u}_t)\right].$$

Hence, since the cost function c is a positive definite quadratic form in $\mathbf{u}_t$ and $\mathbf{z}_t$ and $\widetilde{\mathbf{\Pi}}_{t+1}$ is positive definite, $\mathcal{L}\phi(\mathbf{z}_t)$ is non-negative and therefore $\mathbf{\Pi}_t$ must be positive semi-definite. As in T $\mathbf{\Pi}_T$ is equal to $\mathbf{0}$, by induction we prove that in any period t the $\mathcal{L}$-recursion has the solution discussed in the proof of Theorem 2, as the second order condition of the minimization is always respected, while the matrix $\mathbf{\Pi}_t$ is positive semi-definite.

That the cost function c is a positive definite quadratic form in $\mathbf{u}_t$ and $\mathbf{z}_t$ is a *sufficient* condition for the DLEQG problem to have the recursive solution presented in Theorem 2, but it is not *necessary*. If this assumption is abandoned, it will be necessary to verify that the matrix $\mathbf{Q} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}$ is positive definite in any period t.

The $\tilde{\mathcal{L}}$-Recursion with Deterministic Disturbances to the Plant Equation. Suppose $\phi(\mathbf{z}) = \delta\mathbf{z}'\mathbf{\Pi}\mathbf{z} - 2\delta\boldsymbol{\vartheta}'\mathbf{z}$, so that $\tilde{\mathcal{L}}\,\phi(\mathbf{z}) \;=\; \max_{\boldsymbol{\epsilon}}\,[(\mathbf{z}+\boldsymbol{\epsilon})'\delta\mathbf{\Pi}(\mathbf{z}+\boldsymbol{\epsilon}) - 2\delta\boldsymbol{\vartheta}'\,(\mathbf{z}+\boldsymbol{\epsilon}) - \frac{1}{\rho}\boldsymbol{\epsilon}'\mathbf{N}^{-1}\boldsymbol{\epsilon}]$. Taking first derivatives, we find that

$$2\left(\delta\mathbf{\Pi} - \frac{1}{\rho}\mathbf{N}^{-1}\right)\boldsymbol{\epsilon} + 2\delta\,(\mathbf{\Pi}\,\mathbf{z} - \boldsymbol{\vartheta}) \;=\; 0 \;\;\Leftrightarrow\;\; \tilde{\boldsymbol{\epsilon}} \;=\; -\left(\delta\mathbf{\Pi} - \frac{1}{\rho}\mathbf{N}^{-1}\right)^{-1}\delta\mathbf{\Pi}\,\mathbf{z} + \left(\delta\mathbf{\Pi} - \frac{1}{\rho}\mathbf{N}^{-1}\right)^{-1}\delta\boldsymbol{\vartheta}\,,$$

ie. $\tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\epsilon}}_o + \tilde{\boldsymbol{\epsilon}}_e$, with $\tilde{\boldsymbol{\epsilon}}_o = -\breve{\mathbf{\Pi}}^{-1}\delta\mathbf{\Pi}\mathbf{z}$ and $\tilde{\boldsymbol{\epsilon}}_e = \breve{\mathbf{\Pi}}^{-1}\delta\boldsymbol{\vartheta}$. Plugging this formula in the expression for

$\tilde{\mathcal{L}}\phi(\mathbf{z})$, we find that

$$\begin{aligned}
\tilde{\mathcal{L}}\phi(\mathbf{z}) &= -\frac{1}{\rho}\,(\tilde{\boldsymbol{\epsilon}}_o + \tilde{\boldsymbol{\epsilon}}_e)'\,\mathbf{N}^{-1}\,(\tilde{\boldsymbol{\epsilon}}_o + \tilde{\boldsymbol{\epsilon}}_e) \;+\; (\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o + \tilde{\boldsymbol{\epsilon}}_e)'\,\delta\,\boldsymbol{\Pi}\,(\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o + \tilde{\boldsymbol{\epsilon}}_e) \;+\; -2\,\delta\,\boldsymbol{\vartheta}'\,(\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o + \tilde{\boldsymbol{\epsilon}}_e) \\
&= \underbrace{-\frac{1}{\rho}\,\tilde{\boldsymbol{\epsilon}}_o'\,\mathbf{N}^{-1}\,\tilde{\boldsymbol{\epsilon}}_o \;+\; (\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o)'\,\delta\,\boldsymbol{\Pi}\,(\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o)}_{\mathbf{z}'\,\tilde{\boldsymbol{\Pi}}\,\mathbf{z}\ \ \text{with}\ \ \tilde{\boldsymbol{\Pi}}\,=\,((\delta\boldsymbol{\Pi})^{-1}-\rho\,\mathbf{N})^{-1}} \underbrace{-\frac{1}{\rho}\,\tilde{\boldsymbol{\epsilon}}_e'\,\mathbf{N}^{-1}\,\tilde{\boldsymbol{\epsilon}}_e \;+\; \tilde{\boldsymbol{\epsilon}}_e'\,\delta\,\boldsymbol{\Pi}\,\tilde{\boldsymbol{\epsilon}}_e \;-\; 2\,\delta\,\boldsymbol{\vartheta}'\,\tilde{\boldsymbol{\epsilon}}_e}_{\text{independent of}\,\mathbf{z}} \\
&\quad -\frac{2}{\rho}\,\tilde{\boldsymbol{\epsilon}}_e'\,\mathbf{N}^{-1}\,\tilde{\boldsymbol{\epsilon}}_o \;+\; 2\,\tilde{\boldsymbol{\epsilon}}_e'\,\delta\,\boldsymbol{\Pi}\,(\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) \;-\; 2\,\delta\,\boldsymbol{\vartheta}'\,(\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o)\,.
\end{aligned}$$

The last three terms can be re-written as follows,

$$-2\left[\frac{1}{\rho}\,\tilde{\boldsymbol{\epsilon}}_e'\,\mathbf{N}^{-1} \;-\; \tilde{\boldsymbol{\epsilon}}_e'\,\delta\,\boldsymbol{\Pi} \;+\; \delta\,\boldsymbol{\vartheta}'\right](\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) \;+\; \frac{2}{\rho}\,\tilde{\boldsymbol{\epsilon}}_e'\,\mathbf{N}^{-1}\,\mathbf{z}\,.$$

Given the expression for $\tilde{\boldsymbol{\epsilon}}_e$ we find that the sum in the squared brackets is equal to

$$\begin{aligned}
&-2\left[\frac{1}{\rho}\,\delta\,\boldsymbol{\vartheta}'\,\check{\boldsymbol{\Pi}}^{-1}\,\mathbf{N}^{-1} \;-\; \delta\,\boldsymbol{\vartheta}'\,\check{\boldsymbol{\Pi}}^{-1}\,\delta\,\boldsymbol{\Pi} \;+\; \delta\,\boldsymbol{\vartheta}'\right](\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) \\
= \quad &-2\,\delta\boldsymbol{\vartheta}'\left[\mathbf{I} \;-\; \check{\boldsymbol{\Pi}}^{-1}\underbrace{\left(\delta\,\boldsymbol{\Pi} \;-\; \frac{1}{\rho}\,\mathbf{N}^{-1}\right)}_{\check{\boldsymbol{\Pi}}}\right](\mathbf{z} + \tilde{\boldsymbol{\epsilon}}_o) \;=\; 0\,,
\end{aligned}$$

while

$$\frac{2}{\rho}\,\tilde{\boldsymbol{\epsilon}}_e'\,\mathbf{N}^{-1}\,\mathbf{z} \quad = \quad -\,2\,\delta\,\boldsymbol{\vartheta}'\,\check{\boldsymbol{\Pi}}^{-1}(-\,\rho\mathbf{N})^{-1}\,\mathbf{z}\,,$$

so that $\boldsymbol{\mathcal{L}}\phi(\mathbf{z}) = \mathbf{z}'\widetilde{\boldsymbol{\Pi}}\mathbf{z} - 2\,\widetilde{\boldsymbol{\vartheta}}'\mathbf{z} + \cdots$, with $\widetilde{\boldsymbol{\vartheta}} = \delta\,(-\,\rho\mathbf{N})^{-1}\,\check{\boldsymbol{\Pi}}^{-1}\,\boldsymbol{\vartheta}$ and $\cdots$ indicating terms independent of $\mathbf{z}$. Now

$$\begin{aligned}
(-\,\rho\mathbf{N})^{-1}\,\check{\boldsymbol{\Pi}}^{-1} &= \delta\,(-\,\rho\mathbf{N})^{-1}\,(\delta\boldsymbol{\Pi} \;-\; (\rho\mathbf{N})^{-1})^{-1} \\
&= (-\,\rho\mathbf{N})^{-1}\,[(\mathbf{I} \;+\; \delta\,(-\,\rho\mathbf{N})\,\boldsymbol{\Pi})(-\rho\mathbf{N})^{-1})]^{-1} \\
&= (\mathbf{I} \;-\; \delta\,\rho\,\mathbf{N}\,\boldsymbol{\Pi})^{-1}\,,
\end{aligned}$$

so that

$$\begin{aligned}
\widetilde{\boldsymbol{\vartheta}} &= \delta\,(\mathbf{I} \;-\; \delta\,\rho\,\mathbf{N}\,\boldsymbol{\Pi})^{-1}\,\boldsymbol{\vartheta} \\
&= \delta\,((\delta\boldsymbol{\Pi})^{-1} \;-\; \rho\,\mathbf{N})^{-1}\,(\delta\,\boldsymbol{\Pi})^{-1}\,\boldsymbol{\vartheta} \\
&= ((\delta\boldsymbol{\Pi})^{-1} \;-\; \rho\,\mathbf{N})^{-1}\,\boldsymbol{\Pi}^{-1}\,\boldsymbol{\vartheta} \;=\; \widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\Pi}^{-1}\,\boldsymbol{\vartheta}\,.
\end{aligned}$$

The Continuous-time Limit. To derive the continuous-time limit of the Markovian DLEQG problem divide the interval of time $[0, T]$ in n sub-periods of length $\Delta = T/n$. Then, the per-period cost function is Δc_t, the plant equation for the state variable is $\mathbf{z}_t = (\mathbf{I}+\mathbf{A}\Delta)\mathbf{z}_{t-1}+\mathbf{B}\Delta\mathbf{u}_{t-1}+\boldsymbol{\epsilon}_t$, with $\boldsymbol{\epsilon}_t \sim \boldsymbol{\mathcal{N}}(\mathbf{0}, \Delta\mathbf{N})$, and the discount factor is δ^Δ. Under perfect state observation, the discounted total stress function is redefined as follows $\boldsymbol{\mathcal{S}}_t = \Delta c_t - (1/\rho)\boldsymbol{\epsilon}'_{t+1}(\Delta\mathbf{N})^{-1}\boldsymbol{\epsilon}_{t+1} + \delta^\Delta \boldsymbol{\mathcal{V}}_{t+1}$. Lemma 3 and Theorem 1 still hold, and so does Lemma 4. This means that we can proceed along the proof of Theorem 2 to reformulate its statement.

Consider hence that in the $\tilde{\boldsymbol{\mathcal{L}}}$-recursion, with $\phi(\mathbf{z}) = \mathbf{z}'\boldsymbol{\Pi}\mathbf{z}$, a maximum is found for

$$\tilde{\boldsymbol{\epsilon}} \;=\; -\left(\delta^\Delta\,\boldsymbol{\Pi} - \frac{1}{\rho}(\Delta\,\mathbf{N})^{-1}\right)^{-1}\delta^\Delta\,\boldsymbol{\Pi}\,\mathbf{z}\,,\ \text{so that}\ \tilde{\boldsymbol{\mathcal{L}}}\phi(\mathbf{z}) \;=\; \mathbf{z}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{z}\,,\ \text{with}\ \widetilde{\boldsymbol{\Pi}} \;=\; ((\delta^\Delta\,\boldsymbol{\Pi})^{-1} - \rho\Delta\,\mathbf{N})^{-1}\,.$$

Importantly, in the limit for $\Delta \downarrow 0$, for $\boldsymbol{\epsilon}_{t+1} \neq \mathbf{0}$, $-(1/\rho)\boldsymbol{\epsilon}'_{t+1}(\Delta\mathbf{N})^{-1}\boldsymbol{\epsilon}_{t+1} \downarrow -\infty$ and hence a maximum for the stress function $\boldsymbol{\mathcal{S}}_t$ with respect to $\boldsymbol{\epsilon}_{t+1}$ is found for $\boldsymbol{\epsilon}_{t+1} = \mathbf{0}$, irrespective of c_t and $\boldsymbol{\mathcal{V}}_{t+1}$. Similarly, whatever $\boldsymbol{\Pi}$, in the limit, for $\Delta \downarrow 0$, a maximum in the $\tilde{\boldsymbol{\mathcal{L}}}$-recursion above is found for $\tilde{\boldsymbol{\epsilon}} = \mathbf{0}$.

Clearly, for $\Delta \downarrow 0$, $\widetilde{\boldsymbol{\Pi}} \to \boldsymbol{\Pi}$. As for the $\boldsymbol{\mathcal{L}}$-recursion, consider that we need to solve

$$\min_{\mathbf{u}}\ [\Delta c \;+\; \phi((\mathbf{I} \,+\, \mathbf{A}\,\Delta)\mathbf{z} \,+\, \mathbf{B}\,\Delta\,\mathbf{u})]\ ,$$

where $\phi(\mathbf{z}) = \mathbf{z}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{z}$ and $\Delta c = \mathbf{u}'\,\mathbf{Q}\,\Delta\,\mathbf{u} + \mathbf{z}'\,\mathbf{R}\,\Delta\,\mathbf{z} + 2\mathbf{u}'\,\mathbf{S}\,\Delta\,\mathbf{z}$. The first order condition is

$$2\,\Delta\left(\mathbf{Q}\,\mathbf{u} \;+\; \Delta\,\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{B}\,\mathbf{u} \;+\; \mathbf{S}\,\mathbf{z} \;+\; \mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}\,(\mathbf{I} \,+\, \mathbf{A}\,\Delta)\,\mathbf{z}\right) \;=\; \mathbf{0}\,,\ \text{so that}$$

$$\mathbf{u} \;=\; \mathbf{K}\,\mathbf{z}\ \text{with}\ \mathbf{K} \;=\; -\,(\mathbf{Q} \,+\, \Delta\,\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{B})^{-1}\left(\mathbf{S} \,+\, \mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}\,(\mathbf{I} \,+\, \mathbf{A}\,\Delta)\right).$$

For $\Delta \downarrow 0$, as $\widetilde{\boldsymbol{\Pi}} \to \boldsymbol{\Pi}$, $\mathbf{K} \to -\,\mathbf{Q}^{-1}\,(\mathbf{S}+\mathbf{B}'\,\boldsymbol{\Pi})$. Thus, in the limit a *sufficient* condition for the DLEQG problem to have the recursive solution discussed in Theorem 6 is that $\mathbf{Q}$ being positive definite. Inserting the expression for $\mathbf{u}$ in the minimizing function we find $\boldsymbol{\mathcal{L}}\phi(\mathbf{z}) = \mathbf{z}'\,\boldsymbol{\Pi}_-\,\mathbf{z}$, where

$$\begin{aligned}\boldsymbol{\Pi}_- \;&=\; \widetilde{\boldsymbol{\Pi}} \,+\, \Delta\left(\mathbf{R} \,+\, \mathbf{K}'\,\mathbf{Q}\,\mathbf{K} \,+\, 2\mathbf{S}'\,\mathbf{K} \,+\, \mathbf{A}'\,\boldsymbol{\Pi} \,+\, \widetilde{\boldsymbol{\Pi}}\,\mathbf{A} + 2\boldsymbol{\Pi}\,\mathbf{B}\,\mathbf{K}\right)\\ &\qquad + \Delta^2\left(\mathbf{A}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{A} \,+\, \mathbf{K}'\,\mathbf{B}\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{B}\,\mathbf{K} \,+\, 2\mathbf{A}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{B}\,\mathbf{K}\right),\end{aligned}$$

so that applying the double recursion in period t, $F_t = \boldsymbol{\mathcal{L}}\tilde{\boldsymbol{\mathcal{L}}}F_{t+1}$, we find that

$$\boldsymbol{\Pi}_t \;=\; \widetilde{\boldsymbol{\Pi}}_{t+1} \,+\, \Delta\left(\mathbf{R} \,+\, \mathbf{K}'_t\,\mathbf{Q}\,\mathbf{K}_t \,+\, 2\mathbf{S}'\,\mathbf{K}_t \,+\, \mathbf{A}'\,\widetilde{\boldsymbol{\Pi}}_{t+1} \,+\, \widetilde{\boldsymbol{\Pi}}_{t+1}\,\mathbf{A} + 2\widetilde{\boldsymbol{\Pi}}_{t+1}\,\mathbf{B}\,\mathbf{K}_t\right) + o(\Delta^2)\,,$$

where $\widetilde{\mathbf{\Pi}}_{t+1} = ((\delta^{\Delta}\,\mathbf{\Pi}_{t+1})^{-1} - \rho\Delta\,\mathbf{N})^{-1} = \delta^{\Delta}\,\mathbf{\Pi}_{t+1}\,(\mathbf{I} - \rho\Delta\,\mathbf{N}\,\delta^{\Delta}\,\mathbf{\Pi}_{t+1})^{-1}$. This implies that

$$\begin{aligned}
\frac{\mathbf{\Pi}_t - \mathbf{\Pi}_{t+1}}{\Delta} &= \frac{\widetilde{\mathbf{\Pi}}_{t+1} - \mathbf{\Pi}_{t+1}}{\Delta} + \Big(\mathbf{R} + \mathbf{K}_t'\,\mathbf{Q}\,\mathbf{K}_t + 2\mathbf{S}'\,\mathbf{K}_t + \mathbf{A}'\,\widetilde{\mathbf{\Pi}}_{t+1} + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{A} + 2\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\,\mathbf{K}_t\Big) + \frac{o(\Delta^2)}{\Delta} \\
&= \frac{\widetilde{\mathbf{\Pi}}_{t+1} - \mathbf{\Pi}_{t+1}}{\Delta} + \mathbf{R} + \mathbf{A}'\,\widetilde{\mathbf{\Pi}}_{t+1} + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{A} + \Big(\mathbf{K}_t'\,\mathbf{Q}\,\mathbf{K}_t + 2\mathbf{S}'\,\mathbf{K}_t + 2\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\,\mathbf{K}_t\Big) + \frac{o(\Delta^2)}{\Delta}\,.
\end{aligned}$$

Now, given the expression for $\mathbf{K}_t$,

$$\begin{aligned}
&\Big(\mathbf{K}_t'\,\mathbf{Q} + 2\,(\mathbf{S}' + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})\Big)\,\mathbf{K}_t = (\mathbf{S}' + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})\,\Big(2\mathbf{I} - (\mathbf{Q} + \Delta\,\mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\,\mathbf{Q} + o(\Delta)\Big)\,\mathbf{K}_t = \\
&-(\mathbf{S}' + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})\,\Big(2\mathbf{I} - (\mathbf{Q} + \Delta\,\mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\,\mathbf{Q} + o(\Delta)\Big)\,(\mathbf{Q} + \Delta\,\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\,\Big(\mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,(\mathbf{I} + \mathbf{A}\,\Delta)\Big) = \\
&-(\mathbf{S}' + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})\,\Big(2\mathbf{I} - (\mathbf{Q} + \Delta\,\mathbf{B}'\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\,\mathbf{Q}\Big)\,(\mathbf{Q} + \Delta\,\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})^{-1}\,\Big(\mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\Big) + o(\Delta) + o(\Delta^2)\,.
\end{aligned}$$

Therefore, considering that $\lim_{\Delta\downarrow 0}\widetilde{\mathbf{\Pi}}_{n+1} = \lim_{\Delta\downarrow 0}\mathbf{\Pi}_{t+1} = \mathbf{\Pi}(t)$, we conclude that

$$\lim_{\Delta\downarrow 0}\Big(\mathbf{K}_t'\,\mathbf{Q} + 2\,(\mathbf{S}' + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B})\Big)\,\mathbf{K}_t = -(\mathbf{S}' + \mathbf{\Pi}(t)\,\mathbf{B})\,\mathbf{Q}^{-1}\,(\mathbf{S} + \mathbf{B}'\,\mathbf{\Pi}(t))\,.$$

In addition,

$$\lim_{\Delta\downarrow 0}\frac{\mathbf{\Pi}_t - \mathbf{\Pi}_{t+1}}{\Delta} = -\frac{d\,\mathbf{\Pi}(t)}{d\,t} \quad \text{and} \quad \lim_{\Delta\downarrow 0}\mathbf{A}'\,\widetilde{\mathbf{\Pi}}_{t+1} + \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{A} = \mathbf{A}'\,\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\,\mathbf{A}\,.$$

Finally, consider that

$$\widetilde{\mathbf{\Pi}}_{t+1} - \mathbf{\Pi}_{t+1} = \mathbf{\Pi}_{t+1}\left[\delta^{\Delta}\,(\mathbf{I} - \rho\Delta\,\mathbf{N}\,\delta^{\Delta}\,\mathbf{\Pi}_{t+1})^{-1} - \mathbf{I}\right].$$

For Δ small we can write $(\mathbf{I} - \rho\Delta\,\mathbf{N}\,\delta^{\Delta}\,\mathbf{\Pi}_{t+1})^{-1} = \mathbf{I} + \rho\Delta\,\mathbf{N}\,\delta^{\Delta}\,\mathbf{\Pi}_{t+1} + o(\Delta^2)$, so that

$$\widetilde{\mathbf{\Pi}}_{t+1} - \mathbf{\Pi}_{t+1} = \mathbf{\Pi}_{t+1}\left[\delta^{\Delta}\,(\mathbf{I} + \rho\Delta\,\mathbf{N}\,\delta^{\Delta}\,\mathbf{\Pi}_{t+1}) - \mathbf{I}\right] + o(\Delta^2)\,.$$

Therefore,

$$\frac{\widetilde{\mathbf{\Pi}}_{t+1} - \mathbf{\Pi}_{t+1}}{\Delta} = \left(\frac{\delta^{\Delta} - 1}{\Delta}\right)\mathbf{\Pi}_{t+1} + \rho\,\delta^{2\Delta}\,\mathbf{\Pi}_{t+1}\,\mathbf{N}\,\mathbf{\Pi}_{t+1} + o(\Delta)\,.$$

In conclusion, we find that

$$\lim_{\Delta\downarrow 0}\frac{\widetilde{\mathbf{\Pi}}_{t+1} - \mathbf{\Pi}_{t+1}}{\Delta} = \log(\delta)\,\mathbf{\Pi}(t) + \rho\,\mathbf{\Pi}(t)\,\mathbf{N}\,\mathbf{\Pi}(t)$$

and hence the continuous-time counterpart of the Riccati equation is

$$\frac{d\,\mathbf{\Pi}(t)}{d\,t} + \mathbf{R} + \mathbf{A}'\,\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\,\mathbf{A} - (\mathbf{S}' + \mathbf{\Pi}(t)\,\mathbf{B})\,\mathbf{Q}^{-1}\,(\mathbf{S} + \mathbf{B}'\,\mathbf{\Pi}(t)) + \rho\,\mathbf{\Pi}(t)\,\mathbf{N}\,\mathbf{\Pi}(t) + \log\delta\,\mathbf{\Pi}(t) \;\;=\;\; \mathbf{0}\,.$$

The Continuous-time Limit with Pre-determined Disturbances. Suppose that $\mathbf{z}_t = (\mathbf{I} + \mathbf{A}\Delta)\mathbf{z}_{t-1} + \mathbf{B}\Delta\mathbf{u}_{t-1} + \boldsymbol{\mu}_t\Delta + \boldsymbol{\epsilon}_t$. In discussing the Markovian DLEQG with pre-determined disturbances we have seen that $\mathbf{u}_t = \mathbf{u}_{t,o} + \mathbf{u}_{t,e}$, where the former component corresponds to the solution in the time-homogeneous formulation, while the latter is now given by

$$(\mathbf{Q} + \Delta\mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\Delta\mathbf{B})^{-1}\mathbf{B}'(\widetilde{\boldsymbol{\vartheta}}_{t+1} - \widetilde{\mathbf{\Pi}}_{t+1}\boldsymbol{\mu}\Delta) \;\;\text{where}$$

$$\widetilde{\mathbf{\Pi}}_{t+1} \;=\; ((\delta^{\Delta}\mathbf{\Pi}_{t+1})^{-1} - \rho\,\Delta\,\mathbf{N})^{-1} \quad\text{and}\quad \widetilde{\boldsymbol{\vartheta}}_{t+1} \;=\; \widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{\Pi}_{t+1}^{-1}\,\boldsymbol{\vartheta}_{t+1}\,.$$

Then, consider that

$$\begin{aligned}
&\mathbf{z}_t'\,\mathbf{\Pi}_t\,\mathbf{z}_t - 2\,\boldsymbol{\vartheta}_t'\,\mathbf{z}_t + \cdots = \mathbf{z}_t'\,\mathbf{R}\Delta\,\mathbf{z}_t + (\mathbf{u}_{t,o} + \mathbf{u}_{t,e})'\mathbf{Q}\Delta\,(\mathbf{u}_{t,o} + \mathbf{u}_{t,e}) + 2\,\mathbf{z}_t'\,\mathbf{S}'\,\Delta\,(\mathbf{u}_{t,o} + \mathbf{u}_{t,e}) + \\
&[(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + \mathbf{B}\Delta(\mathbf{u}_{t,o} + \mathbf{u}_{t,e}) + \boldsymbol{\mu}_{t+1}\Delta]'\widetilde{\mathbf{\Pi}}_{t+1}\,[(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + \mathbf{B}\Delta(\mathbf{u}_{t,o} + \mathbf{u}_{t,e}) + \boldsymbol{\mu}_{t+1}\Delta] + \\
&-2\,\widetilde{\boldsymbol{\vartheta}}_{t+1}'\,[(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + \mathbf{B}\Delta(\mathbf{u}_{t,o} + \mathbf{u}_{t,e}) + \boldsymbol{\mu}_{t+1}\Delta] + \cdots = \mathbf{z}_t'\,\mathbf{R}\Delta\,\mathbf{z}_t + \underbrace{\mathbf{u}_{t,o}'\mathbf{Q}\Delta\,\mathbf{u}_{t,o}}_{\text{function of } \mathbf{u}_{t,o}} + \\
&\underbrace{2\,\mathbf{z}_t'\,\mathbf{S}'\,\Delta\,\mathbf{u}_{t,o} + [(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + \mathbf{B}\Delta\mathbf{u}_{t,o}]'\widetilde{\mathbf{\Pi}}_{t+1}\,[(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + \mathbf{B}\Delta\mathbf{u}_{t,o}]}_{\text{function of } \mathbf{u}_{t,o}} + \\
&\underbrace{-2\,\widetilde{\boldsymbol{\vartheta}}_{t+1}'\,[(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + \mathbf{B}\Delta\mathbf{u}_{t,o}]}_{\text{function of } \mathbf{u}_{t,o}} + \underbrace{\mathbf{u}_{t,e}'\mathbf{Q}\Delta\,\mathbf{u}_{t,e} + \mathbf{u}_{t,e}'\mathbf{B}'\,\Delta\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\,\Delta\,\mathbf{u}_{t,e} - 2\,\widetilde{\boldsymbol{\vartheta}}_{t+1}'\,\mathbf{B}\Delta\mathbf{u}_{t,e}}_{\text{independent of } \mathbf{z}_t} + \\
&\underbrace{-2\,\widetilde{\boldsymbol{\vartheta}}_{t+1}'\,\boldsymbol{\mu}_{t+1}\,\Delta + \boldsymbol{\mu}_{t+1}'\,\Delta\,\widetilde{\mathbf{\Pi}}_{t+1}\,\boldsymbol{\mu}_{t+1}\,\Delta + 2\boldsymbol{\mu}_{t+1}'\,\Delta\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{B}\,\Delta\mathbf{u}_{t,e}}_{\text{independent of } \mathbf{z}_t} + 2\mathbf{u}_{t,e}'\mathbf{Q}\Delta\,\mathbf{u}_{t,o} + \\
&2\mathbf{u}_{t,e}'\mathbf{S}\Delta\,\mathbf{z}_t + 2\mathbf{u}_{t,e}'\,\mathbf{B}'\,\Delta\,\widetilde{\mathbf{\Pi}}_{t+1}\,(\mathbf{I} + \mathbf{A}\,\Delta)\mathbf{z}_t + 2\boldsymbol{\mu}_{t+1}'\,\Delta\,\widetilde{\mathbf{\Pi}}_{t+1}\,[\mathbf{I}\,\mathbf{z}_t + \underbrace{\mathbf{A}\,\Delta\,\mathbf{z}_t + \mathbf{B}\,\Delta\,\mathbf{u}_{t,o}}_{o(\Delta^2)}]\,.
\end{aligned}$$

So that, considering that $\mathbf{u}_{t,o} = \mathbf{K}_t\mathbf{z}_t$, we see that $\mathbf{z}_t'\mathbf{R}\Delta\mathbf{z}_t$ plus the terms tagged as "functions of $\mathbf{u}_{t,o}$" correspond to $\mathbf{z}_t'\,\mathbf{\Pi}_t\,\mathbf{z}_t$. Hence, considering that $\mathbf{u}_{t,o} = \mathbf{K}_t\mathbf{z}_t$,

$$\begin{aligned}
&\mathbf{z}_t'\,\mathbf{\Pi}_t\,\mathbf{z}_t - 2\,\boldsymbol{\vartheta}_t'\,\mathbf{z}_t + \cdots = \mathbf{z}_t'\,\mathbf{\Pi}_t\,\mathbf{z}_t - 2\,\widetilde{\boldsymbol{\vartheta}}_{t+1}'\,[(\mathbf{I} + \mathbf{A}\Delta) + \mathbf{B}\,\Delta\,\mathbf{K}_t]\,\mathbf{z}_t + \\
&+2\Delta\,\mathbf{u}_{t,e}'[\mathbf{Q}\mathbf{K}_t + \mathbf{S} + \mathbf{B}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,(\mathbf{I} + \mathbf{A}\,\Delta)]\,\mathbf{z}_t + 2\Delta\,\boldsymbol{\mu}_{t+1}'\,\widetilde{\mathbf{\Pi}}_{t+1}\,\mathbf{z}_t + o(\Delta^2) + \cdots
\end{aligned}$$

where $\mathbf{\Pi}_t$ respects the recursion identified in the time-homogeneous formulation. To determine $\boldsymbol{\vartheta}_t$

consider that it respects the recursion

$$\boldsymbol{\vartheta}_t' \;=\; \widetilde{\boldsymbol{\vartheta}}_{t+1}'\left[\mathbf{I} + \Delta\left(\mathbf{A}+\mathbf{B}\,\mathbf{K}_t\right)\right] \;-\; \Delta\,\mathbf{u}_{t,e}'\left[\mathbf{Q}\,\mathbf{K}_t + \mathbf{S} + \mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\left(\mathbf{I} + \mathbf{A}\,\Delta\right)\right] \;-\; \Delta\boldsymbol{\mu}_{t+1}'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\,.$$

This is equivalent to

$$\begin{aligned}\frac{\boldsymbol{\vartheta}_t - \boldsymbol{\vartheta}_{t+1}}{\Delta} &= \frac{\widetilde{\boldsymbol{\vartheta}}_{t+1} - \boldsymbol{\vartheta}_{t+1}}{\Delta} + \left(\mathbf{A} + \mathbf{B}\,\mathbf{K}_t\right)'\,\widetilde{\boldsymbol{\vartheta}}_{t+1} \\ &\quad - \left[\mathbf{Q}\,\mathbf{K}_t + \mathbf{S} + \mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\left(\mathbf{I} + \mathbf{A}\,\Delta\right)\right]'\,\mathbf{u}_{t,e} - \widetilde{\boldsymbol{\Pi}}_{t+1}\,\boldsymbol{\mu}_{t+1}\,.\end{aligned}$$

Now, $\lim_{\Delta\downarrow 0}\widetilde{\boldsymbol{\Pi}}_{t+1} = \lim_{\Delta\downarrow 0}\boldsymbol{\Pi}_t = \boldsymbol{\Pi}(t)$, $\lim_{\Delta\downarrow 0}\widetilde{\boldsymbol{\vartheta}}_{t+1} = \lim_{\Delta\downarrow 0}\boldsymbol{\vartheta}_t = \boldsymbol{\vartheta}(t)$, $\lim_{\Delta\downarrow 0}\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}(t)$, $\lim_{\Delta\downarrow 0}\frac{\boldsymbol{\vartheta}_t - \boldsymbol{\vartheta}_{t+1}}{\Delta} = -\frac{d\,\boldsymbol{\vartheta}(t)}{d\,t}$. In addition, $\lim_{\Delta\downarrow 0}\mathbf{K}_t = -\mathbf{Q}^{-1}(\mathbf{S} + \mathbf{B}'\boldsymbol{\Pi}(t))$. We then conclude that

$$\lim_{\Delta\downarrow 0}\widetilde{\boldsymbol{\Pi}}_{t+1}\,\boldsymbol{\mu}_{t+1} = \boldsymbol{\Pi}(t)\,\boldsymbol{\mu}(t)\,, \qquad \lim_{\Delta\downarrow 0}\left(\mathbf{A} + \mathbf{B}\,\mathbf{K}_t\right) = \mathbf{A} - \mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{S} - \mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{B}'\,\boldsymbol{\Pi}(t)\,,$$

$$\lim_{\Delta\downarrow 0}\mathbf{Q}\,\mathbf{K}_t = -\left(\mathbf{S} + \mathbf{B}'\,\boldsymbol{\Pi}(t)\right), \qquad \lim_{\Delta\downarrow 0}\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\left(\mathbf{I} + \mathbf{A}\,\Delta\right) = \mathbf{B}'\,\boldsymbol{\Pi}(t) \;\text{ and hence}$$

$$\lim_{\Delta\downarrow 0}\left[\mathbf{Q}\,\mathbf{K}_t + \mathbf{S} + \mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\left(\mathbf{I} + \mathbf{A}\,\Delta\right)\right] = \mathbf{0}\,.$$

Since $\widetilde{\boldsymbol{\vartheta}}_{t+1} = \widetilde{\boldsymbol{\Pi}}_{t+1}\,\boldsymbol{\Pi}_{t+1}^{-1}\,\boldsymbol{\vartheta}_{t+1}$,

$$\widetilde{\boldsymbol{\vartheta}}_{t+1} - \boldsymbol{\vartheta}_{t+1} = \left(\widetilde{\boldsymbol{\Pi}}_{t+1}\,\boldsymbol{\Pi}_{t+1}^{-1} - \mathbf{I}\right)\boldsymbol{\vartheta}_{t+1}\,.$$

In addition, $\widetilde{\boldsymbol{\Pi}}_{t+1} = (\mathbf{I} - \rho\,\Delta\,\delta^{\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N})^{-1}\,\delta^{\Delta}\,\boldsymbol{\Pi}_{t+1}$, so that

$$\left(\widetilde{\boldsymbol{\Pi}}_{t+1}\,\boldsymbol{\Pi}_{t+1}^{-1} - \mathbf{I}\right) = \left(\mathbf{I} - \rho\,\Delta\,\delta^{\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N}\right)^{-1}\delta^{\Delta} - \mathbf{I}\,.$$

Since for Δ small we can write $(\mathbf{I} - \rho\,\Delta\,\delta^{\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N})^{-1} = \mathbf{I} + \rho\,\Delta\,\delta^{\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N} + o(\Delta^2)$ we find that

$$\begin{aligned}\left(\widetilde{\boldsymbol{\Pi}}_{t+1}\,\boldsymbol{\Pi}_{t+1}^{-1} - \mathbf{I}\right) &= \left(\mathbf{I} + \rho\,\Delta\,\delta^{\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N} + o(\Delta^2)\right)\delta^{\Delta} - \mathbf{I} \\ &= \left(\delta^{\Delta} - 1\right)\mathbf{I} + \rho\,\Delta\,\delta^{2\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N} + o(\Delta^2)\,. \;\text{ Thus,} \\ \frac{\widetilde{\boldsymbol{\vartheta}}_{t+1} - \boldsymbol{\vartheta}_{t+1}}{\Delta} &= \left[\frac{(\delta^{\Delta} - 1)}{\Delta}\,\mathbf{I} + \rho\,\delta^{2\Delta}\,\boldsymbol{\Pi}_{t+1}\,\mathbf{N} + o(\Delta)\right]\boldsymbol{\vartheta}_{t+1}\end{aligned}$$

and hence

$$\lim_{\Delta\downarrow 0}\frac{\widetilde{\boldsymbol{\vartheta}}_{t+1} - \boldsymbol{\vartheta}_{t+1}}{\Delta} = \log\delta\,\boldsymbol{\vartheta}(t) + \rho\,\boldsymbol{\Pi}(t)\,\mathbf{N}\,\boldsymbol{\vartheta}(t)\,.$$

Summing up terms we find that

$$\frac{d\boldsymbol{\vartheta}(t)}{dt} \;+\; \log\delta\,\boldsymbol{\vartheta}(t) \;+\; \rho\,\boldsymbol{\Pi}(t)\,\mathbf{N}\,\boldsymbol{\vartheta}(t) \;+\; [\mathbf{A} \;-\; \mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{S} \;-\; \mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{B}'\,\boldsymbol{\Pi}(t)]'\,\boldsymbol{\vartheta}(t) \;-\; \boldsymbol{\Pi}(t)\,\boldsymbol{\mu}(t) \;=\; \mathbf{0}\,,$$

which can also be written as

$$\frac{d\boldsymbol{\vartheta}(t)}{dt} \;+\; (\log\delta \;+\; \widetilde{\boldsymbol{\Gamma}}(t)')\,\boldsymbol{\vartheta}(t) \;-\; \boldsymbol{\Pi}(t)\,\boldsymbol{\mu}(t) \;=\; \mathbf{0}\,, \quad \text{with}$$

$$\widetilde{\boldsymbol{\Gamma}}(t) \quad=\quad \mathbf{A} \;-\; \mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{S} \;-\; (\mathbf{B}\,\mathbf{Q}^{-1}\,\mathbf{B}' \;-\; \rho\,\mathbf{N}\,)\boldsymbol{\Pi}(t)\,.$$

Optimal Monetary Policy. In the stationary solution,

$$\begin{aligned}
\widetilde{\boldsymbol{\Pi}} \quad &= \quad ((\delta\,\boldsymbol{\Pi})^{-1} \;-\; \rho\mathbf{N})^{-1} \;=\; \delta\boldsymbol{\Pi}\,(\mathbf{I}_2 \;-\; \delta\,\rho\,\mathbf{N}\,\boldsymbol{\Pi})^{-1} \\
&= \quad \delta\boldsymbol{\Pi}\begin{pmatrix} 1-\delta\rho\sigma_\pi^2\boldsymbol{\pi}_1 & -\delta\rho\sigma_\pi^2\boldsymbol{\pi}_{1,2} \\ -\delta\rho\sigma_y^2\boldsymbol{\pi}_{1,2} & 1-\delta\rho\sigma_y^2\boldsymbol{\pi}_2 \end{pmatrix}^{-1} \\
&= \quad \frac{\delta}{\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})}\begin{pmatrix} \boldsymbol{\pi}_1 & \boldsymbol{\pi}_{1,2} \\ \boldsymbol{\pi}_{1,2} & \boldsymbol{\pi}_2 \end{pmatrix}\begin{pmatrix} 1-\delta\rho\sigma_y^2\boldsymbol{\pi}_2 & \delta\rho\sigma_\pi^2\boldsymbol{\pi}_{1,2} \\ \delta\rho\sigma_y^2\boldsymbol{\pi}_{1,2} & 1-\delta\rho\sigma_\pi^2\boldsymbol{\pi}_1 \end{pmatrix} \\
&= \quad \frac{\delta}{\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})}\begin{pmatrix} (1-\delta\rho\sigma_y^2\boldsymbol{\pi}_2)\boldsymbol{\pi}_1 \;+\; \delta\rho\sigma_y^2\boldsymbol{\pi}_{1,2}^2 & \boldsymbol{\pi}_{1,2} \\ \boldsymbol{\pi}_{1,2} & (1-\delta\rho\sigma_\pi^2\boldsymbol{\pi}_1)\boldsymbol{\pi}_2 \;+\; \delta\rho\sigma_\pi^2\boldsymbol{\pi}_{1,2}^2 \end{pmatrix} \\
&= \quad \frac{\delta}{\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})}\,\widehat{\boldsymbol{\Pi}}\,.
\end{aligned}$$

where

$$\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi}) \quad=\quad 1-\delta\rho(\sigma_\pi^2\boldsymbol{\pi}_1 \;+\; \sigma_y^2\boldsymbol{\pi}_2) \;+\; \delta^2\,\rho^2\,\det(\boldsymbol{\Pi})\,\sigma_\pi^2\sigma_y^2\,.$$

It is immediate to check that $\mathbf{B}'\widehat{\boldsymbol{\Pi}}\mathbf{B} = \gamma^2\,\hat{\boldsymbol{\pi}}_2$, so that

$$(\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1} \;=\; \frac{1}{\delta}\,\frac{1}{\gamma^2}\,\frac{1}{\hat{\boldsymbol{\pi}}_2}\det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})\,, \qquad \mathbf{B}(\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1}\mathbf{B}' \;=\; \det(\mathbf{I}_2-\delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\delta}\,\frac{1}{\hat{\boldsymbol{\pi}}_2} \end{pmatrix}.$$

Hence,

$$\mathbf{B}(\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}} \;=\; \begin{pmatrix} 0 & 0 \\ \frac{1}{\delta}\,\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} & 1 \end{pmatrix}, \qquad \mathbf{I}_2 \;-\; \mathbf{B}(\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}} \;=\; \begin{pmatrix} 1 & 0 \\ -\frac{1}{\delta}\,\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} & 0 \end{pmatrix}.$$

In the modified Riccati equation we have

$$
\begin{aligned}
\mathbf{\Pi} \;&=\; \mathbf{R} \;+ \mathbf{A}'\widetilde{\mathbf{\Pi}}\left(\mathbf{I}_2 \;-\; \mathbf{B}(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\,\widetilde{\mathbf{\Pi}}\right)\mathbf{A} \\
&=\; \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;+\; \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho \mathbf{N}\,\mathbf{\Pi})} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\pi}}_1 & \hat{\boldsymbol{\pi}}_{1,2} \\ \hat{\boldsymbol{\pi}}_{1,2} & \hat{\boldsymbol{\pi}}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&=\; \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;+\; \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho \mathbf{N}\,\mathbf{\Pi})} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} \frac{\det(\hat{\mathbf{\Pi}})}{\hat{\boldsymbol{\pi}}_2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&=\; \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \;+\; \delta\,\frac{\det(\hat{\mathbf{\Pi}})}{\det(\mathbf{I}_2 - \delta\,\rho \mathbf{N}\,\mathbf{\Pi})}\,\frac{1}{\hat{\boldsymbol{\pi}}_2} \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix}.
\end{aligned}
$$

Then we can define $W = \frac{1}{\det(\mathbf{I}_2 - \delta\,\rho \mathbf{N}\,\mathbf{\Pi})}\left(\hat{\boldsymbol{\pi}}_1 - \frac{\hat{\boldsymbol{\pi}}_{1,2}^2}{\hat{\boldsymbol{\pi}}_2}\right)$ and conclude that

$$
\boldsymbol{\pi}_1 \;=\; 1 + \delta\,W\,, \quad \boldsymbol{\pi}_{1,2} \;=\; \alpha\,\delta\,W\,, \quad \boldsymbol{\pi}_2 \;=\; \lambda + \alpha^2\,\delta\,W\,.
$$

Now,

$$
\begin{aligned}
\hat{\boldsymbol{\pi}}_1 - \frac{\hat{\boldsymbol{\pi}}_{1,2}^2}{\hat{\boldsymbol{\pi}}_2} \;&=\; \boldsymbol{\pi}_1 \;-\; \delta\,\rho\det(\mathbf{\Pi})\,\sigma_y^2 \;-\; \frac{\boldsymbol{\pi}_{1,2}^2}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2} \\
&=\; \frac{(\boldsymbol{\pi}_1 \;-\; \delta\,\rho\det(\mathbf{\Pi})\,\sigma_y^2)\,(\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2) - \boldsymbol{\pi}_{1,2}^2}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2} \\
&=\; \frac{\det(\mathbf{\Pi})\left[1 - \delta\rho(\sigma_\pi^2\boldsymbol{\pi}_1 \;+\; \sigma_y^2\boldsymbol{\pi}_2) \;+\; \delta^2\,\rho^2\det(\mathbf{\Pi})\,\sigma_\pi^2\sigma_y^2\right]}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2} \\
&=\; \frac{\det(\mathbf{\Pi})\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2}\,,
\end{aligned}
$$

so that $W = \frac{\det(\mathbf{\Pi})}{\boldsymbol{\pi}_2 - \delta\,\rho\det(\mathbf{\Pi})\,\sigma_\pi^2}$. Given the expressions for $\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_{1,2}$ and $\boldsymbol{\pi}_2$, we have that $\det(\mathbf{\Pi}) = \lambda + \delta(\alpha^2 + \lambda)W$, so that

$$
W \;=\; \frac{\lambda \;+\; \delta\,(\alpha^2 + \lambda)\,W}{\lambda\,(1 - \delta\rho\sigma_\pi^2) \;+\; \delta\left(\alpha^2 \;-\; \delta\,(\alpha^2 + \lambda)\,\rho\,\sigma_\pi^2\right)W}\,.
$$

Rearranging we find that

$$
\delta\left(\alpha^2 \;-\; \delta(\alpha^2 + \lambda)\rho\sigma_\pi^2\right)W^2 \;-\; \left(\delta(\alpha^2 + \lambda) \;-\; \lambda \;+\; \delta\lambda\rho\sigma_\pi^2\right)W - \lambda \;=\; 0
$$

whose roots are

$$W^{\pm} \quad = \quad \frac{\delta(\alpha^2+\lambda) \;-\; \lambda\,(1-\delta\rho\sigma_\pi^2) \;\pm\; \Delta^{1/2}}{2\,\delta\Big(\alpha^2 \;-\; \delta(\alpha^2+\lambda)\rho\sigma_\pi^2\Big)} \qquad \text{where}$$

$\Delta = \Big(\delta(\alpha^2+\lambda)-\lambda(1-\delta\rho\sigma_\pi^2)\Big)^2 + 4\delta\lambda(\alpha^2-\delta(\alpha^2+\lambda)\rho\sigma_\pi^2\Big)$. For $\rho = 0$, $\Delta = \Big(\delta\alpha^2-(1-\delta)\lambda)\Big)^2 + 4\alpha^2\delta\lambda$, while

$$W^{\pm} \quad = \quad \frac{1}{2}\left(1 - \frac{(1-\delta)\lambda \pm \Delta^{1/2}}{\alpha^2\delta}\right) \quad = \quad = \quad \frac{1}{2}\left(1 \;-\; \frac{(1-\delta)\lambda}{\alpha^2\delta} \;\pm\; \sqrt{\left(1 \;+\; \frac{(1-\delta)}{\alpha^2\delta}\right)^2 \;+\; \frac{4\,\lambda}{\alpha^2}}\right) .$$

Only the positive root will be coherent with the conditions that the matrix $\widetilde{\mathbf{\Pi}}$ is positive definite. This means that there is no indeterminacy in the stationary solution. To determine $\mathbf{K}$ consider that

$$\begin{aligned}
\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{A} \quad &= \quad \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}(0 \quad -\gamma) \begin{pmatrix} \hat{\boldsymbol{\pi}}_1 & \hat{\boldsymbol{\pi}}_{1,2} \\ \hat{\boldsymbol{\pi}}_{1,2} & \hat{\boldsymbol{\pi}}_2 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \\
&= \quad -\frac{\delta\,\gamma}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}\;(\hat{\boldsymbol{\pi}}_{1,2} \quad \alpha\hat{\boldsymbol{\pi}}_{1,2} \;+\; \beta\hat{\boldsymbol{\pi}}_2)\;.
\end{aligned}$$

Given that $\mathbf{K} = -(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{A}$, we find that

$$\mathbf{K} \quad = \quad \frac{1}{\gamma}\left(\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} \quad \alpha\frac{\hat{\boldsymbol{\pi}}_{1,2}}{\hat{\boldsymbol{\pi}}_2} \;+\; \beta\right) .$$

Finally, since $\hat{\boldsymbol{\pi}}_{1,2} = \boldsymbol{\pi}_{1,2} = \alpha\delta W$ and $\hat{\boldsymbol{\pi}}_2 = \boldsymbol{\pi}_2 - \delta\det(\mathbf{\Pi})\rho\sigma_\pi^2 = \lambda + \alpha^2\delta W - \delta(\lambda + \delta(\alpha^2+\lambda)W)\rho\sigma_\pi^2$, we find that

$$\mathbf{K} \quad = \quad \frac{1}{\gamma}\left(\frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \delta(\lambda+\delta(\alpha^2+\lambda)W)\rho\sigma_\pi^2} \qquad \beta \;+\; \frac{\alpha^2\delta W}{\alpha^2\delta W + \lambda - \delta(\lambda+\delta(\alpha^2+\lambda)W)\rho\sigma_\pi^2}\right) .$$

To reach a minimum $\delta\mathbf{\Pi}_{t+1} - (1/\rho)\mathbf{N}^{-1}$ must be negative definite. This corresponds to the double condition that

$$\delta\boldsymbol{\pi}_1 - \frac{1}{\rho}\frac{1}{\sigma_\pi^2} \;<\; 0\;, \qquad (\delta\boldsymbol{\pi}_1 - \frac{1}{\rho}\frac{1}{\sigma_\pi^2})\,(\delta\boldsymbol{\pi}_2 - \frac{1}{\rho}\frac{1}{\sigma_y^2}) \;-\; \delta^2\,\boldsymbol{\pi}_{1,2} \;>\; 0\,.$$

The Recursive Optimization Criterion and the Inflation Forecast. Given the plant equation for π_t we immediately see that $\pi_{t+1|t} = \pi_t + \alpha y_t$. Then, consider that

$$\begin{aligned}
\mathbf{z}_t'\mathbf{\Pi}\mathbf{z}_t &= (\pi_t \; y_t) \begin{pmatrix} 1+\delta W & \alpha\delta W \\ \alpha+\delta W & \lambda+\alpha^2\delta W \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= (\pi_t \; y_t) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + (\pi_t \; y_t) \begin{pmatrix} \delta W & \alpha\delta W \\ \alpha\delta W & \alpha^2\delta W \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= \pi_t^2 + \lambda y_t^2 + \delta\, W\, (\pi_t \; y_t) \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} \\
&= \pi_t^2 + \lambda y_t^2 + \delta\, W\, (\pi_t + \alpha y_t)^2 \,.
\end{aligned}$$

Immediately it follows that

$$\exp\left(\frac{\rho}{2}\boldsymbol{\mathcal{V}}_t\right) = \exp\left(\frac{\rho}{2}[\kappa + \pi_t^2 + \lambda\, y_t^2 + \delta\, W \pi_{t+1|t}^2]\right).$$

Inflation and Output Gap Forecast. Since $\pi_{t+1|t} = \pi_t + \alpha y_t$ we find that

$$r_t = \frac{1}{\gamma}\left(\beta\, y_t + \frac{\alpha\delta W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\, \pi_{t+1|t}\right)$$

Inserting this into the plant equation for output gap, we find that

$$y_{t+1|t} = -\frac{\alpha\,\delta\, W}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\pi_{t+1|t}\,.$$

Since $\pi_{t+2|t} = \pi_{t+1|t} + \alpha y_{t+1|t}$ and $\pi_{t+1|t} = -\frac{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}{\alpha\delta W}\, y_{t+1|t}$, we conclude that

$$\pi_{t+2|t} = -\frac{1}{\alpha\delta W}\left(\lambda - \theta\,\rho\,\sigma_\pi^2\right) y_{t+1|t} = \left(\frac{\lambda - \theta\,\rho\,\sigma_\pi^2}{\alpha^2\delta W + \lambda - \theta\rho\sigma_\pi^2}\right)\pi_{t+1|t}\,.$$

Unconditional Variance of Inflation, Output Gap and Short-term Interest Rate. By definition, considering that $\kappa_y = \beta/\gamma + \alpha\kappa_\pi$,

$$\begin{aligned}
\mathbf{\Gamma} &= \mathbf{A} + \mathbf{B}\,\mathbf{K} = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 \\ -\gamma \end{pmatrix} \begin{pmatrix} \kappa_\pi & \frac{\beta}{\gamma} + \alpha\,\kappa_\pi \end{pmatrix} \\
&= \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\gamma\kappa_\pi & -\beta - \alpha\gamma\kappa_\pi \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\gamma\kappa_\pi & -\alpha\gamma\kappa_\pi \end{pmatrix}.
\end{aligned}$$

Hence,

$$\mathbf{I}_2 - \boldsymbol{\Gamma} = \begin{pmatrix} 0 & -\alpha \\ \gamma\kappa_\pi & 1 + \alpha\gamma\kappa_\pi \end{pmatrix} \quad \text{so that} \quad \boldsymbol{\Lambda} = (\mathbf{I}_2 - \boldsymbol{\Gamma})^{-1} = \frac{1}{\alpha\gamma\kappa_\pi} \begin{pmatrix} 1 + \alpha\gamma\kappa_\pi & \alpha \\ -\gamma\kappa_\pi & 0 \end{pmatrix}.$$

Now, $\mathrm{Var}[\mathbf{z}_t] = \boldsymbol{\Lambda}\mathbf{N}\boldsymbol{\Lambda}'$. Then, consider that

$$\boldsymbol{\Lambda}\mathbf{N} = \frac{1}{\alpha\gamma\kappa_\pi} \begin{pmatrix} 1 + \alpha\gamma\kappa_\pi & \alpha \\ -\gamma\kappa_\pi & 0 \end{pmatrix} \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} = \frac{1}{\alpha\gamma\kappa_\pi} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 & \alpha\,\sigma_y^2 \\ -\gamma\kappa_\pi\,\sigma_\pi^2 & 0 \end{pmatrix}$$

so that

$$\begin{aligned} \boldsymbol{\Lambda}\mathbf{N}\boldsymbol{\Lambda}' &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 & \alpha\,\sigma_y^2 \\ -\gamma\kappa_\pi\,\sigma_\pi^2 & 0 \end{pmatrix} \begin{pmatrix} 1 + \alpha\gamma\kappa_\pi & -\gamma\kappa_\pi \\ \alpha & 0 \end{pmatrix} \\ &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\,\sigma_\pi^2 + \alpha^2\sigma_y^2 & -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 \\ -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 & \gamma^2\kappa_\pi^2\sigma_\pi^2 \end{pmatrix}, \end{aligned}$$

i.e.

$$\mathrm{Var}[\pi_t] = \frac{(1 + \alpha\gamma\kappa_\pi)^2}{(\alpha\gamma\kappa_\pi)^2}\,\sigma_\pi^2 + \frac{1}{(\gamma\kappa_\pi)^2}\,\sigma_y^2, \qquad \mathrm{Var}[y_t] = \frac{1}{\alpha^2}\,\sigma_\pi^2.$$

Finally, $\mathrm{Var}[r_t] = \mathbf{K}\mathrm{Var}[\mathbf{z}_t]\mathbf{K}'$. Then, consider that

$$\begin{aligned} \mathrm{Var}[\mathbf{z}_t]\,\mathbf{K}' &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\,\sigma_\pi^2 + \alpha^2\sigma_y^2 & -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 \\ -\gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 & \gamma^2\kappa_\pi^2\sigma_\pi^2 \end{pmatrix} \begin{pmatrix} \kappa_\pi \\ \frac{\beta}{\gamma} + \alpha\,\kappa_\pi \end{pmatrix} \\ &= \frac{1}{(\alpha\gamma\kappa_\pi)^2} \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\,\kappa_\pi\,\sigma_\pi^2 + \alpha^2\,\kappa_\pi\sigma_y^2 - \gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)\sigma_\pi^2 \\ -\gamma\kappa_\pi^2(1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 + \gamma^2\kappa_\pi^2\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)\sigma_\pi^2 \end{pmatrix} \end{aligned}$$

while

$$
\begin{aligned}
\mathbf{K}\mathrm{Var}[\mathbf{z}_t]\mathbf{K}' &= \frac{1}{(\alpha\gamma\kappa_\pi)^2}\begin{pmatrix} \kappa_\pi & \frac{\beta}{\gamma} + \alpha\,\kappa_\pi \end{pmatrix} \times \\
&\quad \begin{pmatrix} (1 + \alpha\gamma\kappa_\pi)^2\,\kappa_\pi\,\sigma_\pi^2 + \alpha^2\,\kappa_\pi\sigma_y^2 - \gamma\kappa_\pi(1 + \alpha\gamma\kappa_\pi)\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)\sigma_\pi^2 \\ -\gamma\kappa_\pi^2(1 + \alpha\gamma\kappa_\pi)\,\sigma_\pi^2 + \gamma^2\kappa_\pi^2\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)\sigma_\pi^2 \end{pmatrix} \\
&= \frac{1}{(\alpha\gamma\kappa_\pi)^2}\left(\alpha^2\kappa_\pi^2\sigma_y^2 + \left[(1+\alpha\gamma\kappa_\pi)^2\kappa_\pi^2 - 2\,(1+\alpha\gamma\kappa_\pi)\gamma\kappa_\pi^2\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)\right.\right. \\
&\quad \left.\left. + \gamma^2\kappa_\pi^2\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)^2\right]\sigma_\pi^2\right) \\
&= \frac{1}{\gamma^2}\,\sigma_y^2 + \frac{1}{(\alpha\,\gamma)^2}\left[(1+\alpha\gamma\kappa_\pi) - \gamma\left(\frac{\beta}{\gamma} + \alpha\,\kappa_\pi\right)\right]^2\sigma_\pi^2 \\
&= \frac{1}{\gamma^2}\left[\sigma_y^2 + \left(\frac{1-\beta}{\alpha}\right)^2\sigma_\pi^2\right].
\end{aligned}
$$

Optimal Monetary Policy with Imperfect State Observation. In the stationary solution, we find that

$$
\breve{\mathbf{z}}_t = (\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})^{-1}\hat{\mathbf{z}}_t, \quad \text{where}
$$

$$
(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})^{-1} = \frac{1}{\det(\mathbf{I}_2 - \rho\mathbf{N}\,\mathbf{\Pi})}\begin{pmatrix} 1 - \rho\sigma_y^2\boldsymbol{\pi}_2 & \rho\sigma_\pi^2\boldsymbol{\pi}_{1,2} \\ \rho\sigma_y^2\boldsymbol{\pi}_{1,2} & 1 - \rho\sigma_\pi^2\boldsymbol{\pi}_1 \end{pmatrix},
$$

so that

$$
\begin{aligned}
\breve{\pi}_t &= \left(\frac{1 - \rho\sigma_y^2\boldsymbol{\pi}_2}{\det(\mathbf{I}_2 - \rho\,\mathbf{N}\mathbf{\Pi})}\,\hat{\pi}_t + \frac{\rho\,\sigma_\pi^2\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \rho\,\mathbf{N}\mathbf{\Pi})}\hat{y}_t\right), \\
\breve{y}_t &= \left(\frac{\rho\,\sigma_y^2\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \rho\,\mathbf{N}\mathbf{\Pi})}\,\hat{\pi}_t + \frac{1 - \rho\sigma_\pi^2\boldsymbol{\pi}_1}{\det(\mathbf{I}_2 - \rho\,\mathbf{N}\mathbf{\Pi})}\hat{y}_t\right).
\end{aligned}
$$

Given that

$$
\begin{aligned}
\frac{1 - \rho\sigma_y^2\boldsymbol{\pi}_2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} &= 1 + \frac{\boldsymbol{\pi}_1 - \det(\mathbf{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \rho\mathbf{\Pi})}\,\rho\,\sigma_\pi^2, \\
\frac{1 - \rho\sigma_\pi^2\boldsymbol{\pi}_1}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})} &= 1 + \frac{\boldsymbol{\pi}_2 - \det(\mathbf{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\mathbf{\Pi})}\,\rho\,\sigma_y^2,
\end{aligned}
$$

we conclude that the MTSE is

$$\breve{\pi}_t \quad = \quad \hat{\pi}_t \; + \; \left(\frac{\boldsymbol{\pi}_1 - \det(\boldsymbol{\Pi})\rho\sigma_y^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})}\, \hat{\pi}_t \; + \; \frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})} \hat{y}_t \right) \rho\, \sigma_\pi^2 \, ,$$

$$\breve{y}_t \quad = \quad \hat{y}_t \; + \; \left(\frac{\boldsymbol{\pi}_{1,2}}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})}\, \hat{\pi}_t \; + \; \frac{\boldsymbol{\pi}_2 - \det(\boldsymbol{\Pi})\rho\sigma_\pi^2}{\det(\mathbf{I}_2 - \rho\mathbf{N}\boldsymbol{\Pi})} \hat{y}_t \right) \rho\, \sigma_y^2 \, .$$

Unconditional Variance of Inflation, Output Gap and Short-term Interest Rate Under Imperfect State Observation. By definition $\boldsymbol{\Psi} = \mathbf{B}\mathbf{K}_I$. As $\mathbf{B} = (0, -\gamma)$, we can write

$$\boldsymbol{\Psi} \; = \; \begin{pmatrix} 0 & 0 \\ -\gamma\kappa_\pi^I & -\gamma\kappa_y^I \end{pmatrix} .$$

Hence,

$$\mathbf{I}_2 \; - \; \boldsymbol{\Psi} \; = \; \begin{pmatrix} 1 & 0 \\ \gamma\kappa_\pi^I & 1 \, + \, \gamma\kappa_y^I \end{pmatrix} \quad \text{and} \quad (\mathbf{I}_2 \; - \; \boldsymbol{\Psi})^{-1} \; = \; \frac{1}{1 \, + \, \gamma\kappa_y^I} \begin{pmatrix} 1 \, + \, \gamma\kappa_y^I & 0 \\ -\gamma\kappa_\pi^I & 1 \end{pmatrix}$$

while

$$\boldsymbol{\Phi} \quad = \quad (\mathbf{I}_2 \; - \; \boldsymbol{\Psi})^{-1}\, \mathbf{A} \; = \; \frac{1}{1 \, + \, \gamma\kappa_y^I} \begin{pmatrix} 1 \, + \, \gamma\kappa_y^I & 0 \\ -\,\gamma\kappa_\pi^I & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \; = \; \begin{pmatrix} 1 & \alpha \\ \frac{-\,\gamma\kappa_\pi^I}{1+\gamma\kappa_y^I} & \frac{\beta\,-\,\alpha\gamma\kappa_\pi^I}{1+\gamma\kappa_y^I} \end{pmatrix} .$$

Then,

$$\boldsymbol{\Psi}\,\boldsymbol{\Phi} \; = \begin{pmatrix} 0 & 0 \\ -\,\gamma\kappa_\pi^I & -\,\gamma\kappa_y^I \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ \frac{-\,\gamma\kappa_\pi^I}{1+\gamma\kappa_y^I} & \frac{\beta\,-\,\alpha\,\gamma\kappa_\pi^I}{1+\gamma\kappa_y^I} \end{pmatrix} \; = \; \frac{1}{1 \, + \, \gamma\kappa_y^I} \begin{pmatrix} 0 & 0 \\ -\,\gamma\kappa_\pi^I & -\,\alpha\gamma\kappa_\pi^I \, - \, \beta\gamma\kappa_y^I \end{pmatrix} .$$

Therefore,

$$\begin{aligned} \mathbf{I}_2 \; - \; \mathbf{A} - \boldsymbol{\Psi}\,\boldsymbol{\Phi} \quad &= \quad \begin{pmatrix} 0 & -\,\alpha \\ 0 & 1 - \beta \end{pmatrix} \; - \; \frac{1}{1 \, + \, \gamma\kappa_y^I} \begin{pmatrix} 0 & 0 \\ -\,\gamma\kappa_\pi^I & -\,\alpha\gamma\kappa_\pi^I \, - \, \beta\gamma\kappa_y^I \end{pmatrix} \\ &= \quad \frac{1}{1 \, + \, \gamma\kappa_y^I} \begin{pmatrix} 0 & -\alpha(1 \, + \, \gamma\kappa_y^I) \\ \gamma\kappa_\pi^I & 1 \, - \, \beta \, + \, \alpha\gamma\kappa_\pi^I \, + \, \gamma\kappa_y^I \end{pmatrix} \end{aligned}$$

and

$$\boldsymbol{\Lambda}_I \; = \; (\mathbf{I}_2 \; - \; \mathbf{A} - \boldsymbol{\Psi}\,\boldsymbol{\Phi})^{-1} \; = \; \begin{pmatrix} 1 \, + \, \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I} & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I} \\ -\,\frac{1}{\alpha} & 0 \end{pmatrix} .$$

Then,

$$\mathbf{\Lambda}_I\,\mathbf{N} \;=\; \begin{pmatrix} 1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I} & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I} \\ -\frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right)\sigma_\pi^2 & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I}\sigma_y^2 \\ -\frac{1}{\alpha}\,\sigma_\pi^2 & 0 \end{pmatrix}$$

and hence

$$\begin{aligned} \mathrm{Var}^I[\mathbf{z}_t] \;&=\; \mathbf{\Lambda}_I\,\mathbf{N}\,\mathbf{\Lambda}_I' \;=\; \begin{pmatrix} \left(1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right)\sigma_\pi^2 & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I}\sigma_y^2 \\ -\frac{1}{\alpha}\,\sigma_\pi^2 & 0 \end{pmatrix} \begin{pmatrix} 1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I} & -\frac{1}{\alpha} \\ \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I} & 0 \end{pmatrix} \\ &=\; \begin{pmatrix} \left(1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right)^2\sigma_\pi^2 + \left(\frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I}\right)^2\sigma_y^2 & -\frac{1}{\alpha}\left(1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right)\sigma_\pi^2 \\ -\frac{1}{\alpha}\left(1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right)\sigma_\pi^2 & \frac{1}{\alpha^2}\,\sigma_\pi^2 \end{pmatrix}, \end{aligned}$$

from which we immediately conclude that $\mathrm{Var}^I[y_t] = \mathrm{Var}[y_t] = (1/\alpha^2)\sigma_\pi^2$. For the unconditional variance of the short-term interest rate, consider first that

$$\begin{aligned} \mathbf{\Phi}\,\mathbf{\Lambda}_I \;&=\; \begin{pmatrix} 1 & \alpha \\ \frac{-\gamma\kappa_\pi^I}{1+\gamma\kappa_y^I} & \frac{\beta-\alpha\gamma\kappa_\pi^I}{1+\gamma\kappa_y^I} \end{pmatrix} \begin{pmatrix} 1 + \frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I} & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I} \\ -\frac{1}{\alpha} & 0 \end{pmatrix} \\ &=\; \begin{pmatrix} \left(\frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right) & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I} \\ -\frac{1}{\alpha} & -1 \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned} \mathbf{K}_I\,\mathbf{\Phi}\,\mathbf{\Lambda}_I \;&=\; \begin{pmatrix} \kappa_\pi^I & \kappa_y^I \end{pmatrix} \begin{pmatrix} \left(\frac{1+\gamma\kappa_y^I-\beta}{\alpha\gamma\kappa_\pi^I}\right) & \frac{1+\gamma\kappa_y^I}{\gamma\kappa_\pi^I} \\ -\frac{1}{\alpha} & -1 \end{pmatrix} \\ &=\; \begin{pmatrix} \frac{1}{\alpha\gamma}\,(1-\beta) & \frac{1}{\gamma} \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbf{K}_I\,\mathbf{\Phi}\,\mathbf{\Lambda}_I\,\mathbf{N} \;&=\; \begin{pmatrix} \frac{1}{\alpha\gamma}\,(1-\beta) & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} \sigma_\pi^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \\ &=\; \begin{pmatrix} \frac{1}{\alpha\gamma}\,(1-\beta)\,\sigma_\pi^2 & \frac{1}{\gamma}\,\sigma_y^2 \end{pmatrix} \quad \text{and hence that} \end{aligned}$$

$$
\begin{aligned}
\mathrm{Var}^I[r_t] &= \mathbf{K}_I\,\boldsymbol{\Phi}\,\boldsymbol{\Lambda}_I\,\mathbf{N}\,\boldsymbol{\Lambda}_I'\,\boldsymbol{\Phi}'\,\mathbf{K}_I \\
&= \begin{pmatrix} \frac{1}{\alpha\gamma}\,(1-\beta)\,\sigma_\pi^2 & \frac{1}{\gamma}\,\sigma_y^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha\gamma}\,(1-\beta) \\ \frac{1}{\gamma} \end{pmatrix} \\
&= \frac{1}{(\alpha\gamma)^2}\,(1-\beta)^2\,\sigma_\pi^2 + \frac{1}{\gamma^2}\,\sigma_y^2\,,
\end{aligned}
$$

so that $\mathrm{Var}^I[r_t] = \mathrm{Var}[r_t]$.

Optimal Monetary Policy with Positive Inflation Target. For $\boldsymbol{\mu}_t = \boldsymbol{\mu}$,

$$
\boldsymbol{\vartheta}_t = \boldsymbol{\Gamma}_t'\,\widetilde{\boldsymbol{\Pi}}_{t+1}\,(\boldsymbol{\Pi}_{t+1}^{-1}\,\boldsymbol{\vartheta}_{t+1} - \boldsymbol{\mu})\,.
$$

In steady state,

$$
\begin{aligned}
\boldsymbol{\vartheta} &= \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,(\boldsymbol{\Pi}^{-1}\,\boldsymbol{\vartheta} - \boldsymbol{\mu}) \\
&= \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\Pi}^{-1}\,\boldsymbol{\vartheta} - \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\mu}\,, \quad \text{so that} \\
&= -\,(\mathbf{I} - \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\Pi}^{-1})^{-1}\,\boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\mu} \\
&= -\,[(\boldsymbol{\Pi} - \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}})\,\boldsymbol{\Pi}^{-1}\,]^{-1}\,\boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\mu} \\
&= -\,\boldsymbol{\Pi}\,(\boldsymbol{\Pi} - \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}})^{-1}\,\boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\mu}\,.
\end{aligned}
$$

For $\mathbf{Q} = \mathbf{0}$, $u_t = \mathbf{K}\,\mathbf{z}_t + (\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\boldsymbol{\Pi}}(\boldsymbol{\Pi}^{-1}\boldsymbol{\vartheta} - \boldsymbol{\mu})$, where

$$
\begin{aligned}
\boldsymbol{\Pi}^{-1}\,\boldsymbol{\vartheta} - \boldsymbol{\mu} &= -\,(\boldsymbol{\Pi} - \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}})^{-1}\,\boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,\boldsymbol{\mu} - \boldsymbol{\mu} \\
&= -\,[\mathbf{I} - (\boldsymbol{\Pi} - \boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}})^{-1}\,\boldsymbol{\Gamma}'\,\widetilde{\boldsymbol{\Pi}}\,]\,\boldsymbol{\mu}\,,
\end{aligned}
$$

so that $u_t = \mathbf{K}\,\mathbf{z}_t + u_e$, where $u_e = -(\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\boldsymbol{\mu} - (\mathbf{B}'\widetilde{\boldsymbol{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\boldsymbol{\Pi}}(\boldsymbol{\Pi} - \boldsymbol{\Gamma}'\widetilde{\boldsymbol{\Pi}})^{-1}\boldsymbol{\Gamma}'\widetilde{\boldsymbol{\Pi}}\boldsymbol{\mu}$. Now,

$$
(\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}}\,\mathbf{B})^{-1} = \frac{1}{\delta}\,\frac{1}{\gamma^2}\,\frac{1}{\hat{\pi}_2}\,\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})\,,
$$

while

$$
\begin{aligned}
\mathbf{B}'\,\widetilde{\boldsymbol{\Pi}} &= (0 \quad -\gamma)\,\frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})} \begin{pmatrix} \hat{\pi}_1 & \hat{\pi}_{1,2} \\ \hat{\pi}_{1,2} & \hat{\pi}_2 \end{pmatrix} \\
&= \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\boldsymbol{\Pi})} \begin{pmatrix} -\gamma\hat{\pi}_{1,2} & -\gamma\hat{\pi}_2 \end{pmatrix}\,,
\end{aligned}
$$

so that

$$-(\mathbf{B}'\,\widetilde{\mathbf{\Pi}}\,\mathbf{B})^{-1}\,\mathbf{B}'\,\widetilde{\mathbf{\Pi}} \;=\; \frac{1}{\gamma}\begin{pmatrix} \frac{\hat{\pi}_{1,2}}{\hat{\pi}_2} & 1 \end{pmatrix} \;=\; \begin{pmatrix} \kappa_\pi & \frac{1}{\gamma} \end{pmatrix},$$

in that $\kappa_\pi = \hat{\pi}_{1,2}/(\gamma\hat{\pi}_2)$. Hence, for $\boldsymbol{\mu} = \begin{pmatrix} 0 \\ \gamma\pi^* \end{pmatrix}$, $-(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\mathbf{\Pi}}\boldsymbol{\mu} = \pi^*$. In addition, consider that

$$\begin{aligned} \mathbf{\Gamma}'\widetilde{\mathbf{\Pi}} &= \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}\begin{pmatrix} 1 & -\gamma\,\kappa_\pi \\ \alpha & -\alpha\,\gamma\,\kappa_\pi \end{pmatrix}\begin{pmatrix} \hat{\pi}_1 & \hat{\pi}_{1,2} \\ \hat{\pi}_{1,2} & \hat{\pi}_2 \end{pmatrix} \\ &= \frac{\delta}{\det(\mathbf{I}_2 - \delta\,\rho\mathbf{N}\,\mathbf{\Pi})}\begin{pmatrix} \hat{\pi}_1 - \gamma\,\kappa_\pi\,\hat{\pi}_{1,2} & \hat{\pi}_{1,2} - \gamma\,\kappa_\pi\,\hat{\pi}_2 \\ \alpha\,(\hat{\pi}_1 - \gamma\,\kappa_\pi\hat{\pi}_{1,2}) & \alpha\,(\hat{\pi}_{1,2} - \gamma\,\kappa_\pi\,\hat{\pi}_2) \end{pmatrix}. \end{aligned}$$

Since $\kappa_\pi = \hat{\pi}_{1,2}/(\gamma\hat{\pi}_2)$, $\mathbf{\Gamma}'\widetilde{\mathbf{\Pi}} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. Then,

$$\mathbf{\Gamma}'\,\widetilde{\mathbf{\Pi}}\,\boldsymbol{\mu} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and hence}$$

$-(\mathbf{B}'\widetilde{\mathbf{\Pi}}\mathbf{B})^{-1}\mathbf{B}'\widetilde{\mathbf{\Pi}}(\mathbf{\Pi} - \mathbf{\Gamma}'\widetilde{\mathbf{\Pi}})^{-1}\mathbf{\Gamma}'\widetilde{\mathbf{\Pi}}\boldsymbol{\mu} = 0$. We then conclude that $u_e = \pi^*$ and that $\boldsymbol{\vartheta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This means that

$$\iota_t \;=\; \kappa_\pi\,\varsigma_t + \kappa_y\,y_t + \pi^* \quad \text{and} \quad F(\mathbf{z}_t) \;=\; \mathbf{z}_t'\,\mathbf{\Pi}\,\mathbf{z}_t\,.$$

MLE for the state vector $\mathbf{z}_t$ with Positive Inflation Target. Given the plant equation, $\hat{\mathbf{z}}_t = \mathbf{A}\,\mathbf{z}_{t-1} + \mathbf{B}\,\mathbf{u}_{t-1} + \boldsymbol{\mu}_t$, where $\mathbf{z}_t' = (\pi_t - \pi^*\;\, y_t)$ and $\boldsymbol{\mu}_t' = (0\;\, \gamma\pi^*)$. Given $\mathbf{A}$ and $\mathbf{B}$ we have that

$$\begin{aligned} \widehat{\pi_t - \pi^*} &= \pi_{t-1} - \pi^* + \alpha\,y_t\,, \\ \hat{y}_t &= \beta\,y_{t-1} - \gamma\,\iota_{t-1} + \gamma\,\pi^*\,. \end{aligned}$$

Since $\iota_t = r_t + \pi^*$, we conclude that

$$\begin{aligned} \hat{\pi}_t &= \pi_{t-1} + \alpha\,y_t\,, \\ \hat{y}_t &= \beta\,y_{t-1} - \gamma\,r_{t-1}\,. \end{aligned}$$

Optimal Production Policy for Risk-averse Monopolist. Given that $\mathbf{Q} = q$, $\mathbf{R} = 0$, $\mathbf{S} = -1/2$, $\mathbf{A} = a$, $\mathbf{B} = b$, $\mathbf{z} = p$ and $\mathbf{u} = x$, the continuous-time risk-sensitive Riccati equation is

$$\frac{d\,\pi(t)}{d\,t} + 2\,a\,\pi(t) - \frac{1}{q}\left(b\,\pi(t) - \frac{1}{2}\right)^2 + \rho\,\sigma_\epsilon^2\,\pi(t)^2 + \log\delta\,\pi(t) \;=\; 0$$

which we can re-write as $\frac{d\,\pi(t)}{d\,t} = h_0 + h_1\pi(t) + h_2\pi(t)^2$, with $h_0 = 1/(4q)$, $h_1 = -(2a + b/q + \log\delta)$, $h_2 = b^2/q - \rho\sigma_\epsilon^2$, and transformed into a homogeneous ordinary differential equation of order two,

$$\frac{d^2\,y(t)}{d^2\,t} \;-\; h_1\,\frac{d\,y(t)}{d\,t} \;+\; h_0\,h_2\,y(t) \;=\; 0\,, \quad \text{with} \quad \pi(t) \;=\; -\frac{1}{h_2}\,\frac{\frac{d\,y(t)}{d\,t}}{y(t)}\,.$$

Assume then that $y(t) = m\,\exp(\zeta\,t)$. We have a solution of the ODE iff

$$\begin{aligned} \zeta^2\,m\,\exp(\zeta\,t) \;-\; \zeta\,h_1\,m\,\exp(\zeta\,t) \;+\; h_0\,h_2\,m\,\exp(\zeta\,t) &\;=\; 0\,, \quad \text{i.e. iff} \\ m\,\zeta^2 \;-\; m\,h_1\,\zeta \;+\; m\,h_0\,h_2 &\;=\; 0\,. \end{aligned}$$

This admits two roots equal to $\zeta \;=\; \begin{cases} \zeta_1 \;=\; \frac{1}{2}\,h_1 \;+\; \frac{1}{2}\,\sqrt{\mathcal{D}} \\ \zeta_2 \;=\; \frac{1}{2}\,h_1 \;-\; \frac{1}{2}\,\sqrt{\mathcal{D}} \end{cases}$, with $\mathcal{D} = h_1^2 - 4h_0h_2$. Thus, $y(t) = m_1\,\exp(\zeta_1\,t) + m_2\,\exp(\zeta_2\,t)$. Given that $\pi(t) = -\frac{1}{h_2}\,\frac{\frac{d\,y(t)}{d\,t}}{y(t)}$, we can write that

$$\pi(t) \;=\; -\;\frac{m_1\,\zeta_1\,\exp(\zeta_1\,t) + m_2\,\zeta_2\,\exp(\zeta_2\,t)}{(\frac{b^2}{q} \;-\; \rho\,\sigma_\epsilon^2)\,(m_1\,\exp(\zeta_1\,t) + m_2\,\exp(\zeta_2\,t))}\,.$$

We can impose the terminal condition $\pi(T) = 0$ to find that

$$m_1\,\zeta_1\,\exp(\zeta_1\,T) + m_2\,\zeta_2\,\exp(\zeta_2\,T) \;=\; 0 \quad\Leftrightarrow\quad m_2 \;=\; -\frac{\zeta_1}{\zeta_2}\,m_1\,\exp((\zeta_1 - \zeta_2)\,T) \;=\; -\frac{\zeta_1}{\zeta_2}\,m_1\,\exp(\sqrt{\mathcal{D}}\,T)\,.$$

Re-inserting this expression in that for $\pi(t)$ we find that

$$\begin{aligned} \pi(t) &\;=\; -\;\frac{1}{(\frac{b^2}{q} \;-\; \rho\,\sigma_\epsilon^2)}\left(\frac{\zeta_1\,\exp(\zeta_1\,t) \;-\; \zeta_1\,\exp(\sqrt{\mathcal{D}}\,T)\,\exp(\zeta_2\,t)}{\exp(\zeta_1\,t) \;-\; \frac{\zeta_1}{\zeta_2}\,\exp(\sqrt{\mathcal{D}}\,T)\,\exp(\zeta_2\,t)}\right) \\ &\;=\; -\;\frac{\zeta_1}{(\frac{b^2}{q} \;-\; \rho\,\sigma_\epsilon^2)}\left(\frac{1 \;-\; \exp(\sqrt{\mathcal{D}}\,T)\,\exp(-(\zeta_1 - \zeta_2)\,t)}{1 \;-\; \frac{\zeta_1}{\zeta_2}\,\exp(\sqrt{\mathcal{D}}\,T)\,\exp(-(\zeta_1 - \zeta_2)\,t)}\right) \\ &\;=\; -\;\frac{\zeta_1}{(\frac{b^2}{q} \;-\; \rho\,\sigma_\epsilon^2)}\left(\frac{1 \;-\; \exp(-\sqrt{\mathcal{D}}\,(t - T))}{1 \;-\; \frac{\zeta_1}{\zeta_2}\,\exp(-\sqrt{\mathcal{D}}\,(t - T))}\right) \\ &\;=\; -\;\frac{\zeta_1}{(\frac{b^2}{q} \;-\; \rho\,\sigma_\epsilon^2)}\left(\frac{\exp(\sqrt{\mathcal{D}}\,(T - t)) \;-\; 1}{\frac{\zeta_1}{\zeta_2}\,\exp(\sqrt{\mathcal{D}}\,(T - t)) \;-\; 1}\right), \end{aligned}$$

as $\zeta_1 - \zeta_2 \;=\; \mathcal{D}$. Notice, that for $\frac{b^2}{q} \;>\; \rho\sigma_\epsilon^2$ $0 \;<\; \zeta_2 \;<\; \zeta_1$. It immediately follows that $\pi(t) \;<\; 0$, while for $\frac{b^2}{q} \;<\; \rho\sigma_\epsilon^2$, $\zeta_2 \;<\; 0 \;<\; \zeta_1$. Even in this case $\pi(t) \;<\; 0$ as $\frac{\zeta_1}{(\frac{b^2}{q} - \rho\,\sigma_\epsilon^2)}$ and $\frac{\exp(\sqrt{\mathcal{D}}\,(T-t))-1}{\frac{\zeta_1}{\zeta_2}\,\exp(\sqrt{\mathcal{D}}\,(T-t))-1}$ change sign. Notice that for $\frac{b^2}{q} = \rho\sigma_\epsilon^2$, $\zeta_2 = 0$ and the solution turns degenerate and collapses to the static solution. In fact, in this case $\zeta_2 = 0$ and $\zeta_1 = h_1$, so that $y(t) = m_1\,\exp(h_1 t) + m_2$ and hence $d\,y(t)/d\,t = m_1 h_1\exp(h_1\,t)$. Then, the terminal condition $\pi(T) = 0$ entails that $m_1 = 0$ and this on turn implies that $\pi(t) = 0$.