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State Dependent Monetary Policy

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Abstract

We study the optimal anticipated monetary policy in a flexible-price economy featuring heterogenous agents and incomplete markets which give rise to a business cycle. The optimal policy prescribes monetary expansions in recessions, when insurance is most needed by cash-poor unproductive agents. To minimize the inflationary effect of these expansions the policy prescribes monetary contractions in good times. Although the optimal monetary policy varies greatly through the business cycle it "echoes" Friedman's principle in the sense that the money supply is regulated such that its expected real return approaches the rate of time preference.

JEL classification: E50.

Keywords: Incomplete markets, Heterogenous agents, Liquidity, Precautionary savings, Friedman rule, State dependent policy.

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1 Introduction

We study optimal monetary policy in a competitive flexible-price economy where infinity-lived agents are subject to idiosynchratic productivity shocks and money is valued in equilibrium due to anonymity. The state of this economy is described by the wealth distribution, which evolves through time following the history of shocks, and determines the value of money and aggregate output. Our objective is to characterize how to optimally regulate the money supply as a function of the state of the economy, what we call an optimal state-dependent monetary policy. A key feature of the setup is that monetary policy affects the wealth distribution, as in Grossman and Weiss (1983) or Weil (1991) and many other monetary models whose first principles are explicitly spelled out. Although the propagation of such redistributive effects of monetary policy is often "muted" by means of appropriate assumptions for the sake of tractability, as in Lucas (1990), Shi (1997) or Lagos and Wright (2005), in this paper we use an analytically tractable setup that allows us to study the role of systematic monetary policy taking fully into account the dynamics of the wealth distribution. The key assumption to get tractability, following Scheinkman and Weiss (1986), is that we consider the simplest economy with time varying wealth distribution, namely one with 2 types of agents. We see this as a convenient starting point to study the interactions between the dynamics of the wealth distribution and monetary policy.

We think the question is interesting because it is novel in the theory and because the analysis provides a framework to interpret the large monetary expansions, sometimes observed during deep recessions, with a mechanism that is completely different from the canonical one relying on sticky prices.[1] The properties of an optimal monetary policy in models where incomplete markets and heterogenous agents allow for a potential redistributive role of monetary policy were first studied by Levine (1991). They have since been explored in a variety of contexts by e.g. Kehoe, Levine and Woodford (1990), Molico (2006), Algan, Challe and Ragot (2011), and Tenreyro and Sterk (2013). A common feature of these models is a tension between the benefits of a deflationary policy, one that produces an efficient return on money as under the Friedman's rule, and the benefits of an expansionary policy, which provides partial insurance to cash-poor agents. A novelty of this paper is that while previous models focused on a constant rule, i.e. seeking the optimal constant rate of monetary expansions, we consider a state-dependent monetary policy, in which the rate of monetary expansion depends on the state of the economy.

Our model extends Scheinkman and Weiss's (1986) analysis, which assumes a constant

[1] The assumption of flexible prices is useful to emphasize the workings of the redistributive role of monetary policy, and to distinguish it from the better understood mechanism that arises with sticky prices. Both channels are likely relevant in practice.

money supply, by letting the government control the money supply through lump-sum transfers.[2] We provide a characterization of the price of money and of aggregate production in terms of the policy rule in a competitive equilibrium. We adopt an ex-ante welfare criterion and characterize an efficient monetary policy by solving a Ramsey problem.

Our results cast some light on the interactions between the dynamics of the assets distribution and the optimal anticipated policy. The main policy choice is a tradeoff between providing insurance (through monetary expansions) and ensuring an efficient return on savings (through contractions). We show that in our model the importance of the insurance motive varies with state of the economy, so that a state-dependent monetary policy allows for a significant improvement compared to a constant policy. An expansionary monetary policy turns out to be efficient in recessions, when poor and unproductive economic agents benefit from some wealth redistribution. Surprisingly, in spite of the occasional large monetary expansions in states where the insurance motive is large, the optimal policy neutralizes the inflationary effect of these expansions by contracting the money supply in states where the insurance motive is small. In this way the state-dependent nature of the optimal policy allows for the provision of insurance when mostly needed without severely distorting the return of the asset. Thus, although our setup creates a potentially beneficial role for monetary expansions, the optimal rule prescribes an almost complete undoing of the inflationary effects of those expansions, and implements a policy that brings the expected return on money as close as possible to the constant rate prescribed by the Friedman rule.

Our research question is related to the one analyzed by Molico (2006) who considers a search model of money with a non degenerate distribution of cash holdings, showing that mild monetary expansions can be beneficial. In his model randomly matched agents may exchange goods for money. The price paid by the buyer results from bargaining and depends on the amount of money held by each agent upon entering the pairwise meeting. Therefore, the distribution of money is non-degenerate and monetary injections, via lump-sum transfers, can improve the terms of trade of poor buyers. Related results in the context of search models of money and mechanism design are obtained by Berentsen, Camera and Waller (2005), Green and Zhou (2005), Deviatov and Wallace (2012), Wallace (2012). Our model departs from Molico in a few important ways. First we restrict attention to an economy with only two types of agents, so that the wealth distribution is analytically tractable, while Molico does not restrict the number of types and as a consequence his model must be solved numerically. Second, and perhaps more importantly, we allow the planner to tie its policy to

[2] As in Levine (1991) we assume that the government does not know which agent is productive, so that the transfers are equal across agents. See Kehoe, Levine and Woodford (1990) for a thorough discussion of this assumption and in particular Levine (1991) for a derivation of the equal-treatment restriction from first principles.

the distribution of cash holdings, while Molico considers a constant policy. A key feature of our model is that business cycles, and the magnitude of fluctuations, depend on the tightness of the borrowing constraint. Because the borrowing constraint is tighter in downturns the return on money is high in recessions and low during booms. As a result inflation (the inverse of the return on money) is positively correlated with aggregate activity, thus generating a "Phillips curve". This result relates to Guerrieri and Lorenzoni (2009) and Guerrieri and Lorenzoni (2011) who explore the effects of borrowing constraints on business cycles in a model with liquid assets. Similar to their papers, as the borrowing constraint becomes tighter, economic fluctuations become more severe in our model. The relative simplicity of our setup allows us to investigate the optimal provision of liquidity. Our analysis is also related to Algan, Challe and Ragot (2011) who characterize the output-inflation tradeoff in a flexible price economy with incomplete markets and persistent wealth inequality among agents. While the setups are similar, the focus is different as monetary policy is treated as an exogenous parameter in these models.

The paper is organized as follows. Section 2 presents the set up of the model. Section 3 sets the ground for the analysis of monetary equilibria. Section 4 characterizes the value of money in equilibrium under a state-dependent rule and compares it to the one with a constant money rule. Section 5 defines an ex ante welfare criterion and studies the best state dependent rule for the supply of liquidity. Section 6 concludes.

2 The model

This section describes the model economy: agents' preferences, production possibilities, and markets. Two useful benchmarks are presented: the (efficient) allocation with complete markets and the optimal monetary policy with no uncertainty. To finalize, we also argue that value functions and allocations are homogeneous in the exogenous parameters of interest; this is useful as it allow us to reduce the dimensionality of the problem by doing an appropriate normalization.

We consider two types of infinitely lived agents (with a large mass of agents of each type), indexed by $i = 1, 2$, and assume that at each point in time only one type of agent can produce. We further restrict attention to the case where agents of the same type play the same action at every point in time so that we can discuss the model in terms of two representative agents, one of each type. Because there are two agents in this economy, we can solve the model by looking at the problem from the perspective of agent one. Let $\mathcal{I}_t^s = \{0, 1\}$ denote whether the agent is productive, $\mathcal{I}_t^s = 1$, or unproductive, $\mathcal{I}_t^s = 0$, at time t. When productive, the agent transforms labor into consumption one for one. When unproductive, the agent cannot produce. The productivity of labor is state dependent: the duration of productivity spells is random, exponentially distributed, with mean duration $1/\lambda > 0$. Money is distributed at each time t between the two agents so that $m_t^1 + m_t^2 = M_t$. The growth rate of the money supply at time t is μ_t; then, the money supply follows $M_t = M_0 e^{\int_0^t \mu_i di}$, with M_0 given. As in Scheinkman and Weiss (1986) and Levine (1991), we let the individual state of an agent to be private information, precluding agents from issuing private debt.[3] A key assumption is that agents face a borrowing constraint restricting their unique savings instrument, money, to be non-negative. Because of the assumption of anonymity fiscal policy has limited powers in this setup.[4]

Let $\rho > 0$ denote the time discount rate. The agent chooses consumption c_t, labor supply ℓ_t, and depletion of money balances $\dot{m}_t$, in order to maximize her (time-separable) expected discounted utility,

$$\max_{\{c_t, \ell_t, \dot{m}_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left(\bar{c} \ln c_t \; - \; \mathcal{I}_t^s \; \ell_t \right) \; dt \right\} \tag{1}$$

subject to the constraints

$$\tilde{q}_t \; \dot{m}_t \; \leq \; \ell_t \; + \; \tau_t \; - \; c_t \qquad\qquad \text{if } \mathcal{I}_t^s = 1 \tag{2}$$

$$\tilde{q}_t \; \dot{m}_t \; \leq \; \tau_t \; - \; c_t \quad \text{and} \quad \ell_t \; = \; 0 \qquad\qquad \text{if } \mathcal{I}_t^s = 0 \tag{3}$$

$$m_t \; \geq \; 0 \;\; , \;\; \ell_t \; \geq \; 0 \;\; , \;\; c_t \; \geq \; 0 \;\; , \;\; M_0, \; m_0^1, \; \mathcal{I}_0^s \text{ given} \; , \; \bar{c} > 0 \tag{4}$$

where $\tilde{q}_t$ denotes the price of money, i.e. the inverse of the consumption price level, τ_t denotes a government lump-sum transfer to each agent, and expectations are taken with respect to the productivity process defining $\mathcal{I}_t^s$ and M_t conditional on time $t = 0$.

A monetary policy with $\mu_t > 0$ is called expansionary, a policy with $\mu_t < 0$ is called contractionary. It is immediate that when the money supply is constant for all t (i.e. $\mu_t = 0 \; \forall \; t$) and $\bar{c} = 1$, the economy is the one analyzed by Scheinkman and Weiss (1986). The monetary policy μ_t determines the transfers to the agents τ_t through the government budget constraint,

$$\tilde{q}_t \; \mu_t \; m_t = 2\tau_t \; .$$

The government transfer scheme implies that in the case of a contractionary policy agents must use their money holdings to pay taxes (i.e. $\tau_t < 0$). The "tax solvency" constraint, $m_t^1 \geq 0$, imposes this restriction. Notice that in the continuous time characterization of the

[3]Having a large mass of agents of each type is important for the argument as it implies that a single agent cannot infer the productive state of a different agent given his own state.

[4] Appendix ?? discusses what allocations can be achieved using tax policy under various assumptions about government powers (commitment vs. no commitment), types of available taxes (lump-sum vs. distortionary), and government knowledge about the state (agent's type observable vs. not observable).

model the tax solvency constraint coincides with the borrowing constraint.

Note that the government cannot differentiate transfers across agent types. This follows from the assumption that the identity of the productive type is not known to the government. Levine (1991) shows in a similar setup that, because of anonymity, the best mechanism is linear and resembles monetary policy.

Next we state two important remarks. The first one characterizes a symmetric efficient allocation with complete markets (the proof is standard so we omit it):

Remark 1 *Assume complete markets and an ex-ante equal probability of each productive state. The symmetric efficient allocation prescribes the same constant level of consumption, $c_t = \bar{c}$ for all t.*

Thus without borrowing constraints the efficient allocation solves a static problem, and it encodes full insurance: agents consume a constant amount $\bar{c}$ (since ex-ante agents are equal) and the aggregate output is constant.

The second remark characterizes the optimal monetary policy in the case of no uncertainty. This helps highlighting the essential role of uncertainty in our problem. In particular, consider the case where each agent is productive for T time, and then becomes unproductive for the next T time. Without loss of generality, for the characterization of the stationary equilibria, let us assume that the economy starts in period $t = 0$ with the agent being productive and holding no money, so that $m^1_{t=0} = 0$. We have

Remark 2 *Consider a deterministic production cycle of length T. The symmetric efficient allocation, $c_t = \bar{c}$ for all t is attained by deflating at the rate of time preference $\mu_t = -\rho$ for all t.*

This remark, together with the efficient allocation described in Remark 1, shows that without uncertainty this economy replicates Townsend (1980), Bewley (1980) result on the optimality of the "Friedman rule" (see Appendix **??** for a proof).

To conclude notice that, by inspection of the agent problem presented in equations (1) to (4) and the evolution of money supply ($M_t = M_0 e^{\int_0^t \mu_i di}$), it is easily seen that the problem is homogeneous on $\{\lambda, \rho, \mu_t\}$: allocations (the flows) are homogeneous of degree 0 while prices and values (the stocks) are homogeneous of degree minus 1. This result is a natural consequence of the Poisson rate of changing states λ, the discount rate ρ, and the monetary expansion rate μ_t all being measured with respect to calendar time. This result is useful as it shows that, after normalizing by λ, the model has only three parameters: the normalized discount rate, ρ/λ, the normalized money growth rate, μ_t/λ, and $\bar{c}$. Later on we show that implementing the optimal allocation in this economy is independent of the value of $\bar{c}$ and, as a result, when we treat μ_t as a policy instrument, the model has a unique exogenous parameter given by the normalized discount rate ρ/λ.

3 Characterization of monetary equilibrium

We look for an equilibrium where the price of money depends on the whole history of shocks, as encoded in the current values of the money supply, the distribution of money holdings, and the current state of productivity; that is, we let $\tilde{q}_t = \tilde{q}(M_t, m_t^1, \mathcal{I}_t^s)$. With a slight abuse of notation this implies $c_t = c(M_t, m_t^1, \mathcal{I}_t^s)$, $\ell_t = \ell(M_t, m_t^1, \mathcal{I}_t^s)$, and $\dot{m}_t = (M_t, m_t^1, \mathcal{I}_t^s)$.

As usual the nominal variables are homogenous of degree one in the level of money, so that the state space is simplified by noting that the price of money satisfies $\tilde{q}(M_t, m_t^1, \mathcal{I}_t^s) = \frac{1}{M}\hat{q}(z_t, \mathcal{I}_t^s)$, where $z_t \equiv \frac{m_t^1}{M_t}$. The variable $z_t \in [0,1]$ is the share of total money balances in the hands of the agent, i.e. a measure of the wealth distribution. Likewise the consumption and labor supply rules are homogeneous of degree zero in the level of the money supply, $c(M_t, m_t^1, 1) = c^p(z_t)$, $c^1(M_t, m_t^1, 0) = c^u(z_t)$, and $\ell(M_t, m_t^1, 1) = \ell^p(z_t)$.

Let x_t denote the wealth share in the hands of the unproductive agent. Note that this variable will record discrete jumps every time the identity of the productive type changes: whenever the identity of the productive type switches, the state x jumps. We allow the planner to choose a monetary policy μ_t that is Markovian. As a result, notice that x_t summarizes the whole history of the economy and, without loss of generality, $\mu_t = \mu(x_t)$. Given the symmetry of the problem we let $q(x_t)$ denote the price of money in terms of consumption units which occurs when the unproductive type assets is x_t. Next we define a monetary equilibrium.

Definition 1 *For a given policy rule $\mu(x_t)$, initial level of money supply M_0, initial productivity status $\mathcal{I}_0^s$, and an initial distribution of money holdings x_0, a monetary equilibrium is a price function $\tilde{q}_t = \frac{1}{M_t}\, q(x_t)$, with $q : [0,1] \to \mathbb{R}^+$ and a stochastic process x_t with values in $[0,1]$ such that, for all t, consumers maximize expected discounted utility (equation (1)) subject to (2), (3) and (4), and the market clearing constraint $c^p(1-x_t) + c^u(x_t) = \ell(1-x_t)$ and the government budget constraint are satisfied.*

From now on we omit the time index t to simplify the notation. A straightforward result is that permanent deflations cannot be implemented in equilibrium.[5] We state this result in the next proposition.

[5] This result relates to Bewley (1983) who showed that in a neoclassical growth model with incomplete markets there is no monetary equilibrium if the interest rate is lower than the discount rate.

Proposition 1 *There is no monetary equilibrium where $\mu(x) < 0$ for all x. Moreover, all monetary equilibria must satisfy $\mu(0) \geq 0$.*

See Appendix ?? for a proof. The economics of this result is simple. As the length of the unproductive spell cannot be bounded above, there is a nonzero probability that a poor unproductive agent fails to cover her tax needs. The only way she can fulfill her tax obligations is by keeping half of the money stock and not trading for goods. Because of no trade, money has no value (i.e. $q(x) = 0$ for all x), there is no monetary equilibrium, and the allocation is autarkic. Moreover for any rule, including those that may allow for monetary contractions, the money growth rate cannot be negative at $x = 0$. This is immediate since when $x = 0$ unproductive agents hold no assets and are unable to cover their tax obligations.

Solving the model requires characterizing the marginal value of money given by the Lagrange multipliers for $\dot{m}$ in the problem defined in (1). Let $\tilde{p}(M, m^1)$ and $\tilde{u}(M, m^1)$ denote the (un-discounted) multipliers associated to the constraints in equation (2) and (3), respectively, so that e.g. $\tilde{u}(M, m^1)$ measures the marginal value of money for agent 1 when the money supply is M, her wealth share is m^1/M and she is unproductive. Likewise, $\tilde{p}(M, m^1)$ measures the marginal value for agent 1 when her wealth share is m^1/M and she is productive. Using the homogeneity in the level of money M we can write $\tilde{u}(M, m^1) = u(z)/M$ and $\tilde{p}(M, m^1) = p(z)/M$.

Combining the first order conditions with respect to ℓ and c^u give

$$p(z) \;=\; q(x) \quad , \quad \frac{\bar{c}}{c^p(z)} = \frac{p(z)}{q(x)} \quad \text{and} \quad \frac{\bar{c}}{c^u(z)} \;=\; \frac{u(z)}{q(x)} \;, \tag{5}$$

where z is the share of money in the hands of the agent and x is the share of money in the hands of the unproductive agent. These conditions equate marginal costs and benefits of an additional unit of money. The first two equations applies when the agent is productive (i.e. $\mathcal{I}^s = 1$). The first one states that the marginal benefit of an a additional unit of money, $p(z)$, equals the cost of obtaining that unit, i.e. the disutility of work to produce and sell a consumption amount $q(x)$. The second equation states that the marginal cost of the foregone unit of money, which is $p(z)$, equals the marginal benefit, which is given by the product of the price $q(x)$ (consumption per unit of money) times the marginal utility of consumption $\bar{c}/c^p(z)$. Notice that combining these two equations imply that $c^p(z) = 1$ for all z. Finally, the third equation applies when the agent is unproductive (i.e. $\mathcal{I}^s = 0$) and states that the marginal cost of the foregone unit of money $u(z)$, equals the benefit which is given by the additional units of consumption that can be bought with it: the product of the price $q(x)$ times the marginal utility of consumption $\bar{c}/c^u(z)$.

The following functions define the evolution of money holdings,

$$\dot{z}^u(z) = h^u(z) \qquad \text{and} \qquad \dot{z}^p(z) = h^p(z) \tag{6}$$

where $h^u(z)$ is the change in the share of money holdings of an unproductive agent holding a share z, and $h^p(z)$ is the analogue for a productive agent. It is immediate that $h^u(z) + h^p(1-z) = 0$. Consider the law of motion for z by type 1 when unproductive:

$$h^u(z) = \mu(x)\left(\frac{1}{2} - z\right) - \frac{c^u(z)}{q(x)} = \mu(x)\left(\frac{1}{2} - z\right) - \frac{\bar{c}}{u(z)} \tag{7}$$

where we used the budget constraint of the unproductive agent, equation (3), the government budget constraint, and the first order condition in equation (5).

For any $z \in (0,1)$ the marginal value of money for productive and unproductive agents, $p(z)$, and $u(z)$, solve a system of differential equations, which is the continuous time counterpart of the discrete time Euler equations,

$$(\rho + \mu(x))\ p(z) \quad = \quad p'(z)\ h^p(z) + \lambda(u(z) - p(z)) \tag{8}$$

$$(\rho + \mu(x))\ u(z) \quad = \quad u'(z)\ h^u(z) + \lambda(p(z) - u(z)) \tag{9}$$

The derivation is standard so we omit it. To provide some intuition consider the first equation: when the agent is productive and holds a share of money z the value flow (discounted by the nominal rate $(\rho + \mu(x))$) is equal to the change in the marginal value due to the evolution of her money holdings, $p'(z)h^p(z)$, and to the expectations of the change in value in case the state switches and the agent becomes unproductive: $\lambda(u(z) - p(z))$.

To complete the description of the equilibrium we provide the boundary condition for the marginal value of money $p(z)$ and $u(z)$. The boundary occur when the unproductive agent has no money. In this case an unproductive agent spends the whole money transfer to finance her consumption, so that $h^u(0) = h^p(1) = 0$.[6] The budget constraint gives that the consumption of an unproductive agent with no money is $\lim_{z\to 0} c^u(z) = \tau(0) = q(0)\mu(0)/2$. Using equation (5),

$$\lim_{z\to 0} u(z) = \frac{2}{\mu(0)} \tag{10}$$

with $lim_{z\to 0}u(z) = \infty$ if $\lim_{x\to 0}\mu(x) = 0$, and where the limit obtains because of Inada conditions. This is an important result in our analysis. An expansionary policy provides an upper bound to the marginal utility of money because the agent enjoys a positive consumption even with no wealth. If there is no money growth when the unproductive agent is poor (i.e.

[6] We provide a formal proof of this statement in Section 4.

when $\mu(x) \to 0$ as $x \to 0$), the agent is not able to consume in poverty and therefore Inada conditions imply that her marginal utility diverges. Also, evaluating equation (8) at $z = 1$ and $x = 0$ gives

$$u(1) = \left(1 + \frac{\rho}{\lambda} + \frac{\mu(0)}{\lambda}\right) p(1) \ . \tag{11}$$

Notice that $\mu(0)$, the money growth rate when the unproductive agent has zero wealth, appears in both boundaries. An implication is that the choice of the money growth rate simultaneously affects the insurance needs of the unproductive agents and the production incentives of the productive agents.

Using the first order conditions presented in equation (5) yields an expression for the optimal consumption for an unproductive agent,

$$c^u(z) = \bar{c}\ \frac{p(1-z)}{u(z)} \quad . \tag{12}$$

This is an important object because unproductive agents need to spend money to consume and therefore monetary policy will affect their consumption behavior. Looking at the consumption level of the unproductive agent when she has no money gives an idea of the insurance role of monetary expansions,

$$\lim_{z \to 0} c^u(z) = \bar{c}\ \frac{\mu(0)}{2} p(1) \tag{13}$$

with $\lim_{z \to 0} c^u(z) = 0$ if $\lim_{x \to 0} \mu(x) = 0$, where the limit obtains from the agent's budget constraint. This is important for the welfare analysis that will follow because it shows that monetary transfers provide the unproductive agent with a lower bound to her consumption level. Without transfers, an agent with no money cannot consume.

Another interesting object is the expected real return on money or, equivalently, the expected real interest rate. This object is useful in understanding the workings of the model and illustrates how the market incompleteness affects the economy by generating a risk premium. Let $r(x)$ denote the expected (net) return on money,

$$r(x) \equiv \mathbb{E}\left[\left. \frac{\dot{\tilde{q}}(x)}{\tilde{q}(x)} \right| x_t = x \right]$$

which is just the expected growth rate of the price of money conditional on the unproductive agent money holdings being x. Using that $\tilde{q}(x) = p(1-x)/M$, the expected return can be written as (see Appendix B)

$$r(x) = \rho + \lambda \frac{p(x)}{p(1-x)} \left(\frac{c^u(1-x) - \bar{c}}{c^u(1-x)} \right) \ . \tag{14}$$

In a complete markets setting, such as the one described in Remark 2, consumption is constant (at level $\bar{c}$) and the expected real return equals the time discount, ρ. With incomplete markets the return on money depends on the history of the shocks, as summarized by the wealth distribution x. Recall, by equation (5), that the price of money is determined by the productive agent $q(x) = p(1-x)$. Equation (14) shows that the expected real rate is proportional to the change in the price of money associated to a switch of the state, i.e. $p(x)/p(1-x)$, and is proportional to the productive agent's expected consumption growth.[7]

Finally, noting that $1/\tilde{q}(x)$ denotes the price of consumption in units of money, we can compute the expected inflation rate

$$\pi(x) = -\frac{r(x)}{1+r(x)} \quad . \tag{15}$$

After determining the equilibrium functions $u(z), p(z), c^u(z)$ we will use these formulas to argue that the model displays a "Phillips curve", a positive correlation between the inflation rate and the output level.

4 The value of money in equilibrium

The previous analysis showed that allocations in a monetary equilibrium are fully characterized by the Lagrange multipliers, $u(z)$ and $p(z)$, that solve the system of Euler equations and associated boundary. This section characterizes the properties of these multipliers, measuring the "value of money" to the productive and unproductive agents, under two rules for the money supply. The first rule is state independent, i.e. equal to a constant non-negative money growth value $\mu \in [0, +\infty)$. The second rule assumes that the money growth rate is a continuous function of the wealth share of the unproductive agent: $\mu(x)$.

Using equations (8)-(9) we can define the following system:

$$p'(z) = \frac{(\rho + \lambda + \mu(1-z))p(z) - \lambda u(z)}{h^p(z)}, \tag{16}$$

$$u'(z) = \frac{(\rho + \lambda + \mu(z))u(z) - \lambda p(z)}{h^u(z)}. \tag{17}$$

The first equation describes the marginal value of money for a productive agent holding z, so that $x = 1 - z$ i.e. monetary policy is a function of the wealth of the unproductive agent. The second equation describes the marginal value for an unproductive agent holding z, so that

[7] Consumption remain constant with probability $1-\lambda$ per unit of time, or it changes from $\bar{c}$ to $c^u(1-x)$ with probability λ.

$x = z$. The solution of this system, together with the boundary condition, fully characterizes the value of money in equilibrium.

Next we state a result that is key in characterizing the problem.

Proposition 2 *Assume $\mu(x)$ is continuous in $[0,1]$ and that the forcing terms on the right hand side of (16)-(17) have no singularities in $(0,1)$. Then, for any $\bar{c} > 0$, $h^u(z) < 0$ for all $z \in (0,1)$ and $\lim_{z\to 0} h^u(z) = 0$.*

See Appendix A for a proof. This result, arises from the requirement of continuity and uniform Lipschitz condition in the space $(0,1)$ for the forcing terms of equations (16)-(17). The economic content of the proposition is that unproductive agents deplete their share of money holdings as long as they remain unproductive.

Proposition 2 allows us to characterize some interesting features of the Lagrange multipliers, the marginal value of money $p(z)$ and $u(z)$, by representing their evolution in the corresponding phase diagram. To this end we consider the three-dimensional space $[0,1]\times\mathbb{R}_+\times\mathbb{R}_+$, and define the set $\mathcal{L}_p$ with elements (z,U,P); U is the unproductive agent marginal utility $u(z)$ and $P = \frac{\lambda}{\rho+\lambda+\mu(1-z)}u(z)$. Likewise, we define $\mathcal{L}_u$ as the set with elements (z,U,P); again, U denotes the unproductive agent marginal utility $u(z)$, and $P = \frac{\rho+\lambda+\mu(z)}{\lambda}u(z)$. Notice that, by construction, $p'(z)h^p(z)$ and $u'(z)h^u(z)$ are zero on $\mathcal{L}_p$ and $\mathcal{L}_u$, respectively. Now consider the projections of $\mathcal{L}_p$ and $\mathcal{L}_u$ in the two-dimensional phase plane (U,P), denoted by L_p, L_u and defined as

$$L_p: \quad P = \frac{\lambda}{\rho+\lambda+\mu(1-z)}u(z), \quad \text{and} \quad L_u: \quad P = \frac{\rho+\lambda+\mu(z)}{\lambda}u(z).$$

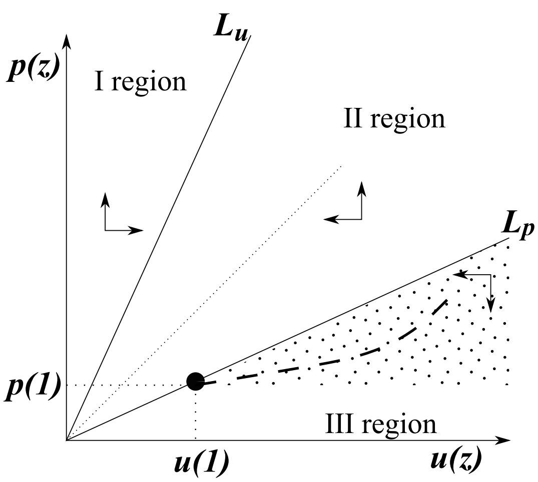
The loci L_p and L_u determine different regions in the phase plane (U,P) where the functions $u(z)$, $p(z)$ change behavior according to the sign of their derivative.

Two examples are shown in the top panels of Figure 1 where the arrows describe the increasing/decreasing behavior for $u(z)$ and $p(z)$. The direction of the arrows is determined taking into account the evolution of money holdings established in Proposition 2. The phase diagrams describe the model dynamics: the first case, plotted in the upper-left panel, describes a constant money rule; the second case, describing a state dependent rule, is plotted in the upper-right panel. The bold dot in each plot corresponds to the boundary condition in equation (11): $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$ which lies on L_p, and the dotted line represents the locus $P = U$. The dashed curve is a possible path for the solution. Next we exploit the phase diagram to investigate some features of the solution under the different type of policies. In particular, our analysis allows us to provide global results for the constant policy. However, when the policy is allowed to vary with the state, the same results can be obtained only locally when x is low enough and the productive agent holds most of the money.

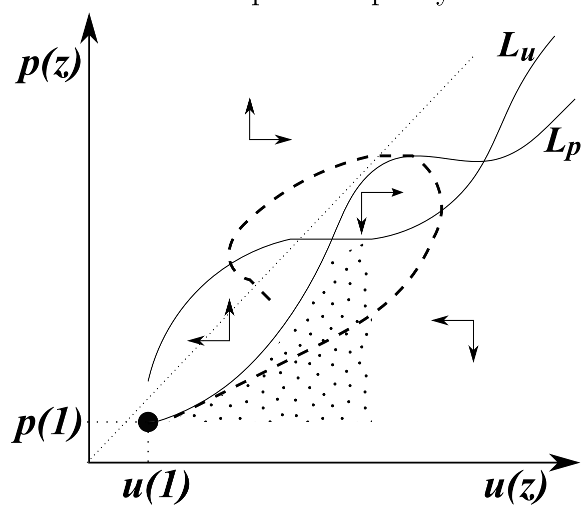
Figure 1: The marginal value of money

Phase diagrams

Constant policy

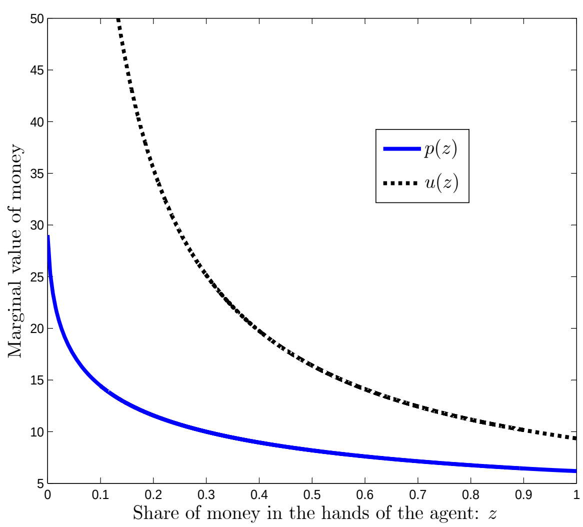


State dependent policy

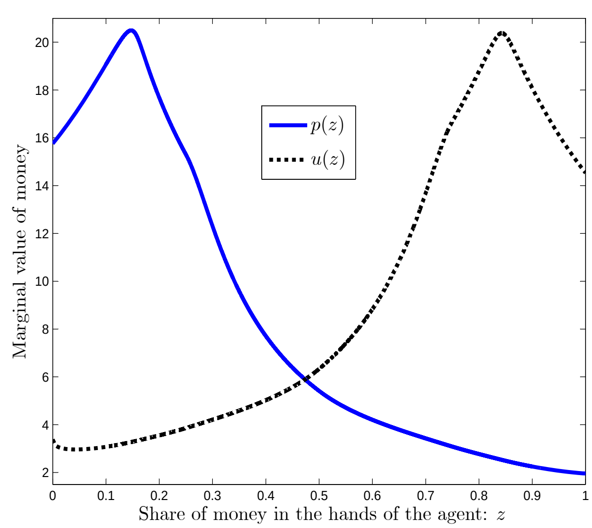


The shadow value of money: Lagrange multipliers

Constant policy



State dependent policy



When μ is constant, so that $\mu(x) = \mu \geq 0$ for every value of the aggregate state x, our results extend those provided in Scheinkman and Weiss (1986) which only considered the case where $\mu = 0$. The phase diagram determines three regions where the functions $u(z)$ and $p(z)$ change behavior according to their derivative sign. It is evident that the only positive solutions of the problem must stay in the third region, where both functions are decreasing on the entire (0,1) interval. More precisely, their path develops in the area which is dotted, as it is shown in the figure. The next proposition characterizes the marginal value of money when μ is constant.

Proposition 3 *Under a constant policy $\mu(x) = \mu \geq 0$ we have that for any $\bar{c} > 0$:*

$$(i) \;\; p(z) < u(z), \qquad \textit{and} \quad (ii) \;\; p'(z) < 0, \quad u'(z) < 0 \qquad \textit{for all} \quad z \in (0,1) \; .$$

See Appendix A.2 for a proof. The first part of the proposition establishes that when the monetary policy is constant the value of money for an unproductive agent is higher than the value of money for a productive agent, $u(z) > p(z)$, at all levels of money holdings $z \in (0,1)$. This property, first highlighted by Scheinkman and Weiss (1986), seems intuitive: because the only difference between productive and unproductive agents holding z is that the productive agent can work, an unproductive agent values more an extra unit of money; as a result, her Lagrange multiplier is higher. Second, the proposition also states that the functions $u(z)$ and $p(z)$ are decreasing in z at all levels of money holdings $z \in (0,1)$. The bottom-left panel in Figure 1 displays these properties and the upper-left panel presents the phase diagram explaining their origin: since region III is the only admissible region for a solution to satisfy the boundary condition, the curves satisfy the properties listed in Proposition 3.

When the monetary policy depends on the state x some interesting new features arise. Inspection of the upper-right panel of Figure 1 shows that the equilibrium Lagrange multipliers must reach the boundary condition, that occurs when $x = 0$ and the productive agent money holdings are $z = 1$ (i.e. the bullet point), in the region which is under L_p and L_u (i.e. the dotted region). As a result, when the productive agent money holdings z are large enough, the solution path develops in the area where both $p(z)$ and $u(z)$ are decreasing functions, as was the case for the constant policy. But crucially this result holds only locally, and in general the functions $p(z)$ and $u(z)$ that solve the problem can take many shapes. The next proposition states this result.

Proposition 4 *Suppose that $\mu(x)$ is continuous and that the following assumptions hold:*

(A_1) the policy $\mu(x)$ satisfies $\frac{\lambda}{\rho+\lambda+\mu(0)} < \frac{\rho+\lambda+\mu(1)}{\lambda}$;

(A_2) the policy $\mu(x)$ is strictly decreasing at $x = 0$, i.e. $\mu'(x) < 0$ for all $0 \leq x < \upsilon$, where υ is a suitable positive number.

Then, there is a neighborhood for the productive agent money holdings at $z = 1$, denoted by the interval I_z, where the multipliers satisfy, for any $\bar{c} > 0$,

$$(i) \quad p(z) < u(z), \qquad \text{and} \quad (ii) \quad p'(z) < 0, \quad u'(z) < 0 \qquad \text{for all} \quad z \in I_z \ .$$

See Appendix A.3 for a proof. The reason for the local similarity of the constant and the state-dependent problem is the following. Start by noting that when the productive agent money holdings are $z = 1$ the unproductive agent money holdings are $x = 0$; in this case, by Proposition 1, $\mu(0) \geq 0$. It follows that, by the continuity of $\mu(x)$, for any $z \in I_z$ it is the case that $\mu(1 - z) \geq 0$. This non-negativity constraint on the money growth rate, together with Assumptions (A_1) and (A_2), produces a "local phase diagram", and implied behavior of the multipliers, that is similar to the behavior under a constant policy. As shown in the the upper-right panel of Figure 1, under the state dependent policy $\mu(x)$ we have that $u(z) > p(z)$ and that both curves are decreasing only when the unproductive agent money holdings are low (i.e. x close to zero) and the productive agent money holdings are high (i.e. z close to one). Assumptions (A_1) and (A_2) are important as they bound the local behavior of the dynamical system. Assumption (A_1) guarantees that in the neighborhood of $z = 1$ the locus L_p lies below the locus L_u, as depicted in the upper-right panel of Figure 1.[8] A sufficient condition for satisfying this Assumption is $\mu(1) \geq -\rho$. Assumption (A_2) guarantees that in the neighborhood of $z = 1$, the locus L_p is decreasing.

Notice that when $z \notin I_z$, money growth $\mu(1 - z)$ can be negative so that $u(z)$ can be below $p(z)$ and one (or both) of them can be increasing. To understand this result it is useful to contrast the state dependent case with the constant case. In the constant case, as the money rule does not depend on x, the only difference between productive and unproductive agents is the production opportunity; as shown in Proposition 3 this immediately implies that $u(z) > p(z)$ for all z. But when the monetary rule varies with the state x, a comparative static across productivity states, i.e. comparison of $u(z)$ vs. $p(z)$, also involves a different path for the monetary rule. To make this point clear we make explicit the dependence of the marginal value of money on the money rule μ. That is, let $\{p(z; \mu(x)), u(z; \mu(x))\}$ denote the marginal value of money for an agent with money holdings z and where the money rule is μ. Consider a productive agent holding money z. Under a constant policy her current value is $p(z; \mu)$ and, if the state switches, her value will be $u(z; \mu)$. Under the state dependent rule

[8] To see this notice that this Assumption can be extended to the inequality $\frac{\lambda}{\rho+\lambda+\mu(1-z)} \leq \frac{\rho+\lambda+\mu(z)}{\lambda}$ so that L_p lies below L_u.

her current value is $p(z;\mu(1-z))$ and, if there is a state switch, her value would become $u(z;\mu(z))$. This shows that when the policy is state dependent the value of money across agent types differs not only because of differences in production opportunities, but also because the money rule is different depending on the wealth distribution. It follows that the restriction that $p(z) < u(z)$ need not to hold over the whole state space under a state dependent rule. This feature, as we discuss next, allows for a substantial welfare improvement.

To understand why relaxing the properties (i) and (ii) of Proposition 3 is important for our problem, notice that the complete markets allocation features a constant consumption for all agents, at $\bar{c} = 1$ (see Remark 1). If a money rule existed to implement a first best allocation, it can be seems using equation (12) that the rule would imply

$$p(1-z) = u(z) \quad \text{for all } z \in [0,1]$$

or, in words, that the functions $u(z), p(z)$ are symmetric around $z = 1/2$. Obviously this feature cannot be achieved by a constant rule, since in that case the functions $u(z), p(z)$ are decreasing. A state dependent policy has a chance of getting closer to this symmetric benchmark, as shown by the lower-right panel of Figure 1. Even though a first best allocation is not achievable in general by the monetary rule, as proven by Proposition 5 below, we show next that the state dependent policy yields a welfare level that is much higher than the one produced by a constant policy.

5 The optimal supply of liquidity

In this section we define a welfare criterion and explore the optimal supply of liquidity. Let $v(x)$ denote the discounted present value of the sum of utilities of both types of agents, where agents are given the same Pareto weight. This is a function of the money share of unproductive agents, x. The continuous time Bellman equation is

$$\rho\, v(x) = \bar{c}\left[\ln c^u(x) + \ln \bar{c} - 1 - \frac{c^u(x)}{\bar{c}}\right] + v_x(x)\, h^u(x) + \lambda\,(\, v(1-x) - v(x)\,) \tag{18}$$

where we made explicit the dependence of the value function, consumption, and evolution of money holdings on the money growth rate $\mu = \mu(x)$. The flow value $\rho\, v(x)$ is given by the sum of the period utility for both agents plus the expected change in the value function. The latter occurs because of the evolution of assets (the change in x) as well as of the possibility that identity of the productive agent will change. Notice that in this case the state, i.e. the wealth of the unproductive agent, switches from x to $1-x$.

We consider the problem from an ex-ante perspective, i.e. assuming that at the beginning of time nature assigns the initial productive states and the planner can choose the initial wealth distribution and a policy rule for money growth. We assume that the planner, once it chooses the policy, commits to it. Note that because individual types are not observable, and given the symmetry of the environment (and identical Pareto weights), the planner will give the same amount of liquidity to every agent and therefore at the beginning of time $x = \frac{1}{2}$. As a result, the planner chooses the function μ in order to maximize $v(1/2)$.

To evaluate the policy it is useful to define the welfare of a given policy using a certainty equivalent compensating variation. Let α denote the consumption equivalent cost of market incompleteness associated with a given policy. That is, α solves the following equation

$$2\bar{c}\,\ln\left(\bar{c}(1-\alpha)\right) - 2\bar{c} = \rho\, v\left(\frac{1}{2}\right) \tag{19}$$

so that α measures the fraction of the consumption under complete markets that agents would be willing to forego to eliminate the volatility of consumption due to market incompleteness for a given policy rule μ.[9]

The next proposition states that, for any given value for the discount rate parameter ρ, there is a range of values for the parameter governing the persistence of the productive state for which there is no monetary rule $\mu(x)$ able to attain the complete markets allocation. This is an important result as it precludes us from imposing the complete markets allocation and backward engineer the policy rule $\mu(x)$ that attains it.

Proposition 5 *Let $\rho > 0$. There exists a value $\bar{\varepsilon} > 0$ such that if $\lambda \in (0, \bar{\varepsilon})$ then there is no policy rule $\mu(x)$ such that the complete markets allocation, where both agents consume the constant value $\bar{c}$, can be attained.*

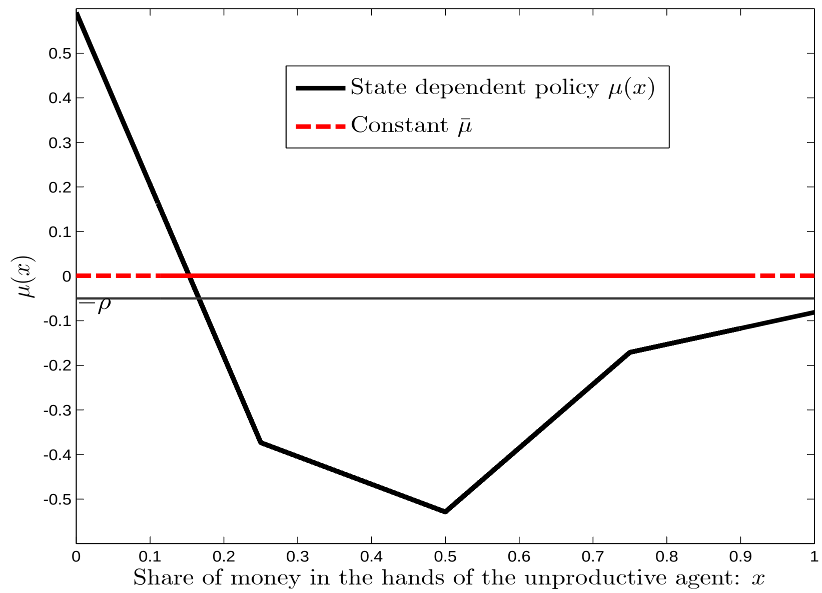
See Appendix C for a proof. The proof of the proposition stems by noting that the complete markets allocation implies a finite, and constant welfare $v(x) = v_c < 0$, while, for $\lambda \in (0, \bar{\varepsilon})$, for any policy $\mu(x)$ the welfare function $v(x)$ cannot be bounded from below. This immediately shows that for $\lambda \in (0, \bar{\varepsilon})$ there is no policy $\mu(x)$ that can support the complete markets allocation.

We look for the optimal policy by searching numerically for the policy that maximizes ex-ante expected welfare $v(1/2)$; we label the optimal policy $\hat{\mu}(x)$. To aid in the understanding of the policy we also compute an alternative policy, $\bar{\mu}$, which maximizes $v(1/2)$ under the restriction that the policy has to be constant. In all computations reported below we set the scaling factor $\bar{c} = 1$. Figure 2 plots both policies obtained for the baseline parametrization of

[9] Recall that under complete markets $c^u(z) = c^p(z) = \bar{c}$ for all z and $l^p(z) = 2\bar{c}$ for all z.

the model where the normalized discount rate ρ/λ equals $1/2$. Given this parameter choice, by setting the discount rate to the standard value of 0.05, implies that λ equals $1/10$, so that the average length of a productive state is 10 years. Qualitatively similar results are obtained for other parameterizations for ρ/λ.

Figure 2: Optimal policy ($\rho/\lambda = 1/2$)



The constant policy $\bar{\mu}$ consists of an expansion of the monetary base of 0.1%; $\bar{\mu}$ is positive because the insurance motives outweighs the inflation costs. However, as depicted in Figure 2, the optimal policy $\hat{\mu}(x)$ is very different: it expands the money supply when x is low and it contracts the money supply when x is high; this happens because the optimal policy, through its state dependent nature, is able to decouple the insurance motives and inflation costs. The welfare cost of market incompleteness α, as defined in equation (19), under the constant policy is 31.7% while under the optimal policy is 3.3%. It can be seen that the optimal policy radically increases welfare.

We now describe the features of the optimal policy and interpret them using the trade-off between insurance motives and the cost of inflation. When the insurance motives are very strong, which happens when $x \approx 0$, the policy prescribes an expansion of the money supply at a rate around 60 per cent. As the insurance motives decrease, which happens as we take x away from zero, the rate at which money is pumped into the economy decreases. Eventually, at $x \approx 0.2$, the cost of inflation (through the production incentives) outweighs the insurance role of money and, as a result, the policy turns to prescribing monetary contractions. When x lies in an intermediate region the policy prescribes monetary contractions at rates ranging from 30% to 50%, way above the rate of discount ρ, value usually associated with Friedman's rule. These extreme contractions are a reflection of the large monetary expansions occurring when x is low: because the latter erodes the return on money, the forward looking nature of agents requires large monetary contractions in order to increase the asset return and thus provide production incentives. As x transits from this intermediate region to 1 the level of the monetary contractions decreases; and they become small as x approaches 1. This is a reflection of the provision of insurance at the other extreme, where x is low enough. When x is high the unproductive agent is rich, and her insurance needs are small. However, the productive agent is poor and, after a state switch the new unproductive agent will be poor and with a high insurance need. As a result, for high values of x the productive agent has strong insurance motives which erode the benefits of monetary contractions. Nevertheless, because a productive agent can acquire money by working, her insurance motives when poor are smaller than those of a poor unproductive agent. This explains why the policy has an asymmetric v-shape form.

Figure 3: Consumption and the return on money under optimal policy rule ($\rho/\lambda = 1/2$)

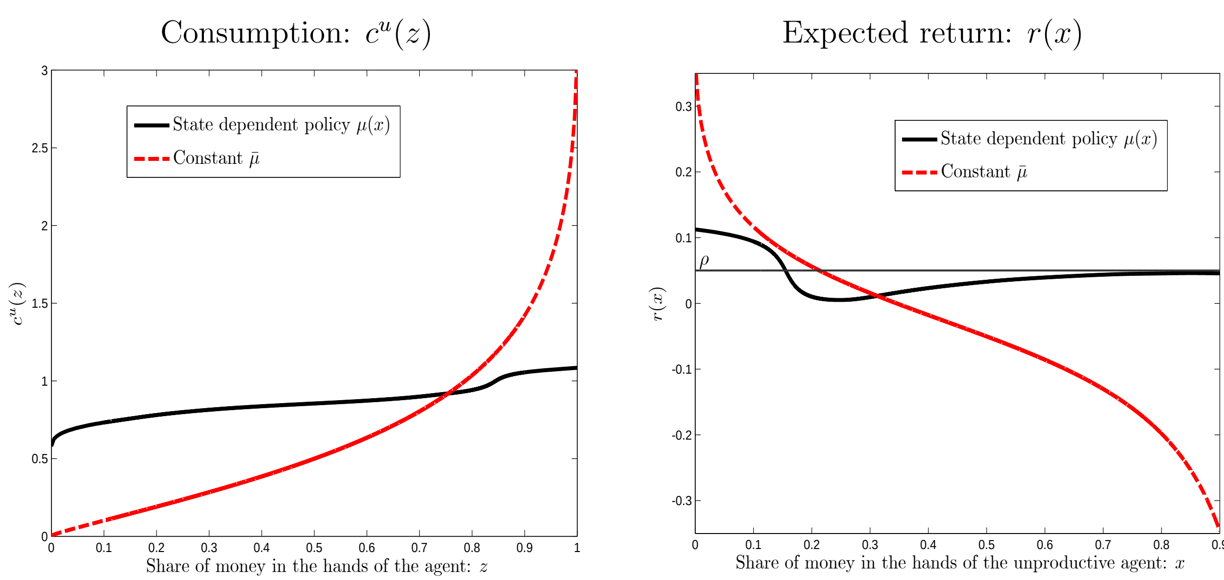


Figure 3 illustrates the profiles of the consumption function and the expected return on money under the optimal policy $\hat{\mu}(x)$ and constant policy $\bar{\mu}$. The left panel of the figure shows that the consumption of the unproductive type under the optimal policy is (i) smoother than under the constant policy, (ii) closer to the consumption implied by the complete markets allocation (i.e. $c^u(z) = \bar{c} = 1$). The smoother consumption profiles of the state dependent rule also yields a flatter profile for the expected return on money $r(x)$, as shown in the right panel of the figure. This flatter profile reflects the fact that the consumption of an unproductive agent is less extreme under $\hat{\mu}(x)$: the smaller expected changes in consumption (hence marginal utilities) associated to a state switch dampen the risk premia and lead to a smoother expected return of the asset. Moreover, because the consumption under the optimal policy is closer to the consumption under complete markets, the expected rate of return is close to the rate of discount ρ, which is the expected rate of return with a complete markets allocation, as shown by equation (14).

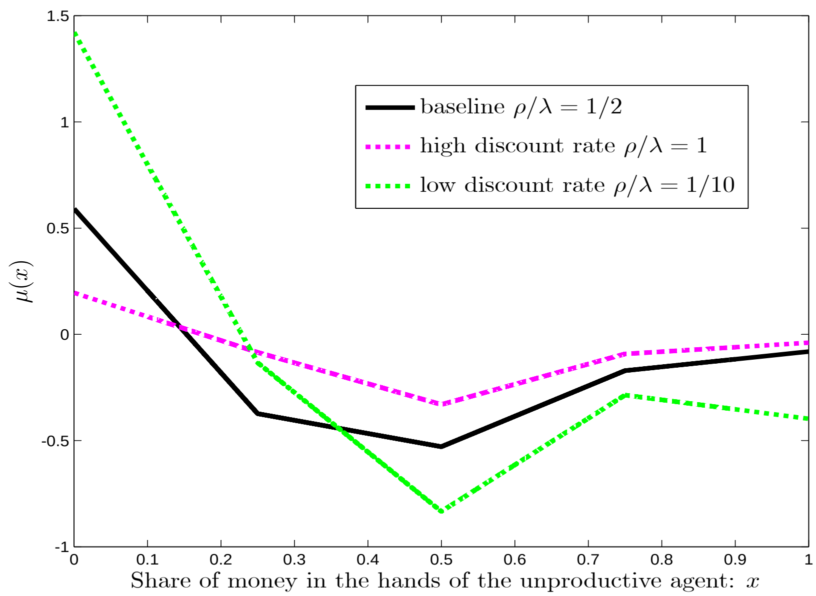
Notice that, because the consumption of the unproductive agent $c^u(x)$ is increasing in x, aggregate production, $c^p(1-x) + c^u(x) = 1 + c^u(x)$, increases with x. Inspection of the right panel of Figure 3 shows that the real return on money is high when x is low. To understand this remember from the discussion of equation (14) that the return on money is proportional to the consumption growth of the productive agent. At $x = 0$ the productive agent is consuming $c = 1$, and expects, in case of a state switch, to consume $c^u(1) > 1$, which explains the high expected return at $x = 0$. An analogous logic explains the low returns when production is high (when x is high). Recall from equation (15) that the expected inflation rate $\pi(x)$ is inversely related to $r(x)$. This implies a positive correlation between the aggregate production and inflation, i.e. the model features a "Phillips curve".

Recall, as discussed at the end of Section 4, that replicating the first best allocation requires that $u(z) = p(1-z)$. This means that, for any wealth level z, the marginal value of money has to be equal for both agent's types; in other words, the marginal value of money has to be symmetric around $z = 1/2$. This symmetry requirement cannot be handled by a constant policy. However when μ is allowed to depend on x, the functions $u(z)$ and $p(z)$ are able, at least partially, to produce the required symmetry. This can be seen in the lower-right panel of Figure 1 where we plot $u(z)$ and $p(z)$ under the optimal money rule $\hat{\mu}(x)$.

Notice that the state dependent pattern of the policy rule can be equivalently interpreted in terms of the business cycle. As aggregate production is increasing in x then the optimal policy $\hat{\mu}(x)$ is such that monetary expansions happen when aggregate production is low and monetary contractions occur when it is high. In other words, the policy is expansionary during recessions and contractionary during expansions.

To conclude this section we explore the effect of changing the normalized discount rate ρ/λ on the optimal policy. We do so in Figure 4 where we present the policy under three different parameter configurations. The first configuration uses our benchmark parameter value, $\rho/\lambda = 1/2$, the second configuration uses a high rate, $\rho/\lambda = 1$, while the third one uses a low one, $\rho/\lambda = 1/10$. As the figure shows, the qualitative features of the policy are similar for the three parameter configurations: high monetary injections when insurance motives are the highest (when x is low), and high monetary contractions when they are the lowest (when x is close to $1/2$). Even though similar in qualitative terms, they do differ in the level of monetary injections. Monetary expansions and contractions are more extreme the lowest the normalized discount rate. Our intuition is as follows. As ρ/λ falls the value of money increases so that the production distortion that follows a monetary expansion is milder, and therefore, when x is small, the policy can provide insurance through liquidity expansions at a higher rate. Moreover, as previously discussed, to handle production incentives the policy prescribes monetary contractions to counterbalance the monetary expansions, which also naturally increase as ρ/λ falls. Finally, as previously discussed, the welfare cost α is 3.3% under the baseline configuration, while it increases to 11.7% when the normalized discount rate is high, and it falls to 0.3% when it is low. The improvement in welfare as ρ/λ falls follows because the the insurance motives decreases as the normalized discount rate decreases which, in turn, decreases the degree of the trade-off between insurance motives and production incentives.

Figure 4: Optimal policy



6 Concluding remarks

Friedman's rule prescribes that the real return on money should be equal to the rate of preference. In many setups, this prescription maps into a liquidity contraction at the rate of time discount. When liquidity is essential for trading, which stems from the private nature of individual histories, an expansionary policy can be desirable due to insurance needs. A natural trade-off arises as expanding the liquidity base dampens the return on the asset, therefore reducing the production incentives. The regulation of liquidity strikes a balance between these two forces.

As we discussed in the paper, insurance needs and production incentives depend on the wealth distribution which evolves through the business cycle. The novelty of our paper is that we acknowledge this dependence and we explore how the state dependent policy balances the costs of anticipated inflation with the need for insurance along the business cycle. This policy allows for a dramatic improvement in welfare compared to a policy that does not respond to the state. The optimal policy expands the supply of liquidity when the unproductive agents are poor (when the insurance needs are large), and it contracts the liquidity base otherwise to maximize production incentives. The principle underlying this prescription is due to the state-dependent redistributive role of monetary policy, and differs from the one arising in sticky-price models. Because aggregate production is low when the unproductive group is poor and high when they are rich, the best policy can be interpreted as counter-cyclical. Interestingly, in spite the policy being far away from contracting the money supply at the rate of preference, the optimal policy "echoes" Friedman's rule as the expected real return of money is close to the rate of preference.

Several interesting extensions are left for future work. An important, and classical, assumption for our results is that the planner has the ability to levy lump-sum taxes, and that agents are not allowed to renege from their obligations. However, as pointed out by Andolfatto (2013), if agents are allowed to voluntary be subject to taxation the degree of monetary contractions has to be limited by the voluntary nature of participation. Therefore, it is possible that the incentive-feasible allocation cannot support the optimal policy we constructed in this paper, as this one requires large monetary contractions. An interesting open question is to use the mechanism design toolkit to find the optimal state dependent policy implementable under voluntary participation. Another interesting extension is to reformulate our model, following Atkeson and Lucas (1992), considering an endowment economy where agents are subject to preference shocks. If only two types of agents are considered, as we did in this paper, the two setup share several similarities and the solution developed for this paper can also be used in this other context. This would be useful to check the robustness of the results. Finally, an extension that seems worth exploring is to extend the model to allow for a larger set of agent types. Although this will come at the cost of losing tractability, as the state space grows with the number of agent types, this model, and the optimal policy, could probably be solved numerically. We conjecture that the main insights of our paper would remain: monetary expansions when unproductive agents are poor counterbalanced by monetary contractions in states where insurance motives are small.

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Appendix

A Mathematical characterization of the equilibrium

A.1 Proof of Proposition 2

The boundary condition (10) yields $\lim_{z\to 0} h^u(z) = 0$. Moreover, we have $h^u(1/2) = -\frac{\bar{c}}{u(1/2)} < 0$. Under the continuity assumption on $\mu(x)$, $h^u(z)$ is a continuous function; then the inequality can be extended over a suitable neighborhood of the asset $z = \frac{1}{2}$. More precisely, it is possible to consider an interval $(\frac{1}{2} - \delta, \frac{1}{2} + \delta) \subset (0, 1)$ where $h^u(z) < 0$. The further assumption related to the lack of singularities yields that the function $h^u(z)$ cannot nullify within the domain and its sign has to be uniform. It follows that the interval $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ overlaps the whole integration interval and the previous inequality holds for all $z \in (0, 1)$. In this way, the result is completely obtained.

The assumptions in the proposition mean that the solution $(u(z), p(z))$ evolves for $z \in (0, 1)$ in a region of $\mathbb{R}^2_+$ where the forcing term has no singularity and is continuous, with the classical uniform Lipschitz condition locally satisfied. We recall that this kind of smoothness is a basic assumption which is needed for investigating the solution of the differential problem. As discussed in Ascher, Mattheij and Russell (1988), smoothness together with the requirement of solvability of the algebraic system related to the boundary conditions are the crucial issues for establishing the existence of a solution.

A.2 Proof of Proposition 3

The proof follows from the inspection of the phase diagram, which is plotted in upper-left Figure 1. We assume the policy $\mu(x) = \mu \geq 0$ is constant and exploit the geometric loci in $\mathbb{R}^2_+$ defined as

$$L_p = \left\{ (U, P) \in \mathbb{R}^2_+ \,|\, P = \frac{\lambda}{\rho + \lambda + \mu} U \right\}, \quad \text{and} \quad L_u = \left\{ (U, P) \in \mathbb{R}^2_+ \,|\, P = \frac{\rho + \lambda + \mu}{\lambda} U \right\}. \tag{20}$$

We first note that, according to equations (16)-(17), the terms $p'(z)h^p(z)$ and $u'(z)h^u(z)$ are zero on L_p and L_u, respectively. Moreover, we have $\frac{\lambda}{\rho+\lambda+\mu} < 1 < \frac{\rho+\lambda+\mu}{\lambda}$; this implies that the locus L_p lies under L_u and the dotted line $P = U$ is between them, as it is shown in the figure. The uniform signs of $h^u(z) < 0$ and $h^p(z) > 0$ allow us to gain an insight about the increasing/decreasing pattern of the Lagrange multipliers $p(z)$ and $u(z)$, as it is shown by the arrows in the figure. It can be seen that region III is the only admissible area for a solution to satisfy the boundary condition $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda} p(1)$, which is on L_p. Indeed the only positive solutions must stay in that region and their path develops in the dotted area. Therefore, both $p(z)$ and $u(z)$ are decreasing functions: it follows that $p'(z) < 0$ and $u'(z)$ everywhere. Moreover, in the same region, we have $p(z) \leq \frac{\lambda}{\rho+\lambda+\mu} u(z) < u(z)$ for all z.

A.3 Proof of Proposition 4

The results in Proposition 4 can be stated by investigating the upper-right plot in Figure 1, where some qualitative features of the solution are represented. Under a continuous state dependent policy, we define the following geometric loci in the three-dimensional space $[0,1] \times \mathbb{R}_+ \times \mathbb{R}_+$:

$$\mathcal{L}_p = \{(z,U,P) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \quad U = u(z) \text{ solves equations (16)-(17) at the state } z \text{ and } P = \frac{\lambda}{\rho+\lambda+\mu(1-z)} u(z) \Big\} ,$$

and

$$\mathcal{L}_u = \{(z,U,P) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \quad U = u(z) \text{ solves equations (16)-(17) at the state } z \text{ and } P = \frac{\rho+\lambda+\mu(z)}{\lambda} u(z) \Big\} .$$

We remark that $\mathcal{L}_p$ and $\mathcal{L}_u$ can be considered as a generalization of the geometric loci L_p and L_u already defined by (20) in Appendix A.2, where the monetary policy is assumed to be state independent. Again, terms $p'(z)h^p(z)$ and $u'(z)h^u(z)$ are zero on $\mathcal{L}_p$ and $\mathcal{L}_u$, respectively.

With the aim to describe the dynamics of the solution in an easy way, we consider the projection of the sets $\mathcal{L}_p$ and $\mathcal{L}_u$ in the two-dimensional space $\mathbb{R}^2_+$: they are denoted by L_p, L_u and defined as

$$L_p = \Big\{(U,P) \in \mathbb{R}^2_+ \mid \quad \exists\, z \in [0,1] \text{ so that } U = u(z) \text{ solves equations (16)-(17) at the state } z \text{ and } P = \frac{\lambda}{\rho+\lambda+\mu(1-z)} u(z) \Big\} ,$$

and

$$L_u = \Big\{(U,P) \in \mathbb{R}^2_+ \mid \quad \exists\, z \in [0,1] \text{ so that } U = u(z) \text{ solves equations (16)-(17) at the state } z \text{ and } P = \frac{\rho+\lambda+\mu(z)}{\lambda} u(z) \Big\} .$$

Both L_p and L_u are represented in the phase plane (U,P). In this respect, the following features are crucial to draw the picture and obtain the proof:

(P_1) The loci may intersect in an even number of points, since the following condition

$$\frac{\lambda}{\rho+\lambda+\mu(1-z)} = \frac{\rho+\lambda+\mu(z)}{\lambda},$$

is equivalent to have $\rho^2 + 2\rho\lambda + (\rho+\lambda)(\mu(z)+\mu(1-z)) + \mu(z)\mu(1-z) = 0$, which is a symmetric formula with respect to $z = 1/2$.

(P_2) Condition $\frac{\lambda}{\rho+\lambda+\mu(0)} < 1 \leq \frac{\rho+\lambda+\mu(1)}{\lambda}$ in assumption (A_1) can be extended in order to have inequality

$$\frac{\lambda}{\rho+\lambda+\mu(1-z)} \leq \frac{\rho+\lambda+\mu(z)}{\lambda},$$

satisfied in a suitable neighborhood of $z = 1$. It follows that, in the same neighborhood, locus L_p is under L_u.

(P_3) Boundary condition $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$ lies on L_p. It has to be reached by the solution in correspondence with z tending toward 1.

(P_4) L_p and L_u determine different regions in the phase plane (U, P) where both functions $u(z)$, $p(z)$ change behavior according to their derivative sign. As a first example, in Figure 5 we show the local behaviour of $u(z)$ and $p(z)$ in a neighborhood of $z = 1$, where L_p lies under L_u. Each arrow describes the increasing/decreasing patterns for $u(z)$ and $p(z)$ according to the signs $h^u(z) < 0$ and $h^p(z) > 0$ which are uniform and established by Proposition 2. By inspection of Figure 5, it is evident that $u(z)$ is decreasing when the money holdings of the productive agent is near $z = 1$, since the solution should reach the boundary condition on L_p according to (P_3).

(P_5) The increasing/decreasing shape for L_p can be established by setting

$$\Gamma(z) = \frac{\lambda}{\rho+\lambda+\mu(1-z)}u(z)$$

for all z. It is straightforward that the set L_p is described in the phase plane (U, P) by the point $(u(z), \Gamma(z))$ as z varies in the domain $[0, 1]$. Notice that for any $z \in [0, 1]$,

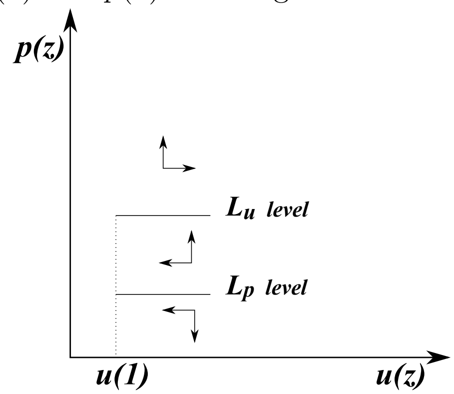
$$\Gamma'(z) = \frac{\lambda}{\rho+\lambda+\mu(1-z)} \left(\frac{u(z)\mu'(1-z)}{u'(z)(\rho+\lambda+\mu(1-z))} + 1 \right) u'(z) \; .$$

We remark that

$$\mu(1-z) \geq 0, \qquad \text{and} \qquad \mu'(1-z) < 0, \tag{21}$$

near $z = 1$. The first condition in (21) is due to the continuity for $\mu(x)$ and $\mu(0) \geq 0$;

Figure 5: Behavior of $u(z)$ and $p(z)$ in a neighborhood of $z = 1$: L_p lies below L_u.



the second relationship arises from assumption (A_2). In this respect, since $u(z)$ is decreasing when z moves near 1 (see Figure 5), then $u'(z) < 0$; in this way, it follows that $\Gamma'(z) < 0$ and the locus L_p decreases in a neighborhood of $z = 1$.

The previous features can be exploited in order to draw the picture in Figure 1, whose inspection lets the proof of Proposition 4 be completed. In the same figure, the dotted line is the locus where $P = U$ and the bold dot corresponds to boundary condition $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$. It is evident that the only solutions, that are positive, must reach that boundary condition in the dotted region, that is under L_p and L_u (the dashed curve represents a possible path for the solution). In this way, the existence of an asset $\bar{z} \in [0, 1]$ can be stated so that, for all $z \geq \bar{z}$, the solution path develops in the area where both $p(z)$ and $u(z)$ are decreasing functions. This is equivalent to have, for all $z \in [\bar{z}, 1]$,

$$p'(z) < 0, \qquad \text{and} \qquad u'(z) < 0.$$

In addition, due to conditions $\mu(0) > 0$ and $u(1) = \frac{\rho+\lambda+\mu(0)}{\lambda}p(1)$, the point $(u(1), p(1)) \in L_p$ lies under the dotted line and $p(1) < u(1)$. Since the function μ is supposed to be continuous, then the inequality can be extended in a suitable neighborhood of $z = 1$. More precisely, there exists an asset $\widetilde{z} \in (0, 1]$ so that

$$p(z) < u(z),$$

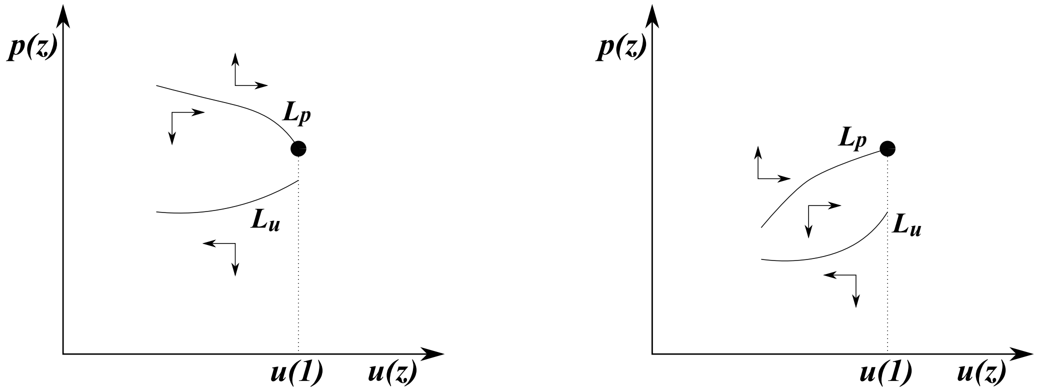
for all $z \in [\widetilde{z}, 1]$. We set $I_z = [\bar{z}, 1] \cap [\widetilde{z}, 1]$, then the proof is completely obtained. $\square$

We remark that

$$\mu(1) \geq -\rho,$$

represents one possible sufficient condition for assumption (A_1) being satisfied. As an alternative, when assuming (A_1) does not hold, then the locus L_p is over L_u in a suitable

Figure 6: Different shapes for L_p and local behaviour of $u(z)$ and $p(z)$ in a neighborhood of $z = 1$, where L_p lies over L_u.



neighborhood of $z = 1$. In that case, following the same argument developed in the previous proof, different behaviours for the solution can be obtained. Actually, $u(z)$ increases near $z = 1$ and the increasing/decreasing pattern for L_p depends on the shape of $\mu(x)$ in a neighborhood of $x = 0$. In particular, the L_p curve is decreasing with respect to z when

$$\mu'(0) < \frac{-u'(1)(\rho + \lambda + \mu(0))}{u(1)} \ , \tag{22}$$

and it is increasing when

$$\frac{-u'(1)(\rho + \lambda + \mu(0))}{u(1)} < \mu'(0) < 0 \ . \tag{23}$$

In Figure 6, the phase diagram is shown in a neighborhood of $z = 1$, in correspondence with the different increasing/decreasing shapes for L_p. We may argue that, although the function $u(z)$ locally increases near $z = 1$ under both equations (22) and (23), the local behavior for $p(z)$ cannot be uniquely established but it depends on the rate $\mu'(0)$. Indeed, equation (22) implies that $p(z)$ is decreasing (i.e. $u'(z) > 0$ and $p'(z) < 0$ in a neighborhood of $z = 1$); alternatively, when equation (23) holds, $p(z)$ is increasing (i.e. $u'(z) > 0$ and $p'(z) > 0$ in a neighborhood of $z = 1$).

B The return on money and inflation

We define the stochastic expected net return on money for a small time interval Δ as

$$r(x)\Delta = \mathbb{E}\left[\frac{\tilde{q}_{t+\Delta}}{\tilde{q}_t} - 1 \middle| x_t = x\right] = \mathbb{E}\left[\frac{q_{t+\Delta}}{q_t} - \Delta\mu_t - 1 \middle| x_t = x\right]$$

where $\tilde{q}_{t+\Delta} = \tilde{q}(x_{t+\Delta})$ and $\tilde{q}_t = q(x_t)/m_t$. Without loss of generality consider the case where at time t agent 1 is productive with money holdings given by z_t^1. Then, using that $q(x) = p(z)$, we have

$$r(x_t)\Delta = \frac{(1-\lambda\Delta)p(z_{t+\Delta}^1) + \lambda\Delta p(z_{t+\Delta}^2)}{p(z_t^1)} - 1 - \Delta\mu_t \ ,$$

where we used equation (5). A first order Taylor expansion of $p(z_{t+\Delta}^i)$ gives

$$r(x_t)\Delta = \frac{p'(z_t^1)h^p(z_t^1)\Delta}{p(z_t^1)} + \lambda\Delta\left(\frac{p(z_t^2) + p'(z_t^2)h^p(z_t^2)\Delta}{p(z_t^1)} - \frac{p(z_t^1) + p'(z_t^1)h^p(z_t^1)\Delta}{p(z_t^1)}\right) - \Delta\mu_t$$

taking the limit as $\Delta \to 0$ gives: $r(x_t) = \frac{p'(z_t^1)h^p(z_t^1)}{p(z_t^1)} + \lambda\left(\frac{p(z_t^2)}{p(z_t^1)} - 1\right) - \mu_t$

We use equation (8), equation (9) and equation (7) to get $r(x_t) = \rho + \lambda\left(\frac{p(z_t^2)}{p(z_t^1)} - \frac{u(z_t^1)}{p(z_t^1)}\right)$ or, using that in this proof we assumed that agent 2 was unproductive at time t, i.e. $x_t = z_t^2$,

$$r(x_t) = \rho + \lambda\left(\frac{p(x)}{p(1-x)} - \frac{u(1-x)}{p(1-x)}\right)$$

which yields the expression in the text. Define expected inflation as $\pi(x_t) = \mathbb{E}\left[\frac{1/\tilde{q}_{t+\Delta}}{1/\tilde{q}_t} - 1 \middle| x_t = x\right]$ is it easily seen that $\pi(x_t) = -\frac{r(x_t)}{1+r(x_t)}$.

C Proof of Proposition 5

Let $v(x;\lambda)$ denote the (ex-ante) welfare on the economy. Here $v(x;\lambda)$ equals the function defined in equation (18) but where we also made explicit the dependence on the parameter λ. Notice that $v(x;0) = -\infty$. This happens because when the productive state is completely persistent, money has no value and as a result there is no trade. Then, the unproductive agent's consumption level is zero, her utility is divergent, and so does the welfare of the economy.

We now want to show that, for some money rule $\mu(x;\lambda)$, $\lim_{\lambda\to 0} v(x;\lambda) = -\infty$. This, together with the fact that $v(x;\lambda)$ is continuous in λ, implies that the welfare function is not bounded from below. We prove the statement by showing first that $\lim_{\lambda\to 0} v(0;\lambda) = -\infty$. Notice that, for a given λ we have that $h^u(0;\lambda) = 0$ and $u(0;\lambda) = 2/\mu(0)$ so that

$$v(0;\lambda) = \bar{c}\left[\ln\left(\bar{c}\frac{p(1;\lambda)}{2}\mu(0)\right) + \ln\bar{c} - \left(1 + \frac{p(1;\lambda)}{2}\mu(0)\right)\right] + \lambda[v(1;\lambda) - v(0;\lambda)] \ .$$

In order to understand the behavior of $v(0;\lambda)$ with respect to λ, we need to understand the behavior of $u(z;\lambda)$ and $p(z;\lambda)$ with respect to λ. Recall that we are requiring that both functions $u(z;\lambda)$ and $p(z;\lambda)$ are continuous in z. This implies that both functions are bounded and finite in $[0,1]$. As a result, both $u(z;\lambda)$ and $p(z;\lambda)$ cannot diverge to $+\infty$ when $\lambda \to 0$. Moreover, recall that the boundary condition requires that

$$p(1;\lambda) = \frac{\lambda}{\rho + \lambda + \mu(x)} u(1;\lambda) \ .$$

Given that we already established that $u(1;\lambda)$ is finite as $\lambda \to 0$, the boundary condition implies that $\lim_{\lambda\to 0} p(1;\lambda) = 0$. As a result, we get $\lim_{\lambda\to 0} v(0;\lambda) = -\infty$. Then, this implies that $\lim_{\lambda\to 0} v(x;\lambda) = -\infty$. Then, because $v(x;\lambda)$ is continuous in λ, for any $K \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all λ satisfying $0 < \lambda < \varepsilon$ we have that $v(x;\lambda) < K$.

Let v_c denote the welfare value associated with the complete markets allocation. Notice that, because the consumption is constant at the level $\bar{c}$ for any set of parameter values and wealth distribution x, v_c is a constant. Moreover, using the welfare definition in equation (18) we get that

$$v_c = 2\bar{c}\frac{\ln \bar{c} - 1}{\rho} \ ,$$

which is finite as long as $\rho > 0$. Now, let $K = v_c$. Then, there exists $\bar{\varepsilon} > 0$ such that for all $\lambda \in (0, \bar{\varepsilon})$ we have that $v(x;\lambda) < v_c$. This immediately shows that, for any $\lambda \in (0, \bar{\varepsilon})$, there is no policy rule $\mu(x)$ which is able to generate the welfare implied by the complete markets allocation.