# Embracing Risk: Hedging Policy for Firms with Real Options*

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**ABSTRACT**

This study analyzes the dynamic risk management policy of a firm that faces a tradeoff between minimizing the costs of financial distress and maximizing financing for investment. Costly external financing of investment discourages hedging because of the option to abandon investment at low profitability and the option to expand investment at high profitability. Our theory generates results consistent with actual policies, without relying on the costs of risk management. First, we show that firms with safe assets and fewer growth options can choose to hedge more aggressively than firms with risky assets. Second, firms prefer to hedge systematic rather than firm-specific risk, even when hedging technologies for both types of risk are available. Third, risk is optimal at lower net worth. Therefore, more constrained firms may appear to hedge less aggressively.

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Contemporary research on determinants of corporate hedging policy has stumbled upon several facts that are seemingly inconsistent with firm optimization. Three often-mentioned observations are especially puzzling: i) hedging activity tends to be concentrated in large mature firms with few growth options (Bartram, Brown, and Fehle (2009), Nance, Smith, and Smithson (1993), Tufano (1996), Haushalter (2000), Graham and Rogers (2002), Mian (1996), Allayannis and Ofek (2001), Gay and Nam (1998)); firms with tighter financing constraints do not appear to hedge more than firms with fewer constraints (Rampini, Sufi, and Viswanathan (2013)); riskier firms tend to have lower hedging ratios (see, e.g., Guay (1999)). These facts seem to run counter to the basic prediction that hedging policy is dictated by firm risk.

The goal of this paper is to reconcile these empirical regularities with theory by building a model of real investment and financing. The simple fact that the exercise of investment options typically requires a large amount of financing goes a long way toward explaining the observed hedging policies. The need to finance investment can rationalize, in particular, the lack of hedging by the low-net-worth firms and can also explain why riskier firms, which inherently have more investment options, hedge less than safer firms.

Intuitively, risk management does not always create value because it may jeopardize a firm's chances to undertake lumpy profitable investment. For example, consider a company holding a single investment option, which can only be financed internally. A low-expected-net-worth firm will lack sufficient funds for a large investment project and will abandon it unless its cash flow turns out to be very high. Because hedging effectively moves cash away from the high- to the low-profitability states, it makes the prospect of undertaking investment even more remote and is value-destroying absent any other benefits of hedging. In contrast, a high-expected-net-worth firm will have sufficient financing for a project in most, but not all, cases. Hedging is desirable for such a firm because it increases the amount of cash in the low-profitability states, in which investment would otherwise be impossible, and at the same time preserves the level of investment in the high-profitability states.

Further, because the value of investment options naturally increases with profitability, an additional "cash flow correlation" effect appears. If options are in-the-money, the cash flows from the existing assets also tend to be high and thus provide a convenient source of investment funds. Over-hedging decreases the correlation between a firm's investment demand and internal funds, thereby necessitating costly external financing. We show that the natural positive correlation between cash flows and investment is higher in riskier firms and in those with more real options, which helps to explain the lower hedging ratios for those firms.

Another way to understand the intuition for why the optimal cash flow risk is higher in this case is by thinking about the firm's financing gap. The usual argument is that risk is undesirable because it causes the firm to have either too much or too little cash to pay for its expenses, assuming that expenses are constant. We find, however, that the firm's investment costs tend to increase with the cash flow. First, the option to abandon investment helps to reduce the investment costs when the cash flow is low. Second, the opportunity to expand investment results in higher investment costs when the cash flow is high. In sum, the firm's revenues and expenses tend to be positively correlated, making the cash flow risk desirable.

Our model exhibits parsimony in most of its assumptions, requiring only that investment cost is significant and that external financing is costly. The optimal risk management strategy balances the costs of financial distress with the ability to finance investment. We first develop intuition for the optimal hedging ratio in the one-period setting and then build the intertemporal model with dynamic cash accumulation.

Using the model, we obtain several new results. First, the relation between the hedging ratio and asset risk is non-monotonic. Firms with riskier assets may want to leave a larger proportion of their risk unhedged because, intuitively, they have more valuable real options. Cash flows and the demand for financing tend to have a higher correlation in these firms. In contrast, safer firms exercise fewer growth options and are mainly concerned with eliminating negative cash flow outcomes, which is accomplished by hedging most of their risk exposure.

Second, and paradoxically, stronger financing constraints can lead to less hedging. In particular, we show that the low-expected-net-worth firms maximize the value of their investment by leaving their profits unhedged. In a sense, this effect is akin to the risk-shifting behavior of levered firms. The difference, however, is that we are discussing the financing effect that arises because a higher risk in cash savings can increase the probability of investment. Shareholders are protected by the option to abandon investment when there is a lack of financing, and the value of this option increases with the riskiness of the firm's cash flow. Therefore, our results provide a new explanation for insufficient risk management that does not require high costs of hedging or agency problems between shareholders and debtholders.

Third, using the dynamic model with cash accumulation, we derive the value of cash inside the firm, examine the incentive to increase the risk of cash savings, and show the benefit of a dynamic risk management policy relative to a one-shot policy. In static models, a fixed hedging ratio must balance losses in some regions with gains in others. One must evaluate the firm's payoff function over the whole domain of the underlying risk variable to determine whether this function is overall concave or convex. Under the dynamic policy, the firm can tailor its hedging strategy to specific circumstances, such as the current liquidity position. For example, we show that when the firm's current cash savings are large, the value function is concave and therefore hedging is optimal. In contrast, firm value exhibits convexity when cash savings are relatively low, and it becomes optimal to increase the volatility of cash savings to maximize the value of investment.

Further, we show that the optimal risk management strategy is influenced by the strategies of other firms in the economy. The model predicts that the optimal hedging ratio will be higher for firms with more systematic risk (i.e., common risk across firms) and lower for firms with more idiosyncratic risk. To arrive at this result, we rely on the view that firms mainly compete for the common component of their profits and therefore derive less value from investment options that are linked to the overall state of economy.[1] In turn, the limit to growth imposed by the interaction between firms decreases the correlation between cash flows and option value. Risk management policy therefore depends not only on the amount but also on the type of risk that firms are subjected to.

[1] The basic idea is that competition erodes the value of a firm's real options (see, e.g., Grenadier (2002)), and that competition matters more if risk is systematic. For example, if the investment opportunities improve uniformly for all firms in an industry, the increase in competition associated with higher aggregate production and new entry into the market will limit each firm's profits (see, e.g., Caballero and Pindyck (1996) for a discussion of competition effects on real options). However, when a firm's success is *unique*, its real options are likely to increase in value, implying that firms that start with unique assets derive a larger component of their market value from real options than firms with generic assets.

The rest of the paper is organized as follows. The next section offers a brief literature review. Section 2 presents a single-period model, which explains how risk management affects investment. Section 3 lays out a general continuous-time model of investment under financing constraints, which allows for cash savings and a dynamic hedging ratio. The last section concludes.

## I. Literature Review

Most of the previous studies focus on the positive effects of risk management. Benefits come from the reduction in expected bankruptcy costs (e.g., Smith and Stulz (1985) and Graham and Smith (1999)); higher debt capacity (Leland (1998) and Graham and Rogers (2002)); convexity in operating costs and/or concavity in the production function (Froot, Scharfstein, and Stein (1993) and Mackay and Moeller (2007)); improvement in contracting terms with firm creditors, customers, and suppliers (Bessembinder (1991)); mitigation of information asymmetry (DeMarzo and Duffie (1995)); and reduction in management overinvestment incentives (Morellec and Smith (2007)).

It is somewhat more difficult to justify a preference for risk, especially given the recent developments in derivatives trading and reductions in transaction costs. Froot, Scharfstein, and Stein (1993) show that firms may choose not to eliminate risk exposure when their cash flow is positively correlated with profitability of investment. The study by Adam, Dasgupta, and Titman (2007) shows that incomplete hedging can arise from competition. They recognize that a firm's optimal risk management can depend on its competitors' strategies. The goal is to increase cash reserves in those states of the world where the competitors lack financing. Fehle and Tsyplakov (2005) present a dynamic model, in which full hedging is suboptimal when the firm is in deep financial distress or very far from it. Their results rely on leverage, the costs of financial distress, and the fixed costs of hedging.

Methodologically, we contribute to the growing literature on dynamic risk management, cash policies, and investment. Bolton, Chen, and Wang (2011) build a structural investment model with adjustment costs. In their model, the firm's investment opportunity set is continuous and does not depend on the current profitability shock; therefore the value function is everywhere concave, inducing hedging. Hugonnier, Malamud, and Morellec (2014) derive the optimal financing and cash policies in a model with lumpy investment. In their setup, the cash policy of a firm without growth options is characterized by a standard "barrier" policy, i.e., the firm distributes cash when its holdings are above the target and raises equity financing when its holdings are below the target. They also show that for a firm with investment options, the barrier policy may no longer be valid. The firm optimally retains cash in the pre-investment region waiting to invest and finance internally when cash reaches a critical value. Therefore, in their model , cash risk may also be beneficial to shareholders.

Decamps, Gryglewicz, Morellec, and Villeneuve (2015) study cash and risk management policies of firms that are subject to permanent and temporary cash flow shocks. Using the setup similar to Decamps, Mariotti, Rochet, and Villeneuve (2011), they show that firms may prefer to increase cash flow volatility because it increases cash flow correlation to permanent profitability shocks. Finally, they show that when cash flow shocks are permanent, hedging (using derivatives) and asset substitution (selecting real assets with different risk) are not equivalent because the former generates immediate cash flow, but the latter does not. Similar result holds in our model, but for a different reason; the value of the option to invest increases with the risk of the assets but not with the risk of cash flows.

Similar to us, Rampini and Viswanathan (2010) obtain the result that financially constrained firms can prefer to hedge less, but using a different mechanism. In their model, both financing and risk management involve promises to pay that need to be collateralized, thereby resulting in a tradeoff between a firm's ability to finance current investment and engage in risk management in order to maximize future investment. Since more constrained firms find it more advantageous to use the available cash for current investment, a negative relation between financial constraints and risk management arises. In our setting, there are no collateral constraints, and hence risk management is not limited by the current amount of cash. Nevertheless, firms with low expected-net-worth can prefer to hedge less because this allows them to maximize the probability of future investment.

Our work is also related to studies on investment and firm liquidity under financing constraints. Boyle and Guthrie (2003), Dasgupta and Sengupta (2007), and Bolton, Wang, and Yang (2013) study the implications of future cash flow uncertainty and the current level of cash for current investment. In their papers, cash flow risk is exogenous, and it matters because firms choose to accelerate or delay investment keeping the possibility of future funding shortfalls in mind. In our model, firms choose their risk management policies optimally.

# II. Single-Period Analysis

To fix ideas, we develop a simple one-period example of risk management under financing constraints that builds on the investment literature. We first provide the solution for the optimal hedging strategy and then proceed with a discussion of how a firm's net worth, firm risk and its composition, product market competition, and investment opportunities affect the optimal hedging ratio. In Section III, we present a full model with cash accumulation and dynamic risk management policy, which extends the simple example.

## A. Preliminaries

There are three dates in the model corresponding to: (1) hedging strategy, (2) cash flow realization, external financing, and investment, and (3) the final payoff.[2] There is no discounting. We assume that the firm can hedge its cash flow risk by buying forward contracts and postpone the discussion of nonlinear hedging strategies to Section II.E. For the firm that adopts hedging ratio $\phi$, the cash flow available at date 2 is

$$w = w_0 + w_1 \left(\phi\bar{\varepsilon} + (1-\phi)\,\varepsilon\right), \tag{1}$$

where $\varepsilon$ is the primitive uncertainty (the cash flow shock) governed by the probability distribution function $g(\varepsilon)$ with mean $\bar{\varepsilon}$ and variance $\sigma^2$. By construction, hedging decreases cash flow variability but leaves the expected cash flow, $w_0 + w_1\bar{\varepsilon}$, unchanged. This assumption, in particular, implies that hedging has no direct costs. Note that if $\phi = 1$, the firm's internal funds are completely independent of the profitability shock.

[2]If the payoff is immediate, the last date is redundant. This notation simply clarifies that the final payoff cannot be used to pay for investment or operating costs.

The cash flow $w$ can be used to pay for firm's operating costs and investment at date 2. Whenever the firm is short of internally generated funds, it can raise external financing $e$ available at a cost $C(e)$. Following Kaplan and Zingales (1997) and Hennessy and Whited (2007), we assume that the external cost function is increasing and convex ($C_e > 0$, $C_{ee} > 0$).

Operating costs $R$ are paid at date 2 to keep the firm running and to retain the claim on the final payoff $f_0$. The payoff is high enough so that liquidating the firm with the goal of avoiding the operating costs is never optimal, i.e., the firm is better off by raising external financing than liquidating.[3]

At the management's discretion, the firm can also invest an additional amount $I - R > 0$ to replace the existing assets and to increase firm profitability. If this investment option is exercised, the payoff at date 3 changes from $f_0$ to $\theta f(I)$, with variable $\theta$ capturing the firm's investment opportunities at date 2. Note that the assumption that new assets will render old ones unproductive (profit $f_0$ is not available if investment is made) is equivalent to assuming that there are fixed costs of investment. Such costs may, for example, originate from new investment cannibalizing profits associated with assets in place (see, e.g., Hackbarth, Richmond, and Robinson (2012)).

To model the fact that firm's investment opportunities are correlated with cash flow shocks, we specify

$$\theta = \alpha (\varepsilon - \overline{\varepsilon}) + \beta, \tag{2}$$

where $\alpha \geq 0$ captures the positive correlation and higher $\beta$ implies better overall investment opportunities. We also assume that investment technology has decreasing returns to scale, $f_I > 0$, $f_{II} < 0$.

[3]The condition for no liquidation is $f_0 - R - C(R - w) > 0$. It is straightforward but unnecessary to model liquidation or bankruptcy costs because the costs of external financing $C()$ already punish the firm for having a low cash flow.

## B. Optimal Investment

The firm decides whether to invest and how much to invest. Should the firm invest, it chooses the optimal amount of investment by solving:

$$\max_{I}\{\theta f(I) - I - C(e)\}, \tag{3}$$

which gives $I^*$ through the first-order condition:

$$\theta f_I\left(I^*\right) - 1 = C_e\left(e\right). \tag{4}$$

In general, investment is optimal only if the profit net of investment and financing costs is higher than the profit associated with running existing assets

$$\theta f(I) - I - C(e) \geq f_0 - R - C(e_0), \tag{5}$$

where $e$ and $e_0$ are the financing gaps given, respectively, by $e = I - w$, $e_0 = R - w$.

Setting (5) to equality gives the implicit condition for the investment threshold $\varepsilon^*$, such that investing in the new technology is optimal only for the cash flow shock above such threshold, i.e., when $\varepsilon > \varepsilon^*$.

The option to abandon investment is an important feature of the model that differentiates our study from the past risk management literature. If the firm *must* invest, its sole concern is minimizing the financing gap, and therefore a low cash flow risk is optimal. Realistically, however, it may be better to drop the investment project altogether if the external financing is too costly and the present value of investment is relatively small. In this case, the cash flow risk may be beneficial.

To summarize, the firm's profit function in the no-investment (inaction) and investment region is given by

$$\Pi\left(\varepsilon\right) = \left\{ \begin{array}{cc} P = \theta f(I^*) - I^* - C(e), \text{ if } \varepsilon \geq \varepsilon^*, & \text{investment region} \\ P_0 = f_0 - R - C(e_0), \text{ if } \varepsilon < \varepsilon^*, & \text{inaction region} \end{array} \right\} \tag{6}$$

where the threshold $\varepsilon^*$ is determined endogenously.

Before we proceed to analyzing the firm's optimal hedging policy, it is worthwhile to examine the incremental effect of hedging on investment. As the following lemma shows, the answer depends on whether the cash from the firm's fully hedged position is sufficient to cover the investment cost.

**Lemma 1.** *Suppose there exists an interior solution for the investment threshold $\varepsilon^*$. Then, we have:*
*(i) If $\varepsilon^* > \bar{\varepsilon}$, then hedging increases the investment threshold (investment is more selective), $\frac{d\varepsilon^*}{d\phi} > 0$, and decreases the optimal investment level, $\frac{dI^*}{d\phi} < 0$, for $\forall \varepsilon$.*
*(ii) If $\varepsilon^* < \bar{\varepsilon}$, then hedging decreases the investment threshold (investment is less selective), $\frac{d\varepsilon^*}{d\phi} < 0$. The effect of hedging on the optimal investment level is ambiguous.*

Intuitively, when a firm decides whether to invest, the firm internalizes the costs of external financing, which increase with the financing gap. By hedging the firm effectively moves cash from the high-profitability states (i.e., from the states above the average, $\varepsilon > \bar{\varepsilon}$) to the low-profitability states ($\varepsilon < \bar{\varepsilon}$). In case (i) in the lemma, the investment is only optimal in the high-profitability states. Therefore, it will require a larger amount of external financing when the firm decides to hedge. This will make investment less profitable and lead to an increase in the investment threshold. In contrast, if investment was feasible in the low-profitability states without hedging, it will require a smaller amount of external financing after hedging, which results in a decrease in the investment threshold (case (ii)).

## C. Optimal Hedging Ratio: Uncorrelated Investment

We first consider a case when the investment payoff is uncorrelated with firm profitability, $\alpha = 0$. For example, one can think of a firm that can invest \$1 million to obtain a guaranteed fixed payoff of \$1.2 million. An unconstrained firm would always invest and pocket \$0.2 million from this deal, while a constrained firm may either invest less than \$1 million or completely abandon the investment if external financing is expensive.

The optimal hedging ratio maximizes the firm's expected profit:

$$\phi^* = \underset{\phi \in [0,1]}{argmax} E\left[\Pi(\varepsilon)\right]. \tag{7}$$

**Proposition 1.** *Suppose $\alpha = 0$ and $\phi \in [0,1]$. Then, the optimal hedging ratio is given by:*

$$\begin{aligned} \phi^* &= 0 \text{ for } \varepsilon^* \in [\varepsilon_L, \varepsilon_H], \\ \phi^* &= 1 \text{ otherwise,} \end{aligned}$$

*where $\varepsilon_L$ and $\varepsilon_H$ are defined in the Appendix.*

Two opposing effects determine the choice of optimal hedging. On the one hand, convexity of costs ($C_{ee} > 0$) and concavity of revenues ($f_{II} < 0$) create an incentive to hedge, as in Froot, Scharfstein, and Stein (1993). On the other hand, because the firm has an investment option, there is a disadvantage to hedging. The benefit of hedging is independent of the investment threshold, but the disadvantage is largest when the option value is large, i.e., when the investment option is neither far out-of-the-money nor deep in-the-money. Therefore, a firm will prefer not to hedge for intermediate investment thresholds $\varepsilon^* \in [\varepsilon_L, \varepsilon_H]$.

Using the proposition, we next examine the comparative statics with respect to profitability of firms investment opportunities, $\beta$, the firm's wealth, $w_0$, and volatility, $\sigma$. Note that a high investment threshold $\varepsilon^*$ means that the firm is highly constrained.

First, we show that hedging ratio can be lower for firms with more valuable investment options.[4]

**Corollary 1.** *Suppose $\varepsilon^* > \varepsilon_L$. Then, firms with more valuable growth options choose to hedge less, i.e., $\frac{\Delta\phi^*}{\Delta\beta} \leq 0$.*

This result follows from the fact that for firms with very poor investment opportunities (low $\beta$), the probability of investment is small. Therefore, the risk management policy in these firms is mostly driven by the desire to minimize the costs of financial distress, which is achieved by choosing high hedging ratios. In contrast, firms with better investment opportunities are concerned about investment financing, which results in choice of lower hedging ratios.

Second, we show that tighter financing constraints (lower $w_0$) can lead to less hedging.

[4]The empirical literature generally finds that hedging activity is concentrated in firms with few growth options, as measured by their book-to-market ratios (see, e.g., Bartram, Brown, and Fehle (2009), Mian (1996), Graham and Rogers (2002), Allayannis and Ofek (2001), Haushalter (2000), Gay and Nam (1998). Two papers by Guay (1999) and Geczy, Minton, and Schrand (1997) find the opposite result.

**Corollary 2.** *Suppose* $\varepsilon^* < \varepsilon_H$. *Then, easing of financing constraints leads weakly to more hedging, i.e.,* $\frac{\Delta\phi^*}{\Delta w_0} \geq 0$.

The intuition is that greater savings lead to lower costs of external financing and therefore a lower investment threshold. In this case, it is optimal for a firm to fully hedge its risk exposure since such action guarantees investment and at the same time decreases the costs of external financing when the profitability is low.

Finally, we investigate the role of volatility. Observing that the effects limiting the value of hedging come from the firm holding investment options, we conjecture that the high volatility firms—i.e., companies holding more valuable options—will be less aggressive in their cash flow risk management.

**Corollary 3.** *There exists* $\overline{\sigma} > 0$*, such that for* $\sigma \in [0, \overline{\sigma}]$ *hedging ratio weakly decreases in cash flow volatility, i.e.,* $\Delta\phi^*/\Delta\sigma \leq 0$.

To understand why riskier firms may prefer lower hedging ratios, consider the case when $\varepsilon^* > \overline{\varepsilon}$ and the volatility of the profitability shock is close to zero. In this case, the probability of ever reaching the investment threshold is negligibly small so that investment does not affect the risk management choice. However, because of the concavity of value function, any risk is undesirable, and the hedging ratio is equal to 1.[5] When the volatility increases, however, the probability that the shock crosses the threshold increases. Additionally, not only the probability but also the level of investment becomes larger because larger cash flow shocks can be drawn from a distribution with a higher volatility. As a result, we have a lower hedging ratio. Therefore, $\phi^*$ must decrease in volatility at least in some continuous range of the volatility parameter values.

[5] More formally, observe that the threshold $\varepsilon^*$ is independent of volatility and that $\varepsilon^* > \overline{\varepsilon}$; therefore $\Pr(\varepsilon > \varepsilon^*) \longrightarrow 0$ when $\sigma \to 0$.

## D. Optimal Hedging Ratio: Positive Investment-Cash Flow Correlation

**Proposition 2.** *There exists a threshold* $\underline{\alpha} > 0$*, a solution to (89) in the Appendix, such that: i) for* $\alpha > \underline{\alpha}$*, the optimal interior hedging ratio is given by*

$$\phi^* = 1 - \frac{\alpha}{w_1} \frac{\int_{\varepsilon^*}^{\infty} f_I k(e) \, dG}{\int_{-\infty}^{\varepsilon^*} C_{ee}(e_0) \, dG - \int_{\varepsilon^*}^{\infty} \theta f_{II} k(e) \, dG}, \tag{8}$$

*where*

$$k(e) = \frac{C_{ee}(e)}{C_{ee}(e) - \theta f_{II}(e)}, \tag{9}$$

*ii) for* $0 < \alpha < \underline{\alpha}$*, the optimal hedging ratio is given by a corner solution, i.e.* $\phi^* = \phi_{\min}$ *or* $\phi^* = \phi_{\max}$.

The proposition shows that the natural tendency of real options to become more valuable when the firm is profitable can discourage hedging. In particular, it follows from (8) that firms with a positive correlation between cash flows and investment opportunities ($\alpha > 0$) can use incomplete hedging and that the amount of optimal hedging decreases when the correlation is greater.

## E. Illustrations and Comparative Statics

Figure 1 illustrates the effect of hedging on financing of investment and cash shortfall in the distress region. The firm has operating costs $R = 20$ and can also invest a fixed amount $I - R = 80$ in the profitable project. For facilitate the exposition, we assume that external financing is prohibitively expensive and investment is possible only out of internal funds. Parameters are chosen in such a way that investment always generates a higher profit than running existing assets. Panels A and B are respectively for the cases of $w_0 = 0$ (low net worth) and $w_0 = 30$ (high net worth). If the low-net-worth firm fully hedges, the investment never occurs. In contrast, the high-net-worth firm always invests when fully hedged. The filled areas show financing slack and financing shortfalls in the regions of investment (on the right) and distress (on the left) when the firms do not hedge.

In Figure 2, we plot the expected investment level as a function of hedging policy when cash flows and investment opportunities are uncorrelated ($\alpha = 0$). For illustration purposes, we assume a logarithmic payoff function and quadratic costs of external financing. Lemma 1 establishes that the effect of risk management on investment depends on the profitability of investment. If the investment option is out-of-the-money when hedging ratio is zero, then hedging decreases the probability of investment. Indeed, Panel A shows that the expected investment steadily drops with hedging, with the probability of investment eventually going down to zero at high hedging ratios. In contrast, if the investment option is in-the-money when hedging ratio is zero, hedging encourages investment. This effect is observed in Panel B. In fact, when hedging ratio is high, the investment reaches the first-best level, i.e., the investment made by an unconstrained firm.

Figure 3, Panels A and B, provide an illustration of the volatility effect on optimal hedging policy. We plot the optimal investment amount and cash flow realizations for a firm with positive correlation between investment opportunities and cash flows. The difference between the investment amount and cash represents the "financing gap," which requires external financing. Because of a positive correlation between investment opportunities and cash flow, the optimal investment amount increases with cash flow. Panel A shows what happens when the firm has high expected distress costs in a low profitability state but does not hedge any risk exposure. For comparison we also provide a corresponding function in Panel B, which shows what happens when the firm hedges completely. It is clear that if the firm fully hedges, it minimizes the costs of financial distress where cash is low (since a constant financing gap minimizes convex costs). However, the financing gap of the firm is not constant in the investment region and increases with higher cash flow shocks. Therefore, the firm with a high hedging ratio will incur, in expectation, large costs of raising external financing for investment. In fact, by comparing Panel A and Panel B, we can see that a fully hedged firm will even decrease the amount of state-contingent investment because of financing costs.

The next pair of panels illustrates how firm risk affects hedging. Panels B and C plot the investment demand and cash relation for the case when the volatility of a firm's cash flow is low (Panel C and D). Intuitively, the probability of state-contingent investment is low, and therefore demand for cash is unchanged across states of profitability. It follows that maximum hedging is optimal in this case.

Next, Figure 4 shows how the value of investment and the optimal hedging ratio change with volatility of the assets. In Panel A of Figure 4, we show the expected investment amount of a constrained firm (solid line). The dashed line displays the expected investment level of an unconstrained firm. It is easy to see that financial constraints affect both the level and probability of investment, and that the expected investment increases in volatility. Finally, in Panel B of Figure 4, we plot the optimal hedging ratio of a firm as a function of volatility. Indeed, we observe that at low volatility, the firm prefers to hedge all of its risk exposure. However, as volatility increases, the optimal hedging ratio drops and can even become negative, which means that a firm may choose to speculate.

Panel A of Figure 5 shows how the optimal hedging ratio changes with financing constraints for a firm with uncorrelated investment opportunities (dashed line) and for a firm with a positive correlation between investment opportunities and cash flows (solid line). The graph shows that the firm with a positive correlation never chooses full hedging and always maintains a lower hedging ratio than the firm with uncorrelated investment opportunities. When financing constraints ease (higher $w_0$), both types of firms may prefer to hedge more aggressively, but the relation between constraints and hedging is non-monotonic.

In Panel B, we plot the optimal hedging ratio as a function of firm's investment opportunities $\beta$. It is clear that firms with higher positive correlation between their cash flows and investment opportunities tend to have lower hedging ratios. Furthermore, the relation between risk management and the value of growth options is non-monotonic, which could explain the mixed empirical results on the relation between hedging ratios and firms' market-to-book ratios.

Finally, Figure 6 shows the relation between the correlation between investment opportunities and cash flows, $\alpha$, and a firm's optimal hedging ratio. Since larger positive $\alpha$ implies a better coordination between firm's investment needs and internal funds, it is intuitive that the optimal hedging ratio decreases with $\alpha$. However, the shape of the function crucially depends on the value of investment options, $\beta$. For example, when $\beta = 1.5$, the hedging ratio function is flat and equal to zero irrespective of $\alpha$ .

## F. Economy with Multiple Firms

In the benchmark model we consider the firm's decisions in isolation. However, both investment and risk management strategies must be determined in conjunction with strategies of other firms.[6] It is intuitive that real options are less valuable in a competitive economy since increased production by other firms can depress the price of output and therefore impose a limit to profitability. Since the profitability of new investment is reduced by competition, firms invest less and are also less concerned with financing investment. Our model then predicts a higher optimal hedging ratio.

The relation between real options and competition is well known in the real options literature; however most of the related studies are concerned with the exercise timing. For example, Grenadier (2002) argues that competition decreases the value of real options and the advantage of waiting to invest. In contrast, Leahy (1993) and Caballero and Pindyck (1996) argue that despite the fact that the option to wait is less valuable in a competitive environment, irreversible investment is still delayed because upside profits are limited by new entry. By focusing on nonlinear production technology, Novy-Marx (2007) shows that firms in a competitive industry may delay irreversible investment longer than suggested by a neoclassical framework. None of these studies, however, analyze hedging incentives.

We formalize the intuition by representing the total cash flow of the firm as the sum of two parts—the component of the profit common across all firms and a firm-specific profit component. We assume that firms are identical aside from the differences in their cash flow composition. The required adjustment to the previous section's model is as follows: we supply the profitability shock with a firm-index "$i$" and separate it into common and idiosyncratic components

$$\varepsilon_i = \beta_i v_m + \sqrt{1-\beta_i^2} v_i, \tag{10}$$

where we denote $0 < \beta_i < 1$ the sensitivity of the total cash flow to the common shock. It is

[6] Zhu (2012) empirically analyzes the relation between hedging policies and competition and concludes that the hedging strategy a firm chooses affects the probability of the exit.

related but not necessarily equal to the market "beta." We assume that the two components have the same mean $\overline{v}$ and are drawn from an identical probability distribution with density function $G(v)$. This assumption allows us to vary the mix of firm-specific risk (by changing $\beta_i$) without changing the volatility of total cash flow.

With competition among different firms, the optimal investment strategy turns out to be a function of both the total cash flow shock $\varepsilon_i$ and the common component of profit $v_m$. Since competitors are more likely to invest with high shock $v_m$, a particular firm's investment is more profitable when $v_m$ is relatively low and the idiosyncratic component $v_i$ is relatively high.

It is important that in our model firms compete to the extent that their profit shocks contain the common component. Therefore, firms in the economy that have a larger proportion of the systematic profit risk are less valuable and have fewer real options. The following proposition summarizes these facts.

**Proposition 3.** *In the economy with multiple firms:*

*(i) The optimal hedging ratio* $\phi^*$ *increases with competition.*

*(ii) The optimal hedging ratio* $\phi^*$ *increases with* $\beta_i$.

The proposition states that the optimal hedging ratio increases with the systematic risk exposure because the probability of exercise decreases. Even if the total cash flow is currently high, firms may not invest heavily because they expect new entry to reduce profitability in the future. In contrast, firms with a high proportion of unique risk possess valuable investment options and expect to invest in the future. Therefore, the correlation between their cash flows and investment is large and leads to their adoption of a lower hedging ratio.[7]

The one-period model provides the basic intuition for the static problem. However, it assumes that firms can only finance investment out of their cash flows or using external

[7] Mello and Ruckes (2005), Adam, Dasgupta, and Titman (2007) analyze optimal hedging in models with product market competition. In their setting, firms choose a hedging policy simultaneously with their rivals in anticipation of the opponent's strategy and with the purpose to increase the chances of its own survival in competition. The mechanism in our model is different. The ex-post competition decreases the value of investment options that are not unique to the firm. This approach is more similar to that in the real options literature (see, e.g., Grenadier (2002), Novy-Marx (2007)).

financing. This may be unrealistic because liquidity issues and a persistent wedge between internal and external financing costs force firms to save. We next consider a dynamic model.

# III. Dynamic Model with Cash Accumulation

This section presents and solves the model with cash savings, investment, and dynamic risk management policy. First, we present the dynamic model of a financially constrained firm with risky cash holdings and a simple investment option. Second, we analyze the firm's incentive to increase or decrease risk and derive the optimal hedging strategy. Third, we present a more general case when the underlying stochastic profitability can affect both the firm's cash holdings and the value of its investment option. The last step is complicated by the fact that firm profitability presents another state variable, in addition to cash savings.

## A. Risky Cash Inventory and Risk Management

We first present the base model, with a single source of risk.

### A.1. Evolution of Cash Inventory

The firm is initially endowed with a stock of cash $C_0$ and can carry the cash balance forward by investing it in a risk-free security that earns a riskless rate of return $r$.[8] Cash inventory is subject to random shocks due to uncertainty in the production process or unexpected expenses. Cash process evolves according to

$$dC_t = rC_t dt + \sigma C_t dB_c. \tag{11}$$

The first term above reflects the interest earned on the running cash balance, whereas the second term reflects the cash inventory risk. It captures the uncertainty in the firm's financing environment, such as unanticipated expenses, revenues, settlement of legal disputes, or proceeds from employee stock option exercises.

[8] We deviate from the analysis in Bolton, Chen, and Wang (2011) by assuming in the base model that carrying cash is costless. Because of this assumption, the voluntary payout to shareholders is never optimal in the base model. The cash cost, such as tax disadvantage to savings on the corporate account, can be easily accommodated in the model by reducing the interest rate. In this case, the optimal payout policy would amount to distributing excess cash when the accumulated cash process reaches an upper boundary.

Because the firm has no leverage, it never finds it optimal to default even if a series of negative shocks cause its cash inventory to decrease to zero.[9] To model the incentive to hedge and also to avoid the singularity problem when the cash variable approaches zero from above, we assume that the firm has to satisfy the minimum working capital requirement. We therefore impose the minimum working capital requirement on savings, $C \geq \underline{C}$. Once the cash level falls below this threshold, the firm must raise more cash at the price $k_d > 1$. The fact that rasing cash entails a cost creates an incentive for the firm to avoid the distress states, similar to the effect created by the bankruptcy costs.[10] If, on opposite, cash reserves become excessively large the marginal value of cash becomes one, meaning that the firm is indifferent between retaining cash or disbursing cash to shareholders as dividends or share repurchases.

### A.2. Option to Invest

The firm may have an opportunity to invest in a new project at time $\tau$. We model the investment option in a simple way to keep the model tractable. In particular, if a Poisson shock with intensity $\lambda$ arrives at time $\tau$, the firm can invest an amount $I$ and obtain an instantaneous payoff of $\Theta$, where $\Theta > I$. If the firm does not have a sufficient cash balance to make investment, it needs to raise external financing and pay the associated financing costs $k\,(I-C)^+$. We allow for the possibility that $k \neq k_d$ because the costs of raising external financing in distress can, in principle, differ from costs incurred at other times. The option is worthless if the costs of financing exceed the benefit of the investment.

For tractability purposes, we assume that after investment is made the firm pays a dividend and is immediately liquidated. The value of the firm at date $\tau$ is then equal to the final payout

$$D(C) = C + \max\left(0, \Theta - I - k\,(I-C)^+\right). \tag{12}$$

[9]It is never optimal to liquidate an unlevered firm because the cash inventory is guaranteed to stay positive. This is the common feature in the cash flow models that use the geometric Brownian motion (e.g., Leland (1994)).

[10]One could interpret such cost as a proportional cost of financial distress or a refinancing cost that is incurred in the low-cash states. We assume that the firm cannot chose $\underline{C}$. In a setup with no financial leverage, the optimal value of $\underline{C}$ is zero because it minimizes the cost of expected recapitalization.

### A.3. Hedging Instrument

To mitigate the cash savings risk the firm can buy financial securities (e.g., futures or customized hedging contracts), that carry payments correlated with the firm's profitability. The firm can optimize dynamically over its hedging policy and is restricted to using linear contracts. In particular, at any point in time, the firm can enter into a short position in $\Phi_t$ futures contracts, subject to a proportional cost $\pi$. The costs associated with hedging can be thought of as either direct transaction fees or the cost of holding cash in a margin account and posting collateral. Transaction costs do not drive any of our results and we will be assuming zero proportional cost for the base case to emphasize this fact. The futures price $F_t$ is driftless under the risk-neutral measure and is assumed to evolve as[11]

$$\frac{dF_t}{F_t} = \sigma_F dB_F, \tag{13}$$

$$E\left[dB_c dB_F\right] = \rho_c dt. \tag{14}$$

Anticipating that the amount of money $\Phi_t$ invested in the hedging portfolio is proportional to the size of cash inventory, we normalize the size of hedging position $\Phi_t$ by cash and define the hedging ratio as $\phi_t = \Phi_t F_t / C_t$.[12] Hence, we can amend the cash process with the net proceeds from the hedging portfolio

$$dC_t = rC_t dt + \sigma C_t dB - \phi_t C_t \sigma_F dB_F - \pi \phi_t C_t dt, \tag{15}$$

where the last two terms represent the hedging portfolio payoff and cost, respectively. The costs associated with hedging can be thought of as either direct transaction fees or the cost of holding cash in a margin account and posting collateral. Note that complete hedging (meaning that savings is a locally deterministic process) is achieved if there is a perfect correlation between cash innovation shocks and the hedging security, $\rho_c = 1$, and the following hedging

[11] Equivalently, the model is solved with the forward price on the same underlying asset (see Cox, Ingersoll, and Ross (1981) for comparison of the futures and forward prices for the nonstochastic interest rate case).

[12] To verify the conjecture that the futures position $\Phi_t$ is proportional to cash $C_t$, it will be sufficient to show that the optimal hedging ratio $\phi_t$ is independent of cash $C_t$.

portfolio is chosen[13]

$$\Phi_t = \frac{\sigma C_t}{\sigma_F F_t} \text{ or } \phi_t = \frac{\sigma}{\sigma_F}. \tag{16}$$

When a complete hedging is impossible because of imperfect correlation, the hedging portfolio (16) trivially minimizes the conditional variance of cash. As defined, the hedging ratio has a meaningful range between zero (no hedging) and $\frac{\sigma}{\sigma_F}$ (minimum risk), we therefore assume going forward $\phi \in [0, \frac{\sigma}{\sigma_F}]$.

### A.4. Firm Value

Applying the Itô's lemma to firm value $V(C)$ and using the dynamics of cash savings (15), we can describe the firm value function as a solution to the Hamilton-Jacoby-Bellman equation

$$\begin{aligned}(r+\lambda)V(C) &= \max_{\phi_t} CV_C(r-\pi F_t\phi_t) + \frac{C^2}{2}V_{CC}\left(\sigma^2+\phi_t^2\sigma_F^2-2\rho_c\phi_t\sigma_F\sigma\right) \\ &\quad +\lambda C+\lambda\max\left(0,\Theta-I-k(I-C)^+\right).\end{aligned} \tag{17}$$

The left-hand side, intuitively, represents the required return on firm assets, including the required return $r$ and the probability of liquidation $\lambda$, while the right-hand side is the expected rate of change in its value. In particular, the first two terms on the right side capture the effect of cash growth and the payments from the hedging portfolio, while the last two terms are due to the expected exercise of the option and the final dividend payment.

Equation (17) is subject to boundary conditions. First, when the cash level reaches the recapitalization threshold $\underline{C}$, we require that the marginal value of cash is equal to the marginal cost of recapitalization, that is

$$V_C(\underline{C}) = 1 + k_d, \tag{18}$$

where $V_C$ is the first-order derivative of firm value with respect to cash. Second, when the cash level is high $C \to \infty$, the value $V(C)$ must approach the value of the fully unconstrained firm, so that we have

$$V_C(\overline{C})\,|_{\overline{C}\to\infty} = 1. \tag{19}$$

[13] To see this, note that the instantaneous variance of cash innovations is given by $E(dC^2) = \sigma^2 - 2\rho_c\phi\sigma\sigma_F + \phi^2\sigma_F^2$.

### A.5. Base Case Solution

The base case assumes $\phi = 0$ (no hedging).[14] Ordinary differential equation (17) can be solved in closed-form. We conjecture the following solution

$$V(C) = AC^{a_1} + BC^{a_2} + EC + D, \tag{20}$$

where $a_1$ and $a_2$ are constants. Substituting the trial solution (20) into the ODE (17) gives

$$\begin{aligned}&(r+\lambda)\left(AC^{a_1} + BC^{a_2} + EC + D\right) = r\left(a_1 AC^{a_1} + a_2 BC^{a_1} + EC\right) \qquad (21)\\ &+\frac{\sigma^2}{2}\left(a_1(a_1-1)AC^{a} + a_2(a_2-1)BC^{a_2}\right) + \lambda C + \lambda \max\left(0, \Theta - I - k\left(I - C\right)^+\right).\end{aligned}$$

It follows that $\{a_1, a_2\}$ are the solutions to the standard quadratic equation

$$a_{1,2} = \frac{1}{2} - \frac{r}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{r}{\sigma^2}\right)^2 + \frac{2\left(r+\lambda\right)}{\sigma^2}}. \tag{22}$$

Note that $a_1 > 1$ and $a_2 < 0$. Because of the last term in (21), we need to consider three regions: no investment (left region), investment using external financing (middle region), and investment out of cash (right region). If $C > I$, then the firm has sufficient financing to cover the full cost of investment, and the last term in (21) is $\lambda\left(\Theta - I\right)$. If $C < I$, the firm may choose to finance investment by raising external financing or may choose not to invest. The cash reserves that sets the payoff from investment net of costs of external financing to zero determines the threshold, $C^*$, at which the firm starts to invest

$$C^* = I - \frac{\Theta - I}{k}. \tag{23}$$

When $C^* < C < I$, the external financing is used, and the last term in (21) is $\lambda\left(\Theta - I - k\left(I - C\right)\right)$. If $C < C^*$, the firm chooses not to invest and the last term is 0. Using (21), we can solve for constants $E$ and $D$

$$\begin{aligned} E &= 1, D = 0 \text{ for } C \leq C^*, \qquad (24)\\ E &= 1 + k, D = \frac{\lambda\left(\Theta - I - Ik\right)}{r+\lambda} \text{ for } C^* < C < I,\\ E &= 1, D = \frac{\lambda\left(\Theta - I\right)}{r+\lambda} \text{ for } C \geq I,\end{aligned}$$

[14] In the Appendix, we solve a simpler case, with internal financing only ($k$ very large). It may be more intuitive because the solution is necessary only for two regions, where cash is sufficient for investment and where investment is dropped.

and the firm value function in each region, $V_L$, $V_M$, and $V_R$, is given by

$$\begin{aligned} V_L(C) &= A_L C^{a_1} + B_L C^{a_2} + C \text{ for } C < C^*, \\ V_M(C) &= A_M C^{a_1} + B_M C^{a_2} + C + kC + \frac{\lambda(\Theta - I - Ik)}{r+\lambda} \text{ for } I > C > C^*, \\ V_R(C) &= B_R C^{a_2} + C + \frac{\lambda(\Theta - I)}{r+\lambda} \text{ for } C \geq I. \end{aligned} \tag{25}$$

Free constants $A_L$, $B_L$, $A_M$, $B_M$, and $B_R$ are determined by the value-matching and smooth-pasting conditions at the boundaries $C^*$ and $I$

$$\begin{aligned} V_M(I) &= V_R(I), \\ V_M'(I) &= V_R'(I), \\ V_L(C^*) &= V_M(C^*), \\ V_L'(C^*) &= V_M'(C^*), \end{aligned} \tag{26}$$

and by the boundary conditions (18) and (19) at the lowest and highest values of cash. Condition (19) implies that $V_R(C)$ in the right region does not contain a positive exponent term, therefore we already used $A_R = 0$ in (25). Condition (18) implies

$$V_L'(\underline{C}) = 1 + k_d. \tag{27}$$

Combing (25), (26), and (27), we obtain a system of linear equations which can be solved for all constants

$$\begin{aligned} A_L &= \frac{\left(k - \frac{kra_2}{r+\lambda}\right)\left(I^{1-a_1} - C^{*1-a_1}\right)}{a_2 - a_1}, \ A_M = \frac{\left(k - \frac{kra_2}{r+\lambda}\right) I^{1-a_1}}{a_2 - a_1}, \\ B_L &= \frac{k_d \underline{C}^{1-a_2}}{a_2} - \frac{\frac{a_1 \underline{C}^{a_1 - a_2}}{a_2}\left(k - \frac{kra_2}{r+\lambda}\right)\left(I^{1-a_1} - C^{*1-a_1}\right)}{a_2 - a_1}, \\ B_M &= \frac{k_d \underline{C}^{1-a_2}}{a_2} - \frac{\frac{a_1 \underline{C}^{a_1 - a_2}}{a_2}\left(k - \frac{kra_2}{r+\lambda}\right)\left(I^{1-a_1} - C^{*1-a_1}\right)}{a_2 - a_1} - \frac{\left(k - \frac{kra_1}{r+\lambda}\right) C^{*1-a_2}}{a_2 - a_1}, \\ B_R &= \frac{k_d \underline{C}^{1-a_2}}{a_2} - \frac{\frac{a_1 \underline{C}^{a_1 - a_2}}{a_2}\left(k - \frac{kra_2}{r+\lambda}\right)\left(I^{1-a_1} - C^{*1-a_1}\right)}{a_2 - a_1} + \frac{\left(k - \frac{kra_1}{r+\lambda}\right)\left(I^{1-a_2} - C^{*1-a_2}\right)}{a_2 - a_1}. \end{aligned} \tag{28}$$

Having solved for the value function, we proceed to investigating the properties of the solution.

### A.6. Incentives to Hedge

Intuitively, the local incentive to increase or decrease cash risk is determined by the convexity or concavity of the value function with respect to cash. We therefore investigate the second derivative of the value function, starting from the left region

$$V_L^{''}(C) = a_1 (a_1 - 1) A_L C^{a_1 - 2} + a_2 (a_2 - 1) B_L C^{a_2 - 2}. \tag{29}$$

The claim is that the value function is convex on the left, $V_L^{''}(C)$, so that higher risk is optimal. Because $a_1 > 1$ and $a_2 < 0$, $a_1 (a_1 - 1) > 0$, $a_2 (a_2 - 1) > 0$, and therefore convexity is guaranteed if both $A_L > 0$ and $B_L > 0$. Further, $A_L > 0$ because $I > C^*$ and $B_L > 0$ if

$$k_d < \frac{k a_1 \underline{C}^{a_1 - 1}}{a_1 - a_2} \left(1 - \frac{r a_2}{r + \lambda}\right) \left(C^{*1 - a_1} - I^{1 - a_1}\right). \tag{30}$$

This is a sufficient condition for the value function in the left region to be convex. The intuition is that the convexity/concavity of the value function (and therefore risk-aversion or risk-loving incentive) is a product of the trade-off. On one hand, higher risk increases the probability to move into the region to the right and be able to exercise the valuable investment option. On the other hand, higher risk also increases the expected costs of recapitalization at low cash levels. Therefore, when recapitalization is less costly (smaller $k_d$), the incentive to increase risk dominates. The described trade-off is akin to the familiar considerations, from the real options literature, that the higher risk speeds up and increases the value from exercises of the option, however it comes at the cost of a higher probability of bankruptcy. By comparison, in our model, risk increases the chances of obtaining sufficient financing, however at the expense of more frequent recapitalizations.

We can also derive the necessary condition for convexity in the whole left region:

$$\begin{aligned} V_L'' &= C^{a_2 - 2} \left(a_1 (a_1 - 1) A_L C^{a_1 - a_2} + a_2 (a_2 - 1) B_L\right) \\ k_d &< \frac{k a_1 \underline{C}^{a_1 - 1}}{1 - a_2} \left(1 - \frac{r a_2}{r + \lambda}\right) \left(C^{*1 - a_1} - I^{1 - a_1}\right). \end{aligned} \tag{31}$$

The right region is concave, and the middle region is concave under mild conditions.

$$
\begin{aligned}
V_M''(C) &= a_1(a_1-1)\frac{\left(k-\frac{kra_2}{r+\lambda}\right)I^{1-a_1}}{a_2-a_1}C^{a_1-2}+a_2(a_2-1)B_M C^{a_2-2}, \quad (32)\\
V_R''(C) &= a_2(a_2-1)B_R C^{a_2-2}.
\end{aligned}
$$

An important implication of the results in this section lies in observation that hedging becomes less desirable when firms become more poor. With low net worth, the incentive to take risk and obtain sufficient financing are especially valuable, and so the risk will be optimally increased to the extent that the negative effect of the risk is limited.

Next, we discuss the optimal hedging portfolio.

### A.7. The Optimal Hedging

From (17), the optimal hedging ratio maximizes

$$
\phi_t^* = \max_{\phi_t}\left(\frac{\phi_t^2\sigma_F^2}{2}-\rho_c\phi_t\sigma_F\sigma\right)V_{CC}C^2-\pi F_t\phi_t V_C C, \quad (33)
$$

where the second term is the negative impact of the cost of creating the hedging portfolio. The fact that optimal $\phi_t^*$ is constant confirms the conjecture in the previous section that the optimal hedging portfolio is proportional to cash. The first-order condition of the above maximization problem produces an interior solution whenever $V_{CC} < 0$

$$
\begin{aligned}
\phi_t^* &= \frac{\rho_c\sigma}{\sigma_F}+\frac{\pi F_t V_C}{\sigma_F^2 C V_{CC}} \text{ if } V_{CC} < 0, \quad (34)\\
SOC &\sim V_{CC} < 0, \quad (35)
\end{aligned}
$$

and we have the corner solution whenever $V_{CC} > 0$

$$
\phi_t^* = 0 \text{ if } V_{CC} > 0,
$$

because the hedging ratio is bounded by zero (we do not allow using the hedging portfolio to speculate). Note that, without the transaction costs, the hedging ratio (34) is the one that minimizes the conditional variance of cash

$$
\phi_t^* = \frac{\rho_c\sigma}{\sigma_F}, \text{ for } V_{CC} < 0, \text{ if } \pi = 0. \quad (36)
$$

Further, the optimal hedging ratio increases in the volatility of cash and the correlation with the hedging instrument, but decreases with the variance of futures.

Finally, note that the hedging strategy changes the exponents in the concave region and therefore the solution for the value function (intuitively, it becomes less volatile). We describe the adjustment in the Appendix.

## B. The Model with Risky Cash and Correlated Investment Option

We now extend the model by explicitly modeling risky cash flows and allowing for the cash to be correlated with the firm's investment. This is an important case, because it leads to the natural correlation between firm demand for investment and the cash used to finance it. As we show below, the profit-cash correlation carries implications for the optimal hedging policy.

To capture the idea that the higher underlying profitability leads to higher cash holdings and higher investment option value, we introduce an additional term in (15), which accounts for contemporaneous profitability and the value of options. Specifically, assume that at every point in time firms observe the stochastic variable, $p_t$, which follows the geometric Brownian motion (GBM) process in the risk-adjusted probability measure

$$dp_t = p_t \mu_p dt + \sigma_p p_t dB_p. \tag{37}$$

Here, $p_t$ can be interpreted as an underlying profitability variable, such per-unit output price. We assume there is no arbitrage in the economy and there exists a stochastic discount factor (SDF) that evolves as $d\Lambda_t/\Lambda_t = -rdt - \varsigma dB_t$, where $E(dB_t dB_p) = \rho_{B\Lambda} dt$. Because of the positive risk premium implied by SDF, the drift $\mu_p$ in the risk-adjusted measure $\mathbb{Q}$ is lower by $\varsigma \sigma_p \rho_{B\Lambda}$ than the drift in the physical measure.

The firm's profit flow resulting from selling units at this price is assumed to be

$$\Pi_t dB_p = \delta p_t^{\gamma} dB_p, \tag{38}$$

where $\delta$ is a positive proportionality constant, and $\gamma$ captures nonlinearity in the profit function. Note that the output flow is homogeneous of degree one in $p_t^{\gamma}$. As we will see shortly, the firm's basic valuation equation, including the present value of firm's profit and costs, is also homogeneous of degree one in $p_t^{\gamma}$.

We rewrite the recapitalization requirement (18) by imposing the minimum requirement on savings as

$$C \geq \underline{c} p^{\gamma}, \tag{39}$$

with constant $\underline{c} > 0$. Once the cash level falls below this threshold, the firm must raise more cash at the marginal cost $k_d$.

As in the base case, to mitigate the cash savings risk the firm can enter into $\Phi_t$ futures contracts; the dynamics of the futures price is given by (13). Because we conjecture (and verify) that the optimal number of hedging contracts will be proportional to the size of cash inventory, we amend the notation to measure the number of contracts in the scaled units, i.e., $\phi_t = \Phi_t / p^{\gamma}$. We allow for the correlation between the hedging security and the firm's profitability

$$E\left[dB_p dB_F\right] = \rho_p dt. \tag{40}$$

It is intuitive to further assume that the correlations $\rho_p$ and $\rho_c$ are of the same sign. This allows us to use a single hedging instrument capable of reducing the cash holdings risk in $dB_c$ and the risk from variations in profit $dB_p$.

Hence, cash savings with a hedging portfolio follow

$$dC_t = rC_t dt + \delta p_t^{\gamma} dB_p + \sigma_c C_t dB_c - p^{\gamma} \phi_t F_t \sigma_F dB_F - \pi \phi_t C_t dt, \tag{41}$$

where the last term, as before, captures the cost of hedging. Note that the conditional variance of cash savings is minimized when

$$\phi^* = \frac{\delta \rho_p + \sigma_c c \rho_c}{F \sigma_F}. \tag{42}$$

where $c = C p^{-\gamma}$. Note that complete hedging is impossible here even if all stochastic shocks have correlation of one.

Finally, we scale the invested amount and the investment payoff by $p^{\gamma}$. This makes investment option more valuable when the firm profitability is larger. In particular, if a Poisson shock with intensity $\lambda$ arrives at time $\tau$, the firm can invest an amount $ip^{\gamma}$ and obtain an instantaneous payoff of $\theta p^{\gamma}$. The investment amount and the payoff from investment are both made proportional to $p^\gamma$ to ensure that the firm's cash flows do not outgrow investment. The value of the firm at date $\tau$ is then equal to the final payout

$$D(C,p) = C + \max\left(0, \theta p^\gamma - ip^\gamma - k\left(ip^\gamma - C\right)^+\right). \tag{43}$$

At any other time, the value of the firm is equal to the present value of cash flows, including the interest earned on cash, plus the expected liquidation value

$$\begin{aligned} V(p_t, C_t) &= C_0 + \int_t^\tau rC_t dt + \int_t^\tau \delta p_t^\gamma dB_p + \int_t^\tau \sigma_c C_t dB_c \\ &\quad + e^{-r\tau} \max\left(0, \theta p^\gamma - ip^\gamma - k\left(ip^\gamma - C\right)^+\right). \end{aligned} \tag{44}$$

Using Itô's lemma and expression (11), we can describe the value function $V(p_t, C_t)$ as a solution to the Hamilton-Jacoby-Bellman equation

$$\begin{aligned} (r+\lambda) V(p,C) &= V_p \mu_p p + V_C rC + \frac{V_{pp}}{2}\sigma_p^2 p^2 + \frac{V_{CC}}{2}\left(\delta^2 p^{2\gamma} + \sigma_c^2 C^2\right) \\ &\quad + V_{pC}\delta p^{\gamma+1}\sigma_p + \lambda C + \lambda \max\left(0, \theta p^\gamma - ip^\gamma - k\left(ip^\gamma - C\right)^+\right). \end{aligned} \tag{45}$$

This equation is subject to boundary conditions. First, when the cash level reaches the threshold $\underline{c}p^\gamma$, we have from (39)

$$V_C\left(p, \underline{c}p^\gamma\right) = 1 + k_d, \tag{46}$$

where $V_C$ is the first-order derivative of firm value with respect to cash savings. Second, when the cash level is high, the value $V(p,C)$ must approach the value of the fully unconstrained firm.

We therefore obtain a second-order partial differential equation (PDE) with respect to the profitability shock $p$ and amount of cash $C$. It should be considered as a more general version of the ODE (17) where we modeled only idiosyncratic cash risk. In general, this PDE equation does not have the analytical solution. In the next section, we show that this particular PDE can be reduced to the ordinary differential equation.

### B.1. The Solution for Firm Value Without Hedging

The model with correlated cash flow and investment admits a closed-form solution (as an Ordinary Differential Equation with two boundary conditions). The value of a firm's assets, including cash, is proportional to $p^\gamma$. One can think of the scaling parameter $p^\gamma$ as a new numeraire or a new currency, in terms of which all values, such as firm value, cash holdings, investment, and payoff, will be computed. Because of this scaling property, the model is identical to a much simpler one, where all values are scaled by a factor of $p^\gamma$.

It is therefore convenient to define the scaled cash variable, which will be the main state variable in the model

$$c = Cp^{-\gamma}. \tag{47}$$

The value of the firm at date $\tau$ is then

$$D(c,p) = cp^\gamma + p^\gamma \left(\theta - i - k\,(i-c)^+\right)^+, \tag{48}$$

where the first term is accumulated cash and the second term is the option payoff. When the cash level is low and investment requires significant external financing, the option is optimally abandoned. We define such a trigger level of scaled cash by

$$c^* \equiv i - \frac{\theta - i}{k}. \tag{49}$$

Finally, if cash savings are higher than the cost of investment when the investment option arrives, i.e.,

$$c \geq i, \tag{50}$$

the firm does not need to raise any external financing and does not incur any costs. Note, however, that even if $c \geq i$ the firm still remains constrained. This is because cash $c$ can fall below $i$ before an option arrives and therefore firm value is lower than that of the unconstrained firm by the amount of expected financing costs. Unlike in Bolton, Wang, and Yang (2013), in our model there is no such level of cash, at which the firm becomes permanently unconstrained. It is always possible for the level of cash to decrease below the investment costs.

We conjecture that the value function can be written in the following separable form

$$V(p,C) \equiv p^\gamma v\left(\frac{C}{p^\gamma}\right) = p^\gamma v(c), \tag{51}$$

where $v(c)$ is the scaled value function. Using definitions in (47) and (51), it is easy to show

$$V_p(p,C) = \gamma p^{\gamma-1}\left(v(c) - cv'(c)\right), \; V_{pC}(p,C) \equiv -\gamma p^{-1} v''(c)\, c \tag{52}$$

$$V_C(p,C) = v'(c), \; V_{CC}(p,C) = v''(c)\, p^{-\gamma}, \tag{53}$$

$$V_{pp}(p,C) = \gamma(\gamma-1)\, p^{\gamma-2}\left(v(c) - cv'(c)\right) + c^2\gamma^2 p^{\gamma-2} v''(c). \tag{54}$$

Substituting (52-54) into the Hamilton-Jacoby-Bellman equation (45), we can write the second-order ordinary differential equation (ODE) for value function $v(c)$, where now $c$ is the only state variable

$$\lambda v = \left(r - \gamma\mu_p - \frac{\sigma_p^2}{2}\gamma(\gamma-1)\right)\left(v'c - v\right) + \frac{v''}{2}\left(\sigma_p^2 c^2\gamma^2 + \delta^2 + \sigma_c^2 c^2 - 2\gamma c\delta\sigma_p\right) \tag{55}$$
$$+\lambda \max\left(c, c + \theta - i - k\,(i-c)^+\right).$$

To properly characterize the solution, we also need to specify two boundary conditions. The first condition comes from the working capital requirement at the lower boundary (46), which we can rewrite in the scaled variables notation as

$$v'(\underline{c}) = 1 + k_d. \tag{56}$$

The second condition applies when cash savings are large, $c \to \infty$. At this point, the firm is not constrained and does not incur any costs of raising external financing. Therefore, the firm is worth its cash holdings, plus the value of cash flows and the investment opportunity

$$v(c) \to c + g, \tag{57}$$

where constant $g$ can be determined endogenously by substituting (57) into (55) and solving for $g$

$$g = \frac{\lambda(\theta - i)}{r + \lambda - \gamma\mu_p - \frac{1}{2}\sigma_p^2\gamma(\gamma-1)}. \tag{58}$$

Note that $g$, which is the value of unconstrained firm net of cash, increases with the profitability of investment $\theta - i$ and the probability of option arrival $\lambda$.

### B.2. Hedging Incentives

At this point, we can describe the scaled value function $v(c)$ as a solution to the ODE (55), subject to the boundary conditions (56) and (57). Because concavity/convexity of the function $v(c)$ determines the incentive to hedge, we proceed to finding the second derivative of this function.

The value function $v(c)$ is concave at high levels of cash, $c \geq i$, and concavity induces hedging. The upper boundary condition requires that the function is linear at high values of cash, implying that $v' \to 1$ as $c \to \infty$, while the lower boundary condition (the refinancing requirement) requires that $v' = 1 + k_d \geq 1$. Therefore, the function is concave in some region. In the Appendix, we prove that function $v(c)$ which satisfies the ODE must be concave in the whole region above $i$ (that is, where the option is exercised using firm's own cash savings only). It follows then that it is always optimal to reduce cash flow risk when the firm becomes unconstrained. Intuitively, for the firm carrying sufficient cash to finance the exercise of the option, the best strategy is to preserve this amount by reducing risk.

However, in the region where $c < i$, the value function can be convex. We prove in the Appendix that if the costs of financial distress are not too large compared to the costs of financing the investment, there exists a convexity region for lower values of $c$. This result is explained by the fact that the firm holds an option to invest, which has a convex payoff in the cash variable. For example, when cash is just below investment threshold $c^*$, it can pay off for a firm to increase the volatility in cash savings since such behavior increases the probability of cash exceeding the threshold $c^*$, and therefore also increases the value of the option. It is important that the additional value of risk in cash flows comes from the option to abandon the investment if the financing is insufficient. If the firm were *always* required to exercise the option, cash flow risk would be value-destroying because it would necessitate large financing costs when cash is very low.

We further show that the higher financial distress costs reduce or eliminate the convexity region in $v(c)$. When cash savings are very low and the probability of exercise is small, a firm may actually prefer to hedge because this allows it to avoid distress and the costly external financing associated with distress. Therefore, for the high values of $k_d$ (high distress costs) we recover the standard intuition. The value function is concave in the whole region because concerns about the potential distress dominate other considerations.

In Figure 7, Panel A, we show the shape of function $v(c)$ for a given set of parameters. The solution to ODE (55) is obtained numerically. The upper dashed line in the graph gives the value of the firm if it were completely unconstrained (i.e., $k_d = k = 0$), in which case the firm would always exercise its investment option when the option arrives. It is clear that firm value approaches this line as the accumulated cash savings increase beyond $i$. The lower dashed line shows the value of the firm that does not have an investment option and is not subject to the costs of financial distress (i.e., $\lambda = 0$ and $k_d = 0$). As predicted, the function exhibits concavity in the region of the high values of cash and convexity in the region of the low values. Panels B helps to evaluate how the first derivative of $v(c)$ change with cash.

Having analyzed the shape of the value function that gives us guidance on where it is optimal to increase or decrease risk of cash savings, we now turn to determining the optimal amount of hedging.

### B.3. Optimal Hedging Policy

We now discuss the optimal hedging portfolio. Using the expression for the cash evolution specified in (41) and also using the previously obtained derivatives (52)-(54), we can write the Hamilton-Jacoby-Bellman equation for the scaled firm value $v(c)$

$$\begin{aligned} (r+\lambda)V(p,C) &= \max_{\phi} V_p\mu_p p + V_C C(r-\pi\phi) + V_{pC}\sigma_p p^{\gamma+1}\left(\delta - \phi F\sigma_F\rho_p\right) + \frac{1}{2}V_{pp}\sigma_p^2 p^2 \\ &\quad + \frac{1}{2}V_{CC}\left(\delta^2 p^{2\gamma} + \sigma_c^2 C^2 - 2\left(\delta p^{2\gamma}\rho_p + \sigma_c\rho_c C p^{\gamma}\right)\phi F\sigma_F + p^{2\gamma}\phi^2F^2\sigma_F^2\right) \\ &\quad + \lambda C + \lambda\max\left(0, \theta p^{\gamma} - ip^{\gamma} - k\left(ip^{\gamma} - C\right)^{+}\right). \end{aligned} \tag{59}$$

where $v(c)$ is the scaled value function. Using definitions in (47) and (51), it is easy to show

$$(r+\lambda)\,v = \max_{\phi}\left(v - cv'\right)\left(\gamma\mu_p + \frac{\sigma_p^2}{2}\gamma(\gamma-1)\right) + v'c\,(r - \pi\phi) - \gamma v'' c\sigma_p\left(\delta - \phi F\sigma_F\rho_p\right) \tag{60}$$
$$+\frac{1}{2}v''\left(c^2\gamma^2\sigma_p^2 + \delta^2 + \sigma_c^2c^2 - 2\left(\delta\rho_p + \sigma_c\rho_c c\right)\phi F\sigma_F + \phi^2F^2\sigma_F^2\right)$$
$$+\lambda c + \lambda\max\left(0, \theta - i - k\,(i-c)^+\right),$$

this ODE corresponds to (55) but includes the hedging portfolio. Differentiating with respect to hedging policy $\phi$ gives the optimal choice of hedging for the concavity region

$$\phi_t^* = \frac{\delta\rho_p + \sigma_c\rho_c c - \gamma\sigma_p\rho_p c}{F\sigma_F} + \frac{\pi c v'}{\sigma_F^2F^2v''}. \tag{61}$$

$$SOC = v''F^2\sigma_F^2 < 0 \tag{62}$$

Note that the second-order condition is satisfied as long as function $v(c)$ is concave. It is worth examining the expression (61) with care. First, observe that the second term is negative, and therefore optimal hedging is less than complete (compare (61) to (42)). Second, observe that the optimal hedging ratio decreases with the volatility of the output price, $\sigma_p$, particularly if the convexity of the firm's profit ($\gamma$) is high. This means that the correlation between cash and investment decreases the amount of hedging as compared to the minimum conditional variance case. Third, $\phi_t$ decreases with the cost of hedging.

When $v(c)$ is convex, there is no interior solution for hedging and the firm chooses between corner solutions 0 and $\phi_{\max}$. In particular, if there are no costs of hedging ($\pi = 0$), the firm will choose $\phi^* = 0$ when for $c < i$

$$-\left(\delta\rho_p + \sigma_c\rho_c c\right)\left[\frac{v'c\pi}{F\sigma_F} + v''\left(\frac{\delta\rho_p}{2} + \frac{\sigma_c\rho_c c}{2} - \gamma\sigma_p\rho_p c\right)\right] < 0. \tag{63}$$

Interestingly, it is possible for the firm to radically change its hedging position from time to time (and even go from hedging to speculation if it is allowed), as firm cash holding increases or decreases.

# IV. Conclusion

In this study, we analyze the relation between optimal risk management policy and investment under financing constraints. In particular, we recognize that hedging policy can affect the probability of option exercise and also the cost of financing. The optimal amount of financial hedging balances the benefits of lower expected financial distress costs with the better ability to finance investment. The model demonstrates importance of real frictions, such as irreversibility of investment or fixed costs.

The predictions of the model are consistent with the empirical findings: firms with less financing constraints operate with higher hedging ratios, and firms with more risky cash flows operate with lower hedging ratios. The hedging ratio is linked theoretically to the value of growth options, the ratio of firm-specific to systematic risk, and the costs of forming a hedging portfolio. Therefore, the model generates additional empirical predictions for future work. Our results offer an alternative explanation for the observed hedging policies that does not rely on the cost of hedging.

The analytical solution for the one-period model and the dynamic model with cash accumulation can be used in other applications. For example, it would be relatively easy to extend the model to study optimal dividend policy and the implications of a minimum cash balance requirement.

# V. Appendix A: Proposition Proofs

*Proof of Lemma 1.* Using the implicit function theorem for (5), we obtain

$$\frac{d\varepsilon^*}{d\phi} = \frac{\left(C_e\left(e\right) - C_e\left(e_0\right)\right)\left(\varepsilon^* - \overline{\varepsilon}\right)}{\frac{\alpha f}{w_1} + \left(C_e\left(e\right) - C_e\left(e_0\right)\right)\left(1 - \phi\right)}. \tag{64}$$

Our assumptions imply that $I^* > R$, $\alpha \geq 0$, $C_{ee} > 0$, and $e > e_0$.

Therefore (i) $\frac{d\varepsilon^*}{d\phi} > 0$ for $\varepsilon^* > \overline{\varepsilon}$, and (ii) $\frac{d\varepsilon^*}{d\phi} < 0$ for $\varepsilon^* < \overline{\varepsilon}$. Additionally, using condition (4) yields

$$\frac{dI^*}{d\phi} = -\frac{w_1\left(\varepsilon - \overline{\varepsilon}\right)C_{ee}}{C_{ee} - \theta f_{II}}. \tag{65}$$

Because $C_{ee} < 0$, it follows that $\frac{dI^*}{d\phi} < 0$ for all states $\varepsilon$ in the investment region, $\varepsilon^* > \overline{\varepsilon}$ .

□

*Proof of Proposition 1.* If $\alpha = 0$, the optimization function does not have an interior maximum; we show this formally in the proof of Proposition 2. Therefore, the optimal hedging ratio is either a minimum $\phi^* = 0$ or a maximum $\phi^* = 1$.

If the firm hedges completely ($\phi^* = 1$), its cash savings at date 2 are independent of the state and the profit is given by either $P_0\left(\overline{\varepsilon}\right)$ or $P\left(\overline{\varepsilon}\right)$ depending on whether investment is optimal at $\overline{\varepsilon}$

$$P_0\left(\overline{\varepsilon}\right) = f_0 - R - C\left(R - w_0 - w_1\overline{\varepsilon}\right), \tag{66}$$

$$P\left(\overline{\varepsilon}\right) = \beta f\left(I^*\right) - I^* - C\left(I^* - w_0 - w_1\overline{\varepsilon}\right). \tag{67}$$

where the optimal investment $I^*$ is fixed (does not vary with the profitability state.

If the firm does not hedge, there exists a threshold $\varepsilon^*$, above which the firm invests. To determine $\phi$ that maximizes expected profit, consider two cases: (1) $\varepsilon^* < \overline{\varepsilon}$ (the firm would invest at the average profitability, $P\left(\overline{\varepsilon}\right) > P_0\left(\overline{\varepsilon}\right)$); and (2) $\varepsilon^* > \overline{\varepsilon}$ (the firm would not invest at the average profitability, $P\left(\overline{\varepsilon}\right) < P_0\left(\overline{\varepsilon}\right)$).

If $\varepsilon^* < \overline{\varepsilon}$, the difference between the expected profit in the case of full hedging and the case of no hedging is

$$
\begin{aligned}
& P(\overline{\varepsilon})-\int_{-\infty}^{\varepsilon^*} P_0(\varepsilon)\, dG(\varepsilon)-\int_{\varepsilon^*}^{\infty} P(\varepsilon)\, dG(\varepsilon) \\
= \; & P(\overline{\varepsilon})-E(P(\varepsilon))+\int_{-\infty}^{\varepsilon^*}(P(\varepsilon)-P_0(\varepsilon))\, dG(\varepsilon).
\end{aligned} \tag{68}
$$

Because the profit function is concave it must be that $P(\overline{\varepsilon}) > E(P(\varepsilon))$ and the first term in (68) is positive. However, the second term is negative and captures the value of abandonment option lost by hedging.[15] Since the first term in (68) is independent of $\varepsilon^*$ and the second term increases in magnitude with $\varepsilon^*$, there must exist such $\varepsilon_L$ that for $\varepsilon^* > \varepsilon_L$ hedging destroys value ($\phi^* = 0$) and for $\varepsilon^* < \varepsilon_L$ hedging creates value ($\phi^* = 1$). Such $\varepsilon_L$ can be found by setting expression (68) to zero and plugging $\varepsilon_L$ in place of $\varepsilon^*$.

Similarly, if $\varepsilon^* > \overline{\varepsilon}$, the difference between the expected profit under full hedging and no hedging is

$$
\begin{aligned}
& P_0(\overline{\varepsilon})-\int_{-\infty}^{\varepsilon^*} P_0(\varepsilon)\, dG(\varepsilon)-\int_{\varepsilon^*}^{\infty} P(\varepsilon)\, dG(\varepsilon) \\
= \; & P_0(\overline{\varepsilon})-E(P_0(\varepsilon))-\int_{\varepsilon^*}^{\infty}(P(\varepsilon)-P_0(\varepsilon))\, dG(\varepsilon).
\end{aligned} \tag{69}
$$

The first term is positive because of the concavity of the profit function, whereas the second term is negative and captures the value of the investment option lost by hedging. Since the first term is independent of $\varepsilon^*$ and the second one decreases with $\varepsilon^*$, there must exist $\varepsilon_H$ such that for any $\varepsilon^* > \varepsilon_H$ hedging creates value ($\phi^* = 1$) and for $\varepsilon^* < \varepsilon_H$ hedging destroys value ($\phi^* = 0$). The value of $\varepsilon_H$ is found by setting (69) to zero at $\varepsilon^* = \varepsilon_H$.

□

*Proof of Corollary 1.* From investment condition (5) we obtain

$$
\frac{d\varepsilon^*}{d\beta} = -\frac{f}{\alpha f + w_1(1-\phi)(C_e(e)+C_e(e_0))} < 0,
$$

which implies that firms with more valuable options start investing at lower thresholds. Therefore, from Proposition 1 it follows that $\frac{\Delta\phi^*}{\Delta\beta} \leq 0$.

□

[15] Intuitively, hedging decreases the volatility in cash flow and hence makes the external financing cost constant. Because $\alpha = 0$ and $\varepsilon^* < \overline{\varepsilon}$, the only reason not to invest is low cash flow and expensive financing, but this situation never happens with full hedging. Therefore, the option not to invest has no value when $\phi = 1$.

*Proof of Corollary 2.* Using the definition of the investment threshold (5), we obtain the comparative statics with respect to the firm's initial cash position $w_0$

$$\frac{d\varepsilon^*}{dw_0} = -\frac{C_e(e) - C_e(e_0)}{\alpha f + w_1(1-\phi)(C_e(e) - C_e(e_0))} < 0. \tag{70}$$

Therefore, as $w_0$ increases and the firm becomes less constrained, the investment threshold $\varepsilon^*$ decreases. Proposition 1 shows that the optimal hedging ratio depends on the investment threshold.

If initially $\varepsilon^* < \varepsilon_L$, then full hedging remains optimal as $w_0$ increases since the firm remains in the same region (see Proposition 1). If initially $\varepsilon^* \in [\varepsilon_L, \varepsilon_H]$, then the hedging ratio either remains unchanged or increases to $\phi^* = 1$. Therefore, an increase in the firm's internal cash reserves can result in more hedging.

□

*Proof of Corollary 3.* Recall from the proof of Proposition 1 that when $\varepsilon^* < \bar{\varepsilon}$, the difference between the expected profit in the case of full hedging and the case of zero hedging is

$$P(\bar{\varepsilon}) - E(P(\varepsilon)) + \int_{-\infty}^{\varepsilon^*} (P(\varepsilon) - P_0(\varepsilon))\, dG(\varepsilon). \tag{71}$$

Note that the investment threshold $\varepsilon^*$ is independent of volatility. When the volatility is small, the firm never reaches the region below $\varepsilon^*$. Therefore, the second term in the formula above disappears, while the first term is positive and induces hedging. A firm with low volatility and $\varepsilon^* < \bar{\varepsilon}$ chooses $\phi^* = 1$. As volatility increases, the second term starts to create a greater disadvantage to hedging, with a resulting decrease in the hedging ratio.

Similarly, if $\varepsilon^* > \bar{\varepsilon}$, the difference between the expected profit under full hedging and no hedging is

$$P_0(\bar{\varepsilon}) - E(P_0(\varepsilon)) - \int_{\varepsilon^*}^{\infty} (P(\varepsilon) - P_0(\varepsilon))\, dG(\varepsilon). \tag{72}$$

If the volatility is small, the second term is zero and thus $\phi^* = 1$. As volatility increases, the second term becomes more important and hedging ratio drops.

□

*Proof of Proposition 2.* From (7) we can use Leibniz's rule to obtain the first order condition

$$\int_{\varepsilon^*}^{\infty} P_w \frac{\partial w}{\partial \phi} dG(\varepsilon) + \int_{-\infty}^{\varepsilon^*} P_{0w} \frac{\partial w}{\partial \phi} dG(\varepsilon) + \frac{d\varepsilon^*}{d\phi}\left(P_0(\varepsilon^*) - P(\varepsilon^*)\right) g(\varepsilon^*) = 0. \tag{73}$$

Because the profit functions $P_0(\varepsilon^*)$ and $P(\varepsilon^*)$ match at $\varepsilon^*$, the last term in condition (73)is zero, so that we have

$$E\left[\Pi_w \frac{\partial w}{\partial \phi}\right] = 0. \tag{74}$$

By applying (1), we can further simplify the first order condition (74) to

$$cov(\Pi_w, \varepsilon) = 0. \tag{75}$$

Using Stein's lemma for normally distributed profitability shocks, $g(\varepsilon) \sim N\left(\overline{\varepsilon}, \sigma^2\right)$, and using the expression (75), we have

$$E(\Pi_{w\varepsilon})\sigma^2 = 0. \tag{76}$$

Alternatively, if the distribution is not normal, the same expression can be obtained from the second-order Taylor expansion around $\overline{\varepsilon}$. In the investment region, $\varepsilon > \varepsilon^*$, we obtain

$$\Pi_{w\varepsilon} = P_{w\varepsilon} = (\alpha f_I + \theta f_{II} I_\varepsilon) I_w + (\theta f_I - 1 - C_e) I_{w\varepsilon} - C_{ee}(I_\varepsilon - w_1(1-\phi))(I_w - 1), \tag{77}$$

which simplifies using the first order condition for investment (4) to

$$P_{w\varepsilon} = [\alpha f_I + \theta f_{II} I_\varepsilon - C_{ee} I_\varepsilon + C_{ee} w_1(1-\phi)] I_w + C_{ee}[I_\varepsilon - w_1(1-\phi)]. \tag{78}$$

Differentiating implicitly equation (4),

$$I_\varepsilon = \frac{\alpha f_I + C_{ee} w_1(1-\phi)}{C_{ee} - \theta f_{II}}, \tag{79}$$

$$I_w = \frac{C_{ee}}{C_{ee} - \theta f_{II}}, \tag{80}$$

and substituting these expressions in (78) we obtain

$$P_{w\varepsilon} = \frac{C_{ee}\alpha f_I + \theta f_{II} C_{ee} w_1(1-\phi)}{C_{ee} - \theta f_{II}}. \tag{81}$$

In the inaction region, $\varepsilon < \varepsilon^*$, we have

$$\Pi_{w\varepsilon} = P_{0w\varepsilon} = -C_{ee}(e_0) w_1(1-\phi). \tag{82}$$

Substituting (78), rewrite (76) as

$$E\left[\Pi_{w\varepsilon}\right]=0=\int_{\varepsilon^*}^{\infty}P_{w\varepsilon}dG\left(\varepsilon\right)+\int_{-\infty}^{\varepsilon^*}P_{0w\varepsilon}dG\left(\varepsilon\right). \tag{83}$$

Finally, solving this equation for the optimal hedging ratio $\phi^*$ yields

$$\phi^*=1-\frac{\alpha}{w_1}\frac{\int_{\varepsilon^*}^{\infty}\frac{f_I C_{ee}}{C_{ee}-\theta f_{II}}dG\left(\varepsilon\right)}{\int_{-\infty}^{\varepsilon^*}C_{ee}\left(e_0\right)dG\left(\varepsilon\right)-\int_{\varepsilon^*}^{\infty}\frac{\theta f_{II}C_{ee}}{C_{ee}-\theta f_{II}}dG\left(\varepsilon\right)}. \tag{84}$$

The second order condition with respect to $\phi$ is

$$\begin{aligned}&\int_{\varepsilon^*}^{\infty}P_{ww}\left(\frac{\partial w}{\partial\phi}\right)^2dG\left(\varepsilon\right)+\int_{-\infty}^{\varepsilon^*}P_{0ww}\left(\frac{\partial w}{\partial\phi}\right)^2dG\left(\varepsilon\right)\\&+\frac{d\varepsilon^*}{d\phi}\left(P_{0w}\left(\varepsilon^*\right)-P_w\left(\varepsilon^*\right)\right)w_1\left(\overline{\varepsilon}-\varepsilon^*\right)g\left(\varepsilon^*\right)<0.\end{aligned} \tag{85}$$

Since

$$P_{ww} = \frac{\theta f_{II}C_{ee}}{C_{ee}-\theta f_{II}}<0, \tag{86}$$

$$P_{0ww} = -C_{ee}<0, \tag{87}$$

the first two terms in (85) are negative. The last term in (85) is positive and is equal to

$$\frac{\left(C_e\left(e\right)-C_e\left(e_0\right)\right)^2\left(\varepsilon^*-\overline{\varepsilon}\right)^2w_1g\left(\varepsilon^*\right)}{\frac{\alpha f}{w_1}+\left(C_e\left(e\right)-C_e\left(e_0\right)\right)\left(1-\phi\right)}>0. \tag{88}$$

For a sufficiently large $\alpha$ (i.e., for $\alpha>\underline{\alpha}$), the condition (85) is satisfied, where $\underline{\alpha}$ is a solution to the following equation

$$-\int_{\varepsilon^*}^{\infty}P_{ww}\left(\varepsilon-\overline{\varepsilon}\right)^2dG-\int_{-\infty}^{\varepsilon^*}P_{0ww}\left(\varepsilon-\overline{\varepsilon}\right)^2dG=\frac{\left(C_e\left(e\right)-C_e\left(e_0\right)\right)^2\left(\varepsilon^*-\overline{\varepsilon}\right)^2g\left(\varepsilon^*\right)}{\underline{\alpha}f+\left(C_e\left(e\right)-C_e\left(e_0\right)\right)w_1\left(1-\phi^*\right)}. \tag{89}$$

Note that when $\alpha\to 0$, from (84) we have $\phi^*\to 1$. The denominator in (88) is linear in $\alpha$, and therefore the last term in the second order condition is infinite for a very small $\alpha$. Therefore, when $\alpha\to 0$ the solution entails either maximum or minimum value for the hedging ratio.

□

*Proof of Proposition 3.* Similarly to the steps in Proof of Proposition 2, we obtain the first order condition for the optimal hedging ratio as

$$E\left[\Pi_{w\varepsilon}(\phi^*)\right]=0. \tag{90}$$

In the investment region, we have

$$P_{w\varepsilon} = \frac{C_{ee}\alpha f_I + \theta f_{II} C_{ee} w_1 (1-\phi)}{C_{ee} - \theta f_{II}}, \tag{91}$$

whereas in the region where the firm operates its existing assets, we obtain

$$P_{0w\varepsilon} = -C_{ee}(e_0)\, w_1 (1-\phi). \tag{92}$$

Substituting these expressions into the first-order condition (90) and rewriting the expectation yields

$$\begin{aligned} E[\Pi_{w\varepsilon}] &= 0 = \int_{-\infty}^{\widehat{v}_m} \left( \int_{\frac{\varepsilon_i^* - \beta_i v_m}{\sqrt{1-\beta_i^2}}}^{\infty} P_{w\varepsilon} dG(v_i) \right) dG(v_m). \\ &+ \int_{-\infty}^{\widehat{v}_m} \left( \int_{-\infty}^{\frac{\varepsilon_i^* - \beta_i v_m}{\sqrt{1-\beta_i^2}}} P_{0w\varepsilon} dG(v_i) \right) dG(v_m). \end{aligned} \tag{93}$$

Finally, solving this equation for the optimal hedging ratio $\phi^*$ yields

$$\phi^* = 1 - \frac{\alpha}{w_1} \frac{\int_{-\infty}^{\widehat{v}_m} \left( \int_{\frac{\varepsilon_i^* - \beta_i v_m}{\sqrt{1-\beta_i^2}}}^{\infty} \frac{C_{ee} f_I dG(\varepsilon_i)}{C_{ee} - \theta f_{II}} \right) dG(v_m)}{\int_{-\infty}^{\widehat{v}_m} \left( \int_{-\infty}^{\frac{\varepsilon_i^* - \beta_i v_m}{\sqrt{1-\beta_i^2}}} C_{ee}(e_0)\, dG(v_i) - \int_{\frac{\varepsilon_i^* - \beta_i v_m}{\sqrt{1-\beta_i^2}}}^{\infty} \frac{\theta f_{II} C_{ee} dG(v_i)}{C_{ee} - \theta f_{II}} \right) dG(v_m)} \tag{94}$$

To show that $\frac{d\phi^*}{d\beta_i} > 0$, note that $\phi^*$ depends on $\beta_i$ only through the limits of the integration

$$v_i^*(v_m) = \frac{\varepsilon_i^* - \beta_i v_m}{\sqrt{1-\beta_i^2}}, \tag{95}$$

which has the meaning of the minimum idiosyncratic shock $v_i$ which warrants new investment at the current realized value of the systematic shock $v_m$. Note that $v_i^*(v_m) \to +\infty$ when $\beta_i \to 1$, i.e., an infinitely large idiosyncratic shock is required to trigger investment if almost all risk comes from the systematic component. Therefore, it follows from (94) that $\phi^* \to 1$ as $\beta_i \to 1$, and $\phi^* < 1$ if $\beta_i < 1$.

□

*Derivation of the value function and prove of convexity for the case with cash financing only.* Consider a simple case with $\pi \to 0$ (no transaction costs for simplicity) and $\phi = 0$ (we are looking for the value function before hedging is in place). Then we have the following solution to the left ($L$) and to the right ($R$) of the threshold $I$

$$\begin{aligned} V_L(C) &= A_L C^{a_1} + B_L C^{a_2} + C \text{ for } C < I, \\ V_R(C) &= A_R C^{a_1} + B_R C^{a_2} + C + \frac{\lambda(\Theta - I)}{r+\lambda} \text{ for } C \geq I, \end{aligned} \tag{96}$$

subject to the value-matching and smooth-pasting condition at the boundary between two regions at $C = I$, and subject to the recapitalization threshold $C = \underline{C}$:

$$\begin{aligned} V_L(I) &= V_R(I), \\ V_L'(I) &= V_R'(I), \\ V_L'(\underline{C}) &= 1 + k_d, \end{aligned} \tag{97}$$

From the requirement that $V_R(\infty)$ is finite, we must set $A_R = 0$. The boundary conditions pin the other three free constants in left and right regions

$$\begin{aligned} A_L &= \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_2 I^{-a_1}}{a_2 - a_1} > 0, \\ B_R &= \frac{k_d}{a_2}\underline{C}^{1-a_2} - \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_1 I^{-a_1}}{a_2 - a_1}\underline{C}^{a_1 - a_2} + \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_1 I^{-a_2}}{a_2 - a_1}, \\ B_L &= \frac{k_d}{a_2}\underline{C}^{1-a_2} - \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_1 I^{-a_1}}{a_2 - a_1}\underline{C}^{a_1 - a_2}. \end{aligned} \tag{98}$$

Further, we check the second derivative $V_L''(C)$ to establish convexity in some lower region, $V_L''(C) > 0$.

$$\begin{aligned} V_L''(C) = & \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_2 a_1 (a_1 - 1) I^{-a_1}}{a_2 - a_1} C^{a_1 - 2} \\ & + \left( (a_2 - 1) k_d \underline{C}^{1-a_2} - \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_2 a_1 (a_2 - 1) I^{-a_1}}{a_2 - a_1}\underline{C}^{a_1 - a_2} \right) C^{a_2 - 2} \end{aligned} \tag{99}$$

Let $C \to \underline{C}$

$$k_d < \frac{\lambda(\Theta - I)}{r+\lambda}\frac{(-a_2) a_1}{1 - a_2} I^{-a_1}\underline{C}^{a_1 - 1} \tag{100}$$

Thus convexity at $\underline{C}$ is guaranteed as long as $k_d$ is not too large (note that it does not have to be zero). Convexity is also guaranteed for any $C < I$ as long as

$$k_d < \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_2 a_1}{a_2 - a_1} I^{-a_1}\underline{C}^{a_1 - 1} \tag{101}$$

This second condition is more binding (as it should be), but note that it is sufficient, but not necessary condition too.

$$a_1 > 1, a_2 < 0$$

Check the shape on the right

$$V_R''(C) = (a_2 - 1)\,\underline{C}^{-a_2}\left(\underline{C}k_d + \frac{\lambda(\Theta - I)}{r+\lambda}\frac{a_1 a_2}{a_2 - a_1}\left(\frac{I}{\underline{C}}\right)^{-a_1}\left(\left(\frac{I}{\underline{C}}\right)^{a_1 - a_2} - 1\right)\right) C^{a_2 - 2} < 0 \tag{102}$$

It is always negative, i.e. the function is always concave. □

*Derivation of value functions under optimal hedging.* Substituting the trial solution into the ODE gives

$$(r+\lambda)\left(AC^{b_1} + BC^{b_2} + EC + D\right) = \max_{\phi_t} r\left(b_1 AC^{b_1} + b_2 BC^{b_2} + EC\right) \tag{103}$$
$$+\frac{1}{2}\left(b_1(b_1-1)AC^{b_1} + b_2(b_2-1)BC^{b_2}\right)\left(\sigma^2 + \phi_t^2\sigma_F^2 - 2\rho_c\phi_t\sigma_F\sigma\right)$$
$$+\lambda C + \lambda(\Theta - I) * Ind(C > I).$$

It follows that $\{b_1,b_2\}$ are solutions to the standard quadratic equation

$$r + \lambda = \max_{\phi_t}\frac{1}{2}b(b-1)\left(\sigma^2 + \phi_t^2\sigma_F^2 - 2\rho_c\phi_t\sigma_F\sigma\right) + rb. \tag{104}$$

which given the interior solution $\phi = \frac{\rho_c\sigma}{\sigma_F}$ simplifies to

$$r + \lambda = rb + \frac{1}{2}b(b-1)\sigma^2\left(1 - \rho_c^2\right) \tag{105}$$

We define the exponents as

$$b_{1,2} = -\left(\frac{r}{\sigma^2(1-\rho_c^2)} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{r}{\sigma^2(1-\rho_c^2)} - \frac{1}{2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2(1-\rho_c^2)}}. \tag{106}$$

Note $b_1 > 1$, $b_2 < 0$. Free constants $A$ and $B$ are determined by the boundary conditions (18) and (19) and constants $E$, and $D$ are given by

$$E = 1;\ D = \frac{\lambda(\Theta - I)}{r+\lambda} \text{ for } C \geq I, \text{ and } D = 0 \text{ for } C < I. \tag{107}$$

We hypothesize that value function is convex for $C < I$ and concave for $C > I$.

Then we have

$$\begin{aligned} V_L(C) &= A_L C^{a_1} + B_L C^{a_2} + C \text{ for } C < I, \qquad (108) \\ V_R(C) &= B_R C^{b_2} + C + \frac{\lambda(\Theta - I)}{r+\lambda} \text{ for } C \geq I, \end{aligned}$$

subject to boundary conditions

$$\begin{aligned} V_L(I) &= V_R(I), \qquad (109) \\ V_L'(I) &= V_R'(I), \\ V_L'(\underline{C}) &= 1 + k_d, \end{aligned}$$

Solving these system of equations yields:

$$\begin{aligned} A_L &= \frac{k_d \underline{C}^{1-a_2} I^{a_2}(b_2 - a_2) - a_2 b_2 \frac{\lambda(\Theta - I)}{r+\lambda}}{a_2(a_1 - b_2) I^{a_1} - a_1(a_2 - b_2)\underline{C}^{b_1 - a_2} I^{a_2}}, \qquad (110) \\ B_L &= \frac{k_d \underline{C}^{1-a_2}(a_1 - b_2) I^{a_1} \underline{C}^{a_1 - b_1} + a_1 b_2 \frac{\lambda(\Theta - I)}{r+\lambda}}{a_2(a_1 - b_2) I^{a_1} \underline{C}^{a_1 - b_1} - a_1(a_2 - b_2) I^{a_2}}, \\ B_R &= \frac{a_1}{b_2} A_L I^{a_1 - b_2} + \frac{a_2}{b_2} B_L I^{a_2 - b_2}. \end{aligned}$$

The case with external financing is very similar and not shown here. □

*Proof of concavity of $v(c)$ function when $c \geq i$.* We have shown that function $v(c)$ must have at least some region of $c$ values where it is concave. The following argument shows that the value function contains no convex region when $c \geq i$. The ODE for the value function is

$$\begin{aligned} (r+\lambda) v(c) &= \big(\gamma v(c) - \gamma c v'(c)\big)\mu_p + \frac{\sigma_p^2}{2}\gamma(\gamma - 1)\big(v(c) - c v'(c)\big) \qquad (111) \\ &\quad + \frac{\sigma_p^2}{2} c^2 \gamma^2 v''(c) + v'(c)(rc + \delta) + \lambda d(c), \end{aligned}$$

with function $d(c)$ being weakly concave for $c \geq i$.

We proceed with the proof by contradiction. Suppose function $v(c)$ is convex at point $c_2$. Then it should be possible to pick such values $c_1$ and $c_3$, with $c_1 < c_2 < c_3$ and $c_2 = \alpha c_1 + (1-\alpha) c_3$, that

$$v'(c_1) = v'(c_2) = v'(c_3) = b, \qquad (112)$$

and

$$v''(c_1) < 0,\ v''(c_2) > 0,\ v''(c_3) < 0. \tag{113}$$

Using these conditions and letting $v_1 = v(c_1)$, $v_2 = v(c_2)$ and $v_3 = v(c_3)$, we can then write the following inequalities

$$\begin{aligned}
(r+\lambda) v_1 &< \gamma\left(\mu_p + \frac{\sigma_p^2}{2}(\gamma-1)\right)(v_1 - c_1 b) + b(rc_1 + \delta) + \lambda d(c_1), && (114)\\
(r+\lambda) v_3 &< \gamma\left(\mu_p + \frac{\sigma_p^2}{2}(\gamma-1)\right)(v_3 - c_3 b) + b(rc_3 + \delta) + \lambda d(c_3), && (115)\\
(r+\lambda) v_2 &> \gamma\left(\mu_p + \frac{\sigma_p^2}{2}(\gamma-1)\right)(v_2 - c_2 b) + b(rc_2 + \delta) + \lambda d(c_2). && (116)
\end{aligned}$$

Let $\widehat{v}_2 \equiv \alpha v_1 + (1-\alpha) v_3$. If the function $v(c)$ is convex at point $c_2$, then it must be that $\widehat{v}_2 > v_2$. Using this fact, we can rewrite (116) as

$$\begin{aligned}
(r+\lambda)\widehat{v}_2 &< \gamma\left(\mu_p + \frac{\sigma_p^2}{2}(\gamma-1)\right)(\widehat{v}_2 - c_2 b) + b(rc_2 + \delta) \\
&\quad + \lambda\left[\alpha d(c_1) + (1-\alpha) d(c_3)\right].
\end{aligned} \tag{117}$$

Taking the difference, we obtain

$$\begin{aligned}
(r+\lambda)(\widehat{v}_2 - v_2) &< \left[\gamma\mu_p + \frac{\sigma_p^2}{2}\gamma(\gamma-1)\right](\widehat{v}_2 - v_2) \\
&\quad + \lambda\left[d(c_2) - \alpha d(c_1) - (1-\alpha) d(c_3)\right].
\end{aligned} \tag{118}$$

Because

$$r + \lambda > \gamma\mu_p + \frac{\sigma_p^2}{2}\gamma(\gamma-1),\ \text{and}\ d(c_2) \geq \alpha d(c_1) + (1-\alpha) d(c_3),$$

we get a contradiction. □

*Proof of existence of convexity of $v(c)$ function when $c < i$.* Suppose the distress costs are small ($k_d \to 0$). Then, the slope of the value function $v(c)$ is the same at the lower boundary and the upper boundary, i.e.,

$$v'(\underline{c}) = v'(c \to \infty) = 1, \tag{119}$$

At the lower boundary, $c \to \underline{c}$, the investment option is far out-of-the-money and hence the value of the firm is the same as of a firm without option. However, at the upper boundary, the firm always exercises the option, that is the firm value approaches the value of the unconstrained firm *with* the investment option. To have the same slope at both boundaries, but a higher value at the right boundary, the function $v(c)$ must have convexity on the left and concavity on the right. The inflection point may or may not coincide with the exercise threshold. We have shown that function $v(c)$ must have at least some region of $c$ values where it is convex. The following argument shows that the value function contains no concave region when $c < i$. The ODE for the value function is

$$\lambda v = \left( r - \gamma \mu_p - \frac{\sigma_p^2}{2} \gamma (\gamma - 1) \right) (v'c - v) + \frac{v''}{2} \left( \sigma_p^2 c^2 \gamma^2 + \delta^2 + \sigma_c^2 c^2 - 2\gamma c \delta \sigma_p \right) + d(c) . \quad (120)$$

with function $d(c)$ being weakly convex for $c < i$. Note that for any $c$ function $f(c) > 0$

$$f(c) = \frac{1}{2} \left( \sigma_p^2 c^2 \gamma^2 + \delta^2 + \sigma_c^2 c^2 - 2\gamma c \delta \sigma_p \right) > 0. \quad (121)$$

The way to show this is to find minimum of this function with respect to $c$ and show that function evaluated at its minimum is positive.

$$\begin{aligned} c_{\min} &= \frac{\gamma \delta \sigma_p}{\gamma^2 \sigma_p^2 + \sigma_c^2} \quad (122) \\ f(c_{\min}) &= \frac{1}{2} \frac{\delta^2 \sigma_c^2}{\gamma^2 \sigma_p^2 + \sigma_c^2} \end{aligned}$$

We proceed with the proof by contradiction. Suppose function $v(c)$ is concave at point $c_2$. Then it should be possible to pick such values $c_1$ and $c_3$, with $c_1 < c_2 < c_3$ and $c_2 = \alpha c_1 + (1 - \alpha) c_3$, that

$$v'(c_1) = v'(c_2) = v'(c_3) = b, \quad (123)$$

and

$$v''(c_1) > 0, \; v''(c_2) < 0, \; v''(c_3) > 0. \quad (124)$$

Using these conditions and letting $v_1 = v(c_1)$, $v_2 = v(c_2)$ and $v_3 = v(c_3)$, we can then write the following inequalities

$$\begin{aligned}
\lambda v_1 &> \left(r - \gamma\mu_p - \frac{\sigma_p^2}{2}\gamma\left(\gamma - 1\right)\right)\left(bc_1 - v_1\right) + d\left(c_1\right) \qquad (125) \\
\lambda v_3 &> \left(r - \gamma\mu_p - \frac{\sigma_p^2}{2}\gamma\left(\gamma - 1\right)\right)\left(bc_3 - v_3\right) + d\left(c_3\right) \\
\lambda v_2 &< \left(r - \gamma\mu_p - \frac{\sigma_p^2}{2}\gamma\left(\gamma - 1\right)\right)\left(bc_2 - v_2\right) + d\left(c_2\right)
\end{aligned}$$

Let $\widehat{v}_2 \equiv \alpha v_1 + (1-\alpha) v_3$. If the function $v(c)$ is concave at point $c_2$, then it must be that $\widehat{v}_2 < v_2$. Using this fact, we can write

$$\lambda \widehat{v}_2 > \left(r - \gamma\mu_p - \frac{\sigma_p^2}{2}\gamma\left(\gamma - 1\right)\right)\left(bc_2 - \widehat{v}_2\right) + \alpha d\left(c_1\right) + \left(1-\alpha\right) d\left(c_3\right) \qquad (126)$$

Taking the difference, we obtain

$$\left(r + \lambda - \gamma\mu_p - \frac{\sigma_p^2}{2}\gamma\left(\gamma - 1\right)\right)\left(\widehat{v}_2 - v_2\right) > \alpha d\left(c_1\right) + \left(1-\alpha\right) d\left(c_3\right) - d\left(c_2\right) \qquad (127)$$

Because

$$r + \lambda > \gamma\mu_p + \frac{\sigma_p^2}{2}\gamma\left(\gamma - 1\right), \text{ and } d\left(c_2\right) \leq \alpha d\left(c_1\right) + \left(1-\alpha\right) d\left(c_3\right) \qquad (128)$$

we get a contradiction.

$\square$

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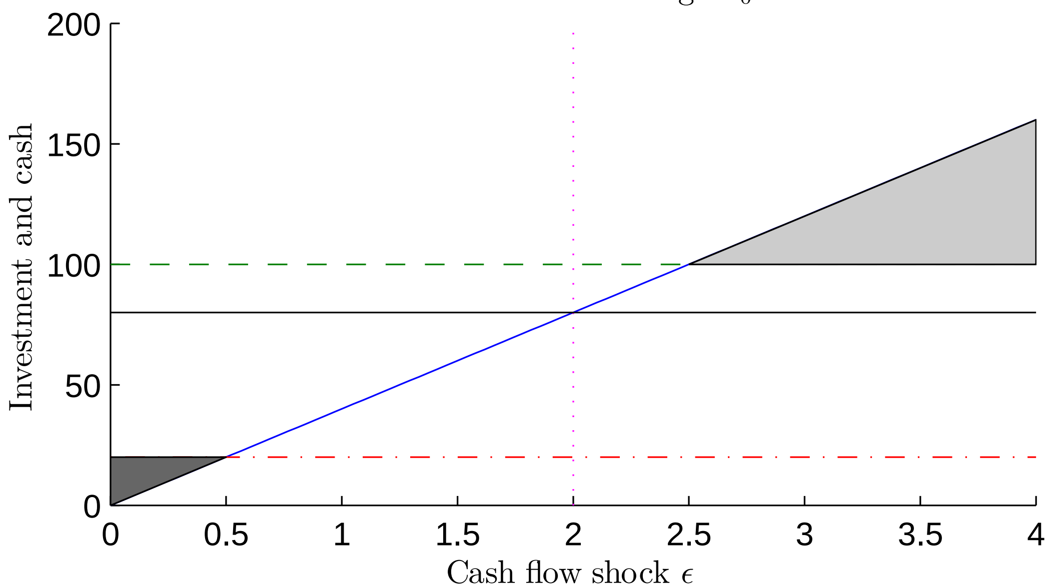
**Figure 1.** Investment and Distress Regions

The firm has operating costs, $R = 20$, and a profitable option to invest a fixed amount, $I - R = 80$. All expenses must be financed internally. Shock $\varepsilon$ is distrubuted uniformly on $[0, 4]$, and $w_1 = 40$. The upper dashed line marks the sum of investment and operating costs, $I = 100$, whereas the lower dashed line marks the operating costs. The solid upward-sloping (flat) line shows the firm's internal funds under no hedging (full hedging). The filled area on the left of the figure represents the financing shortfall in distress; the filled area on the right side represents the financing slack available after the firm invests. The firm invests only when cash exceeds 100. Panels A and B are for the cases of $w_0 = 0$ and $w_0 = 30$, respectively.

Panel A: Low cash savings $w_0 = 0$




Panel B: High cash savings $w_0 = 30$

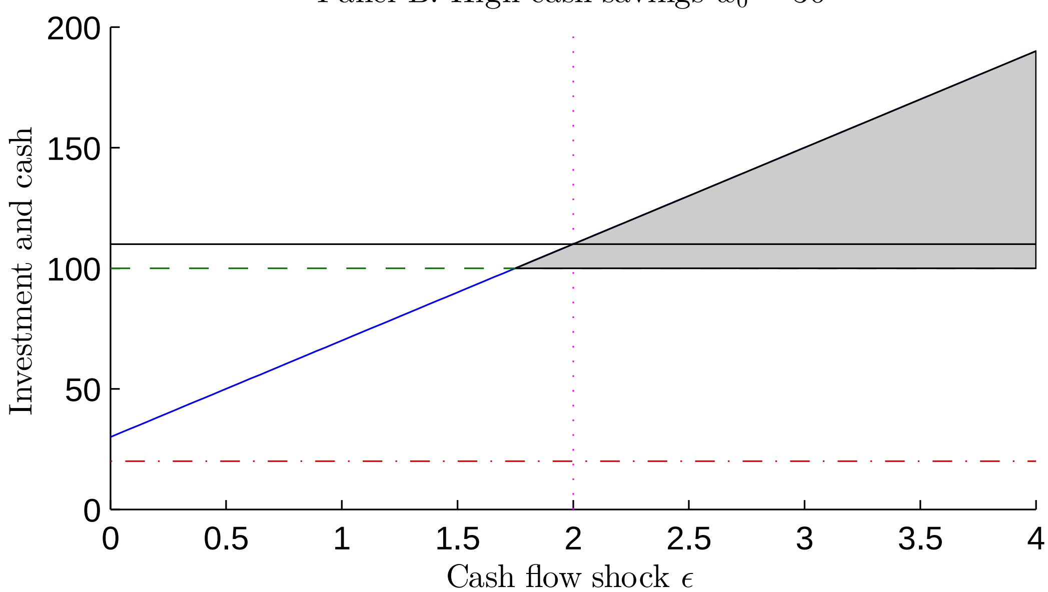


**Figure 2.** Expected Investment and Hedging

This figure shows the expected investment level as a function of hedging ratio $\phi$. We assume $f(I) = b \log I$ and $C(e) = \frac{ke^2}{2}$ and set parameters as follows: $w_1 = 24$, $w_0 = 0$, $R = 50$, $b = 300$, $\alpha = 0$, $f_0 = 1900$, $k = 0.08$, $\overline{\varepsilon} = 2$. Panels A is for the case when the investment option is out-of-the-money at zero hedging, $\beta = 1.45$. Panel B is for the case when the investment option is in-the-money at zero hedging, $\beta = 1.5$.

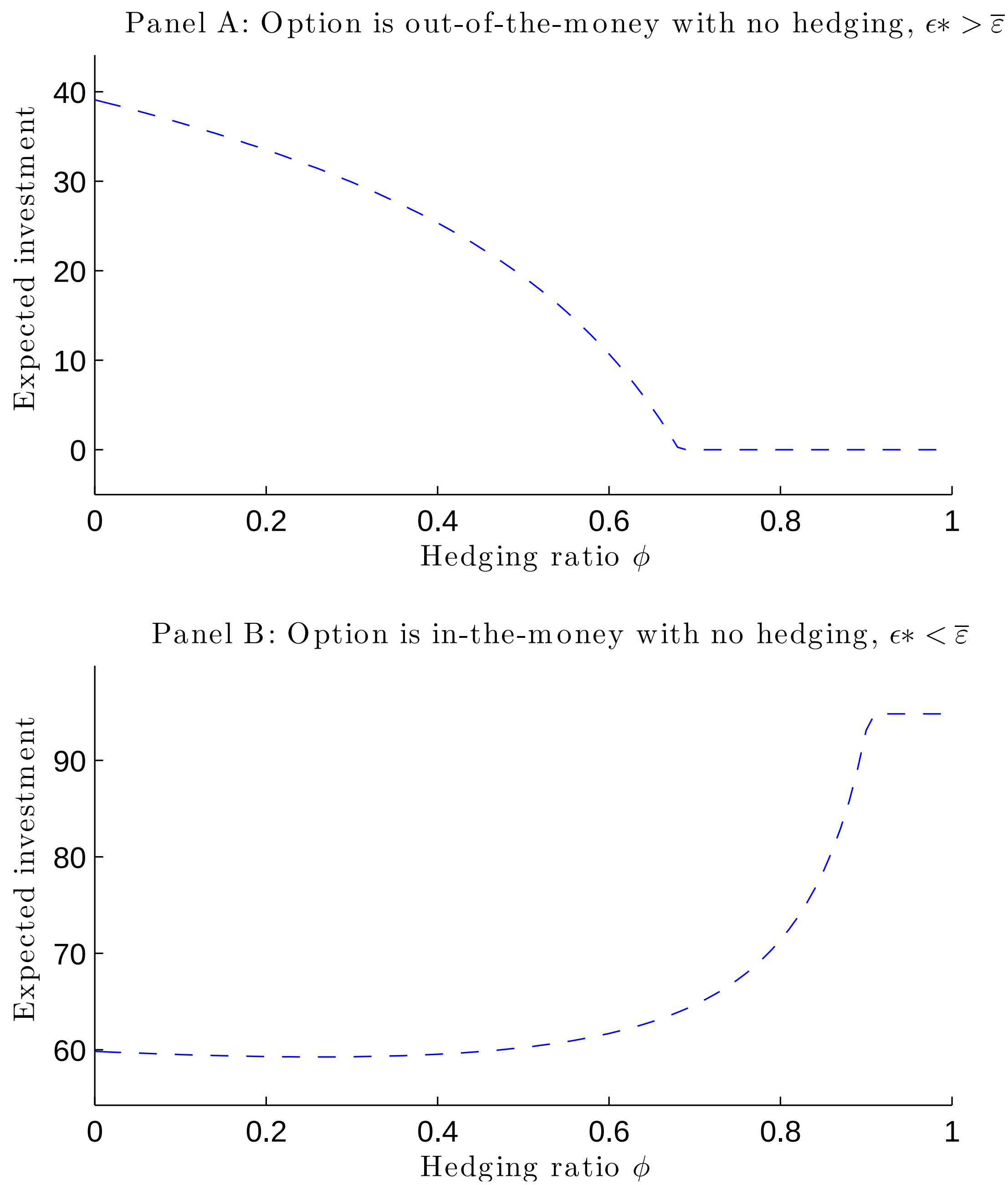


**Figure 3.** Financing Gap and Hedging

This figure shows a firm's cash flow (solid line) and optimal investment level (dashed line) as a function of the primitive uncertainty shock $\varepsilon$. We assume $f(I) = b \log I$ and $C(e) = \frac{ke^2}{2}$ and set parameters as follows: $w_1 = 24$, $w_0 = 0$, $R = 50$, $b = 300$, $\alpha = 1$, $\beta = 1$, $f_0 = 1900$, $k = 0.08$, $\overline{\varepsilon} = 2$. Panels A and B are for high volatility $\sigma = 1.2$; Panels C and D are for low volatility $\sigma = 0.2$.

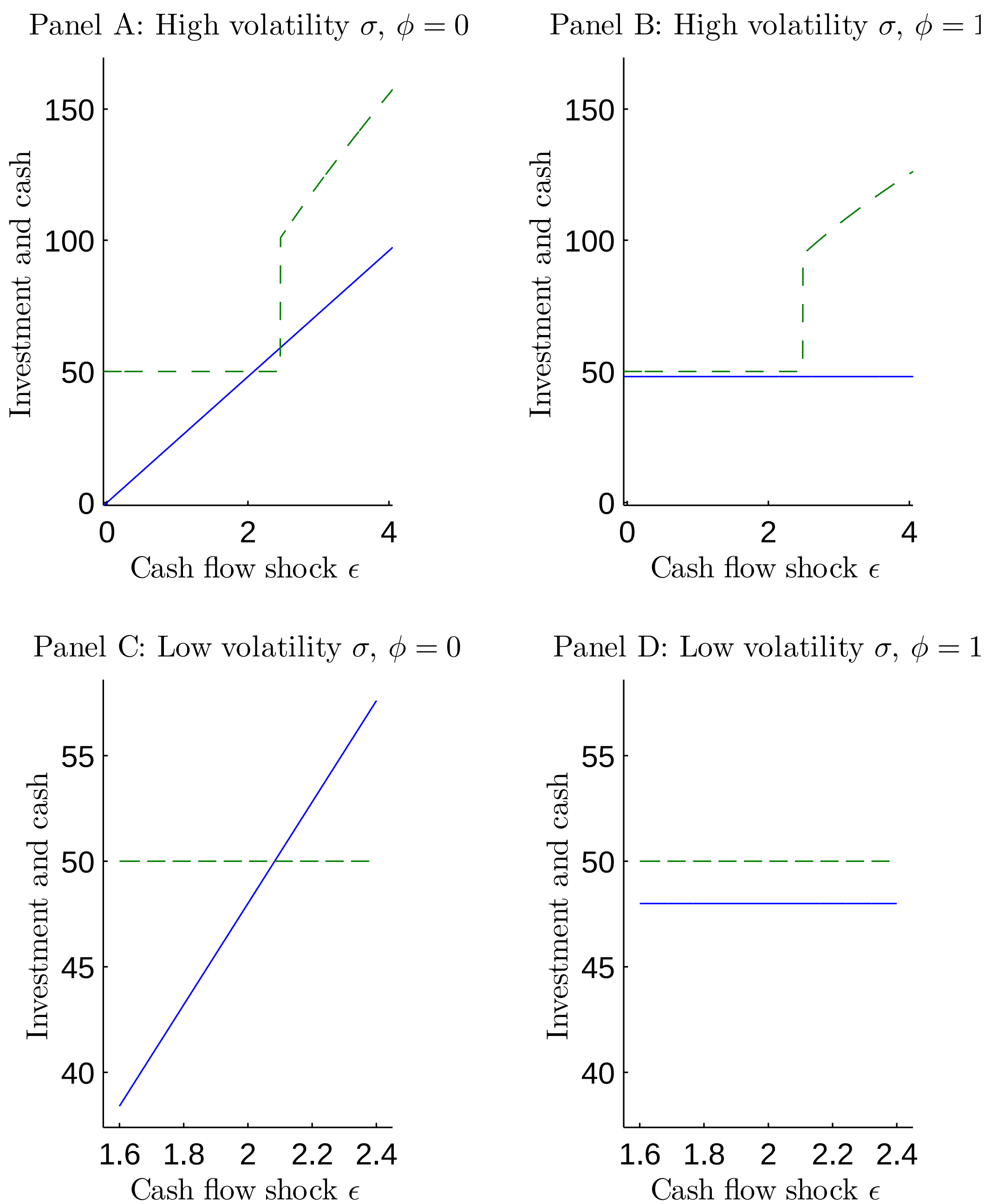
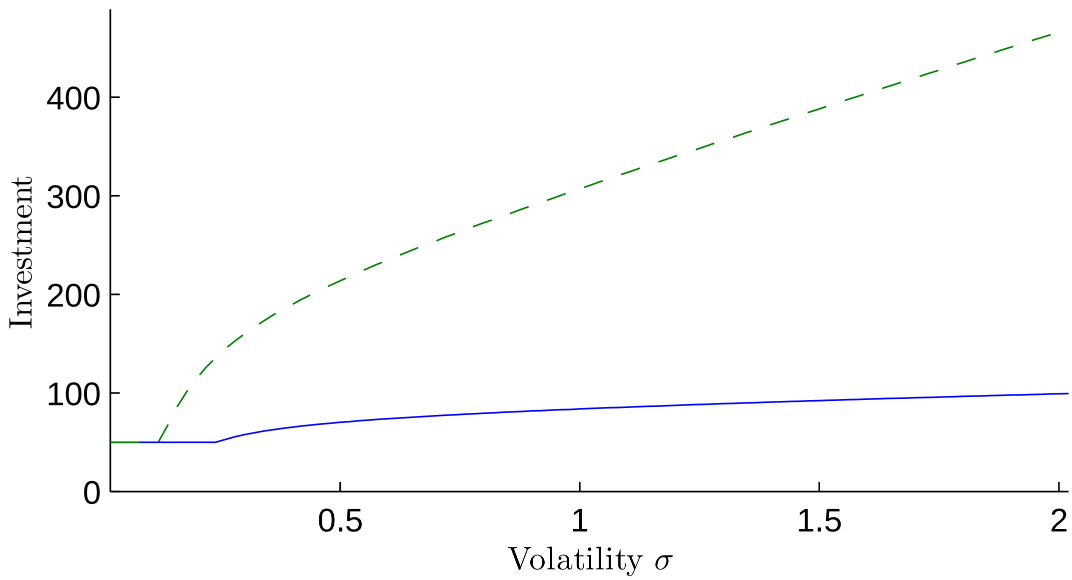


**Figure 4.** Expected Investment and Optimal Risk Management

Panel A shows the expected investment level of a constrained firm (solid line) and an unconstrained firm (dashed line) as a function of volatility $\sigma$. Panel B displays the optimal hedging ratio $\varphi$ (solid line) as a function of volatility $\sigma$. Hedging ratios below zero (dashed line) indicate speculation. Whenever investment exceeds cash flow, the firm raises external financing. We assume $f(I) = b \log I$ and $C(e) = \frac{ke^2}{2}$ and set the parameters as follows: $w_1 = 24$, $w_0 = 0$, $R = 50$, $b = 300$, $\alpha = 1$, $\beta = 1$, $f_0 = 1900$, $k = 0.08$, $\bar{\varepsilon} = 2$.

Panel A: Expected Investment Level




Panel B: Optimal Hedging Ratio

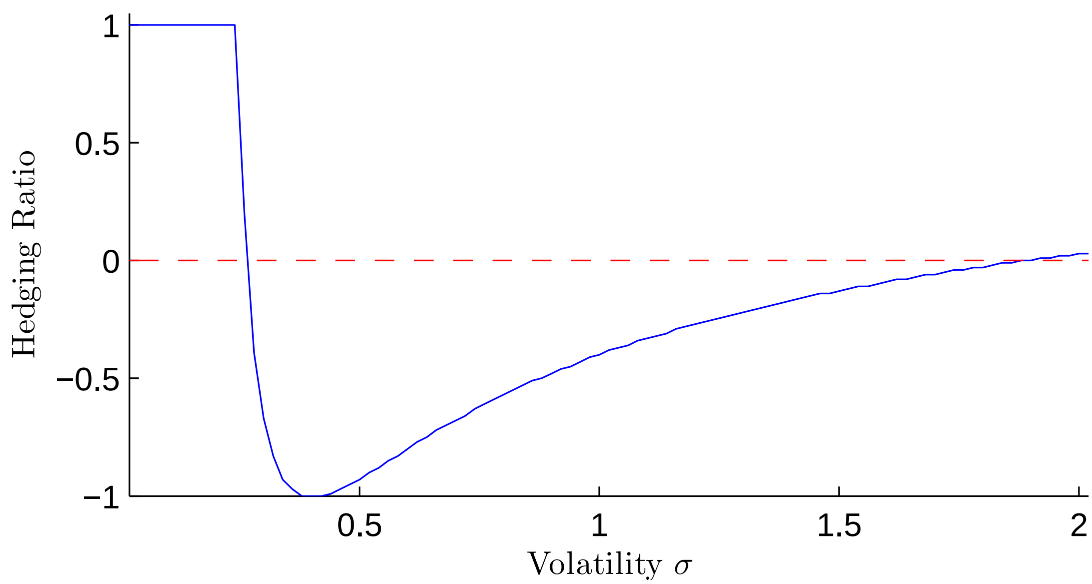


**Figure 5.** The Effect of Financing Constraints and Growth Options on Hedging Ratio

This figure shows the optimal hedging ratio for a firm with zero correlation between investment opportunities and cash flows (dashed line) and for a firm with positive correlation (solid line). Panel A plots the hedging ratio as a function of financing constraints $w_0$, and Panel B as a function of a firm's investment opportunities $\beta$. We assume $f(I) = b \log I$ and $C(e) = \frac{ke^2}{2}$ and set the parameters as follows: $w_1 = 24$, $w_0 = 0$, $R = 50$, $b = 300$, $\beta = 1$, $\sigma$=0.8, $f_0 = 1900$, $k = 0.08$, $\bar{\varepsilon} = 2$.

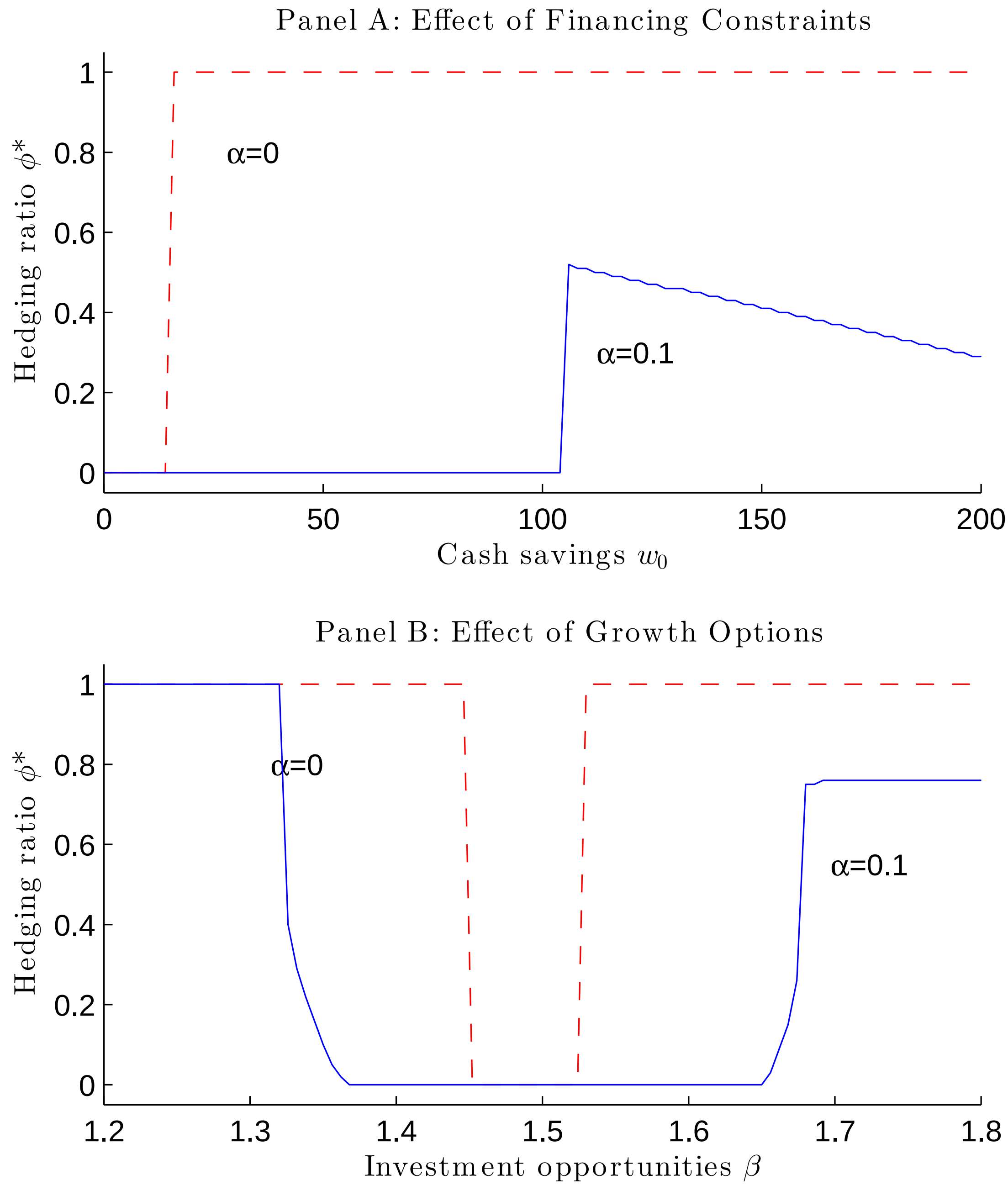


**Figure 6.** The Effect of Positive Correlation on Hedging Ratio

This figure shows the optimal hedging ratio as a function of correlation between investment opportunities and cash flows, $\alpha$. We assume $f(I) = b \log I$ and $C(e) = \frac{ke^2}{2}$ and set the parameters as follows: $w_1 = 24$, $w_0 = 0$, $R = 50$, $b = 300$, $\sigma$=0.8, $f_0 = 1900$, $k = 0.08$, $\bar{\varepsilon} = 2$.

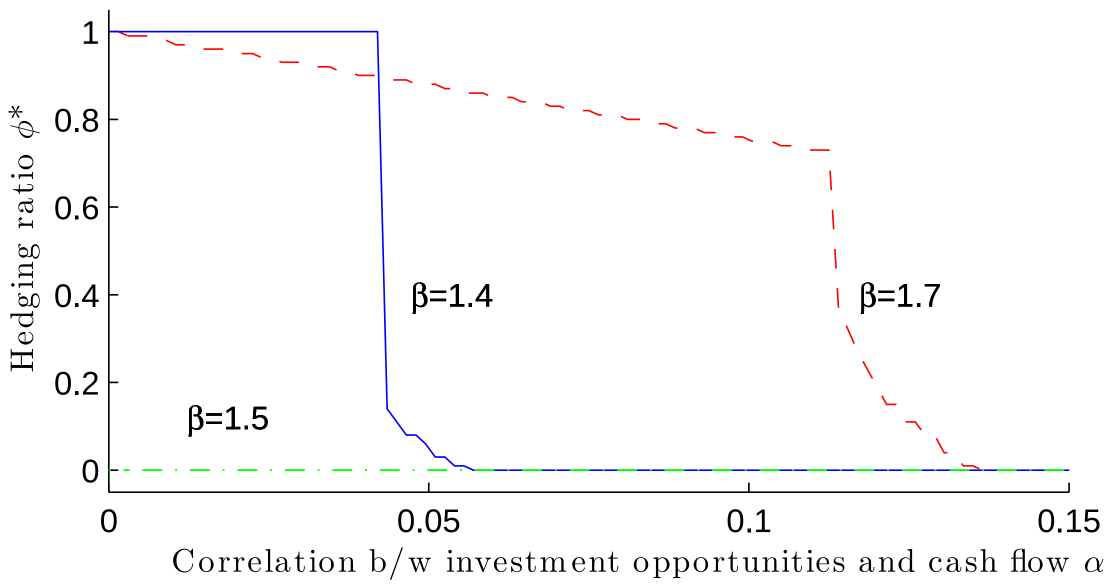
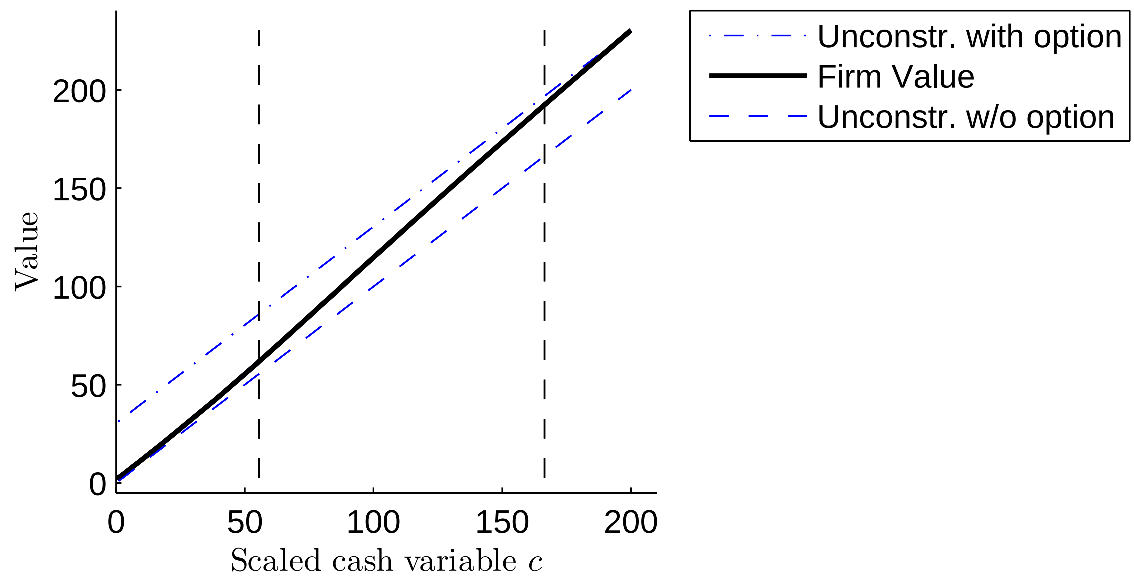


**Figure 7.** Value function $v(c)$ and its first derivative

This figure shows the shape of value function $v(c)$ (Panel A) and its first derivative (Panel B). The solution to ODE (55) is obtained numerically. The upper dashed line in Panel A gives the value of the firm if it were completely unconstrained (i.e., $k_d = k = 0$). The lower dashed line shows the value of the firm that does not have an investment option and is not subject to the costs of financial distress (i.e., $\lambda = 0$ and $k_d = 0$). Vertical lines denote the option exercise threshold and the value of $c$ at which the firm can finance the exercise internally.

Panel A: Value function $v(c)$ and its asymptotes




Panel B: First derivative of $v(c)$