Precautionary Savings and Pecuniary Externalities: Analytical Results for Optimal Capital Income Taxation in OLG Economies

Dirk Krueger* Alexander Ludwig†

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Abstract

In this paper we characterize the optimal linear tax on capital in the standard Overlapping Generations model with neoclassical production and capital accumulation as well as idiosyncratic labor income risk and incomplete markets. For logarithmic utility we provide a complete analytical solution of the optimal Ramsey tax policy problem for arbitrary social welfare weights across generations. The Ramsey allocation is characterized by a constant (over time) aggregate saving rate that is independent of the extent of idiosyncratic income risk. The Ramsey government internalizes that an increase in the household saving rate impacts wages and interest rate in general equilibrium; we show that with logarithmic this pecuniary externality general equilibrium effect exactly cancels out the standard precautionary savings effect in partial equilibrium. The constant tax on capital implementing this saving rate is increasing in the extent of income risk, but might be positive or negative, depending on how the Ramsey government values current and future generations. We also show that if it is positive, then a government implementing this tax rate generates a Pareto-improving, policy induced transition from the unregulated steady equilibrium even if this equilibrium is dynamically efficient. We then generalize our results to arbitrary Epstein-Zin utility and show that the optimal steady state savings rate is increasing in the amount of income risk if and only if the intertemporal elasticity of substitution is smaller than 1. The associated tax rate is increasing in income risk unless both the IES and risk aversion are large.

Keywords: Idiosyncratic Risk, Capital Overaccumulation, Capital Income Taxation, Overlapping Generations

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*University of Pennsylvania, CEPR and NBER

†SAFE, Goethe University Frankfurt

1 Introduction

How should a benevolent government tax capital in a neoclassical production economy when households face uninsurable idiosyncratic labor income risk. Partial answers to this question have been given in Bewley-Huggett-Aiyagari style general equilibrium model with neoclassical production and infinitely lived consumers, starting from Aiyagari (1995), and continuing with recent work by Davila et al. (2012), Acikgoz (2016), Panousi and Reis (2015), Dyrda and Pedrono (2016), Hagedorn et al (2017) and Chen et al. (2017), Chien and Wen (2017).

In this paper we instead provide a complete analytical characterization of this question in a canonical Diamond (1965) Overlapping Generations model, enriched by uninsurable idiosyncratic labor income risk. In this environment we characterize the optimal linear tax on capital chosen by a Ramsey government that places arbitrary Pareto weights on different generations born into this economy, and has to respect equilibrium behavior of households. For logarithmic utility we provide a complete analytical solution of the optimal Ramsey tax policy problem.

The Ramsey allocation is characterized by a constant (over time) aggregate saving rate that is independent of the extent of idiosyncratic income risk. We show that this constant saving rate is shaped by three forces i) a standard precautionary savings force, ii) a general equilibrium pecuniary externality effect that recognizes that changes in the household saving rate impacts wages and interest rate, and iii) a future generations effect that recognizes that higher saving rates by current generations increase the future capital stock, future wages and thus welfare of future generations in the economy. We characterize all three effects in closed form and show that with logarithmic this pecuniary externality general equilibrium effect exactly cancels out the standard precautionary savings effect in partial equilibrium.

The constant tax on capital implementing this saving rate is increasing in the extent of income risk, but might be positive or negative, depending on how the Ramsey government values current and future generations. We also show that if it is positive, then a government implementing this tax rate generates a Pareto-improving, policy induced transition from the unregulated steady equilibrium even if this equilibrium is dynamically efficient.

We then generalize our results to arbitrary Epstein-Zin utility and show that the optimal steady state savings rate is increasing in the amount of income risk if and only if the intertemporal elasticity of substitution is smaller than 1. The associated tax rate is increasing in income risk unless both the IES and risk aversion are large.

1.1 Relation to the Literature

Our paper combines, and contributes to two strands of the literature. First, the literature on optimal taxation in models with idiosyncratic income risk, Aiyagari (1995), Davila, Hong, Krusell and Rios-Rull (2012), and recently, ans especially relevant, Acikgoz (2015), Panousi (2015), Panousi and Reis (2016, 2017), Dyrda and Pedrono (2016), Hagedorn et al (2017) and Chen et al. (2017) as well as Chien and Wen (2017). We share the key modeling trick that permits a closed form solution of the optimal Ramsey policy with Panousi and Reis (2017): in both papers the private saving rate is chosen prior to the realization of idiosyncratic, uninsurable income risk, in our case due to the life cycle structure and the fact that idiosyncratic income risk only hits households in the second period of their lives. As a consequence, there is no heterogeneity in the saving rate across households despite the presence of idiosyncratic income risk. The key difference between both papers is that Panousi and Reis (2017) permit idiosyncratic risk to affect both capital and labor income, but abstract from precautionary saving, whereas the capital tax results we obtain are driven precisely by this force.

Second, we contribute to and extend the literature on optimal (capital income) taxation in life cycle economies, see Garriga (2001), Erosa and Gervais (2003) as well as Conesa, Kitao and Krueger (2009). Neither of these papers is concerned specifically with the impact of idiosyncratic income risk on optimal capital income taxes

2 Model

2.1 Time and Demographics

Time is discrete and extends from $t = 0$ to $t = \infty$. In each period a new generation (indexed by the time of birth t) is born that lives for two periods. Thus at any point in time there is a $y$oung and an $o$ld generation. We normalize household size to 1 for each age cohort. In addition there is an initial old generation that has one remaining year of life.

2.2 Household Preferences and Endowments

2.2.1 Endowments

Each household has one unit of time in both periods, supplied inelastically to the market. Labor productivity when young is equal to $(1-\kappa)$, in the second period labor productivity is given by

$$\kappa\eta_{t+1}$$

where $\kappa \in [0,1)$ is a parameter that captures relative labor income of the old, and η_{t+1} is an idiosyncratic labor productivity shock. We assume that the cdf of η_{t+1} is given by $\Psi(\eta_{t+1})$ in every period and denote the corresponding pdf by $\psi\left(\eta_{t+1}\right)$. We assume that Ψ is both the population distribution of η_{t+1} as well as the cdf of the productivity shock for any given individual (that is, we assume a Law of Large Numbers, LLN henceforth). Whenever there is no scope for confusion we suppress the time subscript of the productivity shock η_{t+1}. We make the following

Assumption 1. *The shock η_{t+1} takes positive values Ψ-almost surely and*

$$\int \eta_{t+1} d\Psi = 1.$$

Each member of the initial old generation is additionally endowed with assets equal a_0, equal to the initial capital stock k_0 in the economy. The asset endowment is independent of the household's realization of the shock η.

2.2.2 Preferences

A household of generation $t \geq 0$ has preferences over consumption allocations $c_t^y, c_{t+1}^o(\eta_{t+1})$ given by

$$V_t = u(c_t^y) + \beta \int u(c_{t+1}^o(\eta_{t+1}))d\Psi \tag{1}$$

Lifetime utility of the initial old generation is determined as

$$V_{-1} = \int u(c_0^o(\eta_0))d\Psi$$

In order to obtain the sharpest analytical results in the first part of the paper we will assume logarithmic utility:

Assumption 2. *The utility function u is logarithmic*

$$u(c) = \log(c) \tag{2}$$

We will generalize our results to general Epstein-Zin-Weil preferences in Section 6 of the paper.

2.3 Technology

The representative firm operates a standard Cobb-Douglas production technology of the form

$$F(K_t, L_t) = K_t^{\alpha} (L_t)^{1-\alpha}.$$

Capital fully depreciates between two (30 year) periods.

2.4 Government

The government levies a (potentially time varying) capital tax τ_t on capital (including interest) and rebates the proceeds in a lump-sum fashion to all members of the current old generation as a transfer T_t. For the optimal tax policy analysis we assume the government has the following social welfare function

$$SWF = \sum_{t=-1}^{\infty} \omega_t V_t$$

where $\{\omega_t\}_{t=-1}^{\infty}$ are the Pareto weights on different generations and satisfy $\omega_t \geq 0$. Since lifetime utilities of each generation will be bounded, so will be the social welfare function as long as $\sum_{t=-1}^{\infty} \omega_t < \infty$. We will also consider the case $\omega_t = 1$ for all t, in which case we will take the social welfare function to be defined as

$$SWF = \lim_{T \to \infty} \frac{\sum_{t=-1}^{T} V_t}{T}$$

which is equivalent to maximizing steady state welfare (as long as a steady state exists).

2.5 Competitive Equilibrium

2.5.1 Household Budget Set and Optimization Problem

The budget constraints in both periods read as

$$c_t^y + a_{t+1} = (1-\kappa) w_t \tag{3}$$

$$c_{t+1}^o = a_{t+1} R_{t+1} (1 - \tau_{t+1}) + \kappa \eta_{t+1} w_{t+1} + T_{t+1} \tag{4}$$

where w_t, w_{t+1} are the aggregate wages in period t and $t+1$, $R_{t+1} = 1 + r_{t+1}$ is the gross interest rate between period t and $t+1$, and T_{t+1} are lump-sum transfers to the old generation. η_{t+1} is the age-2 period-$t+1$ idiosyncratic shock to wages.[1]

2.5.2 Firm Optimization

From the firms first order conditions

$$R_t = \alpha k_t^{\alpha-1} \tag{5a}$$

$$w_t = (1-\alpha) k_t^{\alpha} \tag{5b}$$

where

$$k_t = \frac{K_t}{L_t} = \frac{K_t}{1 - \kappa + \kappa \int \eta_t d\Psi} = K_t$$

is the capital-labor ratio. Since labor supply in the economy is $L_t = 1$, we henceforth do not need to distinguish between the aggregate capital stock K_t and the capital-labor ratio.

[1] Notice that instead of working with a tax on capital τ_t, one could work, completely equivalently, with a (standard) capital income tax τ_t^k given by

$$1 + r_t(1 - \tau_t^k) = (1 + r_t)(1 - \tau_t)$$

and thus

$$\tau_t^k = 1 - \frac{R_t(1-\tau_t) - 1}{R_t - 1}.$$

2.5.3 Equilibrium Definition

Definition 1. *Given the initial condition $a_0 = k_0$ an allocation is a sequence $\{c_t^y, c_t^o(\eta_t), L_t, a_{t+1}, k_{t+1}\}_{t=0}^{\infty}$.*

Definition 2. *Given the initial condition $a_0 = k_0$ and a sequence of tax policies $\boldsymbol{\tau} = \{\tau_t\}_{t=0}^{\infty}$, a competitive equilibrium is an allocation $\{c_t^y, c_t^o, L_t, a_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, prices $\{R_t, w_t\}_{t=0}^{\infty}$ and transfers $\{T_t\}_{t=0}^{\infty}$ such that*

1. *Given prices $\{R_t, w_t\}_{t=0}^{\infty}$ and policies $\{\tau_t, T_t\}_{t=0}^{\infty}$ for each $t \geq 0$, $(c_t^y, c_{t+1}^o(\eta_{t+1}), a_{t+1})$ maximizes (1) subject to (3) and (4) (for each realization of η_{t+1}).*

2. *Consumption $c_0^o(\eta_0)$ of the initial old satisfies (4) (for each realization of η_0):*
$$c_0^o = a_0 R_0 (1 - \tau_0) + \kappa \eta_0 w_0 + T_0$$

3. *Prices satisfy equations (5a) and (5b).*

4. *The government budget constraint is satisfied in every period: for all $t \geq 0$*
$$T_t = \tau_t R_t k_t$$

5. *Markets clear*
$$\begin{aligned} L_t &= L = 1 \\ a_{t+1} &= k_{t+1}. \\ c_t^y + \int c_t^o(\eta_t) d\Psi + k_{t+1} &= k_t^{\alpha} \end{aligned}$$

   Denote by $SWF(\boldsymbol{\tau})$ social welfare associated with an equilibrium for given tax policy $\boldsymbol{\tau}$. As we will show below, for a given tax policy $\boldsymbol{\tau}$ the associated competitive equilibrium in our economy exists and is unique and thus the function $SWF(\boldsymbol{\tau})$ is well-defined.[2]

Definition 3. *Given the initial condition $a_0 = k_0$, a Ramsey equilibrium is a sequence of tax policies $\hat{\boldsymbol{\tau}} = \{\hat{\tau}_t\}_{t=0}^{\infty}$ and equilibrium allocations, prices and transfers associated with $\hat{\boldsymbol{\tau}}$ (in the sense of the previous definition) such that*
$$\hat{\boldsymbol{\tau}} \in \arg\max_{\boldsymbol{\tau}} SWF(\boldsymbol{\tau})$$

3 Analysis of Equilibrium for a Given Tax Policy

3.1 Partial Equilibrium

We first proceed to analyze the household problem for given prices and policies. We will first proceed under the assumption that a unique solution characterized

[2] This requires some restrictions on the τ_t that we need to spell out. I believe we need $\tau_t \leq 1$ and $\tau_t \geq -\bar{\tau}$.

by the Euler equation exists, and then make sufficient parametric assumptions to insure that this is indeed the case.

The optimal asset choice a_{t+1} satisfies

$$1 = \beta(1-\tau_{t+1}) \int \frac{R_{t+1}\left[u'(a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1})\right]}{u'((1-\kappa)w_t - a_{t+1})} d\Psi(\eta_{t+1}).$$

Defining the saving rate as

$$s_t = \frac{a_{t+1}}{(1-\kappa)w_t}$$

we can rewrite the above equation as

$$1 = \beta(1-\tau_{t+1}) \int \frac{R_{t+1}\left[u'(s_t R_{t+1}(1-\tau_{t+1})(1-\kappa)w_t + \kappa\eta_{t+1}w_{t+1} + T_{t+1})\right]}{u'\left[(1-\kappa)w_t(1-s_t)\right]} d\Psi(\eta_{t+1}) \tag{6}$$

which defines the solution

$$s_t = s_t(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \kappa, \Psi)$$

Note by assumption 1 that consumption in the second period is positive Ψ-almost surely. Without further assumptions on the fundamentals we cannot make progress analytical progress. Therefore now invoke assumption 2 that the utility function is logarithmic. Then the Euler equation becomes:

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\kappa w_{t+1}}{(1-\kappa)w_t R_{t+1}}\eta_{t+1} + \frac{T_{t+1}}{(1-\kappa)w_t R_{t+1}}} d\Psi(\eta_{t+1}) \tag{7}$$

Equation (7) implicitly defines the optimal savings rate $s_t = s(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \kappa\Psi)$.

3.2 General Equilibrium

Now we exploit the remaining equilibrium conditions. In equilibrium factor prices and transfers are given by

$$\begin{aligned} w_t &= (1-\alpha)k_t^\alpha &\quad (8a)\\ w_{t+1} &= (1-\alpha)k_{t+1}^\alpha &\quad (8b)\\ R_{t+1} &= \alpha k_{t+1}^{\alpha-1} &\quad (8c)\\ T_{t+1} &= \tau_{t+1}R_{t+1}k_{t+1} &\quad (8d)\end{aligned}$$

From the definition of the saving rate $s_t = \frac{a_{t+1}}{(1-\kappa)w_t}$ and market clearing in the asset market, which implies $a_{t+1} = k_{t+1}$, we find that

$$k_{t+1} = a_{t+1} = (1-\kappa)s_t w_t$$

and thus

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha \tag{9}$$

Sometimes it will be useful to express the saving rate as a function of the capital stocks and write

$$s_t = \frac{k_{t+1}}{(1-\alpha)(1-\kappa)k_t^{\alpha}} \tag{10}$$

In general, for a given sequence of capital income taxes $\{\tau_t\}_{t=0}^{\infty}$ the competitive equilibrium is a sequence of capital stocks $\{k_{t+1}\}_{t=0}^{\infty}$ that solves, for a given initial condition k_0, the first order difference equation (equation 7) when factor prices have been substituted

$$\begin{aligned} 1 &= \alpha\beta(1-\tau_{t+1})k_{t+1}^{\alpha-1}\int\left(\frac{\left[\kappa\eta_{t+1}(1-\alpha)+\alpha\right]k_{t+1}^{\alpha}}{(1-\kappa)(1-\alpha)k_t^{\alpha}-k_{t+1}}\right)^{-1} d\Psi(\eta_{t+1}) \\ 1 &= \alpha\beta(1-\tau_{t+1})\left(\frac{(1-\kappa)(1-\alpha)k_t^{\alpha}-k_{t+1}}{k_{t+1}}\right)\Gamma \end{aligned} \tag{11}$$

where the constant

$$\Gamma = \int\left(\kappa\eta_{t+1}(1-\alpha)+\alpha\right)^{-1} d\Psi(\eta_{t+1}) = \Gamma(\alpha,\kappa;\Psi) \tag{12}$$

fully captures the impact of idiosyncratic income risk on the equilibrium dynamics of the capital stock.

Equation (11) implicitly defines the function $k_{t+1} = \Omega(k_t, \tau_{t+1})$. Alternatively, and often more conveniently, we instead of expressing the solution as $k_{t+1} = \Omega(k_t, \tau_{t+1})$ we can also express it in terms of the saving rate as

$$s_t = \frac{k_{t+1}}{(1-\alpha)(1-\kappa)k_t^{\alpha}} = \frac{\Omega(k_t,\tau_{t+1})}{(1-\alpha)(1-\kappa)k_t^{\alpha}} = \Lambda(k_t,\tau_{t+1}) \tag{13}$$

where the function $s_t = \Lambda(k_t, \tau_{t+1})$ solves (using the definition of the saving rate in equation (11):

$$1 = \alpha\beta(1-\tau_{t+1})\left(\frac{1-s_t}{s_t}\right)\Gamma \tag{14}$$

3.3 Characterization of the Savings Rate

Evidently, equation (14) has a closed form solution for the saving rate s_t in general equilibrium, and we can give a complete analytical characterization of its comparative statics properties.

Proposition 1. *Suppose assumptions 1 and 2 are satisfied. Then for all $k_t > 0$ and all $\tau_{t+1} \in (-\infty, 1]$ the unique savings rate $s_t = \Lambda(k_t, \tau_{t+1}; \Gamma)$ is given by*

$$s_t = \frac{1}{1 + \left[(1-\tau_{t+1})\alpha\beta\Gamma(\alpha,\kappa;\Psi)\right]^{-1}} \tag{15}$$

which is strictly increasing in Γ, strictly decreasing in τ_{t+1} and independent of the beginning of the period capital stock.

The next corollary assures that any desired savings rate $s_t \in [0,1]$ can be implemented as part of a competitive equilibrium by appropriate choice of the capital tax rate τ_{t+1}. This corollary is crucial for our approach of solving the optimal Ramsey tax problem, since we can cast that problem directly in terms of the government choosing savings rates rather than tax rates.

Corollary 1. *For each savings rate $s_t \in [0,1]$ there exists a unique tax rate $\tau_{t+1} \in (-\infty, 1]$ that implements that savings rate s_t as part of a competitive equilibrium.*

Finally we want to determine the influence of income risk on the savings rate in general equilibrium. From proposition 1 we know that the savings rate depends on income risk η exclusively through the constant Γ. Furthermore, Γ is a strictly convex function of income risk η, and thus by Jensen's inequality we have the following:

Observation 1. *Assume that $\alpha \in (0,1)$ and $\kappa > 0$. Then*

1. *The constant $\Gamma(\alpha, \kappa; \Psi)$ is strictly increasing in the amount of income risk, in the sense that if the distribution $\tilde{\Psi}$ over η is a mean-preserving spread of Ψ, then $\Gamma(\alpha, \kappa; \Psi) < \Gamma(\alpha, \kappa; \tilde{\Psi})$.*

2. *Defining the degenerate distribution at $\eta \equiv 1$ as $\bar{\Psi}$, then for any nondegenerate Ψ*
$$1 < \bar{\Gamma} := \Gamma(\alpha, \kappa; \bar{\Psi}) < \Gamma(\alpha, \kappa; \Psi)$$

From the above remark and the previous proposition we then immediately deduce the following:

Corollary 2. *The equilibrium savings rate is strictly increasing in the amount of income risk, where an increase in income risk is defined in the sense of the previous observation 1.*

The proof of this result follows directly from the fact that $s_t = \Lambda(k_t, \tau_{t+1}; \Gamma)$ is strictly increasing in Γ and Γ is strictly increasing in the amount of income risk. Equipped with this full characterization of the competitive equilibrium for a given sequence of tax policies $\{\tau_{t+1}\}_{t=0}^{\infty}$ we now turn to the analysis of *optimal fiscal policy.*

4 The Ramsey Problem

The objective of the government is to maximize social welfare $W(k_0) = \sum_{t=-1}^{\infty} \omega_t V_t$ by choice of capital taxes $\{\tau_{t+1}\}_{t=0}^{\infty}$ where V_t is the lifetime utility of generation t in the competitive equilibrium associated with the sequence $\{\tau_{t+1}\}_{t=0}^{\infty}$. Writing lifetime utility in terms of the savings rate s_t yields

$$V(k_t, s_t) = u((1-s_t)(1-\kappa)(1-\alpha)k_t^{\alpha}) + \beta \int u\left(\kappa \eta_{t+1} w(s_t) + R(s_t) s_t (1-\kappa)(1-\alpha) k_t^{\alpha}\right) d\Psi(\eta_{t+1}) \tag{16}$$

where

$$
\begin{aligned}
w(s_t) &= (1-\alpha)\left[k_{t+1}(s_t)\right]^{\alpha} && (17)\\
R(s_t) &= \alpha\left[k_{t+1}(s_t)\right]^{\alpha-1} && (18)\\
k_{t+1}(s_t) &= s_t(1-\kappa)(1-\alpha)k_t^{\alpha} && (19)
\end{aligned}
$$

We could of course substitute factor prices in the lifetime utility function, but for the purpose of better interpretation of the results we refrain from doing so at this moment.

Finally, remaining lifetime utility of the initial old generation is given by (with factor prices already substituted out)

$$V_{-1} = V(k_0,\tau_0) = \int u\left(\left[\alpha + \kappa\eta_0(1-\alpha)\right]k_0^{\alpha}\right)d\Psi(\eta_0) = V(k_0) \qquad (20)$$

Note that τ_0 is irrelevant for welfare of the initial old generation (and all future generations) and can be set arbitrarily. This is due to the fact that τ_0 is nondistortionary, is lump-sum rebated and (most crucially) that the government is assumed to have a period-by-period budget balance. In fact, expression (20) shows that with the set of policies we consider here lifetime utility of the initial old cannot be affected at all, which is useful since we therefore do not need to include it in the social welfare function.

By corollary 1 the Ramsey government can implement any sequence of savings rates $\{s_t\}_{t=0}^{\infty}$ as a competitive equilibrium and thus can choose private savings rates directly. We therefore can restate the problem the Ramsey government solves as

$$W(k_0) = \max_{\{s_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \omega_t V(k_t, s_t) \qquad (21)$$

subject to (17)-(19).

4.1 Recursive Formulation and Characterization of the Ramsey Problem

The Ramsey problem lends itself to a recursive formulation, under the following assumption on the social welfare weights:

Assumption 3. *The social welfare weights satisfy, for all $t \geq 0, \omega_t > 0$ and*

$$\frac{\omega_{t+1}}{\omega_t} = \theta \in (0,1].$$

Under this assumption, the recursive formulation of the problem reads as

$$
\begin{aligned}
W(k) &= \max_{s\in[0,1)} u((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\
&\quad +\beta \int u\left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Psi(\eta) + \theta W(k'(s)) \\
&\quad s.t. && (22) \\
k'(s) &= s(1-\kappa)(1-\alpha)k^{\alpha} && (23) \\
R(s) &= \alpha\,[k'(s)]^{\alpha-1} && (24) \\
w(s) &= (1-\alpha)\,[k'(s)]^{\alpha} && (25)
\end{aligned}
$$

This perhaps unusual way of writing the problem clarifies the three effects the Ramsey government considers when choosing the savings rate s in the current period.[3] First, there is the direct effect of reduced consumption when young and increased consumption when old, henceforth denoted by $PE(s)$. Second, there is the indirect, general equilibrium effect on the current generation of changed wages and rates of return when old, which we denote as $GE(s)$. And third, there is the impact on future generations from a changed capital stock induced by a change in the current savings rate, denoted by $FG(s)$.

Taking first order conditions yields

$$
\begin{aligned}
0 &= (1-\kappa)(1-\alpha)k^{\alpha}\left[-u'(c^y) + R(s)\beta\int u'\left(c^o(\eta)\right) d\Psi(\eta)\right] \\
&\quad +\beta\int u'\left(c^o(\eta)\right)\left[\kappa\eta w'(s) + (1-\kappa)(1-\alpha)k^{\alpha}R'(s)s\right] d\Psi(\eta) \\
&\quad +\theta W'(k'(s))\frac{dk'(s)}{ds} \\
&= PE(s) + GE(s) + FG(s)
\end{aligned}
$$

We make the following observations:

1. Denote by s^{CE} the savings rate household would choose in the competitive equilibrium with zero capital taxes. Then $PE(s^{CE}) = 0$.

2. In Appendix A we show that the general equilibrium effect can be written as

$$
GE(s) = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}[s]^{\alpha-1}\,\beta\int u'\left(c^o(\eta)\right)[\kappa\eta - 1]\,d\Psi(\eta)
$$

   and thus the sign of the general equilibrium benefit of an extra unit of saving for the current generation is determined by the term

$$
\int u'\left(c^o(\eta)\right)[\kappa\eta-1]\,d\Psi(\eta) = \int u'\left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right)[\kappa\eta - 1]\,d\Psi(\eta)
$$

[3] Or equivalently, when choosing the tax rate τ' that then induces private households to choose the savings rate s.

If $\kappa = 0$, then the old do not have labor income, and thus the impact of higher saving and consequently a larger capital stock is unambiguously negative, due to a lower return on saving when old. If, on the other hand, κ is large, wages when old are important for this generation which calls, ceteris paribus, for a larger saving rate. Note that whereas the magnitude of a change in factor prices induced by a change in savings rates is purely determined from the production side of the economy, the utility value to the household and thus the Ramsey government of these factor movements depends on the utility function since it determines the size of the covariance between $u'(c^o(\eta))$ and η (which is negative). If households are risk-neutral (or there is no risk), then the sign of $GE(s)$ is given by $\kappa - 1$ which is negative, leading to a reduced incentive to save due to general equilibrium effects, and an associated extra incentive to tax capital income.[4] With risk the sign of $GE(s)$ is determined by the sign of

$$\begin{aligned} E\left[u'(c^o(\eta))(\kappa\eta - 1)\right] &= (\kappa - 1)E\left[u'(c^o(\eta))\right] + Cov\left[u'(c^o(\eta)), (\kappa\eta - 1)\right] \\ &< (\kappa - 1)E\left[u'(c^o(\eta))\right] < 0 \end{aligned}$$

and thus there is an extra disincentive to save from the general equilibrium effect: higher wages exacerbate idiosyncratic income and thus consumption risk and thus it is optimal for the social planner to reduce labor income risk by reducing savings incentives, other things equal.

3. The effect of a higher savings rate today on future generations through a higher capital stock from tomorrow on, $k'(s)$ is encoded in the term

$$FG(s) = \theta W'(k'(s))\frac{dk'(s)}{ds} = (1-\kappa)(1-\alpha)k^{\alpha}\theta W'(k'(s)) > 0$$

and depends on the relative social welfare weights of future generations $\theta = \frac{\omega_{t+\tau}}{\omega_t}$.

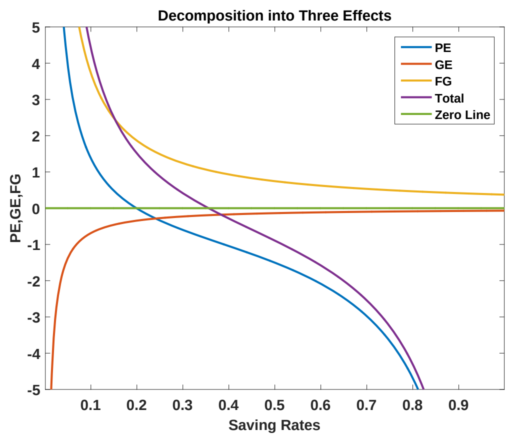
The following figure plots the terms $PE(s), GE(s), FG(s)$ as well as their sum against the savings rate s for a parametric example, and fixing a current (or initial) capital stock k.[5] We observe that, as expected, $FG(s)$ is always positive (the marginal benefit from a higher saving rate on future generations through a higher capital stock is always positive). Also, as argued in item 2. above, $GE(s)$ is always negative, and thus calls for a lower savings rate and higher capital income tax rate. Finally, the $PE(s)$ line shows where the competitive equilibrium savings rate absent government policies is located (at the intersection between $PE(s)$ and the zero line, and the sum $PE(s)+GE(s)+FG(s)$ displays the optimal Ramsey savings rate s (intersection with the zero line). In this example the FG effect dominates the GE effect and the savings rate

[4] Unless $\kappa = 1$ and the old hold all labor income, in which case the general equilibrium effect on optimal saving is exactly equal to zero.

[5] We will show below that for the logarithmic case the Ramsey savings rate is independent of the current capital stock, and since we display an example with $\sigma = 1$ in the plot, the dependence of s on k is actually moot here.

s^* chosen by the Ramsey government exceeds that emerging in the unregulated competitive equilibrium s^{CE}. Of course this is not a general result; for example, if $\theta = 0$ and future generations are not valued at all, one would obtain $s^* < s^{CE}$.

Figure 1: Decomposition of Optimal Savings Rate Determination



4.2 Explicit Solution of the Ramsey Tax Problem

We now provide a complete analytical characterization of the Ramsey optimal policy problem under the assumption that utility is logarithmic. We can do so for arbitrary social welfare weights $\{\omega_t\}_{t=0}^{\infty}$ using the sequential formulation of the problem, and we do so in appendix B. Here we exploit the recursive formulation of the problem, which requires assumption 3, but allows us to arrive at the solution rather immediately.

As in the standard neoclassical growth model, the recursive version of the Ramsey problem with log-utility has a unique closed-form solution, which can be obtained by the method of undetermined coefficients. To this end, guess that the value function takes the following log-linear form:

$$W(k) = \Psi_0 + \Psi_1 \log(k)$$

Using this guess and equations (23)-(25) we can rewrite the Bellman equation (22) as:

$$
\begin{aligned}
W(k) &= \Psi_0 + \Psi_1 \log(k) \\
&= \max_{s\in[0,1]} \Big\{ \log((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\
&\quad + \beta \int \log\left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Psi(\eta) + \theta W(k') \Big\} \\
&= \log((1-\kappa)\,(1-\alpha)) + \alpha\beta \log((1-\kappa)(1-\alpha)) \\
&\quad + \int \log\left(\kappa\eta(1-\alpha)+\alpha\right) d\Psi(\eta) + \theta\Psi_0 + \theta\Psi_1 \log\left[(1-\kappa)(1-\alpha)\right] \\
&\quad + \left[\alpha + \alpha^2\beta + \alpha\theta\Psi_1\right] \log(k) + \max_{s\in[0,1]} \left\{\log(1-s) + (\alpha\beta+\theta\Psi_1)\log(s)\right\}
\end{aligned} \tag{26}
$$

We immediately observe that for the Bellman equation to hold, the coefficient Ψ_1 has to satisfy

$$\Psi_1 = \alpha + \alpha^2\beta + \alpha\theta\Psi_1$$

or

$$\Psi_1 = \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)}$$

We also immediately recognize that the optimal saving rate chosen by the Ramsey planner is independent of the capital stock k and determined by the first order condition

$$\frac{1}{1-s} = \frac{\alpha\beta + \theta\Psi_1}{s}$$

and thus

$$s^* = \frac{\alpha\beta+\theta\Psi_1}{1+\alpha\beta+\theta\Psi_1} = \frac{\alpha(\beta+\theta)}{1+\alpha\beta} \tag{27}$$

Plugging in s^* and Ψ_1 into the Bellman equation (26) yields a linear equation in the constant Ψ_0 whose solution completes the full analytical characterization of the Ramsey optimal taxation problem, summarized in the following

Proposition 2. *Suppose assumptions 1 and 2 are satisfied. Then the solution of the Ramsey problem is characterized by a constant saving rate*[6]

$$s_t = s^* = \frac{\alpha(\beta+\theta)}{1+\alpha\beta}$$

a sequence of capital stocks that satisfy

$$k_{t+1} = s^*(1-\kappa)(1-\alpha)k_t^{\alpha}$$

[6] In appendix we show, using the sequential formulation of the problem, that for arbitrary welfare weights the optimal savings rate is still independent of the capital stock and given by

$$s_t = \frac{1}{1+\left(\alpha\beta + \alpha\,(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}$$

which implies the savings rate in the proposition under the assumption that $\frac{\omega_{t+1}}{\omega_t} = \theta$ for all t.

with initial condition k_0. The associated value function is given by

$$W(k) = \Psi_0 + \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)} \log(k)$$

with derivative

$$W'(k) = \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)k}$$

The Ramsey allocation is implemented with constant capital taxes $\tau = \tau(\beta, \theta, \kappa, \alpha; \Psi)$ satisfying

$$1 - \tau = \frac{(\theta + \beta)}{(1-\alpha\theta)\,\beta\Gamma(\alpha, \kappa; \Psi)} \tag{28}$$

where Γ is a positive constant that was defined in equation (12) and just depends on parameters.

Corollary 3. *The optimal savings rates are independent of the extent of income risk in the economy.*

Corollary 4. *The optimal capital tax rates are strictly increasing in the extent of income risk (as measured by Γ), strictly decreasing in the social discount factor θ, strictly increasing in the individual discount factor β and strictly decreasing in the labor income share κ of the old.*

It is noteworthy that not only is the optimal savings rate constant and does not depend on the level of the capital stock, but it also is independent of the extent of income risk η. This is true despite the fact that for a *given* tax policy higher income risk induces a larger individually optimal savings rate, as shown in section 3.3. The Ramsey government finds it optimal to exactly offset this effect with a capital tax that is increasing in the amount of income risk, cancelling out exactly the partial equilibrium incentive to save more as income risk increases.

One advantage of the complete characterization of the recursive problem is that we can now give a cleaner decomposition of the three forces determining the optimal Ramsey saving rate. We now find that

$$\begin{aligned}
PE(s) &= \frac{-1}{(1-s)} + \frac{\alpha\beta}{s}\Gamma(\alpha, \kappa; \Psi) \\
GE(s) &= \frac{\alpha\beta}{s}\left[1 - \Gamma(\alpha, \kappa; \Psi)\right] \\
FG(s) &= \frac{\theta\alpha(1+\alpha\beta)}{(1-\alpha\theta)s}
\end{aligned}$$

where we note that that

$$\Gamma(\alpha, \kappa; \Psi) > \frac{1}{\kappa(1-\alpha)+\alpha} \geq 1$$

and where the first inequality is strict as long as Ψ is nondegenerate and $\kappa > 0$, and the second inequality is strict as long as $\kappa < 1$. Thus $[1 - \Gamma(\alpha, \kappa; \Psi)] \leq 0$, with strict inequality if $\kappa < 1$.

$$\begin{aligned} PE(s) &\gtreqless 0, PE'(s) < 0 \\ GE(s) &< 0, GE'(s) > 0 \\ FG(s) &> 0, FG'(s) < 0 \end{aligned}$$

Recall that the savings rate s^{CE} in the competitive equilibrium with zero taxes satisfies $PE(s^{CE}) = 0$. This implies that, starting from zero taxes, the only reason to tax capital is the general equilibrium, pecuniary externality, effect which unambiguously pushes the desired savings rate down and the tax rate up (i.e. makes it positive). Against this works the future generations effect (whose size is controlled by θ) and calls unambiguously for a higher savings rate and thus a lower (i.e. negative) tax rate.

Also note that

$$\begin{aligned} PE(s) + GE(s) &= \frac{-1}{(1-s)} + \frac{\alpha\beta}{s}\Gamma(\alpha, \kappa; \Psi) + \frac{\alpha\beta}{s}\left[1 - \Gamma(\alpha, \kappa; \Psi)\right] \\ &= \frac{-1}{(1-s)} + \frac{\alpha\beta}{s} \qquad (29) \end{aligned}$$

and thus the partial equilibrium incentive to save more when income risk rises is *exactly* cancelled out by the general equilibrium effect on factor prices. Thus the simple solution with log-utility of the Ramsey problem masks the presence of a partial equilibrium and a general equilibrium effect that turn out to exactly cancel each other out.

4.3 Discussion of Optimal Tax Rates

In this section we use the sharp characterization of optimal Ramsey savings rates and capital taxes from equation (28) to discuss further properties of the optimal Ramsey capital tax rates. The following proposition, which follows immediately from the inspection of (28) gives conditions under which the optimal Ramsey capital tax is positive, and, in contrast, conditions under which capital is subsidized. For the next proposition, recall that for $\theta = 0$ only the utility of the first generation receives weight in the social welfare function, whereas $\theta = 1$ amounts to the Ramsey government maximizing steady state welfare.

Proposition 3. *There is a threshold social discount factor $\bar{\theta}$ such that for all $\theta \geq \bar{\theta}$ capital is subsidized in every period whereas for all $\theta < \bar{\theta}$ it is taxed in every period. This threshold is explicitly given as*

$$\bar{\theta} = \frac{(\Gamma - 1)\beta}{1 + \alpha\beta\Gamma} > 0$$

Corollary 5. *If $\bar{\theta} \geq 1$, then capital is taxed even when the Ramsey government maximizes steady state welfare. If $\bar{\theta} < 1$ then the government should subsidize* *capital when the Ramsey government maximizes steady state welfare. If the government maximizes welfare of only the initial generation ($\theta = 0$) it should unambiguously tax capital.*

Note that these results also apply to the model without income risk. In that case, which provides a useful benchmark to interpret the general findings, note that the optimal Ramsey capital tax from equation (28) is given by

$$\tau = 1 - \frac{(\theta/\beta + 1)\,(1 - (1-\kappa)(1-\alpha))}{(1-\alpha\theta)}$$

If $\theta = 0$ and the Ramsey government only values the first generation (as effectively, in the simple model of Krusell et al. (2012)), the future generations term $FG(s)$ is absent, and their optimal capital tax is given by

$$\tau = (1-\kappa)(1-\alpha)$$

Thus capital is taxed at a weakly positive rate, and at a strictly positive rate unless $\kappa = 1$ (the old receive all labor income).[7] Since taxes with income risk are higher than without, the capital tax rate τ is strictly positive for any degenerate distribution of the income shock if $\theta = 0$.

At the other extreme, suppose that $\theta = 1$. Then

$$\tau = 1 - \frac{(1/\beta + 1)\,(1 - (1-\kappa)(1-\alpha))}{(1-\alpha)}$$

and we show in appendix C.3 that in this case $\tau < 0$ if and only if the competitive equilibrium without taxes is dynamically efficient (i.e. has an interest rate $R > 1$, or equivalently, a capital stock below the golden rule capital stock k^{GR}).

This suggests the possibility that without income risk the competitive economy is dynamically efficient and the government optimally *subsidizes* capital in the steady state, but with sufficiently large income risk the result reverses and the Ramsey government finds it optimal to tax capital in the steady state. The following proposition, again proved in appendix C.3, shows that this is indeed the case.

Proposition 4. *Let $\theta = 1$ such that the Ramsey government maximizes steady state welfare, and denote by s^* the associated optimal savings rate. Furthermore denote by $s_0(\eta)$ the steady state equilibrium savings rate in the absence of government policy and s^{GR} the golden rule savings rate that maximizes steady state aggregate consumption. Finally assume that $\beta < \left[(1-\alpha)\bar{\Gamma} - 1\right]^{-1}$.*

1. *Let income risk be large: $\Gamma > \frac{1}{\beta\left[(1-\alpha)-1/\bar{\Gamma}\right]}$. Then the steady state competitive equilibrium is dynamically inefficient, $s^{GR} < s_0(\eta)$, and $s^* < s_0(\eta)$, and the optimal capital tax rate has $\tau > 0$.*

[7] If $\kappa = 1$, then in the absence of income risk $\bar{\Gamma} = 1$, and the general equilibrium effect is $GE(s) = 0$, and thus in the absence of also the future generations effect $FG(s) = 0$, and thus the Ramsey optimal saving rate coincides with the competitive equilibrium savings rate for $\tau = 0$.

2. *Let income risk be intermediate:*

$$\Gamma \in \left(\frac{1+\beta}{(1-\alpha)\,\beta}, \frac{1}{\left[(1-\alpha) - 1/\bar{\Gamma}\right]\beta} \right)$$

Then the steady state competitive equilibrium is dynamically efficient, $s^* < s_0(\eta) < s^{GR}$*, but optimal capital taxes are nevertheless positive.*

3. *Let income risk be small:*

$$\Gamma \in \left[\bar{\Gamma}, \frac{1+\beta}{(1-\alpha)\,\beta} \right)$$

Then the steady state competitive equilibrium is dynamically efficient, $s_0(\eta) < s^{GR}$*, and* $s_0(\eta) < s^*$*, and optimal capital taxes are negative (capital is subsidized).*

Note that if condition $\beta < \left[(1-\alpha)\bar{\Gamma} - 1\right]^{-1}$ is violated, then the steady state competitive equilibrium is dynamically inefficient and the optimal capital tax rate is positive *for all degrees of income risk.* The interesting and perhaps unexpected result is case 2: in the presence of income risk the Ramsey government maximizing steady state welfare might want to tax capital even though this reduces aggregate consumption (since the equilibrium capital stock is not inefficiently high) because of the GE effect: a lower capital stock shifts away income from risky labor income to non-risky capital income, and for moderate income risk this effect dominates the future generations effect as parametrized by θ. Note that the bounds in the previous proposition can of course be directly be defined in terms of the variance of the idiosyncratic income shock η, to a second order approximation of the integral defining Γ for general distributions, and without any approximation necessary in case the distribution of η is log-normal.

4.4 Implications for Dynamics of the Capital Stock and Capital *Income* Taxes

The discussion in the previous section concerned the optimal, time-invariant savings rate. The savings rate, together with the law of motion for the capital stock

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^{\alpha} = \frac{\alpha(\theta+\beta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}k_t^{\alpha}$$

and the initial condition k_0 determine the entire time path for the capital stock. That sequence $\{k_t\}_{t=1}^{\infty}$ is independent of the amount of income risk and converges monotonically to the steady state

$$k^* = \left[\frac{\alpha(\theta+\beta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}\right]^{\frac{1}{1-\alpha}},$$

either from above if $k_0 > k^*$ or from below, if $k_0 < k^*$. Again, of course the optimal tax policy that implements this allocation does depend on the extent of income risk, as shown above.

With this sharp analytical characterization of the optimal sequence of capital, we can now also make precise the relation between the capital taxes τ_t studied thus far, and the implied optimal capital *income* taxes τ_t^k. These are related by the equation

$$1 + (R_t - 1)(1 - \tau_t^k) = R_t(1 - \tau_t)$$

and thus

$$\tau_t^k = \frac{R_t}{R_t - 1}\tau_t$$

where the gross return is given by

$$R_t = \alpha\,(k_t)^{\alpha - 1}$$

As long as $R_t > 1$ for all t, capital taxes and capital income taxes have the same sign, and the two instruments are fully equivalent. A sufficient condition for this to be the case is

Assumption 4. *The initial capital stock satisfies*

$$k_0 < \alpha^{\frac{1}{1-\alpha}}$$

and the model parameters satisfy

$$\frac{1 + \alpha\beta}{(\theta + \beta)(1 - \kappa)(1 - \alpha)} > 1$$

This assumption assures that net returns are strictly positive at all times in the Ramsey equilibrium, since $R_0 = \alpha\,(k_0)^{\alpha-1} > 1$ and $R^* = \alpha\,(k^*)^{\alpha-1} > 1$, (and because the sequence of R_t along the transition is monotone) and thus the Ramsey allocation can be supported by capital income taxes of the same sign as the original wealth taxes. Under assumption 4 therefore all interpretations and qualitative results extend without change to capital income taxes. If instead assumption 4 is not satisfied, and (ignoring the knife edge case $R_t = 1$) thus for some t we have $R_t < 1$ as part of the Ramsey allocation, then, since capital income is negative (the net return $R_t - 1 < 0$), the capital tax τ_t and associated capital income tax τ_t^k are of opposite signs.

5 Efficiency Properties of the Ramsey Equilibrium

In this section we discuss the welfare properties of the Ramsey equilibrium characterized thus far. By construction, the Ramsey allocation is the best allocation, given the weights in the social welfare function, that a government that needs to respect equilibrium behavior of households and is restricted to proportional taxes on capital can implement. In this section we establish three main results.

First, the Ramsey equilibrium is generically not Pareto efficient, even if idiosyncratic income risk is absent. Second, defining constrained efficient allocations as those chosen by a social planner that cannot directly transfer consumption across households of different ages and with different idiosyncratic shocks (as in Davila et al., 2012), we show that the Ramsey equilibrium is constrained efficient in this precise sense. And third, we prove that if the optimal Ramsey savings rate $s^*(\theta = 1)$ that maximizes steady state welfare is smaller than $s_0(\eta)$, the steady state savings rate in the competitive equilibrium without government, then implementing $s^*(\theta = 1)$ through positive capital taxes yields a Pareto-improving transition from the initial steady state equilibrium without government policy towards the steady state associated with $s^*(\theta = 1)$. This is true even if $s_0(\eta) < s^{GR}$ and thus the steady state equilibrium capital stock is smaller than the golden rule capital stock.

5.1 Ramsey Equilibria and Pareto Efficient Allocations

In section C.1 of the Appendix we fully characterize the set of Pareto efficient allocations, restricting attention to those associated with welfare weights satisfying $\frac{\omega_{t+1}}{\omega_t} = \theta \leq 1$.[8] The next proposition summarizes the results:

Proposition 5. *For a given social discount factor $\theta \in [0, 1]$ and a given initial condition k_0, the Pareto efficient allocation $\{c_t^y, c_t^o(\eta_t), s_t, k_{t+1}\}_{t=0}^{\infty}$ is characterized by a constant savings rate*

$$s^{SP} := \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^{\alpha}} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}$$

and associated sequence of capital stocks

$$k_{t+1} = \alpha\theta k_t^{\alpha}$$

and consumption levels

$$\begin{aligned} c_t^y &= \frac{\theta(1-\alpha\theta)}{\theta+\beta}k_t^{\alpha} \\ c_t^o(\eta_t) &= \frac{\beta(1-\alpha\theta)}{\theta+\beta}k_t^{\alpha} \end{aligned}$$

If $\theta = 1$, then the optimal saving rate

$$s^{SP} = \frac{\alpha}{(1-\kappa)(1-\alpha)} := s^{GR} \tag{30}$$

implements the golden rule capital stock

$$k^{GR} = \lim_{t\to\infty} k_t = (\alpha)^{\frac{1}{1-\alpha}} \tag{31}$$

that maximizes steady state aggregate consumption.

[8] It is not difficult to relax this assumption, but since the objective here is a comparison to Ramsey equilibria associated with the same set of social welfare weights, we impose this restriction to simplify the exposition.

Comparing the characterization of Pareto efficient allocations with the results in the previous section on the optimal Ramsey allocation we immediately have the following obvious

Corollary 6. *The Ramsey equilibrium is not Pareto efficient because it does not provide full consumption insurance against idiosyncratic income risk.*

Of course this result is fully expected and not noteworthy at all, since the Ramsey government has no powers to affect or offset the market incompleteness inherent in our model. What is more remarkable is that even though the optimal Ramsey savings rate is independent of income risk (and the same as in a model where income risk is absent), it is in general different from the savings rate optimally chosen by the social planner (who fully insures the idiosyncratic income risk). This result is summarized in the next

Corollary 7. *For a fixed social discount factor $\theta \in [0,1]$, the optimal Ramsey savings rate equals to the saving rate chosen by the social planner if and only if the following knife edge condition is satisfied:*

$$(1-\kappa) = \frac{\theta(1+\alpha\beta)}{(1-\alpha)(\beta+\theta)}$$

Note that the Ramsey government can surely implement the saving rate desired by the social planner through an appropriate choice of taxes, but unless the condition above is satisfied, it is suboptimal to do so. The reason is that the Ramsey government has no instruments to transfer resources across generations and thus forcing the planner saving rate onto households (by appropriate choice of the capital tax rate) results in an equilibrium allocation of consumption across the young and the old that is typically suboptimal.[9]

5.2 Constrained Efficiency of Ramsey Equilibria

Can the Ramsey government implement constrained efficient allocations with the set of instruments it has? A constrained efficient allocation is an allocation of capital and consumption that maximizes social welfare subject to the constraint that the allocation does not permit transfers across currently old households with different η realizations. Define the set of allocation that are feasible for the *constrained* planner

$$c_t^y + \int c_t^o(\eta_t)d\Psi + k_{t+1} = k_t^\alpha \tag{32}$$

$$c_t^o(\eta_t) = k_t MPK(k_t) + \kappa\eta_t MPL(k_t) \tag{33}$$

[9]Finally note that if one were to treat the social discount factor θ as a *free parameter*, then one concludes that the Ramsey optimal savings rate is efficient, in that it is identical to the choice of the social planner with a *different social discount rate* θ^{SP}

$$\theta^{SP} = \frac{(\beta+\theta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}.$$

The first constraint is simply the resource constraint. The second constraint has extra bite as it restricts transfers across different η households: old age consumption is required to equal capital income plus an η household's share of labor income, where the returns to capital and labor are equal to the factors' relative productivities. The constrained planner might find it optimal, however, to manipulate factor prices by choosing a different sequence of capital stocks, relative to that of a competitive equilibrium (without or with tax policy).

Note that these constraints also imply that

$$\begin{aligned}\int c_t^o(\eta_t)d\Psi &= k_t MPK(k_t) + \kappa MPL(k_t) \\ c_t^y &= (1-\kappa)MPL(k_t) - k_{t+1}\end{aligned}$$

so that no *intergenerational* transfers are permitted either, relative to the competitive equilibrium.

A constrained efficient allocation is one that maximizes

$$SWF = \sum_{t=-1}^{\infty} \omega_t V_t$$

subject to (32) and (33).

The social planner may want to manipulate the capital stock so as to change relative factor prices, relative to the competitive equilibrium without taxes. The question is whether the simple tax policy we consider here is sufficient to offset the pecuniary externality and implement the constrained efficient allocation as defined above. The answer is yes, as the following proposition (proved in appendix C.2) shows.

Proposition 6. *The Ramsey equilibrium, for a given set of social welfare weights, implements the constrained-efficient allocation for exactly that set of social welfare weights.*

5.3 Pareto-Improving Tax Transitions

In this section we show that under certain condition, starting from the *steady state competitive equilibrium without taxes* as initial condition, switching to the Ramsey optimal savings and tax policy that maximizes *steady state welfare* yields a Pareto improvement, that is, *all generations*, including those along the transition, are better off. This is true, again under certain parametric restrictions, even if the original competitive steady state equilibrium is dynamically efficient in the sense of satisfying $k_0 < k^{GR}$, (and thus $R_0 > 1$) where k^{GR} is the golden rule capital stock characterized above.

Proposition 7. *Let $s_0(\eta)$ denote the savings rate in a steady state competitive equilibrium with zero taxes. Assume that $s_0(\eta) > s^*$. Then a government policy that sets $\tau_t = \tau^* > 0$ leads to a Pareto improving transition from the initial steady state with capital $k_0(\eta)$ towards the new steady state associated with tax policy τ^*.*

We provide the proof of this proposition in appendix C.4. The proof shows that *all generations* benefit from the government implementing a saving rate that is lower than the initial competitive equilibrium rate despite the fact that it lowers the capital stock, thus aggregate production, wages and consumption along the transition. The key step is to argue that this adverse effect of a lower capital stock is most severe in the long run (the new steady state), and to show that by choice of s^* the government insures that even generations in the new steady state benefit, in terms of lifetime utility, from the higher tax rate and associated lower saving rate. Note that this argument is independent of the specific form of the utility function and thus the result holds for arbitrary CRRA utility, although the conditions on fundamentals that guarantee that the equilibrium savings rate $s_0(\eta)$ exceeds the optimal Ramsey steady state savings rate s^* evidently will depend on the specific form of the utility function.

Note that from proposition 4 the assumption $s_0(\eta) > s^*$ is satisfied if and only if income risk is sufficiently large, in the sense that $\Gamma > \frac{1+\beta}{(1-\alpha)\beta}$. The result in the previous proposition is of course not surprising if $s_0(\eta) \geq s^{GR}$ and the initial steady state competitive equilibrium is dynamically inefficient to start with. However, for intermediate risk, i.e. for

$$\Gamma \in \left(\frac{1+\beta}{(1-\alpha)\,\beta}, \frac{1}{\left[(1-\alpha) - 1/\bar{\Gamma}\right]\beta} \right)$$

the same proposition shows that $s^* < s_0(\eta) < s^{GR}$, and thus the steady state equilibrium is dynamically efficient yet setting $\tau^* > 0$ implements a Pareto-improving transition.

Finally, it is important to note that the converse of proposition 7 is not true: even if $s_0(\eta) < s^*$, implementing the Ramsey optimal (for $\theta = 1$) savings subsidy $\tau^* < 0$ and associated higher saving rate s^* does *not* lead to a Pareto improving transition. We demonstrate this in Appendix C.5 by showing that the generation born into the first period of this hypothetical policy-induced transition will lose from this policy innovation. In fact, not only is implementing $\tau^* < 0$ not Pareto improving if $s_0(\eta) < s^*$, any policy reform that induces a savings rate in period 1 above the competitive savings rate with zero taxes, $s_0(\eta)$, will not result in a Pareto improvement (since it will make the first generation strictly worse off).

6 General Intertemporal Elasticity of Substitution ρ and Risk Aversion σ

In this section we generalize our *steady state results* to a more general utility function with intertemporal elasticity of substitution ρ and risk aversion σ. The main text contains the essential results, and the details of the derivations are relegated to Appendix D.

We now consider a utility function of the form

$$V_t = \frac{(c_t^y)^{1-\frac{1}{\rho}} - 1}{1 - \frac{1}{\rho}} + \beta \frac{\left\{ \left[\int c_{t+1}^o (\eta_{t+1})^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}} \right\}^{1-\frac{1}{\rho}} - 1}{1 - \frac{1}{\rho}}$$

The parameter ρ measures the IES and the parameter σ governs risk aversion.[10]

If $\sigma = \frac{1}{\rho}$ then the utility function takes the standard CRRA form, and as the IES $\rho \to 1$, the utility function becomes

$$V_t = \ln(c_t^y) + \frac{\beta}{1-\sigma} \ln \left(\int c_{t+1}^o (\eta_{t+1})^{1-\sigma} d\Psi \right)$$

As in section 4, equation 16 we can write lifetime utility of a generation born in period t, in general equilibrium, as a function of the beginning of the period capital stock k_t and the saving rate s_t chosen by the Ramsey government and implemented by the appropriate choice of the capital tax τ_{t+1}. In addition, in the steady state the savings rate and the associated capital stock are related by:

$$k = ((1-\kappa)(1-\alpha)s)^{\frac{1}{1-\alpha}} \tag{34}$$

In Appendix D we show that the objective function of the Ramsey government is to maximize, by choice of the steady state saving rate, steady state lifetime utility, which is given (for $\rho \neq 1$) by

$$V(s) = \tilde{\phi} \left((1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta \tilde{\psi} \tilde{\Gamma}_2 \right) s^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \tag{35}$$

where $\tilde{\phi}$ and $\tilde{\psi} > 0$ and $\tilde{\Gamma} > 0$ are constants that depend on parameter values. Taking first order conditions and rearranging we find that optimal steady state savings rate is defined implicitly as

$$s = \frac{\alpha}{1-\alpha} \left[(1-s) + \beta \tilde{\psi} \tilde{\Gamma}_2 (1-s)^{\frac{1}{\rho}} \right] \tag{36}$$

---

[10] Note that V_t is ordinally equivalent, i.e. represents the same preference ordering over consumption c_t^y when young and the certainty equivalent over utility tomorrow, $\left[\int c_{t+1}^o (\eta_{t+1})^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}}$ as the more commonly used specification

$$\tilde{V}_t = \left\{ (1-\tilde{\beta})(c_t^y)^{1-\frac{1}{\rho}} + \tilde{\beta} \left[\int c_{t+1}^o (\eta_{t+1})^{1-\sigma} d\Psi \right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}} \right\}^{\frac{1}{1-\frac{1}{\rho}}}$$

since one is a monotone transformation of the other:

$$V_t = \frac{\tilde{V}_t^{1-\frac{1}{\rho}}}{(1-\tilde{\beta})\left(1-\frac{1}{\rho}\right)} - \frac{(1+\beta)}{1-\frac{1}{\rho}}$$

where $\beta = \frac{\tilde{\beta}}{1-\tilde{\beta}}$. We should note, however, since the Ramsey problem is stated in terms of the weighted sum of cardinal utilities, a monotone transformation of the utility function will in general alter the Ramsey problem. When focusing on a steady state analysis, this concern does not arise, however, since the same saving rate (and associated tax rate) maximizes steady state V and its monotone transformation $\tilde{V}$.

Inspection of equation 36 (with the formal argument presented in Appendix D) we immediately obtain

Proposition 8. *Suppose $\theta = 1$ and thus the Ramsey government maximizes steady state welfare. There exists a unique optimal Ramsey saving rate $s^\star \in (0,1)$ solving equation 36. This savings rate can be implemented with a capital tax rate $\tau^\star$ determined by the competitive equilibrium Euler equation:*

$$1 = (1-\tau^\star)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s^\star)^{\frac{1}{\rho}}}{s^\star}\tilde{\Gamma} \tag{37}$$

Note that all comparative statics results, especially those with respect to an increase in income risk, can be deduced from an analysis of equations $(36, 37)$. Income risk affects the optimal Ramsey savings rate $s^\star$ and associated implementing tax rate $\tau^\star$ only through the constants $\tilde{\Gamma}, \tilde{\Gamma}_2$ which are given as:

$$\tilde{\Gamma} = ce(\eta)^{\left(\sigma-\frac{1}{\rho}\right)}\Gamma \tag{38}$$

$$\tilde{\Gamma}_2 = ce(\eta)^{\left(1-\frac{1}{\rho}\right)} \tag{39}$$

where we had defined Γ above for the log-case, and is now given by:

$$\Gamma = \int \left(\kappa\eta_{t+1}(1-\alpha)+\alpha\right)^{-\sigma} d\Psi(\eta_{t+1}) \tag{40}$$

and where the certainty equivalent of η is defined as, for $\sigma \neq 1$

$$ce(\eta) = \left[\int \left(\alpha+(1-\alpha)\kappa\eta\right)^{1-\sigma} d\Psi(\eta)\right]^{\frac{1}{1-\sigma}} \tag{41}$$

and for $\sigma = 1$

$$ce(\eta) = \exp\left(\int \ln\left(\alpha+(1-\alpha)\kappa\eta\right) d\Psi(\eta)\right). \tag{42}$$

In appendix F we prove the following result that relates income risk and the constants $\tilde{\Gamma}, \tilde{\Gamma}_2$ which are in turn crucial for the comparative statics results we will provide in section 6.2.

Lemma 1. *An increase in income risk (again in the sense of a mean-preserving spread of η), unambiguously reduces $ce(\eta)$, increases $\tilde{\Gamma}_2$ if and only if $\rho < 1$ and increases $\tilde{\Gamma}$ if $\rho < 1$ or $\rho > 1$ and $\sigma < 1/\rho$.*

Note that the condition that characterizes $\tilde{\Gamma}_2$ is necessary and sufficient whereas the two alternative conditions that characterize the relation between income risk and $\tilde{\Gamma}$ are only sufficient.

6.1 Unit Elasticity of Substitution $\rho = 1$

Recognizing that for an IES of $\rho = 1$ we have $\tilde{\psi} = \tilde{\Gamma}_2 = 1$ direct calculations reveal

Proposition 9. *Suppose that the IES $\rho = 1$. Then the solution of the Ramsey problem is identical to that of the log-utility case analyzed in section 5. That is, the optimal, constant saving rate is given by*

$$s = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta}$$

The optimal tax rate τ that implements this saving rate as a competitive equilibrium is given by

$$1 = (1 - \tau)\left(\frac{1 - s}{s}\right)\alpha\beta\tilde{\Gamma}$$

and thus is strictly increasing in income risk measured by $\tilde{\Gamma}$.

Note that the optimal Ramsey savings rate does neither depend on income risk nor risk aversion, but that the optimal capital tax rate τ implementing this savings rate is increasing in income risk, and *does* depend on risk aversion through the constant $\tilde{\Gamma} = \frac{\int(\kappa\eta(1-\alpha)+\alpha)^{-\sigma} d\Psi(\eta)}{\int(\kappa(1-\alpha)\eta+\alpha)^{1-\sigma} d\Psi(\eta)}$ since σ controls the degree of precautionary saving in the competitive equilibrium that needs to be offset with capital taxes.

Also note that although here we state this result for steady states only, Appendix D.2 shows that the entire analysis of section 5 with log-utility (including the dynamic programming formulation and the analysis of the transition path) goes through completely unchanged (and only replacing Γ by $\tilde{\Gamma}$) for general Epstein-Zin utility as long as the IES $\rho = 1$.

6.2 The Impact of Risk on the Optimal Saving and Tax Rate: Disentangling Risk Aversion and IES: $\frac{1}{\sigma} \neq \rho \neq 1$

In the previous section we demonstrated that an intertemporal elasticity of substitution of 1 was sufficient (and, as will turn out, necessary) for the result that the optimal Ramsey saving rate can be solved in closed form, is constant over time and independent of the extent of income risk in the economy. In this section we investigate how income risk impacts the optimal Ramsey savings rate and implementing capital tax rate when we allow for general IES and risk aversion (ρ, σ) where the standard CRRA case is nested as a special case $\rho = 1/\sigma$.

From equation 36 we immediately observe that the optimal steady state saving rate s is strictly increasing in the constant $\tilde{\Gamma}_2$ that fully summarizes the impact of income risk. The response of s to income risk then immediately follows from the impact of an increase in income risk on $\tilde{\Gamma}_2$ stated in Lemma 1. Thus we have

Proposition 10. *An increase in income risk, that is, a mean-preserving spread in the distribution of η, increases the optimal steady state Ramsey saving rate s if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$.*

Thus the direction of the change in s with respect to income risk is exclusively determined by the intertemporal elasticity of substitution ρ, with the log-case emphasized thus far acting as natural watershed. Of course how strongly the savings rate responds to an increase in income risk is also controlled by risk aversion through the term $\tilde{\Gamma}_2$.

What is the intuition for this result? Consider the following thought experiment. Suppose the economy is in the steady state associated with a given extent of income risk and the optimal Ramsey tax policy, and now consider an increase in income risk. Private households will adjust their savings behavior, but the Ramsey government can neutralize this behavior by appropriate adjustment of the tax rate on capital to implement the new desired (by the Ramsey government) savings rate.[11]

The question is then how the saving rate desired by the Ramsey government itself changes. Households (and thus the Ramsey government) obtain utility from safe consumption when young and risky consumption when old, and the desire for smoothness between safe consumption when young and the certainty equivalent of consumption when old is determined by the IES ρ. As risk increases, the certainty equivalent of old-age consumption declines, for a given consumption allocation. In effect, old age consumption is now a less effective (because more risky) way to generate utils, and whether the Ramsey government wants to prop up old-age consumption (by increasing the saving rate) or reduce it (by lowering the saving rate) depends on how much households value smoothness between consumption when young and the certainty equivalent of consumption when old. In the log-case the two forces exactly balance out and the Ramsey saving rate does not respond to income risk at all. In contrast, if households strongly desire a smooth path of (the certainty equivalence of) consumption, then the Ramsey government compensates for the loss of old-age certainty equivalent consumption by saving at a higher rate. Thus s increases with income risk if the IES ρ is small. The reverse is true for a high IES.

Finally, we can also determine the impact of income risk on optimal steady state capital taxes. We can rewrite equation (37) characterizing the optimal steady state tax rate as

$$1 = (1 - \tau^*)\nu \frac{(1 - s^*)^{\frac{1}{\rho}}}{s^*} \tilde{\Gamma} \tag{43}$$

where $\nu > 0$ is a constant. From equation (43) we see that income risk affects the Ramsey optimal tax rate in two ways. First, for a given target saving

[11] We saw this explicitly in the decomposition of the first order condition of the Ramsey government in section 4.2, where the risk term Γ from the competitive equilibrium optimality condition dropped out because the government chooses, through taxes and the associated changes in factor prices, to exactly offset the impact of higher risk on private household savings decision. In the logic of that section, an increase in Γ increases $PE(s)$ but reduces $GE(s)$ by the same factor.

rate s^* to implement, the direct impact of income risk depends on $\tilde{\Gamma}$. Second, a change in income risk changes the optimal Ramsey tax rate s^* through $\tilde{\Gamma}_2$, as characterized in the previous proposition. The next proposition, proved in section D.4 of the appendix, gives conditions on the IES and risk aversion (ρ, σ) under which the optimal capital tax rate is increasing in income risk, and an example under which the it is strictly decreasing in income risk.

Proposition 11. *If $\rho \leq 1$, then an increase in income risk increases the optimal tax rate on capital. Similarly, if $\rho > 1$ and $\sigma \leq 1/\rho$, then an increase in income risk increases the optimal tax rate on capital.*

Proposition 12. *If $\rho > 1$ and $\sigma > 1/\rho$ then an increase in income risk might lead to a strict reduction in the optimal tax rate on capital, but only if the private saving rate in competitive equilibrium for given tax rate $\tau \in (-\infty, 1)$ is strictly decreasing in income risk. Specifically, if $\rho \to \infty$ and $\sigma \to \infty$, then the optimal tax rate decreases with an increase in income risk.*

6.3 Numerical Exploration of Optimal Ramsey Tax Transitions for General IES $\rho \neq 1$ and Risk Aversion σ

In the previous section we provided a theoretical characterization of the optimal Ramsey savings rate under the assumption that the government maximized steady state utility, i.e. $\theta = 1$. Since no analytical results are available *outside the steady state* (unless $\sigma = \rho = 1$) in this section we show numerically how the optimal savings rate depends on the capital stock and the degree of income risk, and the optimal savings and capital tax rate evolves over time in this case. Our main focus is on the dynamics of the optimal capital tax, and how that dynamics depends on the magnitude of income risk and households' attitudes towards that risk (as again measured by σ).

Therefore now consider a CRRA utility function with a coefficient of relative risk aversion of $\sigma \in \{2, 0.25\}$, i.e., an $IES \in \{0.5, 4\}$. Furthermore, we choose $\alpha = 0.2$, $\beta = 0.7$, $\kappa = 0.3$. We assume that η is lognormally distributed with $\sigma_{\ln \eta} \in \{0.0, 0.75, 1.5\}$, and we refer to these alternative calibrations of risk as "no risk", "medium risk" and "high risk" economies, respectively.[12] The government discount factor is set to equal the private discount factor at $\theta = \beta = 0.7$.

For these parameterizations of the model, we analyze policy functions as well as transitional dynamics of the economy from an initial steady state to the final steady state.[13] To compute the initial allocation in the respective economy, we characterize an initial steady state with zero capital income taxes. Transitional dynamics in this economy are then induced by a Ramsey planner

[12] We approximate the distribution by Gaussian quadrature methods choosing $n = 11$ integration nodes.

[13] We solve for the dynamic problem by first computing the Ramsey planner's optimal long-run steady state saving rate (i.e., $s^\star(\theta = 1)$). We then construct a capital grid where the lowest grid point is at $k_0 = 0.01$ and the highest grid point is at 4 times the value of the respective steady state capital stock. Finally, we solve the dynamic program using first-order methods, see the Appendix of the paper for details.

that optimally sets saving rates along the transition of the economy (starting with implementing the optimal policy in period 0, taking as given the initial allocation).

Saving rates in the long-run optimum for $\theta = 1$, $s^\star(\theta = 1)$, the competitive equilibrium saving rate for a given risk calibration, $s_0(\eta)$, and the golden rule saving rate s^{GR} for the 3x2 scenarios of income risk and risk aversion are summarized in Table 1. From this table, we make three observations. In line with our theoretical analysis, first, the competitive equilibrium saving rate is strictly increasing in risk and second, the optimal Ramsey long-run saving rate is increasing in risk for high risk aversion (dominance of the precautionary savings effect) and decreasing in low risk aversion (dominance of the pecuniary externality effect). Third, the competitive equilibrium is dynamically efficient in all cases but for the high risk aversion / high risk calibration with $\sigma = 2$ and $\sigma_{\ln\eta} = 1.5$.

Table 1: Saving Rates: Long-Run Optimum, Initial CE, Golden Rule

$\sigma = 2$	$s^\star(\theta = 1)$	$s_0(\eta)$	s_{GR}
no risk	0.2899	0.2362	0.3571
med. risk	0.3002	0.2936	0.3571
high risk	0.3199	0.3787	0.3571
$\sigma = 0.25$	$s^\star(\theta = 1)$	$s_0(\eta)$	s_{GR}
no risk	0.3066	0.2473	0.3571
med. risk	0.305	0.2528	0.3571
high risk	0.3001	0.2647	0.3571

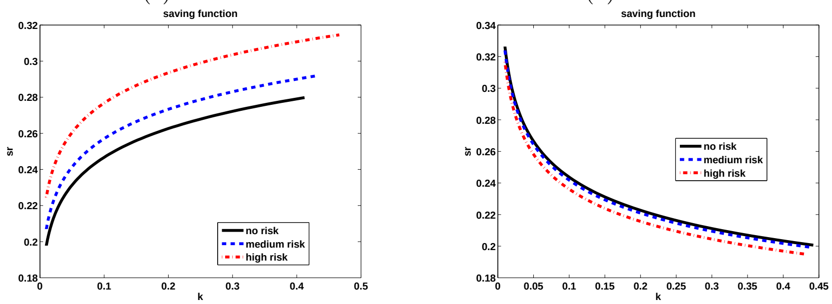
Notes: Simulated saving rates in long-run optimum, $s^\star(\theta = 1)$, competitive equilibrium, $s_0(\eta)$ and golden rule s^{GR} for $\alpha = 0.25$, $\beta = 0.7$, $\kappa = 0.3$, $\sigma \in \{0.25, 2\}$, and $\sigma_{\ln\eta} \in \{0.0, 0.75, 1.5\}$.

Figure 2 shows the policy function for the saving rate for the high risk aversion economy in Panel (a) and the low risk aversion economy in Panel (b). These policy functions reflect the motives we characterized in our previous theoretical analysis, see Theorem 1: for all capital stocks the saving rate is increasing in risk for $\sigma = 2$ and decreasing in risk for $\sigma = 0.25$. Furthermore, the policy functions are upward (downward) sloping in k for $\sigma = 2$ ($\sigma = 0.25$) because with increasing k wages increase leading to higher earnings risk and therefore a higher (lower) saving rate because the precautionary savings effect (the pecuniary externality effect) dominates.

Turning to the transitional dynamics induced by the Ramsey government that optimizes the social welfare function taking as given the initial allocation as of period 0, Figures 3, 4 and 5 show the transitional dynamics of the saving rate s_t, the capital stock k_t, and the capital tax rate τ_t, respectively. Along the entire transition, the same ranking can be observed as for the policy functions shown in Figure 2: the saving rate and the capital stock increase (decrease) in risk for the high (low) risk aversion calibration. For $\sigma = 2$ we further observe

Figure 2: Policy Function: Saving Rate $s(k)$

(a) $\sigma = 2$ (b) $\sigma = 0.25$



Notes: Simulated saving policy function for $\theta = 0.7$ and $\alpha = 0.25$, $\beta = 0.7$, $\kappa = 0.3$, $\sigma \in \{0.25, 2\}$, and $\sigma_{\ln \eta} \in \{0.0, 0.75, 1.5\}$.

that the government finds it optimal to decrease the capital stock along the transition under medium and high risk. The case with medium risk is particularly noteworthy because, as documented in Table 1, the economy is initially dynamically efficient. The reduction of the saving rate (and thereby the capital stock) is welfare improving because it reduces the risk exposure to households.

For $\sigma = 0.25$, the dynamics of the saving rate are non-monotone. Starting from an initial allocation with saving rates of $0.24, 0.25, 0.26$ for no, medium and high risk, respectively, see Table 1, the Ramsey planner finds it optimal to implement in period 0 saving rates of about $0.25, 0.24, 0.23$ in period 0 leading to an increase of the capital in the no risk calibration and a decrease of the capital stock in the medium and high risk calibrations. Thereafter the saving rates revert slightly so that they still remain above (below) in the no risk (in the medium/high risk) calibrations compared to the initial period 0 allocation. As for $\sigma = 2$ the capital stock therefore decreases along the transition in the medium and high risk economies despite the fact that these economies are dynamically efficient.

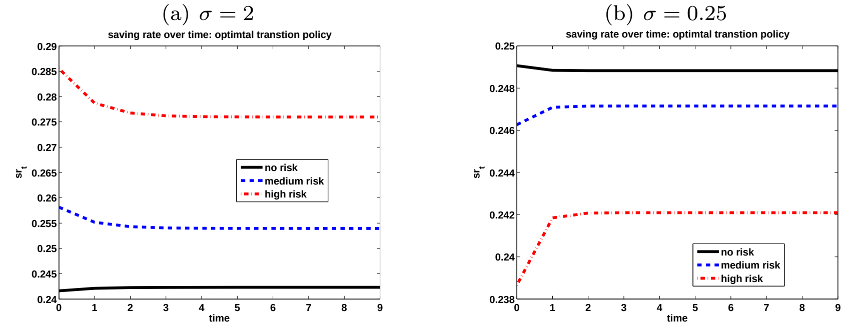
The capital taxes required to implement these allocations, shown in Figure 5, are time varying (which is hard to see from the figures as changes over time show up in the third digit after the comma) and generally increasing in risk (in fact, in the no risk economies, the Ramsey planner achieves the increase of the capital stock through a savings subsidy).

7 Conclusion

In this paper we have analyzed optimal capital taxes in an OLG model with idiosyncratic labor income risk. The problem is fully analytically tractable and we obtain a complete characterization of the Ramsey allocation and associated tax policy along the transition to a steady state in case the intertemporal elas-

Figure 3: Saving Rate $s(t)$ in Transition

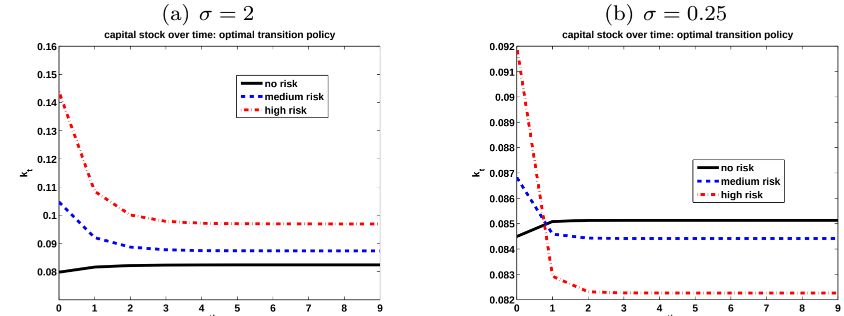
(a) $\sigma = 2$ (b) $\sigma = 0.25$



Notes: Simulated saving rate in the transition for $\theta = 0.7$ and $\alpha = 0.25$, $\beta = 0.7$, $\kappa = 0.3$, $\sigma \in \{0.25, 2\}$, and $\sigma_{\ln\eta} \in \{0.0, 0.75, 1.5\}$.

Figure 4: Capital Stock $k(t)$ in Transition

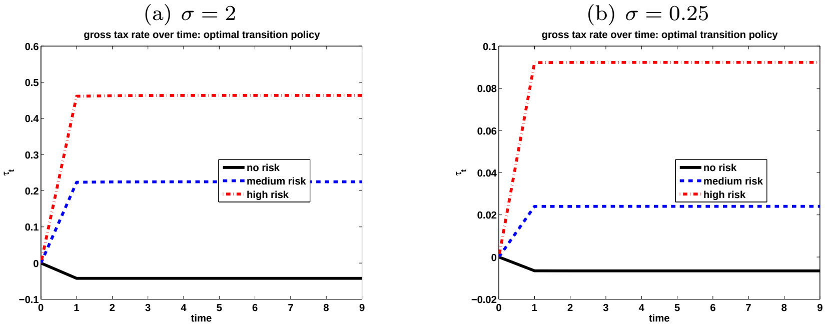
(a) $\sigma = 2$ (b) $\sigma = 0.25$



Notes: Simulated capital stock in the transition for $\theta = 0.7$ and $\alpha = 0.25$, $\beta = 0.7$, $\kappa = 0.3$, $\sigma \in \{0.25, 2\}$, and $\sigma_{\ln\eta} \in \{0.0, 0.75, 1.5\}$.

Figure 5: Capital Tax Rate $\tau(t)$: Optimal Ramsey Transition Policy ($\theta = 0.7$)

(a) $\sigma = 2$ (b) $\sigma = 0.25$



Notes: Simulated capital tax rate in the transition for $\theta = 0.7$ and $\alpha = 0.25$, $\beta = 0.7$, $\kappa = 0.3$, $\sigma \in \{0.25, 2\}$, and $\sigma_{\ln \eta} \in \{0.0, 0.75, 1.5\}$.

ticity of substitution is unity. The optimal aggregate saving rate is independent of idiosyncratic income risk, and is implemented by a tax rate that is increasing in income risk, and positive if and only if income risk is sufficiently large.

Our paper confirms that capital taxation is the appropriate tool of the government to deal with the precautionary externality induced by private precautionary saving behavior of households that are subject to uninsurable idiosyncratic income risk. However, we also demonstrate that capital should not necessarily be taxed, and it might well be subsidized, especially when the government cares strongly about future generations. By making these points fully analytically our paper is complementary to the literature characterizing optimal taxation numerically in large-scale life cycle economies.

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A Details of the General Ramsey Problem

From equations (24) and (24) we find that

$$w'(s) = (1-\alpha)\alpha\,[k'(s)]^{\alpha-1}\,\frac{dk'(s)}{ds} = (1-\alpha)\alpha\,[(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha}\,[s]^{\alpha-1} \tag{44}$$

$$R'(s) = \alpha(\alpha-1)\,[k'(s)]^{\alpha-2}\,\frac{dk'(s)}{ds} = \alpha(\alpha-1)\,[(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha-1}\,[s]^{\alpha-2} \tag{45}$$

and thus

$$(1-\kappa)(1-\alpha)k^{\alpha}R'(s)s = \alpha(\alpha-1)\,[(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha}\,[s]^{\alpha-1} \tag{46}$$

$$\kappa\eta w'(s) + (1-\kappa)(1-\alpha)k^{\alpha}R'(s)s = (1-\alpha)\alpha\,[(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha}\,[s]^{\alpha-1}\,[\kappa\eta-1] \tag{47}$$

which leads to the equation in the main text:

$$GE(s) = (1-\alpha)\alpha\,[(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha}\,[s]^{\alpha-1}\,\beta\int u'\left(c^{o}(\eta)\right)[\kappa\eta-1]\,d\Psi(\eta)$$

B Derivation for Log-Utility

In this section we provide a full solution to the Ramsey optimal taxation problem for the case of logarithmic utility in its sequential formulation, for an arbitrary set of social welfare weights. We first recognize from the aggregate law of motion that

$$\begin{aligned}
\log(k_{t+1}) &= \log(1-\alpha) + \log(1-\kappa) + \alpha\log(k_t) + \log(s_t) \\
&= \varkappa + \log(s_t) + \alpha\left[\alpha\log(k_{t-1}) + \log(s_{t-1})\right] \\
&= \varkappa + \sum_{\tau=0}^{t}\alpha^{\tau}\log(s_{t-\tau}) + \alpha^{t+1}\log(k_0) \\
&= \varkappa_{t+1} + \sum_{\tau=0}^{t}\alpha^{\tau}\log(s_{t-\tau})
\end{aligned}$$

or

$$\log(k_t) = \varkappa_t + \sum_{\tau=0}^{t-1}\alpha^{\tau}\log(s_{t-1-\tau}) = \varkappa_t + \sum_{\tau=1}^{t}\alpha^{\tau-1}\log(s_{t-\tau})$$

Therefore the objective of the Ramsey is given by (suppressing maximization-irrelevant constants)

$$\begin{aligned}\sum_{t=0}^{\infty}\omega_t V(k_t,s_t) &= \sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\alpha\beta\log(s_t)+\alpha\left(1+\alpha\beta\right)\log(k_t)\right]\\ &= \chi+\sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\alpha\beta\log(s_t)+\alpha\left(1+\alpha\beta\right)\sum_{\tau=1}^{\infty}\alpha^{\tau-1}\log(s_{t-\tau})\right]\\ &= \chi+\sum_{t=0}^{\infty}\left[\omega_t\log(1-s_t)+\log(s_t)\left(\alpha\beta\omega_t+\alpha\left(1+\alpha\beta\right)\sum_{\tau=t+1}^{\infty}\omega_\tau\alpha^{\tau-(t+1)}\right)\right]\end{aligned}$$

and thus the social welfare function can be expressed purely in terms of savings rates

$$SWF(\{s_t\}_{t=0}^{\infty})=\chi+\sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\log(s_t)\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\right]$$

where χ is a constant that depends positively on the initial capital stock k_0, but is again irrelevant for maximization.

Taking first order conditions with respect to s_t and setting it to zero delivers the optimal savings rate

$$s_t=\frac{1}{1+\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}$$

C Characterization of (Unconstrained) Efficient Allocations

In this section we study whether the Ramsey government implement Pareto efficient allocations. The set of obvious answer is no, since an unconstrained social planner would provide full insurance against idiosyncratic η shocks, which, given the market structure, is ruled out in any competitive equilibrium.

C.1 Characterization of Pareto Efficient Allocations

In this section we derive the solution to the unconstrained social planner problem. The planner maximizes social welfare

$$\omega_{-1}\int\log(c_0^o(\eta_0))d\Psi(\eta_0)+\sum_{t=0}^{\infty}\omega_t\left[\log(c_t^y)+\beta\int\log(c_{t+1}^o(\eta_{t+1}))d\Psi(\eta_{t+1})\right]$$

subject just to the sequence of resource constraints

$$c_t^y+\int c_t^o(\eta_t)d\Psi(\eta_t)+k_{t+1}=k_t^{\alpha}.$$

As before, we restrict attention to geometrically declining welfare weights such that $\omega_{t+1}/\omega_t = \theta \leq 1$.

Trivially, the social planner provides full insurance against idiosyncratic income risk so that $c_t^o(\eta) = c_t^o$ for all η and all t. Thus the problem simplifies to

$$\begin{aligned} & \max_{\{c_t^y, c_t^o, k_{t+1}\}} \omega_{-1} \int \log(c_0^o) + \sum_{t=0}^{\infty} \omega_t \left[\log(c_t^y) + \beta \log(c_{t+1}^o)\right] \\ & s.t. \\ c_t^y + c_t^o + k_{t+1} &= k_t^\alpha \end{aligned}$$

with $k_0 > 0$ given. The first order conditions are given by

$$\begin{aligned} \frac{\omega_t}{c_t^y} &= \lambda_t \\ \frac{\beta \omega_{t-1}}{c_t^o} &= \lambda_t \\ \lambda_t &= \lambda_{t+1} \alpha k_{t+1}^{\alpha-1} \\ c_t^y + c_t^o + k_{t+1} &= k_t^\alpha \end{aligned}$$

Thus the optimal allocation of consumption across the two generations at a given point of time is given by

$$\frac{c_t^o}{c_t^y} = \frac{\beta \omega_{t-1}}{\omega_t} = \frac{\beta}{\theta}$$

and thus from the resource constraint

$$\begin{aligned} c_t^y &= \frac{\theta}{\theta+\beta}\left(k_t^\alpha - k_{t+1}\right) \\ c_t^o &= \frac{\beta}{\theta+\beta}\left(k_t^\alpha - k_{t+1}\right) \end{aligned}$$

Define, analogously with the Ramsey problem, the savings rate of the social planner as

$$s_t = \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^\alpha}$$

or

$$(1-\kappa)(1-\alpha)s_t = \frac{k_{t+1}}{k_t^\alpha}$$

Then from the first order conditions

$$\begin{aligned} \frac{1}{c_t^y} &= \frac{\beta}{c_{t+1}^o} \alpha k_{t+1}^{\alpha-1} \\ \frac{k_{t+1}}{(k_t^\alpha - k_{t+1})} &= \frac{\alpha \theta k_{t+1}^\alpha}{\left(k_{t+1}^\alpha - k_{t+2}\right)} \\ (1-(1-\kappa)(1-\alpha)s_{t+1}) &= \alpha\theta \left(\frac{1}{(1-\kappa)(1-\alpha)s_t} - 1\right) \end{aligned}$$

As in the neoclassical growth model we can show that the only solution to the first order difference equation that does not eventually violate the non-negativity constraint of consumption and does not violate the TVC is the constant saving rate s solving

$$(1-(1-\kappa)(1-\alpha)s) = \alpha\theta\left(\frac{1}{(1-\kappa)(1-\alpha)s}-1\right)$$

Define $\tilde{s} = (1-\kappa)(1-\alpha)s$ then we have

$$1-\alpha\theta = \alpha\theta\left(\frac{1}{\alpha\theta}-1\right)$$

with solutions $\tilde{s} = 1$ and $\tilde{s} = \alpha\theta$ and thus

$$s^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}$$

the optimal sequence of capital stocks, starting from initial capital stock k_0, is given by

$$\begin{aligned} k_{t+1} &= (1-\kappa)(1-\alpha)s_t k_t^{\alpha} \\ &= \alpha\theta k_t^{\alpha} \end{aligned}$$

Since

$$k_t^{\alpha} - k_{t+1} = (1-\alpha\theta)k_t^{\alpha}$$

we immediately have

Proposition 13. *The solution to the social planner problem, any $k_0 > 0$, is given by a constant savings rate*

$$\frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^{\alpha}} = s^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}$$

and associated sequence of capital stocks

$$k_{t+1} = \alpha\theta k_t^{\alpha}$$

and consumption levels

$$\begin{aligned} c_t^y &= \frac{\theta(1-\alpha\theta)k_t^{\alpha}}{\theta+\beta} \\ c_t^o &= \frac{\beta(1-\alpha\theta)k_t^{\alpha}}{\theta+\beta} \end{aligned}$$

Corollary 8. *If $\theta = 1$ (associated with a steady state analysis), then the social planner chooses the golden rule savings rate*

$$s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}$$

and the capital stock converges, in the long run, to

$$k^{GR} = \alpha^{\frac{1}{1-\alpha}}$$

which satisfies

$$\alpha \left[k^{GR}\right]^{\alpha-1} = 1$$

and associated consumption levels

$$\begin{aligned} c^y &= \frac{(1-\alpha)}{1+\beta}\alpha^{\frac{\alpha}{1-\alpha}} \\ c_t^o &= \frac{\beta(1-\alpha)}{1+\beta}\alpha^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Thus the social planner chooses the golden rule capital stock k^{GR} that maximizes net output $y^{GR} = \left(k^{GR}\right)^{\alpha} - k^{GR}$ and splits it efficiently between c^y and c^o according to the rule $c^o = \beta c^y$.

C.2 Proof of Constrained Efficiency of Ramsey Allocation

Proof. Define the savings rate of the constrained planner as

$$s_t = \frac{k_{t+1}}{(1-\kappa)MPL(k_t)} = \frac{k_{t+1}}{(1-\alpha)(1-\kappa)k_t^{\alpha}}$$

and thus the law of motion for the effective capital stock for the constrained planner is

$$k_{t+1} = s_t(1-\alpha)(1-\kappa)k_t^{\alpha}$$

as in the Ramsey problem. Furthermore, from the constraints on the constrained planner

$$\begin{aligned} c_t^y &= (1-\kappa)MPL(k_t) - k_{t+1} = (1-s_t)(1-\kappa)(1-\alpha)k_t^{\alpha} \\ c_{t+1}^o(\eta_{t+1}) &= k_{t+1}MPK(k_{t+1}) + \kappa\eta_{t+1}MPL(k_{t+1}) \\ &= \alpha k_{t+1}^{\alpha} + \kappa\eta_{t+1}(1-\alpha)k_{t+1}^{\alpha} \\ &= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right] k_{t+1}^{\alpha} \\ &= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right] \left[s_t(1-\alpha)(1-\kappa)k_t^{\alpha}\right]^{\alpha} \end{aligned}$$

and thus consumption levels are the same as in the Ramsey equilibrium. Thus the solution, in terms of savings rates, of the constrained planner problem is identical to that of the Ramsey equilibrium. □

C.3 Dynamic Inefficiency of the Competitive Equilibrium and Positive Capital Taxation

In this section we provide the details of the steady state analysis of the Ramsey problem and its connection with the dynamic efficiency of the steady state equilibrium absent government policy. First, recall that the golden rule capital stock, savings rate and associated gross real interest rate are given by (see proposition 8)

$$\begin{aligned} k^{GR} &= \alpha^{\frac{1}{1-\alpha}} \\ s^{GR} &= \frac{\alpha}{(1-\kappa)(1-\alpha)} \\ R^{GR} &= 1 \end{aligned}$$

Second, we note that the steady state gross interest rate is given by

$$R = \alpha k^{\alpha-1}$$

and from the law of motion of capital (equation 9) we have

$$k = s(1-\kappa)(1-\alpha)k^{\alpha}$$

and thus

$$R = \frac{\alpha}{s(1-\kappa)(1-\alpha)}.$$

The steady state savings rate in turn is given by (see equation 15)

$$s_{\tau}(\eta) = \frac{1}{1+[(1-\tau)\alpha\beta\Gamma]^{-1}} = \frac{(1-\tau)\alpha\beta\Gamma}{1+(1-\tau)\alpha\beta\Gamma}$$

Thus we have a steady state relation between the real interest rate and the tax rate determined by

$$R = \frac{\frac{1}{(1-\tau)\beta\Gamma}+\alpha}{(1-\kappa)(1-\alpha)} = R(\tau;\Gamma)$$

and thus a higher tax rate reduces the savings rate, thus the capital stock and thus increases the real interest rate. Furthermore, for a given τ, the steady state interest rate is decreasing in the amount of income risk (unless $\beta = 0$). Therefore the steady state interest rate in the absence of government policy is given by

$$R(\tau=0;\Gamma) = \frac{\frac{1}{\beta\Gamma}+\alpha}{(1-\kappa)(1-\alpha)}$$

and thus the steady state competitive equilibrium without taxes is dynamically inefficient, i.e.

$$R(\tau=0;\Gamma) < 1$$

if and only if

$$\frac{\frac{1}{\beta\Gamma}+\alpha}{(1-\kappa)(1-\alpha)} < 1$$

or if and only if

$$\begin{aligned} \frac{1}{[(1-\kappa)(1-\alpha)-\alpha]\Gamma} &< \beta \\ \Psi_1 &= \frac{1}{(1-\alpha)\Gamma-\Gamma/\bar{\Gamma}} < \beta \end{aligned}$$

where $\bar{\Gamma} \leq \Gamma$, with equality if η is generate at $\eta = 1$, and thus there is no income risk.

The optimal Ramsey steady state (i.e. $\theta = 1$) savings and tax rates (see equations 27 and 28) are given by

$$\begin{aligned} s^* &= \frac{\alpha(1+\beta)}{1+\alpha\beta} \\ 1-\tau &= \frac{1+\beta}{(1-\alpha)\,\beta\Gamma} \end{aligned}$$

and thus the optimal Ramsey tax rate is positive, $\tau > 0$, if and only if

$$\frac{(1+\beta)}{(1-\alpha)\,\beta\Gamma} < 1$$

or if and only if

$$\Psi_2 := \frac{1}{(1-\alpha)\,\Gamma - 1} < \beta$$

Therefore

$$\Psi_2(\Gamma) = \frac{1}{(1-\alpha)\,\Gamma - 1} \leq \frac{1}{(1-\alpha)\Gamma - \Gamma/\bar{\Gamma}} = \Psi_1(\Gamma)$$

with equality if and only if η is generate at $\eta = 1$. Comparing savings rates we have

$$\begin{aligned} s^* &= \frac{\alpha(1+\beta)}{1+\alpha\beta} \\ s_0(\eta) &= \frac{1}{1+[\alpha\beta\Gamma]^{-1}} \\ s^{GR} &= \frac{\alpha}{(1-\kappa)(1-\alpha)} \end{aligned}$$

and thus $s_0(\eta) > s^{GR}$ if and only if

$$\beta > \frac{1}{[(1-\kappa)(1-\alpha)-\alpha]\,\Gamma}$$

and thus if and only if the steady state equilibrium is dynamically inefficient. Furthermore $s^* < s_0(\eta)$ if and only if $\Psi_2 < \beta$ and thus if and only if $\tau > 0$.

We thus have the following results stated in the main text, which directly follow from comparing $\Psi_1(\Gamma)$ and $\Psi_2(\Gamma)$ with β.

Proposition 14. *Let $\theta = 1$. If the steady state competitive equilibrium is dynamically inefficient, then the optimal Ramsey tax rate τ is positive. If in addition η is degenerate at $\eta = 1$, then the reverse is true as well: $\tau > 0$ only if the steady state competitive equilibrium is dynamically inefficient.*

Proposition 15. *Let $\theta = 1$ such that the Ramsey government maximizes steady state welfare, and denote by s^* the associated optimal savings rate. Furthermore denote by $s_0(\eta)$ the steady state equilibrium savings rate in the absence of government policy and s^{GR} the golden rule savings rate that maximizes steady state aggregate consumption. Finally assume that $\beta < \left[(1-\alpha)\bar{\Gamma} - 1\right]^{-1}$.*

1. *Let income risk be large: $\Gamma > \frac{1}{\beta\left[(1-\alpha)-1/\bar{\Gamma}\right]}$. Then the steady state competitive equilibrium is dynamically inefficient, $s^{GR} < s_0(\eta)$, and $s^* < s_0(\eta)$, and the optimal capital tax rate has $\tau > 0$.*

2. *Let income risk be intermediate:*

$$\Gamma \in \left(\frac{1+\beta}{(1-\alpha)\,\beta}, \frac{1}{\left[(1-\alpha) - 1/\bar{\Gamma}\right]\beta} \right)$$

   Then the steady state competitive equilibrium is dynamically efficient, $s^* < s_0(\eta) < s^{GR}$, and optimal capital taxes are nevertheless positive, $\tau > 0$.

3. *Let income risk be small:*

$$\Gamma \in \left[\bar{\Gamma}, \frac{1+\beta}{(1-\alpha)\,\beta}\right)$$

   Then the steady state competitive equilibrium is dynamically efficient, $s_0(\eta) < s^{GR}$, and $s_0(\eta) < s^*$, and optimal capital taxes are negative (capital is subsidized).

C.4 Proof of Pareto-Improving Tax-Induced Transition

Proof of Proposition 7. The capital stock evolves according to the law of motion

$$k_{t+1} = s(1-\kappa)(1-\alpha)k_t^{\alpha}$$

Therefore if the Ramsey government implements s^* through positive capital taxes in the first period of the transition will lead to a falling capital stock along the transition. Recall from (1) that utility of a generation born in period t is given by

$$V_t = \ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi.$$

Now, suppose that the policy is implemented (as a surprise) in period 1 where $k_1 = k_0$. The initial old are unaffected by and thus indifferent to the tax reform. Now we need to characterize the utility consequences for all generations born along the transition. Denoting by $s_0 = s_0(\eta)$ the equilibrium savings rate in the initial steady state, we have

$$\Delta V_t = V_t(s^*) - V_t(s_0) = \ln(c_t^y(s^*)) - \ln(c_t^y(s_0)) + \beta \int \left(\ln(c_{t+1}^o(s^*)) - \ln(c_{t+1}^o(s_0))\right) d\Psi.$$

where the consumption allocations are given as functions of a given saving rate:

$$\begin{aligned}
c_t^y(s_t) &= (1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha \\
c_{t+1}^o(\eta_{t+1}; s_t) &= s_t(1-\kappa)(1-\alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1} + \kappa\eta_{t+1}(1-\alpha)k_{t+1}^\alpha \\
&= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right] k_{t+1}^\alpha
\end{aligned}$$

Thus

$$\Delta V_t = \underbrace{\ln(1-s^*) - \ln(1-s_0)}_{=\Lambda_1>0} + \underbrace{\alpha\left(\ln(k_t(s^*)) - \ln(k_0)\right) + \alpha\beta\left(\ln(k_{t+1}(s^*)) - \ln(k_0)\right)}_{=\Lambda_{2,t}<0}$$

Term Λ_1 is positive because $s^* < s_0$, and constant in t. Term $\Lambda_{2,t}$ is negative for all generations t because $k_t(s^*)) \leq k_0$ for all t and because $k_{t+1}(s^*) < k_0$ for all t. Furthermore $\Lambda_{2,t}$ monotonically declines and converges from above to its minimum for $t \to \infty$ when the economy reaches the optimal steady state capital allocation and the loss term is given as

$$\lim_{t\to\infty} \Lambda_{2,t} = \alpha(1+\beta)\left(\ln(k^*) - \ln(k_0)\right)$$

But because s^* maximizes steady state utility and $s_0 \neq s^*$, we know that

$$\lim_{t\to\infty} \Delta V_t = \Lambda_1 + \lim_{t\to\infty} \Lambda_{2,t} > 0$$

It then follows that

$$\Delta V_t \geq \lim_{\tau\to\infty} \Delta V_\tau > 0$$

and thus all transition generations strictly benefit from the tax reform. □

C.5 Savings Subsidy Does Not Induce Pareto Improvement

In this section we show that even if $s_0(\eta) < s^*$, implementing the Ramsey optimal (for $\theta = 1$) savings subsidy $\tau^* < 0$ and associated higher saving rate s^* does *not* lead to a Pareto improving transition.

Exploiting the fact that in the first period of the transition the capital stock $k_1 = k_0$ is predetermined, and the capital stock in period 2 satisfies

$$k_2 = s(1-\alpha)(1-\kappa)k_0^\alpha$$

for any savings rate implemented by a given tax policy. Thus we can calculate lifetime utility of the first transition generation, as a function of an implemented savings rate s, as

$$\begin{aligned}
V_1(s) &= \ln\left((1-s)(1-\kappa)(1-\alpha)k_0^\alpha\right) + \beta\int \ln\left(\alpha + \kappa\eta_2(1-\alpha)\right)\left(s(1-\alpha)(1-\kappa)k_0^\alpha\right)^\alpha d\Psi(\eta) \\
&= \ln(1-s) + \beta\alpha\ln(s) + \ln\left((1-\kappa)(1-\alpha)k_0^\alpha\right) \\
&+ \beta\int \ln\left(\alpha + \kappa\eta_2(1-\alpha)\right)\left((1-\alpha)(1-\kappa)k_0^\alpha\right)^\alpha d\Psi(\eta)
\end{aligned}$$

and thus

$$\begin{aligned} V_1'(s) &= -\frac{1}{1-s} + \frac{\alpha\beta}{s} \\ V_1''(s) &= -\frac{1}{(1-s)^2} - \frac{\alpha\beta}{s^2} < 0 \end{aligned}$$

and thus $V_1(s)$ is strictly concave in s. Therefore, if $V_1'(s = s_0(\eta)) \leq 0$, then $V(s = s_0(\eta)) > V(s)$ for all $s > s_0(\eta)$. But

$$\begin{aligned} V_1'(s &= s_0(\eta)) = -\frac{1}{1-s_0(\eta)} + \alpha\beta\frac{1}{s_0(\eta)} \leq 0 \\ &\Leftrightarrow s_0(\eta) \geq \frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

which is satisfied, exploiting expression (15) for the optimal competitive equilibrium saving rate (with zero taxes).

Thus not only is implementing $\tau^* < 0$ not Pareto improving if $s_0(\eta) < s^*$, but in fact any policy reform that induces a savings rate in period 1 above the competitive savings rate with zero taxes, $s_0(\eta)$, will not result in a Pareto improvement (since it will make the first generation strictly worse off).

D Analysis of General Epstein-Zin Utility

Now consider general Epstein-Zin preferences, applied to our two period OLG model. Households have preferences over deterministic consumption when young, c_t^y, and the (deterministic) certainty equivalent over utility from consumption tomorrow, $\int c_{t+1}^o(\eta_{t+1})^{1-\sigma} d\Psi$. We assume that these preferences are represented by the lifetime utility function

$$V_t = \frac{(c_t^y)^{1-\frac{1}{\rho}} - 1}{1-\frac{1}{\rho}} + \beta \frac{\left\{\left[\int c_{t+1}^o(\eta_{t+1})^{1-\sigma} d\Psi\right]^{\frac{1}{1-\sigma}}\right\}^{1-\frac{1}{\rho}} - 1}{1-\frac{1}{\rho}} \tag{48}$$

Note that the limit, as the IES $\rho \to 1$, the (transformed) utility function becomes

$$V_t = \ln(c_t^y) + \frac{\beta}{1-\sigma} \ln\left(\int c_{t+1}^o(\eta_{t+1})^{1-\sigma} d\Psi\right)$$

We should point out that often in the literature Epstein-Zin preferences of the form

$$\tilde{V}_t = \left\{(1-\tilde{\beta})(c_t^y)^{1-\frac{1}{\rho}} + \tilde{\beta}\left[\int c_{t+1}^o(\eta_{t+1})^{1-\sigma} d\Psi\right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}}\right\}^{\frac{1}{1-\frac{1}{\rho}}} \tag{49}$$

are used. When it comes to ordinal rankings of allocations, we can take monotonic transformations of (49) without changing preference rankings. Thus for all *ordinal* purposes the formulations in (48) and (49) are equivalent since

$$V_t = \frac{\tilde{V}_t^{1-\frac{1}{\rho}}}{(1-\tilde{\beta})\left(1-\frac{1}{\rho}\right)} - \frac{(1+\beta)}{1-\frac{1}{\rho}}$$

where $\beta = \frac{\tilde{\beta}}{1-\tilde{\beta}}$.

However, also note that, in contrast to any analyses that only require ordinal rankings (such as studying competitive equilibrium for a given tax system) when moving from utility form V_t to $\tilde{V}_t$ we are changing the Ramsey problem, since this problem is based on cardinal weighted lifetime utilities. It is therefore *not* innocuous whether we choose (49) or (48) when formulating the *Ramsey optimal tax problem.*[14] If, however, we restrict attention to a steady state analysis, then this last concern does not emerge, since the first order condition characterizing the optimal Ramsey steady state saving rate is identical under any monotone transformation of steady state lifetime lifetime utility (as effectively the only generation that is relevant for the maximization is the long-run, steady state generation).

D.1 Competitive Equilibrium for Given Tax Policy

Household maximization delivers

$$\begin{aligned}
(c_t^y)^{-\frac{1}{\rho}} &= \beta(1-\tau_{t+1})R_{t+1}\left[\int c_{t+1}^o(\hat{\eta}_{t+1})^{1-\sigma}d\Psi\right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}-1}\int c_{t+1}^o(\eta_{t+1})^{-\sigma}d\Psi(\eta_{t+1}) \\
1 &= \beta(1-\tau_{t+1})R_{t+1}\left[\int\left(\frac{c_{t+1}^o(\eta_{t+1})}{c_t^y}\right)^{1-\sigma}d\Psi(\eta_{t+1})\right]^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(\frac{c_{t+1}^o(\eta_{t+1})}{c_t^y}\right)^{-\sigma}d\Psi(\eta_{t+1})
\end{aligned}$$

Now, as before, we need to work out the ratio

$$\begin{aligned}
\frac{c_{t+1}^o(\eta_{t+1})}{c_t^y} &= \frac{s_tR_{t+1}(1-\tau_{t+1})(1-\kappa)w_t+\kappa\eta_{t+1}w_{t+1}+T_{t+1}}{(1-\kappa)w_t(1-s_t)} \\
&= \left[\alpha+(1-\alpha)\kappa\eta_{t+1}\right]\frac{s_t}{1-s_t}k_{t+1}^{\alpha-1}
\end{aligned}$$

[14] For both formulations the household is not indifferent to the resolution of income risk as long as $\sigma \neq \frac{1}{\rho}$.

and thus

$$
\begin{aligned}
1 &= \beta(1-\tau_{t+1})R_{t+1}\left[\int\left(\frac{c^o_{t+1}(\eta_{t+1})}{c^y_t}\right)^{1-\sigma}d\Psi(\eta_{t+1})\right]^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(\frac{c^o_{t+1}(\eta_{t+1})}{c^y_t}\right)^{-\sigma}d\Psi(\eta_{t+1})\\
1 &= \beta(1-\tau_{t+1})R_{t+1}\left(\frac{s_t k^{\alpha-1}_{t+1}}{1-s_t}\right)^{-\frac{1}{\rho}}\tilde{\Gamma}\\
1 &= \alpha\beta(1-\tau_{t+1})\left(k^{\alpha-1}_{t+1}\right)^{\left(1-\frac{1}{\rho}\right)}\left(\frac{s_t}{1-s_t}\right)^{-\frac{1}{\rho}}\tilde{\Gamma}\\
1 &= \alpha\beta(1-\tau_{t+1})\left(s_t(1-\kappa)(1-\alpha)k^{\alpha}_t\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{s_t}{1-s_t}\right)^{-\frac{1}{\rho}}\tilde{\Gamma}\\
1 &= \alpha\beta(1-\tau_{t+1})\left(s_t(1-\kappa)(1-\alpha)k^{\alpha}_t\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{s_t}{1-s_t}\right)^{-\frac{1}{\rho}}\tilde{\Gamma}\\
1 &= \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau_{t+1})k_t^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)}s_t{}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}}\tilde{\Gamma}.
\end{aligned}
$$

In the main text we are mainly concerned with characterizing the Ramsey steady state savings and associated tax rate. In steady state the Euler equation reads as

$$1=\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau)k^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)}s^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s}{s}\right)^{\frac{1}{\rho}}\tilde{\Gamma}$$

where

$$k=\left[(1-\kappa)(1-\alpha)s\right]^{\frac{1}{1-\alpha}}$$

is the steady state capital stock. Inserting the steady state capital stock into the Euler equation delivers

$$1=(1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}s^{\left(\frac{1}{\rho}-1\right)}\left(\frac{1-s}{s}\right)^{\frac{1}{\rho}}\tilde{\Gamma} \tag{50}$$

or

$$1=(1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s)^{\frac{1}{\rho}}}{s}\tilde{\Gamma} \tag{51}$$

where

$$
\begin{aligned}
\tilde{\Gamma} &= \frac{\int\left(\alpha+(1-\alpha)\kappa\eta\right)^{-\sigma}d\Psi(\eta)}{\left[\int\left(\alpha+(1-\alpha)\kappa\eta\right)^{1-\sigma}d\Psi(\eta)\right]^{\frac{\frac{1}{\rho}-\sigma}{1-\sigma}}}=ce(\eta)^{\left(\sigma-\frac{1}{\rho}\right)}\Gamma.\\
\Gamma &= \int\left(\alpha+(1-\alpha)\kappa\eta\right)^{-\sigma}d\Psi(\eta)
\end{aligned}
$$

and where the certainty equivalent of η is defined as, for $\sigma \neq 1$

$$ce(\eta) = \left[\int (\alpha + (1-\alpha)\kappa\eta)^{1-\sigma}\, d\Psi(\eta)\right]^{\frac{1}{1-\sigma}}$$

and for $\sigma = 1$

$$ce(\eta) = \exp\left(\int \ln\left(\alpha + (1-\alpha)\kappa\eta\right) d\Psi(\eta)\right).$$

Note that this result is precisely the generalization of the log-case where $\rho = \sigma = 1$, and where the Euler equation was given as

$$\begin{aligned} 1 &= (1-\tau)\alpha\beta\left(\frac{1-s}{s}\right)\Gamma \\ \Gamma &= \int (\kappa\eta(1-\alpha)+\alpha)^{-1}\, d\Psi(\eta) \end{aligned}$$

Thus our previous analysis for log-utility is just a special case. Also note that if $\rho = 1$ but $\sigma \neq 1$, then the steady state Euler equation is given by

$$1 = (1-\tau)\alpha\beta\left(\frac{1-s}{s}\right)^{\frac{1}{\rho}}\tilde{\Gamma}$$

but

$$\tilde{\Gamma} = \frac{\int (\alpha + (1-\alpha)\kappa\eta)^{-\sigma}\, d\Psi(\eta)}{\left[\int (\alpha + (1-\alpha)\kappa\eta)^{1-\sigma}\, d\Psi(\eta)\right]} \neq \int (\kappa\eta(1-\alpha)+\alpha)^{-1}\, d\Psi(\eta) = \Gamma_{\sigma=1}$$

D.1.1 Precautionary Savings Behavior in the Competitive Equilibrium

In order to aid with the interpretation of the optimal Ramsey tax rate it is useful to establish conditions under which, for a fixed tax rate constant, the savings rate in a competitive general equilibrium is increasing in income risk.

Proposition 16. *If $\tilde{\Gamma}$ is strictly increasing in income risk, then for any given tax rate $\tau \in (-\infty, 1)$ the steady state saving rate $s^{CE}(\tau)$ in competitive equilibrium is strictly increasing in income risk. If $\tilde{\Gamma}$ is strictly decreasing in income risk, then so is $s^{CE}(\tau)$.*

Proof. Rewrite equation

$$f(s) = (1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s)^{\frac{1}{\rho}}}{s} - \frac{1}{\tilde{\Gamma}}$$

Then the steady state saving rate $s^{CE}(\tau)$ satisfies $f(s^{CE}(\tau)) = 0$. We readily observe that f is continuous and strictly decreasing in s, with

$$\begin{aligned} \lim_{s} f(s) &= \infty \\ f(1) &= -1 \end{aligned}$$

and thus for each $\tau \in (-\infty, 1)$ there is a unique $s = s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$. Inspection of f immediately reveals that $s^{CE}(\tau)$ is strictly increasing in $\tilde{\Gamma}$, from which the comparative statics results immediately follow. □

Corollary 9. *For any given $\tau \in (-\infty, 1)$, the steady state saving rate $s^{CE}(\tau)$ increases in income risk if either $\rho \leq 1$, or $1 < \rho < \frac{1}{\sigma}$.*

Proof. Proof: Follows directly from the previous proposition and Lemma 1 characterizing the behavior of $\tilde{\Gamma}$ with respect to income risk. □

Proposition 16 establishes a sufficient condition for the private saving rate to increase in income risk. But, for $\rho > \frac{1}{\sigma} > 1$ it might be possible that the combination of individual precautionary savings behavior and general equilibrium factor price movements lead to the result that, for *fixed* government policy, the equilibrium savings rate is decreasing in income risk.[15] We will show below that this in turn is a necessary condition for the optimal Ramsey tax rate to decrease in income risk.

Figure illustrates how the competitive equilibrium saving rate (in the steady state, with $\tau = 0$) varies with income risk for different parameterizations of ρ and σ as a function of risk. We choose $\alpha = 0.2$ and $\kappa = 0.5$. The figure shows that, consistent with the proposition, the saving rate in the competitive equilibrium is increasing with income risk for all preference specifications such that $\rho \leq 1$ or $1 < \rho < \frac{1}{\sigma}$. It also displays an example with very high IES ($\rho = 50$) and very high risk aversion ($\sigma = 50$) where the previous proposition does not apply and we indeed observe that the competitive equilibrium saving rate *declines* with income risk, at least as long as risk is sufficiently large to start with.[16]

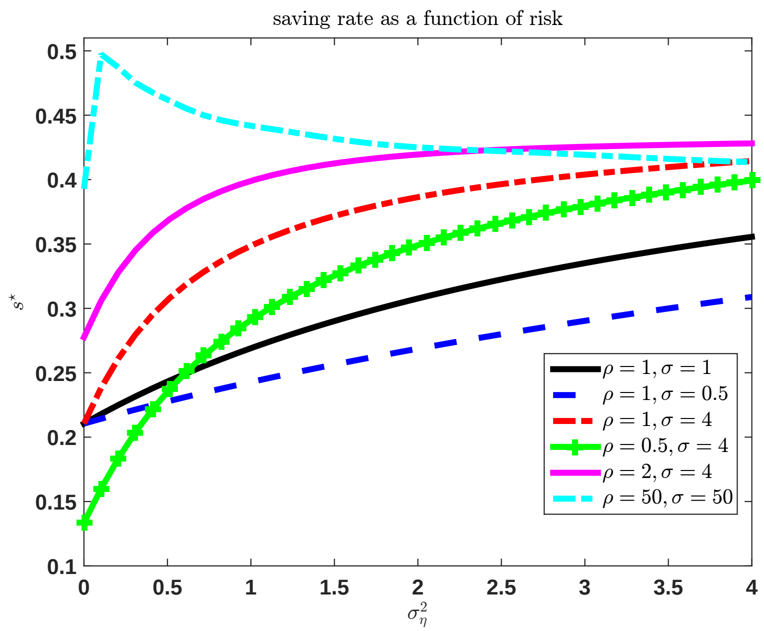
D.2 Ramsey Problem for Unit IES

Now suppose we use the formulation of lifetime utility in equation (48). Then it is straightforward to show that for $\rho = 1$ the analysis of the Ramsey problem

[15] Also observe that a parameter constellation $1 < \rho < \frac{1}{\sigma}$ pairs a high IES with a preference for a *late* resolution of risk in a multi-period (more than two periods) model. Interestingly, the competitive equilibrium savings rate may therefore decrease in income risk precisely when we pair a high IES with a preference constellation for *early* resolution of risk.

[16] The phenomenon already shows up for $\rho = \sigma = 10$.

Figure 6: Saving Rate in Competitive Equilibrium



Notes: Saving rate s in competitive equilibrium for Epstein-Zin utility. Lognormally distributed shocks, approximated with $n = 11$ integration nodes for $\kappa = 0.5$, $\alpha = 0.2$, $\sigma^2_{\ln\eta} = [0, \ldots, 4]$, $\rho \in [0.5, \ldots, 50]$, $\sigma \in [0.5, \ldots, 50]$.

proceeds exactly as before,

$$\begin{aligned}
W(k) &= \Psi_0 + \Psi_1 \log(k) \\
&= \max_{s\in[0,1]} \{\log((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\
&\quad + \frac{\beta}{1-\sigma} \log \int (\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha})^{1-\sigma}\, d\Psi(\eta) + \theta W(k')\Big\} \\
&= \max_{s\in[0,1]} \{\log((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\
&\quad + \frac{\beta}{1-\sigma} \log \int ([\kappa\eta(1-\alpha)+\alpha]\,[s(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha})^{1-\sigma}\, d\Psi(\eta) + \theta W(s(1-\kappa)(1-\alpha)k^{\alpha})\Big\} \\
&= \alpha\,[1+\theta\Psi_1+\alpha\beta]\log(k) + \log\,[(1-\kappa)\,(1-\alpha)] + \theta\Psi_0 + \theta\Psi_1\log((1-\kappa)(1-\alpha)) \\
&\quad + \beta\alpha\log\,[(1-\kappa)(1-\alpha)] + \frac{\beta\log\int[\kappa\eta(1-\alpha)+\alpha]^{1-\sigma}\, d\Psi(\eta)}{1-\sigma} \\
&\quad + \max_{s\in[0,1]} \{\log(1-s) + \alpha\beta\log(s) + \theta\Psi_1\log(s)\}
\end{aligned}$$

with an optimal savings rate as in the main text:

$$s = \frac{\alpha(\beta+\theta)}{1+\alpha\beta}$$

These results clarify that the closed form solution, and the fact that the optimal saving rate is constant over time and independent of the level of capita, is driven by an $IES = \rho = 1$ (and obtained for arbitrary risk aversion), whereas the size of the capital tax needed to implement the optimal Ramsey allocation does depend on risk aversion σ, since this parameter determines the degree of precautionary saving in the competitive equilibrium that needs to be offset with capital taxes, see Section D.1.1.

D.3 Steady State Analysis for Arbitrary IES

In the steady state we seek to maximize

$$\begin{aligned}
V(s) &= \frac{(c_t^y)^{1-\frac{1}{\rho}} + \beta\left\{\left[\int c_{t+1}^o(\eta_{t+1})^{1-\sigma} d\Psi\right]^{\frac{1}{1-\sigma}}\right\}^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-s)(1-\alpha)k^{\alpha})^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&\quad + \frac{\beta\,[s(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha\left(1-\frac{1}{\rho}\right)}\left\{\left[\int\{[\kappa\eta(1-\alpha)+\alpha]\}^{1-\sigma}\, d\Psi\right]^{\frac{1}{1-\sigma}}\right\}^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-\alpha))^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}}(1-s)^{\left(1-\frac{1}{\rho}\right)}k^{\alpha\left(1-\frac{1}{\rho}\right)} + \frac{\beta\,[(1-\kappa)(1-\alpha)]^{\alpha\left(1-\frac{1}{\rho}\right)}\,\tilde{\Gamma}_2}{1-\frac{1}{\rho}} s^{\alpha\left(1-\frac{1}{\rho}\right)} k^{\alpha^2\left(1-\frac{1}{\rho}\right)}
\end{aligned}$$

where

$$\tilde{\Gamma}_2 = \left[\int \{[\kappa\eta(1-\alpha)+\alpha]\}^{1-\sigma} \, d\Psi \right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \Gamma_2$$

Exploiting that

$$k = ((1-\kappa)(1-\alpha)s)^{\frac{1}{1-\alpha}}$$

yields

$$\begin{aligned}
V(s) \quad &= \quad \frac{((1-\kappa)(1-\alpha))^{1-\text{‘}\frac{1}{\rho}}}{1-\frac{1}{\rho}} (1-s)^{\left(1-\frac{1}{\rho}\right)} ((1-\kappa)(1-\alpha)s)^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&\quad + \frac{\beta\left[(1-\kappa)(1-\alpha)\right]^{\alpha\left(1-\frac{1}{\rho}\right)} \tilde{\Gamma}_2}{1-\frac{1}{\rho}} (s)^{\alpha\left(1-\frac{1}{\rho}\right)} ((1-\kappa)(1-\alpha)s)^{\frac{\alpha^2\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&= \quad \frac{((1-\kappa)(1-\alpha))^{\frac{1-\frac{1}{\rho}}{1-\alpha}}}{1-\frac{1}{\rho}} \left[(1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\left[(1-\kappa)(1-\alpha)\right]^{-\left(1-\frac{1}{\rho}\right)} \tilde{\Gamma}_2 \right] s^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&= \quad \tilde{\phi} \left((1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\tilde{\psi}\tilde{\Gamma}_2 \right) s^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\phi} \quad &= \quad \frac{((1-\kappa)(1-\alpha))^{\frac{1-\frac{1}{\rho}}{1-\alpha}}}{1-\frac{1}{\rho}} \\
\tilde{\psi} \quad &= \quad \left(\frac{1}{(1-\kappa)(1-\alpha)} \right)^{\left(1-\frac{1}{\rho}\right)} > 0 \\
\tilde{\Gamma}_2 \quad &= \quad \left(\left[\int \{[\kappa\eta(1-\alpha)+\alpha]\}^{1-\sigma} \, d\Psi \right]^{\frac{1}{1-\sigma}} \right)^{1-\frac{1}{\rho}} > 0
\end{aligned}$$

as before. Thus the steady state analysis in the main text carries through to Epstein-Zin utility almost entirely unchanged, but with the constant that maps earnings risk into the optimal savings rate now being affected both by risk aversion and the IES.

Hence, the optimal steady state savings rate is defined implicitly as

$$\frac{s}{(1-s)^{\frac{1}{\rho}}} = \frac{\alpha}{1-\alpha}(1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\frac{\alpha}{1-\alpha}\tilde{\psi}\tilde{\Gamma}_2 \tag{52}$$

and rewriting this equation yields

$$LHS(s) = s = \frac{\alpha}{1-\alpha}\left[(1-s) + \beta\tilde{\psi}\tilde{\Gamma}_2(1-s)^{\frac{1}{\rho}} \right] = RHS(s) \tag{53}$$

We observe that the left hand side is linearly increasing in s, with $LHS(0) = 0$ and $LHS(1) = 1$ and the right hand side is strictly decreasing in s, with

$RHS(0) > 0$ and $RHS(1) = 0$. Since both sides are continuous in s, from the intermediate value theorem it follows that there is a unique $s^* \in (0,1)$ solving the first order condition of the Ramsey problem (53). Since $RHS(s)$ is strictly increasing in $\tilde{\Gamma}_2$, the Ramsey saving rate is strictly increasing in $\tilde{\Gamma}_2$. The comparative statics of s^* with respect to income risk in the main text then directly follow from the properties of $\tilde{\Gamma}_2$ stated in Lemma 1.

D.4 Implementation

The optimal steady state capital tax rate τ^* satisfies, as in equation (51)

$$1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)} \frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma} \tag{54}$$

We observe that the optimal tax rate is strictly increasing in $\tilde{\Gamma}$ and strictly decreasing in the Ramsey savings rate s^* that is to be implemented. Further, recall that that the Ramsey saving rate s^* itself satisfies the first order condition (52)

$$\frac{s^*}{(1-s^*)^{\frac{1}{\rho}}} = \frac{\alpha}{1-\alpha}\left[(1-s^*)^{\left(1-\frac{1}{\rho}\right)} + \beta\tilde{\psi}\tilde{\Gamma}_2\right] \tag{55}$$

and is impacted by income risk through $\tilde{\Gamma}_2$. To analyze the impact of a change on risk on the optimal capital tax rate, in light of Lemma 1 it is useful to consider 3 cases.

D.4.1 Case $\rho > 1$ and $\sigma \leq 1/\rho$

In this case $\tilde{\Gamma}$ is strictly increasing in risk (Lemma 1) and s^* is strictly decreasing in risk (see Proposition 10) It then directly follows from equation (54) that τ^* is strictly increasing in income risk.

D.4.2 Case $\rho \leq 1$

This case is more difficult, since the direct effect of risk on τ^* is positive, but at the same time s^* is decreasing in risk. Now rewrite (55) as

$$\frac{\frac{1}{\alpha}-\frac{1}{s^*}}{\beta\tilde{\psi}\tilde{\Gamma}_2} = \frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}$$

and plugging this into (54) and exploiting that $\tilde{\psi} = ((1-\kappa)(1-\alpha))^{\left(\frac{1}{\rho}-1\right)}$ we find

$$1 = (1-\tau^*)\left(1-\frac{\alpha}{s^*}\right)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}. \tag{56}$$

Lemma 1 establishes that the ratio $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in risk, and since $\rho \geq 1$ the saving rate s^* is increasing in risk (strictly so if $\rho < 1$). Thus the term $\left(1-\frac{\alpha}{s^*}\right)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in risk, and thus τ^* is strictly increasing in risk.

D.4.3 Case $\rho > 1$ and $\sigma > 1/\rho$

In this case we know that the Ramsey saving rate s^* is strictly decreasing in income risk (which by itself calls for a tax rate that is strictly increasing in income risk). However, now the direct impact of income risk on taxes through the term $\tilde{\Gamma}$ might call for lower taxes since $\tilde{\Gamma}$ might now be decreasing in income risk. If $\tilde{\Gamma}$ is weakly increasing in income risk, then so is τ^*. Thus a necessary condition for τ^* to decrease with income risk is $\tilde{\Gamma}$ to be strictly decreasing with income risk. This in turn is a necessary and sufficient condition for the private saving rate in competitive equilibrium to decrease with income risk. Thus the Ramsey tax rate τ^* is strictly decreasing in income risk *only if* the private saving rate $s^{CE}(\tau)$ is strictly decreasing in income risk (see Proposition 16). The corresponding *if* statement is not necessarily true, as the numerical illustrations in the main text show.

Now consider the case $\rho \to \infty$ and $\sigma \to \infty$. Our objective is to show that in this case the optimal Ramsey Tax rate is indeed strictly decreasing with income risk. The case of $\rho = \infty$ corresponds to preferences

$$V_t = c_t^y + \beta \left[\int c_{t+1}^o(\eta_{t+1})^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}}$$

where households have linear indifference curves between consumption today and the certainty equivalence of consumption tomorrow. Also note that $\tilde{\Gamma}_2 = ce(\eta)$. We can determine the optimal Ramsey saving rate in closed form as

$$s^* = \alpha \left[\frac{(1-\kappa)(1-\alpha) + \beta ce(\eta)}{(1-\kappa)(1-\alpha)} \right]$$

and the optimal tax rate is then given, by (56), as

$$\begin{aligned} 1 &= (1-\tau^*)\left(1 - \frac{\alpha}{s^*}\right)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2} \\ 1 &= (1-\tau^*)\left(\frac{\beta ce(\eta)}{(1-\kappa)(1-\alpha)+\beta ce(\eta)}\right)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2} \\ 1-\tau^* &= \left(1 + \frac{(1-\kappa)(1-\alpha)}{\beta ce(\eta)}\right)\frac{\tilde{\Gamma}_2}{\tilde{\Gamma}} \end{aligned} \tag{57}$$

This expression is valid for any σ as long as $\rho = \infty$. Now we note that

$$\frac{\tilde{\Gamma}_2}{\tilde{\Gamma}} = \frac{\int (\kappa\eta(1-\alpha)+\alpha)^{1-\sigma} d\Psi(\eta)}{\int (\kappa\eta(1-\alpha)+\alpha)^{-\sigma} d\Psi(\eta)}$$

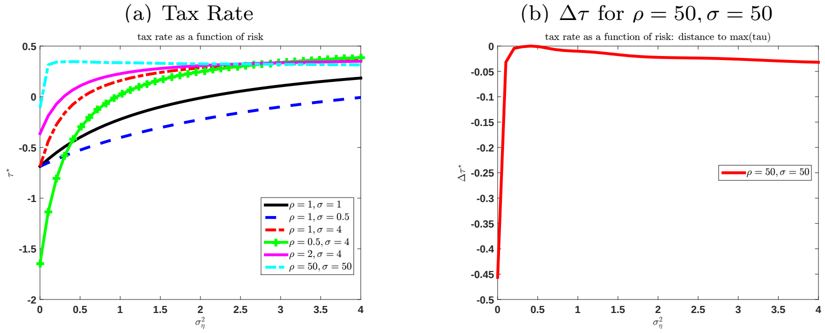
as thus $\lim_{\sigma\to\infty} \frac{\tilde{\Gamma}_2}{\tilde{\Gamma}} = 1$ and since $ce(\eta)$ is strictly decreasing in income risk for all σ, the right hand side of equation (57) is strictly increasing in income risk, and thus the optimal Ramsey tax rate is strictly decreasing in income risk as $\sigma \to \infty$ as long as $\rho = \infty$.

In this case the government reduces the tax rate on capital (or increases the subsidy on capital) as income risk increases since the Ramsey government wants a *smaller reduction of the saving rate* than what households find optimal to choose in the competitive equilibrium, in response to an increase in income risk.

Numerical Illustration Figure 7 plots the tax rate that implements the optimal Ramsey allocation in a competitive equilibrium at the steady state in Panel (a). For all scenarios except for $\rho = \sigma = 50$ the tax rate is monotonically increasing in risk. Panel (b) illustrates this more clearly by showing the distance to the maximum tax rate for $\rho = \sigma = 50$.

Figure 7: Optimal Tax Rate

(a) Tax Rate   (b) $\Delta\tau$ for $\rho = 50, \sigma = 50$



Notes: Tax rate τ in Ramsey equilibrium for Epstein-Zin utility. Lognormally distributed shocks, approximated with $n = 11$ integration nodes for $\kappa = 0.5$, $\alpha = 0.2$, $\sigma^2_{\ln\eta} = [0, \ldots, 4]$, $\rho \in [0.5, \ldots, 50]$, $\sigma \in [0.5, \ldots, 50]$. Panel (a): tax rate, panel (b) distance to maximum tax rate for $\rho = 50, \sigma = 50$.