Topics In Empirical Industrial Organisation

Demand Systems for Highly Differentiated Products

An Introduction to Demand Models in Characteristics Space

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Outline I

Market Level: Observe allocation of market demand across products

$$U_{ij} = \alpha_j + \mathbf{v}_j' \boldsymbol{\omega} + \varepsilon_{ij} \quad (1)$$

Observe s_j; Predict ς_j

Product Level: Observe individual choice across products

$$U_{ij} = \alpha_j + \mathbf{v}_j(\mathbf{x}_i)' \boldsymbol{\omega} + \varepsilon_{ij} \quad (2)$$

Observe s_{ij} (y_{ij}); Predict P_{ij}

$\mathbf{v}_j$ (α_j) observed (unobserved) attributes

$\mathbf{x}_i$ vector of individual characteristics; $\boldsymbol{\omega}$ - taste parameters.

Market Level: May supplement market demands with distribution of demographics in the population.

Market Shares Most frequently observed and the object we most care about

Microeconomic Models Models which allow us to incorporate what we know about individual behaviour. Allow consumer behaviour to depend on characteristics such as income and other demographics

Disjunction Between aggregate market shares and individual-level behaviour.

Simulation Utilising the distribution of consumer characteristics within some population of interest, sum up the demands generated by the microeconomic model - share predictions - and confront these with observed market shares

Estimation Estimate model parameters by minimising some function over the distance between observed and predicted shares

Distribution of ε

- ε generates selected purchase conditional on other variables and taste parameters
- ε - unobservable taste parameters
- Mean utility: $\alpha_j \equiv \mathbf{v}_j' \omega + \xi_j$
- Set of consumer unobservables that generate consumption of alternative j

$$s_j(\boldsymbol{\alpha}) \equiv \boldsymbol{\varepsilon} : \alpha_j + \varepsilon_{ij} > \alpha_l + \varepsilon_{il}, \quad , \forall l \neq j \tag{3}$$

- Distribution of ε_i and implied rankings over Ω_J
- p_j increases. Logit model: rankings over Ω_j are expressed through consumer preferences $\boldsymbol{\varepsilon}$ which are i.i.d. -

Consider the utility function for for consumer i.

$$U_{ij} = \alpha_j + H_{ij} + \boldsymbol{\tau}_{ij} + \varepsilon_{ij} \tag{4}$$

Additive Separability in Market-Level Data

$$\begin{aligned} U_{ij} &= \sum_{l=1}^{L} \omega_{jl} v_{jl} - \omega_p v_p + \xi_j + \varepsilon_{ij} \\ &= \alpha_j + \varepsilon_{ij} \end{aligned}$$

Additive Separability and type 1 extreme value

$\varepsilon_{ij} \sim$ Type 1 Extreme Value delivers the **Logit Model**

The utility of individual i for good j if it purchases the j^{th} good is given by

$$U_j(\mathbf{v}_j, \xi_j, \boldsymbol{\tau}_i, \varepsilon_{ij}; \boldsymbol{\theta}) = H_j(\mathbf{v}_j, \xi_j, \boldsymbol{\tau}_i; \boldsymbol{\theta}) + \varepsilon_{ij}$$

$\mathbf{v}_j$ is a vector of observed product characteristics for product j

ξ_j denotes a product characteristic that is observed by the consumer but unobserved by the econometrician

$\boldsymbol{\theta}$ is a vector of parameters.

$\mathbf{t}_i = (\tau_{i1}, ..., \tau_{iL}, \varepsilon_i, ..., \varepsilon_{iJ})$ describes the consumer's 'type'.

Each consumer i solves

$$\max_{j=\{1,...,J\}} U_j(\mathbf{v}_j, \xi_j, \boldsymbol{\tau}_i, \varepsilon_{ij}; \boldsymbol{\theta})$$

A Canonical Model

We assume that types are distributed in the population according to a distribution, $f(\tau_{i1}, ..., \tau_{iL}, \varepsilon_i, ..., \varepsilon_{iJ})$.

ε_{ij} is an additive random term specific to both consumer and product.

Given that ε_{ij} are i.i.d type 1 extreme value then *individual* share functions are given by

$$s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta}) = \frac{e^{H_j(v_j, \xi, \tau; \theta)}}{\sum_{k=1}^{J} e^{H_k(v_k, \xi_k, \tau; \theta)}}.$$

Aggregate market shares may be written as

$$s_j(\mathbf{v}, \xi, p; \boldsymbol{\theta}) = \int s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta}) df(\tau)$$

own and cross price elasticities own and cross price elasticities are

$$\begin{aligned} E_j p_k &= \frac{p_k}{s_j(\mathbf{v}, \xi, p; \boldsymbol{\theta})} \frac{\partial s_j(\mathbf{v}; \xi, p; \boldsymbol{\theta})}{\partial p_k} \\ &= \frac{p_k}{s_j(\mathbf{v}, \xi, p; \boldsymbol{\theta})} \int \frac{\partial s_j(\boldsymbol{\tau}, \mathbf{v}; \xi, p; \boldsymbol{\theta})}{\partial p_k} df(\boldsymbol{\tau}) \\ &= \frac{p_k}{s_j(\mathbf{v}, \xi, p; \boldsymbol{\theta})]} E_\tau \left[\frac{\partial s_j(\boldsymbol{\tau}, \mathbf{v}; \xi, p; \boldsymbol{\theta})}{\partial p_k} \right] \end{aligned}$$

where $\frac{\partial s_j(\boldsymbol{\tau}, \mathbf{v}; \xi, p; \boldsymbol{\theta})}{\partial p_k}$ is given by

$$\frac{\partial H_j(v_j; \xi_j, \mathbf{v}; \boldsymbol{\theta})}{\partial p_k} s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\{\mathbf{1}(j = k) - s_k(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\}$$

own and cross price elasticities - cont

$$E_j p_k = \frac{p_k}{s_j(\mathbf{v}, \xi, p; \boldsymbol{\theta})} \int \frac{\partial s_j(\boldsymbol{\tau}, \mathbf{v}; \xi, p; \boldsymbol{\theta})}{\partial p_k} df(\boldsymbol{\tau})$$

$\frac{\partial s_j(\boldsymbol{\tau}, \mathbf{v}; \xi, p; \boldsymbol{\theta})}{\partial p_k}$ is given by

$$\begin{aligned} \frac{\partial H_j(v_j; \xi_j, \mathbf{v}; \boldsymbol{\theta})}{\partial p_k} s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\{\mathbf{1}(j &= k) - s_k(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\} \\ \frac{\partial H_j(v_j; \xi_j, \mathbf{v}; \boldsymbol{\theta})}{\partial p_k} s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\{1 &- s_k(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\} \quad \text{j=k} \\ \frac{\partial H_j(v_j; \xi_j, \mathbf{v}; \boldsymbol{\theta})}{\partial p_k} s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\{ &- s_k(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta})\} \quad j \neq k \end{aligned}$$

Predicted own and cross-price (attribute) elasticities in this class of model depends on:

a) assumptions regarding the distribution of $\boldsymbol{\tau}$
b) the shape of the MNL market share functions
c) the derivative function $\frac{\partial H_k(\boldsymbol{\tau}, v_k, \xi; \boldsymbol{\theta})}{\partial p_k}$

the distribution of $\boldsymbol{\tau}$ In the vanilla logit model the coefficients are *fixed* and the distribution on the consumer unobservables $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2}, ..., \varepsilon_J)$ is known and does not involve any unknown parameter.

In *random coefficient* models the principle extension is to introduce consumer types through a non i.i.d. distribution of preferences $(\boldsymbol{\tau}_i)$.

Random coefficient models with assumptions on the distribution of the coefficient i.e. MVN, log-normal, independence
dependence across random coefficients.

the shape of the MNL market share functions Given ε_{ij} ∼i.i.d. Type 1 extreme value the shares have the usual logit form

$$s_j(\boldsymbol{\tau}, \mathbf{v}, \xi, p; \boldsymbol{\theta}) = \frac{e^{H_j(v_j, \xi, \boldsymbol{\tau}; \boldsymbol{\theta})}}{\sum_{k=1}^{J} e^{H_k(v_k, \xi_k, \boldsymbol{\tau}; \boldsymbol{\theta})}}.$$

This is a *logit* probability conditional on known value of $\boldsymbol{\theta}$.

A Canonical Model

the derivative function $\frac{\partial H_k(\boldsymbol{\tau}, v_k, \xi; \boldsymbol{\theta})}{\partial p_k}$

Remark

The form of the derivative function $\frac{\partial H_k(\boldsymbol{\tau}, v_k, \xi; \boldsymbol{\theta})}{\partial p_k}$ *will depend upon the linearity of the index function, whether price is interacted with observed consumer characteristics and/or allowed to vary randomly in the population.*

Marginal Effects and Cross Elasticities

V_j denotes the linear index

$$\begin{aligned}\frac{\partial s_j}{\partial p_j} &= \frac{\partial (e^{V_j} / \sum_l e^{V_l})}{\partial p_j} \\ &= [e^{V_j} / \sum_l e^{V_l}] \frac{\partial V_j}{\partial p_j} - [e^{V_j} / (\sum_l e^{V_l})^2] e^{V_l} \frac{\partial V_j}{\partial p_j} \\ &= \frac{\partial V_j}{\partial p_j} (s_j - s_j^2) \\ &= \frac{\partial V_j}{\partial p_j} s_j (1 - s_j)\end{aligned}$$

Own Price Elasticities

Elasticity of demand with respect to p_j

$$\begin{aligned}E_{jp_j} &= \frac{\partial s_j}{\partial p_j} \frac{p_j}{s_j} \\ &= \frac{\partial V_j}{\partial p_j} s_j (1 - s_j) \frac{p_j}{s_j} \\ &= \frac{\partial V_j}{\partial p_j} p_j (1 - s_j)\end{aligned}$$

If representative utility V_j is linear in p_j then

$$E_{jp_j} = \beta_p p_j (1 - s_j) \quad (5)$$

Cross Marginal Effects

$$\begin{aligned}\frac{\partial s_j}{\partial p_k} &= \frac{\partial(e^{V_j}/\sum_l e^{V_l})}{\partial p_k} \\ &= -[e^{V_j}/(\sum_l e^{V_l})^2]e^{V_{ih}}\frac{\partial V_{ih}}{\partial p_k} \\ &= -\frac{\partial V_{ih}}{\partial p_k}s_j s_k\end{aligned}$$

If representative utility V_j is linear in p_j then the cross marginal effect is

$$-\beta_p p_j s_j s_k.$$

Cross Elasticities

Denote E_{jp_k} as the cross-elasticity of demand with respect to p_k

$$\begin{aligned}E_{jp_k} &= \frac{\partial s_j}{\partial p_k}\frac{p_k}{s_j} \\ &= -\frac{\partial V_{ih}}{\partial p_k}s_j s_k[\frac{p_k}{s_j}] \\ &= -\frac{\partial V_{ih}}{\partial p_k}s_k p_k \\ &= -\beta_p s_k p_k \qquad (6)\end{aligned}$$

and (3) follows if representative utility V_j is linear in p_j.

Summary:

$$\eta_{jk} = \frac{\partial s_j}{\partial p_k}\frac{p_k}{s_j} = \left\{ \begin{array}{ll} -\beta_p p_j(1-s_j) & j=k \\ \beta_p p_k s_k & \text{else} \end{array} \right\}$$

Remark

For any set of J goods the true **own-price elasticities** *can take on up to J arbitrarily different values.*
In the MNL model the specific functional form generates tractability but constrains elasticities to vary systematically as a function of prices and market shares, i.e. $E_j p_j = \beta_p p_j(1-s_j)$.

Remark

Likewise true cross-elasticities among J products can take up to $J(J-1)$ different values.

In the extreme case considered with the functional form of the MNL model, cross-elasticities can take on a maximum of J values; these values are again constrained to vary systematically with prices and market shares.

The cost of estimating a single parameter to fit a $J \times J$ matrix of own and cross-price elasticities of demand

Two Specific problems:

own elasticities In case of a large number of heterogeneous products market share are small.

$\beta_p(1 - s_k)$ is close to β_p. Therefore own-price elasticities are proportional and close to own price.

Implications: a lower elasticity (absolute value) implies that in standard pricing models there will be a higher markup for lower-price brands.

This is a direct implication of the logit functional form.

Logit | Heterogenous Products and Restrictive Elasticities

Cross-price elasticities

$$\eta_{jk} = \frac{\partial s_j}{\partial p_k}\frac{p_k}{s_j} = \beta_p p_k s_k \qquad j \neq k.$$

Logit cross-price elasticities implies that if two products have similar market share - say l, m, this substitution from k following a change in p_k will be the same.

Logit therefore restricts substitution patterns towards other products to be proportional to market share, independent of the 'distance' between products in the space of attributes.

Remark

The problem of unobserved product characteristics creates a problem for estimation given that endogeneity appears in a non-linear model. Berry (1994) solutions is to base estimation on the following relationship

$$s_j = \varsigma_j(\mathbf{v}, p, \xi; \boldsymbol{\theta})$$

s_j $(\varsigma_j(.))$ observed (predicted shares).

A Logit Simplification:
Distribution of *consumer* unobservables known (ε_{ij}). Let mean utility be given by α_j

$$s_j = \varsigma_j(\boldsymbol{\alpha}), \quad j = 1, ..., J \tag{7}$$

For true value of $\boldsymbol{\alpha}$, J equations in (7) hold exactly.

Berry (1994) exploits this exact fit to develop an estimation

If given an assumed distribution for **tau** and the market shares, the vector valued equations $\mathbf{s} = \varsigma(\boldsymbol{\alpha})$ can be inverted then

$$\boldsymbol{\alpha} = \varsigma^{-1}(s). \tag{8}$$

the observed market shares and the distribution of consumer unobservables uniquely determine $\boldsymbol{\alpha}$

Proposition

For every possible vector $\mathbf{s}$ $\exists\ \alpha \in \mathbb{R}^J$ *ST* $\mathbf{s} = \varsigma(\boldsymbol{\alpha})$.

Proof See Appendix in Berry(1994)
Market shares determined by no other unknowns outside of vector $\boldsymbol{\alpha}$, given distribution of consumer unobservables known.

Estimator: standard IV

Instruments: Later.

$$\varsigma_j(\boldsymbol{\alpha}_s) = e^{\alpha_j} / \sum_{k=1}^{J} e^{\alpha_k}$$
$$\ln(s_j) - \ln(s_o) = \alpha_j \equiv \Sigma\omega_{jl} v_{jl} - \omega_p v_p + \xi_j$$

Issues:

1. Outside good s_o, or if not outside good an arbitrary element of the choice set.
2. Density of τ known, no unknown parameters.
3. Density of τ depends on a vector of unknown parameters to be estimated.

- Data on sales of over-the-counter pain medication for 3 brands (Tylenol, Advil and Bayer) in Chicago supermarkets has been collected for a number of stores over a xx week period.
- Available information includes the number of customers, product sales (for 3 package sizes - 25, 10, and 100), retail prices, and wholesale prices (the retailer's costs).
- Data on a generic store brand for 50, 100 tablets is also available.
- We also have access to demographic data on *income distributions* in the store regions.
  Assumption: prices are set at the brand (firm) level, where each brand (Tylenol, Advil, Bayer, Store) sets price to maximise its total profits.

A Logit Model Consider the utility function for product j in store-week t for consumer i.

$$U_{ijt} = \mathbf{x}_{jt}\boldsymbol{\beta} + \omega_p p_{jt} + \xi_{jt} + \epsilon_{ijt}$$

ϵ_{ijt} is an i.i.d. logit draw, p_{jt} denotes price, $\mathbf{x}_{jt}$ are observed product characteristics

ξ_{jt} are unobserved product characteristics.

The mean utility from product j in week t is

$$\alpha_{jt} = \mathbf{x}_{jt}\boldsymbol{\beta} + \omega_p p_{jt} + \xi_{jt}. \tag{9}$$

A random coefficient model

$$U_{ijt} = \mathbf{x}_{jt}\boldsymbol{\beta} + \beta_{iB} B_{jt} + \omega_{p,i} p_{jt} + \xi_{jt} + \epsilon_{ijt},$$

random coefficients are $\beta_{iB} = \sigma_B v_i \sim N(0,1)$;

$\omega_{p,i} = \omega_p + \sigma_I I_i$

I_i the observed income.

Collecting the terms that represent the mean utility of product j, we write the model as

$$U_{ijt} = \alpha_{jt} + \sigma_B v_i B_{jt} + \sigma_I I_i p_{jt} + \epsilon_{ijt},$$

Mean utility from product j in week t is

$$\alpha_{jt} = \mathbf{x}_{jt}\beta + \omega_p p_{jt} + \xi_{jt}.$$

Identifying assumption

Identifying assumption: $E[\xi_{jt}^{0\prime}\mathbf{z}_{jt}] = 0\ \forall j, t$ for instruments $\mathbf{z}_{jt}$ and true unobservables $\xi_{jt}^{0\prime}$.

We don't know true unobservables so we use estimates $\xi_{jt}^{\prime}(\boldsymbol{\theta})$

Parameter estimates minimize standard GMM criterion function

$$\widehat{\boldsymbol{\theta}} = \underset{\theta}{\text{argmin}} \xi^{\prime}\mathbf{z}(\mathbf{z}^{\prime}\mathbf{z})^{-1}\mathbf{z}^{\prime}\xi \tag{10}$$

Two things needed: $\mathbf{z}$ and ξ.

Calculating ξ

1. Given expression for the mean utility from product j in week t then
$$\xi_{jt} = \alpha_{jt} - \mathbf{x}_{jt}\beta + \omega_p p_{jt} \tag{11}$$
$\mathbf{x}$ and p are data and following Berry (1994), we know there is a unique vector $\varsigma(\boldsymbol{\alpha}_{jt})$ that solves $s_{jt} = \varsigma(\boldsymbol{\alpha}_{jt}) \quad \forall j, t$ - i.e. observed = predicted shares.
2. The contraction mapping that solves for $\boldsymbol{\alpha}$
$$\alpha_{jt}^{l+1} = \alpha_{jt}^{l} + \log(s_{jt}) - \log(\varsigma(\boldsymbol{\alpha}_{jt}^{l}, \boldsymbol{\theta})) \tag{12}$$
3. Need $\varsigma(\boldsymbol{\alpha}_{jt}^{l}, \boldsymbol{\theta})$ for given $\boldsymbol{\theta}$ – l indexes iteration of optimisation routine.
4. In this model $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \omega_p, \sigma_\beta, \sigma_I\}$ and unlike the logit model the contraction mapping has no analytic solution given that there is no chosen form expression for $\varsigma(\boldsymbol{\alpha}_{jt}^{l}, \boldsymbol{\theta})$.

Market share implied by model is the share of consumers who choose good j at time t - assuming consumers maximise utility

$$\varsigma(\boldsymbol{\alpha}^{I}_{jt}, \boldsymbol{\theta}) = \int\int \frac{\exp(\alpha_j + \sigma_B v_i B_j + \sigma_I I_i p_j)}{1 + \Sigma_m \exp(\alpha_m + \sigma_B v_i B_m + \sigma_I I_i p_m)} df(v) df(I) \tag{13}$$

No analytic solution:

Simulate nS "individuals" (for each mkt t) each characterised by a pair (τ_i, I_i) drawn from the appropriate distributions. We then use a simulator:

$$\varsigma(\boldsymbol{\alpha}^{I}_{jt}, \boldsymbol{\theta}) = \frac{1}{nS} \sum_{i=1}^{nS} \frac{\exp(\alpha_j + \sigma_B v_i B_j + \sigma_I I_i p_j}{1 + \Sigma_m \exp(\alpha_m + \sigma_B v_i B_m + \sigma_I I_i p_m)} \tag{14}$$

nS elements of the sum are the probability that i chooses j in market t

- Given $\varsigma(\boldsymbol{\alpha}_{jt}, \boldsymbol{\theta})$ and "inversion" to get $\boldsymbol{\alpha}_{jt}(s, \boldsymbol{\alpha})$ we again have a linear model with residuals $\xi^*_{jt} = \alpha_{jt}(.) - \mathbf{x}_{jt}\beta + \alpha p_{jt}$.
- For true values of parameters and true market share $\xi^*_{jt} = \xi_{jt}$.
- Now we can use GMM, interacting ξ^*_{jt} with the instruments - the exogenous elements of $\mathbf{x}$ and instrumental variables.
- These are typically attributes of other products by the same firm or of competing products - assuming Nash prices.

For fixed market t the price elasticities for product j with respect to a price change in product k in the same market, η_{jk}, $\frac{\partial s_j/s_j}{\partial p_k/p_k}$ is given by

$$
\begin{aligned}
&= \left(\frac{p_k}{s_j}\right)\frac{\partial}{\partial p_k}\left(\int\int \frac{\exp(\alpha_j + \sigma_B v_i B_j + \sigma_I I_i p_j}{1 + \Sigma_m \exp(\alpha_m + \sigma_B v_i B_m + \sigma_I I_i p_m)} df(v) df(I)\right) \\
&\approx \frac{p_k}{s_j}\sum_{i=1}^{ns}(\alpha + \sigma_I I_i)(-s_{ij}s_{ik} + \mathbf{1}_{\{k=j\}}s_{ik})
\end{aligned}
$$

(t subscript dropped for convenience).