Three-Stage Semi-parametric Estimation of T-Copulas: Asymptotics, Finite-Samples Properties and Computational Aspects

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Summary Genest et al. (1995) proposed a two-stages semi-parametric estimation procedure for bivariate Archimedean copulas. A three stage semi-parametric estimation method based on Kendall's tau has been recently proposed in the financial literature to estimate the Student's T copula, too. Its major advantage is to allow for greater computational tractability when dealing with high dimensional issues, where two-stage procedures are no more a viable choice. We develop the asymptotic properties of this methodology and we examine its finite-sample behavior via simulations. We then analyze the pros and cons of this methodology in terms of numerical convergence and positive definiteness of the estimated T-copula correlation matrix.

Keywords: *Copulas, Maximum Likelihood, Two-stage estimation, Three-stage estimation, Semi-parametric estimation*

1. INTRODUCTION

The theory of copulas dates back to Sklar (1959), but its application in financial modelling is far more recent and dates back to the late '90, instead.

A copula is a function that embodies all the information about the dependence structure between the components of a random vector. When it is applied to marginal distributions which do not necessarily belong to the same distribution family, it results in a proper multivariate distribution. As a consequence, this theory enables us to incorporate a flexible modelling of the dependence structure between different variables, while allowing them to be modelled by different marginal distributions.

A semi-parametric estimation procedure for bivariate Archimedean copulas was initially proposed by Genest et al. (1995), while a method-of-moment estimation procedure based on Kendall's tau was discussed by Genest and Rivest (1993). However, recent empirical financial literature combined these two procedures to estimate the elliptical Student's T copula, too (see Bouyé et al. (2001), Marshall and Zeevi (2002), Cherubini et al. (2004), McNeil et al. (2005)). This new methodology entails three stages: a first stage where a non-parametric estimation of the marginal empirical distribution functions is performed; a second stage where a method-of-moment estimator based on Kendall's tau is used for the T-copula correlation matrix, and a third stage which considers Maximum Likelihood methods for the degrees of freedom ν, instead.

This has become a common procedure among financial practitioners when dealing with high-dimensional portfolios and standard Maximum Likelihood methods cannot be used. Nevertheless, neither the asymptotic properties of such a multi-step procedure have been studied, nor the finite-sample properties. The former are important in order to build statistical tests for the structural parameters, while the latter are of interest to compare the small sample efficiency and bias of this estimator with other estimators, such as the one-stage ML estimator.

What we do in this paper is to provide the asymptotic distribution for this recent semi-parametric method, and use simulations with different Data Generating Processes to examine the behavior of this estimator in small samples. The Monte Carlo study shows that this semi-parametric estimator is more efficient and less biased than the one-stage ML estimator when small samples and copulas with low degrees of freedom are of concern.

We then analyze the pros and cons of this methodology in terms of numerical convergence and positive definiteness of the estimated T-copula correlation matrix. When small samples are of concern and ν is high, the number of times when the numerical maximization of the log-likelihood fails to converge is much higher for the ML method than for the KME-CML method. Yet, while the coverage rates at the 95% level for the ML estimates for ν do not show any particular bias or trend, the KME-CML estimates show very low rates when ν becomes close to 30 and the correlations are not too strong. However, this drop in the coverage rates is large with bivariate t-copulas, only, while it is much lower when dealing with higher dimensional t-copulas, which is the usual case for real managed financial portfolios. Besides, both the ML and the KME-CML methods show high mean and median biases for the estimated correlations when the true ones are close to zero. Nevertheless, the effects on the coverage rates for the correlations are rather limited in this case. Interestingly, when the sample dimension increases, the previous biases decrease but they still remain quite high for the ML method when ν is low.

Finally, we show that the eigenvalue method by Rousseeuw and Molenberghs (1993) has to be used to obtain a positive definite correlation matrix only when dealing with very small samples ($n \leq 100$) and when the true underlying process has the lowest eigenvalue close to zero. This fix induces a positive bias in the estimate of ν, but the effects on the coverage rates are rather limited. Besides, the number of times when this method has to be used quickly decreases when ν increases.

The rest of the paper is organized as follows. We review semi-parametric copula estimators in Section 2. We introduce a recent semi-parametric estimation method of Student's T copulas based on Kendall's tau in Section 3 and we provide its asymptotics in Section 4. We present in Section 5 the results of a Monte Carlo study of the small sample properties of this estimator, while we conclude in Section 6.

2. SEMI-PARAMETRIC COPULA ESTIMATORS: A REVIEW

The study of copulae has originated with the seminal papers by Hoeffding (1940) and Sklar (1959) and has seen various applications in the statistics and financial literature. Examples include Clayton (1978), Schweizer and Wolff (1981), Genest and Rivest (1986a,b), Genest and Rivest (1993), Rosenberg (1998, 2003), Bouyé et al. (2001), Embrechts et al. (2003a,b), Patton (2004, 2005), Granger et al. (2006), Fantazzini (2007), Dalla Valle et al. (2007). For more details, we refer the interested reader to the recent methodological overviews by Joe (1997) and Nelsen (1999), while Cherubini et al. (2004) and McNeil et al. (2005) provide a comprehensive and detailed discussion of copula techniques for financial applications.

Genest et al. (1995) were the first to analyze a semi-parametric estimation of a bivariate Copula with i.i.d. observations and to develop its asymptotic properties Their Canonical Maximum Likelihood (CML) method differs from full Maximum Likelihood methods because no assumptions are made about the parametric form of the marginal distributions. We now review this semi-parametric method since it constitutes the building block for our following analysis.

Let us consider a multivariate random sample represented by the time series $X = (x_{1t}, \ldots, x_{nt})$, and $t = 1, \ldots, T$, where n stands for the number of variables included and T represents the number of observations available. Let f_h be the density of the joint distribution of X. Then, by using Sklar's theorem (1959) and the relationship between the distribution and the density function we have:

$$f_h(x_i; \alpha_1, \ldots, \alpha_n, \gamma) = c(F_1(x_{1;}\alpha_1), \ldots, F_1(x_{n;}\alpha_n); \gamma) \cdot \prod_{i=1}^{n} f_i(x_i; \alpha_i) \tag{1}$$

where f_i is the univariate density of the marginal distribution F_i, c is the copula density, $\alpha_i, i = 1, \ldots, n$ is the vector of parameters of the marginal distribution F_i, while γ is the vector of the copula parameters. The CML estimation process is performed in two steps:

Definition 2.1. (CML copula estimation) *1 Transform the dataset* $(x_{1t}, x_{2t}, \ldots, x_{nt})$, $t = 1, \ldots, T$ *into uniform variates* $(\hat{u}_{1t}, \hat{u}_{2t}, \ldots, \hat{u}_{nt})$ *using the empirical distributions* $F_{iT}(\cdot)$ *defined as follows:*

$$F_{iT}(x_{it}) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{\{x_{it} \leq \, x_i\}}, \; i = 1 \ldots n \tag{2}$$

where $\mathbb{1}_{\{x \leq \bullet\}}$ *represents the indicator function.*

2 Estimate the copula parameters by maximizing the log-likelihood:

$$\hat{\gamma}_{\, CML} = \arg\max \sum_{t=1}^{T} \log(c(F_{1T}(x_{1t}), \ldots, F_{nT}(x_{nt})); \gamma) \tag{3}$$

To assess the asymptotic properties of this estimator, we consider the bivariate case for sake of simplicity (see Genest et al. (1995), § 4, for the multivariate case). Let $l(\hat{u}_1, \hat{u}_2, \gamma) = \log c(\hat{u}_1, \hat{u}_2; \gamma)$, and use indices $1, 2, \gamma$ to denote partial derivatives of l with respect to u_1, u_2, γ, respectively. Besides, name S_T and H_T the first and second derivative of l with respect to γ, respectively:

$$\begin{aligned} S_T &= \frac{1}{T} \sum_{t=1}^{T} l_\gamma \left(F_{1T}(x_{1t}), F_{2T}(x_{2t}); \gamma\right) \\ H_T &= \frac{1}{T} \sum_{t=1}^{T} l_{\gamma,\gamma} \left(F_{1T}(x_{1t}), F_{2T}(x_{2t}; \gamma)\right) \end{aligned} \tag{4}$$

Since $\hat{\gamma}_{\, CML}$ solves

$$\frac{1}{T} \frac{\partial l(\cdot; \gamma)}{\partial \gamma} = \frac{1}{T} \sum_{t=1}^{T} l_\gamma \left(F_{1T}(x_{1t}), F_{2T}(x_{2t}); \gamma\right) = S_T = 0 \tag{5}$$

if we take a first order Taylor expansion around $\hat{\gamma}_{CML}$, we get

$$0 = \frac{1}{T}\frac{\partial l(\cdot;\gamma)}{\partial\gamma}\|_{\gamma=\hat{\gamma}_{CML}} \simeq S_T + H_T(\hat{\gamma}_{CML}-\gamma)$$
$$\sqrt{T}(\hat{\gamma}_{CML}-\gamma) \simeq \sqrt{T}\frac{S_T}{-H_T} \tag{6}$$

Under the regularity conditions reported in Ruymgaart et al. (1972), Genest et al. (1995) show in Proposition 2.1 that $\hat{\gamma}_{CML} \stackrel{a.s.}{\to} \gamma_0$ as $n \to \infty$. We report those steps of the proof that will be useful for our subsequent analysis. See Genest et al. (1995) for the full proof.

Let us consider a statistics of the form

$$R_T = \frac{1}{T}\sum_{t=1}^{T} J\left(F_{1T}(x_{1t}), F_{2T}(x_{2t})\right) = \int J(u_1,u_2)d\mathfrak{C}(u_1,u_2) \tag{7}$$

where $F_{iT}(x_{1t})$ are the empirical distribution functions, $\mathfrak{C}(u_1,u_2)$ is the empirical copula function of the bivariate sample, given by

$$\mathfrak{C}_t(u_1,u_2) = \frac{1}{T}\sum_{t=1}^{T}\mathbb{1}_{F_1(x_{1t})\le u_1}\mathbb{1}_{F_2(x_{2t})\le u_2}\ , t=1,\ldots,T \tag{8}$$

and $J(u_1,u_2)$ is a continuous function from $(0,1)^2$ into $\mathbb{R}$. These kind of statistics are usually known as multivariate rank order statistics.

Genest et al. (1995) show in Proposition A.1 that under certain regularity conditions,

$$R_T \stackrel{a.s.}{\to} \mu, \quad where$$
$$\mu \ = \ \mathrm{E}\left[J\left(F_1(X_1), F_2(X_2)\right)\right] = \int J(u_1,u_2)dC(u_1,u_2) \tag{9}$$

Now take J equal to $l_\gamma(\cdot)$: then by the theorem of Consistency of Maxima (see Ruud 2000, § 15.8, for example), it follows that $\hat{\gamma}_{CML} \stackrel{a.s.}{\to} \gamma_0$ as $n \to \infty$.

Under the same assumptions, Genest et al. (1995) show that the semiparametric estimator $\hat{\gamma}_{CML}$ has the following asymptotic distribution:

$$\sqrt{T}(\hat{\gamma}_{CML}-\gamma_0) \stackrel{d}{\to} \mathrm{N}\left(0,\frac{\sigma^2}{h^2}\right) \tag{10}$$

where

$$\sigma^2 = var[l_\gamma\left(F_1(X_1),F_2(X_2);\gamma\right) + W_1(X_1) + W_2(X_2)] \tag{11}$$
$$W_i(x_i) = \int \mathbb{1}_{F_i(X_i)\le u_i}\ l_{\gamma,i}\left(u_1,u_2;\gamma\right)c(u_1,u_2;\gamma)du_1du_2 \quad i=1,2 \tag{12}$$
$$h = -E[l_{\gamma,\gamma}\left(F_1(x_{1t}),F_2(x_{2t});\gamma\right)] \tag{13}$$

and where $W_i(x_{it})$ can have this alternative expression too, upon integrating by parts with respect to u_i $(i=1,2)$:

$$W_i(x_{it}) = -\int \mathbb{1}_{F_i(X_i)\le u_i}\ l_\gamma\left(u_1,u_2;\gamma\right)l_i\left(u_1,u_2;\gamma\right)c(u_1,u_2;\gamma)du_1du_2 \tag{14}$$

In order to prove this result, Genest et al. (1995) show in Proposition A.1 that

$$\sqrt{T}(R_T - \mu) \xrightarrow{d} N(0, \sigma^2),$$
$$\sigma^2 = var\left[J\left(F_1(X_1), F_2(X_2)\right) + \sum_{i=2}^{2} \int \mathbb{1}(X_i \leq x_i) J_i\left(F_1(X_1), F_2(X_2)\right) dF(x_1, x_2)\right] \tag{15}$$

following the argument used by Ruymgaart et al. (1972) to prove their theorem 2.1, and where R_T was defined before. Take J equal to l_γ, that is $R_T = S_T$, apply Sklar's Theorem to $dF(x_1, x_2)$ and perform a change of variables, so that (11) and (12) follow. Furthermore, take J equal to $l_{\gamma,\gamma}(\cdot)$ this time: since $R_T \overset{a.s.}{\to} \mu$, we have that $H_T \overset{a.s.}{\to} h$. Finally, apply these asymptotic results to (6) and the asymptotic normality follows.

3. THE THREE-STAGE KME-CML METHOD

Since the seminal work by Genest et al. (1995), it has become common practice to use semi-parametric methods with high-dimensional elliptical Student's T copulas too; see Cherubini et al. (2004) and McNeil et al. (2005) for a detailed discussion about their financial applications. Particularly, after the marginal empirical distribution functions are computed in a first stage, the correlation matrix is estimated in a second stage using a method-of-moment estimator based on Kendall's tau, while the degrees of freedom are estimated in a third stage using Maximum Likelihood methods. Despite the widespread use of this procedure, its asymptotic properties have not been developed yet. Before doing that and properly define this estimation method, we have to introduce the Kendall's tau, its relation with linear correlation for elliptical copulas and the method-of-moment estimator based on it.

A method for estimating copula parameters based on Kendall's tau has been suggested in Genest and Rivest (1993), Lindskog (2000), Lindskog et al. (2002), Cherubini et al. (2004) and McNeil et al. (2005). This dependence measure can be defined as follows:

Definition 3.1. (Kendall's tau) *If we have $(X_1; X_2)$ and $(\tilde{X}_1; \tilde{X}_2)$ two independent and identically distributed random vectors, the population version of Kendall's tau $\tau(X_1; X_2)$ is (see Kruskal (1958)):*

$$\tau(X_1, X_2) = \mathrm{E}\left[sign\left((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2)\right)\right] \tag{16}$$

The generalization of Kendall's tau to $n > 2$ dimensions is analogous to the procedure for linear correlation, where we have a $n \times n$ matrix of pairwise correlations (see Clemen and Jouini (1996)). The main properties of this dependence measure and relative proofs are reported in Embrechts et al. (2002). Besides, Kendall's tau can be expressed in terms of copulas, thus simplifying calculus, see e.g. Nelsen (1999), p.127.

$$\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1 \tag{17}$$

Evaluating Kendall's tau requires the evaluation of a double integral and for elliptical copulas like the T-copula this is not an easy task: this problem was solved by Lindskog, McNeil, and Schmock (2002) who proved that Kendall's tau for elliptical distributions is given by

$$\tau(X_1, X_2) = \frac{2}{\pi} \arcsin \rho_{X_1 X_2} \tag{18}$$

where $\rho_{X_1 X_2}$ is the copula correlation parameter.

The Kendall's tau moment estimator can now be defined:

Definition 3.2. (Copula estimation with Kendall's tau) *Let us consider the population version of Kendall's τ (16) and its relationship with copula parameters (17) to build a moment function of the type*

$$\mathrm{E}\left[\psi\left(X_1, X_2; \gamma_0\right)\right] = 0 \tag{19}$$

Then we can construct an empirical estimate of the Kendall's tau pairwise correlation matrix and use relationship (17) to infer an estimate of the relevant parameters of the copula.

This is a method of moments estimate because the true moment (16) is replaced by its empirical analogue,

$$\binom{T}{2}^{-1} \sum_{1 \le t < s < T} \mathrm{sign}\left((x_{1,t} - \tilde{x}_{1,s})(x_{2,t} - \tilde{x}_{2,s})\right) \tag{20}$$

and (17) is then used to estimate the copula parameters. For example, when using elliptical copulas and eq. (18), this moment function becomes

$$\mathrm{E}\left[\psi\left(X_1, X_2; \rho_{X_1,X_2}\right)\right] = \mathrm{E}\left[\rho_{X_1,X_2} - \sin(\pi\tau(X_1, X_2)/2)\right] = 0 \tag{21}$$

where $\rho_{X_1 X_2}$ is the copula correlation parameter.

There are cases when the copula parameter vector has different kinds of parameters and only some of them can be expressed as a function of the Kendall's tau. This is the case for the *T-copula* which is parameterized by the correlation matrix Σ and the degrees of freedom ν, but only the former has a direct relationship with Kendall's tau. This is the copula of the multivariate Student's T-distribution and we can derive its density function by using Sklar's theorem (1959) and the relationship between the distribution and the density function:

$$c(t_\upsilon(x_1), \dots, t_\upsilon(x_n)) = \frac{f^{Student}(x_1, \dots, x_n)}{\prod_{i=1}^{n} f_i^{Student}(x_i)} =$$

$$= |\Sigma|^{-1/2} \frac{\Gamma\left(\frac{\upsilon+n}{2}\right)}{\Gamma\left(\frac{\upsilon}{2}\right)} \left[\frac{\Gamma\left(\frac{\upsilon}{2}\right)}{\Gamma\left(\frac{\upsilon+1}{2}\right)}\right]^n \frac{\left(1 + \frac{\zeta'\Sigma^{-1}\zeta}{\upsilon}\right)^{-\frac{\upsilon+n}{2}}}{\prod_{i=1}^{n}\left(1 + \frac{\zeta_i^2}{2}\right)^{-\frac{\upsilon+1}{2}}} \tag{22}$$

where $\zeta = (t_\upsilon^{-1}(u_1), \dots, t_\upsilon^{-1}(u_n))^\top$ is the vector of univariate Student's T inverse distribution functions, ν are the degrees of freedom, $u_i = t_\nu(x_i)$ is the univariate Student's T cumulative distribution function, while Σ is the correlation matrix.

In this situation, Bouyé et al. (2001) and McNeil et al. (2005) have suggested the following estimation procedure:

Definition 3.3. (Three-stage KME - CML copula estimation)

1 *Transform the dataset* $(x_{1t}, x_{2t}, \ldots x_{nt})$*, into uniform variates* $(F_{1T}(x_{1t}), F_{2T}(x_{2t}), \ldots, F_{nT}(x_{nt}))$*, using the empirical distribution function.*

2 *Collect all pairwise estimates of the sample Kendall's tau given by (20) in an empirical Kendall's tau matrix* $\hat{R}^{\tau}$ *defined by* $\hat{R}^{\tau}_{jk} = \tau(F_{jT}(X_j), F_{kT}(X_k))$*, and then construct the correlation matrix using this relationship* $\hat{\Sigma}_{j,k} = \sin(\frac{\pi}{2}\hat{R}^{\tau}_{j,k})$*, where the estimated parameters are the* $q = n \cdot (n-1)/2$ *correlations* $[\hat{\rho}_1, \ldots \hat{\rho}_q]'$*. Since there is no guarantee that this componentwise transformation of the empirical Kendall's tau matrix is positive definite, when needed,* $\hat{\Sigma}$ *can be adjusted to obtain a positive definite matrix using a procedure such as the eigenvalue method of Rousseeuw P. and G. Molenberghs (1993) or other methods.*

3 *Look for the CML estimator of the degrees of freedom* $\hat{\nu}_{CML}$ *by maximizing the log-likelihood function of the T-copula density:*

$$\hat{\nu}_{CML} = \arg\max \sum_{t=1}^{T} \log c_{T-copula}(F_{1T}(x_{1t}), \ldots, F_{nT}(x_{nt}); \hat{\Sigma}, \nu) \tag{23}$$

4. ASYMPTOTIC PROPERTIES OF THE THREE-STAGE METHOD

The second step in the previous definition 3.3 corresponds to a method-of-moments estimation based on q moments and Kendall tau rank correlations estimated with empirical distribution functions. We can therefore generalize eq. (21) and build a $q \times 1$ moments vector ψ for the parameter vector $\theta_0 = [\rho_1, \ldots, \rho_q]'$ as reported below:

$$\psi(F_1(X_1), \ldots, F_n(X_n); \theta_0) = \begin{pmatrix} \mathrm{E}[\psi_1(F_1(X_1), F_2(X_2);\ \rho_1)] \\ \vdots \\ \mathrm{E}[\psi_q(F_{n-1}(X_{n-1}), F_n(X_n);\ \rho_q)] \end{pmatrix} = 0 \tag{24}$$

Then these theorems follow[1]:

Theorem 4.1. (Consistency of $\hat{\theta}$) *Let assume that* $(x_{1t}, \ldots, x_{nt})$ *are i.i.d random variables with dependence structure given by* $c(u_{1,t}, \ldots, u_{n,t}; \Sigma_0, \nu_0)$*. Suppose that*

(i) *the parameter space* Θ *is a compact subset of* $\mathbb{R}^q$*,*

(ii) *the q-variate moment vector* $\psi(F_1(X_1), \ldots, F_n(X_n); \theta_0)$ *is continuous in* θ_0 *for all* X_i*,*

(iii) $\psi(F_1(X_1), \ldots, F_n(X_n); \theta)$ *is measurable in* X_i *for all* θ *in* Θ*,*

(iv) $\mathrm{E}[\psi(F_1(X_1), \ldots, F_n(X_n); \theta)] \neq \mathbf{0}$ *for all* $\theta \neq \theta_0$ *in* Θ*,*

(v) $\mathrm{E}[\sup_{\theta \in \Theta} \|\psi(F_1(X_1), \ldots, F_n(X_n); \theta)\|] < \infty$*,*

Then $\hat{\theta} \xrightarrow{p} \theta_0$ *as* $n \to \infty$*.*

Theorem 4.2. (Consistency of $\hat{\nu}_{CML}$) *Let the assumptions of the previous theorem hold, as well as the regularity conditions reported in Proposition A.1 in Genest et al.(1995). Then* $\hat{\nu}_{CML} \xrightarrow{p} \nu_0$ *as* $n \to \infty$*.*

The asymptotic normality is not straightforward, since we use a 3-step procedure where we perform a different kind of estimation at the second and third stage. A possible

[1]The proofs are reported in Appendix A

solution is to consider the CML used in the 3^{rd} stage as a special method-of-moment estimator. Just note that the CML estimator is defined by the derivative of the log-likelihood function with respect to the degrees of freedom:

$$\frac{\partial l(\cdot;\nu)}{\partial \nu} = \sum_{t=1}^{T} l_{\nu}\left(F_{1T}(x_{1,t}), \ldots, F_{nT}(x_{n,t}); \hat{\Sigma}\ , \hat{\nu}\right) = 0 \tag{25}$$

Dividing both sides by T yields the definition of the method of moments estimator:

$$\frac{1}{T}\sum_{i=1}^{T} l_{\nu}\left(F_{1T}(x_{1,t}), \ldots, F_{nT}(x_{n,t}); \hat{\Sigma}\ , \hat{\nu}\right) = \frac{1}{T}\sum_{i=1}^{n} \psi_{\nu}(F_{1T}(x_{1,t}), \ldots, F_{nT}(x_{n,t}); \hat{\Sigma}\ , \hat{\nu}) = 0$$

Thus, the CML estimator is a simple MM estimator with the score as its moment function, where the sample mean of the score is equal to the population mean of the score. We can now apply the method-of-moments asymptotics together with multivariate rank statistics.

Let define the sample moments vector $\Psi_{\ KME-CML}$ for the parameter vector $\hat{\Xi} = [\hat{\rho}_1, \ldots \hat{\rho}_q, \hat{\nu}]'$ as follows:

$$\Psi_{\ KME-CML}\left(F_{1T}(x_{1,t}), \ldots, F_{nT}(x_{n,t}); \hat{\Xi}\right) = \begin{pmatrix} \frac{1}{T}\sum_{i=1}^{T} \psi_1\left(F_{1T}(x_{1,t}), F_{2T}(x_{2,t});\ \hat{\rho}_1\right) \\ \vdots \\ \frac{1}{T}\sum_{i=1}^{T} \psi_q\left(F_{n-1,T}(x_{n-1,t}), F_{nT}(x_{n,t});\ \hat{\rho}_q\right) \\ \frac{1}{T}\sum_{i=1}^{T} \psi_{\nu}\left(F_{1T}(x_{1,t}), \ldots, F_{nT}(x_{n,t}); \hat{\Sigma}, \hat{\nu}\right) \end{pmatrix} = 0$$

Let also define the population moments vector with a correction to take the non-parametric estimation of the marginals into account, together with its variance (see Genest et al. (1995), § 4):

$$\Delta_0 = \begin{pmatrix} \psi_1\left(F_1(X_1), F_2(X_2);\ \rho_1\right) \\ \vdots \\ \psi_q\left(F_{n-1}(X_{n-1}), F_n(X_n);\ \rho_q\right) \\ \psi_{\nu}\left(F_1(X_1), \ldots, F_n(X_n); \Sigma_0, \nu_0\right) + \sum_{i=1}^{n} W_{i,\nu}(X_i) \end{pmatrix} = 0 \tag{26}$$

$$\Upsilon_0 \equiv var\left[\Delta_0\right] = \mathrm{E}\left[\Delta_{\ KME-CML}\, \Delta_{\ KME-CML}\ '\right] \tag{27}$$

where

$$W_{i,\nu}(X_i) = \int \mathbb{1}_{\,F_i(X_i)\leq u_i}\ \frac{\partial^2}{\partial\nu\partial u_i}\log c(u_1, \ldots u_n) dC(u_1, \ldots u_n) \tag{28}$$

Note that the population moments used to estimate the correlations are not affected by the marginals empirical distribution functions, since the Kendall's tau is invariant under strictly increasing marginal transformations. Then this theorem follows:

Theorem 4.3. (Asymptotic Distribution 3-stages KME-CML Method) *Let the assumptions of the previous theorems hold. Assume further that* $\frac{\partial \mathbf{\Psi}_{KME-CML}(\cdot;\mathbf{\Xi})}{\partial \mathbf{\Xi}'}$ *is* $O(1)$ *and uniformly negative definite, while* $\mathbf{\Upsilon}_0$ *is* $O(1)$ *and uniformly positive definite. Then,* *the three-stages KME-CML estimator verifies the properties of asymptotic normality:*

$$\sqrt{T}(\hat{\Xi} - \Xi_0) \xrightarrow{d} \mathrm{N}\left(0, \mathrm{E}\left[\frac{\partial \Psi_{\ KME-CML}}{\partial \Xi'}\right]^{-1} \Upsilon_0 \, \mathrm{E}\left[\frac{\partial \Psi_{\ KME-CML}}{\partial \Xi'}\right]^{-1\prime}\right) \tag{29}$$

The previous asymptotic properties are still valid when dealing with multivariate heteroscedastic time series models, where one first obtains consistent estimates of the parameters of each univariate marginal time-series, and computes the corresponding residuals. These are then used to estimate the joint distribution of the multivariate error terms, which is specified using a copula. Such a result is a straightforward application of Theorems 1 and 2 in Gunky et al. (2007), who make use of, and build upon, recent elegant results of Koul and Ling (2006) and Koul (2002) for these models. Besides, we remark that essentially the same result of Gunky et al. (2007) is also used in Chen and Fan (2006) but without proofs.

THEOREM 4.4. (ASYMPTOTIC DISTRIBUTION 3-STAGES KME-CML METHOD FOR MULTIVARIATE HETEROSCEDASTIC TIME SERIES MODELS) *Let the regularity conditions (i)-(v) reported in theorem 4.1 hold, together with conditions A.1 and A.9 in Gunky et al. (2007). Then, the three-stages KME-CML estimator verifies the properties of asymptotic normality defined in (29).*

5. FINITE-SAMPLE PROPERTIES AND COMPUTATIONAL ASPECTS

In this section we present the results of a Monte Carlo study of the small-sample properties of the estimator discussed above for a representative collection of Data Generating Processes (DGPs). Furthermore, we analyze the pros and cons of this methodology in terms of numerical convergence and positive definiteness of the estimated T-copula correlation matrix.

We consider the following possible DGPs:

1 We examine the case that two variables have a bivariate Student's T copula, with the copula linear correlation ρ ranging between -0.9 to 0.9. We examine different values for the degree of freedom ν, too, ranging between 3 up to 30. The former corresponds to a case of strong tail dependence, that is there is a high probability of observing an extremely large observation on one variable, given that the other variable has yielded an extremely large observation. The latter exhibits low tail dependence, and corresponds to the case where the t-copula becomes almost indistinguishable from a Normal copula. We remark that the Student's t copula generates positive tail dependence, while the normal copula generates zero tail dependence, instead. See Cherubini et al. (2004) for more details. We consider two possible data situations: $n = 50$ and $n = 500$.

2 We examine the case that ten variables have a multivariate Student's T copula, with the copula correlation matrix equal to:

1	-0.15	-0.15	-0.15	-0.15	-0.14	-0.09	-0.03	0.05	0.13
-0.15	1	-0.15	-0.15	-0.15	-0.13	-0.08	-0.02	0.06	0.14
-0.15	-0.15	1	-0.15	-0.15	-0.12	-0.07	-0.01	0.07	0.15
-0.15	-0.15	-0.15	1	-0.15	-0.11	-0.06	0.01	0.08	0.15
-0.15	-0.15	-0.15	-0.15	1	-0.10	-0.05	0.02	0.09	0.15
-0.14	-0.13	-0.12	-0.11	-0.10	1	-0.04	0.03	0.10	0.15
-0.09	-0.08	-0.07	-0.06	-0.05	-0.04	1	0.04	0.11	0.15
-0.03	-0.02	-0.01	0.01	0.02	0.03	0.04	1	0.12	0.15
0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	1	0.15
0.13	0.14	0.15	0.15	0.15	0.15	0.15	0.15	0.15	1

Table 1. Correlation matrix T-copula with lowest eigenvalue equal to 0.0768

We choose this correlation matrix because its lowest eigenvalue is very close to zero (0.0786) and it allows us to study the effect that the eigenvalue method by Rousseeuw and Molenberghs (1993) has on the limiting distribution of the KME-CML estimator. Furthermore, we examine different values for the degree of freedom ν, too, ranging between 3 up to 30, as well as two possible data situations: $n = 50$ and $n = 500$.

3 We examine the case that ten variables have a multivariate Student's T copula, with the copula correlation matrix equal to:

1	0.21	0.33	0.22	0.36	0.30	0.37	0.34	0.31	0.47
0.21	1	0.20	0.15	0.27	0.18	0.18	0.31	0.20	0.21
0.33	0.20	1	0.16	0.32	0.28	0.40	0.33	0.17	0.42
0.22	0.15	0.16	1	0.20	0.16	0.18	0.20	0.27	0.20
0.36	0.27	0.32	0.20	1	0.32	0.33	0.55	0.33	0.35
0.30	0.18	0.28	0.16	0.32	1	0.28	0.32	0.26	0.31
0.37	0.18	0.40	0.18	0.33	0.28	1	0.35	0.23	0.40
0.34	0.31	0.33	0.20	0.55	0.32	0.35	1	0.31	0.35
0.31	0.20	0.17	0.27	0.33	0.26	0.23	0.31	1	0.30
0.47	0.21	0.42	0.20	0.35	0.31	0.40	0.35	0.30	1

Table 2. Correlation matrix T-copula - returns Dow Jones Industrial Index

This is the correlation matrix of the returns of the first 10 stocks belonging to the Dow Jones Industrial Index, observed between the 18/11/1988 and the 20/11/2003. Furthermore, we examine different values for the degree of freedom ν, too, ranging between 3 up to 30, as well as two possible data situations: $n = 50$ and $n = 500$.

The estimators considered are the 3-stage KME-CML method described in section 3, and the Maximum Likelihood estimator computed with given marginals, in order to assess the loss in efficiency associated with absence of knowledge of the marginals. We also considered the 2-stage CML method which delivered results in-between the KME-CML and ML methods, as expected. Therefore, we do not report its results for sake of interest and space.

The simulation results are organized as follows:

- *Bivariate T-copula*:
  - Figures 1-2 report the Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation and degrees of freedom parameters, respectively, for the KME-CML method, across different correlation levels and degrees of freedom parameters, as well as different data samples. Figure 3 reports the coverage rate (in %) for a large sample 95% confidence interval based on a normal

approximation for ν and ρ, together with the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom.
- Figures 4-5 and 6 report the same simulation statistics but for the ML method.

- *10-variate t-copula with correlation matrix reported in table 1*:
  - Figure 7 report the Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the KME-CML method, across different correlation levels and degrees of freedom as well as different data samples. The same statistics for the degrees of freedom are reported in figure 8, together with the % of convergence failures when maximizing the log-likelihood (23) across different degrees of freedom. Figure 8 also contains the % of times when the correlation matrix was not positive definite and the eigenvalue method was used, again across different degrees of freedom. Finally, the coverage rate for the 95% confidence intervals of the T-copula parameters based on a normal approximation are reported in figure 9.
  - Figures 10-11 and 12 report the same simulation statistics but for the ML method.

- *10-variate t-copula with correlation matrix reported in table 2*:
  - Figure 13 report the Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the KME-CML method, across different correlation levels and degrees of freedom as well as different data samples. The same statistics for the degrees of freedom are reported in figure 14, together with the % of convergence failures when maximizing the log-likelihood (23) across different degrees of freedom. Figure 14 also contains the % of times when the correlation matrix was not positive definite and the eigenvalue method was used, again across different degrees of freedom. Finally, the coverage rate for the 95% confidence intervals of the T-copula parameters based on a normal approximation are reported in figure 15.
  - Figures 16-17 and 18 report the same simulation statistics but for the ML method.

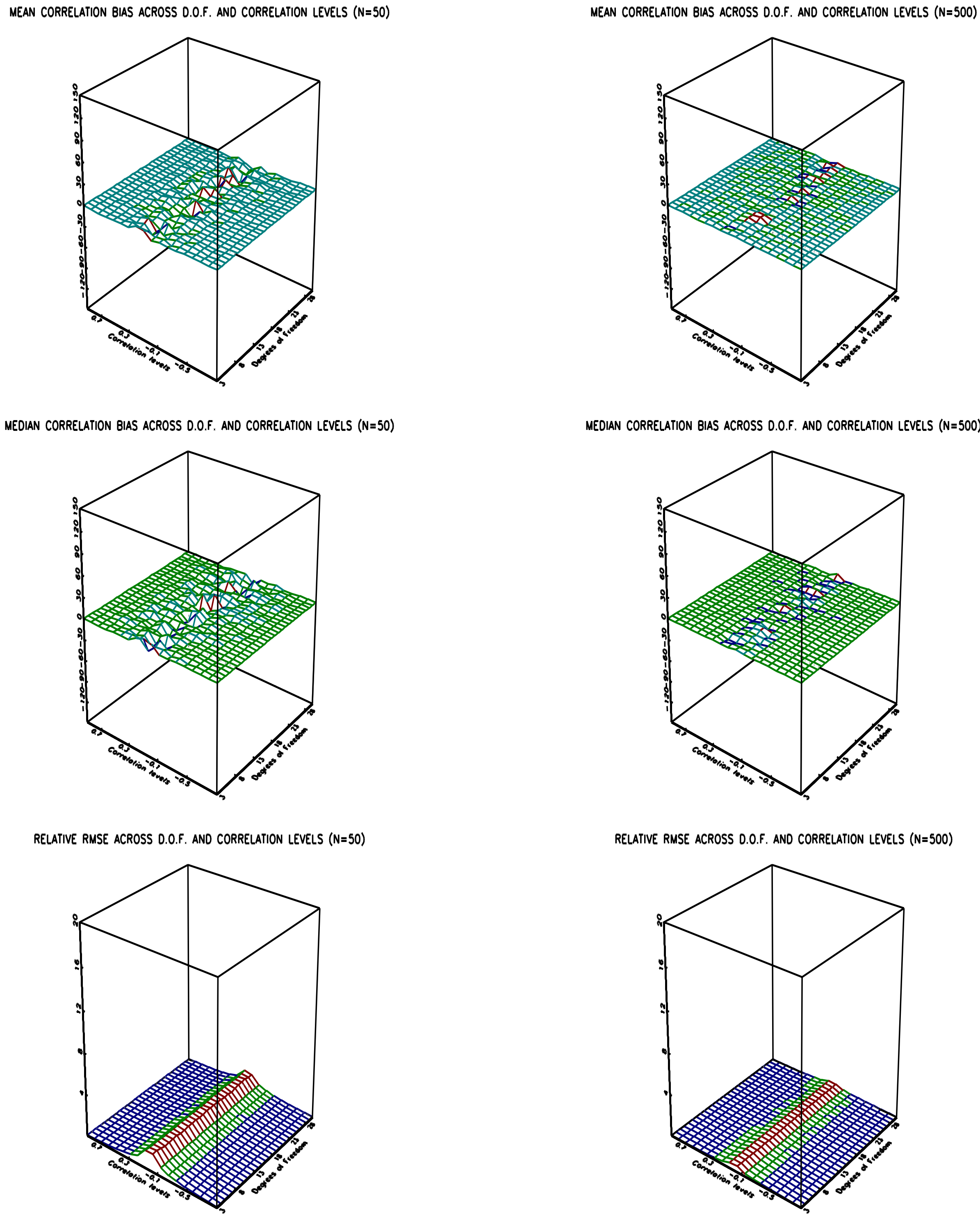

Figure 1. Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the KME-CML method for the bivariate Student's T copula, across different correlations and degrees of freedom parameters

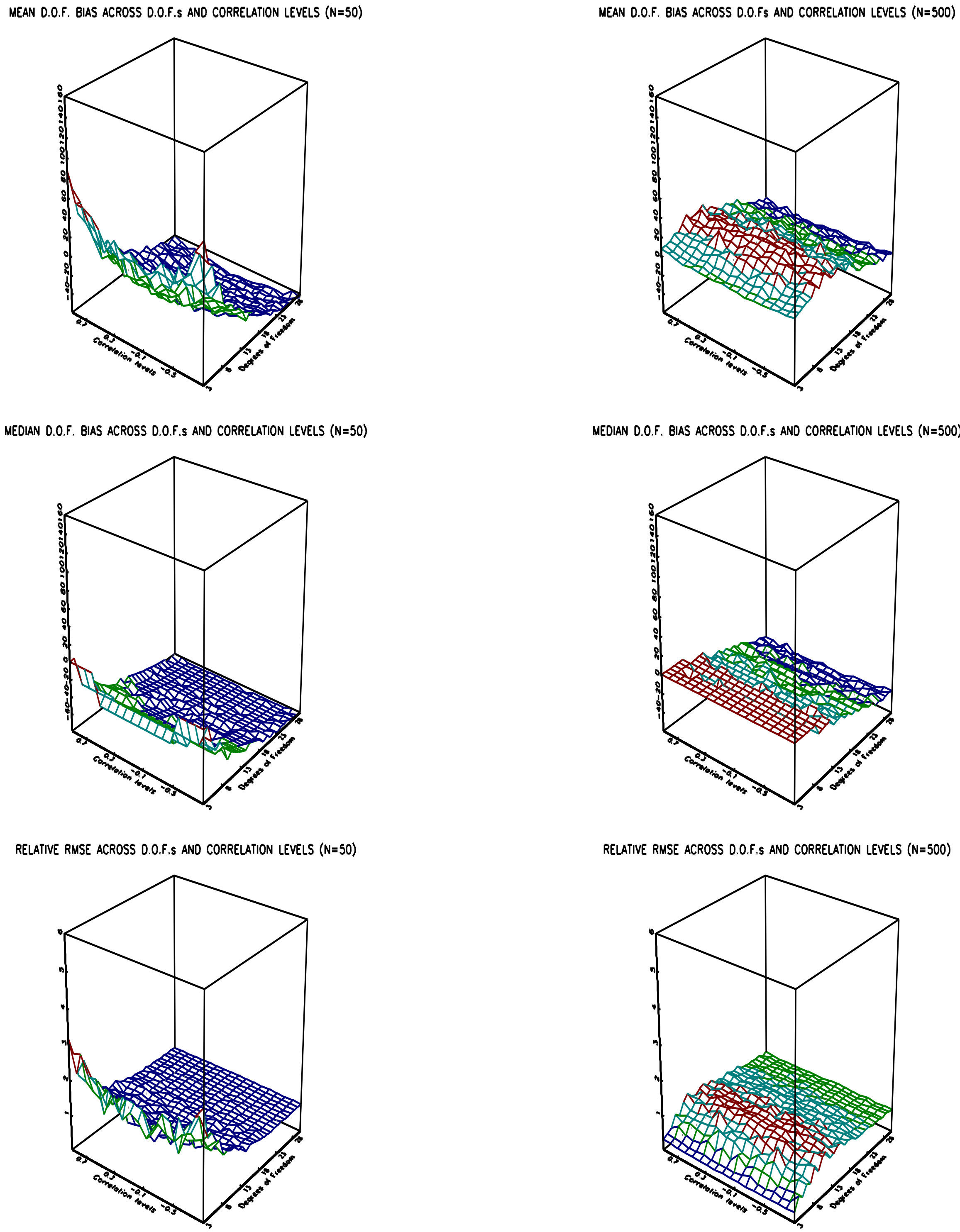

Figure 2. Mean bias (in %), Median bias (in %) and Relative RMSE of the degrees of freedom parameters, for the KME-CML method for the bivariate Student's T copula, across different correlations and degrees of freedom parameters

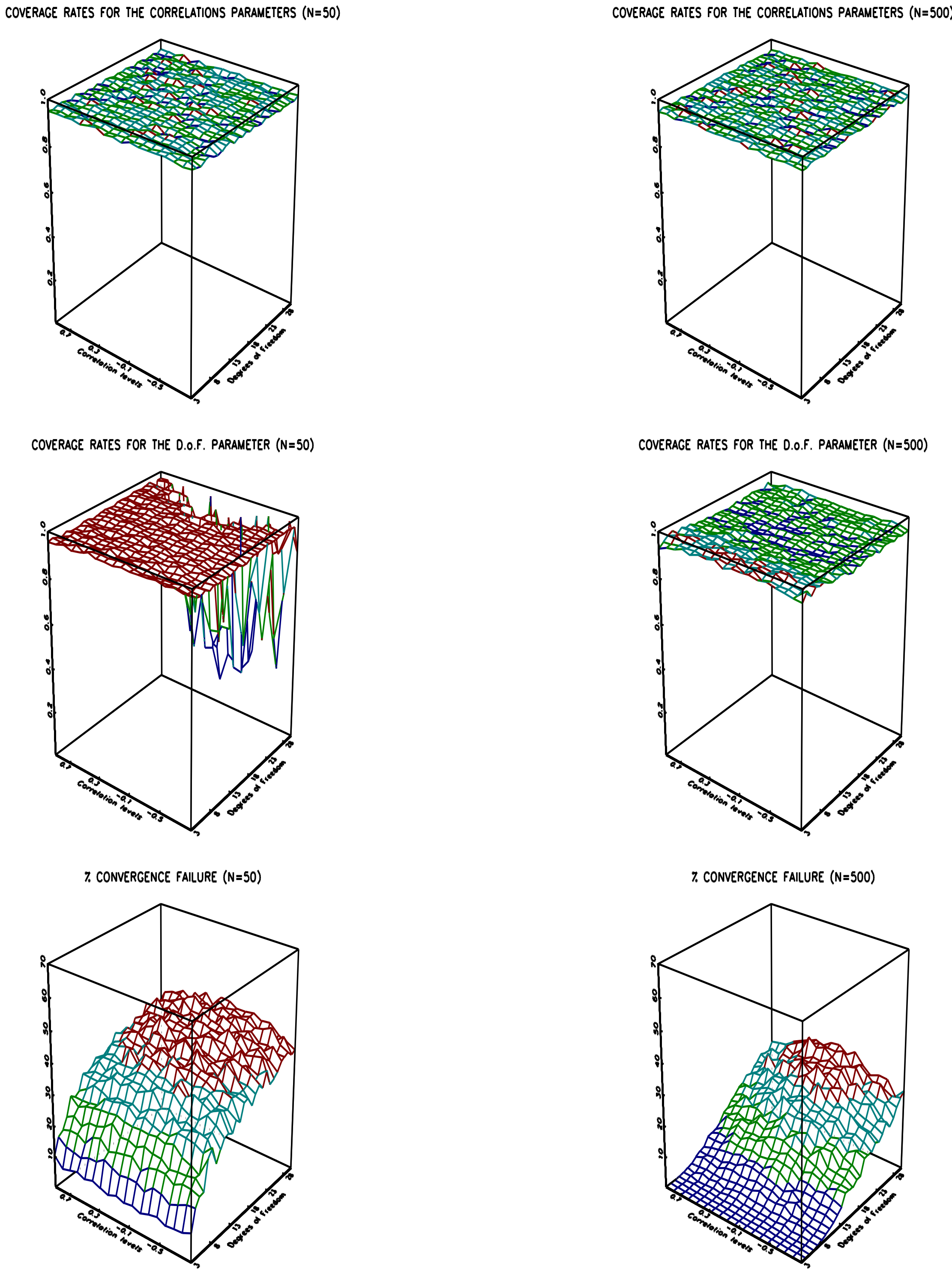

Figure 3. The first four plots report the coverage rate for the 95% confidence intervals based on a normal approximation for different sample distributions of the bivariate T-copula parameters, i.e. the correlation and the degrees of freedom (T-copula estimated with the KME-CML method). The last two plots report the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom.

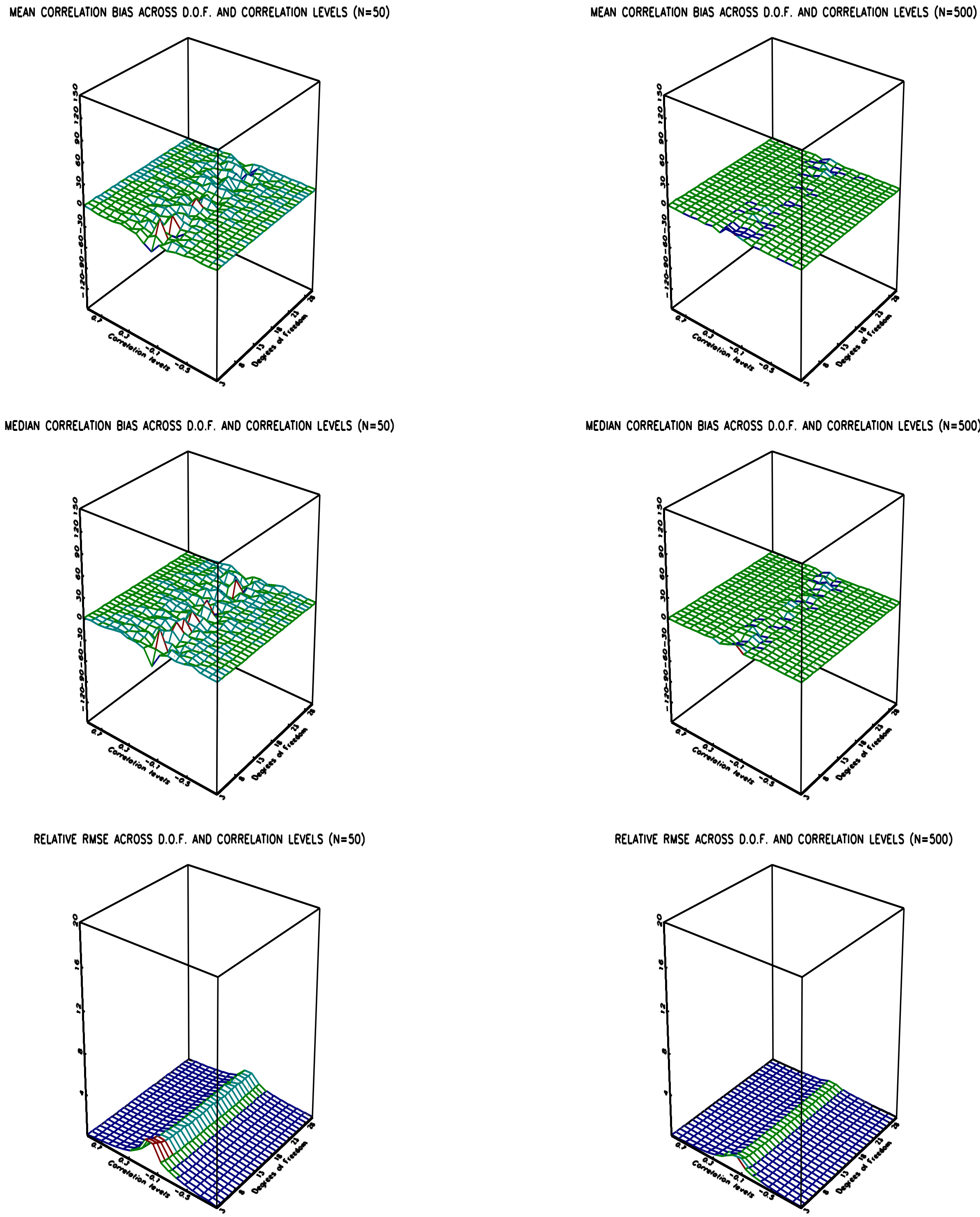

Figure 4. Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the ML method for the bivariate Student's T copula, across different correlations and degrees of freedom parameters

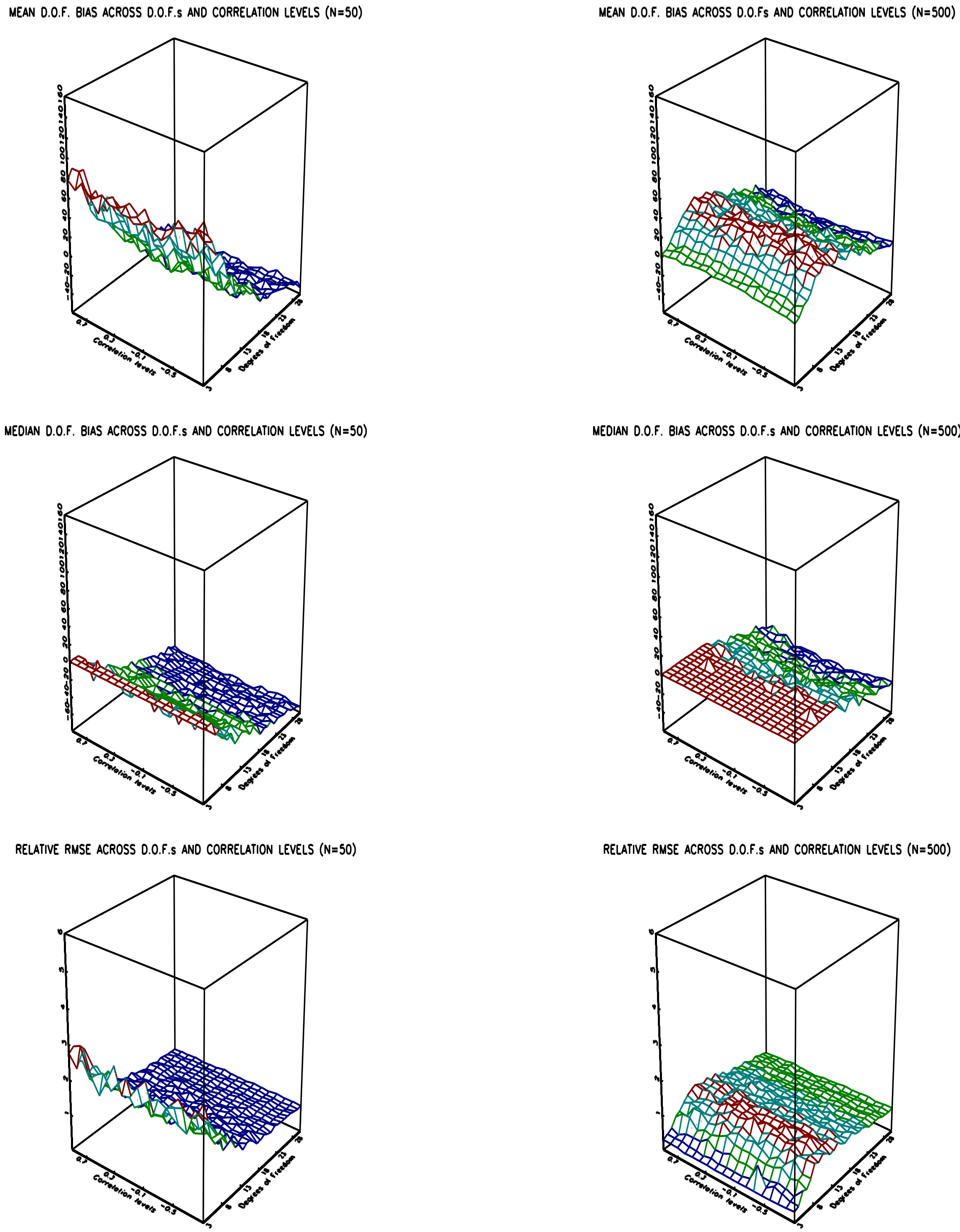

Figure 5. Mean bias (in %), Median bias (in %) and Relative RMSE of the degrees of freedom parameters, for the ML method for the bivariate Student's T copula, across different correlations and degrees of freedom parameters

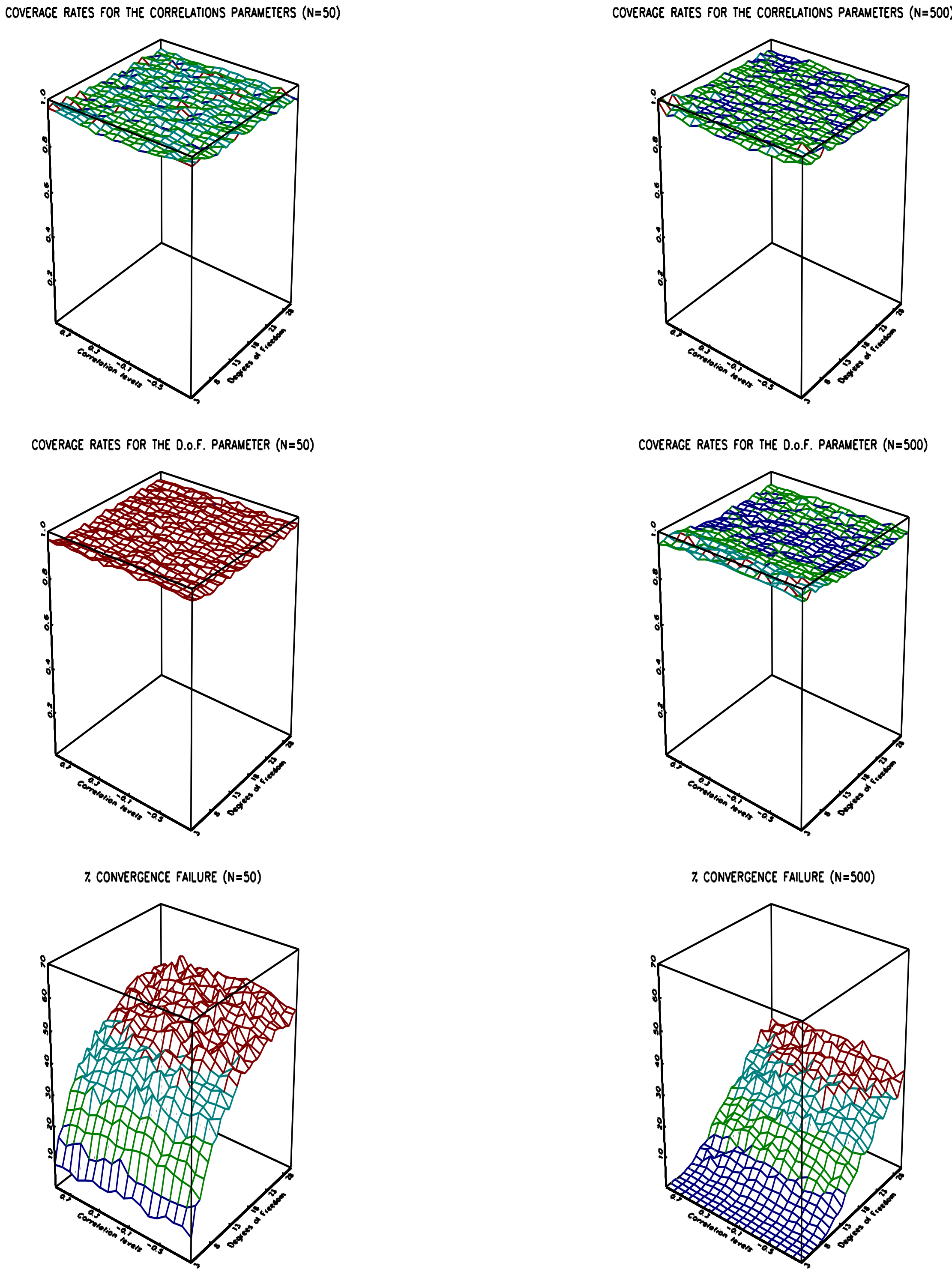

Figure 6. The first four plots report the coverage rate for the 95% confidence intervals based on a normal approximation for different sample distributions of the bivariate T-copula parameters, i.e. the correlation and the degrees of freedom (T-copula estimated with the ML method). The last two plots report the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom.

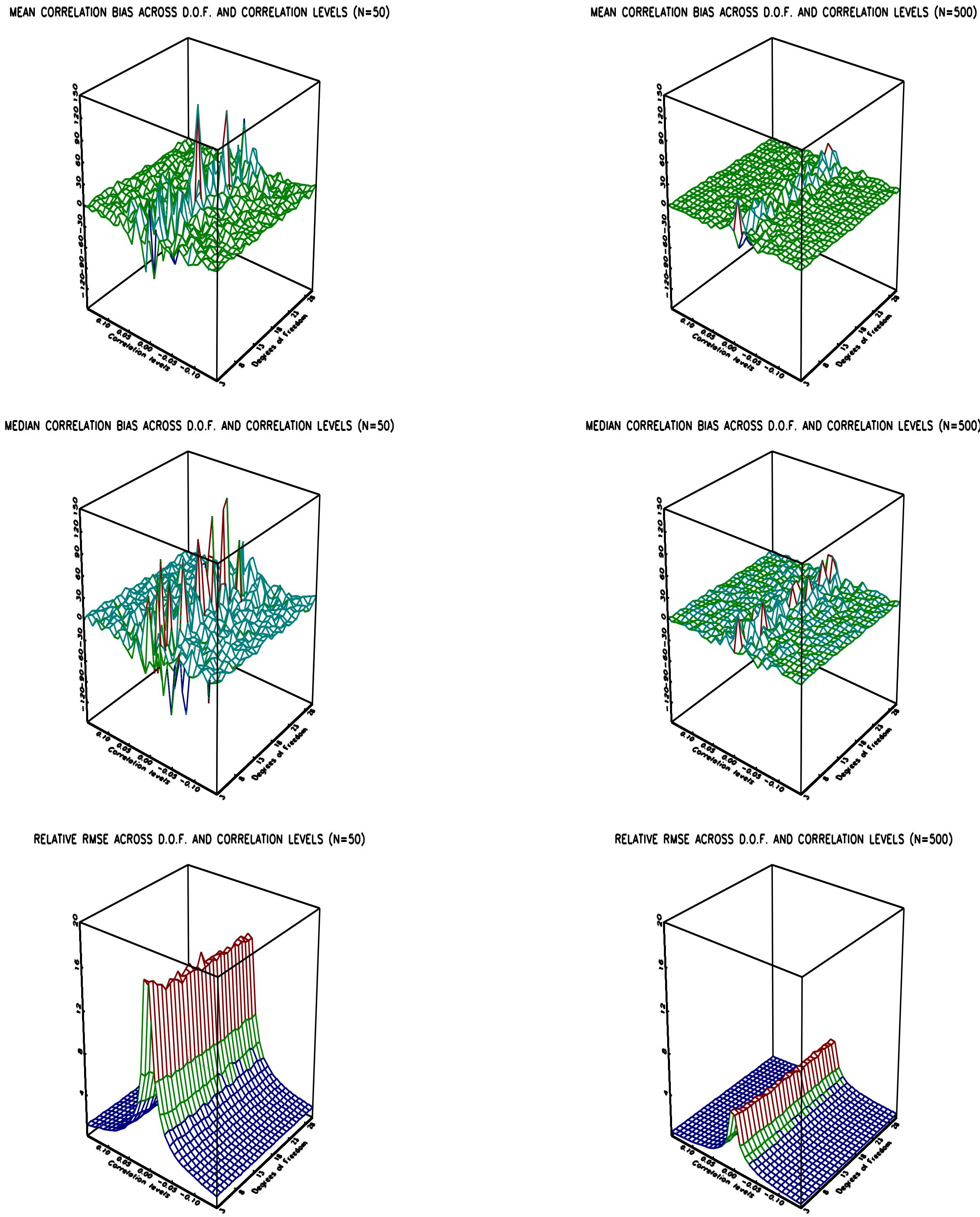

Figure 7. Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the KME-CML method for the 10-variate t-copula reported in table 1

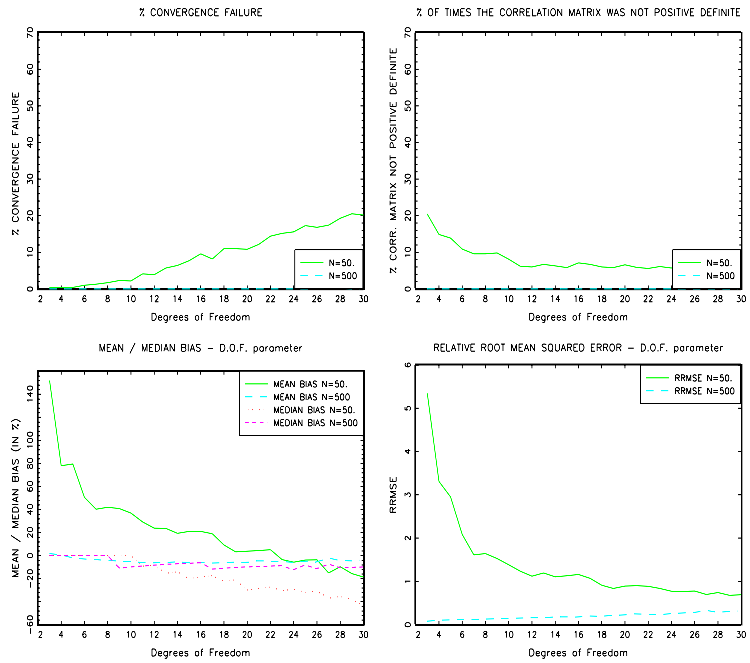

Figure 8. The first plot reports the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom. The second reports the % of times when the correlation matrix was not positive definite and the eigenvalue method was used. The third and the fourth plots report the Mean bias (in %), the Median bias (in %) and the Relative RMSE of the d.o.f. parameter, for the KME-CML method for the 10-variate t-copula with correlation matrix reported in table 1.

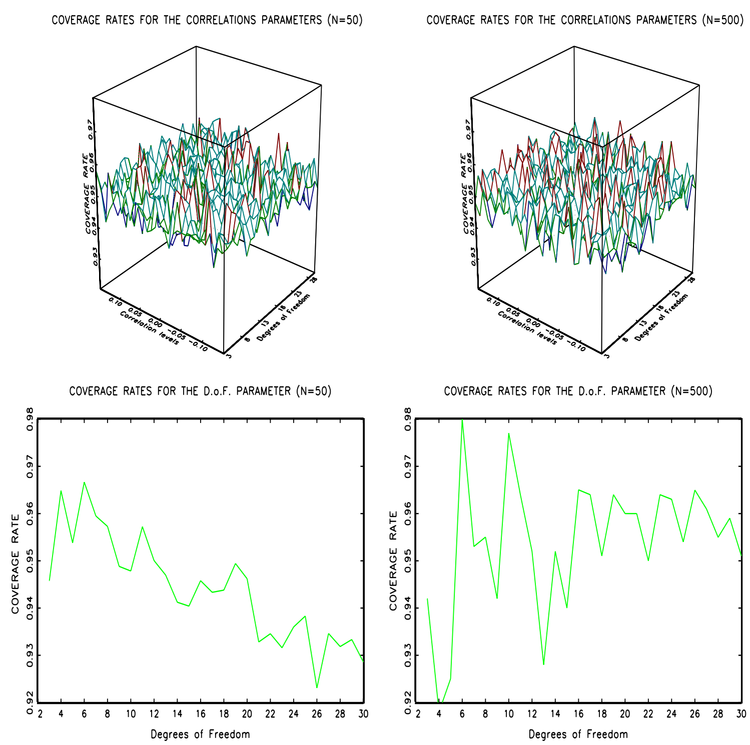

Figure 9. The four plots report the coverage rate for the 95% confidence intervals based on a normal approximation for different sample distributions of the T-copula parameters, i.e. the correlations and the degrees of freedom. T-copula with correlation matrix reported in table 1 estimated with the KME-CML method

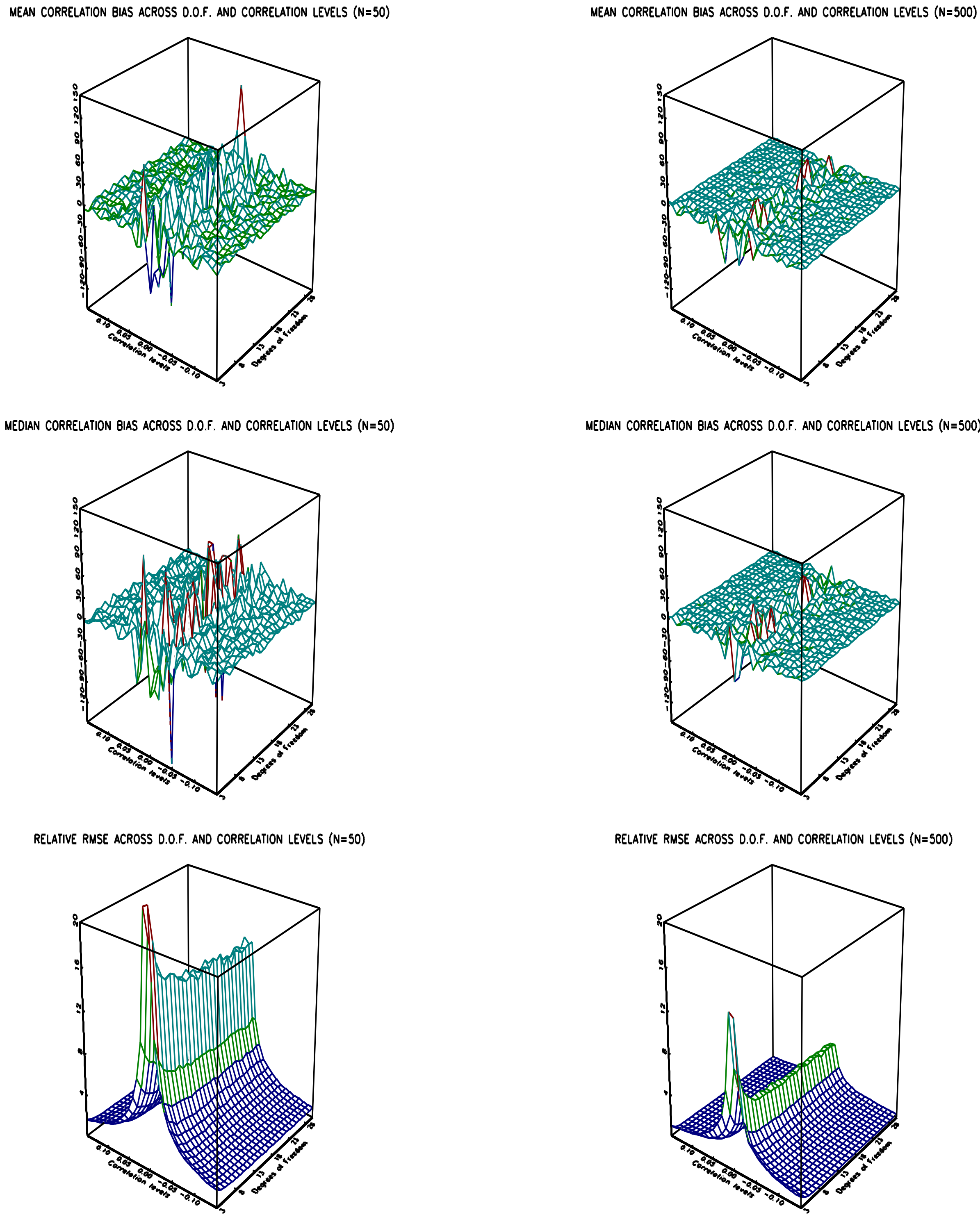

Figure 10. Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the ML method for the 10-variate t-copula reported in table 1

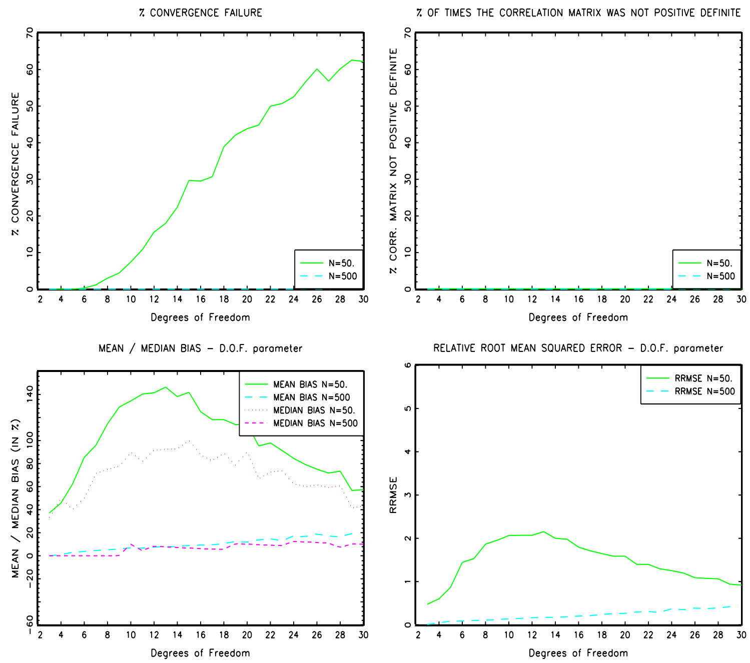

Figure 11. The first plot reports the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom. The second reports the % of times when the correlation matrix was not positive definite and the eigenvalue method was used. The third and the fourth plots report the Mean bias (in %), the Median bias (in %) and the Relative RMSE of the d.o.f. parameter, for the ML method for the 10-variate t-copula with correlation matrix reported in table 1.

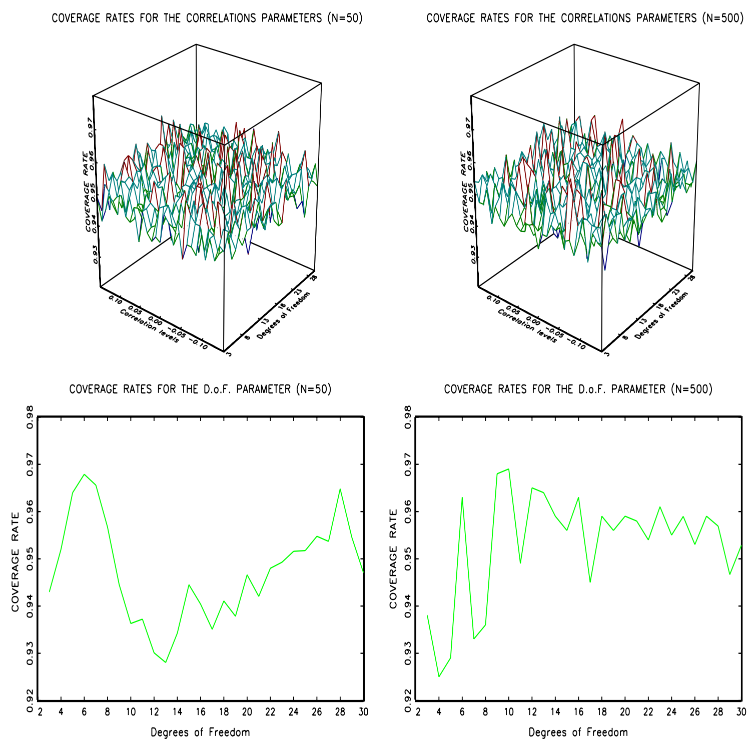

Figure 12. The four plots report the coverage rate for the 95% confidence intervals based on a normal approximation for different sample distributions of the T-copula parameters, i.e. the correlations and the degrees of freedom. T-copula with correlation matrix reported in table 1 estimated with the ML method

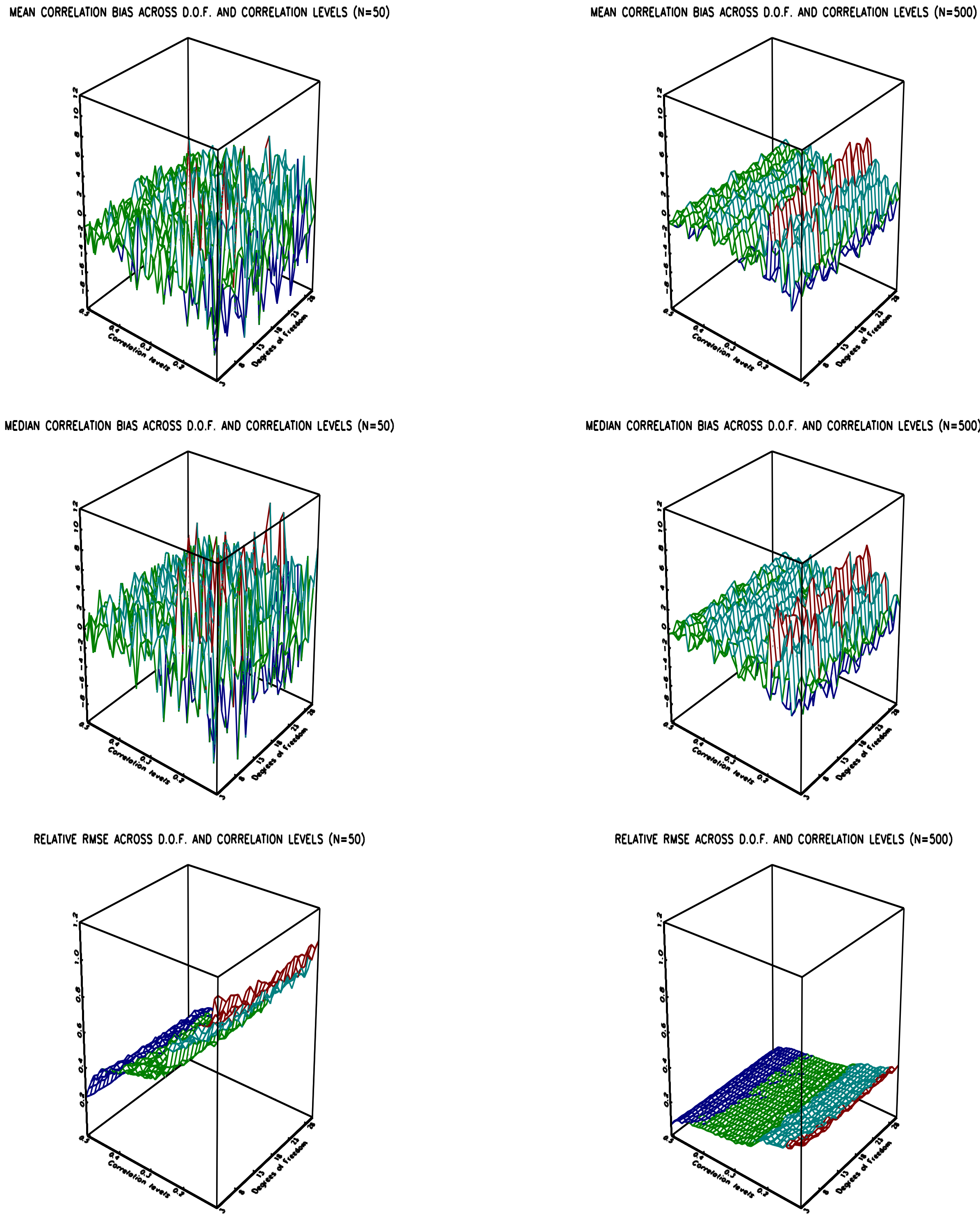

Figure 13. Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the KME-CML method for the 10-variate t-copula reported in table 2

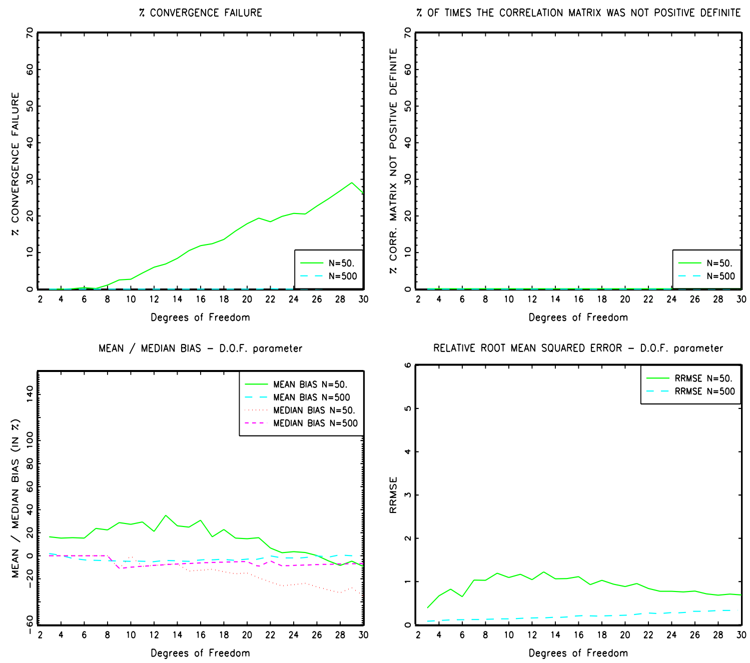

Figure 14. The first plot reports the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom. The second reports the % of times when the correlation matrix was not positive definite and the eigenvalue method was used. The third and the fourth plots report the Mean bias (in %), the Median bias (in %) and the Relative RMSE of the d.o.f. parameter, for the KME-CML method for the 10-variate t-copula with correlation matrix reported in table 2.

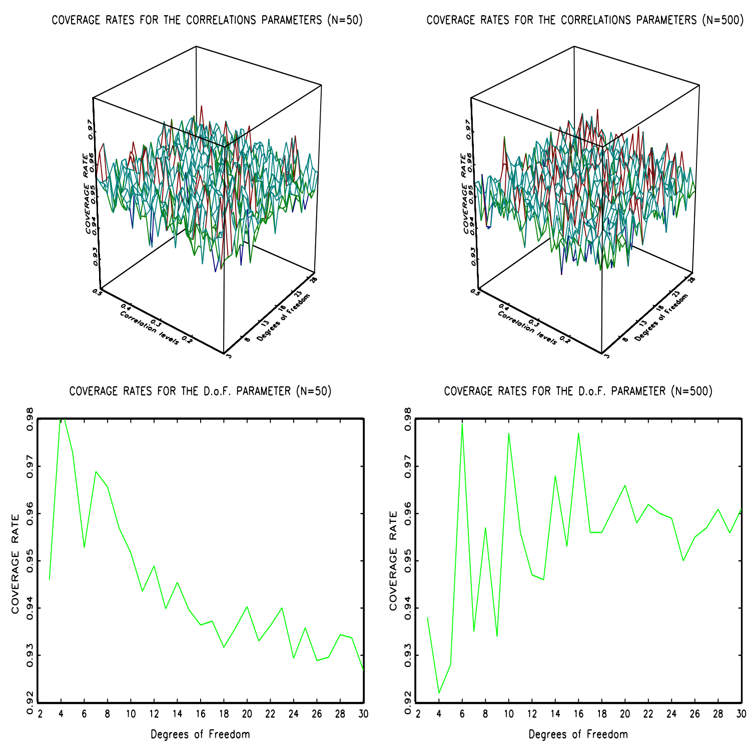

Figure 15. The four plots report the coverage rate for the 95% confidence intervals based on a normal approximation for different sample distributions of the T-copula parameters, i.e. the correlations and the degrees of freedom. T-copula with correlation matrix reported in table 2 estimated with the KME-CML method

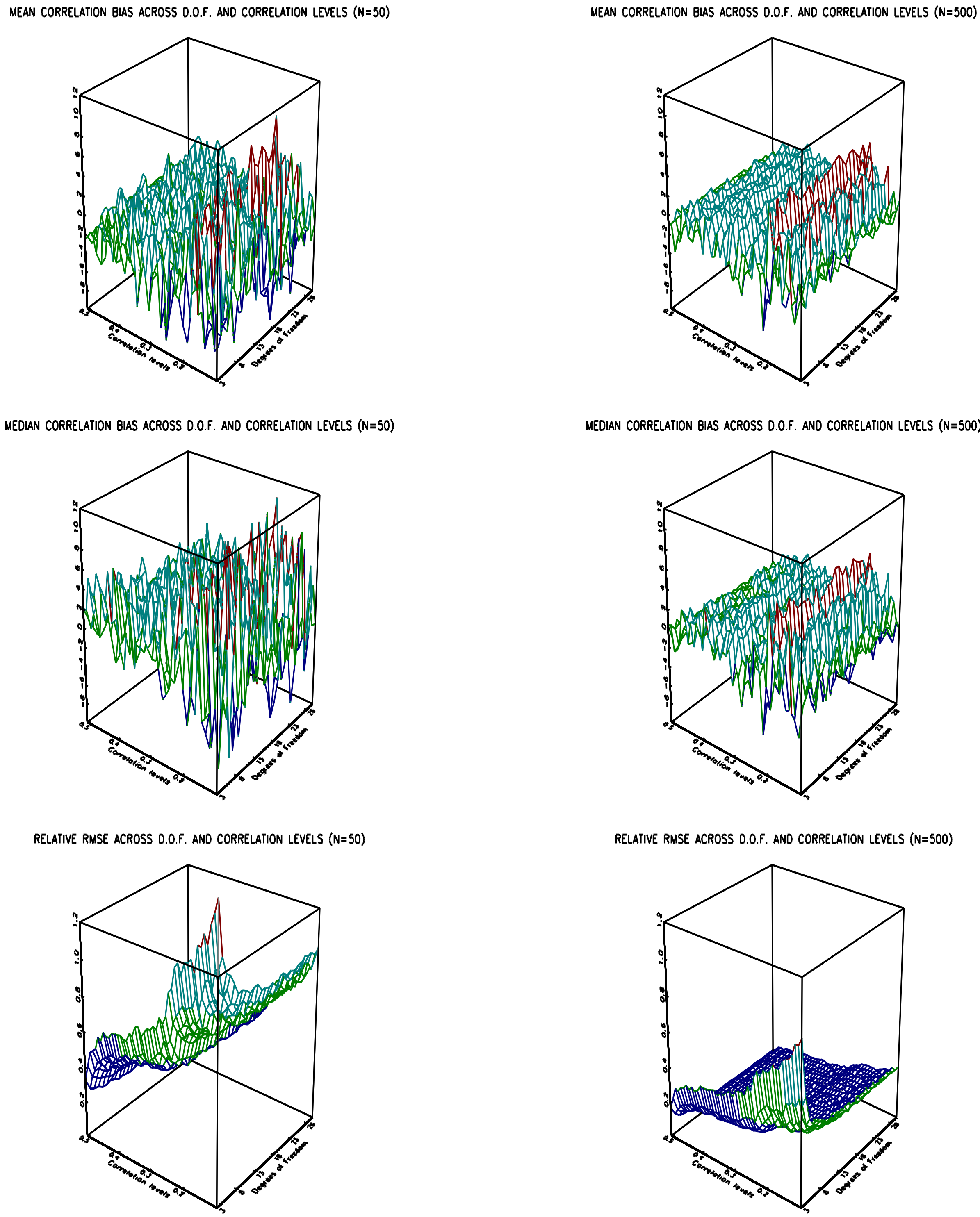

Figure 16. Mean bias (in %), Median bias (in %) and Relative RMSE of the correlation parameters, for the ML method for the 10-variate t-copula reported in table 2

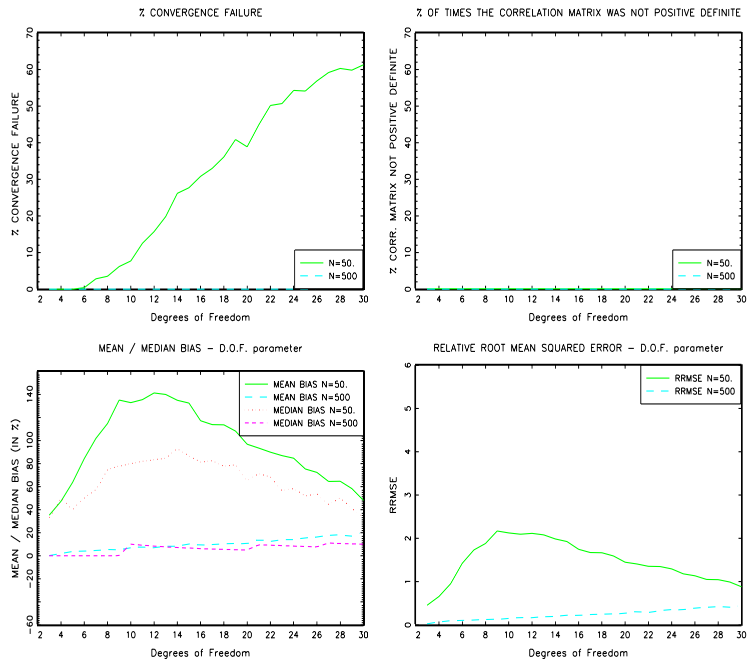

Figure 17. The first plot reports the % of convergence failures when maximizing the log-likelihood (23) for the degrees of freedom. The second reports the % of times when the correlation matrix was not positive definite and the eigenvalue method was used. The third and the fourth plots report the Mean bias (in %), the Median bias (in %) and the Relative RMSE of the d.o.f. parameter, for the ML method for the 10-variate t-copula with correlation matrix reported in table 2.

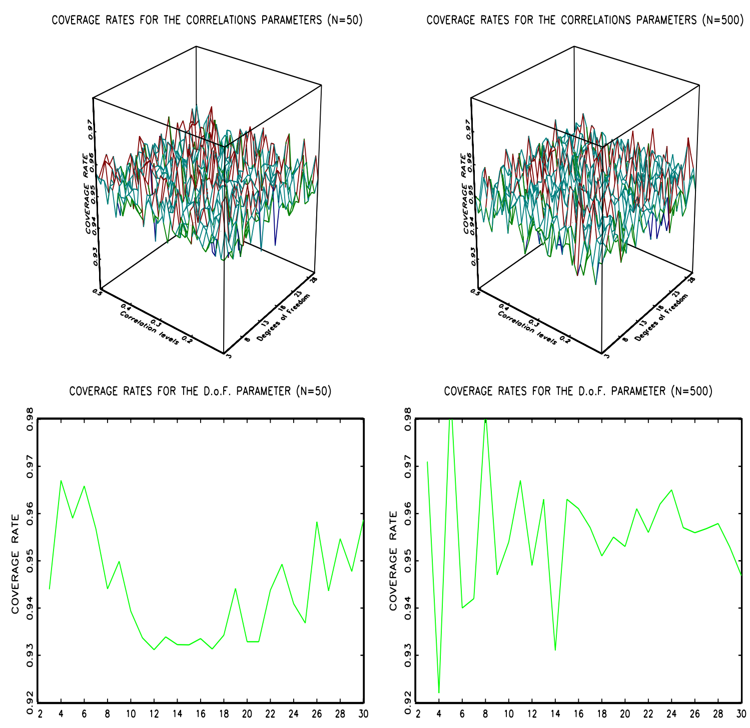

Figure 18. The four plots report the coverage rate for the 95% confidence intervals based on a normal approximation for different sample distributions of the T-copula parameters, i.e. the correlations and the degrees of freedom. T-copula with correlation matrix reported in table 2 estimated with the ML method

The simulation studies show some interesting results:

- *Bivariate T-copula*:
  - *Degrees of Freedom "ν":* when considering a small sample with $n = 50$ observations, both the ML and the KME-CML method show poor results, with high mean and median biases, as well as high percentage of numerical convergence failures. However, if a low value for ν is used, the KME-CML method shows slight better results than the ML estimator, while the reverse is true if ν is high, that is if the t-copula becomes no more distinguishable from a Normal copula[2].
    When n=50 and ν is high, the % of time when the numerical maximization of the log-likelihood failed to converge is much higher for the ML method than for the KME-CML method (over 50% vs 30%, respectively). However, while the coverage rates at the 95% level for the ML estimates do not show any particular bias or trend, the KME-CML estimates for ν show very low rates when ν become close to 30 and the correlation is not too strong (between -0.5 and 0.5). Therefore, the lower number of convergence failures for the KME-CML method comes at a cost: the estimates show much stronger negative mean (and median) biases than the ML method, around the 70% of the true value vs. the 50%, respectively. As a consequences, the confidence intervals based on a normal approximation are very poor.
    When a larger sample with $n = 500$ observations is considered, the two methods perform rather well and the ML estimator shows better properties, as expected. Moreover, the problem with the coverage rates for the KME-CML method disappears.
  - *Correlation "ρ"*: the two methods produce consistent estimates already with very small samples and the analysis reveals no major difference between the two. However, if the variables are close to be uncorrelated, both methods present small biases and RRMSE that decrease when n increases (see Figure 1 and 4).
  - *Computational aspects*: As anticipated in the previous points, the ML method shows much higher convergence failures than the the KME-CML method when n=50 and ν is higher than 10, while when $n = 500$ the numerical performances of the two methods are quite close. Furthermore, the analysis shows that when ν is high and the KME-CML is employed, the % of convergence failures is higher when t-copulas with *weaker* correlations are considered, reaching the maximum when the variables are uncorrelated (see the lower graphs in Figure 3). This pattern is not present for the ML method, instead. This result together with the stronger negative biases of $\hat{\nu}_{KME-CML}$, helps us explain the U-shape drop in coverage rates observed in Figure 3 for the KME-CML method.

- *10-variate t-copula with the correlation matrix reported in table 1, i.e. with the lowest eigenvalue close to zero*:
  - *Degrees of Freedom "ν":* When $n = 50$, *ν is low* and the KME-CML method is employed, the analysis shows a strong positive mean bias and high RRMSE (see Figure 8). However, the median bias is close to zero and the coverage rates

[2] The T-copula tends to the Normal copula when $\nu \to \infty$.

are similar to the case with the ML method. This difference is due to the high % rates when the correlation matrix is not positive definite (between 10 and 20%) and the eigenvalue method by Rousseeuw and Molenberghs (1993) is used, see the second plot in Figure 8. The analysis shows that this particular ad hoc fix has the effect to introduce a positive mean bias in $\hat{\nu}$, but the effect on the median is rather limited as well on the coverage rates. Besides, this bias quickly disappears when ν increases.

Instead, when ν *is high* and the KME-CML method is employed, the simulation results show a negative median bias that has the effect to decrease the coverage rates below the 95% level. However, we remark that the drop in the coverage rates is much lower than those observed with bivariate t-copulas.

The ML method shows low positive mean and median biases as well as low RRMSE when ν is low. Then, these biases increase when ν increases, reaching the maximum around $\nu = 15$, after which they finally decrease (see Figure 11). The effect on the 95% coverage rate is just specular: it is high when ν is low, then is decreases and finally it converges to the true value when ν is higher than 20,

When a larger sample with $n = 500$ observations is considered, the two methods perform rather well and the properties of the two estimators are quite similar. Interestingly, when $n = 500$ the correlation matrix is always positive definite for all ν, and the eigenvalue method by Rousseeuw and Molenberghs (1993) is not needed.

- *Correlation matrix*: When $n = 50$, both the ML and the KME-CML method show mean and median biases as well as high RRMSE when the correlations are close to zero, see figure 7 and 10. However, the effects on the coverage rates are rather limited (see upper plots in figure 9 and 12). Besides, the KME-CML method shows slight better results than the ML method.
  When $n = 500$ the previous biases decrease but they still remain quite high for the ML method when ν is low and the variables are close to be uncorrelated.
- *Computational aspects*: Similarly to the bivariate case, the ML method shows higher convergence failures than the the KME-CML method when n=50 and ν is higher than 10, while when $n = 500$ the numerical performances of the two methods are equal and there are no convergence failures. Furthermore, as previously discussed, when $n = 50$, ν is low and the KME-CML method is employed, the correlation matrix is not positive definite in the 20% of cases and the eigenvalue method by Rousseeuw and Molenberghs (1993) has to be used. This fix induces a positive bias in the estimate of ν, but the effects on the coverage rates are rather limited. Besides, the number of times when this method has to be used quickly diminish when ν increases. Interestingly, the correlation matrix is always positive definite already with $n = 500$.

- *10-variate t-copula with correlation matrix reported in table 2*, i.e. returns Dow Jones Industrial Index 1988-2003:
  - *Degrees of Freedom "ν":* when $n = 50$, both the KME-CML and the ML method show a inverted U-shape for the positive mean and median biases as well as for the RRMSE, as reported in Figure 14 and 17. However, while the biases for the ML method are very close to those observed for the ill-specified t-copula reported in Figure 11, the ones for the KME-CML method are much lower

than the ML method (20-120% vs 0-30%, respectively). Furthermore, the mean biases observed for the KME-CML method are different from those observed for the same technique when dealing with the ill-specified t-copula (i.e. with the lowest eigenvalue close to zero), and reported in Figure 8. This difference is due to the complete lack of negative definite correlation matrices (see figure 14), so that the eigenvalue method by Rousseeuw and Molenberghs (1993) is not needed.
Instead, both methods show median biases that are very close to those observed for the ill-specified t-copula, and similarly the 95% coverage rates reported in figure 15 and 18 are very close to those observed in figure 9 and 12, respectively: that is a U-shape that converges to the true rate when ν increase for the ML method, while a decreasing trend for the KME-CML method. However, we remark that the drop in the coverage rates for the KME-CML method is much lower than those observed with bivariate t-copulas.
When a larger sample with $n = 500$ observations is considered, the two methods perform rather well and the properties of the two estimators are quite similar.

- *Correlation matrix*: the main results are close to those observed for the 10-variate T-copula with ill-specified correlation matrix: when $n = 50$, both the ML and the KME-CML method show mean and median biases as well as high RRMSE when the correlations are close to zero, as shown in figures 13 and 16. However, the effects on the coverage rates are rather limited (see the upper plots in figures 15 and 18). Besides, the KME-CML method shows slight better results than the ML method. Differently from the ill-specified t-copula, the biases are much more lower, ranging between -10/+10% instead of -100%/+100%.
  When $n = 500$ the previous biases decrease but they still remain quite high for the ML method when ν is low and the variables are close to be uncorrelated.
- *Computational aspects*: Similarly to the previous analysis, the ML method shows higher convergence failures than the the KME-CML method when n=50 and ν is higher than 10, while when $n = 500$ the numerical performances of the two methods are equal and there are no convergence failures. However, as previously discussed, the eigenvalue method by Rousseeuw and Molenberghs (1993) is not needed already with $n = 50$: this simulation evidence confirms previous empirical evidence in Demarta and McNeil (2005) and McNeil et al. (2005, chapter 5) who claim that the componentwise transformation of the empirical Kendall's tau matrix is positive definite in most cases.

Therefore, the previous results suggest to use the KME-CML method when dealing with small samples and low degrees of freedom, while the ML method is a better choice otherwise. A possible strategy would be to first use the KME-CML method: if the estimated degrees of freedom are higher than 20, then one should try to use the ML method if it converges. Otherwise, an alternative solution would be to use the simple normal copula, given that the T-copula tends to the Normal copula when $\nu \to \infty$, and the two copulas are already quite close when $\nu \geq 20$. Besides, the drop in the 95% coverage rates highlighted by our analysis for the KME-CML method when ν is high, is quite low if the number of variables is high: this is the usual case for financial professionals, whose managed portfolios are rarely bivariate, but include a large number of assets to diversify financial risk.

6. CONCLUSIONS

We developed the asymptotics of a recent semi-parametric estimation method used in the financial literature with the multivariate Student's T-copula, which involves empirical distribution functions, method of moments and maximum likelihood methods. We examined the finite-sample properties of this estimator via a Monte Carlo study designed to replicate different Data Generating Processes, and we found that this estimator was more efficient and less biased than the one-stage ML estimator when small samples and t-copulas with low degrees of freedom were of concern.

We then analyzed the pros and cons of this methodology in terms of numerical convergence and positive definiteness of the estimated T-copula correlation matrix. When small samples were of concern and ν was high, the number of times when the numerical maximization of the log-likelihood failed to converge was much higher for the ML method than for the KME-CML method. Yet, while the coverage rates at the 95% level for the ML estimates for ν did not show any particular bias or trend, the KME-CML estimates showed very low rates when ν became close to 30 and the correlations were not too strong. However, this drop in the coverage rates was large with bivariate t-copulas, only, while it was much lower when dealing with higher dimensional t-copulas, which is the usual case for real managed financial portfolios. Besides, both the ML and the KME-CML methods showed high mean and median biases for the estimated correlations when the true ones were close to zero. Nevertheless, the effects on the coverage rates for correlations were rather limited in this case. Interestingly, when the sample dimension increased, the previous biases decreased but they still remained quite high for the ML method when ν was low.

Finally, we showed that the eigenvalue method by Rousseeuw and Molenberghs (1993) has to be used to obtain a positive definite correlation matrix only when dealing with very small samples ($n \leq 100$) and when the true underlying process has the lowest eigenvalue close to zero. This fix induces a positive bias in the estimate of ν, but the effects on the coverage rates are rather limited. Besides, the number of times when this method has to be used quickly decreases when ν increases.

ACKNOWLEDGEMENTS

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A. PROOFS

Proof of Theorem 4.1

Since Kendall's tau (16) is invariant under strictly increasing marginal transformations of the underlying random vector, $(x_{1t}, \ldots, x_{nt})$ are i.i.d random variables and the reported regularity conditions (i)-(v) hold, by the weak uniform law of large numbers (see Ruud (2000) § 15.8), the theorem follows. ■

Proof of Theorem 4.2

See the proof of Proposition 2.1 in Genest et al. (1995) that makes use of the asymptotics of multivariate rank statistics as highlighted in Section 2, together with Theorem 3.10 of White (1994). Then the theorem follows. ■

Proof of Theorem 4.3

Perform a standard first-order Taylor expansion of the moments vector $\Psi_{\ KME-CML}$ around the true parameter value Ξ_0 and solve for the stabilizing distribution:

$$\sqrt{T}(\hat{\Xi}-\Xi_0) = -\left(\frac{1}{T}\sum_{i=1}^{T}\frac{\partial\Psi_{\ KME-CML}\left(F_{1T}(x_{1,t}),\ldots,F_{nT}(x_{n,t});\Xi^*\right)}{\partial\Xi'}\right)^{-1}\cdot$$
$$\cdot\left(\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\Psi_{\ KME-CML}\left(F_{1T}(x_{1,t}),\ldots,F_{nT}(x_{n,t});\Xi_0\right)\right) = -{S_T^*}^{-1}b_T$$

where $\Xi^* = \Xi_0 + \lambda(\hat{\Xi}-\Xi_0), 0 \le \lambda \le 1$. For the random vector b_T we have

$$\sqrt{T}\left(\frac{1}{T}\sum_{i=1}^{T}\Psi_{\ KME-CML}\left(F_{1T}(x_{1,t}),\ldots,F_{nT}(x_{n,t});\Xi_0\right)-\right.$$
$$\left.-\ \mathrm{E}[\Psi_{\ KME-CML}\left(F_1(X_1),\ldots,F_n(X_n);\Xi_0\right)]\right)\xrightarrow{d}\mathrm{N}\left(0,\Upsilon_0\right) \tag{30}$$

by the invariance of Kendall's tau under strictly increasing marginal transformations and by the multivariate versions of the assumptions of Proposition A.1 in Genest et al. (1995), concerning the asymptotic distribution of multivariate rank statistics with zero mean[3], where Υ_0 was defined above in (27). Using again proposition A.1 in Genest et al. (1995), the random matrix S_T^* converges almost surely to a finite non-stochastic matrix.

$$S_T^* \equiv \frac{1}{T}\sum_{i=1}^{T}\frac{\partial\Psi_{\ KME-CML}\left(F_{1T}(x_{1,t}),\ldots,F_{nT}(x_{n,t});\Xi^*\right)}{\partial\Xi'}\xrightarrow{a.s.}\mathrm{E}\left[\frac{\partial\Psi_{\ KME-CML}}{\partial\Xi'}\right] \tag{31}$$

Then the theorem follows. ■

Proof of Theorem 4.4

See the previous proof of Theorem 4.3 but now make use of the regularity conditions A.1-A.9 in Gunky et al. (2007), instead of those of Proposition A.1 in Genest et al. (1995). The theorem follows from theorem 1 and 2 in Gunky et al. (2007). ■

[3]See Genest et al. (1995) p. 547 and pp. 549-550.

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