Growth and Crisis, Unavoidable Connection?

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Abstract

In emerging economies periods of rapid growth and large capital inflows can be followed by sudden stops and financial crises. The paper, abstracting from business cycles aspects, shows that the process of long run growth can be a key element in accounting for these facts. I study a growth model for a small open economy where decreasing marginal returns to capital appear only after the country has reached a threshold level of development, which is uncertain. Limited enforceability of contracts allows default on international debt. International investors can optimally choose to stop lending when the appearance of decreasing marginal returns slows down the growth of the economy, which then defaults and enters a financial crisis.

JEL Nos.: F32, F34, O41, O47. Keywords: growth, sudden stop, default, financial crisis, credit crunch.

1 Introduction

This paper studies the effects of international capital mobility on the growth process of a developing country and on the possibility of financial crises.

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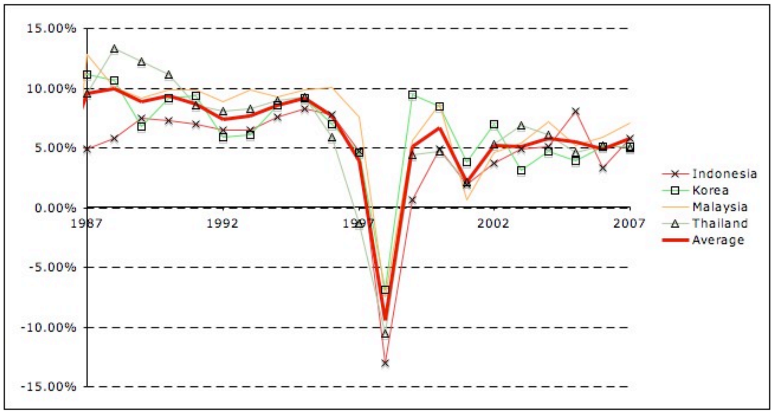

Figure 1: Growth Rates Selected Asian Countries

Simple extensions of the neoclassical growth model to the case of a small open economy (Barro et al. [1995], Lucas [1990]) conclude that, as long as the marginal return to capital is higher than the international interest rate, a developing country borrows from the rest of the world to finance investments. As the economy grows, decreasing marginal returns eventually drive to zero the spread between the international interest rate and the return on the country's capital. The GDP stops growing and positive current accounts ensure that the debt accumulated with the rest of the world is repaid. However, imperfections in international financial markets can seriously impair the borrowing ability of a country, and expose it to possible financial crises. For instance, the theoretical analysis of Bulow and Rogoff (1989) emphasizes that limited enforceability of debt contracts can have the extreme consequence of completely preventing international lending. On the empirical side, financial openness and capital flows to developing countries are also accompanied by sudden stops and financial crises (Calvo [1998]), especially for countries with very high growth rates (Ranciere et. al. [2005]). A striking example of this situation is the case of the Asian crisis of the late '90s, which is considered one of the deepest financial crises of the past 20 years. Figure 1 shows the GDP growth rate of four Asian countries in the years 1987-2007. The year of the crisis is clearly marked by a fall in the GDP of about 10% in most of the economies. The capital inflows, that had financed current account deficits and fueled high growth in the first half of the decade, stopped (Figure 2). The post-crisis years see a permanent reduction in the growth rate, from an yearly 8% to about 5%, and a collapse in the investment rate (Figure 3). These patterns are confirmed in wider

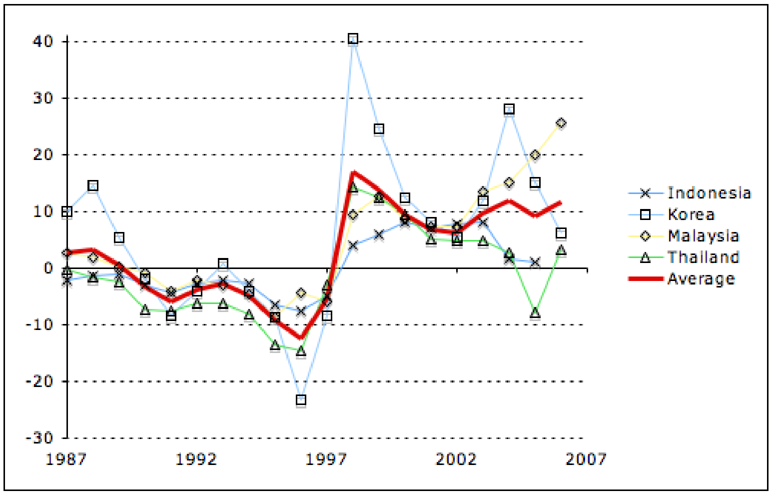

Figure 2: Current Account Selected Asian Countries

samples of countries that have experienced financial crises (Lee and Rhee [2000] and Ranciere, Tornell and Westermann [2005]).

While sudden stops seem to be a recurrent characteristic of emerging countries, much of the existing literature have confined the study of these events to stationary economies. The goal of this paper is to provide an alternative and complementary framework, which emphasizes that financial crises can be the natural outcome of the process of long run growth, rather than the effect of of pathological features of the investors behavior, such as moral hazard or herding, or of business-cycle shocks (see for instance Arellano [2008], Gopinath [2004], Mendoza [2001]).

In particular, I construct a model where the growth path of a country is divided into two successive stages: in the first, marginal returns to capital are constant, while in the second they decrease over time. The threshold level of development where decreasing marginal returns appear is uncertain and thus the "turning point", i.e. the moment when the economy transitions from stage 1 to stage 2, is also uncertain. A faster growth during stage 1 translates into a higher probability that, at each point in time, the economy moves to stage 2. Intuitively, in a poor country there are large amounts of underutilized resources and wide possibilities of efficiency gains to be exploited. Consequently, the marginal product of capital starts to fall only after the economy has reached a high enough level of development, which might be uncertain and difficult to predict.

Households that live in the country have access to an international financial

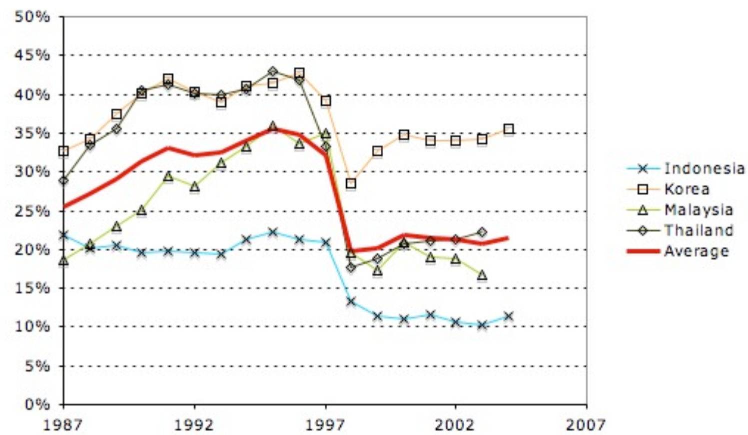

Figure 3: Investment-GDP Ratios Selected Asian Countries

market and can choose to default on their debt obligations. The punishment for default is stochastic autarky during which there is a reduction in income. The possibility of default determines the existence of borrowing constraints, which can be chosen endogenously to represent, for each state of the economy, the highest amount of debt that a household can commit to repay. I show that if the marginal product of capital during stage 1 is high, the borrowing constraint is large and the country experiences strong capital inflows (credit boom) which fuel high growth and investment rates. However, when decreasing marginal returns appear and growth slows down, lenders realize that households would eventually default on their debt. The country faces a sudden stop, a financial crisis and a credit crunch which permanently reduces growth and investment rates. If the credit crunch is sufficiently strong, the current account is reversed. Instead, if the marginal product of capital is initially low, the endogenous borrowing constraint is very tight during stage 1, and the lower availability of credit reduces the growth rate. However, at the turning point, there is no credit crunch nor financial crisis, and the economy moves smoothly to its second stage of development.

As a special case, I find that there can be equilibria where the economy grows at a rate bigger than the interest rate paid on its international debt. The evolution of the country's debt follows bubble-like dynamics and non-autarkic equilibria can exist even in the context of Bulow and Rogoff (1989).

The remaining of the paper is organized as follows. Section 2 presents the problem of the household and of the international financial investors. Section 3 introduces the aggregate technology and defines an equilibrium. Section 4 derives endogenous borrowing constraints. Section 5 shows the connection between bubbles, debt sustainability and dynamics of the balanced of payments. Section 6 concludes.

2 Households and International Investors

A small open economy is inhabited by a unit measure of identical households labeled by $i \in [0, 1]$, who can write debt contracts with competitive international investors. Time flows continuously. At each instant, the aggregate state of the economy is determined by the current marginal product of capital in the small economy. Specifically, the economy can be either in stage 1 of growth, or at some time $\tau \geq 0$ since the beginning of stage 2. The moment $\tilde{t}$, which marks the end of stage 1 and the beginning of stage 2 (corresponding to $\tau = 0$), is a stochastic event with arrival rate π and is called "turning point". As we will see, during stage 1 the marginal product of capital is constant, while it keeps decreasing in stage two as time τ goes by after the turning point. Even though π and the evolution of the marginal product of capital during stage 2 are eventually determined, in equilibrium, by rate of aggregate growth of the economy, they are nonetheless taken as given by the households and the international investors. With this definition of the aggregate state we move to defining first the problem of the household and of then the role of international investors. At each instant, an household can be either in the default (D) or in the non default (ND) state. In addition, if the state is ND then the household's individual state is also characterized by her capital stock k and debt stock b, which is financed by the international investors. In the case D the individual state is additionally characterized only by the capital stock k. Since international debt contracts are not entirely enforceable, an household in state ND can choose to renege on her current debt and opt for the default state D. Corresponding to the possible individual and aggregate states, the streams of instantaneous utility derived by the household are the following. If the economy is in stage 1, then $V^{1,ND}(k, b)$ and $V^{1,D}(k)$ represent, respectively, the value to a household in the non default state who owns a capital k and chooses to repay her debt b, and the value to a household in the default state with capital k. Similarly, if the economy is at time τ since the beginning of stage 2, the values are, respectively, $V^{2,ND}(k, b, \tau)$ and $V^{2,D}(k, \tau)$. Starting from stage 2, the values are obtained as the solution to the following problems

$$\rho V^{2,ND}(k,b,\tau) = \max_{x,d,h} \log c + V_k^{2,ND}\dot{k} + V_b^{2,ND}\dot{b} + V_\tau^{2,ND} \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad & c = Ak^\alpha h^{1-\alpha} - w_2(\tau)h_t + d - x \\ & \dot{k} = -\delta k + x \\ & \dot{b} = r_2(k,b,\tau)b + d \\ & b \leq m_2(\tau)k \end{aligned}$$

$$(\rho+\theta)V^{2,D}(k,\tau) = \max_{x^d,h} \log c^d + \theta V^{2,ND}(k,0,\tau) + V_k^{2,D}\dot{k} + V_\tau^{2,D} \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad & c^d = [1-\xi_2(\tau)][Ak^\alpha h^{1-\alpha} - w_2(\tau)h] - x \\ & \dot{k} = -\delta k + x^d \end{aligned}$$

An household with access to the financial market and who chooses to repay her debt (ND) can issue an amount d of new net debt. International lenders charge, on the outstanding debt b, an interest rate r, which I allow to depend on both the aggregate and individual state, in order to take into account the possible default incentives that the household might have. Incentives to default set also a limit on the maximum amount of debt that the household can issue. In fact, as we will see, if the stock of debt b is too high with respect to the wealth k, the household has incentives to borrow and immediately default on her debt. The borrowing constraint $m_2(\tau)$ rules out this possibility, by limiting the amount of debt per unit of capital, or "leverage", that the household can achieve. Similarly to the evolution of the marginal product of capital, the borrowing constraint $m_2(\tau)$ is also taken as given by the household, but will be determined endogenously, since it represents the maximum amount of debt that the household can credibly commit to repay at any state τ since the beginning of stage 2. The household can operate a constant return to scale technology which combines capital and an intermediate good h, with $\alpha \in (0,1)$. The input h is bought from an intermediate good sector at a price $w_2(\tau)$. An extensive interpretation of h is given in section 3; for the time being, it is enough to point out that it represents a composite input whose aggregate supply becomes more and more costly as the economy grows in stage 2 and larger amounts of h are employed into production. The consequent rise in the aggregate price $w_2(\tau)$ is responsible for the progressively decreasing marginal returns to capital characterizing stage 2. The household uses the output from the production activity to pay for the intermediate input, to make net payments $-d$ to the international investors, to finance her current consumption c and to invest x in the accumulation of capital, which depreciates at a rate $\delta \geq 0$. The use of log-utility is done for analytical tractability, and $\rho > 0$ is a discount rate. The notation V_j indicates the derivative of the value function with respect to the state variable $j \in \{k, b, \tau\}$. Finally, whenever the optimal value of the control variables x and d is not finite we say that the individual state of the household has a "jump".[1]

The household in the default (D) state faces two types of punishments: inability to borrow from the financial market and exogenous output costs which reduce by a fraction $\xi_2(\tau) \in [0, 1)$ the net output of the production activity. The punishment state D ends at a rate $\theta \geq 0$ and, since default is assumed to be complete, the household is readmitted to the financial market with zero debt. Working backwards, we obtain the value functions $V^{1,ND}(b, k)$ and $V^{1,D}(k)$ corresponding, respectively, to the non default and default state in stage 1 as the solution to the following problems

$$(\rho + \pi)V^{1,ND}(k, b) = \max_{x,d,h} \log c + \pi V^{TP}(k, b) + V_k^{1,ND}\dot{k} + V_b^{1,ND}\dot{b} \tag{3}$$

$$\begin{aligned} \text{s.t.} \quad & c = Ak^{\alpha}h^{1-\alpha} - w_1 h + d - x \\ & \dot{k} = -\delta k + x \\ & \dot{b} = r_1(k, b)b + d \\ & b \leq m_1 k \end{aligned}$$

$$(\rho + \theta + \pi)V^{1,D}(k) = \max_{x^d,h} \log c^d + \theta V^{1,ND}(k, 0) + \pi V^{2,D}(k, 0) + V_k^{1,D}\dot{k} \tag{4}$$

[1]As shown in Appendix A, a technically more precise definition of the household's problem is obtained by requiring that $d \leq \bar{D}$ for some arbitrarily large bound $\bar{D}$, so that the maximization problem is always well-defined and the optimal path for k and b is continuos. The possibility of jumps in the individual state is then introduced by computing the optimal program and the corresponding value function for $\bar{D} \to \infty$.

$$\text{s.t.} \quad c^d = (1-\xi_1)(Ak^\alpha h^{1-\alpha} - w_1 h) - x^d$$
$$\dot{k} = -\delta k + x^d$$

From the definition of the aggregate state, time is not a state variable in stage 1. Consequently, the interest rate schedule r_1 and the borrowing constrain m_1 are constant. More importantly, the price w_1 of the intermediate good is constant. The optimal program requires that the household equates the marginal product of h to its constant price w_1. I show below that, given the constant return to scale nature of the technology, the marginal product of capital has also to equal some constant r_1^k during stage 1. With arrival rate π, the turning point is reached, stage 1 ends and stage 2 begins. If the household is in the D state, then her value at the turning point is simply $V^{2,D}(k, \tau = 0)$. If the household is in the ND state the value $V^{TP}(k, b)$ at the turning point is determined as follows. Recall that our final goal is to define borrowing constraints that represent, at each state, the maximum amount of debt that a household can find optimal to to repay. If $b \leq m_2(0)k$ then the debt with which the household enters the turning point satisfies the new borrowing constraint $m_2(0)$ and the household repays her debt at the turning point. However, if the borrowing constraints are as in Figure 4 it might happen that $b > m_2(0)k$. In this case, that I call *credit crunch*, the large stock of debt accumulated in stage 1 exceeds the borrowing constraint at the turning point, which means that the household prefers to renege on her debt and enter the default state. With this intuitions in mind I set

$$V^{TP}(k, b) = \begin{cases} V^{2,ND}(k, b, 0) & \text{if } b \leq m_2(0)k \\ V^{2,D}(k, 0) & \text{if } b > m_2(0)k \end{cases} \tag{5}$$

Having laid out the problem of the household we are ready to introduce the role of the international investors, who are are competitive, risk neutral, and discount time at a rate ρ, which then represents the international risk-free rate. I will now show that a consistent definition of the borrowing constraints requires that incentives for the household to default can arise only at the turning point. In the previous section we have assumed that the household can issue any level of debt up to the borrowing constraints at a given interest rate schedule r. However, we have to make sure that it is indeed optimal for the international investors to buy such debt, provided that the household's incentives to default are taken into account . For example, consider any state $\tau > 0$ in stage 2 and suppose that, given a level of debt $b \leq m_2(\tau)k$, the household is better off by defaulting, i.e. the value of the state D is strictly greater than the value of the state ND. Default here is deterministic, since it takes place exactly when time τ has elapsed since the arrival of the turning

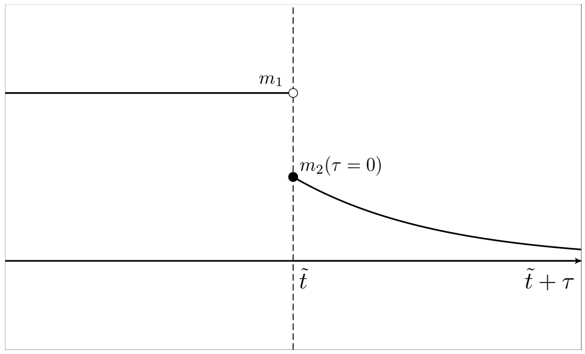

Figure 4: Credit crunch at the turning point

point. Consequently, a moment before τ no international investor would lend to the household, and thus any $m_2(\tau) > 0$ cannot represent levels of debt that the household could finance from international investors. A consistent definition of $m_2(\tau)$ then requires that the borrowing constraint is *sustainable*, i.e.

$$V^{2,ND}(k, b, \tau) \geq V^{2,D}(k, \tau) \quad b \leq m_2(\tau)k \tag{6}$$

for all $\tau \geq 0$ and all $k > 0$.
The situation prior to the turning point is only slightly different. Choose any time t in stage 1 when the household has a debt $b \leq m_1 k$. Assume that default is optimal at t in stage 1 and consider time $t - \epsilon$ with $\epsilon > 0$. International investors at time $t-\epsilon$ in stage 1 know that, if at t the economy is still in stage 1, the household will default on her debt. This event is realized with probability approximately equal to $1 - \pi\epsilon$. If we take ϵ small then default occurs almost surely, and no international investor would be willing to lend at time $t-\epsilon$ at a finite interest rate r_1. Again, a consistent definition of the borrowing constraint requires that m_1 it is *sustainable*, i.e.

$$V^{1,ND}(k, b) \geq V^{1,D}(k) \quad b \leq m_1 k \tag{7}$$

for all $k > 0$.
We conclude that if the borrowing constraints are sustainable, default is never optimal for the household whenever the debt is within the borrowing constraint. Default, however, can take place when the debt is above the borrowing constraint, which might happen at the turning point in the presence of a credit crunch (Figure 4). To justify the intuition behind the definition of V^{TP} in (5), we restrict out attention to the class of sustainable borrowing

constraints with the property that

$$V^{2,ND}(k, m_2(0)k, 0) = V^{2,D}(k, 0) \quad \text{if } m_1 > m_2(0) \tag{8}$$

for all $k > 0$. If condition (8) holds then the borrowing constraint at the turning point makes the household exactly indifferent between repaying a stock of debt $b = m_2(0)k$ or default. Consequently if thanks to the fact that $m_1 > m_2(0)$ the household enters the turning point with a debt $b > m_2(0)k$, then she is strictly better off by choosing the default option, as in fact stated in (5).

When the borrowing constraints are sustainable and satisfy (8), the interest rate schedules r_1 and r_2 offered by the competitive international investors are easily found. Since default never occurs in stage 2 we have that for any $b \leq m_2(\tau)k$

$$r_2(k, b, \tau) = \rho \tag{9}$$

During stage 1, investors anticipate that default takes place at the arrival of the turning point if and only if the outstanding debt of the household is such that $b > m_2(0)k$, therefore

$$r_1(k, b) = \begin{cases} \rho & \text{if } b \leq m_2(0)k \\ \rho + \pi & \text{if } b > m_2(0)k \end{cases} \tag{10}$$

The arrival rate π is then also the risk premium in equilibria with default.

Once we plug into the household's problem the borrowing constraints m_1, m_2 and corresponding interest rate schedules r_1 and r_2, characterized in this section, we can derive the implied evolution over time of the aggregate capital, debt and intermediate input K, B and H, for any initial distribution of capital and debt across households. In particular, to keep the problem simple, I assume that the households share the same initial condition and the borrowing constraint is binding at time zero,

$$k_0^i = k_0 \quad b_0^i = m_0 k_0 \tag{11}$$

for all $i \in [0, 1]$. The solution to the problem of the household gives policy rules $k(t, \tilde{t})$, $b(t, \tilde{t})$, $h(t, \tilde{t})$ which give, at any time $t \geq 0$ and for any realization of $\tilde{t} \geq 0$, the optimal level of the capital and debt stocks, and the optimal amount of intermediate input used into production.

If, for some $\tilde{t}$, the optimal policies imply $b(\tilde{t}, \tilde{t}) > m_2(0)k(\tilde{t}, \tilde{t})$, then the household defaults if the turning point arrives at time $\tilde{t}$. In this case, for any $t > \tilde{t}$ the policy rules are random variables, since they depend on realization of the stochastic moment $\tilde{\tau} \geq 0$ that determines the time $\tilde{t} + \tilde{\tau}$ at which a household is readmitted to the financial market. The realization of $\tilde{\tau}$, with arrival rate θ, is assumed to be i.i.d. across households. For any pair $(t, \tilde{t}) \geq 0$ the aggregate quantities K, B and H implied by the optimal individual policies are

$$\begin{aligned} K(t,\tilde{t}) &= E[k(t,\tilde{t})] \\ B(t,\tilde{t}) &= E[b(t,\tilde{t})] \\ H(t,\tilde{t}) &= E[h(t,\tilde{t})] \end{aligned} \tag{12}$$

where, by the law of large numbers, the expectation aggregates the possible idiosyncratic uncertainty mentioned above. Define also

$$\begin{aligned} G(t,\tilde{t}) &= \frac{\dot{K}(t,\tilde{t})}{K(t,\tilde{t})} \\ G^H(t,\tilde{t}) &= \frac{\dot{H}(t,\tilde{t})}{H(t,\tilde{t})} \end{aligned} \tag{13}$$

We are now ready to explain how the arrival rate π and the price function $w_2(\tau)$ are determined endogenously by the evolution of the aggregate capital in the economy. Section 4 is devoted to show how to fully endogenize also the borrowing constraints.

3 Equilibrium

In the previous sections I derived the evolution of the aggregate variables, given π, w_1, $w_2(\tau)$ and appropriately chosen borrowing constraints. Here, in turn, I consider the general equilibrium effects that, given the motion of the aggregate variables, pin down the rate π at which the turning point is reached and the evolution of the price $w_2(\tau)$. To close the model I need then to give primitives on the aggregate production function available to the country for the supply of the intermediate good H.

The intuition which guides my specification of the aggregate production function is that, even though decreasing marginal returns to the reproducible factor K eventually slow down the growth process, decreasing returns might not appear rightaway. At the early stage of development, the growth of a country is not driven only by the accumulation of capital, but also by a higher (or more efficient) utilization of other resources, which allow the marginal product of capital not to fall. In particular, assume that the marginal cost w, in terms of the final good, of producing a level H of the intermediate good is

$$w(H) = \begin{cases} w_1 & H \leq \bar{H} \\ w_1 + \gamma \left[\left(\frac{H}{\bar{H}}\right)^{1+\gamma} - 1 \right] & H > \bar{H} \end{cases} \tag{14}$$

for $\gamma > 0$ and some constant $\bar{H} > 0$.
When the economy is relatively little developed, and output in the final good sector is low, the amount of H used into production is small and can be supplied at a constant marginal cost w_1. As the economy grows, and the amount of input H increases above the threshold $\bar{H}$, the marginal cost w starts to rise. The turning point is then defined as follows. For a given evolution of the aggregate variable $H(t, \tilde{t})$ derived in the previous section, we say that the the turning point is realized at time $\tilde{t}$ if and only if $H(\tilde{t}, \tilde{t}) = \bar{H}$. As explained below, uncertainty on $\tilde{t}$ is obtained by assuming that $\bar{H}$ is unknown, so that the economy is endowed with a stochastic aggregate technology. To see that the turning point separates a first stage of growth with constant marginal returns to capital from a second stage with decreasing returns, consider the following observation. Regardless the individual and aggregate states, a household with capital k facing a cost w for the intermediate good always chooses h to equate its marginal product to w,

$$w = (1-\alpha)A\left(\frac{k}{h}\right)^{\alpha}$$

Since w is common to all the households, we conclude that, at every instant when the aggregate capital in the economy is K, the aggregate input H satisfies

$$w = (1-\alpha)A\left(\frac{K}{H}\right)^{\alpha} \tag{15}$$

At any moment in time, the marginal product of the capital of any household in the economy is then

$$r^k \equiv \alpha A\left(\frac{k}{h}\right)^{\alpha-1} = \alpha A\left(\frac{K}{H}\right)^{\alpha-1} = \alpha A\left[\frac{(1-\alpha)A}{w}\right]^{\frac{1-\alpha}{\alpha}} \tag{16}$$

Consider aggregate functions $K(t, \tilde{t})$ and $H(t, \tilde{t})$ derived form the household's problem and assume that $K(t, \tilde{t})$ is increasing in t. Combining (14)-(15) we easily see that also $H(t, \tilde{t})$ is increasing in t. At any t prior to the turning $H(t, \tilde{t}) \leq \bar{H}$ and the marginal product of capital is constant and equal to

$$r_1^k = \alpha A\left[\frac{(1-\alpha)A}{w_1}\right]^{\frac{1-\alpha}{\alpha}} \tag{17}$$

An important property of the stage 1 of growth is that constant marginal returns to capital will allow us to construct a balanced growth path where all the aggregate and individual variables grow at the common rate $\bar{G}$.
At any time $t = \tilde{t} + \tau$ after the turning point the marginal cost of the intermediate good is $w_2(\tau) \equiv w(H(\tilde{t}+\tau, \tilde{t}))$. Then,

$$r_2^k(\tau) = \alpha A \left[\frac{(1-\alpha)A}{w_2(\tau)}\right]^{\frac{1-\alpha}{\alpha}} \tag{18}$$

Since $H(t, \tilde{t})$ is increasing in t it follows that $w_2(\tau)$ is increasing and $r_2(\tau)$ is decreasing for any $\tau \geq 0$. In particular, using the growth rates of the aggregate inputs, equations (14) and (15) imply that for any $\tau \geq 0$,

$$\dot{w}_2(\tau) = (1+\gamma)[w(\tau) - w_1 + \gamma]G^H(\tilde{t}+\tau, \tilde{t}) \tag{19}$$

$$\frac{\dot{r}_2^k(\tau)}{r_2^k(\tau)} = \frac{1-\alpha}{\alpha}\frac{\dot{w}_2(\tau)}{w_2(\tau)} = (1-\alpha)[G(\tilde{t}+\tau, \tilde{t}) - G^H(\tilde{t}+\tau, \tilde{t})]$$

and then

$$\frac{\dot{r}_2^k(\tau)}{r_2^k(\tau)} = -a(\tau)G(\tilde{t}+\tau, \tilde{t}) \tag{20}$$

where

$$a(\tau) = \frac{(1-\alpha)(1+\gamma)[w_2(\tau) - w_1 + \gamma]}{\alpha w_2(\tau) + (1+\gamma)[w_2(\tau) - w_1 + \gamma]} > 0$$

The final assumption that we need to make regards the structure of uncertainty on the level $\bar{H}$. It is common knowledge that $\bar{H}$ was drawn at the beginning of time from a Pareto distribution with parameters $H_0 > 0$ and $\eta \in (0, 1)$[2]. Consequently, agents have an unconditional belief on the realization of $\bar{H}$ given by

$$Prob(\bar{H} \leq H) = 1 - \left(\frac{H_0}{H}\right)^\eta$$

for $H \geq H_0$[3]. During the first stage of growth the amount of the aggregate intermediate good H increases at a balanced rate $\bar{G}$. The turning point $\tilde{t}$ is then random variable defined implicitly by

$$\bar{H} = H_0 e^{\bar{G}\tilde{t}}$$

[2]For this parametrization, the Pareto distribution has no finite first and second moment. This is of no consequence in the model. Low values for η create a distribution with a fat tail, which create beliefs that place a lot of mass on large realizations of $\bar{H}$.

[3]As initial condition on the information set of the agents I assume that $w_1 = (1-\alpha)A\left(\frac{K_0}{H_0}\right)^\alpha$.

Suppose that at time t during stage 1 the turning point has not been reached. Conditional on the observation of the amount H_t of aggregate input currently used into production, the probability that the turning point is reached before time $t + \epsilon$ is

$$Prob\{\bar{H} \leq H_t e^{\bar{G}\epsilon} | H_t\} = 1 - e^{-\eta \bar{G}\epsilon} = \eta \bar{G}\epsilon + o(\epsilon) \tag{21}$$

As ϵ goes to zero we obtain that the instantaneous arrival rate of the turning point is

$$\pi = \eta \bar{G} \tag{22}$$

The evolution of the marginal product of capital that I have derived above is aimed at modelling the following intuition. When a poor country is at the early stage of a growth process it may not experience right-away the existence of decreasing marginal returns to capital. This can be the case, for instance, because there is an initially a large pool of underutilized resources. The unemployed input can be a mix of natural resources, labor (Lewis [1954]), slow human capital and technological accumulation (Chari and Hopenhayn [1991]). However, as more of the initially unemployed inputs are used into production, the marginal cost of their provision starts to increase, and this generates a reduction in the marginal product of capital. Another interpretation is that, at the turning point, the process of growth is slowed down because the economy hits a country-specific technological barrier (Parente and Prescott [1994]). The faster an economy grows, the quicker the economy will hit its barrier, whose *level* is nonetheless uncertain. Different countries can have different levels of the barrier. While decreasing marginal returns eventually appear and countries converge to a steady state, a economy that has drawn a higher level of $\bar{H}$ features a higher steady state level of output. Seen from a different angle, this is indeed a model of *conditional convergence*, where the steady state level of output is uncertain.

The assumption that $\bar{H}$ has a Pareto distribution has the following implication. At any time t, the (instantaneous) conditional probability of reaching the turning point depends only on the balanced growth rate $\bar{G}$. The current level of H does not matter per se, since it doesn't say anything about the conditional probability of reaching $\bar{H}$. The Pareto distribution has therefore a sort of "memoryless" property which allows the structure of uncertainty to be unchanged as the economy grows at a constant rate.

Before giving the formal definition of equilibrium, notice that the assumption of a constant arrival rate π implicitly requires that in equilibrium the economy grows along a balanced growth path prior to the turning point, i.e.

$G(t,\tilde{t}) = \bar{G}$ for $t < \tilde{t}$ and all t. Moreover (20) implies that aggregate growth rates in state 2 must not depend on the particular realization of $\tilde{t}$ but just on the time $\tau = t - \tilde{t}$ elapsed since the turning point, or $G(t,\tilde{t}) = G(t-\tilde{t},0)$ for all $t \geq \tilde{t}$ and all $\tilde{t}$.

Definition 1. *An equilibrium is given by borrowing constraints m_1, m_2, prices r_1, r_2 and w_1, w_2, an arrival rate π, initial conditions (11) and policies $k(t,\tilde{t})$, $b(t,\tilde{t})$, $h(t,\tilde{t})$ such that*

i) The borrowing constraints are sustainable and satisfy (8).

ii) Optimality for the household: given prices, initial conditions and π, the policies are obtained from the solution to the household's problem.

iii) Consistent cost function: $w_2(0) = w_1$ and $w_2(\tau)$ satisfies (19) for any $\tau \geq 0$ and any $\tilde{t} \geq 0$, where $G^H(t,\tilde{t})$ is given by (13).

iv) Consistent arrival rate: $\pi = \eta G(t,\tilde{t}) = \eta\bar{G}$ where, $\forall (t,\tilde{t})$ with $t < \tilde{t}$, $G(t,\tilde{t})$ is given by (13).

v) Risk neutral pricing: $r_1(k,b)$ satisfies (10) and $r_2(k,b) = \rho$.

The existence of an equilibrium is not really an issue here. Consider the borrowing constraints $m_1 = m_2(\tau) = 0$, which are sustainable and for which we don't need to check condition (8), since they create no credit crunch at the turning point. In this case the economy grows in autarky, and standard arguments for the existence of an equilibrium for the closed economy can be used. The problem instead is that, in principle, there can be many equilibria, corresponding to different choices of the borrowing constraints. This raises the following question: is there a criterion for selecting some borrowing constraints because they appear to be more meaningful, from an economic point of view, than others? After all, in the example given above it seems that requiring autarky for the country imposes borrowing constraints that are "too tight". The next section answers this question by defining and characterizing an equilibrium with *endogenous borrowing constraints* which have the property that they represent, at any state, the maximum amount of debt that a household can credibly commit to repay.

4 The endogenous borrowing constraints

Endogenous borrowing constraints m_1^* and $m_2^*(\tau)$ satisfy this definition,

Definition 2. *The borrowing constraints* m_1^*, $m_2^*(\tau)$ *are endogenous if they characterize an equilibrium with the following properties*

i) For any alternative sustainable borrowing constraint $m_2(\tau)$ *we have*

$$m_2^*(\tau) \geq m_2(\tau) \quad \forall \tau \geq 0$$

ii) Consider any equilibrium characterized by alternative borrowing constraint $\hat{m}_1$, $\hat{m}_2(\tau)$ *such that* $\hat{m}_2(\tau)$ *satisfies i). Then,*

$$m_1^* \geq m_1$$

Notice that, contrary to models where borrowing constraints are derived endogenously for a given endowment process (see, for instance, Alvarez and Jermann [2000]), here we have a further complication. To assess sustainability of the debt levels we have to take into account the feed-back effect that different borrowing constraints have on the income process, i.e. on the path of growth of the economy. Hence, different borrowing constraints are in general associated with different evolutions of the GDP (Chattejee et. al. [2007]).

Lemma 1. *If* m_1^* *and* $m_2^*(\tau)$ *are endogenous borrowing constraints then, for all* $k > 0$ *and* $\tau \geq 0$

$$V^{2,ND}(k, m_2^*(\tau)k, \tau) = V^{2,D}(k, \tau) \tag{23}$$

Moreover, if $m_1^* > m_2^*(\tau)$*, for all* $k > 0$

$$V^{1,ND}(k, m_1^*k) = V^{1,D}(k) \tag{24}$$

Proof. Suppose that there is an interval of time $\tau \in [\underline{\tau}, \bar{\tau}]$ where

$$V^{2,ND}(k, m_2^*(\tau)k, \tau) > V^{2,D}(k, \tau)$$

From time $\underline{\tau}$ define a new borrowing constraint as follows. For $\tau \in (\underline{\tau}, \bar{\tau})$, $m_2(\tau) > m_2^*(\tau)$ with $\sup_{\tau \in (\underline{\tau}, \bar{\tau})} m_2(\tau) - m_2^*(\tau) = \epsilon > 0$. For $\tau = \underline{\tau}$ or $\tau \geq \bar{\tau}$, $m_2(\tau) = m_2^*(\tau)$. Under the alternative borrowing constraint we obtain new value functions $\tilde{V}^{2,ND}$, $\tilde{V}^{2,D}$ and by continuity we have for $\tau \in [\underline{\tau}, \bar{\tau}]$

$$\tilde{V}^{2,ND}(k, m_2(\tau)k, \tau) > \tilde{V}^{2,D}(k, \tau)$$

and for $\tau \geq \bar{\tau}$

$$\tilde{V}^{2,ND}(k, m_2(\tau)k, \tau) = \tilde{V}^{2,ND}(k, m_2^*(\tau)k, \tau) \geq V^{2,D}(k, \tau) = V^{2,D}(k, \tau)$$

From time $\underline{\tau}$ the new borrowing constraint is larger and sustainable. We may also set $m(\tau) = 0$ for $\tau < \underline{\tau}$ and obtain a sustainable borrowing constraint, alternative to $m_2^*(\tau)$, which violates requirement *i)*. The second property will made clear in (33), which shows that we have no equilibrium for m_1 larger than m_1^*, since with the leverage ratio m_1 the household is better off by defaulting. □

As pointed out in the introduction, this paper wants to show that it is possible for a country to accumulate a large stock of debt in the early stage of its development and then optimally default on it when the economy starts to slow down. The default at the turning point is due to a permanent credit crunch. The country could also follow a smoother path of growth, with initially less debt and no default. I would like to avoid the situation where the equilibrium with no default arises only because I have artificially set the borrowing constraint in stage 1 to a level "too tight", and thus no build up of debt can take place. At the same time, I want to avoid the creation of a credit crunch at the turning point only by restricting too much the maximum levels of debt $m_2(0)$ in stage 2. By focusing on the endogenous borrowing constraints I rule out these two types of problems. For the latter, the definition of m_2^* ensures that a single household is not allowed to exceed the borrowing constraint only because otherwise she wouldn't be able to credibly commit to repay her debt, as required by the sustainability of the borrowing constraint, given the equilibrium evolution of the marginal product of capital. For the former, m_1^* makes sure that the economy follows the equilibrium with the largest sustainable borrowing constraint in stage 1, provided that borrowing constraints are endogenous in stage 2.

Proposition 1. *There is an equilibrium with endogenous borrowing constraints, which are always binding. Depending on the parameters values, either $m_1^* = m_2^*(0)$ or $m_1^* > m_2^*(0)$. Moreover, $\dot{m}_2^*(\tau) < 0$ whenever $m_2^*(\tau) > 0$.*

If the endogenous borrowing constraints are such that $m_1^* = m_2^*(0)$, then no default takes place at the turning point, and the path of growth smoothly decreases after the turning point until the economy reaches the steady state. The remaining of this section focuses on the other possibility, which is that there is a credit crunch ($m_1^* > m_2^*(0)$) and a default event. The

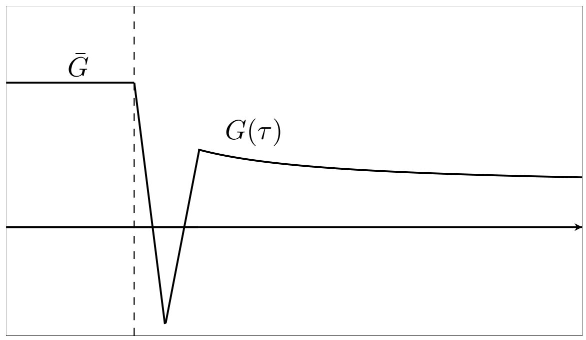

Figure 5: Balanced growth $\bar{G}$ and permanent reduction in growth rate for an equilibrium with credit crunch.

associated growth process displays a boom-bust cycle. Figure 5 shows that the economy grows at a fast pace during stage 1, due to the large availability of credit. At the turning point, the households cannot credibly commit to repay their stock of debt anymore. The consequent rational behavior of the international investors creates a credit crunch which suddenly reduces the borrowing ability of the country which defaults. The recession at the time of the turning point is created by the output costs. Even though households eventually regain access to the financial market, the growth and investment rates are permanently reduced compared to stage 1 because, as seen in Figure 4, the credit crunch is permanent ($m_2^*(\tau) < 0$).

I now derive the main results needed to prove Proposition 1 and characterize the equilibrium with default, while leaving some details to Appendix C. Whenever it does not create confusion, I omit the indication of τ for variables in stage 2, for instance I may write m_2^* for $m_2^*(\tau)$.

First of all, using the definitions (17), (18) for the marginal product of capital r^k we find that, both in stage 1 and at any time in stage 2, the optimal net output for a household with capital k and in the ND state is

$$Ak^\alpha h^{1-\alpha} - wh = r^k k \tag{25}$$

Similarly, for a household in the D state,

$$(1-\xi)(Ak^\alpha h^{1-\alpha} - wh) = (1-\xi) r^k k \tag{26}$$

To make the problem interesting I assume that $r_1^k - \delta > \rho$. Starting from $r_2^k(0) = r_1^k$, the marginal product of capital in stage 2 decreases until the steady state is reached, where $r^k - \delta = \rho$. Therefore the spread $r_2^k(\tau) - \rho$

between the marginal product of capital and the interest rate ρ charged on loans in stage 2 is always strictly positive before the steady state. It is not surprising then that the borrowing constraint $m_2^*(\tau)$ has to be binding in equilibrium. Since default takes place at the turning point, in stage 1 households pay in equilibrium an interest rate $r_2 = \rho + \eta\bar{G}$ on their debt. We now guess, and verify later, that $m_1^* < 1$ and that the equilibrium balanced growth rate $\bar{G}$ equals

$$\bar{G} = \frac{r_1^k - \delta - \rho}{1 - (1-\eta)m_1^*} \tag{27}$$

In this case

$$(r_1^k - \delta) - r_1 = (r_1^k - \delta - \rho)\left[1 - \frac{\eta}{1-(1-\eta)m_1^*}\right] > 0 \tag{28}$$

Again the spread $(r_1^k - \delta) - r_1$ is strictly positive and the borrowing constraint must bind in stage 1.[4] Using (25)-(26) we see that the constraints in the household's problem are linear in both the state k, b and in the control variables x, d. Since the utility is homothetic we can detrend the household's problem, which can be written in terms of leverage ratios $\frac{b}{k}$. In particular,

$$\begin{array}{ll} V^{2,D}(k,\tau) = V^{2,D}(1,\tau) + \frac{\log k}{\rho}; & V^{2,ND}(k,b,\tau) = V^{2,ND}\left(1,\frac{b}{k},\tau\right) + \frac{\log k}{\rho} \\ V^{1,ND}(k,b) = V^{1,ND}\left(1,\frac{b}{k}\right) + \frac{\log k}{\rho}; & V^{1,D}(k) = V^{1,D}(1) + \frac{\log k}{\rho} \end{array} \tag{29}$$

Consider the situation where, at some state, the household has zero debt, which happens for instance at the moment of readmission to the financial market after default. Since borrowing constraints are always binding, the leverage ratio jumps immediately from zero up to it's maximum value m^*. In particular, the capital of the household jumps from a level k up to $\frac{1}{1-m^*}k$, a multiplicative effect due to the possibility of leveraging the capital accumulation through the international financial markets[5]. Therefore,

$$\begin{aligned} V^{2,ND}(1,0,\tau) &= V^{2,ND}(1,m_2^*(\tau),\tau) + \frac{1}{\rho}\log\frac{1}{1-m_2^*(\tau)} \\ V^{1,ND}(1,0) &= V^{1,ND}(1,m_1^*) + \frac{1}{\rho}\log\frac{1}{1-m_1^*} \end{aligned} \tag{30}$$

[4]More precisely, this shows that, provided that the household pays an interest rate $\rho + \pi$ she prefers borrowing up to m_1^*. However, she could borrow only up to $m_2^*(0)$ and being charged the lower rate $r_2(k, m_2^*(0)k) = \rho$. However, in equilibrium, this is never optimal.

[5]See Appendix A for more details.

The household's problem can then be fully characterized in its detrended form (omitting the indication $k = 1$) by (30) and

$$\rho V^{2,ND}(m_2^*, \tau) = \max_g \log c + V_m^{2,ND}(m_2^*, \tau)\dot{m}_2^*(\tau) + V_\tau^{2,ND} + \frac{g}{\rho}$$
$$\text{s.t.} \quad c = r_2^k(\tau) - \delta + \dot{m}_2^*(\tau) - g - m_2^*(\tau)(\rho - g)$$

$$(\rho + \theta + \pi)V^{1,D} = \max_{g^d} \log c^d + \theta V^{1,ND}(0) + \pi V^{2,D}(0) + \frac{g^d}{\rho}$$
$$\text{s.t.} \quad c^d = (1 - \xi_1)r_1^k - \delta - g^d$$

$$(\rho + \pi)V^{1,D}(m_1^*) = \max_g \log c + \pi V^{2,D}(0) + \frac{g}{\rho}$$
$$\text{s.t.} \quad c = r_1^k - \delta - g - m_1^*(r_1^k - g)$$

$$(\rho + \theta)V^{2,D}(\tau) = \max_{g^d} \log c^d + \theta V^{2,ND}(0, \tau) + V_\tau^{2,D}(\tau) + \frac{g^d}{\rho}$$
$$\text{s.t.} \quad c^d = [1 - \xi(\tau)]r_2(\tau) - \delta - g^d$$

with $g = \frac{\dot{k}}{k}$ the growth rate of individual capital. The convenience of using log utility lies in the property that the detrended value functions turn out to be linear in the growth rate of capital. Only for analytical simplicity, I assume that output costs are not too large and the net return of capital $(1-\xi)r^k - \delta$ in the default stage never falls below the international risk free rate ρ,

$$1 - \xi_2(\tau) = \max\left\{1 - \xi_1, \tfrac{\rho+\delta}{r_2^k(\tau)}\right\} \quad \tau \geq 0 \tag{31}$$

with $(1 - \xi_1)r_1^k > \rho + \delta$. For any τ, the first order conditions in stage 2 are then

$$c = \rho(1 - m_2^*)$$
$$g = \frac{r_2^k - \delta - \rho + \dot{m}_2^*}{1 - m_2^*}$$
$$c^d = \rho$$
$$g^d = (1 - \xi_2)r_2^k - \delta - \rho$$

and in stage 1 are

$$
\begin{aligned}
c &= \rho(1 - m_1^*) \\
g &= \frac{r_1^k - \delta - \rho - \pi m_1^*}{1 - m_1^*} \\
c^d &= \rho \\
g^d &= (1 - \xi_1) r_1^k - \delta - \rho
\end{aligned}
$$

If we substituted $g = \bar{G}$ and $\pi = \eta \bar{G}$ we easily verify the balanced growth rate (27).
Conditions (23)-(24) are equivalent to

$$\log c + \frac{g}{\rho} = \log c^d + \frac{g^d}{\rho} + \theta \log \frac{1}{1 - m_2^*(\tau)}$$

$$\log c + \frac{g}{\rho} = \log c^d + \frac{g^d}{\rho} + \theta \log \frac{1}{1 - m_1^*}$$

If we substitute for the first order conditions and write $r_1 = \rho + \pi$ and $r_2 = \rho$ we obtain, after rearranging,

$$\frac{m_2^*}{1 - m_2^*}(r_2^k - \delta - r_2) + \xi_2 r_2^k = (\rho + \theta) \log \frac{1}{1 - m_2^*} - \frac{\dot{m}_2^*}{1 - m_2^*} \tag{32}$$

$$\frac{m_1^*}{1 - m_1^*}(r_1^k - \delta - r_1) + \xi_1 r_1^k = (\rho + \theta) \log \frac{1}{1 - m_1^*} \tag{33}$$

The terms on the left hand side of the equations can be considered the benefit for non defaulting and maintaining access to the financial market, while the terms on the right hand side are the opportunity costs of not defaulting. The endogenous borrowing constraints exactly equate these costs and benefit. When the endogenous borrowing constraint is low most of of the benefits from not defaulting are due to the fact that the household avoids the output costs ξr. Instead, when the endogenous borrowing constraints have a large value, most of the benefits are due to the amplification effect $\frac{m^*}{1-m^*}$ that access to the financial market has on the profitability for the household to exploit the spread $(r^k - \delta) - r$ between the net return on capital at home and the international borrowing rate. Turning to the costs, the first term on the right hand side represents the utility value of the wealth gain accruing to the household from the default choice. A part of this gain, proportional to ρ, is a direct effect from the elimination of the stock of debt from the household's budget constraint, while the part proportional to θ accounts for the jump in the household's capital occurring if she is instantaneously readmitted, with zero debt, to the financial market. Since, as we will see, the borrowing constraint is progressively tightened in stage 2, an additional term takes into account the costs for the household stemming from this deleveraging process. Two properties should be emphasized. The first is that m_1^* is bounded above by one, since we can see that the spread (28 goes to zero as m_1^* approaches one. Intuitively, if leverage is high the growth rate (27) is also very high, then the arrival rate of the turning point is endogenously increased, the risk premium rises and the spread disappears. Second, in stage 2 of growth marginal returns to capital fall over time and thus the benefits of not defaulting decrease for any given leverage level. To keep up with the fall of incentives to repay the debt, the leverage ratio has also to decrease, i.e. $\dot{m}_2^* < 0$. If we bind below the opportunity cost of default in (32) by considering only the wealth gain, we obtain a picture like Figure 6. At any time τ the marginal return to capital is given by an equilibrium function $r^k(\tau)$ to which corresponds a region for the borrowing constraint m_2^* where the benefits from not defaulting are smaller than the lower bound on the opportunity costs. This default region becomes larger over time as marginal returns decrease. In a typical equilibrium with credit crunch, intermediate values of leverage lie inside the default region because they are too high to be supported just by output costs and too low for the amplification effect to be strong enough. As the figure shows, the economy might enter stage 2 with a high leverage and large stock of debt accumulated in stage 1. Initially marginal returns are still very high, the amplification effect is strong and the household might find it valuable to keep repaying the debt. However, as time goes by and marginal returns fall, the leverage level eventually enters the default region. At the turning point, international investors anticipate this future event and create a credit crunch. The equilibrium borrowing constraints permanently adjust to lowers levels for which the deleveraging process is credible.

Equation (32) gives the endogenous borrowing constraints in stage 2 given the equilibrium evolution of the marginal product of capital r_2^k. In turn r_2^k is defined from the motion of m_2^* in the following way. First of all, recall that after default households regain access to the financial market at a rate θ. At any time τ a certain fraction $\phi(\tau)$ of the capital is allocated to households in the default state. The linearity in k of the household's problem allows us to aggregate capital across households in the D state and across those in

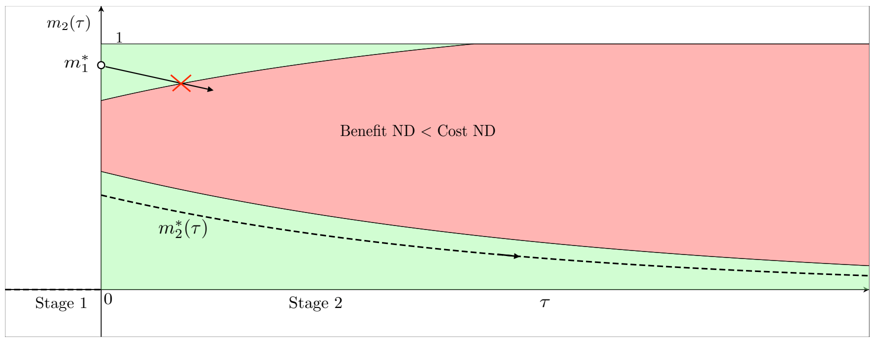

Figure 6: Default area and credit crunch.

the ND state to write the aggregate growth rate of capital at time τ as[6]

$$G_\tau = (1-\phi_\tau)g_\tau + \phi_\tau g_\tau^d + \phi_\tau \theta \frac{m_\tau^*}{1-m_\tau^*} \tag{34}$$

where the last term takes into account the fact that, at each instant, a fraction of households in the default state regain access to the financial market and have their capital stock jump proportionally to the borrowing constraint. The motion of ϕ is

$$\dot{\phi}_\tau = -\phi_\tau(\theta + G_\tau - g_\tau^d) \tag{35}$$

Combining (34) and (35) into (18) we can calculate $r_2^k(\tau)$ given $m_2^*(\tau)$. The system of differential equations is completed by adding (32) and initial conditions $r_2^k(0) = r_1$ and $\phi(0) = 1$. In Appendix C I show that a solution to this system exists and is uniquely determined by the endogenous borrowing constraint $m_2^*(0)$ at the turning point.

It is interesting to notice that the equilibrium with the credit crunch arises only if the initial marginal product of capital r_1^k is sufficiently high[7], otherwise the economy follows the smoother path with $m_1^* = m_2^*(0)$. In fact remember that if $m_1^* > m_2^*(0)$ the household pays an interest rate which incorporates a risk premium, which could be avoided if she were to choose a lower leverage ratio $b = m_2^*(0)$. Borrowing more and paying the premium is optimal only if r_1 is high enough to justify strong the amplification effect that the household can exploit by choosing the higher leverage m_1^*. Moreover, a stronger punishment for default (lower θ and higher ξ), which can

[6]Equations (34) and (35) below are derived in Appendix B.

[7]This point is made clear in the next section when I discuss the application of the Bulow and Rogoff (1989) result.

be interpreted as providing a higher degree of debt enforceability, can have the result of increasing the volatility of the growth process. For instance, in the extreme case of $\theta \to \infty$, punishment for default disappears and the equilibrium converges to the autarky case where no credit crunch and no boom-bust take place, since the economy follows the smooth path of transitions from stage 1 to stage 2.
Notice that business cycle considerations play no role in the causal explanation of the crises, which is entirely driven by a long run process of growth which displays decreasing marginal returns to capital. However, a naive observer may rationalize the crisis as due to a bad shock to the TFP. Over time, the weighted average of the Solow residuals (TFP) across the production technologies of the households is

$$TFP(\tau) = [1 - \phi(\tau)]A + \phi(\tau)[1 - \xi(\tau)]A = [1 - \xi(\tau)\phi(\tau)]A$$

The Solow residuals that I compute are just the average of the individual TFP of the households in non default default and default state, weighted by the fraction ϕ of capital allocated those in default. The time series of the TFP would show a constant value A up to the crises, then a sharp fall down to $(1-\xi)A$ at the turning point ($\phi(0) = 1$) and a subsequent smooth reversion to the long run value A, ($\phi(\tau), \xi(\tau)$ decreasing, $\phi(\infty) = 0$). However, it would be clearly a mistake to attribute the financial crises to a business cycle type of shock. The TFP fall *is caused* by the financial crises and not viceversa (Mendoza [2008] makes a similar point). Output costs and fall in the measured TFP can be due, for instance, to the lack of access to working capital (Neumayer and Perri [2005]).
In the next section I show that, will a small modification in the basic framework of the model, we can obtain a permanent reversal in the balance of payment after the financial crisis.

5 Bubbles, debt and the current account

In their influential paper, Bulow and Rogoff (1989) showed that no contingent debt contract can be enforced between a lender and a borrower whenever the only punishment for a defaulting borrower is the permanent exclusion from future borrowing (but not from lending). The model that I have constructed gives an instructive example of how the Bulow-Rogoff (from now BR) result is strictly connected to a transversality condition that rules out bubbles.
The formulation of BR is easily replicated here by setting $\theta = 0$ and $\xi = 0$.

It is straightforward to see that the assumption that the household can lend abroad after defaulting is inconsequential, since at each point in time $r^k - \delta \geq \rho$.
We can show that, consistently with the BR result[8],

$$m_2^*(\tau) = 0$$

However, before the turning point, the endogenous borrowing constrain can be strictly positive. Substituting for $r_1 = \rho + \eta \bar{G}$ we rewrite (33) as

$$f(m_1^*) = (\rho + \theta) \log(1 - m_1^*) + \frac{(1-\eta)m_1^*}{1-(1-\eta)m_1^*}(r_1^k - \delta - \rho) = 0 \quad (36)$$

Notice that,

$$\frac{df(0)}{dm_1^*} = (1-\eta)(r_1^k - \delta - \rho) - \rho$$

If r_1^k is sufficiently large $\frac{df(0)}{dm_1^*} > 0$ which implies that m_1^*, the largest solution to $f(m) = 0$, is strictly positive. Why does the BR result seem to fail here? Recall that $r_1 = \rho + \eta \bar{G}$ and $\bar{G}$ is given by (27). We can write $f(m_1^*) = 0$ as

$$\rho[\log(1 - m_1^*) + m_1^*] - m_1^*(r_1 - \bar{G}) = 0$$

It is easy to see that, whenever $m_1^* > 0$ we must have

$$r_1 - \bar{G} < 0 \quad (37)$$

The balanced growth path prior to default is non autarkic and strictly positive debt can be sustained only if the "effective interest rate" $r_1 - \bar{G}$ is negative. In other words, if the marginal product of capital r_1^k is high enough, the aggregate capital and the aggregate debt grow at a rate bigger that the interest rate. In this situation the household continuously rolls over her current debt obligations by issuing new debt and has no incentive to default.
To follow the notation of BR define $D^0(T)$ as the time zero *value of the net payments* $-d_t$ up to time T that international investors receive from the small open economy. After the turning point $\tilde{t}$ the economy is in autarky, then $d_t = 0$. The law of motion of the stock of debt prior to the turning point can be written in a detrended form as $\dot{\hat{b}}_t = (r_1 - g_t)\hat{b}_t + \hat{d}_t$, where $b_t = \hat{b}_t k_t$

[8] After the tuning point the growth rate of capital, and thus the growth rate of debt, eventually converge to zero. Then, conditional on being in stage 2 the transversality condition, presented below, on the evolution of the stock of debt has to hold and the application of the BR result implies that no debt can be sustained.

and $d_t = \hat{d}_t k_t$. Since $\hat{b}_t = m_1^*$ and $g_t = \bar{G}$ we have $-\hat{d}_t = (r_1 - \bar{G})m_1^*$ for all t in stage 1, then

$$D^0(T) = E\left[-\int_0^T e^{-\rho t}\hat{d}_t k_t dt\right] = (r_1 - \bar{G})m_1^* k_0 \int_0^T e^{-(r_1-\bar{G})t}dt < 0$$

where I used the fact that the arrival rate π needed to compute the expectation is $\pi = \eta\bar{G} = r_1 - \rho$. The net present value at time zero of the households payments is then $D^0(\infty) = -\infty$. Any equilibrium where the present value of the payments is negative is implicitly ruled out by BR. Debt contracts having a negative net present value for the lenders might be considered non optimal. Nonetheless, we can show that the debt contract we are dealing with is of a particular sort. Recall that rationality for the lenders is here imposed in part *v)* of Definition 1 only by requiring a risk neutral pricing of the debt asset. However, optimal investment plans often requires that a transversality condition, which rules out bubbles, has also to hold. Define $p^0(t)$ the time zero price of an Arrow security that pays one unit of consumption good at time t if the turning point $\tilde{t}$ has not been reached at t, and zero otherwise. Risk neutral pricing implies that

$$p^0(t) = e^{-(\rho+\pi)t} = e^{-r_1 t}$$

The equilibrium *face value* b_t of an household's debt at time t equals $b_0 e^{\bar{G}t}$ if the turning point has not been reached at t, and zero otherwise (default). The price at time zero of b_t is then $P^0(t) = p^0(t)b_t > 0$. We have

$$\lim_{t\to\infty} P^0(t) = \lim_{t\to\infty} e^{-(r_0-\bar{G})t}b_0 = +\infty$$

The value $P^0(t)$ of the wealth invested by the international lenders into the economy goes to infinity as t increases. The usual transversality condition, according to which such value goes to zero, does not hold. The possible source of non optimality can be seen in fact that $P^0(t) > D^0(t)$ for all t, i.e. the price P^0 of the debt asset is always greater than its "fundamental" D^0. I name this situation a *bubble equilibrium.* Without a more precise specification of the problem solved by the international lenders, the simple requirement of a risk neutral pricing is not enough to rule out bubble equilibria. Indeed, equilibria with rational bubbles can be obtained in a number of ways (see for instance Weil [1989], Jovanovic [2007], Hellwig and Lorenzoni [2007]). For the existence of an equilibrium with rational bubbles it is generally necessary that the bubble asset has a bounded supply (Jovanovic [2007]). Debt does not seem to fall in such a category of assets. Interestingly, in my model, the bound on the amount of debt that the country can

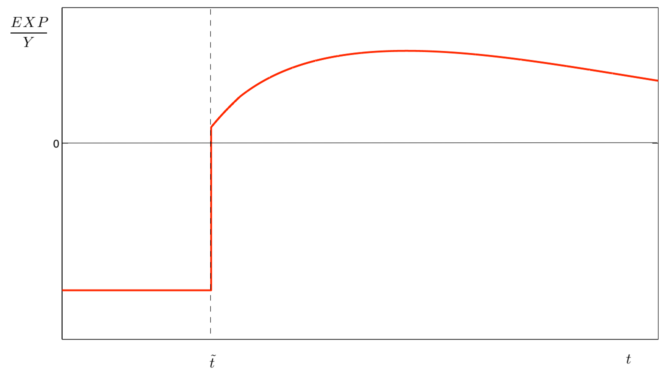

Figure 7: Net exports detrended by GDP for an equilibrium with default and debt renegotiation.

issue is endogenously obtained because incentives to default don't allow the leverage ratio to be greater than m_1^*. At the same time, the very existence of a bubble provides incentives, through the amplification effect, for the households not to default before the turning point, the moment where the bubble bursts. This result resembles Hellwig and Lorenzoni (2007). The discussion on the bubble equilibria introduces us to the problem of assessing whether the model can qualitatively account for the type of dynamics of the current account that we see in Figure 2. The answer to this question is affirmative, as I show in the remaining of this section.

The probelm is addressed in a cleaner way if we make a straightforward extension of the model and we allow for partial default in the form of debt renegotiation. Assume that, at the turning point, an exogenous fraction $1 - \phi > 0$ of households have the option of choosing between complete default and renegotiating their current total debt $b_{\tilde{t}} = m_1^* k_{\tilde{t}}$ to a new level $\bar{m} k_{\tilde{t}}$. Renegotiation has the advantage of sparing the household the punishment inflicted to agents that default on their entire debt. If we assume that the renegotiation process extracts all the surplus from the borrowers in favor of international lenders, the level of the renegotiated debt is exactly $\bar{m} k_{\tilde{t}} = m_2^*(0) k_{\tilde{t}}$. In fact, by construction of the endogenous borrowing constraints, $V^{2,ND}(m_2^*(0), 0) = V^{2,D}(0)$, and the houshold is exactly indifferent between accepting and rejecting the renegotiation offer. The endogenous borrowing constraints $m_2^*(\tau)$ solve the usual system of differential equation

with initial condition[9] $\phi(0) = \phi$. Also, for simplicity, assume that $\theta = 0$ so that the fraction ϕ of households that default disappear forever from any interaction with the financial market. On the contrary, for households that renegotiated the debt, borrowing and repayment continues.
The equilibrium interest rate r_1 takes into account that default is only partial and that international lenders are able to recover a fraction Δ of their investment,

$$\Delta = (1-\phi)\frac{m_2^*(0)}{m_1^*}$$

Therefore,

$$r_1 = \rho + \pi(1-\Delta) = \rho + \eta\bar{G}(1-\Delta) = \rho + \eta'\bar{G}$$

where $\eta'(m_1^*) = \eta(1-\Delta)$ is now a parameter that changes with m_1^*, given $m_2^*(0)$ (which is independent on m_1^*). The endogenous borrowing constraint m_1^* is found as before as the largest solution to $f(m) = 0$, where the function f in (36) is now constructed with an endogenous parameter η'. This creates no major departure from the characterization of the equilibrium presented above for the case of complete default.
The dynamics of the current account of the country mirror that of the households with access to the financial market. Prior to default, the (detrended) net exports are

$$-\hat{d} = m_1^*(r_1 - \bar{G})$$

As we have seen above, it can be the case that $\bar{G} > r_1$, the country is a net importer and experiences a capital inflow. The detrended version of the motion of debt at any time τ after the turning point gives net exports

$$-\hat{d}_\tau = m_2^*(\tau)(\rho - g_\tau) - \dot{m}_2^*(\tau)$$

If the sudden contraction in the borrowing constraint after the tuning point is sufficiently sharp, the permanent reduction in the growth rate (Figure 5) can be strong enough that $g_\tau < \rho$ for all $\tau \geq 0$. Moreover $\dot{m}(\tau) \leq 0$, and then $-\hat{d}_\tau > 0$. This situation is shown in Figure 7.

6 Conclusions

I constructed a model that links together, along a path of development of a small open economy, the growth rate of output, the direction of capital flows

[9]The usual system of differential equations is now solved for $\phi(0) = \phi$.

and the possibility of a financial crisis. Households of the small economy borrow from international investors to finance the capital accumulation which drives the economic development. Default on the international debt is possible, which determines the presence of endogenous borrowing constraints. The borrowing constraints interact with the growth process giving rise to interesting dynamics whenever we introduce a simple element of uncertainty. Marginal returns to capital are initially constant and they start to decrease when the economy has reached an exogenously given level of development that I call "turning point". Agents have only a probabilistic knowledge of the level of the turning point, and the speed at which it is reached depends on the growth rate of the economy, which in turn is determined by how tight are the endogenous borrowing constraints. Two paths of development are possible. In one case, the borrowing constraints are large, the economy grows very rapidly until the turning point is reached. Following the initial credit expansion the economy is hit by a sudden stop and a financial crisis. The country eventually recovers from the distressed periods, but the tightness of the borrowing constraints permanently reduces the growth of output and the investment rate. If the credit crunch is sufficiently severe the balance of payment is reversed. This path if followed by economies with initial large marginal product of capital or that face strong punishments for default. The second possibility is that the endogenous borrowing limits are tight from the very beginning. This produces a slower but more stable growth, since no crisis takes place at the turning point. I also show that there is a strict connection between debt sustainability and the existence of equilibria featuring the presence of bubbles. The growth rate of the stock of debt can be larger than the interest rate during the first stage of growth, generating a bubble-like equilibrium which provides incentives to the borrowers not to default. The bubble bursts at the turning point, when the rate of growth of the debt is set on a path of permanent decrease.

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Appendix A

First of all I show that the value function $V^{2,ND}$ can be detrended, without explicitly addressing the problem of jumps. Then in, the detrended setting, I derive the solution to the model with jumps as the limit of solutions with continuous paths. These property hold for the other value functions and the proof is similar, so it's left to the reader. Define

$$\hat{b}_t = \frac{b_t}{k_t} \quad \hat{c}_t = \frac{c_t}{k_t}, \quad \hat{d}_t = \frac{d_t}{k_t}, \quad g_t = \frac{\dot{k}_t}{k_t}$$

We start by guessing that $\rho V^{2,ND}(k,b,\tau) = \rho \hat{V}(\hat{b},\tau) + \log k$ for some function $\hat{V}(\cdot,\cdot)$ that solves

$$\rho \hat{V}(\hat{b},\tau) = \max_{g,\hat{d}} \log \hat{c} + \hat{V}_{\hat{b}}(\hat{b},\tau)\dot{\hat{b}} + \hat{V}_\tau(\hat{b},\tau) + \frac{g}{\rho} \tag{38}$$

$$\begin{aligned} \text{s.t.} \quad & \hat{c} = r_2^k(\tau) - \delta + \hat{d} - g \\ & \dot{\hat{b}} = \rho\hat{b} + \hat{d} - \hat{b}g \\ & \hat{b} \leq m_2(\tau) \end{aligned}$$

Notice that,

$$\dot{\hat{b}} = \frac{\dot{b}}{k} - \hat{b}g$$

Then our guess implies

$$V_k^{2,ND}(k,b,\tau)\dot{k} = -\hat{V}_{\hat{b}}(\hat{b},\tau)\hat{b}g + \frac{g}{\rho}$$

$$V_b^{2,ND}(k,b,\tau)\dot{b} = \hat{V}_{\hat{b}}(\hat{b},\tau)\frac{\dot{b}}{k}$$

From which,

$$V_k^{2,ND}(k,b,\tau)\dot{k} + V_b^{2,ND}(k,b,\tau)\dot{b} + V_\tau^{2,ND}(k,b,\tau) = \hat{V}_{\hat{b}}(\hat{b},\tau)\dot{\hat{b}} + \hat{V}_\tau(\hat{b},\tau) + \frac{g}{\rho}$$

Substituting into the definition of $V^{2,ND}(k,b,\tau)$ we obtain

$$\rho V^{2,ND}(k,b,\tau) = \max_{g,\hat{d}} \log \hat{c} + \log k + \hat{V}_{\hat{b}}(\hat{b},\tau)\dot{\hat{b}} + \hat{V}'_\tau(\hat{b},\tau) + \frac{g}{\rho}$$

$$
\begin{aligned}
\text{s.t.} \quad & \hat{c} = r_2^k(\tau) - \delta + \hat{d} - g \\
& \dot{\hat{b}} = \rho \hat{b} + \hat{d} - \hat{b} g \\
& \hat{b} \leq m_2(\tau)
\end{aligned}
$$

which verifies our initial guess. By noticing that $V^{2,ND}(1, b, \tau) = \hat{V}(\hat{b}, \tau)$ we have shown how to detrend the problem of the household.
From the discussion in the paper we know that the borrowing constraint have always to bind whenever $r^k(\tau) > \rho$. In this case, if $\hat{b} < m_2(\tau)$ the leverage ratio $\hat{b}$ jumps immediately to the bound $m_2(\tau)$. I now show how this discontinuous process for the leverage ratio is obtain as the limit of a continuous path. For notational simplicity I drop the hat from the variables in the detrended problem. Let us bind above the instantaneous growth of the leverage ratio by some constant $\bar{D} > 0$ so that the problem becomes

$$
\rho \hat{V}(b, \tau) = \max_{g,d} \log c + \hat{V}_{\hat{b}}(b, \tau) \dot{b} + \hat{V}_{\tau}(b, \tau) + \frac{g}{\rho}
$$

$$
\begin{aligned}
\text{s.t.} \quad & c = r_2^k(\tau) - \delta + d - g \\
& \dot{b} = \rho b + d - b g \\
& b \leq m_2(\tau) \\
& \dot{b} \leq \bar{D}
\end{aligned}
$$

If at some time τ we have $b_\tau < m_2(\tau)$ the new constraint $\dot{b} \leq \bar{D}$ is binding. By continuity of $m_2(\tau)$ we have

$$
m_2(\tau + \epsilon) = m_2(\tau) + u(\epsilon)
$$

for some function $u(\epsilon)$ such that $u(\epsilon) \to 0$ as $\epsilon \to 0$. Given the leverage b_τ at time τ, set the value of $\bar{D}$ to

$$
\bar{D} = \frac{m_2(\tau) + u(\epsilon) - b_\tau}{\epsilon}
$$

It is easy to check that

$$
b_{\tau+\epsilon} = b_\tau + D\epsilon = m_2(\tau + \epsilon)
$$

It takes exactly a span of time ϵ for the leverage ratio to reach the borrowing constraint starting from a value $b_\tau < m_2(\tau)$ at time τ. If $\bar{D}$ is sufficiently large the constraint on $\dot{b}$ is not binding anymore from time $\tau + \epsilon$ since the

borrowing constraint is binding.
For any $\tilde{\tau}$ such that $\tau \leq \tilde{\tau} < \tau + \epsilon$ the first order condition with respect to $g_{\tilde{\tau}}$ imply

$$c_{\tilde{\tau}} = \rho(1 - b_{\tilde{\tau}}) = r_2^k(\tilde{\tau}) - \delta - g_{\tilde{\tau}} + d_{\tilde{\tau}} = r_2^k(\tilde{\tau}) - \delta - (1 - b_{\tilde{\tau}})g_{\tilde{\tau}} + \bar{D} - \rho b_{\tilde{\tau}}$$

which gives

$$g_{\tilde{\tau}} = \frac{r_2^k(\tilde{\tau}) - \delta - \rho + \bar{D}}{(1 - b_{\tilde{\tau}})} = \frac{r_2^k(\tilde{\tau}) - \delta - \rho + \bar{D}}{1 - b_\tau - (\tilde{\tau} - \tau)\bar{D}}$$

Integrating with respect to $\tilde{\tau}$ from τ to $\tau + \epsilon$ we obtain

$$\begin{aligned} \log \frac{k_{\tau+\epsilon}}{k_\tau} &= -\frac{r_2^k(\tau) - \delta - \rho + \bar{D}}{\bar{D}} \log \frac{1 - b_\tau - \epsilon\bar{D}}{1 - b_\tau} \\ &= -\frac{r_2^k(\tau) - \delta - \rho}{m_2(\tau) + u(\epsilon)} \epsilon \log \frac{1 - m_2(\tau + \epsilon)}{1 - b_\tau} - \log \frac{1 - m_2(\tau + \epsilon)}{1 - b_\tau} \end{aligned}$$

Taking limits for $\epsilon \to 0$ we finally have

$$\frac{k_{\tau+\epsilon}}{k_\tau} \to \frac{1 - b_\tau}{1 - m_2(\tau)}$$

From the continuity of the value function and the property derived above

$$\begin{aligned} V^{2,ND}(k_\tau, b_\tau k_\tau, \tau) &= V^{2,ND}(k_{\tau+\epsilon}, m_2(\tau + \epsilon)k_{\tau+\epsilon}, \tau + \epsilon) + o(\epsilon) \\ &= \hat{V}(m_{t+\epsilon}, \tau + \epsilon) + \frac{\log k_{t+\epsilon}}{\rho} \end{aligned}$$

Since $\lim_{\epsilon \to 0} k_{t+\epsilon} = \frac{1-b_\tau}{1-m_2(\tau)} k_\tau$ we finally have

$$\lim_{\epsilon \to 0} V^{2,ND}(k_\tau, b_\tau k_\tau, \tau) = V(m_{t+\epsilon}, \tau + \epsilon) + \frac{1}{\rho} \log \frac{1 - b_\tau}{1 - m_2(\tau)} k_\tau$$

In the paper we use the special case $b_\tau = 0$ and the detrended model with $k_\tau = 1$.

Appendix B

I derive the equations determining the evolution of the aggregate growth rate of capital and of the fraction ϕ of capital allocated to households in default.
Total capital K evolves according to

$$K_{t+\epsilon} = (1-\phi_t)K_t e^{\int_0^\epsilon g_{t+\tau}d\tau} + \phi_t K_t \left[e^{-\theta\epsilon} e^{\int_0^\epsilon g^d_{t+\tau}d\tau} + \int_0^\epsilon \frac{e^{\int_0^\tau g^d_{t+T}dT + \int_\tau^\epsilon g_{t+T}dT}}{1-m_{t+\tau}} \theta e^{-\theta\tau} d\tau \right]$$

which gives

$$K_{t+\epsilon} = K_t[(1-\phi_t)(1+g_t\epsilon) + \phi_t \left[1 + (g^d_t - \theta)\epsilon + \frac{\theta}{1-m_t}\epsilon\right] + o(\epsilon)$$

Define $G_t = \lim_{\epsilon\to 0} \frac{K_{t+\epsilon-K_t}}{\epsilon K_t}$ and equation (34) is obtained.
The evolution of the share ϕ_t is the following,

$$1 - \phi_{t+\epsilon} = \frac{K_{t+\epsilon} - \phi_t K_t e^{-\theta\epsilon} e^{\int_0^\epsilon g^d_{t+\tau}d\tau}}{K_{t+\epsilon}}$$

$$\phi_{t+\epsilon} = \phi_t \frac{e^{-\theta\epsilon + \int_0^\epsilon g^d_{t+\tau}d\tau}}{e^{\int_0^\epsilon G^d_{t+\tau}d\tau}}$$

$$\phi_{t+\epsilon} = \phi_t[1 + (g^d_t - \theta - G_t)\epsilon] + o(\epsilon)$$

from which we get (35).

Appendix C

For brevity, I give the proof of Proposition 1 for the case were r_1^k is not too high, i.e. there is at least one solution to the equation $f_2(m, r_1^k) = 0$ where

$$\frac{f_2(m, r^k)}{1-m} = (\rho + \theta)\log(1-m) + m\frac{r^k - \delta - \rho}{1-m} + \min\{\xi_1 r^k, r^k - \delta - \rho\}$$

Notice that in general, for τ large the marginal product $r_2^k(\tau)$ has fallen enough that it satisfy the condition above. When $f_2(m, r_1^k) = 0$ has a solution there are (generically) two roots. Call m_2 the smaller solution. To conserve on notation I use $r(\tau)$ and $m(\tau)$ instead of $r_2(\tau)$ and $m_2(\tau)$. The proof is organized in two steps.

Step 1. I show that for any initial condition $\phi(0) \in [0, 1]$ the system of differential equations (20), (32) and (35) has a solution. The system in implicit form is

$$\begin{bmatrix} 0 \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} (\rho+\theta)\log(1-m) + g - g^d \\ -a(r)rG \\ -\phi\left[\theta + (1-\phi)(g-g^d) + \theta\phi_t\frac{m}{1-m}\right] \end{bmatrix} \tag{39}$$

where $a(\cdot)$ can be written as a function of r instead of w by using (18). The explicit form is

$$\begin{bmatrix} \dot{m} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -f_2(m,r) \\ -a(r)rG \\ -\phi\left[\theta + (1-\phi)(g-g^d) + \theta\phi\frac{m}{1-m}\right] \end{bmatrix} \tag{40}$$

Recall that

$$\begin{aligned} g(\tau) &= \frac{r(\tau) - \delta - \rho + \dot{m}(\tau)}{1-m(\tau)} \\ g^d(\tau) &= \max\{\xi r(\tau) - \delta - \rho, 0\} \\ G(\tau) &= [1-\phi(\tau)][g(\tau) - g^d(\tau)] + g^d(\tau) + \phi(\tau)\theta\frac{m(\tau)}{1-m(\tau)} \end{aligned}$$

Write the system as $\dot{y} = z(y)$ with $y = (m, r, \phi)$ and be $z_n(y)$ the n-th element in the vector $z(y)$. rny solution to the system of differential equations must have the following property.

Property 1. *Suppose that for some* $\tau, \tau_1 \in [t, T]$*, with* $\tau < \tau_1$

$$m(\tau) > 0, \quad r(\tau) > \delta + \rho, \quad m(\tau_1) < 0$$

then, for some $\tau_2 \in (\tau, T)$

$$m(\hat{\tau}) \begin{cases} > 0 & \hat{\tau} \in [\tau, \tau_2) \\ = 0 & \hat{\tau} = \tau_2 \\ < 0 & \hat{\tau} \in (\tau_2, T] \end{cases}$$

$$r(\hat{\tau}) > \delta + \rho \quad \forall \hat{\tau} \in [\tau, T]$$

The property is proven as follows. The existence of τ_2 where $m(\tau_2) = 0$ is a consequence of the continuity of $m(\cdot)$. Now suppose that $m(\hat{\tau}) > 0$ for $\hat{\tau} \in [\tau, \tau_2)$. We can show that $r(\tau_2) > \delta + \rho$. In fact, assume that $r(\tau_2) = \delta + \rho$. We then have

$$g^d(\tau_2) = 0$$

$$g(\tau_2) - g^d(\tau_2) = g(\tau_2) = 0$$

and thus

$$\dot{m}(\tau_2) = 0, \quad G(\tau_2) = 0$$

so that $m(\hat{\tau}) = 0$, $r(\tau_2) = \delta + \rho$ for all $\hat{\tau} \in [\tau_2, T]$, which is a contradiction. Suppose, instead, that $r(\tau_2) < \delta + \rho$. Then,

$$\dot{m}(\tau_2) > 0$$

again a contraction with $m(\tau_2) = 0$. We conclude that $r(\tau_2) > \delta + \rho$ and thus

$$\dot{m}(\tau_2) < 0$$

Notice that, as long as $r(\hat{\tau}) > \delta + \rho$ we have $\dot{m}(\hat{\tau}) < 0$. Suppose that, at some τ_3, we have $r(\hat{\tau}_3) = \rho + \delta$ and $r(\hat{\tau}) > \rho + \delta$ for $\hat{\tau} \in [\tau_2, \tau_3)$. Then, since $m(\tau_3) < 0$ and

$$g^d(\tau_3) = 0$$

$$g(\tau_3) - g^d(\tau_3) = g(\tau_3) < 0$$

it follows that

$$G(\tau_3) < 0$$

which is a contradiction with $r(\hat{\tau}_3) = \rho + \delta$. Therefore we must have

$$m(\tau) < 0 \quad \tau \in (\tau_2, T]$$

and we have established the results of Property 1.
I now show the existence of a global solution to the system (40). Define the set Y as

$$Y = \{(m, r, \phi) | 0 \leq m \leq m_2, 0 \leq r \leq r_0, 0 \leq \phi \leq 1\}$$

For $p > 0$ and small, denote with $B(y; p)$ the open ball centered at y with radius p and define $\bar{Y}$ as

$$\bar{Y} = \bigcup_{y \in Y} B(y, p)$$

It is easy to show that there exists a constant $L > 0$

$$|z_n(y)| \leq L \quad \forall y \in \bar{Y}$$

with $n = 1, 2, 3$. Moreover, call $J_{i,j}(y)$ the typical element of the the Jacobian matrix of $z(y)$ at any differentiable point y . We can show that there is a constant $L_1 > 0$ such that

$$|J_{i,j}(y)| \leq L_1 \quad \forall y \in \bar{Y}$$

$i = 1, 2, 3$, $j = 1, 2, 3$. Then the following Lipschitz condition holds

$$\sum_n |z_n(y) - z_n(x)| \leq L_1 \sum_n |y_n - x_n|$$

$y, x \in \bar{Y}$. Define T_1 as

$$T_1 = \frac{p}{L}$$

General arguments on the existence of local solutions[10] to the system (40) imply that, for any $y^0 \in Y$ there exists one and only one continuous solution $y(\tau)$ to the system (40), for $0 \leq \tau \leq T_1$ and

$$y(0) = y^0$$

$$|y_n(\tau) - y_n(0)| \leq p \quad 0 \leq \tau \leq T_1$$

[10]See, for instance, Franklin (1954).

$n = 1, 2, 3$. The existence and uniqueness result proves the existence of a continuous function $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that, if $y(\tau)$ is a solution to the system (40) for $0 \le \tau \le T_1$, then

$$y(T) = F(y(t))$$

Define the set M_0 as

$$M_0 = [0, m_2]$$

Consider the two initial states $\underline{y}_0 = (0, r_0, \phi_0)$ and $\bar{y}_0 = (m_2, r_0, \phi_0)$, with $\phi_0 \in \{0, 1\}$ to represent the case of default and of no default at the turning point $\tau = 0$. Notice that $\underline{y}_0, \bar{y}_0 \in Y$ and that $\dot{\underline{m}}_0(0) < 0$, $\dot{\bar{m}}_0(0) \ge 0$ which implies

$$\underline{m}_0(T_1) = F_1(\underline{y}_0) < 0$$

$$\bar{m}_0(T_1) = F_1(\bar{y}_0) > m_2$$

The first conclusion is drawn from Property 1, while the second is due to the observation that $f_2(m, r) < 0$ for $r \le r_1^k$ and m in a right neighborhood of m_2. By continuity of $F(\cdot)$, there exists an interval $M_1 = [\underline{m}_1, \bar{m}_1] \subseteq M_0$ such that

$$F_1(\underline{m}_1, r_0, \phi_0) = 0$$

$$F_1(\bar{m}_1, r_0, \phi_0) = m_2$$

$$0 < F_1(m, r_0, \phi_0) < m_2 \quad \forall \underline{m}_1 < m < \bar{m}_1$$

For $\tau \in [0, T_1]$ let us call $\underline{y}_1(\tau) = (\underline{m}_1(\tau), \underline{r}_1(\tau), \underline{\phi}_1(\tau))$ the solution with initial condition $\underline{y}_1(0) = (\underline{m}_1, r_0, \phi_0)$. By Property 1 we have that $F_2(\underline{y}_1(0)) \ge \rho + \delta$. Then, for all $\tau \in [0, T_1]$

$$\underline{m}_1(\tau) \ge 0$$

$$\dot{\underline{r}}_1 \ge 0$$

which imply

$$\begin{aligned} \underline{m}_1(\tau) &\in [0, m_2) \\ \underline{r}_1(\tau) &\in [\rho + \delta, r_0] \end{aligned} \tag{41}$$

for $\tau \in (0, T_1)$.

If $F_2(\underline{y}_1(0)) = \rho + \delta$ then the functions $\underline{m}_1(\tau)$ and $\underline{r}_0(\tau)$ are constant from time T_1 on and, because of (41), we have found a solution to the system (40) with the required restrictions on m, r and ϕ.

If $F_2(\underline{y}_1(0)) > \rho + \delta$ repeat the procedure above as follows. Define the set Y_{T_1} as

$$Y_{T_1} = \{F(m, r_0, \phi_0) | m \in M_1\}$$

Notice that

$$Y_1 \subseteq Y$$

Define $T_2 = T_1 + \frac{p}{L}$. We then know that, for any $y_{T_1} \in Y_{T_1}$ there exists a unique solution $y(\tau)$ to (40) for $\tau \in [T_1, T_2]$ with $y(T_1) = y_{T_1}$. For any $y_{T_1} \in Y_{T_1}$ we can write

$$y(T_2) = F(y_{T_1}) = F(F(y_0))$$

with $y_0 = (m, r_0, \phi_0)$ and for one and only one $m \in M_1$. rgain, notice that

$$F_1(F(\underline{m}_1)) < 0$$

$$F_1(F(\bar{m}_1)) > m_2$$

Hence, using the continuity of $F(F(\cdot))$, we construct a new interval $M_2 = [\underline{m}_2, \bar{m}_2]$ with the usual properties. If $F_2(F(\underline{y}_2)) = \rho + \delta$ then we have found a solution to the system (refsystem) with initial condition $\underline{y}_2$. Otherwise, if $F_2(F(\underline{y}_2)) > \rho + \delta$, we construct a new set $M_3 \subseteq M_2$ and repeat the process. It could be the case that the procedure never stops, and we have then to construct an infinite sequence of nested closed intervals $M_1, M_2, ...$ For $n = 1, 2, ...$ define the restrictions $\bar{M}_n \subseteq M_n$ in the following way. Consider any $m \in M_n$ and an initial condition (m, r_0, ϕ_0). Call $y(\tau) = (m(\tau), r(\tau), \phi(\tau))$ the unique solution with $m(0) = m$ for $\tau \in [0, T_n]$. Then $m \in \bar{M}_m$ if and only if the following holds

$$\begin{aligned} m(\tau) &\in [0, m_2] \quad \tau \in [0, T_n] \\ r(\tau) &\in [\rho + \delta, r_0] \quad \tau \in [0, T_n] \end{aligned} \tag{42}$$

Because of (41) the sets $\bar{M}_n$ are non empty, since they always contain $\underline{m}_n$. By continuity of $y(\tau)$ in $m(0)$ the sets $\bar{M}_n$ are closed. Moreover, it is also easy to see that $M_0 \supseteq \bar{M}_1 \supseteq \bar{M}_2 \supseteq ....$ Since M_0 is compact and $\bar{M}_n$ is an infinite sequence of closed subsets of M_0 having the finite intersection property we conclude that

$$\bar{M} \equiv \bigcap_{n=1}^{\infty} \bar{M}_n \neq \emptyset$$

The set $\bar{M}$ is the set defining initial conditions $m(0)$ to which we can associate a unique solution to the system of differential equations.

Step 2. I have shown that borrowing constraints that satisfy condition *i)* exist in stage 2. In particular there exists a solution $\hat{m}(0$ for the initial condition $\phi(0) = 0$. The borrowing constraints $\hat{m}_1 = \hat{m}(0)$ and $\hat{m}(\tau)$ are an equilibrium. If there is no other equilibrium which satisfies condition *i)* and *ii)* for endogenous borrowing constraints, then $m_1^* = \hat{m}_1$ and $m_2^*(\tau) = m_2(\tau)$ so that an equilibrium with endogenous borrowing constraints always exists.