

Home Ownership, Savings, and Mobility Over The Life Cycle*

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Abstract

Despite the importance of understanding home ownership, there exists little consensus at to why many young households rent despite the lower user-cost of owning. In a Bewley model with endogenous price volatility, mobility, and home ownership, we assess the contribution of financial constraints, housing illiquidities and house price risk to home ownership over the life cycle. We show the existence of a finite dimensional state space, steady state equilibrium with stochastic prices. The calibrated economy is able to explain most of the rise in home ownership over the life cycle. We find that, while some young households rent due to borrowing constraints in the mortgage market, the profile of earnings and desire for mobility are more important determinants of the ownership rate.

Keywords: Home Ownership, Incomplete Markets, Illiquid Assets, Financial Constraints

JEL Classification: C62, E21, J61, R21.

1 Introduction

Borrowing constraints in the mortgage market can explain why many young households rent their housing despite the lower user-cost of owning. However, so too can the illiquidity of housing or changes in the hedging motives of households when housing is risky. Each explanation has

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supporting evidence in the data. Young households are more mobile: they are more likely to move to a new home, to move to a new U.S. state, and to move for self-reported “job reasons”. Similarly, young renters are more mobile than young owners. Young households are also poorer, with lower wealth and income, on average, than middle-aged households.

Understanding the determinants of home ownership is important for our understanding of, among others, the response of consumption to changes in housing wealth (Case et al. (2005), Campbell and Cocco (2007)), household portfolios (Flavin and Yamashita (2002)), investment volatility (Fisher and Gervais (2007)), the regional mobility of households and the propensity to default (Ferreira et al. (2008)), and house price dynamics (Ortalo-Magne and Rady (2006)). Despite the many reasons to study home ownership, there is little consensus on which of the several potential determinants of the relative value of owning versus renting offered are meaningful. We provide a dynamic, stochastic, general equilibrium model which can measure the relative importance of these explanations. We find that while the borrowing constraints in the mortgage market¹ only deter a few households from owning, the inability to insure against changes to earnings or borrow against future earnings together with the illiquidities in the housing market lead many more households to rent.

Models of housing ownership choices over the life cycle in general equilibrium use incomplete markets in the tradition of Aiyagari (1994) and Bewley (1984). Heretofore, these models had no endogenous volatility in house prices and, because households only moved to change their housing consumption, counterfactually low household mobility. Volatility is obviously key to generating hedging motives and without properly accounting for mobility, there is no way to measure the importance of housing illiquidity.

To generate volatility and mobility, we situate a Bewley-type model of earnings shocks in incomplete markets in a Lucas and Prescott (1971) island model of housing and labor markets. Exogenous, stochastic variation in the quality of the local labor market will create endogenous household mobility and movements in house prices and rents. We calibrate the model to U.S. data from 1970-1993. We find that the relative value to the household of owning versus renting depends primarily on their relative user costs, the household’s expected horizon of stay in the house, the riskiness of housing equity and the transactions costs of buying and selling a house. Households that expect to move soon, either for family or career (earnings related) reasons, rent to avoid paying

¹Hereafter we refer to these constraints as the down payment constraint.

the high transactions costs for buying and selling a house. Furthermore, because housing is risky, owning a house means accruing equity in it and thus wealth and young households may not wish to save and so do not own.

In the model's equilibrium, younger households have a lower expected horizon of stay for four reasons. First, from a career perspective, the benefits of moving to a location that offers a higher salary are greater when the agent is younger. Second, young households expect their wages to increase by a lot in the near future, but are unable to borrow against future income and smooth housing consumption over their life cycle. So young households expect to move into larger houses in the future. Third, expected future earnings comprise a large part of a young household's total wealth. Since the earnings are risky, a young household faces uncertainty over the size of the house it will want to inhabit later. Last, relative to middle-aged households, younger households are also smaller in size and thus inhabit smaller houses, making it easier to move.

Our contribution is on two fronts. On the quantitative front, we evaluate consumer behavior in the presence of housing and location choice using the baseline model. Using data from the PSID, SCF, NIPA and the CPS, we calibrate the model to fit key macroeconomic moments. Our model is successful in replicating several aspects of the home ownership profile and mobility over the life cycle. We conduct a series of counterfactual experiments to evaluate the relative impact of various factors to the ownership choice decision². We find that households that are financing constrained are more likely to adjust along the intensive margin; twice as many first-time home buyers in the model choose to buy a smaller house rather than delay owning when forced to make a down payment. The home ownership rate would be only 2.9 percentage points higher for young households (age 21-35) if there were no down payment constraint. However, if households had flat expected earnings profile over the life cycle, the home ownership rate for the young would be 6.8 percentage points lower, and would be 40.8 percentage points lower if households had no permanent idiosyncratic earnings risk.

On the qualitative front, we extend the literature by endogenously incorporating location choice, ownership and house price risk into a dynamic, GE model of housing. Heterogeneous agent, incomplete-market models with non-constant prices typically feature infinite dimensional state variables in the agents' decision problems, and thus afford only approximate solutions (for instance, Krusell and Smith Jr. (1998)). We prove that there is a stationary equilibrium in our economy in

²that is, the choice of whether to rent or to own

which the price of housing on an island is dependent only on its productivity. This allows us to characterize prices and allocations without having to keep track of distributions over households on every island. Our proof, which uses Kakutani's Fixed Point Theorem, is amenable to other Bewley-type models with discrete and continuous choices.³

With recent advances in computing capacity, many dynamic OLG models incorporate housing. The issues addressed range from the ownership choice decision (Chambers et al. (Forthcoming) and Diaz and Luengo-Prado (2008)), the evolution of consumer debt (Scoccianti (2008)), portfolio choice in the presence of housing (Cocco (2005), Yao and Zhang (2005)) and the consumption of durables over the life cycle (Fernandez-Villaverde and Krueger (2005), Gruber and Martin (2003)). Fernandez-Villaverde and Krueger (2005) argue that the hump-shaped pattern of durable consumption (of which housing is a large part) is due to incentives to accrue collateral. Cocco (2005), Yao and Zhang (2005) and Diaz and Luengo-Prado (2008) each use partial equilibrium models where the price of housing is correlated to a household's labor income. Chambers et al. (Forthcoming), Scoccianti (2008), Fernandez-Villaverde and Krueger (2005) and Gruber and Martin (2003) have GE models where the price of housing is constant. Van Nieuwerburgh and Weill (2006) uses an island model of renter-workers to examine the changes in the spatial distribution of house prices and wages in the U.S.. They find that changes in the cross-sectional dispersion of wages can explain the changes in the cross-sectional dispersion of house prices in an economy with housing supply constraints. Ortalo-Magne and Rady (2006) deals with the impact on ownership choice of financial constraints. They analyze a 4-period model where households adjust to income shocks on the internal and extensive margin, that is, by changing the size of the house they buy, or by delaying the purchase of a house.

There are many papers that find, individually, financial market constraints, demographics and career concerns to be significant factors affecting the ownership choice decision. There is a large literature on credit constraints and its impact on ownership choice. We quote Linneman et al. (1997), Haurin et al. (1996) and Zorn (1989) as representative of this strand. Papers that focus demographics or career concerns implicitly concern transactions costs, since changes in the former primarily affect the ownership decision through expected duration. Clark and Onaka (1993) and

³Most other Bewley-type models with discrete choices assume existence (e.g. Chambers et al. (Forthcoming), Chang and Kim (2006), and Kitao (2008)). Chatterjee et al. (2007) proves existence in an economy with discrete choices (in their case, among others, to default or not); however their method only works for economies with discrete choice spaces.

Quigley and Weinberg (1977) find positive results on the significance of the family life cycle to housing consumption and the ownership choice. Finally, Cameron and Tracy (1997) emphasize the effect of career concerns on the mobility and ownership choice decisions of younger households.

Lastly, Sinai and Souleles (2005) focuses on home ownership as insurance against changes in the rental price of housing. In their model, increases in expected duration increases the likelihood that a household owns because changes in the spot price of housing instantaneously changes rental prices, a fortiori, but an owner-occupier only capitalizes the resultant change in the value of their housing stock when it sells in the future. In our model with incomplete markets, ceterus paribus, owner-occupies will optimally hold more wealth than renters for insurance reasons. As a household's intertemporal marginal rate of substitution changes as it ages, so to will its willingness to hold this extra wealth and thus to own.

The rest of the paper is organized as follows: section 2 presents the data on ownership and mobility in the U.S., section 3 presents the model and section 4 discusses the calibration. Section 5 compares the model to the data. In section 6, we discuss the determinants of a household's expected duration in its residence. Section 7, we conduct counterfactual experiments to assess the importance of family size, down payment constraints and career concerns in the determining the ownership rate. Section 8 concludes. Appendix A contains a proof of existence. Appendices B and C contain further details on our computation procedure and calibration, respectively.

2 Data

2.1 Ownership and mobility

Data on ownership and mobility are from the Panel Study of Income Dynamics (PSID) waves 1969-1993. For generating the mobility graphs we only include households that are in the data set for at least two consecutive waves⁴.

We observe that the home ownership rate of younger households is low; the average ownership of households aged 21-25 is 33%. The proportion of owners increases sharply to reach 80% by age 40. The data plateaus at around 85%, reaching a high of 91%. The proportion of home owners remains at that level until retirement (age 65).

Figure 2 shows the pattern of mobility over the life cycle, plotting the proportion of households

⁴For all sample selection criteria, see Appendix C.1.

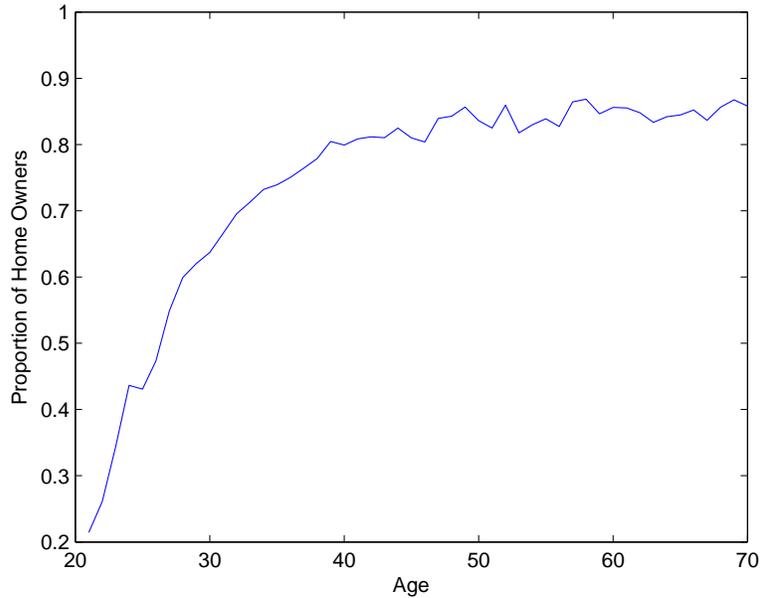


Figure 1: Ownership over the life cycle (PSID)

that move per year conditional on age. Renters are more mobile than owners. The aggregate mobility curve moves from the renter mobility curve to the owner mobility curve as a result of the change in the ownership rate over the life cycle, and (iii) both the renters and owners are more mobile when young, with mobility falling as households get older.

In order to examine the impact of relocating to different job markets for career reasons, we plot the profile of inter-state moving and career-related moving over the life-cycle (figures 3 and 4).⁵ The pattern of inter-state moving is closely related to the pattern of career-related moving; the correlation between the two is .6. The overall level of inter-state migration is about an order of magnitude less than for overall moving, but the life cycle pattern remains substantially the same. Younger households are significantly more mobile than older ones, as are renters compared to owners. The graph of owners' inter-state mobility shows only a small downward trend over the life cycle. Career-related mobility (movers that move for self-reported "job reasons") shows similar patterns over the life cycle.

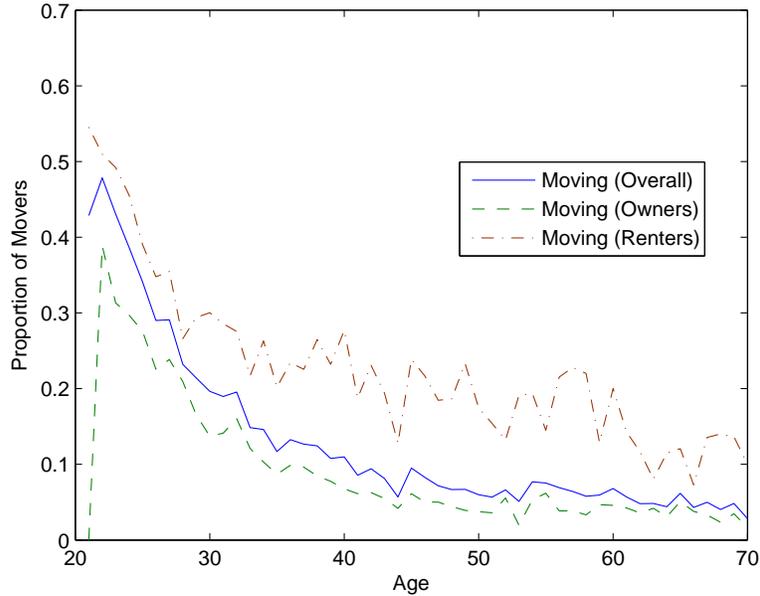


Figure 2: Mobility over the life cycle (PSID)

2.2 Household financials

Household financial data is collected from the Survey of Consumer Finances (SCF). The SCF is conducted once every three years, and we use waves from 1986 to 1998. We look at the pattern of wealth of home owners over the life cycle. Figure 5 shows that household financial wealth starts low and increases at an increasing rate. Wealth peaks in the mid- to late-50's. Net wealth is defined as the sum of financial and housing wealth less the outstanding mortgage debt. We see that housing wealth changes less over the life cycle compared to financial wealth. The widening gap over age between net wealth and financial wealth shows the larger role of housing assets in the household's portfolio. The graph also shows that younger households tend to hold a relatively high level of mortgage debt when they own their home.

2.3 Average income over the life cycle

Figure (6) shows the base wage of households from the Current Population Survey (CPS). *Base wage* is defined as the *exponential of the average log labor earnings* controlling for year effects. It is computed in the following way: we take the log earnings from the CPS and regress it on year and

⁵Relocation would probably be better analyzed at the MSA or district level, but the lack of publicly available data on those variables in the PSID precludes that possibility.

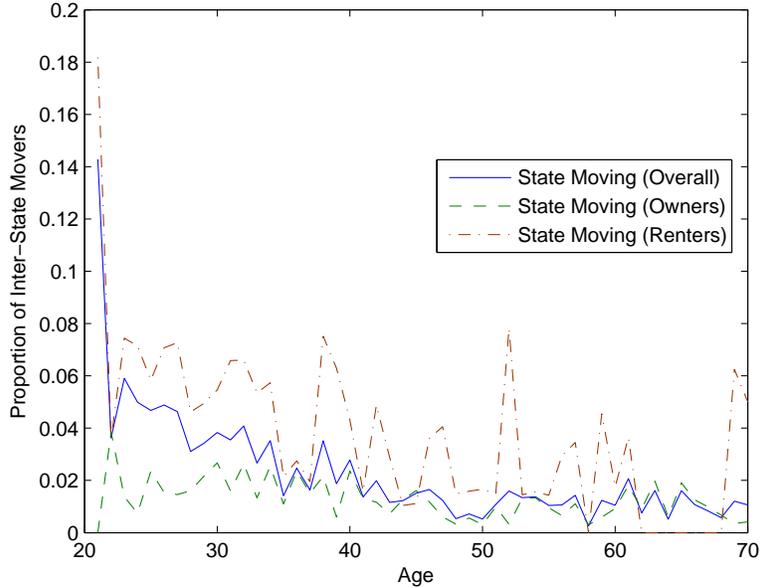


Figure 3: Inter-state mobility (PSID)

age dummies, with the household weights supplied by the CPS. The coefficients of the age dummies are the base wage for that age. The CPS has more detailed data on household income and a larger sample size, which makes it the data set of choice. We see that earnings climb steeply in the early part of life, more than doubling between age 21 and age 40. This creates a strong incentive for households to borrow or spend down their wealth in the early part of life.

2.4 Family evolution

We compute the average family size from the CPS. Figure (7) shows the evolution of family size over the life cycle. We see the familiar hump-shape, though the peak is earlier than for earnings and appears at around age 45. Renters have smaller households on average than owners from early until the middle of life.

3 Model

We consider an OLG island model of household consumption choice. There are a continuum of measure 1 of agents and islands each in the economy. Agents are indexed by $\iota \in (0, 1]$ and islands are indexed by $\epsilon \in (0, 1]$.

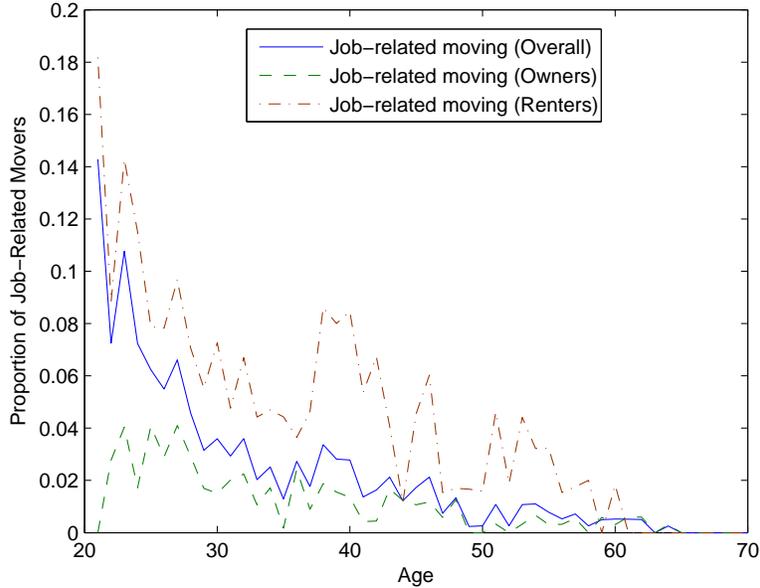


Figure 4: Career-related mobility (PSID)

Time is discrete and each period in the economy corresponds to one year in the data. Households are born at age A and live no longer than to age T . In every period, the household survives to next period with a probability, $\lambda : \mathcal{A} \rightarrow [0, 1]$, which is a function of the age of the head, $a \in \mathcal{A} = \{A, A + 1, \dots, T\}$. We assume that $\lambda(a)$ is not only the probability for a particular individual of survival, but also the deterministic fraction of agents that survive until age $a + 1$ having already survived until age a . Each period, a measure $\mu_1 = (1 + \sum_{\kappa=A}^T \prod_{a=A}^{\kappa} \lambda(a))^{-1}$ is born; so the population of agents in the economy is stationary.

3.1 Technology

There are two goods: a non-durable, globally available, consumption good and a durable, housing good. The housing good is island specific and in fixed supply, $H(\epsilon) = \bar{H} \quad \forall \epsilon \in (0, 1]$, on each island. The consumption good is produced according to a globally available production function $F(K, L)$, where K is the aggregate capital stock and L is the stock of available efficiency labor units in the entire economy⁶. F is assumed to be strictly increasing in both inputs, strictly concave with diminishing marginal products which obey the Inada conditions and is homogeneous of degree one. With constant returns to scale, the number and size of firms in equilibrium will be indeterminate.

⁶i.e. capital is fully mobile across islands

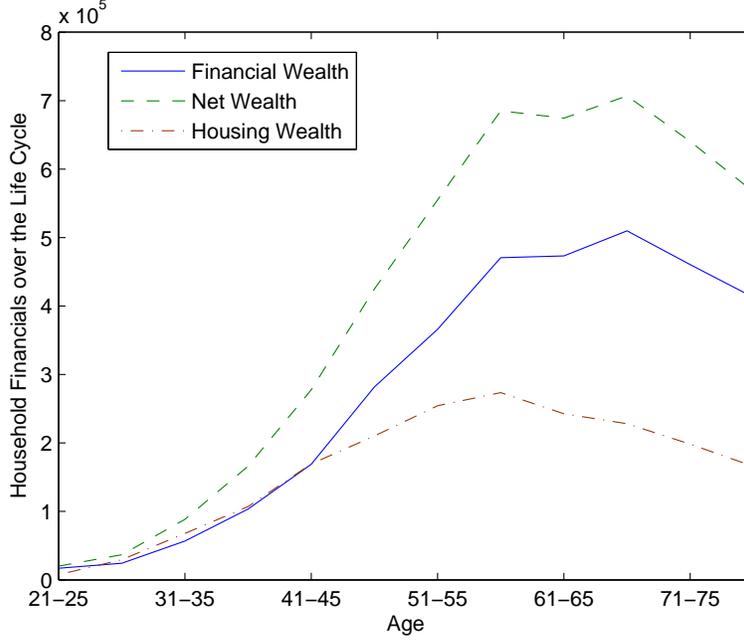


Figure 5: Household financials over the life-cycle

So, without loss of generality, we assume a single, representative firm.

A consumption good produced can be consumed in that period, converted into capital next period, K' , spent on government consumption, used to maintain the housing stock or used up in transaction costs. Capital depreciates at rate δ . The aggregate resource constraint for the consumption good is:

$$C + G + K' - (1 - \delta)K = F(L, K)$$

where C refers to consumption used for all purposes except investment in capital goods and government consumption, G .

Households choose housing $h \in \mathcal{H} \equiv [0, \bar{h}]$.

3.2 Preferences and bequests

The household gets utility from housing and the consumption good, which is denoted the numeraire good. Preferences are described by

$$U(c^{T-a}, h^{T-a}, a) = E_a \sum_{t=a}^T \beta^{t-a} \frac{(\prod_{t'=a}^t \lambda(t'))}{\lambda(a)} u(c_t, h_t, t)$$

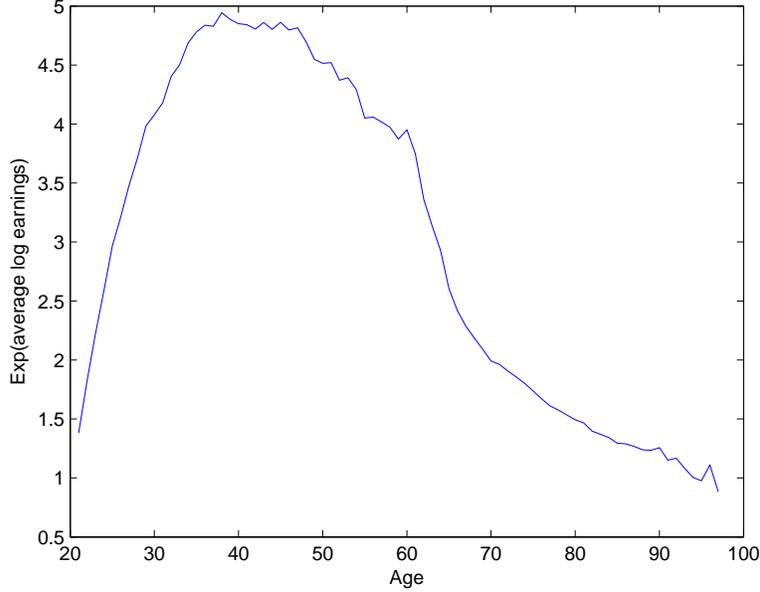


Figure 6: Income over the life cycle (CPS - yearly bins)

where β is the time discount factor and a is the age at which we want to calculate the total expected utility. $x^T = (x_1, x_2, \dots, x_T)$, the sequence from 1 to T . The instantaneous utility function $u(\cdot, \cdot)$ is

$$u(c, h, t) = \frac{\left(\left(\frac{c}{S(t)}\right)^{1-\sigma} \left(\frac{h}{S(t)}\right)^\sigma\right)^{1-\gamma}}{(1-\gamma)}$$

The path for the family size adjustment factor, $S : \mathcal{A} \rightarrow \mathbb{R}_{++}$, is exogenous, constant across households and known to the household at birth.

3.3 Labor productivity and job offers

Each island has a productivity, indexed by j , which follows a finite state Markov chain with state space $j \in \mathcal{J} = \{1, \dots, J\}$ and transition probabilities given by the matrix $\pi_J(j'|j)$. Let Π_J denote the unique invariant measure associated with π_J .

Similarly each household has an ability, indexed by i , which follows a Markov chain with state space $i \in \mathcal{I} \subset [-I, I]$ and transition probabilities given by the matrix $\pi_I(i'|i)$. The initial realization of a newborn household's ability is assumed to be drawn from the distribution Π_I for all households.

Households are endowed with one unit of time per period. If a household chooses to move in the current period, moving occupies θ_m units of time. All other time is supplied inelastically in

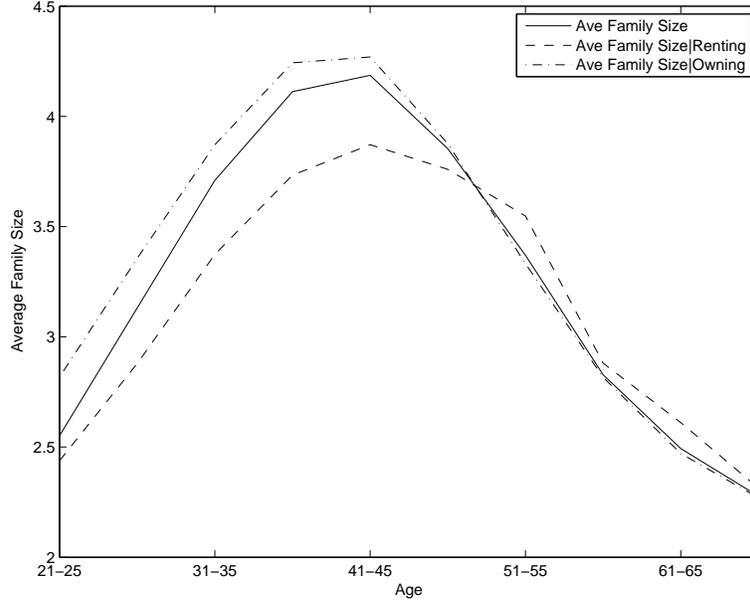


Figure 7: Family size over the life cycle

the labor market. A household's effective labor supply $l(a, i, j)$ in any period is the product of four elements: whether the household moves, the household's age, the household's ability, and the productivity of the island on which it chooses to work:

$$l(a, i, j) = (1 - 1_m \theta_m) l_a(a) l_i(i) l_j(j)$$

where $l_a : \mathcal{A} \rightarrow \mathbb{R}_{++}$, $l_j : \mathcal{J} \rightarrow \mathbb{R}_{++}$, and $l_i : \mathcal{I} \rightarrow \mathbb{R}_{++}$ are known functions and 1_m is a dummy variable which equals 1 if the household chooses to move in the current period. $l_j(\cdot)$ and $l_i(\cdot)$ are assumed to be increasing functions of their arguments.

There is a proportional income tax rate t_y . The tax is levied on a household's total net income from labor earnings and any interest income or payments.

3.4 Assets and Prices

There exists a one-period, risk-free asset b which pays a gross interest rate $R = 1 + r$. All firms and households may borrow or lend at this rate. Households choose asset holdings from the set $b' \in \mathcal{B} \equiv [\underline{b}, \bar{b}]$ subject to a collateral constraint.⁷ Wages per unit of effective labor supply are w .

⁷We set \bar{b} and \underline{b} so that they never bind in equilibrium.

3.5 Institutional structure of the housing market

Housing as an asset is distinct from both the consumption good and the risk-free asset in the following ways: housing enters the utility function, and at the same time is an asset. It is immovable and indivisible. There is a transaction cost associated with changing the stock of housing. The transaction cost of buying and selling a house is a proportion θ_h of the purchased or sold house value.

Housing may be either rented or bought from a risk-neutral, competitive real estate industry. If rented, a household pays $q(j)$ per unit of housing h on island ϵ of productivity j . The household can buy and sell housing on island ϵ of productivity j at the price $p(j)$. In turn, every house sold by a household is bought by a real estate agency.

We assume that all houses require upkeep in consumption goods in an amount proportional to the house value as maintenance and property tax: $\delta_h p(j)h$. In addition, we assume that this keeps the house at a constant quality over time. That is, the maintenance exactly covers the depreciation of the asset.

Owner-occupiers and landlords must pay a property tax, t_p , on the value of the house (assessed each period). Interest payments on mortgages are income tax deductible.

A household cannot rent housing that it owns but does not use and it can only consume housing on the island on which it works. A household cannot simultaneously consume owner-occupied and rental housing and cannot short-sell housing.

Finally, in addition to providing a stream of services, housing is the sole form of collateral for agents in the economy. We model this by giving households a home equity line of credit⁸. When purchasing a house, working agents can borrow up to $(1 - d(a))$ of the value of the house, where $d(a)$ is the down payment constraint for an agent of age a . Thereafter, as long as they continue to be home owners, agents may borrow up to $(1 - d(a))$ of the value of the house. They may also choose to roll over their debt after making an interest payment.

$$b' \geq \min\{-(1 - d(a))\tau'p(j)h', (1 - 1_m)b\},$$

where τ' , ownership, is an indicator variable which equals one if the household chooses to own in

⁸We also call this a mortgage throughout.

the period.⁹

If the household chooses to sell its house, it must pay off all existing debt, though another loan can be taken out if another house is purchased. A household that does not have positive total cash-in-hand (housing wealth plus financial wealth plus current income) will not be able to pay off the mortgage it has (the debt it owes) on its house and will not choose to move in this period. We do not allow the household to choose to default (see Jeske and Krueger (2005) for a model with mortgage default), but households can default implicitly by dying with negative net worth.

3.6 Real Estate Industry

Real estate firms act as intermediaries and market makers in the housing market, since households have to buy, sell or rent housing through the real estate firms. They are risk-neutral and can borrow at the interest rate, r . The real estate industry is competitive, so the size and number of individual firms is indeterminate.

Real estate firms pay income tax on any net earnings less maintenance, interest and property taxes, but including capital gains from housing (however, capital losses may be carried forward). Since real estate is competitive, firms make zero profit on average and pay no net income tax. The zero-profit condition is:¹⁰

$$q(j) = (\delta_h + t_p + \frac{r}{1+r})p(j) - \frac{1}{1+r}E(p(j') - p(j)|j)$$

3.7 Birth and Death

Newborn agents are born with no housing. Their birth location is determined by drawing from a uniform distribution over \mathcal{E} with density f_e . When households are born, they draw their initial wealth from a distribution Π_b , which is a probability distribution on \mathcal{B} . Their initial wealth draw is independent of all other initial state variables. When agents die, any accidental bequests are made to the government and the government makes whole the financial sector on any outstanding loans

⁹This borrowing constraint is different from the more typical one which restricts borrowing to be weakly less than some percentage of the house value ($b' \geq -(1-d)\tau'p(j)h'$). With risky house prices, for an agent near the typical borrowing constraint, a fall in the value of a house results in a "call on the mortgage principle" - the agent must reduce the amount borrowed. If house price volatility is large enough, the effective down payment constraint (the amount the agent could borrow and still be able to repay in any state of the world next period) may be much tighter than the actual (d).

¹⁰We allow households to move frictionlessly between rental houses of the same size on the same island, so ruling out any loss-leader, teaser-rate rental prices.

to dead households. Any existing debt at the time of death is paid off by the government as well. Newborn households receive their initial wealth from the government.¹¹

3.8 Taxes

Proportional taxes are collected on property, labor earnings and interest income for households. However, households are allowed to deduct any interest payments. The government collects taxes and accidental bequests and funds households' initial wealth endowments; any excess amount funds government spending G which is thrown into the ocean.

3.9 Timing

The timing within a period is as follows:

1. Some households die.
2. A household of age a enters the period, observes its ability i and its island's productivity j . The household has housing h , ownership τ , on island ϵ and assets b and any accidental bequests if it is newborn. All of the dead households' housing stock is sold to the real estate agency.
3. The household chooses to locate/work on island ϵ' which is of type \tilde{j} . If the household moves, the household sells τh and chooses how much housing h' and consumption goods c to consume this period and its ownership choice (rent or own), τ' , and next period's financial assets b' . If it chooses to stay in its current housing, the household only chooses c and b' .
4. Efficiency labor units and capital are supplied.
5. Factor payments are made, and consumption and housing services are consumed.

All information is commonly known and all decisions are publicly observable.

3.10 States and actions

The following are state variables of the model economy: $i \in \{1, \dots, I\} \subseteq \mathbb{N}$ is the idiosyncratic ability of the household; $j \in \mathcal{J} = \{1, \dots, J\} \subseteq \mathbb{N}$ is the island type; $\tau \in \{0, 1\}$ is the ownership choice

¹¹In the calibration section, we discuss how Π_b is chosen to match certain aspects of the data. In our counterfactuals, we assume that Π_b remains the same as in the baseline. We leave to further research a joint examination of housing policy, bequests and the wealth of young households.

of the household; $h \in \mathcal{H} = [0, \bar{h}] \subset \mathbb{R}^+$ is last period's housing consumption; $b \in \mathcal{B} = [\underline{b}, \bar{b}] \subset \mathbb{R}$ is the household's financial wealth; $a \in \mathcal{A} = \{1, \dots, A\} \subset \mathbb{N}$ is the age of the household; and $\epsilon \in \mathcal{E} = [0, 1]$ is the island index.

Definition. The state space $S = \mathcal{A} \times \mathcal{I} \times \mathcal{J} \times \{0, 1\} \times \mathcal{H} \times \mathcal{B} \times \mathcal{E}$. A state can be written as $s = (a, i, j, \tau, h, b, \epsilon) \in \mathcal{S}$. $\bar{j} : \mathcal{E} \rightarrow \mathcal{J}$ is the function that maps an island to its productivity. The vector of house prices is $(p)_J = (p_1, \dots, p_J)$. The vector of rents is $(q)_J = (q_1, \dots, q_J)$. The price vector, $\vec{p} = ((p)_J, (q)_J, r, \omega) \in \mathcal{P} \subset \mathbb{R}^{2J+2}$.

The choice variables for the household are as follows: $\tau' \in \{0, 1\}$ is the ownership choice; $h' \in \mathcal{H}$ is the housing consumption choice; $b' \in \mathcal{B}$ is the savings choice; $c \in \mathcal{C}$ is consumption, and $\epsilon' \in \mathcal{E}$ and $\tilde{j} \in \mathcal{J}$ is the island and island productivity choices.

Definition. $\mathcal{Y} = \{0, 1\} \times \mathcal{H} \times \mathcal{B} \times \mathcal{J} \times \mathcal{E} \times \mathcal{C} \subset \mathbb{R}^5$ is the choice space. $y = (\tau', h', b', \tilde{j}, \epsilon', c) \in \mathcal{Y}$ is a particular choice vector.

3.11 Mover's Problem

The problem of the mover is to choose house size and ownership, savings, and consumption, given an age, ability, location productivity, cash-in-hand, and location:

$$V^m(a, i, \tilde{j}, b^m, \epsilon') = \sup_{c, h', \tau', b'} u(c, h', a) + \lambda(a)\beta E[V(a+1, i', j', \tau', h', b', \epsilon')]$$

s.t.

$$\begin{aligned} c + b' + h'((1 - \tau')q(\tilde{j}) + \tau'p(\tilde{j})(\delta_h + t_p + 1 + \theta_h)) &\leq b^m \\ b' &\geq -(1 - d(a))p(\tilde{j})\tau'h' \\ c \geq 0 \quad h' \geq 0 \quad \tau' &\in \{0, 1\} \end{aligned}$$

3.12 Stayer's Problem

The problem of the stayer is to choose savings and consumption, given age, ability, location productivity, ownership, house size, assets and location:

$$V^s(a, i, j, \tau, h, b, \epsilon) = \sup_{c, b'} u(c, h, a) + \lambda(a)\beta E[V(a + 1, i', j', \tau, h, b', \epsilon)]$$

s.t.

$$c + (p(j)\tau(\delta_h + t_p) + q(j)(1 - \tau))h + b' \leq b(1 + r(1 - t_y)) + wl(a, i, j)(1 - t_y)$$

$$b' \geq \min\{(1 - d(a))\tau hp(j), b\}$$

$$c \geq 0$$

3.13 Household's Problem

The household's problem is to choose whether to stay or move to a different location, given its state¹²:

$$V(a, i, j, \tau, h, b, \epsilon) = \max\{V^s(a, i, j, \tau, h, b, \epsilon), \sup_{\tilde{j}, \epsilon'} V^m(a, i, \tilde{j}, b^m, \epsilon')\}$$

s.t.

$$b^m \equiv b(1 + r(1 - t_y)) + w(1 - \theta_m)l_a(a)l_i(i)l_j(\tilde{j})(1 - t_y) + p(j)h\tau(1 - \theta_h) \geq 0$$

$$\tilde{j} = f_j(\epsilon') \quad \epsilon' \in \mathcal{E}$$

Since our model has both discrete and continuous state variables the proof of existence of an equilibrium correspondingly differs from the one in Aiyagari (1994). Our proof involves a selection of state-contingent action plans in areas of indifference. In order to formalize this, we introduce mixed allocations which will serve as tie-breaking criteria. Since our economy is populated by a continuum of agents, there is no aggregate uncertainty using a mixed allocation.

¹²For an "all-in-one" version of the household's problem, see Appendix A.

3.14 Mixed allocations and the distribution of households

Since $\mathcal{Y} \in \mathbb{R}^n$, $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ is a measure space¹³ where $\mathcal{B}(\mathcal{Y})$ is the standard Borel space on \mathcal{Y} . We can then define the probability space, and a mixed allocation as an element of the probability space.

Definition. Let $\tilde{\Lambda}$ be the set of probability measures on \mathcal{Y} . $\lambda^y : \mathcal{B}(\mathcal{Y}) \rightarrow [0, 1]$ is a probability measure on \mathcal{Y} . Let Δ be the space of functions $f : \mathcal{S} \rightarrow \tilde{\Lambda}$.

Now we can define a mixed allocation as a state-contingent distribution over optimal choices.

Definition. A *mixed allocation*, $\alpha : \mathcal{S} \times \mathcal{P} \rightarrow \tilde{\Lambda}$ is map that specifies the probability distribution over the optimal choice set given by $Y(s, \vec{p})$.

$$\alpha(s, \vec{p}) \in \{\tilde{\alpha} \in \tilde{\Lambda} : \text{supp}(\tilde{\alpha}) \subseteq Y(s, \vec{p})\}$$

Define $\alpha_{\vec{p}} : \mathcal{S} \rightarrow \tilde{\Lambda}$ to be a *price-specific mixed-allocation* given \vec{p} :

$$\alpha_{\vec{p}}(s) \in \{\tilde{\alpha} \in \tilde{\Lambda} : \text{supp}(\tilde{\alpha}) \subseteq Y(s, \vec{p})\}$$

Let $\Lambda_{\vec{p}}$ be the space of price-specific mixed allocations and Λ be the space of mixed allocations.

In this paper we consider only stationary competitive equilibria. Before we define the equilibrium we set out the notion of the distribution of households over the state space. Our stationary equilibrium requires that this distribution does not change over time.

Definition. The *household distribution over states*, $\mu : \mathcal{B}(\mathcal{S}) \rightarrow [0, 1]$ is a probability measure on \mathcal{S} . Let \mathcal{M} be the space of probability distributions on \mathcal{S} .

3.15 Stationary competitive equilibrium

Definition. A stationary competitive equilibrium is a vector of strictly positive prices, \vec{p}^* , a set of correspondences

$$Y^*(s; \vec{p}^*) = (H^*(s, \vec{p}^*), B^*(s, \vec{p}^*), \tau^*(s, \vec{p}^*), J^*(s, \vec{p}^*), \epsilon^*(s, \vec{p}^*), C^*(s, \vec{p}^*)),$$

¹³For a set $X \subseteq \mathbb{R}^n$, we assume that the standard Borel space is used in constructing measure and probability measure spaces. That is, the statement “ μ is a probability measure on X ” implies that $(X, \mathcal{B}(X), \mu)$ is a probability measure space.

a mixed allocation $\alpha^* \in \Lambda$, a probability measure μ^* , firm capital and labor holdings K^* and L^* , government expenditures G^* , and a $J^*(s; \vec{p}^*)$ such that:

- (i) y^* solves the household's problem for each $y^* \in Y^*(s, \vec{p}^*)$
- (ii) K^* and L^* solve the firm's optimization problem:

$$r^* + \delta = F_K(K^*, L^*)$$

$$w^* = F_L(K^*, L^*)$$

- (iii) Goods market clears:

$$F(K^*, L^*) = \delta K^* + G^* + \int_{\mathcal{S}} \int_{Y^*(s, \vec{p}^*)} (c^* + 1_m^* (\tau^* h^* p^*(j^*) \theta_h + \tau h p^*(j) \theta_h) + p^*(j^*) h^* \delta_h) d\alpha^*(s, \vec{p}^*, y^*) d\mu^*$$

- (iv) Capital market clears:

$$K^* = \int_{\mathcal{S}} \int_{Y^*(s, \vec{p}^*)} (b^* + (q^*(j^*) - (1 + t_p + \delta_h) p^*(j^*)) (1 - \tau^*) h^*) d\alpha^*(s, \vec{p}^*, y^*) d\mu^*$$

- (v) Labor market clears:

$$L^* = \int_{\mathcal{S}} \int_{Y^*(s, \vec{p}^*)} l(a, i, j^*) d\alpha^*(s, \vec{p}^*, y^*) d\mu^*(s)$$

- (vi) Housing market clears:

$$\bar{H} = H(\epsilon) = \int_{\mathcal{S}} \int_{Y^*(s, \vec{p}^*)} h^* \cdot \mathbb{1}\{\epsilon^* = \epsilon\} d\alpha^*(s, \vec{p}^*, y^*) d\mu^*(s) \quad \forall \epsilon \in [0, 1]$$

(vii) Government budget constraint holds:

$$G^* = t_y w^* L^* + \int_{s \in \mathcal{S}_{y^*}} \int_{\bar{p}^* \in Y^*(s, \bar{p}^*)} [t_y r^* b^* + h^* p^*(j^*) t_p + \frac{1 - \lambda(a-1)}{\lambda(a-1)} ((1+r^*)b^* + \tau^* p^*(j^*) h^*)] d\alpha^*(s, \bar{p}^*, y^*) d\mu^*(s) - \mu_1 \int_{\mathcal{B}} b d\Pi_b$$

(viii) No arbitrage in the real estate sector:

$$q^*(j) = (\delta_h + t_p + \frac{r^*}{1+r^*}) p^*(j) - \frac{1}{1+r^*} E(p^*(j') - p^*(j) | j), \quad \forall j \in J \quad (1)$$

(ix) Steady-state distribution:

$$\mu^* = \Upsilon_{p, \alpha} \mu^*$$

where $\Upsilon_{p, \alpha}$ is the transition function generated by the optimal choice correspondence of the household and the mixed allocation, α .

Theorem 1. *A stationary competitive equilibrium exists.*

Proof. See appendix. □

4 Calibration

4.1 Data sets

We calibrate our model with data from the CPS, SCF, PSID and the NIPA. We use the annual Social and Economic Supplement to the Current Population Study (CPS) conducted by the Bureau of the Census. We use data from the annual supplement from 1970 to 1993. The CPS is used for calibrating the family size and income. The PSID is a panel data set which we use for measuring housing ownership and mobility and calibrating the wage process, since this requires consecutive observations of the same household. We use data from the PSID waves 1970 – 1993. The Survey of Consumer Finances (SCF) is a triennial survey of the balance sheet, pension, income and other demographic characteristics of US households. The strength of the SCF is that it has detailed information on household finances. The wealth supplement of the PSID is collected once every five years, so comparatively the SCF contains more detailed financial information at higher frequencies.

We use data from the 1989 – 2001 waves of the SCF. Data from the National Income and Product Accounts (NIPA) are used to estimate macroeconomic variables. To be consistent with the rest of the data, we use data from 1970 – 1993 for the estimation.

4.2 Computing output growth and population growth

To estimate the growth rate of output and population in the economy, we use the *Real Annual GDP in chained 2000 US\$* from the NIPA for the period 1970 – 1993 to compute the growth rate of GDP. The average growth rate $g = 2\%$. The population growth rate, $\eta = 1\%$, is computed using total population figures from the Census.

4.3 Household life-cycle and preferences

We set $A = 21$ and $T = 100$, with retirement set at age 65. After retirement the household receives income, but is not allowed to take out any additional loans on the house. Our goal is not to replicate with any degree of exactitude the end of life properties in the population, but to ensure that the end-of-life behavior of the households in the model does not affect their pre-retirement behavior too greatly¹⁴.

4.4 Initial wealth distribution

We calibrate the wealth distribution of newborns using the distribution of wealth among 21-25 year olds in the SCF. We drop top-coded observations and households with negative wealth and students from the sample. We use the sample weights provided by the SCF. We parametrize the initial wealth distribution as an exponential distribution. That gives us one parameter that we have to match.

$$f(b_0) = \lambda_w e^{-\lambda_w b_0}$$

where b_0 is the initial wealth, and λ_w is the parameter to estimate in the exponential distribution.

We estimate λ_w by matching the mean of the initial wealth distribution.

$$\lambda_w = \frac{1}{\bar{b}_0}$$

¹⁴For instance, households with a short expected lifetime would have a higher tendency to rent rather than own. This leads to a counterfactually low home ownership rate compared to the data. Setting the maximum lifetime high enough mitigates this problem.

This gives us $\lambda_w = 0.00589$. We convert the initial wealth distribution in the data to model terms by scaling by the ratio of average income at age 21 in the model to average income at age 21 in the data.

4.5 The probability of survival

The survival probability, $\lambda(a)$, is taken from the National Center for Health Statistics, United States Decennial Life Tables for 1989-1991. This table gives the mortality rate of the population as measured in the 1990 Census. We use the life table for the whole population. The share of households of age a in the population is given by $\prod_{t=1}^a (1 - \lambda(t))$. We use the 1989 table as the measure of the probability of survival in our model.

4.6 Evolution of the family size

In the model, households evolve exogenously in terms of size and composition. We use the CPS to estimate the life-cycle profile of family size. See 7.1 for details.

| Parameter | Description | Value |
|--------------|---------------------------------|------------------------------|
| T | Maximum lifespan | 100 |
| T_r | Retirement age | 65 |
| A | Newborn household age | 21 |
| $\lambda(a)$ | Survival probability | NCHS Life Tables (1989-1991) |
| λ_w | Initial wealth dist. parameter | 0.00589 |
| F_{iat} | Family size over the life cycle | CPS (1970-1993) |

Table 1: Household life cycle parameters

4.7 Technology

| Parameter | Description | Value | Source |
|-----------|----------------------------|-------|------------|
| g | Growth rate of the economy | 0.02 | NIPA |
| η | Population growth rate | 0.01 | NIPA |
| δ | Capital depreciation rate | 0.13 | NIPA |
| α | Capital share of output | 0.34 | Literature |

Table 2: Technology parameters

We assume that the production technology is Cobb-Douglas:

$$F(K, L) = AK^\alpha L^{1-\alpha}$$

and capital depreciates at rate δ . Following Cooley and Prescott (1995), we calibrate δ using the law of motion for the capital stock:

$$K' = K(1 - \delta) + I$$

where I is the investment. In the steady state (adjusting for growth), the capital stock remains constant and investment is used only to replace depreciated capital:

$$\delta = \frac{I}{K} - g - \eta$$

We calculate K from the *Historical-cost Stock of Private Non-Residential Fixed Assets* in the NIPA. I is calculated from the *Historical-cost Investment in Private Non-Residential Fixed Assets*. We use data from the period 1970 – 1993. δ is computed as the average of $\frac{I}{K} - g - \eta$ over this period. This gives us $\delta = 0.13$.

We set the capital share of output, α , at 0.34.¹⁵

4.8 Housing

Analogous to capital depreciation, we calibrate housing depreciation using the law of motion of housing capital. In the model we assume that housing supply is fixed and that home owners pay a maintenance cost to replace depreciated housing capital. So, the (growth-adjusted) relation between housing depreciation and housing investment is

$$\delta_h = \frac{I_h - \Delta(pH)}{pH}$$

For the value of housing, pH , we use non-farm owner-occupied housing from NIPA’s *Historical-Cost Net Stock of Residual Fixed Assets* table. Investment in housing is computed using non-farm owner-occupied housing from NIPA’s *Historical-cost Investment in Residential Fixed Assets*. This gives $\delta_h = 0.02$.

This data focuses on owner-occupiers, whereas there have been papers that stress the importance of the moral hazard in renter-occupied housing. Campbell and Cocco (2007) and Henderson and Ioannides (1983) are two representatives of this literature. Further, Chambers et al. (Forthcoming),

¹⁵Heathcote et al. (2007) set the value of α at 0.33 after surveying the literature. Cooley and Prescott (1995) set $\alpha = 0.4$. Greenwood et al. (1995) set $\alpha = 0.29$, which is followed by Gervais (2002).

for instance, find that a depreciation rate of owner-occupied housing of 3.4%, and a depreciation rate of tenant-occupied housing of 7.49% from their GMM estimation process. This suggests that the difference between the two could be significantly different. In order to examine the difference in housing depreciation between owner-occupied and rental housing we use the *Current-cost Net Stock of Residential Fixed Assets* and *Current-cost Depreciation of Residential Fixed Assets* tables in the NIPA. The rate of depreciation of non-farm owner-occupied housing is 0.0143, and for tenant-occupied housing the rate of depreciation is 0.0164. These are sufficiently close together that we set the depreciation rate to be the same for both owner-occupied and tenant-occupied housing.

We use data from the SCF to set the down payment constraint. We consider households that have purchased a home in the last year. The down payment is computed as the mean of the ratio of the original purchase price to the amount originally borrowed on the mortgage. The result of this calculation in the SCF is 19.2%, which we round to 20%. This is in line with Chambers et al. (Forthcoming), who calculate the down payment ratio using the American Housing Survey. So we set $d(a < 65) = .2$. Furthermore, we do not allow retired households to take out new loans: $d(a \geq 65) = 1$. In the calibrated equilibrium, this is sufficient to ensure that no one at age T has any debt.

Martin (2003) finds that the average monetary cost involved in a housing transaction is 7–11%. We conservatively set total moving costs to 7% and divide the costs evenly between buyers and sellers so that $\theta_h = 0.035$.

| Parameter | Description | Value | Source |
|-------------|---------------------------------|-------|------------|
| δ_h | Housing stock depreciation rate | 0.02 | NIPA |
| $d(a < 65)$ | Down payment rate | 0.2 | SCF |
| θ_h | Transaction cost of housing | 0.035 | Literature |

Table 3: Housing parameters

4.9 Taxes

There are two forms of taxes in the model economy - income tax, t_y , and property tax, t_p . Piketty and Saez (2007) use public use micro-files of tax return data from the Internal Revenue Service, which have the advantage of being aggregated to the household level already. The income tax rate we choose, $t_y = 0.2$, is in the same range that they compute for the US economy¹⁶.

¹⁶See Table 1, page 6 in their paper

We use data from the Integrated Public Use Microdata Sample (IPUMS) 1990 5% sample. The variables used are the amount of property tax paid and the estimated value of the house. We remove top-coded variables from the sample, and consider only owner-occupiers. Sample observations are weighted using the household weights given in the data set. The weighted average of the ratio of the amount of property tax paid to the estimated value of the house is 0.012. In the model we set $t_p = 0.01$.

| Parameter | Description | Value | Source |
|-----------|-------------------|-------|-------------------------|
| t_y | Income tax rate | 0.2 | Piketty and Saez (2007) |
| t_p | Property tax rate | 0.01 | 1990 IPUMS |

Table 4: Tax parameters

4.10 The productivity process

Labor earnings are given by $l(a, i, j)w$, where w is the wage per efficiency unit of labor, and $l(a, i, j)$ is the number of efficiency units of labor supplied by a household of age a on an island of productivity j and of idiosyncratic ability i . We follow Storesletten et al. (2004) and develop a model of labor earnings over the life cycle. The basic structure of the model is that a household receives a base wage that is conditional on age, its idiosyncratic ability and location of the household. We assume that regional productivity follows an AR1 process and assume that shocks to ability contain a permanent component and a purely temporary component. Further, households are also born with a certain amount of ability, which we incorporate as a fixed effect in the model. See C.3 for details.

| Parameter | Description | Value |
|----------------|---|--------|
| σ_f | Std. dev. of the fixed effect shock | 0.5 |
| σ_ν | Std. dev. of the temporary idiosyncratic shock | 0.25 |
| σ_ι | Std. dev. of the persistent idiosyncratic shock | 0.098 |
| σ_e | Std. dev. of the regional productivity shock | 0.026 |
| ρ_i | Persistence of the idiosyncratic shock | 1 |
| ρ_j | Persistence of the regional shock | 0.9839 |

Table 5: Productivity parameters

Table 5 shows the estimated parameter values. We discretize the regional productivity AR(1) process following Tauchen (1986). Due to computational constraints we pick a 5-point distribution. Since the persistent idiosyncratic shock is a unit root process, we discretize e_{iat} with a 3-point distribution. ν_{iat} is discretized with a 3-point distribution as well.

4.11 Setting macroeconomic variables

The remaining variables are $p(j), q(j), r, w, \sigma, \beta, \gamma, \theta_m$. We normalize $w = 1$. Given the house price vector p and r , the rental price vector q , is determined by the real estate sector's no-profit condition. The house price vector, p , is set so that the housing market clears. The relative risk aversion parameter, γ , is difficult to estimate. We follow Piazzesi et al. (2007) and set $\gamma = 5$, so that the intertemporal elasticity of substitution is .2.

4.11.1 σ, β and macroeconomic moments

Finally, we pick σ, β, θ_m so that the simulated economy matches the data in four moments: the capital stock-output ratio ($\frac{K}{Y}$), the housing stock-output ratio ($\frac{pH}{Y}$), the average annual percent change in financial wealth for households aged 35-55, and the average moving rate. See Appendix C.3.5 for more details.

The capital stock, K , is calculated using the *Current-cost Stock of Net Fixed Assets* table from the NIPA. We set K to be equal to *non-residential private and government fixed assets*.

The output, Y , is computed from the *Personal Consumption Expenditure* table in the NIPA. We calculate Y as personal consumption of non-durable goods + personal consumption of services + gross private domestic investment + government consumption expenditure and gross investment - housing services + services from durable consumption.

The housing stock, pH , is taken from the *Current-cost Stock of Residential Fixed Assets* table in the NIPA. We calculate pH as non-farm owner occupied housing + tenant-occupied housing.

The average moving rate for all households in our PSID sample is 12.42%.

We use NIPA data from 1970 – 1993. We get $\frac{K}{Y} = 2$ and $\frac{pH}{Y} = 1.22$. The capital-output ratio also pins down the interest rate: $r = .04$.¹⁷ From our equilibrium, we get that $\beta = 1.055$ ¹⁸, $\sigma = .12$, $\gamma = 6$, and $\theta_m = .025$.

| Parameter | Value | Moment | Data | Model |
|------------|-------|----------------|--------|-------|
| β | 1.055 | $\frac{K}{Y}$ | 2.00 | 1.97 |
| σ | .12 | $\frac{pH}{Y}$ | 1.22 | 1.26 |
| θ_m | .025 | Moving rate | 12.42% | 11.6% |

Table 6: Parameters calibrated internally

¹⁷This is the balanced-growth interest rate and not the level interest rate.

¹⁸Adjusting for growth, this is equivalent to $\beta = .975$ in a level steady-state economy.

5 Comparing the model to the data

5.1 Ownership over the life cycle

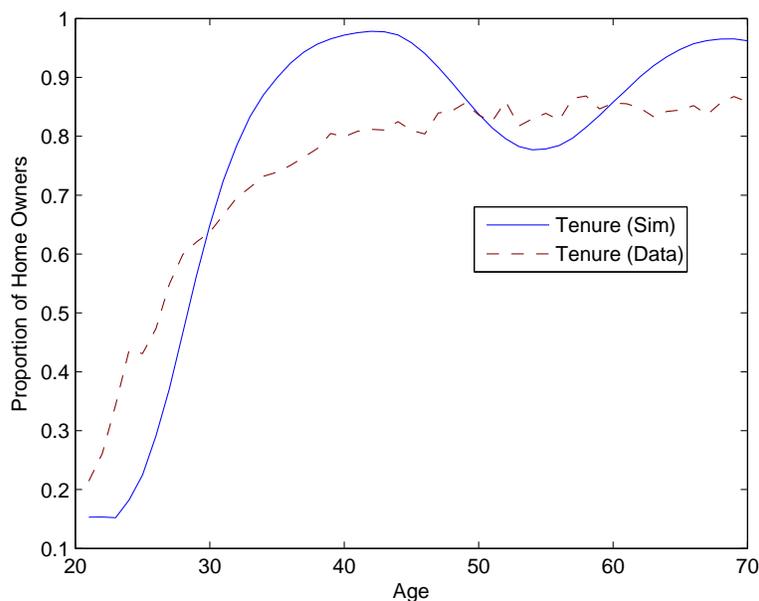


Figure 8: Home ownership: model and data

Figure 8 shows the proportion of home ownership over the life cycle generated by the simulations and from the data. The model economy generates a pattern of ownership over the life cycle that is close to the actual economy. At its peak, ownership in the economy is higher than the data, but well below 100%. Table 7 shows that the model matches the data on ownership level well for young and middle-aged households. Importantly, the model matches the data on the rate of increase of home ownership for younger households (see table 8).

| | 21-35 | 36-50 | 51-65 | Overall |
|-------|-------|-------|-------|---------|
| Model | 0.49 | 0.94 | 0.84 | 0.73 |
| Data | 0.54 | 0.81 | 0.84 | 0.701 |

Table 7: Proportion of owners over the life cycle

| | 21-35 | 36-50 | 51-65 |
|-------|-------|--------|-------|
| Model | 0.05 | -0.006 | 0.009 |
| Data | 0.03 | 0.006 | 0.001 |

Table 8: Yearly rate of increase in average home ownership by age over the life cycle

5.2 Moving

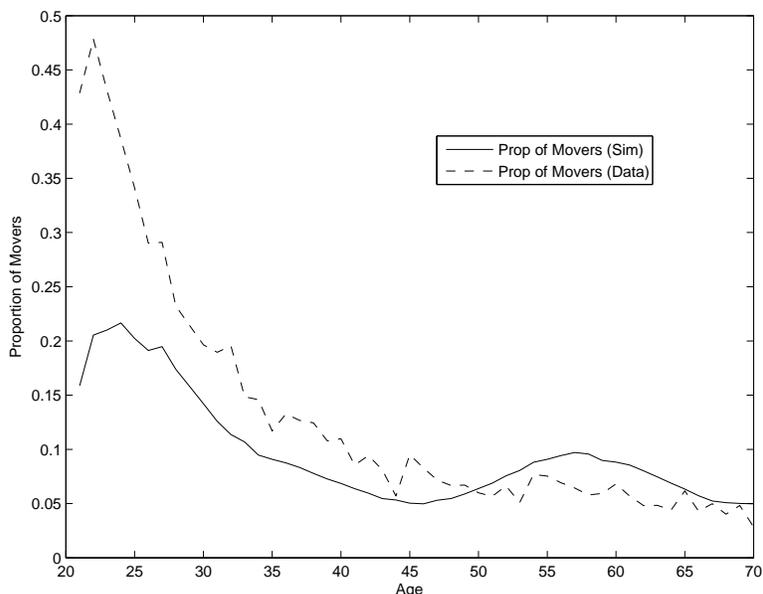


Figure 9: Household moving over the life cycle: model and data

The model matches the moving rates over most of the life cycle (figure 9). From table 9 we see that average mobility from age 35 onwards in the model follows the data closely. In the model, the moving rate falls over the life cycle until the household approaches retirement, at which point the average mobility increases. This pattern arises because of two factors: renters are more mobile than owners (figure 10), and the proportion of owners increases over the life cycle (figure 8).

| | 21-35 | 36-50 | 51-65 |
|-------|-------|-------|-------|
| Model | 0.16 | 0.06 | 0.08 |
| Data | 0.27 | 0.09 | 0.06 |

Table 9: Moving over the life cycle

In the model, we count “inter-state” movers as those households that move to an island of a different productivity from the one on which they start the period. In figure 12 and table 10, the calibrated model and data both feature the same pattern of declining “state” moving rates over most of the working-life. The level of “inter-state” moving in the model is clearly higher than the data; higher than the inter-state movers and job-related movers. Most movers in the simulated economy also move to a new island when they move. In the model, the moving costs of moving “next door” (i.e. changing house size or owning mode but not regional location) are the same as

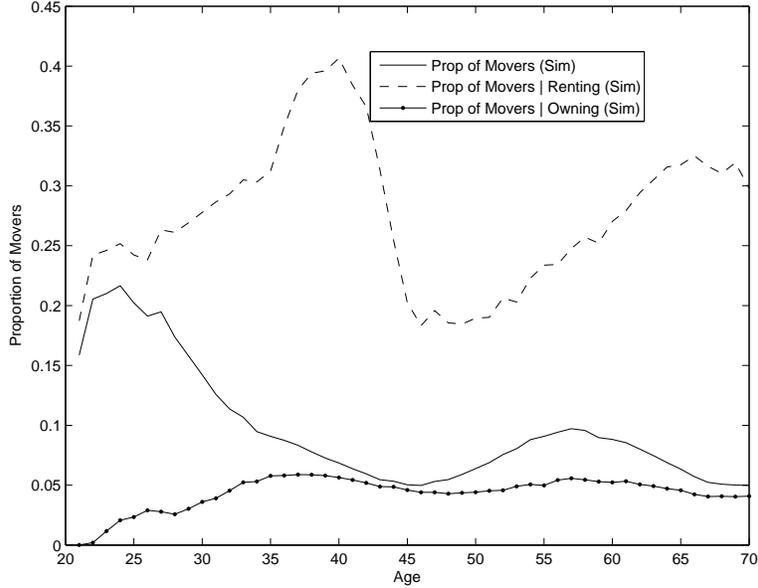


Figure 10: Household moving conditional on ownership choice: model

moving to a completely new location. If there are enough different islands to choose from, nearly every household can find a better “match” with an island that is different from the one that it comes into the period living on. So, conditional on choosing to move to a new house, the household is extremely likely to move to a new island. Evidently, even with only five states for the island productivity process, $\epsilon(j)$, the set of islands to choose from is comprehensive enough that most moving households can find a better match on a different island.

| | 21-35 | 36-50 | 51-65 |
|-------|-------|-------|-------|
| Model | 0.14 | 0.06 | 0.08 |
| Data | 0.05 | 0.02 | 0.01 |

Table 10: Inter-state moving over the life cycle

5.3 Financials

Figure 15 shows the household financial portfolio over the life cycle. We normalize the simulated financial data so that the average net wealth of the simulated economy is equal to the average net wealth observed in the data¹⁹. The net wealth and financial wealth of households over the life cycle of the simulated economy matches closely the patterns in the data. The average house value in the

¹⁹The average net wealth is calculated for the whole economy unconditional on age.

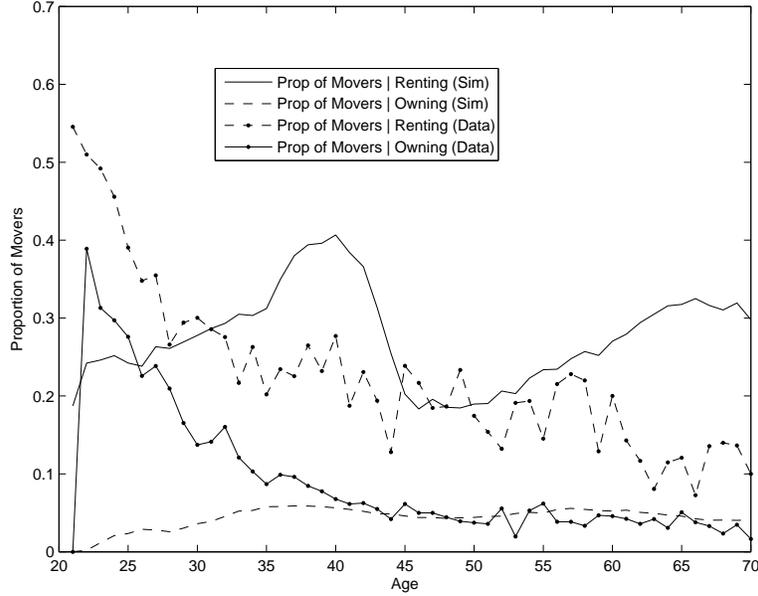


Figure 11: Household moving conditional on ownership choice: model and data

simulated economy rises over the life cycle, but not as far as it does in the data.

6 Understanding the model economy

6.1 Homogeneity

Following our specification of the idiosyncratic ability process in section 4.10, we can separate the idiosyncratic ability thus: $i = (i_t, i_p)$, where i_t is the transitory component of idiosyncratic ability and i_p is the persistent (unit-root) component of idiosyncratic ability. Our model is homothetic in the following sense:

$$V(a, i_p, i_t, j, \tau, h, b, \epsilon) = V(a, 0, i_t, j, \tau, \frac{h}{e^{i_p}}, \frac{b}{e^{i_p}}, \epsilon) \cdot (e^{i_p})^{1-\gamma}$$

This allows us to normalize one state variable in computing the model, along the lines of Grossman and Laroque (1990). Discrete choices are, therefore, determined by the ratios h/e^{i_p} and b/e^{i_p} as opposed to the level values of h, b and i_p . So each household with the same set of discrete states (j, j_o, τ, op, a) and the same ratios $\frac{h}{e^{i_p}}, \frac{b}{e^{i_p}}$ will make the same discrete choices (over ownership and location productivity). In the following subsections we discuss in this light how the household

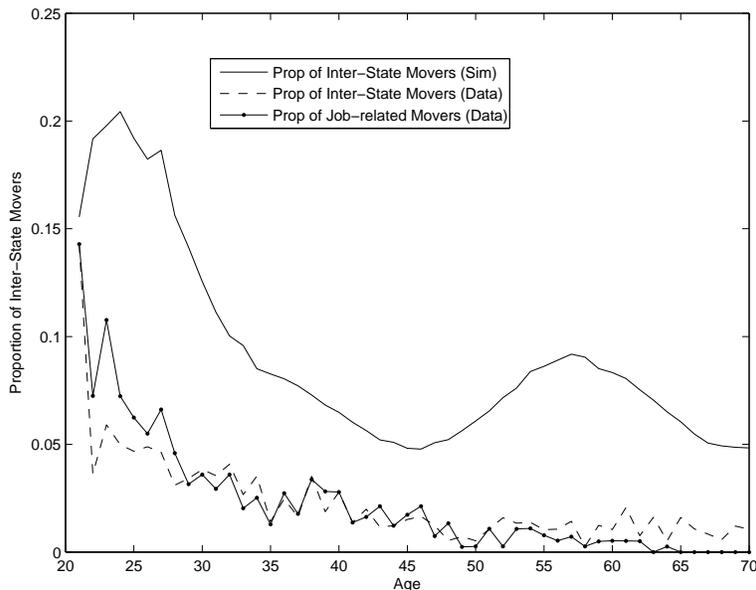


Figure 12: Inter-state moving over the life cycle: model and data

makes owning and renting decisions.

6.2 The tax benefits of ownership

The decision to own or rent is influenced by the tax benefits of ownership. In our model, the household deducts mortgage interest payments from its income tax. This puts a wedge between the user cost of owning and renting.

6.3 Expected human and total wealth

A household's human wealth is the expected discounted value of future earnings over its lifetime. It is a function of the household's ability over time, its location over time, and the age-dependent base wage. Total wealth includes, in addition to human wealth, accumulated savings and housing assets, which we refer to as financial wealth. Given a level of total wealth, higher financial wealth implies that the household's total wealth is less affected by shocks to ability and regional productivity. Hence a household with higher financial wealth compared to human wealth is less likely to want to adjust its housing consumption in the future in response to a shock and is therefore likely²⁰ to have a larger expected duration of stay, and is therefore more likely to own. On the other hand,

²⁰The exception is if the household is poorly matched with its island.

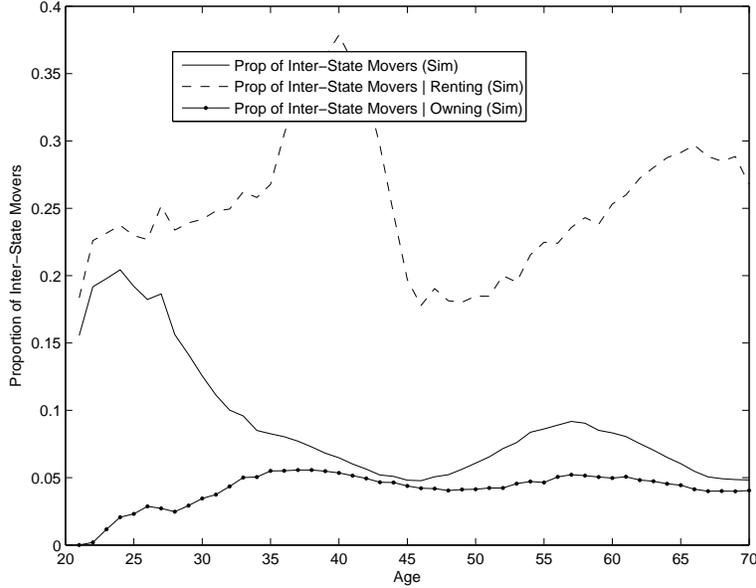


Figure 13: Inter-state moving conditional on ownership choice: model

the higher the share of human wealth in total wealth, the more likely the household will want to adjust its housing consumption or location in the future, and is therefore more likely to rent today.

Permanent idiosyncratic shocks have a qualitatively different effect from transitory idiosyncratic productivity shocks. A positive permanent shock to productivity has two effects: firstly, it raises current income and hence current wealth; secondly, it raises (expected) future income and thus human wealth. Transitory shocks, on the other hand, do not affect future income and can be viewed as wealth shocks.²¹ A positive transitory shock increases the cash-in-hand of the household, while a negative shock reduces it.

Given the homotheticity of the value function, a household with positive idiosyncratic shocks behaves like a household with the appropriately scaled down value of wealth and zero idiosyncratic productivity and transitory shocks. On the other hand, a household with a positive transitory shock behaves like a household with a higher value of wealth. Hence the two shocks have different effects on the ownership decision.

The relative value of human wealth to total wealth would still be an important factor in ownership choice even if we had relaxed the Cobb-Douglas assumption in the utility function by using a more general CES aggregator²². Homogeneity of the value function makes the model computa-

²¹Strictly speaking, a transitory shock can affect future income if the household moves because of the shock.

²²For instance, Piazzesi et al. (2007) find an intratemporal elasticity of substitution between housing and con-

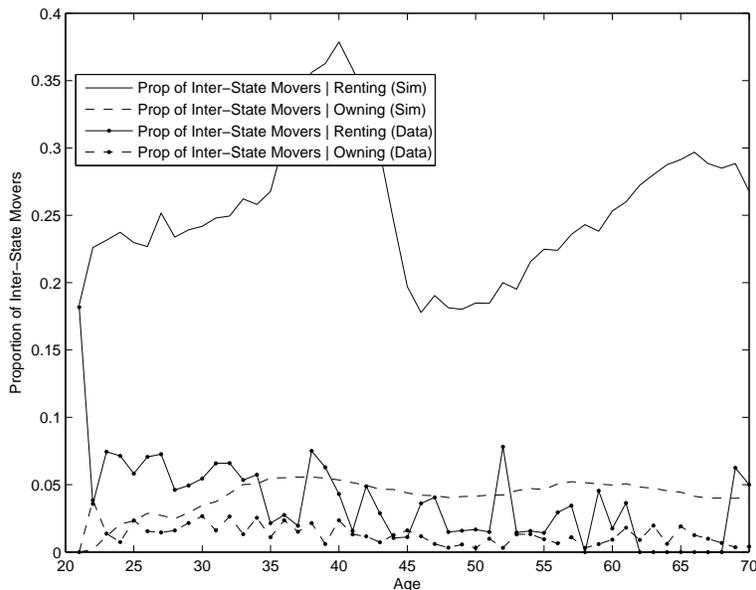


Figure 14: Inter-state moving conditional on ownership: model and data

tionally more tractable and homogeneity of degree 1 of the utility function eliminates the need to separately observe housing values and quantities of service. However homogeneity has the effect of reducing the heterogeneity of ownership and location choice in the model economy, particularly for younger households. Early in life, both in the data and in the model, the wealth of households is close to zero. We further start off the households with no housing assets. All households with zero wealth will then make the same discrete choices conditional on their discrete state. In the model, the initial wealth distribution serves to inject heterogeneity into the economy, so that younger households make choices that are disparate enough to reasonably match the real economy. We point out that all models which are homogeneous of “nearly” degree 1 are likely to have a similar trade-off between tractability and heterogeneity over discrete choices.

6.3.1 Expected duration and mobility

The expected duration of stay can affect the ownership choice decision in two ways - the presence of transaction costs makes ownership “cheaper” over a longer horizon of stay and the price risk of owning a housing asset decreases in the duration of stay while the rental risk increases in the time spent²³.

sumption in the range of 1.04 to 1.25.

²³See Sinai and Souleles (2005).

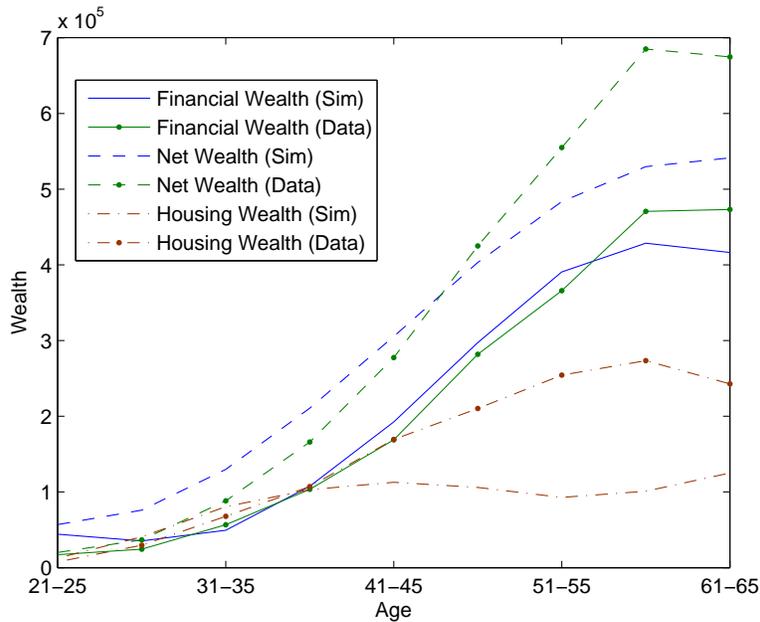


Figure 15: Financials over the life cycle: model and data

Mobility declines over the life cycle in the model economy. This can be attributed to two factors: firstly, renters are more mobile²⁴ than owners and secondly, an increasing proportion of the population owns its home over the life cycle. The higher mobility of renters is a result of the higher moving costs incurred by owners and the fact that the more mobile a household expects to be the riskier it finds ownership. In the model economy, renters are more mobile at all ages (in figure 10). The overall mobility in the economy decreases with age since the proportion of owners increases in the economy.

Figure 16 shows the expected duration of households that have just moved, conditional on ownership as a function of age. Expected duration is calculated here as the average time until the household moves again. The expected duration of stay is higher for owners than for renters. The expected duration of renters does not vary much over the life cycle, but the expected duration of owners shows the following patterns: the expected duration is high for newborn households that own, and then drops in the early part of life. It exhibits a hump-shape following the sharp drop, rising in the period from 25-50, after which it falls off gently.

An artifact of the model is that when households first enter the economy they are all movers. So all newly born households are artificially included in the expected duration calculation. The

²⁴By “more mobile”, we mean more likely to move in the future.

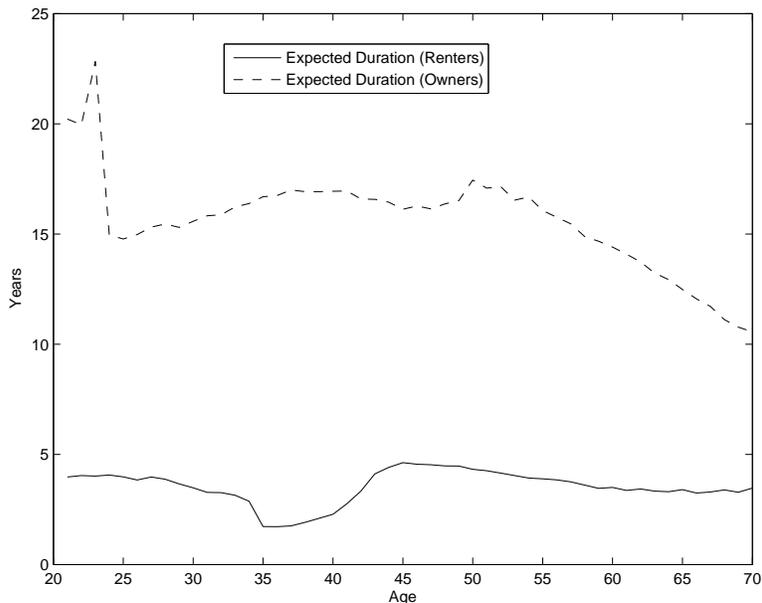


Figure 16: Expected duration of movers over the life cycle in the model (yearly bins)

households that expect to stay for a longer time tend to own in the beginning. The expected duration of stay drops over the first five years as the households with the highest expected duration of stay move to owned housing. The expected duration of stay then increases between ages 25 – 50. This is due to two factors. Firstly, as more uncertainty over human wealth is resolved, households expect to stay in their house longer. Secondly, the composition of movers changes. Even though owners move less often than renters, the number of former owners moving to new owner-occupied homes grows over this part of the life cycle, since the proportion of owners is growing and the proportion of owners moving is relatively constant. Former owners have higher expected durations than former renters since it is more costly for a former owner to move to a new house than a former renter (due to the transaction cost of selling a house). Former owners will only pay their higher cost of moving if the benefit of moving is higher and the benefit is higher when the expected duration is higher. Finally, households near age 65 have a lower expected duration for two reasons. Firstly, their expected lifespan is lower, and secondly, their desired location changes as their earnings fall: they prefer to live on a low-cost island, since human wealth is a small part of their total wealth. Furthermore, “retired” households are unable to take out a new mortgage and so eventually want to sell their house and rent so as to consume their total assets before dying. So most homeowners choose to rent eventually in old age, causing expected duration in owner-occupied houses to fall.

6.3.2 Location choice

Islands in our model are differentiated by their productivity. The choice of location for households is therefore a trade-off between higher earnings due to higher productivity and higher house prices and rental costs. Three factors play a significant role here. The steep slope of the base wage in the early part of life (base wage peaks at around age 45) implies that middle aged households benefit more from living on more productive islands. Secondly, since island and idiosyncratic shocks are persistent, younger households gain more from relocating to more productive islands as they have a longer expected life span. Finally, middle-aged households have more wealth and income and larger families and so tend to have larger houses. This increases their moving cost, so that relocation is less desirable (*ceterus paribus*) for middle-aged households.

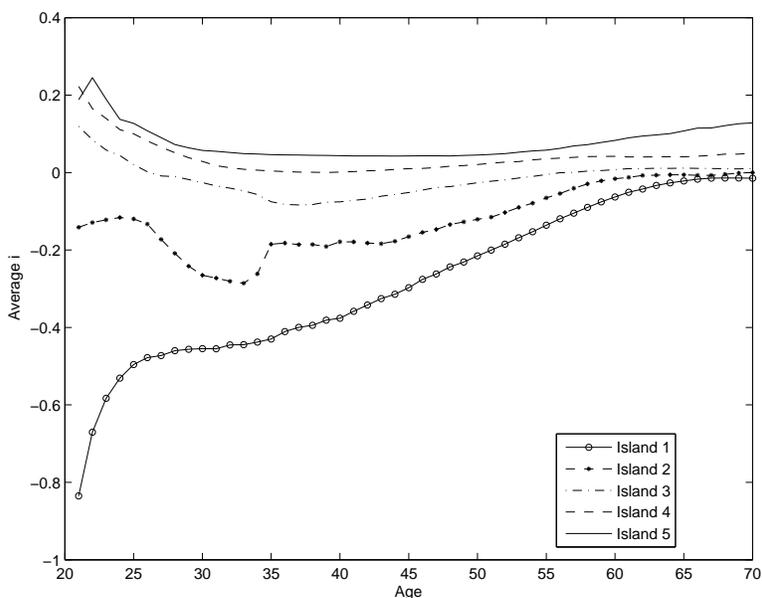


Figure 17: Scaled average permanent idiosyncratic ability over the life cycle by island productivity in the model (yearly bins)

Figure 17 shows the adjusted average idiosyncratic ability of households on each island as a function of age. We scale each household's ability by the cross-sectional standard deviation of the permanent idiosyncratic component for its age so that any spread in the adjusted average reflects the effect of the complementarity between the idiosyncratic and island components²⁵. We see from

²⁵As people sort themselves onto their preferred islands, we expect that the average idiosyncratic ability over each island will diverge. This is clearer if we assume that households are perfectly free to move to any island of their choice. Let us suppose further that the most productive island houses the most able 20% of the population. Then as the variance of idiosyncratic ability increases, the average idiosyncratic ability of the most able 20% of the population

figure 17 that this complementarity is strongest in the first 20 years of life, when the expected life-span is high and the earnings stream is rising.

7 Impact of standard factors

In this section, we look at the impact of the standard factors through a series of experiments where we turn off one of these factors and analyze household behavior. The experiments are comparative statics exercises which are conducted at the same prices as the baseline stationary equilibrium. From the baseline case, we have 4 scenarios, with one factor absent in each.

Financial constraints are turned off by setting $d(a) = 0 \forall a < 65$. Setting the down-payment constraint to zero does not eliminate financial constraints (households still cannot borrow on future income, and cannot hold a negative net asset position), but is a way of relaxing the financial constraint related to the housing market. We also repeat the same experiment, but allow prices to adjust to a new general equilibrium.

The family size adjustment is a life cycle variable. We turn it off by setting $F(a) = 1 \forall a$, which makes the family size profile flat. Since the family size adjustment appears only in the utility function, the level of the family size is not important - the household's decisions remain exactly the same.

Career concerns of households are reflected in their mobility and location choice, where islands are characterized by their productivity. In one counterfactual, we set the household's base wage $l_a(a) = \bar{l}_a$, where \bar{l}_a is the unconditional mean of log base wage in the baseline model. In the last counterfactual, we set the standard deviation of the permanent component of the idiosyncratic component of earnings to zero, $\sigma_{i_p} = 0$.

| | Ownership rate | Change in ownership rate |
|--------------------|----------------|--------------------------|
| Base | 0.4877 | — |
| No fam size | 0.5659 | 0.0782 |
| d=0 | 0.5163 | 0.0286 |
| Flat base wage | 0.4197 | -0.0680 |
| $\sigma_{i_p} = 0$ | 0.0798 | -0.4079 |

Table 11: Ownership rates of younger households: counterfactuals

Table 11 shows the impact on the ownership choice of younger households (ages 21-35) from increases as well. Reasoning analogously, the average idiosyncratic ability of the bottom 20% of households will fall. In order to adjust for this increase in variance, we divide the idiosyncratic ability by its standard deviation.

each of the experiments. The career concerns experiments have the most impact on the ownership rate. The effect of changing the down payment constraint is significant, though less than the effect of career concerns. Interpreting the impact of family size is more subtle: a growing family size pushes the ownership rate up in the earliest part of the life cycle, and down for households age 26-40. These happen to mostly cancel each other out; the discussion in section 7.1 describes the two factors that cause these effects.

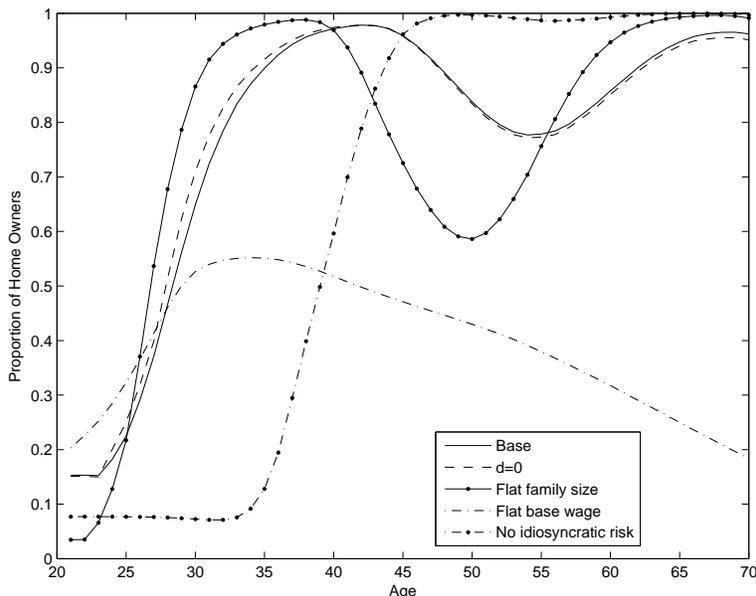


Figure 18: Comparing ownership (from baseline-yearly bins)

7.1 Impact of family size

From figure 18, we see that starting from the baseline case, removing family size reduces ownership in the very young households (until age 26), while it increases ownership of younger households after that, until age 41. Changes in family size affect ownership primarily through two channels. The increase in family size over the early half of the life cycle reduces expected duration and will reduce ownership. Secondly, average earnings rise steeply over the early part of the life cycle. With no changes in family size, younger households want to dis-save as much as possible in order to smooth consumption (see figure 19), while older households want to save more. Owning a house (and putting equity into it) is a form of savings and so more young households prefer to rent. Even if there were no down payment constraints, leveraging completely to buy a house is generally risky

for young households. With the addition of family size, the low propensity to save due to rising wages is mitigated by the fact that the household has to save for the higher family consumption and housing demand. Part of this increased savings is in the form of housing assets, which increases the ownership rate. Though in the model, family size does not differ across households of the same age, in the data, in figure 7, young owners have larger families than young renters but older renters have larger families than older owners, as our counterfactual experiment predicts.

When households are very young, the ones with a higher expected duration of stay will own, and hence the second effect dominates. In the period after that, the reduction in expected duration dominates and we see that ownership rises above the baseline case.

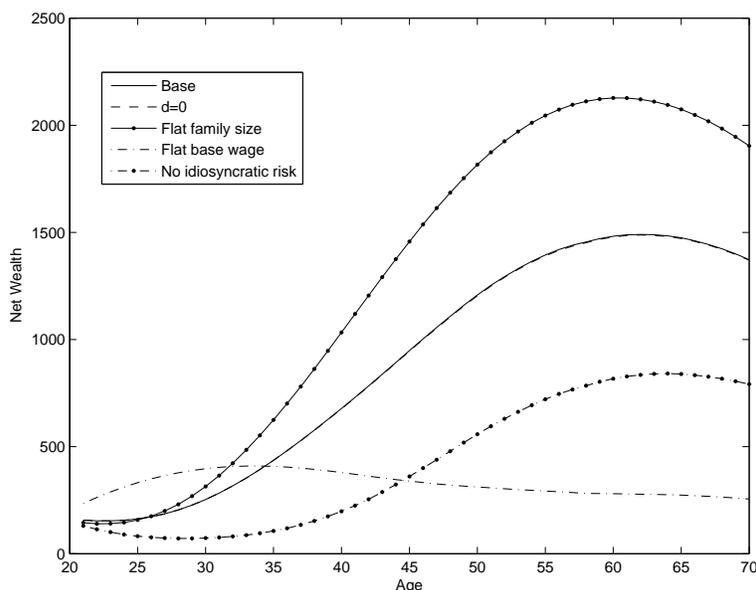


Figure 19: Comparing net wealth (from baseline- yearly bins)

7.2 Relaxing the borrowing constraint

Relaxing the borrowing constraint has a comparatively smaller effect. Home ownership becomes unequivocally more attractive. But in the later part of life, the ownership rate falls very slightly as households with lower equity in their homes choose to move in anticipation of “retirement” to low productivity islands earlier.

Households adjust to the financial constraint on the intensive and extensive margin. On the extensive margin, households that chose to rent because they were financially constrained can now

own. On the intensive margin, the relaxation of the constraint means that households will, on average, own larger houses, because they do not have a down payment that they have to afford. To observe these separate margins, we simulate 200,000 households for their entire life cycle in the baseline economy and in the same economy but with $d(a < 65) = 0$, giving them the same shock values in each economy. The average difference between the age at which households first own in the baseline economy and the age of first ownership in the alternative is 2.06 years. 30.5% of households choose to buy an at least 5% cheaper (and smaller) house in the constrained economy rather than defer buying a house until later in life. Some of the households that are down payment constrained choose to delay their purchase of a house while others choose to buy a smaller home.²⁶ Evidently, many households choose to buy a smaller home rather than delay, which explains the small changes in the ownership rate from changing d . The larger the gap in the user cost of renting over the user cost of owning, the more the households will adjust on the intensive margin rather than delay purchasing a home. The more attractive is owning, the less the down payment constraint effects ownership.²⁷

7.2.1 General Equilibrium Effect

The general equilibrium effect of removing the down payment constraint ($d(a < 65) = 0$) is roughly the same as partial equilibrium. In the partial equilibrium case, ownership rates changed, but total housing demand and savings barely change. So prices need barely change to restore equilibrium.

7.3 No permanent earnings uncertainty

Shutting down the permanent component of idiosyncratic earnings uncertainty causes the ownership rate for young households to drop, while eventually all older households own. There are two effects: precautionary savings drops and expected duration rises with the decrease in earnings uncertainty. For young households, precautionary savings drops the most, since they have the longest horizon of earnings and the lowest total wealth to human wealth ratios. With fewer “rich and dumb” young households in this economy, ownership amongst the young is lower. On the other hand, for middle aged and older households, the rise in expected duration dominates the precautionary savings effect,

²⁶This trade-off is also discussed in the context of the 4-period model by Ortalo-Magne and Rady (2006).

²⁷In the model, the housing choice space \mathcal{H} is a continuous space and does not depend on whether the household chooses to own or rent. To get larger welfare effects from removing down payment constraints, some models (e.g. Chambers et al. (Forthcoming)) have a minimum house size for owner-occupied housing, thus limiting the scope for adjustment along the intensive margin.

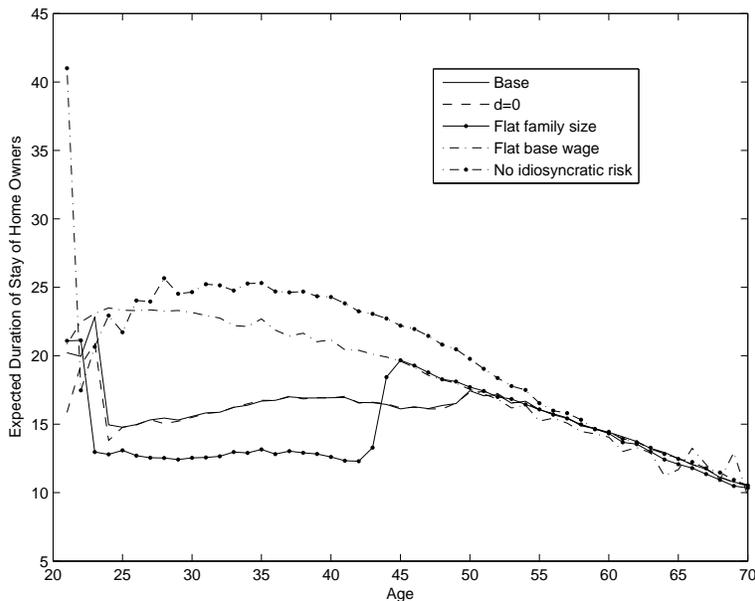


Figure 20: Expected duration of stay of owners (Baseline - yearly bins)

and ownership increases.

7.4 Flat earnings profile

Households with a flat earnings profile over the life cycle are much more likely to own when young. The intuition for the rise in ownership is similar to that of the flat family size case: young households no longer wish to dis-save as much when young and expect to stay in their current residence longer. As family size falls later in life, and households downsize into a smaller house, they become more mobile and are more likely to rent.

There is a common theme that can explain why many, but not all, young households choose to rent their home: owning a home means (but does not necessarily require) saving and the return on savings is too low for young households to want to save. Risky house prices are also important - with risk, it is not optimal to hold highly levered positions in housing for extended periods of time. Households that own want to accrue equity in their home so as to reduce risk.

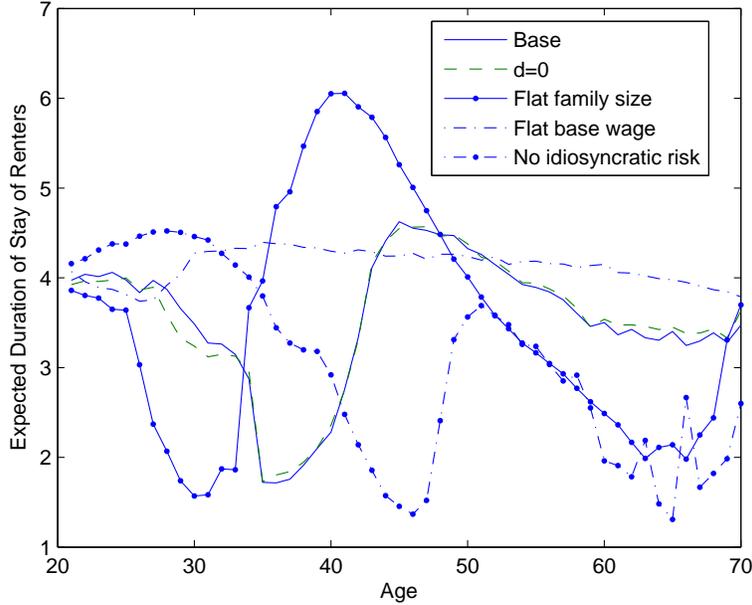


Figure 21: Expected duration of stay of renters (Baseline - yearly bins)

8 Conclusion

The illiquidity and immobility of housing and the inability to borrow against future income or insure against income shocks are at least as important as housing affordability in determining the ownership choice of young households. Illiquidity and immobility make it expensive for households to move, particularly into and out of owner-occupied housing. Factors that affect the expected mobility and desired savings of the household play a large role in determining the ownership choice of the household. Family size and career concerns, which are important determinants of expected mobility and savings, significantly affect the ownership choice of the household.

We construct a general equilibrium, over-lapping generations, incomplete markets (Bewley) model. Our model incorporates risky assets (housing) in a general equilibrium where the households know the exact law of motion for prices. We prove existence of a finite dimensional state-space, stationary equilibrium using Kakutani's Fixed Point Theorem. We then use our calibrated model to measure the relative importance of career concerns, family size change and down payment constraints in determining the ownership rate over the life cycle.

Expected future growth in family size decreases the dis-saving of very young households, and therefore increases the ownership rate. It also makes the household more mobile, which dominates

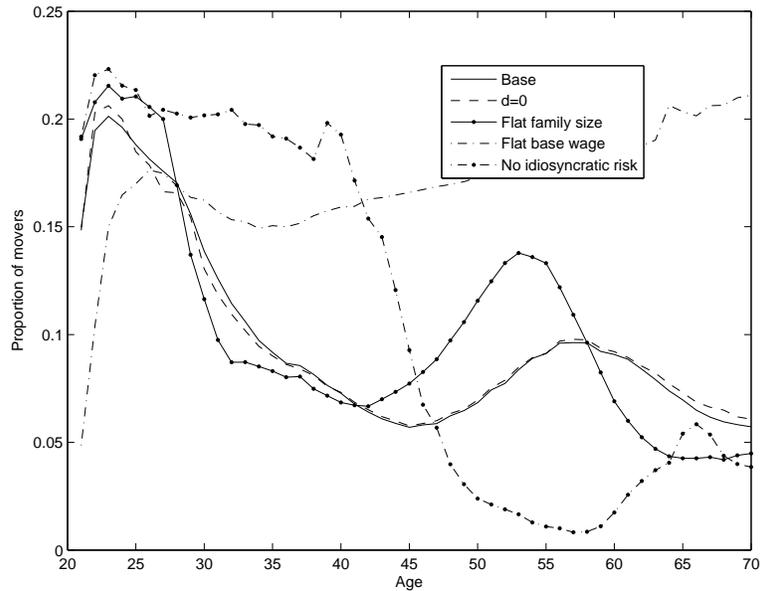


Figure 22: Comparing mobility (from baseline - yearly bins)

the first effect after the first 5 years. Eliminating the down payment needed to buy a house raises the ownership rate of young households by 2.63 percent points and reduces the age at which a household first owns by 2.06 years and hardly changes prices. Twice as many first-time home buyers choose to buy a smaller house rather than delay owning when forced to make a down payment. Our results tell us that while down payment constraints are a factor in determining the ownership choice of younger households, they are not as important as other factors which directly affect mobility, like career concerns.

Given the importance of expected duration in explaining the life cycle pattern of ownership, interesting extensions (which we leave to future research) of the model could include stochastic and/or endogenous family sizes, a search model over job markets, a richer set of mortgages for financing homes, and a deeper investigation into the choices of older households and the role of housing in bequests of the elderly and the role of bequests in housing purchases of the young.

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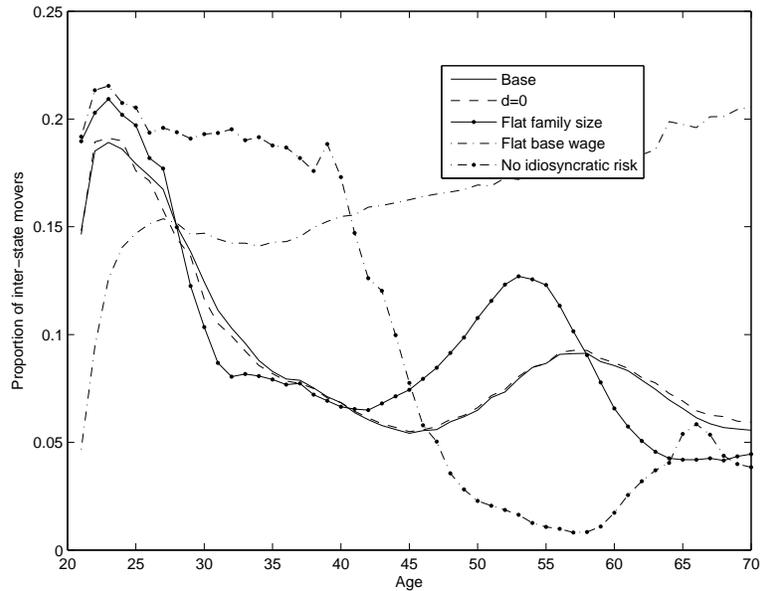


Figure 23: Comparing inter-state mobility (from baseline - yearly bins)

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A Existence of Equilibrium

We use Kakutani’s fixed point theorem in order to establish the existence of a stationary CE. The proof can be broadly divided into four steps:

1. Show that the household’s problem is well-defined and has a solution.

2. The optimal policy function generates a transition function for the household distribution over states. We show that there is a household distribution over states that is invariant with respect to the transition function.
3. Show that the set of stationary household distribution over states is upper hemi-continuous (henceforth uhc) in the price vector.
4. Construct a price transition operator that maps a price vector onto the next period price vector and show that this map has a fixed point using Kakutani's theorem.

Our innovation is to add as an equilibrium object mixed allocations over the optimal choice set, which act as tie-breaking criteria. This gives us a convex (probability) space of optimal choices and a convex set of macroeconomic variables. We show that this is sufficient to satisfy the conditions necessary for Kakutani's theorem to derive a stationary competitive equilibrium.

Notation and conventions

Definition. We use $\exists!$ to denote “there exists a unique”.

Remark. We will be dealing with convergent sequences in \mathbb{R}^n throughout this proof. We follow the convention that if the space in question, $S \subseteq \mathbb{R}^n$, the metric is the standard metric on \mathbb{R}^n . In addition, if the space in consideration is a probability space then the corresponding metric is the sup-norm. Any non-standard metrics will be indicated in the proof.

We start by showing that the household's optimization problem has a solution by using the Maximum Theorem. First, we have to establish that Γ , the feasible correspondence, is continuous in s, \vec{p} . We simplify the feasible correspondence by eliminating c from the budget constraint, using the non-satiation of preferences. We can then define the feasible set of choices.

Definition. $s^c = (h, b, \epsilon) \in \mathcal{S}^c = \mathcal{H} \times \mathcal{B}$ and $s^d = s \setminus s^c \in \mathcal{S} \setminus \mathcal{S}^c$ are the space of continuous state variables and discrete state variables. $y^c = (h', b', \epsilon', c) \in \mathcal{Y}^c = \mathcal{H} \times \mathcal{B} \times \mathcal{E} \times \mathcal{C}$ is the continuous choice vector, and $y^d = (\tau', \tilde{j}) \in \mathcal{Y}^d = \{0, 1\} \times J$ is the discrete choice vector.

The period utility function $u : \mathcal{S} \times \mathcal{Y} \rightarrow \mathbb{R}$ is defined as

$$u(s, y) = \left(\left(\frac{c}{S(a)} \right)^{1-\sigma} \left(\frac{h'}{S(a)} \right)^\sigma \right)^{1-\gamma} / (1-\gamma)$$

where $s = (a, i, j, \tau, h, b, \epsilon)$ and $y = (\tau', h', b', \tilde{j}, \epsilon', c)$.

Before we state the household's problem, we define the household's action plan.

Definition. A *state-contingent action plan* is a function $\tilde{y} : \mathcal{S} \times \mathcal{P} \rightarrow \mathcal{Y}$ that specifies the household's choice at every state given the price vector.

A.1 The household's problem

The value function, $V : \mathcal{S} \times \mathcal{P} \rightarrow \mathbb{R}$ and the optimal policy correspondence, $Y : \mathcal{S} \times \mathcal{P} \rightrightarrows \mathcal{Y}$ are given by the following:

$$V(s; \vec{p}) = \sup_{y \in \Gamma(s; \vec{p})} u(s, y) + \beta \lambda(a) E[V(s') | s, y]$$

where $\Gamma : \mathcal{S} \times \mathcal{P} \rightrightarrows \mathcal{Y}$ is given by

$$\begin{aligned} c + b' + h'((1 - \tau')q(\tilde{j}) + \tau'p(\tilde{j})(\delta_h + t_p + 1 + 1_m\theta_h)) &\leq \\ wl(a, i, \tilde{j})(1 - t_y)(1 - 1_m\theta_m) + b(1 + r(1 - t_y)) + \tau hp(j)(1 - 1_m\theta_h) & \\ b' \geq \min\{-(1 - d)p(\tilde{j})\tau'h', (1 - 1_m)b\} & \\ c \geq 0 & \\ h' \geq 0 & \\ \tau' \in \{0, 1\} & \\ \tilde{j}(\epsilon') = \tilde{j} & \end{aligned}$$

$$1_m = \begin{cases} 0 & \text{if } h' = h, \tau' = \tau, \epsilon' = \epsilon \\ 1 & \text{else} \end{cases}$$

Definition. We define the following for a more parsimonious representation.

$$\begin{aligned}
lhs(s^d, y^d) &= \min\{(1-d)p(\tilde{j})\tau'h', 1_m b\} \\
\bar{w}(s^d, y^d) &= wl(a, i, \tilde{j})(1-t_y) \\
\tilde{r} &= (1+r(1-t_y)) \\
\tilde{p}(s^d) &= p(j)(1-1_m\theta_h) \\
\tilde{x}(y^d) &= ((1-\tau')q(\tilde{j}) + \tau'p(\tilde{j})(\delta_h + t_p + 1 + 1_m\theta_h)) \\
rhs &= \bar{w}(s^d, y^d) + b\tilde{r} + h\tilde{p}(s^d) - h'\tilde{x}(y^d)
\end{aligned}$$

To show that the feasible choice correspondence is continuous we define the set of feasible continuous choices given a discrete choice and show that this is continuous for all discrete choices.

Definition. $\Gamma_d : \mathcal{S} \times \mathcal{P} \times \mathcal{Y}^d \rightrightarrows \mathcal{Y}^c$ is the set of feasible continuous choices given a discrete choice, y^d .

$$\Gamma_d(s, \vec{p}, y^d) = \{y^c \in \mathcal{Y}^c : (y^d, y^c) \in \tilde{\Gamma}(s)\} \cup (0, 0)$$

We note that this ensures that only feasible discrete choices have $\Gamma_d(s) \neq \{(0, 0)\}$. The zero element is added to ensure continuity of the feasible set, and is WLOG, since the element has $u = -\infty$.

Remark. Given the assumptions above, \mathcal{S}, \mathcal{Y} are compact.

We now redefine the feasible correspondence as:

$$\Gamma(s, \vec{p}) = \{y \in \mathcal{Y} : y^c \in \Gamma_d(s, \vec{p}, y^d)\}$$

Lemma 2. If Γ_d is continuous $\forall y^d$, then Γ is continuous.

Proof. First, we show that Γ is uhc. Pick a sequence $(s_n, \vec{p}_n) \rightarrow (s, \vec{p})$, $(s_n, \vec{p}_n) \in \mathcal{S} \times \mathcal{P} \forall n$ and $z_n \in \Gamma(s_n, \vec{p}_n)$. Since \mathcal{Y} is compact, \exists a convergent subsequence $z_n \rightarrow z$.²⁸ The sequence of discrete choices, $z_n^d \rightarrow z^d$. So $\exists N : \forall n > N, z_n^d = z^d$. Take z_n to be the subsequence such that $z_n^d = z^d$. Now, $z_n^c \in \Gamma_d(s, \vec{p}, z^d) \forall n$. Since $\Gamma_d(s, \vec{p}, z^d)$ is uhc, $z^c \in \Gamma_d(s, \vec{p}, z^d) \implies (z^d, z^c) = z \in \Gamma(s, \vec{p})$. Hence Γ is uhc.

²⁸Note the abuse of notation. This does not create any problems because we are not concerned with the original sequence after this point. We will do the same thing through this proof.

To show: Γ is lhc. Pick a sequence $(s_n, \vec{p}_n) \rightarrow (s, \vec{p})$, $(s_n, \vec{p}_n) \in \mathcal{S} \times \mathcal{P} \forall n$ and $z \in \Gamma(s, \vec{p})$. Since Γ_d is lhc, $\exists(z_n) \rightarrow z$ s.t. $z_n \in \Gamma_d(s_n, \vec{p}_n, z^d)$. But this $(z_n) \rightarrow z$, and $z_n \in \Gamma(s_n, \vec{p}_n) \forall n$. Therefore Γ is lhc. \square

Lemma 3. $\Gamma(s, \vec{p}, y^d)$ is continuous.

Proof. We first define a distance function: $D^{y^d, s^d} : \mathcal{S}^c \times \mathcal{Y}^c \rightarrow \mathbb{R}$, where $D(s^c, y^c) = d(\Gamma(s^c; s^d, \vec{p}, y^d), y^c)$, d being the standard distance measure for the \mathbb{R}^N metric space (We will drop the superscripts on D . They will be clear from the context).

Given (s^d, y^d) ,

$$D(s^c, y^c) = \{\{\max(h' - \bar{h}, \underline{h} - h', 0)\}^2 + \{\max(b' - rhs, lhs - b', 0)\}^2\}^{1/2}$$

which is continuous in (s^c, y^c) .

Now we show that $\Gamma_d(s, \vec{p}, y^d)$ is lhc. Pick any $(s_n, \vec{p}_n) \rightarrow s$ and $y^c \in \Gamma_d(s, \vec{p}, y^d)$. WLOG, we consider sequences where $s_n^d = s^d \forall n$. Let $d_n = D(s_n^c, y^c)$. Then $(d_n) \rightarrow 0$. Construct a sequence $(y_n^c) : d(y_n^c, y^c) = D(s^c, y^c)$ and $y_n^c \in \Gamma_d(s_n^c, s^d, \vec{p}, y^d)$, which is possible since $\Gamma_{\vec{p}, y^d}$ is compact-valued. Since $(y_n^c) \rightarrow y^c$, $\Gamma_{\vec{p}, y^d}$ is lhc.

Finally, we show that Γ_d is uhc. Since Γ_d is compact-valued, it is sufficient to show that Γ_d has a closed graph. Pick a sequence $(s_n, \vec{p}_n, y_n^c) \rightarrow (s, \vec{p}, y^c)$ such that $y_n^c \in \Gamma_d(s_n, \vec{p}_n, y^d) \forall n$. WLOG assume that $s_n^d = s^d \forall n$. We want to show that $y^c \in \Gamma_d(s, \vec{p}, y^d)$. Assume not. Then $D(s^c, y^c) > 0 \Rightarrow \exists n : D(s_n^c, y_n^c) > 0$ (by continuity of D). Since this contradicts the assumption that $y_n^c \in \Gamma_d(s_n, \vec{p}_n, y^d) \forall n$, Γ_d has a closed graph, and is uhc. \square

Theorem 4. (*Theorem of the Maximum*). *There is a solution to the household's problem such that*

1. $\exists! V : \mathcal{S} \times \mathcal{P} \rightarrow \mathbb{R}$ that solves the household problem.
2. The optimal policy correspondence, $Y : \mathcal{S} \times \mathcal{P} \rightrightarrows \mathcal{Y}$ is non-empty, compact-valued and uhc.

Proof. $u(\cdot, \cdot)$ is continuous and Γ is compact-valued and uhc. So Berge's maximum theorem applies and the result follows (see Stokey and Lucas, Thm. 3.6). \square

When a household moves to an island of type j , we can see from the household problem that it is indifferent between all islands of that type. We formalize this below.

Fact. For $\forall s, \vec{p}$, let $y(s; \vec{p}) = (\tau'(s; \vec{p}), h'(s; \vec{p}), b'(s; \vec{p}), \tilde{j}(s; \vec{p}), \epsilon'(s; \vec{p})) \in Y(s; \vec{p})$. If $h'(s; \vec{p}) \neq h$ or $\tau'(s; \vec{p}) \neq \tau$ or $\epsilon'(s; \vec{p}) \neq \epsilon$ then

$$\hat{y}(s; \vec{p}) = (\tau'(s; \vec{p}), h'(s; \vec{p}), b'(s; \vec{p}), \tilde{j}(s; \vec{p}), \hat{\epsilon}(s; \vec{p})) \in Y(s; \vec{p}) \forall \hat{\epsilon} : \bar{j}(\hat{\epsilon}) = \tilde{j}$$

The next step is to show that there is a unique invariant household distribution over states given the price, an economy-wide mixed allocation and the optimal policy correspondence. We first define the transition function generated by the optimal policy, with respect to which we show that there is a unique invariant household distribution. There are three parts: the transition function for households that survive in a particular period, the transition function for households that die (this is how newborns are placed in the economy), and the complete transition function.

Definition. Given a price \vec{p} and mixed allocation α , the household transition function for survivors, $GS_{\vec{p}, \alpha} : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1]$ is defined as

$$GS_{\vec{p}, \alpha}(s, S') = \int_{\tilde{y} \in Y(s; \vec{p})} \int_{s' \in S'} 1\{\tilde{h} = h', \tilde{b} = b', \tilde{\tau} = \tau', \tilde{\epsilon} = \epsilon', a' = a+1\} \pi_i(i'|i) \pi_j(j'|\tilde{j}) ds' d\alpha(s, \vec{p}, \tilde{y})$$

The transition function for newborns, $GN : \mathcal{B}(\mathcal{S}) \rightarrow [0, 1]$ is defined as

$$GN(S') = \int_{s' \in S'} 1\{h' = 0, \tau' = 0, a' = 1, j' = \bar{j}(\epsilon')\} \pi_i(i'|i) f_b(b') f_\epsilon(\epsilon') ds'$$

The complete transition function, $G : \mathcal{S} \times \mathcal{B}(\mathcal{S}) \rightarrow [0, 1]$ is defined as

$$G_{\vec{p}, \alpha}(s, S') = \lambda(a) GS_{\vec{p}, \alpha}(s, S') + (1 - \lambda(a)) GN(S')$$

We can now define the operator generated by this transition function.

Definition. Given a price vector \vec{p} and a mixed allocation α , the operator $\Upsilon_{\vec{p}, \alpha} : \mathcal{M} \rightarrow \mathcal{M}$ is defined by the transition function G and gives the household distribution over the next period's states

$$\Upsilon_{\vec{p}, \alpha}(\mu_{\vec{p}, \alpha})(S') = \int_{s \in \mathcal{S}} G_{\vec{p}, \alpha}(s, S') d\mu_{\vec{p}, \alpha}$$

Theorem 5. [Existence of a unique invariant household distribution] For each $\vec{p} \in \mathcal{P}$ and $\alpha \in \Lambda$, $\exists! \mu_{\vec{p}, \alpha} \in \mathcal{M}$ s.t. $\Upsilon_{\vec{p}, \alpha}(\mu_{\vec{p}, \alpha}) = \mu_{\vec{p}, \alpha}$

Proof. We use Theorem 11.10 of Stokey and Lucas. First, we show that $G_{\vec{p},\alpha}$ satisfies Doeblin's condition. From exercise 11.4g of Stokey & Lucas, it is sufficient to show that GN satisfies Doeblin's condition. We must show that there exists a finite measure η on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, an integer $N \geq 1$ and a number $n > 0$ such that if $\eta(z) \leq n$ then $GN^N(s, S') \leq 1 - n \forall s \in \mathcal{S}$. Set $\eta(S') = GN(S')$. Then we can see that GN satisfies Doeblin's condition for $N = 1$ and $n < 1/2$. This guarantees the existence of an invariant distribution.

Observe also that if $\eta(S') > 0$, then $G_{\vec{p},\alpha}(s, S') \geq (1 - \lambda(a))GN(S') > 0$. This implies that the invariant distribution is unique. \square

We now have to construct a price transition function.

Lemma 6. *If $\{(s_n, \vec{p}_n)\}$ is a sequence in $S \times \mathcal{P}$ converging to (s_0, \vec{p}_0) then there exists a sequence $\{\alpha_n\}$ that converges to α_0 such that $G_{\vec{p}_n, \alpha_n}(s_n, \cdot)$ converges weakly to $G_{\vec{p}_0, \alpha_0}(s_0, \cdot)$.*

Proof. GN is independent of \vec{p} , so

$$\lim_{n \rightarrow \infty} GN(S') = GN(S') \forall S' \in \mathcal{B}(\mathcal{S})$$

As before, WLOG we can focus on sequences with $s_n^d = s^d$. Since Y is uhc in s and \vec{p} , $\exists(y_n(s_n, \vec{p}_n)) \rightarrow y(s_0, \vec{p}_0)$. Pick $\alpha(s_n, \vec{p}_n) = 1\{y_n(s_n, \vec{p}_n)\}$. Then,

$$\lim_{n \rightarrow \infty} GS_{\vec{p}_n, \alpha_n}(s_n, S') = GS_{\vec{p}_0, \alpha_0}(s_0, S') \forall S' \in \mathcal{B}(\mathcal{S})$$

\square

Lemma 7. *Take a sequence of $\{\vec{p}_n\} \in \vec{P} \rightarrow \vec{p}_0$. Then, $\exists\{\alpha_n\} \rightarrow \alpha_0$ such that*

$$\mu_{\vec{p}_n, \alpha_n} \rightarrow \mu_{\vec{p}_0, \alpha_0}$$

where $\mu_{\vec{p}_n, \alpha_n} = \Upsilon_{\vec{p}_n, \alpha_n} \mu_{\vec{p}_n, \alpha_n}$ and $\mu_{\vec{p}_0, \alpha_0} = \Upsilon_{\vec{p}_0, \alpha_0} \mu_{\vec{p}_0, \alpha_0}$.

Proof. Theorem 5 and Lemma 6 are sufficient to use Theorem 12.13 of Stokey and Lucas which gives us the result. \square

So far we have shown that a solution to the household problem exists, and that given any price vector, we can find a unique stationary distribution of households over the state space. Significantly,

we have shown that the set of transition functions and the set of household distributions as a function of price are uhc. The next step is to define the aggregate variables and show that they are bounded as well. In what follows, we use $\mu_{\vec{p}, \alpha}$ to represent the invariant household distribution given \vec{p} and α .

The upper bound on rental price, \bar{q} , is set subject to the following condition:

$$\bar{q}\bar{H} \min_j \Pi_j > \bar{b} + \bar{w}\bar{l}$$

This condition states that there is some rental price, \bar{q} , at which all the housing cannot be bought with the maximum wealth in hand. This ensures that there is a price vector where the optimal housing demand is less than the housing supply.

Using equation (1) we get

$$q(j) + E(p(j') - p(j)|j) = (\delta_h + t_p + \frac{r}{1+r})p(j)$$

Now consider $j = J$. On the highest island type the expected capital gain on housing is negative, so that

$$\begin{aligned} q(J) &\geq p(J)(\delta_h + t_p + \frac{r}{1+r}) \\ p(J) &\leq q(J)/(\delta_h + t_p + \frac{r}{1+r}) \\ p(J) &\leq \frac{q(J)}{\delta_h + t_p} \end{aligned}$$

So we set $\bar{p}(J) = \frac{\bar{q}}{\delta_h + t_p}$. But since $p(j) \leq p(J)$, we set the upper bound on the house price space $\bar{p} = \frac{\bar{q}}{\delta_h + t_p}$. So the upper bound on rent gives us the upper bound on the housing price as well.

Definition. The aggregate capital,

$$K(\vec{p}, \alpha) = \int_{\mathcal{S}} \int_{y^* \in Y(s, \vec{p})} (b^* + (q(j^*) - (1 + t_p + \delta_h)p(j^*))(1 - \tau^*)h^*) d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s)$$

$$L(\vec{p}, \alpha) = \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} l(a, i, \tilde{j}) d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s)$$

$$H(j; \vec{p}, \alpha) = \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} h^* 1\{\tilde{j}^* = j\} d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s)$$

Remark. $K(\vec{p}, \alpha), L(\vec{p}, \alpha), H(j; \vec{p}, \alpha)$ are continuous in α given \vec{p} .

Lemma 8. *There exist $\bar{K}, \bar{L}, \bar{H} < \infty$ and $\underline{L} > 0$ such that $K(\vec{p}, \alpha) \leq \bar{K}$, $\underline{L} \leq L(\vec{p}, \alpha) \leq \bar{L}$, $H(j; \vec{p}, \alpha) \leq \bar{H}$ for all (\vec{p}, α) .*

Proof. Since the state space is bounded,

$$\begin{aligned} K(\vec{p}, \alpha) &\leq \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} (\bar{b} + \bar{q}\bar{h}) d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s) \leq \bar{b} + \bar{q}\bar{h} \\ K(\vec{p}, \alpha) &\geq \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} \underline{b} - (1 + \delta_h + t_p)\bar{h} d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s) \leq \underline{b} - (1 + \delta_h + t_p)\bar{h} \\ L(\vec{p}, \alpha) &\leq \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} \bar{l} d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s) \\ &= \bar{l} \\ L(\vec{p}, \alpha) &\geq \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} \underline{l} d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s) \\ &= \underline{l} \\ H(j; \vec{p}, \alpha) &\leq \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} \bar{h} d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s) = \bar{h} \\ H(j; \vec{p}, \alpha) &\geq \int_{\mathcal{S}} \int_{Y^*(s, \vec{p})} \underline{h} 1\{j^* = j\} d\alpha(s, \vec{p}, y^*) d\mu_{\vec{p}, \alpha}(s) = \underline{h}\Pi_j \end{aligned}$$

where $\bar{l} = \max l(a, i, \tilde{j})$ and $\underline{l} = \min l(a, i, \tilde{j})$. □

We are now in a position to define the price transition function and examine its properties.

Definition. $\rho = \frac{1}{1+r}$ is the price of savings.

For the rest of the proof, we will be using the price of savings, ρ , instead of the interest rate, r .

Definition. We define the components of the price transition correspondence as follows:

$$\Omega^w : \vec{P} \times \Lambda \rightarrow \mathbb{R}, \text{ where } \Omega^w(\vec{p}, \alpha) = \begin{cases} F_L(K(\vec{p}, \alpha), L(\vec{p}, \alpha)) & \text{if } K(\vec{p}, \alpha) > 0 \\ F_L(0, L(\vec{p}, \alpha)) & \text{if } K(\vec{p}, \alpha) \leq 0 \end{cases}$$

$$\Omega^\rho : \vec{P} \times \Lambda \rightarrow \mathbb{R}, \text{ where } \Omega^\rho(\vec{p}, \alpha) = \begin{cases} (1 + F_K(K(\vec{p}, \alpha), L(\vec{p}, \alpha)) - \delta)^{-1} & \text{if } K(\vec{p}, \alpha) > 0 \\ 0 & \text{if } K(\vec{p}, \alpha) \leq 0 \end{cases}$$

It is most convenient to define $\Omega^h : \mathcal{P} \rightarrow \mathbb{R}$ recursively. Set $p_0 = \underline{p}$, $p'_0 = \underline{p}$, $p_{J+1} = \bar{p}$, $p'_{J+1} = \bar{p}$. Then, to get p' , we start from $j = 1$, and proceed to $j = J$.

$$p'_j \in [p'_{j-1}, p_{j+1}] = p_j + 1\{H_j > \bar{H}_j\}(p_{j+1} - p_j) \cdot \frac{H_j - \bar{H}_j}{H_j} - 1\{H_j \leq \bar{H}_j\}(p_j - p'_{j-1}) \cdot \frac{\bar{H}_j - H_j}{\bar{H}_j}; \quad j = \{1, J\}$$

where $\bar{H}_j = \bar{H}\Pi_j(j)$ and $H_j = \int h^* 1\{j^* = j\} d\mu^*$.

Definition. $\Omega^h : \mathcal{P} \times \Lambda \rightarrow \mathbb{R}^J$ is defined as $\Omega_j^h(\vec{p}, \alpha) = p'_j$

$\Omega^p : \mathcal{P} \times \Lambda \rightarrow \mathcal{P}$ is defined as $\Omega^p(\vec{p}, \alpha) = (\Omega^w(\vec{p}, \alpha), \Omega^\rho(\vec{p}, \alpha), \Omega^h(\vec{p}, \alpha))$

We are now in a position to define the price transition correspondence.

Definition. The *price transition correspondence*, $\Omega : \mathcal{P} \rightarrow \mathbb{R}^{J+2}$ is defined as

$$\Omega(\vec{p}) = \{\Omega^p(\vec{p}, \alpha) : \alpha \in \Lambda\}$$

Define $\tilde{\Omega}^x(\vec{p}) = \{\Omega^x(\vec{p}, \alpha) : \alpha \in \Lambda\}$ for $x \in \{w, \rho\}$ and $\tilde{\Omega}_j^h(\vec{p}) = \{\Omega_j^h(\vec{p}, \alpha) : \alpha \in \Lambda\} \forall j \in \mathcal{J}$.

These are corresponding components of $\Omega(\vec{p})$.

We now show that the price transition correspondence is a self-map. That is, $range(\Omega) \subseteq \mathcal{P}$.

Definition. The bounds on labor supply are set as follows: $\bar{L} = \bar{l} = \max(l(a, i, j))$ and $\underline{L} = \underline{l} = \min(l(a, i, j))$.

Since $b_0 \in [\underline{b}_0, \bar{b}_0]$ and $b \in [\underline{b}, \bar{b}]$, we can set bounds on capital supply.

$$K(\vec{p}, \alpha) \in [\underline{b}, \bar{b} + \bar{q}\bar{h}] \forall \alpha, \vec{p}, \text{ so that } \bar{K} = \bar{b} + \bar{q}\bar{h}; \quad \underline{K} = \underline{b} - (1 + \delta_h + t_p)\bar{h}$$

-Given bounds on capital and labor supply, we get that

$$\bar{w} = F_L(\bar{b}, \underline{L}) \text{ and } \underline{w} = F_L(0, \bar{L})$$

The bounds on ρ are set as follows:

$$\bar{\rho} = (1 + F_K(\bar{K}, \underline{L}) - \delta)^{-1}$$

so that $\rho \in [0, \bar{\rho}]$

Let $\tilde{p} = (\bar{p}, \dots, \bar{p})$ and $\tilde{H} = \max_{r, \alpha} H((\tilde{p}, r, \underline{w}), \alpha)$. Since the model is homothetic, doubling the house price vector will halve housing demand. So,

$$\exists \underline{p} > 0 : \tilde{H} \frac{\bar{p}}{\underline{p}} > \bar{h}$$

Lemma 9. *Under the conditions above the price transition function, Ω , is a self-map.*

Proof. First we look at Ω^w . Since $\underline{K} \leq K(\vec{p}, \alpha) \leq \bar{K}$ and $\underline{L} \leq L(\vec{p}, \alpha) \leq \bar{L}$,

$$\underline{w} = F_L(0, \bar{L}) \leq F_L(K(\vec{p}, \alpha), L(\vec{p}, \alpha)) \leq F_L(\bar{K}, \underline{L}) = \bar{w}$$

Hence $range(\Omega^w) \subseteq [\underline{w}, \bar{w}]$.

By construction, $\Omega^\rho(\vec{p}, \alpha) \geq 0 \forall \vec{p}, \alpha$. Further,

$$\Omega^\rho(\vec{p}, \alpha) \leq (1 + F_K(\bar{K}, \underline{L}) - \delta)^{-1} = \bar{\rho} \forall \vec{p}, \alpha$$

Therefore, $range(\Omega^\rho) \subseteq [0, \bar{\rho}]$.

By construction, $\Omega_{j-1}^h \leq \Omega_j^h \leq \Omega_{j+1}^h \forall j \in \{2, \dots, J-1\}$ and $\underline{p} \leq \Omega_1^h \leq \Omega_2^h$ and $\Omega_{j-1}^h \leq \Omega_j^h \leq \bar{p}$.

Hence $range(\Omega^h) \subseteq [\underline{p}, \bar{p}]$.

From the definition of Ω^p , we get that $range(\Omega^p) \subseteq \mathcal{P}$. This implies that $\Omega(\vec{p}) \subseteq \mathcal{P} \forall \vec{p} \in \mathcal{P}$. □

Before moving on to the final results, we state and prove a useful lemma that says that the continuous transformation of a uhc correspondence is also uhc.

Lemma 10. *Let Γ be a correspondence- $\Gamma : X \rightrightarrows Y$, $f : X \times Y \rightarrow Z$, f continuous in y , and $\Gamma' : X \rightrightarrows Z$, where $\Gamma'(x) = \{z : \exists y : y \in \Gamma(x) \text{ and } z = f(x, y)\}$. Then the following holds:*

1. If Γ is compact-valued, then Γ' is compact-valued also.
2. If Γ is uhc, then Γ' is uhc also.

Proof. Using Ok (2004), Prop. 3, Ch. D3, Pg 222, f takes compact sets to compact sets. Hence Γ' is compact-valued.

Pick $(x_m) \rightarrow x$ and $(z_m) \in \Gamma'(x_m) \forall m$. We want to show that there is a subsequence $(z_{m_k}) \rightarrow z \in \Gamma'(x)$. Since f is a function, for every $z_m \exists y_m : z_m = f(x_m, y_m)$, which implies that $y_m \in \Gamma(x_m) \forall m$. Since Γ is uhc, \exists a subsequence $(y_{m_k}) \rightarrow y \in \Gamma(x)$. From continuity of f , the subsequence $z_{m_k} = f(x_{m_k}, y_{m_k}) \rightarrow z = f(x, y)$. So $z \in \Gamma'(x)$, hence Γ' is uhc. \square

Definition. $K^p : \mathcal{P} \rightrightarrows \mathbb{R}$ is defined as $K^p(\vec{p}) = \{k : k = K(\vec{p}, \alpha) \alpha \in \Lambda\}$. H_j^p and L^p are defined analogously.

Lemma 11. $K^p, \{H_j^p\}_{j=1}^J, L^p$ are uhc and close-valued.

Proof. We show the result for $H_j(\vec{p})$. The proof for the remaining is analogous. From Theorem (4), Y changes continuously in \vec{p} . Now let $\lambda_p : \mathcal{P} \rightrightarrows \Delta$, where $\lambda_p(\vec{p}) = \Lambda_{\vec{p}}$.

First, we show that λ_p is uhc. Since λ_p is closed, it is sufficient to show that λ_p has a closed graph. Pick a sequence $\vec{p}_n \rightarrow \vec{p}$ and $\alpha_n \rightarrow \alpha$ with $\alpha_n \in \lambda_p(\vec{p}_n) \forall n$. We want to show that $\alpha \in \lambda_p(\vec{p}) \implies \text{supp}(\alpha) \subseteq Y(\vec{p})$. Suppose not. Then $\exists y \in \text{supp}(\alpha) : y \notin Y(\vec{p})$. Construct an open set $O : Y(\vec{p}) \subseteq O, y \notin O$. Since Y is uhc, $\exists \delta : Y(N_\delta(\vec{p})) \subseteq O$. Since $\vec{p}_n \rightarrow \vec{p}$, $\exists N : \forall n \geq N, y \notin Y(\vec{p}_n) \implies d^s(\alpha_n, \alpha) \geq \alpha(y) > 0$, where $d^s(\cdot, \cdot)$ refers to the sup norm. This contradicts the fact that $\alpha_n \rightarrow \alpha$. Hence λ_p is uhc.

$\lambda_p : \vec{P} \rightrightarrows \Delta$ and $H_j(\vec{p}, \alpha) : \mathcal{P} \times \Delta \rightarrow \mathbb{R}$. From the definition of H_j , H_j is continuous. Hence, using Lemma (10), $H_j^p(\vec{p}) = H_j(\vec{p}, \Lambda(\vec{p}))$ is uhc. Since λ_p is close-valued and $H_j(\vec{p}, \alpha)$ is continuous, H_j^p is close-valued. \square

Lemma 12. Ω is uhc, convex-valued and close-valued.

Proof. To show that Ω is convex-valued, we show that $\tilde{\Omega}^w, \tilde{\Omega}^\rho$, and $\tilde{\Omega}_j^h$ are convex-valued. We prove this for $\tilde{\Omega}_\alpha^w$; the proof for the rest is analogous.

$$\Omega^w(\vec{p}, \alpha) = \begin{cases} F_L(K^*(\vec{p}, \alpha), L^*(\vec{p}, \alpha)) & \text{if } K^*(\vec{p}, \alpha) > 0 \\ F_L(0, L^*(\vec{p}, \alpha)) & \text{if } K^*(\vec{p}, \alpha) \leq 0 \end{cases}$$

Now, K^* and L^* are continuous in α , and F_L is continuous in both its arguments. Therefore Ω^w is continuous in α . Since Λ is a convex set,

$$\Omega^w(\vec{p}) = \text{range}(\Omega^w(\alpha))$$

is a convex set. The same reasoning applies for the other components of $\Omega(\vec{p})$. Therefore, $\Omega(\vec{p})$ is convex-valued.

From the definition of $\Omega(\vec{p})$, it is a continuous transformation of Δ . Using Lemma (10), we get that $\Omega(\vec{p})$ is uhc. Since Δ is compact and Ω is a continuous transformation, the image is compact as well. \square

Theorem 13. *A competitive equilibrium exists.*

Proof. \mathcal{P} is a convex and compact space. Ω is convex-valued, and since it is uhc and compact-valued, it has a closed graph. Using Kakutani's FPT, $\exists \vec{p} \in \mathcal{P} : \vec{p} \in \Omega(\vec{p})$. This implies that $\exists \alpha : \vec{p} = \Omega^p(\vec{p})$. The aggregate capital and labor supply equations are satisfied by $K^*(\vec{p}, \alpha)$ and $L^*(\vec{p}, \alpha)$. The aggregate housing demand on island j is $H_j(\vec{p}, \alpha)$. Households are indifferent between islands with the same type when moving (Fact (A.1)). Let $\hat{\alpha}$ be such that

$$\int_0^1 1\{\bar{j}(\epsilon)\} = \bar{j}(\epsilon') \} GS_{\vec{p}, \hat{\alpha}}(s, S') d\epsilon = \int_0^1 1\{\bar{j}(\epsilon)\} = \bar{j}(\epsilon') \} GS_{\vec{p}, \alpha}(s, S') d\epsilon$$

and

$$\int_0^1 1\{\bar{j}(\epsilon)\} = \bar{j}(\epsilon') \} GN_{\vec{p}, \hat{\alpha}}(S') d\epsilon = \int_0^1 1\{\bar{j}(\epsilon)\} = \bar{j}(\epsilon') \} GN_{\vec{p}, \alpha}(s, S') d\epsilon$$

for all $s \in S$ and $S' \in B(S)$. Then, if $\alpha : \vec{p} = \Omega^p(\vec{p})$, then $\hat{\alpha} : \vec{p} = \Omega_{\alpha'}(\vec{p})$. So $\exists \alpha : \vec{p} = \Omega_{\alpha}(\vec{p})$ and

$$\bar{H} = H(\epsilon) = \int_{s \in \mathcal{S}} \int_{y^* \in Y^*(s, \vec{p}^*)} h^* \cdot 1\{\epsilon^* = \epsilon\} d\alpha'(y^*) d\mu^*(s) \quad \forall \epsilon \in [0, 1]$$

By Walras' Law, the goods market clears. Hence all the conditions for a competitive equilibrium are satisfied. \square

B Computation

B.1 Computing the value function

B.1.1 Setting up the state and control grids

Rather than include consumption as a control variable and the budget constraint as a test of feasibility, we save on computation cost by excluding consumption from the control space and computing consumption as the value that causes the budget constraint to bind. The two variables of interest are then h and b . The remaining variables are discrete, while these two variables have to be discretized. We take b_{max} to be the maximum net wealth that the household can hold. Given \vec{p} , h_{max} is set at that value such that $h_{max}p_J = b_{max}$. We arbitrarily set the number of points on the housing grid to 121 with the housing grid points evenly spaced.

The wealth grid is set conditional on the housing grid and the discrete choices such that the net wealth of the household does not exceed b_{max} . The lower bound on the wealth grid is 0 for renters and given by the borrowing constraint for owners. For the mover's problem, the grid for cash-in-hand, b^m , has its lower bound set to $-\bar{l}$, since a household has to have positive cash-in-hand in order to move. The upper bound is set to b_{max} .

The number of island types is finite, and so is the set of idiosyncratic household types since households are finitely lived. The fixed effect is drawn from the distribution over the fixed effects as given in the calibration section. Finally, the initial wealth of newborn households is drawn from the distribution of b_0 . Section (B.1.2) explains why we do not have to pick the initial wealth from a discrete grid. Rather, we treat it as a continuous variable.

All the grid sizes and densities are tested to ensure that results do not change significantly when grids are made finer.

B.1.2 Using the Golden Section Search routine

The standard solution involves discretizing all continuous variables. We improve the efficiency and the accuracy of the value function computation by considering savings to be a continuous variable. The innovation involves using the Golden Section search to optimize over the savings space given all other states and choices. As the grid size of savings increases, using golden section search is more efficient than a brute force search over all grid points. More importantly, treating savings

as a continuous choice variable means that we can get more accurate results for the optimization procedure.

For the rest of the section on computation, when we refer to *discrete states* (\tilde{s}^d) or *discrete choices* (\tilde{y}^d), we mean all state variables or choice variables excluding savings.

B.1.3 The value of moving and staying

As outlined in the standard method, we use backward iteration to compute the value function. At every state s , the value of staying is calculated using the golden section search routine. Similarly we also compute the value of moving, V^m , at every state, s^m and choice y^m .

Next we compute the value at every (s, \tilde{y}^d) pair. The value of staying comes directly from the calculations above. When calculating the value of moving, the first issue we encounter is that the cash-in-hand of the household need not be on the b^m -grid. We use linear interpolation between states here, which is standard in the literature (see, for example, Li and Yao 2007). The second issue concerns the optimal discrete choice, \tilde{y}^d . The standard method does the following: For the two closest points on the b^m grid, calculate the optimal choice and the value of moving. Then interpolate to get the value at s . We found that for a computationally feasible grid size this technique was too inaccurate. That is, increasing the grid size gave significantly different results for the discrete choice.

Instead, we use the following technique: The value of every discrete choice at b^m is calculated as the linearly interpolated value of discrete choices at the two closest points on the b^m -grid. The optimal discrete choice is then calculated as the argmax of this matrix. We then compute the optimal saving using the actual cash-in-hand, b^m , and the optimal discrete choice. Testing with finer grids shows that this technique gives more accurate and robust results than the standard method.

We note that a useful feature of this technique is that households can be followed over the life cycle. Using the standard technique, when the household has wealth outside the grid, using interpolation to calculate the expected value means that there might be no corresponding optimal choice. This is particularly a concern when the two closest values of b^m imply different optimal discrete choices. Our technique provides an more accurate alternative where we can follow households over the life cycle.

If a state point is infeasible, that is, there are no feasible choices at that state, a value of $-\infty$

is assigned, and the optimal control defaults to the same wealth level as the state. The choice of optimal control does not affect the computation since the expected value at any prior date that includes this state will be $-\infty$ as well.

B.1.4 Simulating the economy

Given the (expected) value function, we simulate the economy by forward iteration. We follow 200,000 households over their life cycle. The initial state (the wealth endowment, b_0 and the fixed effect) as well as the shocks through the households' lifetime are drawn from the distribution as specified in the calibration section. Since we are interested in the economy at a steady state, we observe the following: the distribution of households over states at a particular age a is approximated by the distribution of the 200,000 simulated households at age a . This allows us to get a snapshot of the economy which is composed of different cohorts, at any point in time by just using this set of simulated households. Since the economy is in a steady state, this snapshot does not change over time. We compute the aggregate macroeconomic variables using this set of simulated households.

B.1.5 The equilibrium price vector

The final step is determining the prices in the general equilibrium. Once we have determined the steady state distribution of agents, we can calculate the demand and supply in the housing, labor and capital markets. We iterate to find the equilibrium prices using a simplex search algorithm. This has the advantage of being robust even when the size of the parameter vector is large. We use the relative Euclidean distance between the demand and supply as the criterion function with the identity matrix as the weighting matrix.

C Calibration

C.1 Sample Selection Criteria

For the PSID, we remove the SEO sample ($id > 3000$) and drop those households with top-coded labor income of head or wife or total household income is less than \$500 in any year. For the CPS, we drop households with top-coded data on income from alimony and other sources or if no one in the family is working or if no one in the family is over 60 years old or if total income is less than

\$500. For the SCF, we drop households with financial assets outliers (less than \$0 or greater than \$10 million).

C.2 Family size equivalence

We collect data from the period 1970-1993 in the CPS. An issue with the CPS is that from 1988 onwards age is top-coded at 90. On the other hand, the sample size in the PSID is smaller than the CPS, particularly for households of age above 85. We use the CPS since it has a much larger sample size, and we use the household weights provided by the CPS. We control for year effects by using year dummies. The family size profile is generated by the following regression:

$$F_{iat} = \sum_{k=21}^{81} \beta_k 1_k + \sum_{t'=1970}^{1993} \beta_{t'} 1_{t'} + \epsilon_{iat}$$

where 1_k is a year dummy which takes on value 1 when $a = k$, and $1_{t'}$ is the year dummy that takes on value 1 when $t' = t$.

Figure C.2 shows that the profiles of family size from the CPS. Family size increases sharply when the household is young, peaking at age 38 for both data sets. In the model, we set $F(a) = F(81)$ when $a > 81$, where a is the age of the household.

In order to adjust the household's housing and consumption stream we use a household equivalence scale. The objective of an equivalence scale is to measure the change in consumption needed to keep the welfare of the family constant as the family size varies. Note that using per capita consumption assumes that the family converts consumption expenditure into utility flow following constant returns to scale. Lazear and Michael (1980) point to the existence of family goods, economies of scale and complementarities, which are all factors that they show to be significant. We therefore use a household equivalence scale that is not constant returns to scale. Table 12 lists some equivalence scales. L-M stands for Lazear and Michael (1980), US Dept of Commerce refers to US Department of Commerce (1991) and F-V&K stands for Fernandez-Villaverde and Krueger (2007). Lazear and Michael's scale takes greater account of common or public goods, so that the impact of family size is less than other equivalence scales (compare, for instance, Orshansky (1965)). We use the housing equivalence scale used by Fernandez-Villaverde and Krueger (2007).

All households in the model economy have the same life-cycle profile of family size, which is set to the average family size at each age in the CPS. To account for non-integer family sizes, we

| Family Size | L-M | Orshansky (1965) | US Dept of Commerce | F-V&K |
|-------------|-----|------------------|---------------------|-------|
| 1 | 100 | 100 | 100 | 100 |
| 2 | 106 | 126 | 128 | 134 |
| 3 | 128 | 151 | 157 | 165 |
| 4 | 147 | 189 | 201 | 197 |
| 5 | 169 | 223 | 237 | 227 |

Table 12: Family size equivalence scale

assume that the adjustment factor is linear within the family sizes specified in Table (12). Figure C.2 shows the equivalent family size over the life cycle. The adjusted family size has much less

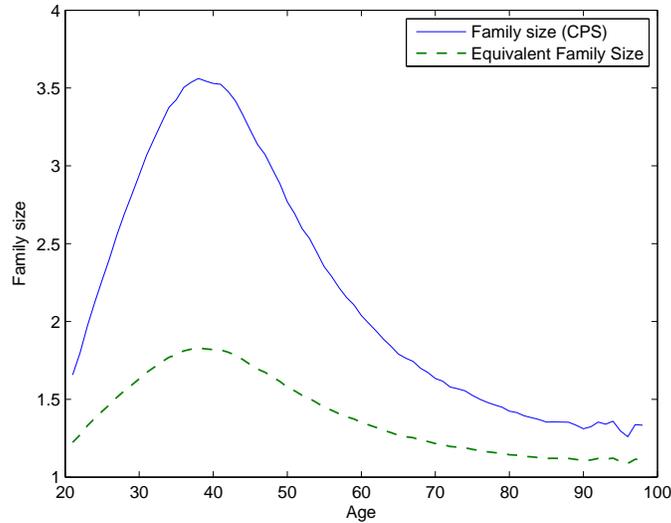


Figure 24: L-M adjusted family size profile (yearly bins)

variation than the actual family size from the CPS.

C.3 Productivity process

For purposes of calibration, we assume that the household does not voluntarily move to a new U.S. state in the data. We then rerun the GMM estimation described below on simulated model data to check this assumption.

C.3.1 Base wage over the life-cycle

The estimation of the productivity process is split into two steps. The first step involves estimating household earnings conditional on age, around which the household gets shocks. The following

regression on log income gets us the base wage, $\exp(d_k)$:

$$y_{ijat} = \sum_{k=1}^A 1_k d_k + \sum_{t'=1970}^{1993} 1_{t'} d_{t'} + w_{ijat}$$

where the subscripts are as follows: i indexes the individual, j indexes the state in which the individual lives, and a is the age of the head of the household. 1_k is the dummy variable that takes value 1 when $k = a$ and 0 otherwise, and d_k is the coefficient on the age-dummy variables. 1_t is the year dummy, and w_{ijat} is a mean-zero residual. The coefficients d_k are the (log) base wage conditional on age.

The base wage is estimated using the CPS. We use a quartic approximation of the age dummies for ages 21 to 80 and then assume that the average log wage declines linearly at the rate of the difference between age 80 and 81 thereafter. Figure (25) shows the evolution of the base wage over the life cycle.

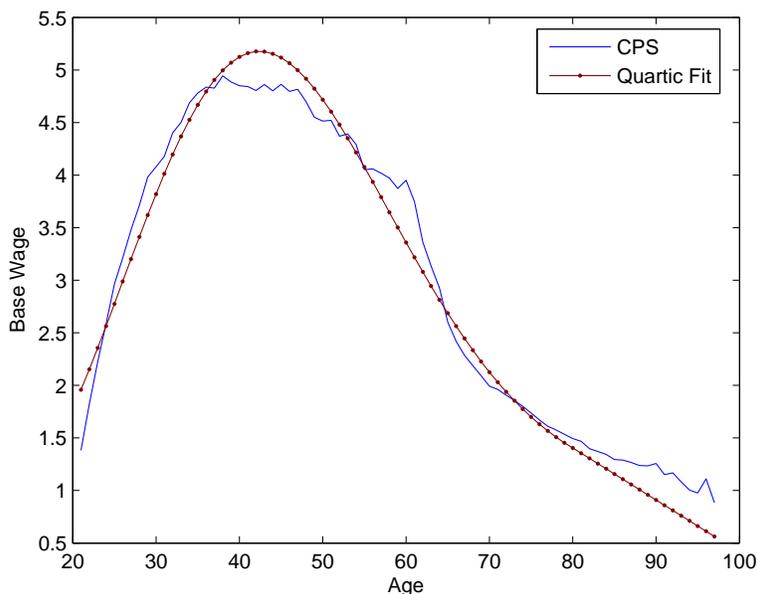


Figure 25: Base wage (quartic-linear polynomial approximation and age dummies)

C.3.2 The earnings residual

The second step involves estimating the idiosyncratic and regional productivity shocks. The shocks are calibrated to match the variance profile of earnings over the life cycle. Since a component of the productivity shocks is persistent we use the PSID, which is a panel data set. Our sample is drawn

from the PSID waves 1970-1993. We drop observations that are top-coded on earnings variables and people who report total household earnings less than \$500. We also drop students from the sample since we are interested in working households.

Our object of analysis is the residual of log earnings after controlling for age, year and family size. Since family size is also a function of age and year we first generate the family size residual by regressing family size on age and year dummies.

$$F_{ijat} = \sum_{k=1}^A 1_k d_k + \sum_{t'=1970}^{1993} 1_{t'} d_{t'} + \phi_{ijat}$$

where ϕ_{ijat} is the family size residual.

The log earnings residual, w_{ijat} , is defined by

$$y_{ijat} = \sum_{k=1}^A 1_k d_k + \sum_{t'=1970}^{1993} 1_{t'} d_{t'} + \beta^F \phi_{ijat} + w_{ijat}$$

and is the residual from the regression of log income on age dummies, year dummies and the family size residual. Henceforth we will refer to w_{ijat} as log earnings residual and log earnings interchangeably.

C.3.3 Parametric model for earnings

We model log earnings as

$$\begin{aligned} w_{ijat} &= \sigma_f f_i + \sigma_\nu \nu_{iat} + \iota_{iat} + \epsilon_{jt} \\ \iota_{it} &= \rho_\iota \iota_{i,a-1,t-1} + \sigma_\iota e_{it} \\ \epsilon_{jt} &= \rho_\epsilon \epsilon_{j,t-1} + \sigma_\epsilon e_{jt} \end{aligned}$$

where i indexes the individual household, j indexes the state where the household resides, a indexes the age of the household, and t indexes time. $\sigma_f f_i$ is the fixed effect, where $f_i \sim \mathcal{N}(0, 1)$ and σ_f is the standard deviation of the fixed effect shock. $\sigma_\nu \nu_{iat}$ is the temporary shock, where $\nu_{iat} \sim \mathcal{N}(0, 1)$ and σ_ν is the standard deviation of the temporary shock. ι_{it} is the persistent idiosyncratic shock and ϵ_{jt} is the persistent regional shock. $e_{it}, e_{jt} \sim \mathcal{N}(0, 1)$.

We assume that the initial values for the persistent shocks are set as follows: $\iota_{i0t} = 0 \forall t$ and ϵ_{j0} is drawn from its ergodic distribution. Storesletten et al. (2004) note that ρ_ι is very close to 1. We set $\rho_\iota = 1$ in our model, so that the persistent idiosyncratic shock follows a random walk. Since we have finitely lived households, we do not face the standard problems associated with non-stationary time series. Our parametric model is well-defined for values of $\rho_\iota > 1$. Secondly, assuming a random walk process for the persistent idiosyncratic shock has the advantage of allowing us to reduce the dimension of the state space of the household's problem. This reduces the state space and makes the computation of the model less time-consuming.

C.3.4 GMM estimation

We estimate the parameters using an overidentified set of moments with the identity matrix as the weighting matrix. The set of moments is listed below:

$$\begin{aligned} m1(k) &\equiv E(w_{ijat}w_{i'ja',t-k}) &= \rho_\epsilon^k \sigma_\epsilon^2 / (1 - \rho_\epsilon^2) \\ m2 &\equiv E(w_{ijat}^2) &= \sigma_f^2 + \sigma_\nu^2 + (a-1)\sigma_\iota^2 + \sigma_\epsilon^2 / (1 - \rho_\epsilon^2) \\ m3(n) &\equiv E(w_{ijat}w_{ija-n,t-n}) &= \sigma_f^2 + (a-1-n)\sigma_\iota^2 + \rho_\epsilon^n \sigma_\epsilon^2 / (1 - \rho_\epsilon^2) \end{aligned}$$

We use $k = \{0, 1, \dots, 15\}$ and $n = \{0, 1, \dots, 9\}$. We use households of ages 26 – 62, where we have the most data, to estimate the above parameters.

The transitory shock term, $\sigma_\nu \nu_{ijat}$, incorporates both the actual transitory shock and the iid measurement error in the PSID data. We make the assumption that 1/2 of our estimate of σ_ν is due to measurement error.

C.3.5 Calibration of β, σ, θ_m

To calibrate β, σ, θ_m , we compute at each point of a coarse three-dimensional grid, the value of each model moment listed in section 4.11.1. As a criterion function, we use the sum of difference in logs of each data moment and the corresponding model moment, and use an identity weighting function for each moment. We then choose the point of the grid which minimizes the criterion function.