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Coordination, Efficiency and Policy Discretion

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Abstract

Would citizens coordinate to punish a government when they observe suspicious behavior? This paper shows that under some circumstances such coordination is impossible. This fact has important implications for policy discretion. We study an environment with the following characteristics: 1) the aggregate productivity (fundamental) is stochastic, 2) only the government observes it and, 3) every agent privately receives a noisy signal about the fundamental. Item 1) implies that the best policy (tax on investment) with commitment is state contingent, while 2) and 3) make information incomplete. The main consequence of incomplete information is to make coordination among small anonymous agents harder. Since coordination is key to punishing the government when it deviates, independently of the accuracy of the signal, the set of sustainable payoffs is drastically reduced. Regardless of the size of the noise, state contingent policies cannot be an equilibrium. Moreover, the best equilibrium policy is independent of the fundamental, i.e., the optimal policy calls for strong rules rather than discretion. Finally, we show that the payoff of the best equilibrium without commitment is uniformly bounded away from the payoff of the equilibrium with commitment.

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1 Introduction

Since the seminal work of Kydland and Prescott (1977), a great deal of research has been devoted to the study of time inconsistency problems. These types of problems arise in a wide variety of policy settings such as capital taxation, monetary policy and default decisions. Their original proposed solution was to tie the hands of policymakers by forcing them to use rules (set ex ante with commitment) rather than allowing discretion. The drawback of this approach is that it undermines the ability of policymakers to react optimally to changes in the fundamentals.

Following up on this idea, and building on Barro and Gordon's (1983) intuition, Chari and Kehoe (1990,1993) (henceforth C&K) showed that if policymakers are sufficiently patient, the optimal policy with commitment can be implemented as an equilibrium when the government is endowed with full discretion. Their proposed solution relies on reputational arguments borrowed from game theory. All (atomistic) individuals would coordinate to punish the government (e.g. by never investing again) if they observed a deviation from a prescribed policy. Therefore, if its discount factor is high enough, the government will never deviate.

Due to its simplicity, tractability and appealing intuition, this modeling strategy has become a standard tool to analyze environments without commitment in macroeconomics. However, this mechanism crucially relies on the complete information assumption. To enforce the equilibrium, not only must all agents know when (if) the government deviates but also that all agents know that everyone knows, that everyone knows and so on. Therefore, coordination is not only possible but also perfect.

Some natural questions arise: would these equilibria survive in the presence of disperse information? Is it possible to approach efficiency as the dispersion of information diminishes? To answer these questions, we study an environment with a continuum of investors who receive an endowment that they can either invest or consume. After the agents have invested, the government observes the aggregate investment and can tax it. The revenue from taxation is used to provide a public good. The fact that the tax is set after the investment decisions have been made generates a time inconsistency problem.

The environment is similar to C&K and many others, but departs from it in sensible way: the informational structure. We assume that the average productivity of investment, the fundamental, is stochastic and observed by the government but not by investors. Instead, every investor privately receives a different signal: the idiosyncratic return on her own investment. Because of the stochastic nature of the fundamental, the optimal tax policy, with commitment, would be contingent on the realization of the fundamental. If investors could observe the the distribution of signals, the modeling strategy proposed by C&K would still implement the optimal policy. Without complete information, the usual approach taken to achieve efficiency fails. Coordination for punishments off the equilibrium path becomes impossible.

The intuition is similar to Bagwell (1995). Suppose there is an equilibrium where the government's strategy depends on the fundamental. For instance, assume that the tax is high after a high realization of the fundamental and low after a low realization. What does an investor who observes a low signal and high tax think? If the government's strategy were an equilibrium, it should be playing a high tax only after observing a high fundamental. Thus, the investor has to necessarily believe that the government has observed a high fundamental and that the distribution of signals is in a high state. Any other belief would be inconsistent with equilibrium behavior. In other words, the investor disregards her own signal, believing that it is a bad realization and fully trusting the government. The same is true for all investors and for any information set. This provides the government with the possibility of deviating without affecting investors' reactions: it deviates by choosing a tax that happens with positive probability on the equilibrium path, and that maximizes its static payoff (Proposition 1).[1]

As a result, any equilibrium strategy for the government must depend in a non-trivial way on public information (the tax history) while the optimal strategy depends only on the fundamental. This creates an inefficiency that may or may not disappear as the size of the noise approaches zero. In order to shed light on this issue, we characterize the best equilibrium without commitment. We show that the optimal tax policy is a constant tax (Proposition 3). That is, the optimal policy is to ban all discretionary policies, which is true for any degree of uncertainty among agents. Therefore, the inefficiency persists in the limit.

An alternative interpretation of the model is to think about investors as citizens who have different and changing beliefs about what the "right" policy is. Hence, if there were an equilibrium in which the government does not succumb to temptation, when there is a change of policy, every agent must believe that the new policy satisfies the majority of people in the economy. This belief among agents generates an opportunity for the government to choose a policy that maximizes its immediate payoff but not the average welfare of society. By choosing an appropriate deviation policy, the government confuses the agents into believing

[1]On any equilibrium path there is always a strictly positive probability of receiving a signal that is distant (including those extremely distant) from the fundamental. The same is true in terms of histories, even if the government never deviates some agents will experience histories that are highly inconsistent with history observed by the government.

that the new policy is optimal for most agents. Even if every agent thinks that the new policy is not optimal from her perspective, everyone believes that most people believe that it is actually the best choice. As a result, coordinating to punish the government appears to be a fruitless endeavor. The best a society can do to deal with this problem is to take away government discretion and impose strong rules, so that when a deviation occurs it is clear to everyone what is happening.

It is important to bear in mind that asymmetry of information between the investors and the government does not necessarily preclude efficiency. If all investors received the same signal[2], there would be room for some coordination: if the investors observe a signal that is inconsistent with the strategy that the government is playing, they know that all other investors have observed the same signal and they could, as in Green & Porter (1984), coordinate to punish the government for a potential deviation. Although this equilibrium specification generates "mistaken punishments" on the equilibrium path, these punishments would become more infrequent as the precision of investors' signals improve and efficiency could be achieved in the limit.

This paper is related to the macroeconomic literature on sustainable plans and the literature on discretionality of optimal policy. Regarding the former, there is an extensive literature following the approach of Barro & Gordon (1983) and C&K and generalized by Phelan & Stacchetti (2001). For the sake of brevity we omit many interesting papers from this area. With respect to the optimality of policy discretion, there are at least three papers that find a characterization of the best policy without commitment similar to what we find in this paper: Sleet (2001), Athey et al. (2005), and Sleet and Yeltekin (2006). The first considers the problem of a monetary authority that receives a private signal about the true state of the economy in an environment where both households and firms have the same information set. The paper shows that under some conditions the optimal policy with commitment is an equilibrium, while in other cases the monetary authority chooses not to use the private signal. Athey et al. (2005), again in an optimal monetary policy context, considers an environment where all agents have the same information set and only the policymaker observes the (random) true state of the economy. They find that if the time inconsistency problem is "severe" the optimal policy is independent of the true state of the economy; otherwise some dependency is allowed. Sleet and Yeltekin (2006) also analyze an economy in which private agents have the same information set, but the government privately observes a taste shock

[2] Or have common beliefs about what the optimal policy should be.

related to the public good consumption.[3]

There are two main differences between our paper and the aforementioned literature. First, we consider heterogenous information sets among agents and analyze the limiting behavior as the uncertainty vanishes. Second, the time inconsistent preferences of the social planner are endogenous. This allows us to provide a more precise and concrete answer on the characterization of optimal policies without commitment.

Finally, some mechanisms generating both the coordination failure and the inefficiency can be found in recent developments in the industrial organization literature on collusion. As previously mentioned, the intuition behind the impossibility of coordination dates back to Bagwell (1995), which is related to the impossibility of pure strategy equilibria in games with private imperfect monitoring.[45] The most closely related paper is Athey et al. (2004). They study an infinitely repeated Bertrand game where prices are public information but firms receive private idiosyncratic shocks to their unit cost of production. They show that if firms are patient enough, the best Perfect Public Equilibrium (PPE) implies full price rigidity with no price wars on the equilibrium path.[6] From this point of view our results are qualitatively similar, but as opposed to Athey et al. (2004) we look at the best Perfect Bayesian Equilibrium rather than at the best PPE. In addition, we focus on the coordination of small anonymous agents rather than coordination of strategic players. From this point of view our paper uses some new results from the industrial organization literature to bring new insights to macroeconomics.

The paper proceeds as follows: Section 2 describes the environment. Section 3 deals with the repeated game. Section 4 defines and characterizes the best equilibrium without commitment and shows the inefficiency result. The last section concludes.

[3]See in addition Amador et al. (2006). All these works consider i.i.d shocks as this paper. Recently Halac & Yared (2012) study a model with a government that has exogenous time inconsistent preferences, through hyperbolic discounting, which creates a temptation to over-spend in each period and a persistent aggregate shock. They find that the degree of flexibility changes with the state of the economy.

[4]See Compte (2002) and Sannikov & Skrzypacz (2007) for more recent studies of coordination failures. However, private imperfect monitoring does not preclude efficiency. See Sekiguchi (1997) and Piccione (2002) for examples where efficiency is asymptotically approached.

[5]Our environment does not exactly fit into the definition of private monitoring because the action of the government is public information. If the tax rate weren't commonly known, for instance, because taxation is noisy, direct applications of the results for imperfect private monitoring games would hold (e.g. Mailath & Samuelson (2006) and Phelan & Skrzypacz (2012)) and the only equilibrium would be a 100% tax every period and zero investment. Making the tax rate publicly known allows for some coordination among agents and therefore provides for a wider range of equilibrium payoffs.

[6]This results contrast with the seminal result of Green & Porter (1984). The expected loses due to price wars is the key treat that makes collusion possible.

2 The Economy

2.1 The Static Game: Actions and Payoffs

The economy is populated by two types of players, a government or policymaker and a continuum of investors indexed by $i \in I = [0, 1]$. There are two goods: a private good that can be used for consumption or investment and a public good. Both the individual and aggregate returns on investments are stochastic. Let R, the fundamental, denote the aggregate return on investment and y^i the individual returns.

At the beginning of the period an outcome R from a finite set Υ is realized with distribution function $P(R)$. Then, conditional on the realization of R, each individual privately observes a draw $y^i \in [1, \bar{y}]$ of a random variable Y with distribution function $F(y|R)$ and density $f(y|R)$.

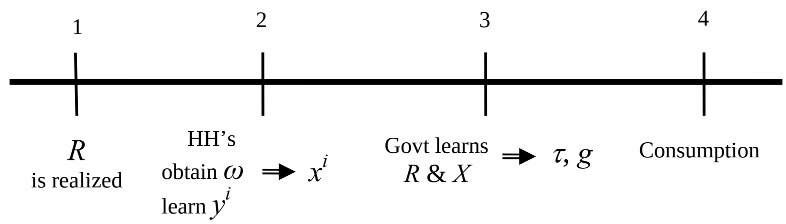
Every individual receives an endowment of ω units of the private good simultaneously with the realization of y^i. Given the endowment and y^i, each investor chooses investment level $x^i \in [0, 1]$. The non-invested portion of the endowment $\omega - x^i$ is automatically consumed by the investors. Given individual investment decisions, aggregate output and investment are $Y(R) = \int_0^1 \int_1^{\overline{y}} y^i x^i(y^i) f(y^i|R) dy^i di$ and $X(R) = \int_0^1 \int_1^{\overline{y}} x^i(y^i) f(y^i|R) dy^i di$, respectively.

After the individual decisions have been made, the government chooses a tax on investment $\tau \in [0, 1]$. The revenue from taxation are used to produce a public good. Because the government's budget constraint must be satisfied, the total investment in production of the public good is given by $g = \tau Y(R)$. However, households do not all receive the same amount of the public good. Instead, every investor receives a share g^i of the total provision of the public good. Thus, if the aggregate state is R and the government sets a tax on investment τ the distribution function of individual shares of public good is given by

$$g^i \sim \xi(g^i|g); \quad \forall g^i \in [\underline{g}, \bar{g}] \tag{1}$$

Each share g^i is uncorrelated through time and uncorrelated with y^i. Given a tax rate τ, at the end of the period each household consumes $c^i = (1 - \tau) y^i x^i + (\omega - x^i)$ of the private good and g^i units of the public good. Figure 1 summarizes the sequence of events and the information sets.

Figure 1: Timing of stage game



Preferences are separable between the private and the public good. If investor i has draw y^i, invests x^i, receives a share g^i of the public good and the tax on investment is τ, her payoff is given by:

$$u(y^i, \tau, x^i, g^i) = (1-\tau)y^i x^i + (\omega - x^i) + v(g^i) \tag{2}$$

where $v : \mathbb{R}_+ \mapsto \mathbb{R}$ is twice continuously differentiable and strictly concave function. Given the realization of R, the profile of individual investment functions and tax τ, the government's payoff is:

$$W(\tau; x^i_{i\in I;R}) = \int_0^1 \left[\int_1^{\overline{y}} [(1-\tau)y^i x^i + (\omega - x^i)]dF(y^i|R) + \int_{\underline{g}}^{\overline{g}} v(g^i) d\xi(g^i|g) \right] di$$

The government is benevolent in the sense that its payoff is the average utility in the economy. Note that the payoff for the government depends only on the aggregate values for investment and output. Since investors are ex ante identical, the decision function is the same for all of them. Hence, the last equation can be rewritten as:

$$W(\tau, X, Y; R) = \int_1^{\overline{y}} [(1-\tau)y^i x^i + (\omega - x^i)]dF(y^i|R) + \int_{\underline{g}}^{\overline{g}} v(g^i) d\xi(g^i|g) \tag{3}$$

In order to simplify the analysis, we assume that the marginal value of the public good is higher than the marginal value of the private consumption in any possible state. With some abuse of notation let $v(g) = \int_1^{\overline{g}} v(g^i(g))d\xi(g^i|g)$, then:

Assumption 1 (Time inconsistency) $v'(g) > 1$ *for all* g.

Consider the situation after which all investors have chosen x^i, and therefore both R and y^i have been realized. Using the government budget constraint, the payoff for the government is:

$$W(\tau, X, Y; R) = (1 - \tau)Y + (\omega - X) + v(\tau Y)$$

If the government increases taxes slightly, say by $d\tau$, the benefit of increasing the public good is $Y(R)v'(\tau Y(R))d\tau$, while the loss of private consumption is $Y(R)d\tau$. Combining these two effects, the government has incentives to increase the current tax as long as $Y(R)[v'(\tau Y(R)) - 1] > 0$, which is guaranteed by Assumption 1.

This assumption creates a time inconsistency problem that is present in every state of nature. It implies that in every state, and for any profile of individual strategies, the static optimal action of the government is a 100% tax. Foreseeing this, investors do not invest and therefore the unique equilibrium of the static (or the finite horizon repeated) game is a 100% tax and zero investment. [7]

Assumption 1 captures the type of incentives that appear in standard games in the macroeconomics literature. For instance, one may think of the classical capital taxation problem where from an static perspective taxing capital has zero efficiency cost and therefore is optimal to tax it fully. Similar incentives appear in monetary policy environments when the monetary authority always has incentives to inflate the economy to reduce unemployment or when a government wants to default on its debt to avoid the distortionary costs of repayment.

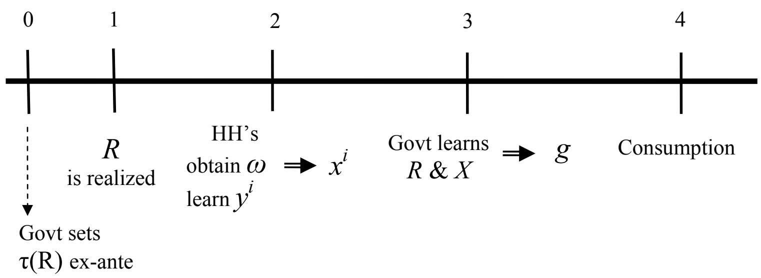
2.2 Optimal Policy With Commitment (Ramsey)

Before proceeding to the repeated game, we first consider the benchmark case in which the government has access to a commitment technology used to bind itself to a tax policy $\tau_R(R)$. When such technology is available, the static nature of the government's problem allows us to restrict the analysis to a one-period game. Following the literature, we call it the Ramsey game.

The introduction of a commitment technology can be formalized changing the timing of the static game, by forcing the policy maker to choose a state contingent tax policy before any information is revealed. Figure 2 illustrates the alternative timing.

[7]Note that assumption 1 implies that in the Pareto optimal solution, all agents invest their entire endowment in every state and transfer their resources to the government to produce the public good. Since agents are anonymous, this cannot be an equilibrium. If that were the case, any particular (measure zero) agent would have an incentive to keep the endowment for herself, consume it at the end of the period and enjoy the same amount of the public good.

Figure 2: Timing of Ramsey Game



There are two differences compared to the stage game in Section 2. First, the government sets a tax policy $\tau_R(R)$ before observing the fundamental R. Second, when the aggregate output is realized, the government learns the realization of R and sets the investment tax according to $\tau_R(R)$. This sequence of events prevents any discussion about communication issues.

Given a a tax policy τ_R, let $E_{\tau_R}(\tau|y^i)$ be the conditional expectation that a household with draw y^i has about the random variable τ_R .

Definition 1 *The Ramsey equilibrium is a collection of functions* $\tau_R : \Upsilon \rightarrow [0,1]$ *and* $x_i : [1, \overline{y}] \rightarrow [0, \omega]$*, one for each* $i \in [0,1]$*, such that:*

a) $$x^i_R(y) = \begin{cases} \omega & \textit{if } y \cdot E_{\tau_R}(1-\tau|y^i) > 1 \\ [0,\omega] & \textit{if } y \cdot E_{\tau_R}(1-\tau|y^i) = 1 \\ 0 & \textit{otherwise} \end{cases}$$

b) τ_R *maximizes* $\int_\Upsilon W(\tau, X, Y; R) dP(R)$ *given* $x^i_R(y)|_{i\in I}$.

It is difficult to characterize the Ramsey policy in a precise way with a general function $v(.)$. All that we can say is that $g(R) < g(R')$ if $R < R'$, that is, that government spending is higher in more productive states.[8] Since aggregate output, $Y(R)$, is increasing in R as well, the magnitude of the taxes is unclear. The properties and relative size of taxes could be very different depending on the shape of $v(.)$. For that reason, in the next lemma we present the set of conditions that the optimal policy must satisfy and then we fully characterize the solution when $v(.)$ is linear. Let $v(g) \equiv b \cdot g$. Because of Assumption 1, $b > 1$. Under these conditions the characterization is intuitive and straightforward. Moreover, it highlights the main complications related to the individual investment decisions.

[8]Here the difficulty is similar to that in the Mirrleesian literature, where in general is not known whether or not workers with higher productivity work more than low productivity workers.

Assumption 2 (Monotone Likelihood Ratio) *For all* $R_H, R_L \in \Upsilon$ *and for all* $\hat{y}, y \in [1, \bar{y}]$ *we have that* $\frac{f(\hat{y}|R_L)}{f(y|R_L)} \leq \frac{f(\hat{y}|R_H)}{f(y|R_H)}$ *if* $\hat{y} \geq y$ *and* $R_H > R_L$*.*

Assumption 3 (Analytic pdf) *For each* $R \in \Upsilon$*,* $f(.|R) : [1, \bar{y}] \rightarrow \mathbb{R}$ *is analytic.*

Assumption 2 simplifies the characterization of aggregate allocations. In this sense is not crucial for the main results of this paper. It implies, for instance, that both aggregate investment and aggregate revenue are increasing in R and decreasing in τ. Without Assumption 2, none of these characteristics are necessarily true. In the appendix we show how to deal with this situation when Assumption 2 fails. Assumption 3 guarantees that for any tax function, the set of individuals indifferent between investing or not has Lesbegue measure zero. It delivers the differentiability of the aggregate allocations with respect to τ that is used in the proof of Lemma 1. This last assumption plays an important role in the repeated game without commitment of Section 3.

The next lemma characterizes the equilibrium of the Ramsey game.

Lemma 1 *Under Assumptions 1-3 , the Ramsey equilibrium is given by:*

a) $\exists\ y^* \in [1, \bar{R}]$,

$$-\frac{E_R[W_{y^*}(\tau_R^*(R), y^*, R)]}{E_R[W_\tau(\tau_R^*(R), y^*, R)]} = \frac{1}{y^*} + \left.\frac{\partial y(1 - E[\tau_R(R)|y])}{\partial y}\right|_{y=y^*}$$

b) such that

$$x_R^i(y^i, \tau_R^*(R)) = \begin{cases} \omega & \text{if } y^i \geq y^* \\ 0 & \text{otherwise} \end{cases}$$

c) For all $R, \hat{R} \in \Upsilon$*,* τ_R^* *satisfies,*

$$\frac{W_\tau(\tau_R^*(R), y^*, R)}{W_\tau(\tau_R^*(\hat{R}), y^*, \hat{R})} = \frac{f(y^*|R)}{f(y^*|\hat{R})}$$

d) Moreover if $v(.)$ *is linear* $(1 - \tau_R^*(\bar{R}))\tau_R^*(R) = 0$ for all $R \in \Upsilon$, $R \neq \max_{R \in \Upsilon}\{R\}$

e) For all $R, \hat{R} \in \Upsilon$*,* $g(R) < g(\hat{R})$ *if* $R < \hat{R}$ *(partial smoothing)*

Proof: Assuming that y^* is unique, the first order conditions imply a)-c). We present a proof of part d) in the appendix. The main complication arises when y^* fails to be unique. In this case, the above characterization requires only minor modifications. Specifically, part a) of the lemma must be modified as shown in the appendix.

Lemma 1 states that the Ramsey problem with linear utility has a corner solution. The government taxes a positive amount in the highest aggregate state. Taxing in the highest state is less costly than taxing in a lower state. For any given average tax, it is always possible to increase the payoff by increasing the tax in the highest state and reducing the tax in a lower state in such a way that the average tax remains the same. The proof exploits this idea. When $v(.)$ is not linear the characterization is not as sharp. However, as part e) states, the government would still like to provide a larger quantity of the public good when the productivity of investment is larger. The optimality of stage contingent policies is what creates a trade off between rules and flexibility.

3 The Repeated Game and Uncertainty

The repeated game is the repetition of the stage game described in Section 2.1. Time is discrete and indexed by $t = 0, 1, 2, ...$ At the beginning of each period t, an outcome R_t of the random variable $\hat{R}$ is realized. The outcome R_t is observed only by the government. Each R_t is distributed i.i.d. over time with probability distribution $P(R)$.

As in the static game, in each period and conditional on the realization of $\hat{R}_t$, each individual privately observes a draw $y_t^i \in [1, \overline{y}]$ of the random variable Y with density function $f(y_t^i|R_t)$. Thus, the idiosyncratic shocks are i.i.d over time. We assume a version of the law of large numbers with a continuum of random variables relying on the construction of Sun (2006).

All agents discount the future with the common factor $\beta \in (0, 1)$. Accordingly we normalize the per period payoff for the government with the constant $1-\beta$. That is, the government payoff is $W(\tau, X, Y; R) = (1 - \beta)[(1 - \tau)Y(R) + (\omega - X(R)) + v(\tau Y(R))]$.

We also assume the following.

Assumption 4 *Stochastic processes.*

1. $f(y_t|R_t) > 0$ *for all* $R_t \in \Upsilon$ *and almost all* $y_t \in [1, \overline{y}]$ *(full support)*

2. $\xi(g_t^i|g_t) > 0$ *for all* g_t *and all* $g_t^i \in [\underline{g}, \bar{g}]$

The first condition is a technical assumption, standard in the literature, that allows us to ignore zero probability events. It ensures that, independently of the realization of $\hat{R}$, every idiosyncratic signal is possible in every period.

The second assumption prevents the reintroduction of complete information. If all agents received the same amount of public good, because of the government budget constraint, every agent would be able to infer the realization of the aggregate return in the economy. In this case, the standard arguments for economies with complete information would apply and therefore the Ramsey policy would be sustainable.

Within each period the timing remains as described in Section 2.1 with all variables, individual and aggregate, now indexed by t. To formally define the payoff of the repeated game we need some additional notation that is presented in the next section.

3.1 Perfect Bayesian Equilibrium

In the repeated game, a public history is a collection of outcomes that have been observed by all players. From now on a particular history will be denoted with a lower case h.[9]. Since the only variable that is publicly known is the tax on investment, a public history is a collection of all past realizations of the tax. Thus, at each period t a public history is $h^{P,t} \equiv \{\tau_0, ..., \tau_{t-1}\}$.

Because households are anonymous there is no loss of generality in defining private histories for the government which do not include the aggregates.[10] Let $h^{g,t} \equiv \{h_0^g, ..., h_t^g\}$ be the private history for the government, with $h_s^g = \{\tau_{s-1}, R_{s-1}\}$ if $t \geq 1$ and $h^{g,0} = \emptyset$. Similarly, a history for individual i is given by $h^{i,t} \equiv \{h_0^i, ..., h_t^i\}$ with $h_s^i = \{\tau_{s-1}, y_{s-1}^i, g_{s-1}^i\}$ if $t \geq 1$ and $h^{i,0} = \emptyset$.

We restrict the analysis to pure strategies on the part of the government. A pure strategy for the government is a sequence $\{\sigma_{G,t}\}_{t=0}^{\infty}$ with $\sigma_{G,t} : H^{g,t} \times \Upsilon \to [0, 1]$. A strategy for investor i is given by $\{\sigma_{i,t}\}_{t=0}^{\infty}$ with $\sigma_{i,t} : H^{i,t} \times [1, \bar{y}] \to [0, \omega]$. Let Σ_G be the set of possible strategy profiles for the government and Σ be the set of all possible strategy profiles $\sigma = (\sigma_G, \{\sigma_i\}_{i \in I})$.

Households must form beliefs about the state of the economy and the history of government. Let $\mu(\cdot|\tilde{h}^{i,t}, y^i)$ be the probability distribution over histories $\hat{h}^{g,t} \in H^{g,t}$ consistent with individual history $\tilde{h}^{i,t}$. A strategy profile σ induces, after any history $h^t \in H^t$, a continuation profile $(\sigma_G|_{h^{g,t}}, \{\sigma_i|_{h^{i,t}}\}_{i \in I}) \in \Sigma$.

Since investors are both anonymous and atomistic, their optimal choice can be reduced to a simple rule. For any σ_G, let $E_{\sigma_G}(\tau|h^{i,t}, y^i)$ be the conditional expectation that an investor

[9]In what follows and upper case H will denote de set of all histories
[10]For a detailed explanation, see Chari and Kehoe (1990).

with history $h^{i,t}$ and idiosyncratic return y^i has about the random variable $\sigma_{G,t}$.[11] This investor will invest a positive amount only if the expected marginal return on investment is positive:

$$x^*(h^{i,t}, y^i) = \begin{cases} \omega & \text{if } y^i E_{\sigma_g}(1-\tau|h^{i,t}, y^i) - 1 > 0 \\ [0,\omega] & \text{if } y^i E_{\sigma_g}(1-\tau|h^{i,t}, y^i) - 1 = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Equation 4 shows a very important feature of investors' strategies. They make their decisions based just on their productivity of investment and the expectation about future taxes. If the investors think that the government has deviated in the past but it is still profitable for them to invest, they will keep investing.

Given a profile σ and a sequence of belief profiles $\mu \equiv \{(\mu(\cdot|h^{i,t}, y^i))_{i\in I}\}_{t=0}^{\infty}$, expected payoffs for the players are naturally defined using the stochastic outcomes that the strategies induce. The payoff for the government at time zero in the repeated game is given by:

$$V(\sigma) = (1-\beta)E\left[\sum_{t=0}^{\infty} \beta^t W(\tau_t, X_t, Y_t; R_t)\right]$$

Notice that since the government observes all the aggregates in the game, it does not need to form beliefs about individuals' actions. It only needs to consider the effects of its own actions on individual beliefs.

Definition 2 *A pair (σ, μ) consisting of a strategy profile and a belief system is a **Perfect Bayesian Equilibrium**(PBE) if:*

(i) Given $\{\mu^i\}_{i\in I}$, $\sigma_{i,t}(h^{i,t}, y^i) = x^*(h^{i,t}, y)$ $\forall i \in I, h^{i,t} \in H^{i,t}, \forall y^i \in [1, \bar{y}]$;

(ii) $V(\sigma_G|_{h^{g,t}}, \{\sigma_i\}_{i\in I}) \geq V(\widetilde{\sigma}, \{\sigma_i\}_i)$ $\forall \widetilde{\sigma} \in \Sigma_G, h^{g,t} \in H^{g,t}, \forall R$;

(iii) Beliefs are given by Bayes' rule whenever possible

Conditions (i) and (ii) capture the difference between the strategic player, the government, and the non-strategic investors. Since each investor has zero measure, they do not perceive their actions as affecting the aggregate outcomes and therefore do not act strategically. Condition (i) formalizes this statement: given their beliefs, investors maximize their utility from

[11] This function is measurable with respect to the sigma-algebra generated by her individual histories.

private consumption in every period.[12] Instead, the government correctly understands that its actions affect the continuation payoffs of the game. Thus, Condition (ii) states that the equilibrium strategy for the government has to generate a payoff that is at least as large as the one generated by any other strategy.

Does an equilibrium exist? As we mentioned in Section 2.1 there is a unique equilibrium in pure strategies in the stage game. This equilibrium is still an equilibrium in the repeated game. In fact, it is the worst PBE as Lemma 2 states.

Lemma 2 (Worst Equilibrium) *The pair of strategies $\sigma_{G,t}^{worst}(h^{g,t}, R_t) = 1$ for all $h^{g,t} \in H^{g,t}$ and all $R_t \in \Upsilon$ together with $\sigma_{i,t}^{worst}(h^{i,t}, y_t^i) = 0$ for all $h^{i,t} \in H^{i,t}$, all $y_t^i \in [1, \overline{y}]$ and all $i \in I$ is a PBE, and yields the lowest payoff among all PBE.*

Proof: See Appendix.

The statements of Lemma 2 are twofold. First, it states that an equilibrium exists, but second, this equilibrium generates the worst among all possible outcomes. How much it is possible to improve over it will be discussed in the the next section. Before that, it is useful to understand what cannot happen in equilibrium. Lemma 3 below states that after any history and for any possible configuration of the strategy profiles the set of indifferent investors has measure zero.

Lemma 3 *Under Assumption 3, for any $\sigma \in \Sigma$ and any $h^t \in H^t$, the set of individuals for which $y^i E_{\sigma_g}(1 - \tau | h^{i,t}, y^i) - 1 = 0$ in (4) has Lebesgue measure zero.*

Proof: See Appendix.

The above lemma discards the possibility of mixed strategy by part of investors. That is, in equilibrium there could be some investors randomizing, but because they have measure zero their strategies will not have an impact on the aggregate. Hence, without loss of generality we focus on pure strategies on the part of investors.

The intuition for this result is straightforward. When the distribution of signals is sufficiently smooth, if an investor is indifferent between investing or not, the investor with a history arbitrarily close to hers will strictly prefer one action or the other. Regardless the shape of the tax function, the government cannot manipulate taxes in order to keep a positive measure of agents indifferent about the investment decision.

[12] This is a anonymous sequential game with uncertainty, hence we rely on Bergin and Bernhardt (1992)-(1995) to show existence.

3.2 Unattainability of Ramsey outcome

In this section we show that there is no strategy profile $(\sigma_G, \sigma_i) \in \Sigma$, together with some belief system, that yields the outcome path of the Ramsey equilibrium (or any state contingent tax policy). If the Ramsey outcome were an equilibrium, on the equilibrium path the government would be playing a strategy that depended only on the current shock R_t and households would assign probability one to the Ramsey strategy independently of their own signal. This feature equips the government with profitable deviations. For instance, every time that the government is supposed to play the lowest tax prescribed by the equilibrium strategy it could choose the highest tax consistent with equilibrium behavior. Since every household believes with probability one that the government plays the Ramsey strategy, those households that received the lowest idiosyncratic returns on investment would believe that they are in the unlucky tail of the distribution of returns. If the Ramsey policy were an equilibrium, receiving "extremely" inconsistent signals happens with positive probability on the equilibrium path.

Given a strategy profile σ and a belief system μ, let $\mathcal{A} = \{(x_t^i(\sigma), c_t^i(\sigma))_{i \in I}, g_t(\sigma), \tau_t(\sigma)\}_{t=0}^{\infty}$ be the outcome path of the strategy profile. As usual, the outcome path is defined as the induced outcome starting from the initial history $h^{g,0}$. More specifically, let $\mathcal{A}^* = \{\{x_t^i(\sigma^*), c_t^i(\sigma^*)\}_{i \in I}, g_t(\sigma^*), \tau_t(\sigma^*)\}$ be the outcome path of the Ramsey allocation, where σ^* is as defined at the end of the last section.

Proposition 1 *Under Assumptions 1 and 3, there is no belief system μ and $\sigma \in \Sigma$ that generates $\mathcal{A}^*$ on the equilibrium path.*

Proof: By contradiction, suppose that there is a pair of strategy profiles $\hat{\sigma} \in \Sigma$ and a belief system μ such that $\mathcal{A}^*$ is an equilibrium outcome. Let $S = \left\{\tau \in [0,1] : \tau = \sigma^*(R_t), R_t \in \Upsilon\right\}$ and define:

$$H^{*,i} = \{h^{i,t} \in H^i : \tau_s \in S, \forall s \leq t-1\}$$

Notice that $H^{*,i}$ is the set of possible histories on the equilibrium path for household i. Because of the full support assumption, all individual histories in $H^{*,i}$ have non-zero measure. Given $\hat{\sigma} \in \Sigma$, let $\mu(\cdot | h^{i,t}, y_t^i)$ be the induced probability distribution over $H^{g,t} \times \Upsilon$ given the history $(h^{i,t}, y_t^i)$. In the appendix we show that for all $h^{i,t} \in H^{*,i}$ the belief system should update as follows:

$$\mu(h^{g,t}, R_t | \tilde{h}^{i,t}, y_t^i) = P(R_t | y_t^i) \frac{\mu(h^{g,t-1} | \tilde{h}^{i,t-1})}{\sum_{\hat{h}^{g,t-1}} \mu(\hat{h}^{g,t-1} | \tilde{h}^{i,t-1})}$$

The above expression can be reduced through recursive calculations to:

$$\mu(h^{g,t}, R_t|\tilde{h}^{i,t}, y_t^i) = f(y_t^i|R_t)\prod_{s=0}^{t-1}\frac{f(y_s^i|R_s)}{\sum_{\hat{R}_s} f(y_s^i|\hat{R}_s)} \tag{5}$$

Equation (5) implies that beliefs are not affected by government actions. In addition, $\mu(\hat{\sigma}(h^{g,t}, R_t)|h^{i,t}, y_t^i) = \mu(\hat{\sigma}(h^{g,t}, R_t)|\tilde{h}^{i,t}, y_t^i)$ for all $h^{i,t}, \tilde{h}^{i,t} \in H^{*,i}$ and all $y_t^i \in [1, \overline{y}]$. Therefore, equation (4) implies that $\hat{x}^i(h^{i,t}, y_t^i) = x^i(\tilde{h}^{i,t}, y_t^i)$ for all $h^{i,t}, \tilde{h}^{i,t} \in H^{*,i}$ and all $y_t^i \in [1, \overline{y}]$. Now take some period t and $R' \in \Upsilon$ such that $\hat{\sigma}(h^{g,t}, R') < \max_{\tau\in S}\tau$. Consider the following one shot deviation on the part of the government:

$$\tilde{\sigma}_G(h^{g,s}, R) = \begin{cases} \tau^D \equiv \max_{\tau\in S}\tau & \text{if } s = t \text{ and R=R'} \\ \hat{\sigma}(h^{g,s}, R) & \text{otherwise} \end{cases}$$

Following history $(h^{g,t}, R')$, the equilibrium strategy generates a government payoff of:

$$(1-\beta)W(\hat{\sigma}(h^{g,t}, R'), X(h^{g,t}), Y(h^{g,t}); R_t) + \beta V(\hat{\sigma}|_{\{h^{g,t},R',\hat{\sigma}(h^{g,t},R')\}}, (\hat{\sigma}_i|_{\{h^{i,t},y_t^i,\hat{\sigma}(h^{g,t},R')\}})_{i\in I}) \tag{6}$$

where $X(h^{g,t})$ and $Y(h^{g,t})$ are the aggregates following the outcome path after history $h^{g,t}$.

The one shot deviation strategy generates a payoff of:

$$(1-\beta)W(\tau^D, X(h^{g,t}), Y(h^{g,t}); R_t) + \beta V(\hat{\sigma}|_{\{h^{g,t},R',\tau^D\}}, (\hat{\sigma}_i|_{\{h^{i,t},y_t^i,\tau^D\}})_{i\in I}) \tag{7}$$

Since $\tau^D \in S$ it follows that $\{h^{i,t}, \tau^D, y_t^i\} \in H^{*,i}$ and therefore $\hat{\sigma}^i|_{\{h^{i,t},\hat{\sigma}(h^{g,t},R'),y_t^i\}} = \hat{\sigma}^i|_{\{h^{i,t},\tau^D,y_t^i\}}$ for all $i \in I$. In addition, $\hat{\sigma}|_{\{h^{g,t},R',\hat{\sigma}(h^{g,t},R')\}} = \tilde{\sigma}|_{\{h^{g,t},R',\tau^D\}}$ by construction. Therefore, the continuation payoffs are equal in (6) and (7). By Assumption 1, $\hat{\sigma}(h^{g,t}, R') < \tau^D$ implies that $W(\tau^D, X(h^{g,t}), Y(h^{g,t}); R_t) > W(\hat{\sigma}(h^{g,t}, R'), X(h^{g,t}), Y(h^{g,t}); R_t)$. Hence the deviation is profitable, a contradiction ■

Assumption 3 plays a role in Proposition 1 only to the extent that it guarantees that a measure zero of agents is indifferent between investing or not and therefore mixing on the part of households is not possible. Notice that Proposition 1 can actually be made stronger. As its proof makes clear, there is nothing special about the Ramsey outcome other than the fact that on the equilibrium path taxes are stochastic. More precisely, no strategy profile in which the government strategy depends only on its private information (including its private history of any length) could be an equilibrium. A crucial feature that prevents any strategy from achieving the Ramsey outcome is the fact that the individual strategies do not depend on history on the equilibrium path. The government then takes advantage by defecting whenever possible.

3.2.1 Discussion

The intuition behind Proposition 1 is similar to Bagwell (1995) and more generally similar to the intuition for the non-existence of pure strategy equilibria in games with private imperfect monitoring. In a game where actions are not observed, when one player plays a pure strategy in equilibrium, the only consistent belief for the other players is to assign probability one to adherence to the prescribed equilibrium strategy. This must be true independently of the signal that any (or all) other players receive about the behavior of that particular player. The full confidence of most of the players in the pure strategy player is what creates the incentive for deviation.

In our environment, there are some differences with respect to the standard private imperfect monitoring setup; government's actions are observed but the object on which the government is conditioning is not. The equilibrium breaks because agents do not know what other agents will do, not because they don't know what the government is doing.

In order to provide some intuition for this point, suppose that instead of receiving idiosyncratic signals all agents receive a common continuous signal y. Note that there is still asymmetric information. However, now it is possible to support state contingent policies with trigger strategies. For instance, suppose there are two states: $R_H > R_L$ with probabilities P_H and P_L. Choose $\epsilon > 0$ and consider the following pair of strategies

$$\sigma_G(R_t, h^{G,t}) = \begin{cases} \tau_L & \text{if } R_t = R_L \text{ and } X_s > 0,\ \forall s = 0, 1, ..., t. \\ \tau_H & \text{if } R_t = R_H \text{ and } X_s > 0,\ \forall s = 0, 1, ..., t. \\ 1 & \text{if } X_s = 0 \text{ some } 0 \leq s \leq t,\ \forall R_t \end{cases}$$

$$\sigma_i(y_t, h^{i,t}) = \begin{cases} x^{i,*}(y_t, h^{i,t}) & \text{if } \forall s \leq t-1 \text{ with } \tau_s = \tau_H,\ \text{then } y_s - R_H \leq \epsilon \\ 0 & \text{if } \exists\ s \leq t-1 \text{ with } \tau_s = \tau_H \text{ and } y_s - R_H > \epsilon \end{cases}$$

In words, this pair of strategies states: let the government play the state contingent policy as long as there has always been positive investment while investors invest optimally as long as all the past signals are ϵ consistent with the equilibrium strategy, otherwise do not invest. We can show that this pair of strategies is an equilibrium if the β and the precision of the signal are large enough. The arguments are similar to Green & Porter's (1984) seminal paper. In this type of equilibria mistakes sometimes are made in the sense that with some strictly positive probability the investors receive an inconsistent signal and stop investing forever. However, as the precision of the signal increases ϵ can be made arbitrarily small and so the probability of a mistake. As a result, as the noise of the signal converges to zero the equilibrium payoff gets arbitrarily close to efficiency.

Note the key difference between the environments with common and idiosyncratic signals. In the former all investors know that the other investors have observed an inconsistent signal and how they would react to that information. Thus, they coordinate to punish the government for a potential deviation. In the latter, investors are not sure about what the other investors have observed, and therefore about what they will do. As a result, for any signal that an investor receives, and if she believes that the government is playing the equilibrium strategy, she should keep investing as if no deviation had happened.

Although the Ramsey payoff cannot be attained, the reader may wonder if it can be approached arbitrarily close for high enough discounting. The answer to this question is negative under circumstances elaborated upon in the next section. Notice that if indeed there is a strategy profile than can approach the (repeated) payoff of the Ramsey equilibrium, such a profile would require some coordination among agents. In other words, a positive measure of agents must have strategies that depend on public histories. If only a measure zero of agents can coordinate, then no punishment that they use can have an effect on the government's payoff.

4 Optimal Policy Without Commitment

Proposition 1 imposes a strong restriction on the set of strategies and payoffs that can be supported in equilibrium, but still there is a wide range of payoffs that can be sustained. Is it possible to approach the Ramsey payoff as the discount factor approaches one and/or the noise converges to zero? The focus of this section is to answer the last questions.

We start by characterizing the solution to a simple static problem. In this problem a benevolent social planner, who has access to arbitrary lump sum transfers, chooses an incentive compatible state contingent tax function and lump sum transfers to maximize the ex-ante expected utility of the investors. We show that the solution to the static problem generates a tax function that implements the best PBE. The key feature of the solution to the static problem that allows us to show that it is also the best PBE is that the optimal tax schedule is independent of the fundamental.

4.1 The static problem

Let $\tau : \Upsilon \to [0, 1]$ be an arbitrary tax function. Given this tax function investors optimally decide how much to invest and how much to consume according to part (b) of definition 1. This in turn implies aggregate investment, $X(\tau, R)$ and aggregate output $Y(\tau, R)$. With some abuse of notation let $W(\tau(R_s), Z(\tau, R_s))$, where $Z(\tau, R) = \{X(\tau, R), Y(\tau, R)\}$, be the implied welfare in the economy and $\mathcal{OP}(\tau, R)$ be the set of all $Z(\tau, R)$ generated by τ. Suppose a benevolent planner who can freely provide the government with utility-transfers as long as they are contained in a set $[\underline{V}, \bar{V}]$. In this static problem transfers (measured in units of utility) are taken as exogenous, with the only constraint being that they have to belong to a compact and convex set. As we see below, the fact that the qualitative properties of the solution are the same for any arbitrary upper and lower bounds allows us to easily implement this solution in the repeated game.

Next we restrict attention to the case in which there are only two possible values for R, $R_L < R_H$. [13] The optimal policy solves the following problem:

$$\hat{T} = \max_{\{\tau,V\}} \sum_{s=L,H} P_{R_s}[(1-\beta)W(\tau(R_s), Z(\tau, R_s)) + \beta V(\tau(R_s))] \tag{PS}$$

subject to;

$$\begin{array}{rl} \text{(IC-ON-S)} & (1-\beta)W(\tau(R_s), Z(\tau, R_s)) + \beta V_s \geq (1-\beta)W(\tau(R_{s'}), Z(\tau, R_s)) + \beta V_{s'}; \forall s, s' \\ \text{(OP-S)} & Z(\tau, R_s) \in \mathcal{OP}(\tau, R_s),\ \forall \tau \in [0,1]^2,\ \forall s = L, H \\ \text{(E-S)} & V_s \in [\underline{V}, \bar{V}],\ \forall s = L, H \end{array}$$

The constraint (IC-ON-S) is a standard incentive compatibility constraint, it imposes truth telling on the part of the government. That is, if the optimal tax schedule calls for a state contingent policy, it has to be in the government's best interest to follow it. The (OP-S) constraint makes sure that the aggregate allocations are generated by the optimal behavior of

[13] In an online appendix we extend the result to the case in which R can take a continuous number of values.

the investors. Finally, the last constraint imposes an upper and lower bound on the possible transfers accessible to the planner. We take the lower and upper bounds as arbitrary values, as long as they are finite.

Proposition 2 *In the solution to PS* $\tau_L = \tau_H = \tau^B$ *and* $V_L = V_H = \bar{V}$*, with:*

$$\tau^B = argmax_{\{\tau, Z(\tau,R_s)\in\mathcal{OP}(\tau,R_s)\}}\left\{E_R\left[W(\tau, Z(\tau, R_s))\right]\right\}$$

Proof: The problem's Lagrangian is

$$\begin{aligned} L &= \sum_{s=L,H} P_s[W(\tau_s, Z(\tau, R_s)) + \beta V_s] + \sum_{s=L,H} \lambda_s[W(\tau_s, Z(\tau, R_s)) + \beta(V_s - V_{-s}) - W(\tau_{-s}, Z(\tau, R_s))] \\ &+ \sum_{s=L,H} \gamma_s \beta[\bar{V} - V_s] \end{aligned}$$

Note that we are disregarding the constraint that $V_s \geq \underline{V}$. Second, we do not need to consider the case in which both $\gamma_s > 0$. Because W is strictly increasing in τ, the (IC-ON-S) would imply $\tau_L = \tau_H$. In the same way, if $\lambda_s > 0$ in both states, then both (IC-ON-S) would be binding, and therefore $\tau_L = \tau_H$. Thus, we only need to consider cases in which only one λ_s and only one γ_s can be strictly positive.

The first order conditions with respect to V_s are,

$$P_L + \lambda_L - \lambda_H - \gamma_L = 0$$
$$P_H + \lambda_H - \lambda_L - \gamma_H = 0$$

If $\gamma_L = \gamma_H = 0$ the equations above imply $P_L = -P_H$ which is not possible. That is, in a best equilibrium, in at least one state, the continuation value has to be a best equilibrium. There are two possible cases.

Case 1: Suppose $\gamma_H > 0$ (hence $V_H = \bar{V}$ and $V_L \leq \bar{V}$), then the above equations imply $\lambda_H = P_L + \lambda_L$ or $\lambda_H > \lambda_L$, since at least one multiplier has to be zero, it follows that $\lambda_H > 0$. One can see by using (IC-ON-S) that in this case $\tau_L \geq \tau_H$ (the best equilibrium requires smaller continuation values for larger taxes).

Case 2: $\gamma_L > 0$ ($V_L = \bar{V}$, $V_H \leq \bar{V}$) and $\lambda_L > 0$. Using a similar argument it follows that $\tau_L \leq \tau_H$ in this case.

Consider case 1. Replacing the binding constraints in the objective function the problem becomes:

$$\hat{T} = \max_{\{\tau_L, \tau_H, V_L\}} P_L W(\tau_L, Z(\tau, R_L)) + P_H W(\tau_L, Z(\tau, R_H)) + \beta V_L$$

subject to;

$$\begin{aligned}
& W(\tau_H, Z(\tau, R_H)) + \beta \bar{V} \geq W(\tau_L, Z(\tau, R_H)) + \beta V_L \\
& Z(\tau, R_s) \in \mathcal{OP}(\tau, R_s) \text{ for all } s \\
& V_L \in [\underline{V}, \bar{V}]
\end{aligned}$$

Again, either $V_L = \bar{V}$ or the (IC-ON-S) is binding, in both cases the solution implies $\tau_L = \tau_H$. A similar argument can be used to show that the second candidate solution implies the same result. Therefore, $V_s = \bar{V}$. Then, maximizing the return function (imposing the additional constraint that taxes are equal) delivers the result. ■

This result is similar to Athey et al. (2005) and Sleet and Yeltekin (2006). Both papers find that it is optimal to take away the discretion from the policy makers (a monetary authority in Athey et al. (2005) and a spending-biased government in Sleet and Yeltekin (2006) in those states in which the time inconsistency problem is severe. In our case we obtain that is optimal to take away the discretion from the policy maker for any possible state. The main difference in our setup is Assumption 1, which entails the government to be tempted to raise taxes in every possible state of nature. Relaxing Assumption 1 we can show that for some values of the fundamental constraint (IC-ON-S) is not binging and therefore the optimal tax policy with commitment is implementable. However, as the discussion after Assumption 1 emphasizes, this assumption is at the heart of the problem. Disposing of it may render the study of the issue meaningless.

If we restricted our attention to government strategies that condition behavior on public histories and the most recent realization of the private shock, but not on the entire history of private shocks, i.e., Public Perfect Equilibria (PPE), we could follow the same approach as Athey et al. (2004) to show that the solution to PS is actually the best PPE.[14] In the next section we show that this result is still true when strategies are allowed to depend on

[14] In a previous version of this paper we present the proof. See Piguillem & Schneider (2009), "Optimal taxation and heterogenous beliefs". See Chang (1998) for an early application of this approach based on Abreu et al. (1990).

the whole history of private information.

4.2 The best PBE

In this section we show that the best PBE has the property that the tax is independent of the fundamental. The proof is by induction showing that in any period t after any public history $h^{P,t}$ the best equilibrium solves a problem like PS. A necessary first step is Lemma 4 which characterizes the set of tax functions that can be implemented as equilibrium strategies.

Take any equilibrium strategy σ with implied histories $h^{j,t}$, for $j = P, g, i$.[15] Let $V(h^{P,t-1}) = E[V(h^{g,t})|R^{t-1}]$ be the ex-ante value of the equilibrium strategy conditional on the realization of R^{t-1}. At period zero the payoff for this strategy is given by:

$$V(\varnothing) = E_R\big[W(\tau(R), Z(\tau, R)) + \beta V(\tau(R))\big] \tag{8}$$

Suppose that the equilibrium strategy prescribes that the set of actions at time one for the government is characterized by the mapping $\tau(R)$ with associated continuation payoffs $V(\tau(R))$, then we have,

Lemma 4 *A pair $\{\tau(R), V(\tau(R))\}$ can be implemented as an equilibrium at time 1 if and only if, for all $R_s \in \Upsilon$ and all $\hat{\tau} \in [0,1]$,*

$$\begin{array}{rl}
\textit{(IC-ON)} & (W(\tau(R_m), Z(\tau, R_m)) + \beta V(\tau(R_m)) \geq W(\tau(R_{-m}), Z(\tau, R_m)) + \beta V(\tau(R_{-m})) \\
\textit{(IC-OFF)} & W(\tau(R_m), Z(\tau, R_m)) + \beta V(\tau(R_m)) \geq W(1, Z(\tau, R_m)) + \beta V^{worst} \\
\textit{(EQ)} & V(\hat{\tau}) \in \Psi_\beta \textit{ for all } \hat{\tau} \in [0,1] \\
\textit{(OP)} & Z(\tau, R_m) \in \mathcal{OP}(\tau, R_m) \textit{ for all } \tau \in [0,1]^2
\end{array}$$

REMARK: There are two differences between the constraints in problem PS and the set of inequalities in Lemma 4. First, the constraint (IC-OFF) is not included in PS. The role of this constraint is to prevent the government from choosing taxes other than those prescribed on the equilibrium path. In other words, constraint (IC-ON) prevents the government from say, taxing high when it is supposed to tax low, but it does not prevent other kinds of deviations

[15] Where histories are constructed in the usual way: $h^{g,t} = \{h^{g,t-1}, R_t, \sigma_G(h^{g,t-1}, R_t)\}$ with $h^{g,0} \in \Upsilon_R$, individual histories $h^{i,t} = \{h^{i,t-1}, y_t^i, \sigma_G(h^{g,t-1})\}$ with $h^{i,0} \in [1, \bar{y}]$ and the public history is $h^{P,t} = \{h^{P,t-1}, \sigma_G(h^{g,t-1})\}$ with $h^{P,t} = \varnothing$.

such as full confiscation. Constraint (IC-OFF) deals with this kind of deviations. If β is large enough the latter would not bind in equilibrium and can be disregarded. Second, the constraint (EQ) in Lemma 4 makes sure that the continuation values belong to the equilibrium value set, while in PS the continuation values are only constrained to belong to a convex and compact set. Thus, renaming the objects one can see that any pair $\{\tau(R), V(\tau(R))\}$ satisfying the inequalities in Lemma 4 is feasible in PS.

Proof: First, it is straightforward to show that if $\{\tau(R), V(\tau(R))\}$ can be implemented as an equilibrium it has to satisfy the above set of inequalities. Suppose $\{\tau(R_1), V(\tau(R_1))\}$ is such a sequence. Since, $V(\tau(R_1)) \in \Psi_\beta\ \forall R_1$, there exist $\tilde{\sigma}_{G,R_1}(h^{g,t}, R_t)$ and $\tilde{\sigma}^i_{R_1}(h^{i,t}, y^i_t)$ for each $R_1 \in \Upsilon$ that together with the belief system $\tilde{\mu}_{R_1}$ are an equilibrium for all $t \geq 2$ and $V(\tau(R_1)) = V(\tilde{\sigma}_G, \tilde{\sigma}^i)$. Let $\tilde{H}^{g,t}_{R_1}$ be the (on the equilibrium path) set of possible histories generated by $\tilde{\sigma}_{G,R_1}(h^{g,t}, R_t)$ after each R_1, $\tilde{H}^{t,i}$ be the set of possible individual histories generated by $\{\tau(R_1), \tilde{\sigma}_{G,R_1}(h^{g,t}, R_t)\}$ and consider the following pair of strategies and belief system,

$$\sigma_{G,t}(h^{g,t}, R_t) = \begin{cases} \tau(R_1) & \text{if } t = 1,\ \forall R_1 \\ \tilde{\sigma}_{G,R_1}(h^{g,t}, R_t) & \text{if } t > 1 \text{ and } \sigma_{G,1}(R_1) = \tau(R_1) \text{ and } h^{g,t} \in \tilde{H}^{g,t}_{R_1},\ \forall R_1 \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma^i_t(h^{i,t}, y^i_t) = \begin{cases} x^{i,*}(h^{i,t}, y^i_t) & \text{if } h^{i,t} \in \tilde{H}^{t,i},\ \forall t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(h^{g,t}|h^{i,t}) = \begin{cases} \frac{f(y^i_1|R)}{\sum_{\hat{R}} f(y^i_1|\hat{R})} & \text{if } t = 1,\ \forall y^i_1 \\ \tilde{\mu}_{R_1}(h^{g,t}|h^{i,t}) & \forall h^{g,t} \in \tilde{H}^{g,t}_{R_1}; \text{ if } t > 1 \text{ and } \tau_1 = \tau(R_1),\ \forall h^{i,t}\ \forall R_1 \\ 1 & \forall h^{g,t} \notin \tilde{H}^{g,t}_{R_1}; \text{ if } t > 1 \text{ and } \tau_1 \notin \tau(R_1),\ \forall h^{i,t} \\ 0 & \text{otherwise} \end{cases}$$

The constructed strategies are standard in repeated games. The government strategy states: play according to $\tau(R)$ satisfying Lemma 4 at period 1, then depending on the realization of R_1 (and therefore of τ_1) choose the appropriated continuation strategy. If the government ever deviates from the equilibrium path it reverts to the worst equilibrium playing $\tau_t = 1$ forever. The strategy for investors states: invest optimally, i.e., according to their consistent beliefs. That is, investing positive amounts if the government has not reverted to the worst equilibrium, otherwise never invest again. The belief system at period 1 is computed using Bayes' rule. In period $t = 2$ and after, beliefs are the same as in the continuation strategy if the government follows $\tau(R)$ and assigns probability one to off-the-equilibrium-path behavior if the government action is not consistent with $\tau(R)$. Notice that this is a well defined probability measure since the complement of $\tilde{H}^{g,t}_{R_1}$ happens with probability zero on the equilibrium path. By construction these strategies and belief system are an equilibrium for $t > 1$. In addition, it is easy to see that given the belief system the behavior of the agents is optimal at period 1 as well. It remains to show that given the belief system the proposed strategy for the government is indeed optimal at period 1. This follows from the fact that $\tau(R_1)$ satisfies the first two inequalities in Lemma 4. ■

Using Lemma 4 in the next proposition we show that the solution PS in fact characterizes the best PBE.

Proposition 3 *There exists $\hat{\beta} \in [0, 1]$ such that for all $\beta \in [\hat{\beta}, 1)$ if the solution to PS implies $\tau(R_s) = \tau^B$, $\forall R_s \in \Upsilon$ the best PBE strategy $\sigma^* = \{\sigma^*_G, \sigma^*_i\}$ is:*

for all $t \geq 0$,

1) $\forall h^{g,t}_B$, $\forall R_t$; $\sigma^*_G(h^{g,t}_B, R_t) = \tau^B$; *if* $\tau^B_s = \tau^B$ *for all* $s < t$ *and* $\sigma^*_G(h^{g,t}_B, R_t) = 1$ *otherwise*

2) $\forall h^{i,t}_B$, $\forall y^i_t$; $\sigma^*_i(h^{i,t}_B, y^i_t) = \omega$; *if* $y^i_t \geq \frac{1}{1-\tau^B}$ *and* $\tau^B_s = \tau^B$ *for all* $s < t$; *and* $\sigma^*_i(h^{i,t}_B, y^i_t) = 0$ *otherwise.*

3) $\forall h^{P,t}_B$, $V(h^{P,t}_B) = \bar{V}$

4) $\bar{V} = E_{R_i}\left[W(\tau^B, Z(\tau^B, R_i))\right] \in \Psi_\beta$

where τ^B is defined in Proposition 2 and $h^{g,t}_B$ and $h^{i,t}_B$ for the histories induced by σ^*.

Proof: The proof is by induction. Notice that at period one, after the empty history, the best equilibrium payoff can be written as in (8). By Lemma 4 the best equilibrium strategy maximizes (8) subject to the set of constraints in Lemma 4. Then, for β large enough $\bar{V} \in \Psi_\beta$, by Proposition 2 the solution is $\tau_s = \tau^B$, for all s and continuation values $V_s = V_{s'}$, for all s and s'. Thus, after period one there is only one continuation value and only one on-path public history. After this public history, and for every realization of R at $t = 1$, the payoff is:

$$V(\tau^B) = E_{R_2}\left[W(\sigma^*_{G,2}(\tau^B, R_1, R_2), Z(\sigma^*_{G,2}(\tau^B, R_1), R_2)) + \beta V(\sigma^*_{G,2}(\tau^B, R_1, R_2), \tau^B)|R_1\right] \quad (9)$$

This equation is similar to (8) and is a best equilibrium payoff as well. In fact, it is easy to see that we can use the same argument as in period one appealing to Lemma 4 by changing only the time index. Thus, after any realization of R_1 if $\sigma^*_{G,2}(\tau^B, R_1, R_2)$ was not a solution to PS there would be another tax function $\hat{\tau}$ and continuation values $\hat{V}$ that solve it and can be implemented as an equilibrium, generating a value $E_{R_2}\left[E[W(\hat{\tau}, Z(\hat{\tau}, R_2))] + \beta\hat{V}\right] > V(\tau^B)$. But this would contradict $V(\tau^B)$ being a best equilibrium.

Therefore, at period 2 it must be the case that $\tau_s = \tau^B$, $\forall s$ and $V_s = V_{s'}$. In order to conclude the induction, suppose that the statement is true at any period $t > 2$, then it must be the case that

$$V(h^{P,t}) = E_{R_t}\left[W(\sigma^*_G(h^{g,t}_B, R_t), Z(\tau^B(h^{g,t}_B), R_t)) + \beta V(\sigma^*_G(h^{g,t}_B, R_t), h^{P,t})|R^{t-1}\right]$$

Then, the argument is the same as for period 2. The strategy $\sigma^*_{G,2}(h^{g,t}_B)$ has to solve PS otherwise there would be a combination of tax function $\hat{\tau}$ and continuation values $\hat{V}$ such that $E[W(\hat{\tau}, Z(\hat{\tau}, R_t))] + \beta\hat{V}\right] > V(h^t)$. Given that by construction the new strategies are incentive-feasible after period t and that for periods before t the continuation values are independent of the state this policy won't violate any equilibrium requirement. Thus, at period t we obtain $\tau_s = \tau^B$, $\forall s$. But if this is true the best tax independent of the state is given by τ^B as defined in Proposition 2, the same as the continuation value. Regarding optimal individual investments, the result is obvious. After any individual history agents know exactly what the tax will be, therefore, given y^i, if the after tax return in investment is positive they invest everything and if the return is negative they do not invest anything (since the set of indifferent agents has measure zero the rule assigned to them is irrelevant). ■

4.3 Asymptotic inefficiency of equilibria

We say that a strategy profile $\sigma \in \Sigma$ is public if for all t, $\tau_{t-1} \in [0,1]$, $h^{g,t} \in H^{g,t}$ and $R, \hat{R} \in \Upsilon$ we have that $\sigma_{G,t}(h^{g,t-1}, \tau_{t-1}, R) = \sigma_{G,t}(h^{g,t-1}, \tau_{t-1}, \hat{R})$. Section 4.2 has shown that the best PBE can be implemented with only public strategies. Since the optimal policy depends on private information one may intuitively think that there is an inefficiency that does not disappear as the economy approaches an environment with perfect information. This intuition turns out to be correct.

The payoff generated by the best PBE is

$$V^*_{PBE} = V(\sigma^*) = (1-\beta)\sum_{t=0}^{\infty}\beta^t E[W(\sigma^*_{G,t}, R_t|\left\{\sigma^*_{i,t}\right\}_{i\in I})] = E[W(\tau^B, Z(\tau^B, R_t))]$$

Wile the best payoff under commitment (the payoff of the Ramsey equilibrium) is:

$$V^R = V(\tau_R(R)) = (1-\beta)\sum_{t=0}^{\infty}\beta^t E[W(\tau_R(R_t), R_t|\left\{x^i_R\right\}_{i\in I})]$$

Proposition 4 *For any for all $f(y|R)$ and $P(R)$ satisfying assumptions 3-4, there exists $\hat{\beta} < 1$ such that for all $\beta \in [\hat{\beta}, 1)$, $V^*_{PBE} < V^R$.*

The statement follows because the Ramsey equilibrium exhibits taxes that are state dependent, while Proposition 3 shows that state contingent policies are not optimal in the game without commitment. A key feature of Proposition 4 is that its statement does not depend on the underlying uncertainty. As long as there is some uncertainty, in the sense of Assumption 3-4, the difference between the best PBE payoff and the Ramsey payoff does not vanish. The inefficiency persist as the economy approaches complete information.

5 Conclusion

In this paper we show how small changes in informational assumptions can have drastic consequences for both the set of equilibrium strategies and the set of equilibrium payoffs in a macro game without commitment. First, we show that coordination is possible only if every equilibrium strategy depends on some information that is publicly known to every agent in the economy and there is no disagreement among them about its value. When we characterize the best equilibrium of the repeated game, we find that the best policy is a constant tax independent of the aggregate state. As a result, the payoff of the best equilibrium without commitment is always strictly smaller than the payoff with commitment. Moreover, this distance does not approach zero as the heterogeneity of information disappears.

The results of this paper can be interpreted both positively and negatively. From the positive point of view, this paper supports the arguments for strong institutions that tie the hands of policymakers. To endow governments with full discretion and to impose the right incentives to avoid deviations from optimal policies could be impossible or too costly. As long as the beliefs about the true value of the optimal policy are heterogeneous the policymaker would always find a way to "trick" individuals into believing that they are doing the right thing no matter what. In this sense, and in spite of being founded in different reasons, the original recommendation of Kydland and Prescott (1977) is still valid.

It is true that under some special circumstances permanent or persistent changes in the fundamentals could require a change of policy, but these changes must be carried out after the individual preferences (or beliefs) have been aggregated. For instance, through voting, see Piketty, (2000), or through public discussions of the proposed changes. Mechanisms that do not allow for belief aggregation could generate sizable inefficiencies.

From the negative point of view the results of this paper cast some doubts on the policy prescriptions arising from models with complete information. Even arbitrarily small departures from the complete information assumption render the results invalid. Abstracting from the heterogeneity of information is by no means without lost of generality.

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6 Appendix

Proof of Lemma 2. First, take an arbitrary agent and history $h^{i,t}$. Suppose that every agent is playing according to σ^{worst}. In this situation the government cannot increase the provision of the public good regardless of the tax it chooses. Therefore it is weakly optimal for the government to fully tax investment. In this way, condition (i) in Definition 2 is met for all histories. Next, suppose the government is playing according to σ_G^{worst}. Then regardless of the individual signals, all the agents assign probability one to full taxation and the optimal action is to choose zero investment. This follows from (4). Therefore condition (ii) in Definition 2 holds and the proposed strategy is an equilibrium. The fact that σ^{worst} yields the the worst equilibrium follows from the observation that the level of provision of the public good is at its minimum value under σ^{worst}. Therefore assumption (1) yields the result ■

Proof of Lemma 3. We start by stating some properties of analytic functions.

Lemma 5 *Suppose that $K(y, R)$ is analytic in y for all $R \in \Upsilon$. Define*

$$m(y) = \int_{\Upsilon} K(y, R) dP(R)$$

and let

$$C = \{y \in [1, \bar{y}] : m(y) = 0\}$$

Then, C has Lebesgue measure zero on $[1, \bar{y}]$.

Proof: See Theorem 10.18 in Rudin (1987) about holomorphic functions.

Lemma 5 states that the integration of analytic functions is analytic itself and that the set of roots is at most countable. Since the expectation operator involves basic operations such as addition, multiplication and potentially integration, which preserve analyticity, there could be at most a countable number of agents that are indifferent between investing or not. ■

Lemma 1: when there are multiple cutoff values. Let $\tau(\hat{R}) \equiv \sigma_G^*(\hat{R})$ and $\tau \equiv \{\tau(R)\}\,|_{R\,\in\Upsilon}$. Given the vector τ, from the individual agent's decision, consider the following function:

$$H(y^i, \tau) = y^i(1 - [E(\tau(R)|y^i)]) - 1 \tag{10}$$

Given Assumption 3, the set of agents $i \in I$ such that $H(y^i, \tau) = 0$ is at most countable (see the proof of Lemma 3). Moreover, the set of points y^i such that $H(y^i, \tau) = 0$ is indeed finite as long as there is a state $R \in \Upsilon$ with tax bounded away from unity. To see this, notice that, for $\hat{y}^i$ high enough, we have $H(\hat{y}^i, \tau) > 0$ whenever $E(\tau(R)|\hat{y}^i) < 1$.

Given the previous reasoning, suppose there are N cutoff points $\{y_i^*\}_{i=1}^N$ satisfying $H(y_i^*, \tau) = 0$. We order them in an ascending order, i.e., $y_i^* \leq y_{i+1}^*$, and let $y_{N+1}^* = \bar{y}$. It is important to keep this in mind since $H(1, \tau) \leq 0$, $\frac{\partial H(y_i^*,\tau)}{\partial y_i^*} > 0$ when i is odd and $\frac{\partial H(y_i^*,\tau)}{\partial y_i^*} < 0$ when i is even.

Notice that, using the implicit function theorem, we have:

$$\begin{aligned} \frac{\partial y_i^*}{\partial \tau(R)} &= \left[\left(1 - E(\tau(R)|y_i^*) - y_i^* \frac{\partial E(\tau(R)|y_i^*)}{\partial y_i^*}\right)(1 - E(\tau(R)|y_i^*))\right]^{-1} P(R|y_i^*) \qquad (11) \\ &= \left(\frac{\partial H}{\partial y_i^*}(1 - E(\tau(R)|y_i^*))\right)^{-1} P(R|y_i^*) \\ &= J(y_i^*)P(R|y_i^*) \end{aligned}$$

where $J(y_i^*) \equiv \left(\frac{\partial H}{\partial y_i^*}(1 - E(\tau(R)|y_i^*))\right)^{-1}$.

By the definition of $\{y_i^*\}_{i=1}^N$, the aggregate investment is given by $X(\tau(R), R) = \sum_{i=1}^N \int_{y_i^*}^{y_{i+1}^*} f(y|R)dy$. In a similar fashion aggregate output is $Y(\tau(R), R) = \sum_{i=1}^N \int_{y_i^*}^{y_{i+1}^*} yf(y|R)dy$. Notice that $\frac{\partial X(\tau(R),R)}{\partial \tau} =$

$\sum_{i=1}^N (-1)^i f(y_i^*|R) \frac{\partial y_i^*}{\partial \tau} < 0$ because $\frac{\partial y_i^*}{\partial \tau} > 0$ when i is odd and $\frac{\partial y_i^*}{\partial \tau} < 0$ when i is even.

In the same way $\frac{\partial Y(\tau(R),R)}{\partial \tau} = \sum_{i=1}^N (-1)^i y_i^* f(y_i^*|R) \frac{\partial y_i^*}{\partial \tau} < 0$ and $\frac{\partial Y(.)}{\partial \tau} - \frac{\partial X(.)}{\partial \tau} < 0$ because $y_i^* \geq 1$ for all i.

Proof of part d) of Lemma 1 We can write the static payoff for the government as:

$$\sum_{R \in \Upsilon} P(R) W(R, \tau(R), \tau(R), Y(\tau(R), R))$$

where

$$W(R, X(\tau(R), R), \tau(R), Y(\tau(R), R)) \equiv (1 - \tau(R))Y(\tau(R), R) + (\omega - \tau(R)) + b\tau(R)\ Y(\tau(R), R)$$

Towards a contradiction, suppose that the solution has $\tau_{\bar{R}} < 1$ and $\tau_R > 0$ for some $R \in \Upsilon$. Then

consider the following perturbation: increase $\tau_{\bar{R}}$ by $d\tau_{\bar{R}} > 0$ and decreases τ_R by $d\tau_R < 0$ such that it keeps y_N^* fixed. Then, at the solution, the change Δ in payoff should be zero:

$$\begin{aligned} \Delta &= (b-1)[P(\bar{R})Y(\bar{R})d\tau_{\bar{R}} + P(R)Y(R)d\tau_R] + \\ &\quad \sum_{i=1}^N (-1)^i \left[\sum_{\hat{R} \in \Upsilon} P(\hat{R}) \left([(1 + \tau(\hat{R})(b-1))y_i^* - 1] f(y_i^*|R)\right)\right] \left(\frac{\partial y_i^*}{\partial \tau_R} d\tau_R + \frac{\partial y_i^*}{\partial \tau_{\bar{R}}} d\tau_{\bar{R}}\right) \\ &= \Delta_1 + \Delta_2 \end{aligned}$$

where

$$\Delta_1 = (b-1)[P(\bar{R})Y(\bar{R})d\tau_{\bar{R}} + P(R)Y(R)d\tau_R]$$

and

$$\Delta_2 = \sum_{i=1}^{N}(-1)^i\Bigg[\sum_{\hat{R}\in\Upsilon} P(\hat{R})\left([(1+\tau(\hat{R})(b-1))y_i^* - 1]f(y_i^*|R)\right)\Bigg]\left(\frac{\partial y_i^*}{\partial \tau_R}d\tau_R + \frac{\partial y_i^*}{\partial \tau_{\bar{R}}}d\tau_{\bar{R}}\right)$$

Let the perturbation described above satisfy:

$$\frac{\partial y_N^*}{\partial \tau_R}d\tau_R + \frac{\partial y_N^*}{\partial \tau_{\bar{R}}}d\tau_{\bar{R}} = 0$$

Using (11) we get that:

$$d\tau_R = -\frac{P(\bar{R})f(y_N^*|\bar{R})}{P(R)f(y_N^*|R)}d\tau_{\bar{R}}$$

For each other y_i^* we have:

$$\frac{\partial y_i^*}{\partial \tau_R}d\tau_R + \frac{\partial y_i^*}{\partial \tau_{\bar{R}}}d\tau_{\bar{R}} = \frac{J(y_i^*)P(\bar{R})}{\sum_{\hat{R}} P(\hat{R})f(y_i^*|\hat{R})}\left[-\frac{f(y_N^*|\bar{R})}{f(y_N^*|R)}f(y_i^*|R) + f(y_i^*|\bar{R})\right]d\tau_{\bar{R}}$$

Assumption 2 implies that $-\frac{f(y_N^*|\bar{R})}{f(y_N^*|R)}f(y_i^*|R) + f(y_i^*|\bar{R}) < 0$ since $y_N^* \geq y_i^*$ for all i. Thus, because $d\tau_{\bar{R}} > 0$, $\frac{\partial y_i^*}{\partial \tau_R}d\tau_R + \frac{\partial y_i^*}{\partial \tau_{\bar{R}}}d\tau_{\bar{R}}$ is negative when i is odd and positive when i is even, and therefore $(-1)^i\left(\frac{\partial y_i^*}{\partial \tau_R}d\tau_R + \frac{\partial y_i^*}{\partial \tau_{\bar{R}}}d\tau_{\bar{R}}\right) \geq 0$ for all i. Therefore $\Delta_2 > 0$. It remains to show that $\Delta_1 > 0$.

$$\begin{aligned}
\Delta_1 &= (b-1)[P(\bar{R})Y(\bar{R})d\tau_{\bar{R}} + P(R)Y(R)d\tau_R] \\
&= (b-1)d\tau_{\bar{R}}\left[P(\bar{R})Y(\bar{R}) - P(R)Y_L\frac{P(\bar{R})f(y_N^*|\bar{R})}{P(R)f(y_N^*|R)}\right] \\
&= P(\bar{R})(b-1)d\tau_{\bar{R}}\left[Y(\bar{R}) - Y(R)\frac{f(y_N^*|\bar{R})}{f(y_N^*|R)}\right]
\end{aligned}$$

Because $d\tau_{\bar{R}} > 0$, it is sufficient to show that $\frac{Y(\bar{R})}{f(y_N^*|\bar{R})} > \frac{Y(R)}{f(y_N^*|R)}$.Notice that $\frac{Y(\bar{R})}{f(y_N^*|\bar{R})} = \omega\sum_{i=1}^{N}\int_{y_i^*}^{y_{i+1}^*} y\frac{f(y|\bar{R})}{f(y_N^*|\bar{R})}dy$.

and $\frac{Y(R)}{f(y_N^*|R)} = \omega\sum_{i=1}^{N}\int_{y_i^*}^{y_{i+1}^*} y\frac{f(y|R)}{f(y_N^*|R)}dy$. Each of these variables represents an integral using a normalized probability distribution function $h(y|R') = \frac{f(y|R')}{f(y^*|R')}$ with $h(y^*|R') = 1$ $\forall R' \in \Upsilon$.Hence

$\frac{Y(\bar{R})}{f(y_N^*|\bar{R})} > \frac{Y(R)}{f(y_N^*|R)}$ and $\Delta > 0$, a contradiction■