Optimal Debt-Maturity Management

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Abstract

A Government wishes to smooth financial expenses and can issue fixed-coupon bonds among a continuum of maturities. The Government takes into account its price impact. It faces income, interest-rate, and liquidity risk. It acknowledges its own temptation to default. We characterize variations of this problem, compute its risky steady state and present applications.

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1   Introduction

The treasury of any government or big corporation faces a large-stakes problem: to design a strategy for its debt-maturity profile. Economics guides that design through a number of theories. Although each theory brings a different insight, all of them develop the same calisthenics: all theories assume an economic environment and solve its corresponding optimal debt-maturity profile. The hope is that the exercise translates into a policy principle. This paper is a new take on that classic exercise.

The paper makes two innovations. The first innovation is conceptual. The paper puts forth the importance of liquidity frictions: the notion that an abrupt adjustment of the debt of a given maturity, can saturate the market for that maturity. The consequent price impact is a common consideration by practitioners, but has been neglected by normative theory.[1] The second innovation is technical. The debt-management problem has been studied under a rich set of shocks, but in contexts where issuances are restricted. The restrictions on the issuances are typically two: first, on the number of maturities allowed—typically two—and, second, on the types of bonds considered—typically *consols* of exponential maturity. By contrast, in practice, Governments issue in many maturities and consols are rarity. This study takes a different route. Here, shocks occur only once, but the Government can issue bonds among a continuum of maturities and of with any arbitrary cash-flow. The model is highly tractable and easy to compute. The paper exploits this characterization to draw new policy principles.

Let us delve into the details. The environment is the following: an impatient Government in a small-open economy chooses the issuance or (re)purchase of bonds among a continuum of maturities. The financial counterparts are international investors. The Government's objective is to smooth expenditures given a revenue path—or, its dual, to smooth financial expenses given an expenditure path. Several features complicate this Government's choice. First, a liquidity friction produces price impact. Second, the Government faces three sources of risk: (i) *income-risk* as revenues are risky, (ii) *interest-rate* risk as interest rates can change unexpectedly, and (iii) *liquidity risk* as prices can suddenly become more elastic. A final complication is a temptation to default.

A general principle emerges from the analysis. Whether there is risk or default in the model, simply changes the details. The principle is that the problem can be studied as if the Government delegates the issuance problem to a continuum of subordinate traders. In this fictitious

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[1] There is however, a broad literature on asset pricing that considers liquidity frictions. Recent microfoundations of the price impact of issuance of different maturity is found Vayanos and Vila (2009). This model is rooted in an earlier tradition that dates back at least to Culbertson (1957) and Modigliani and Sutch (1966); for a classic application to debt management see Modigliani and Sutch (1967). Greenwood and Vayanos (2014) test the implications of a version of the preferred habitat theory of the interest rates, finding that the supply of bonds is a predictor of the interest rates the government pays. Ours is the first to bring those idease into an optimization problem and explain how that shapes the optimal distribution of debt. He and Milbradt (2014) considers xxx, Kozlowski et al. (2017), xxxx.

delegation, each trader manages the issuance of bonds of a single maturity. To do so, each trader computes an internal valuation of the bond that he manages. To compute that valuation, traders use a common discount determined by the Government. The trader then compares his valuation to the market price of that bond, which typically differ. The optimal issuance follows a simple rule:

$$\Delta\% \text{ issuance}/\text{GDP} = \text{liquidity coefficient} \cdot \Delta\% \text{ value gap.}$$

The rule states that, in a given period, the optimal issuances of a bond of a given maturity (relative to GDP) should equal the product of a value gap and a liquidity coefficient. The value gap is the difference between the market price and the internal valuation of a bond, as a percentage of market price. When there is a positive value gap, a trader would otherwise want to issue as much debt as possible, because the market price is higher than the perceived cost. There is a force that contains that desire: the liquidity impact. This force appears as a coefficient in the simple rule. This coefficient be estimated from measures of bond market turn-over rates and intermediation spreads. The higher the liquidity coefficient, the greater the issuances. As we add risk, or default, the principle remains the same, and the effects only appear in the valuations.

For this delegation approach to deliver the Government's *optimal* debt issuance, the Government must assign the *correct* discount factor. This discount factor solves a fixed-point problem in the path of expenditures: An inputed expenditure path maps into a Government discount factor. This discount factor, delivers a path for debt through the issuance rule. Ultimately, the path for debt produces a new expenditure path. In the optimal solution, both expenditure paths must coincide. This fixed point problem can be solved through an efficient algorithm and allows the study of rich transitional dynamics.

The paper analyzes the Government's problem under perfect foresight first. Then, it presents the case with risk, and finally the case with default. The Government's problem under perfect foresight already reveals important policy lessons. Without risk, the simple principle highlighs a trade off between consumption smoothing and liquidity smoothing. At steady state, this trade-off produces an optimal policy that tilts the issuance profile towards longer maturities. This is because the value-gap is higher for bonds of higher maturity. Although the Government prefers higher maturity bonds, it issues bonds of all maturities because all maturities have positive value gaps. In practice we indeed observe issuances at all maturities, and this is something that practitioners refer to as "completing the curve." In reaction to unexpected shocks, expenditure smoothing is limited liquidity smoothing. For example, in response to a low-interest rate episode the Government should issue more debt at all maturities, but tilts the profile towards longer maturities. This prescription is attenuated by a low liquidity coefficient. Another lesson is that with fixed-coupon debt and low liquidity coefficients, unexpected income shocks lead to issuance cycles. The period of those approximately equal to the longest maturity available. An

application of this perfect-foresight model is to compare, quantitatively, if models with consols and fixed-coupon bonds produce different debt levels. This exercise is important because this is the first time we test if the ubiquitous use of consols in models produces a bias in quantitative results.

When we turn to the second layer of complexity, risk, we learn that the only change in our issuance principle appears as a tile in the trader's valuations. With risk, each valuation features an additional penalty captured by a ratio of post- to pre-shock marginal utilities. Shocks that induce a drop in consumption are a force that shrinks all issuances. The ratio of marginal utilities, then governs the effect on the maturity profile. We exploit the model with risk to study to compare a hedging with a limited number of issuances versus a model high liquidity costs.

The final layer of complexity, default, alters valuations and prices because it produces an endogenous risk-premium. The effect of the risk premium is to close the valuation gap. This close-up mechanically lowers issuances at all maturities. However, the force tilts the maturity distribution towards shorter maturities, to the point that an impatient Government can end up accumulating short-term assets. This result is reminiscent of the finding in Aguiar et al. (2016), but here the force is not the debt dilution. Instead, the force that appears here is simply that default makes internal valuations and market prices more similar to each other. We connect with the literature in the next section, before proceeding to the analysis.

Literature Review

Debt management problems are classic problems. They appear in different subfields of economics; in public finance, international finance, household finance and corporate finance. Naturally, our paper relates to studies in each of these areas.

In public finance, debt management is linked to optimal taxation. A first result in Barro (1979) established that with lump-sum transfers, the timing of taxes and debt is irrelevant given an expenditure path. Lucas and Stokey (1983) studied a version of that problem with distortionary taxes and found that a govenment would wan't to structure its debt to smooth distortionary taxes. The desire to smooth distortionary taxes motivates the study introduuce an expenditure-smoothing motive as appears in our study. That classic literature was silent about the optimal-maturity choice of debt, because it assumed that the Government had access to a complete set of Arrow securities. The connection between those problems and maturity management was made by Angeletos (2002) and Buera and Nicolini (2004). Both papers obtained conditions under which a complete markets allocation could be implemented with a discrete number of bonds of different maturity. Buera and Nicolini (2004) showed that the optimal maturity management under complete markets would create unrealistic debt flows. Our paper connects with that literature because the presence of liquidity costs limit the desire to use maturity as insurance mechanisms. We also explain how in a small-open economy, those market-

completion conditions only hold for interest-rate shocks and derive a similar condition for the continuum of bonds. One direction in which the debt-management problem has been extended is to cases where the Government lacks commitment. That case was studied by Aiyagari et al. (2002).

Debt management issues are an recurrent theme in international finance. The focus of that literature has been to analyze how debt manangement is influenced by the possibility of default. Two seminal contributions are Eaton and Gersovitz (1981); Cole and Kehoe (2000) which studied dynamic models endogenous sovering default episodes and self-fullfilling crises.[2] Both papers were silent about maturity. In a three-period environment Bulow and Rogoff (1988) allerted that countries would prefer to issue short-term debt because long-term debt can be dilluted once it is issued. In quantitative models, maturity management were introduced recently by Hatchondo and Martinez (2009); Chatterjee and Eyigungor (2012); Arellano and Ramanarayanan (2012) and Arellano and Ramanarayanan (2012). Hatchondo et al. (2016), for example, studies debt dilution in a model of long-term debt and finds that a substantial amount of spreads is due to debt dilution. Bianchi et al. (2012) study a model where the Government can issue long-term liabilities and hold short-term assets. More recently Aguiar et al. (2016) found a stronger version of the Bulow and Rogoff insight: when there is only default risk, they found that once long-term debt is issued it should only be let to expire, and the Government should only adjust the short-term debt. Bocola and Dovis (2016), build on Cole and Kehoe (2000), and studies the response of the maturity structure with respect to fundamental and self-fulfilling debt crises. They use that model to gauge whether countries perceive the possibility of self-fulfilling crises. For a recent review of the sovereign debt literature see Aguiar and Amador (2013). Broner et al. (2013) study a group of emerging economies and find that the average maturity decreases during recessions.

We also connect with maturity management in corporate finance studies. The seminal contribution in this area is Leland and Toft (1996). Chen et al. (2012) studies optimal debt maturity in the presence of debt dilution and liquidity costs. Our introduction of liquidity costs is similar to theirs. A number of other studies have modeled a price impact of issuances. The preferred habitat theory of the interest rates dates back at least to Culbertson (1957) and Modigliani and Sutch (1966); for a classic application to debt management see Modigliani and Sutch (1967). A recent micro-foundation of price impact for each maturity is in Vayanos and Vila (2009). Greenwood and Vayanos (2014) test the implications of a version of the preferred habitat theory of the interest rates, finding that the supply of bonds is a predictor of the interest rates the government pays. Krishnamurthy and Vissing-Jorgensen (2012) document that changes in the supply of Treasury securities have an impact over a variety of spreads.

On the technical front, there are a number of recent papers that study infinite-dimensional

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[2] Quantitative implementations of those models appear in Aguiar and Gopinath (2006) and Arellano (2008).

control problems. For a general treatment of mean-field games see Bensoussan et al. (2016). Lucas Jr and Moll (2014) study a problem with heterogeneous agents that allocate time between production and technology generation. Nuño and Moll (2015) study optimal policy under incomplete markets and a continuum of agents. Nuño and Thomas (2016) study optimal monetary policy with and without commitment in a model with incomplete markets. Our paper also develops similaar tools to study an infinite-dimensional control problem with another form of limited commitment: the possibility of default.[3]

2 The General Model

We begin with the exhibition of the most general environment first, to give the reader an idea of the challenge that lies ahead. When we move to the characterization, we firs analyze the problem under perfect foresight, then add risk and then add default.

Environment. We consider a continuous-time open economy. There is a single, freely-traded consumption good. The economy features a benevolent Government that trades a continuum of bonds of different maturity. Bonds are issued to foreign investors. The Government can default and is excluded from markets thereafter.

Exogenous Processes. There are four exogenous processes that induce a different sources of risk. The exogenous state, $X(t)$, is the vector of the four exogenous processes: $y(t) \times y^a(t) \times \bar{r}(t) \times \bar{\lambda}(t)$. Each element $x(t) \in X(t)$ is a mean-reverting process with Poisson jumps, and thus is right-continuous.[4] In particular, each $x(t)$ follows a mean-reverting process along its continuous path:

$$\dot{x}(t) = -\alpha^x(x(t) - x_{ss})$$

where α^x captures the speed of reversion to the mean and x_{ss} a the steady-state value. Each process is affected by a common Poisson event with arrival rate ϕ. If the Poisson event occurs, at time t, $x(t^-)$ is immediately reset to some new $x(t^+) \sim F^x(\cdot|X(t^-))$ and F^X, is the joint distribution of the vector $X(t)$. We explain each process:

   a. $y(t)$ the Government revenues and captures income risk.

   b. $\bar{r}(t)$ is international short-term rate and captures interest-rate risk.

   c. $\bar{\lambda}(t)$ is a liquidity cost coefficient and captures liquidity risk.

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[3]In addition, note that the concept of finite-dimensional Markov Perfect Stackelberg Equilibria has been studied both in continuous and discrete time. See for example Başar and Olsder (1998). An example in Economics of Markov Stackelberg equilibrium is Klein et al. (2008).

[4]The stochastic process is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

d. y^D the Government revenues if the Government has defaulted in the past.[5]

Government. Preferences over expenditure paths $c(t)$ are given by

$$\mathcal{V}_0 = \int_0^\infty e^{-\rho t}\mathcal{U}\left(c\left(t\right)\right) dt,$$

where $\rho \in (0,1)$ is the discount factor and $\mathcal{U}\left(\cdot\right)$ is an increasing and concave utility.[6]

The Government trades the continuum of bonds with foreign investors. Bonds differ in their time-to-maturity $\tau \in (0,T]$. Here, T is the maximum maturity available—T is exogenous. Each bond pays a coupon δ per instant of time, prior to maturity, an 1 good when the principal matures—$\tau = 0$. The outstanding stock of bonds owed by the Government at time t with a time-to-maturity τ is $f(\tau, t)$. We call $f(\tau, t)$ the debt profile. The law of motion $f(\tau, t)$ is given by the following Kolmogorov-Forward equation

$$\frac{\partial f}{\partial t} = \iota\left(\tau, t\right) + \frac{\partial f}{\partial \tau}. \tag{2.1}$$

The intuition behind the equation is that, for a maturity τ and time t, the change in the mass of bonds of that maturity, $\partial f/\partial t$, equals the issuance at that maturity, $\iota(\tau, t)$, plus the netflow of pre-existing bonds $\partial f/\partial \tau$ —there's mass outflow towards less maturities and an inflow from the mass at higher maturities.[7]

Issuances, $\iota(\tau, t)$, are chosen from a space of functions $\mathcal{I}: [0,T] \times (0,\infty) \to \mathbb{R}$ that meets some technical conditions.[8] By construction, $f(T^+, t) = f(0^-, t) = 0$. Finally, $f_0(\tau)$ is the initial stock of of debt of maturity τ so $f(\tau, 0) = f_0(\tau)$. The Government's budget constraint is:

$$c\left(t\right) = y\left(t\right) - f\left(0, t\right) + \int_0^T \left[q\left(\tau, t, \iota\right)\iota\left(\tau, t\right) - \delta f\left(\tau, t\right)\right] d\tau. \tag{2.2}$$

Here $f(0, t)$ is the principal repayment, $\delta \int_0^T f d\tau$ are coupon payments and $\int_0^T q\iota d\tau$ the funds received from debt issuances at all maturities. Finally, $q(\tau, t, \iota)$ is the issuance price of bond vintage of maturity τ at date t.

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[5]This process captures the incentives to default as in Arellano (2008) or in Aguiar et al. (2016).

[6]This inpretation follows a public finance view. The curvature in expenditures follows from welfare losses from distortionary taxes. In international finance, the interpretation is that the Government controls national savings and consumption $\mathcal{U}\left(c\left(t\right)\right)$ is the household's utility.

[7]A derivation from it's discrete time analogue is presented in Appendix xxx. The solution to this equation is:

$$f(\tau, t) = \int_\tau^{min\{T, \tau+t\}} \iota(t + \tau - s, s)ds + \mathbb{I}[T > t + \tau] \cdot f(\tau + t, 0),$$

and can be shown via the method of characteristics.

[8]In particular $\mathcal{I} = L^2\left([0,T] \times (0,\infty)\right)$ is the space of functions on $[0,T] \times (0,\infty)$ with a square that is Lebesgue-integrable.

The Government can decide to default. If the Government defaults, it gets a present utility:

$$V^D = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} \mathcal{U}\left(y^D(t)\right) ds \right]. \tag{2.3}$$

Obviously, V^D is a stochastic process fully determined y^D. Default is an absorbing state. Prior to a default, the generic Goverment problem (**GP**) at t is:

Problem 1 *The* **GP** *is:*

$$V([f(\cdot,t)], X_t) = \max_{\{\iota(\cdot)\} \in \mathcal{I}} \mathbb{E}_t \left[\int_t^{t+\tau^D} e^{-\rho(s-t)} \mathcal{U}(c(s)) ds + e^{-\rho\tau^D} V^D(X_t) \right] \tag{2.4}$$

subject to the law of motion of debt (2.1) and the budget constraint (2.2).

Here $V(X_t, [f(\cdot,t)])$ is the optimal value *functional*, which maps a debt profile $f(\cdot,t)$ at time t into a real numbers. The term τ^D is the *default time*.[9] The default time is the first time when the value of default is higher than the value prior to default $\tau^D \equiv \min\left\{\tau, V^D(X_t) > V(X_t, [f(\cdot,t)])\right\}$.

International Investors. The Government sells bonds to competitive risk-neutral international investors at the issuance price $q(\tau,t,\iota)$. This issuance price has two separate components, a market price and a liquidity cost:

$$q(\tau,t,\iota) = \psi(\tau,t) + \lambda(\tau,t,\iota).$$

The first component, $\psi(\tau,t)$, is the market price of the domestic bond. This market price $\psi(\tau,t)$ is has the form:

$$\psi(\tau,t) = \mathbb{E}_t \left[\underbrace{\mathbb{I}_{[\tau<\tau^D]} e^{-\int_t^{t+\tau} \bar{r}(u)du}}_{\text{Principal}} + \underbrace{\int_t^{t+\min\{\tau^D,\tau\}} e^{-\int_t^s \bar{r}(u)du} \delta ds}_{\text{Coupon Payments}} \right]. \tag{2.5}$$

This equation discounts coupons at an interantional rate interest-rate $r(t)$ and accounts for a possible default. To explain the discounting, consider the simpler case when τ^D is large. The equation becomes:

$$\psi(\tau,t) = \mathbb{E}_t \left[e^{-\int_t^{t+\tau} \bar{r}(u)du} + \delta \int_t^{t+\tau} e^{-\int_t^{s+\tau} \bar{r}(u)du} ds \right]. \tag{2.6}$$

Under this equation, the coupons and principal with the stochastic short-term rate $\bar{r}(t)$. Bonds thus satisfy a non-arbitrage condition.

The second component in the issuance price, $\lambda(\tau,t,\iota)$, represents a liquidity cost associated

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[9] The latter is a stopping time with respect to the filtration $\{\mathcal{F}_t\}$.

with issuing—or purchasing— ι bonds of maturity τ at date t. The liquidity cost λ is convex in ι and the idea is that it captures multiple forces. The next section, presents a microfoundation for this cost.[10]

Definition 1 *(Equilibrium) We study a Markov Equilibrium with state variable $f(\tau,t)$; it is defined as follows. A Markov equilibrium is a value functional $V[f(\cdot,t)]$, a issuance policy $\iota(\tau,t,f)$, bond prices $q(\tau,t,\iota,f)$, a stock of debt $f(\tau,t)$ and a consumption path $c(t)$ such that: 1) Given $c(t)$, $q(\tau,t,\iota f)$ and $f(\tau,t)$ the value functional satisfies government problem (2.4) and the optimal control is ; 2) Given $\iota(\tau,t,f)$ the debt stock $f(\tau,t)$ evolves according to the KPE equation (2.1); 3) Given $\iota(\tau,t,f)$, $q(\tau,t,\iota,f)$, $f(\tau,t)$ and $c(t)$ the budget constraint (2.2) of the government is satisfied.*

For the rest of the paper, we adopt a particular notation.

Notation. When we refer to deterministic steady-state, we suppress the time subscripts and denote the steady state values with the sub-index *ss*. We denote asymptotic values as $t \to \infty$ with the sub-index ∞. For example, when the exogenous variables, income, liquidity costs and rates are at steady state we denote them by: $y(t) = y_{ss}$, $\chi(\tau,t) = \chi_{ss}(\tau)$ and $\bar{r}(t) = \bar{r}_{ss}$.

So far, we left a liquidity cost function as general as we can. Next, we present a microfoundation before we proceed to the analysis.

2.1 A Model of Liquidity Costs

In this section we present a microfoundation for the liquidity cost function. The building block is a wholesale-retail model of the bond market. We assume that at each date t, the Government auctions $\iota(\tau,t)$ bonds of the maturity τ. The size of this auction, corresponds to the control variable in the problem of the previous section **GP**.[11] We assume that the participants in that auction are a continuum of investment bankers (henceforth, bankers). Bankers buy large stocks of bonds in the auction (the wholesale market), and then offload their inventories of bonds to international investors (investors) in a secondary (retail) market. The international investors have a discount factor equal to $\bar{r}(t)$, which we introduced earlier. Liquidating the bond inventories takes time, as we explain next.[12] International investors are risk-neutral and have a discount factor equal to $\bar{r}(t)$, the international short-term rate. Thus each investor is willing to pay $\psi(\tau,t)$ for each bond. Bankers, on the other hand, have a higher cost of capital which equals $\bar{r}(t) + \eta$.[13] In the retail market, bankers are continuously contacted by international in-

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[10]Note that the fact that the bond obtains $q(\tau,t,\iota) < \psi(\tau,t)$ does not mean that there is an arbitrage.

[11]One alternative way to provide a micro-foundation is found is the preferred habitat model of Vayanos and Vila (2009).

[12]Implicity, we assume free entry into the auction, but than only investment bankers can participate. There is a continuum of bankers but the vintage is assigned to small number. This is effectively as assuming that bankers participate in only one auction.

[13]One interpretation is that bankers fund the purchase promising a bond of identical payoff structure with short-term rate $\bar{r}(t) + \eta$. Another is that bankers have holding costs.

vestors. In particular, there is a constant flow of contacts μy_{ss} per unit of time.[14] Each contact results in an attempt to purchase bonds. We assume that each investor buys an infinitesimal amount of bonds from the investment bank. This implies that in an interval Δt the total amount of debt sold is $\mu y_{ss} \Delta t$. Upon a contact, we assume that bankers extract all the surplus from the international investors.[15]

The key friction in this microfoundation is that it takes time for investment banks to liquidate their bond portfolios. The larger the auction size, the longer teh average time to sell each bond. Because investment banks have higher discount factors, a bigger issuance depresses the price towards a price that discounts the bond using $\bar{r}(t) + \eta$, instead of $\bar{r}(t)$. As the size of the auction vanishes, the opposite occurs: the price converges to $\psi(\tau, t)$, the price obtained using $\bar{r}(t)$ as a discount.

We present the solution to the auction price in more detail in Appendix A. Although there we present an exact solution to this price, the following first-order approximation yields a convenient functional form for the liquidity-cost function.

Proposition 1 *(Approximation to the Liquidity Cost Function)*

A first-order Taylor expansion around $\iota = 0$ yields a linear auction price:

$$q(\iota, \tau, t) \simeq \psi(\tau, t) - \frac{1}{2} \frac{\eta}{\mu y_{ss}} \psi(\tau, t) \iota.$$

Thus, the approximate liquidity cost function is $\bar{\lambda} = \eta / \mu y_{ss}$ and $\chi(\tau, t) = \psi(\tau, t)$.

The calculations are also found in Appendix A. The main takeaway is that for small issuances relative to the order flow, the liquidity cost function is approximately proportional to the spread and inversely related to the order flows. The formula is remarkably parsimonious. Both spreads and order flows are objects we use in a further calibration.

Discussion. An important assumption is that there are no congestion externalities. This means that the contact rate is independent of the outstanding amount of bonds of a given maturity. For example, a banker that participated in the 10 year auction 5 years ago, is effectively selling a 5 year bond. Our assumption is that the banker's contact rate is independent of how many 5 year bonds are being issued now, or are outstanding. If the Government is part of a much larger international bonds market, or if the Government issues

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[14]That is, we assume that the flow of customers is proportional to the size of the country.

[15] As a side note, the if $\iota < 0$, the government issues assets. The model be entirely reversed as saying that the banker sells the bond to the country, and then the banker closes the position lending at higher rates.

3   Perfect Foresight

We begin with the study of the problem of the Government that faces a deterministic path for $\bar{r}(t)$ and output $y(t)$, and is given an initial condition $f_0(\tau)$. For now, the Government has no option to default—$y_t^D = 0$. This perfect-foresight environment is instructive to understand the richer versions with risk and default, but it is also interesting in its own right.

In this perfect-foresight case, we can characterize the steady-state debt distribution by analytic expressions. We also characterize transitional dynamics as a fix point problem in the expenditure path. We present application in the end of the section.

3.1   Perfect Foresight Problem and its Necessary Conditions

In the deterministic problem, the price of a bond by the international investor is given by:

$$\psi(\tau, t) = e^{-\int_t^{t+\tau} \bar{r}(u)du} + \delta \int_t^{t+\tau} e^{-\int_t^{s} \bar{r}(u)du}ds. \tag{3.1}$$

This price, has a PDE representation:

$$\bar{r}(t)\,\psi(\tau, t) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}$$

with boundary conditions $\psi(0, t) = 1$.[16]

The Government, solves a perfect-foresight problem (**PF**):

Problem 2 *(Perfect Foresight) The **PF** problem is:*

$$V[f] = \max_{\{\iota(\cdot)\} \in \mathcal{I}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho(s-t)} \mathcal{U}\left(c\left(s\right)\right) ds \right] \tag{3.2}$$

subject to the law of motion of debt (2.1), the budget constraint (2.2), an initial condition f_0, and debt prices (3.1).

Although the **PF** problem is a special case of the problem with risk of the next section, we solve the two problems through different techniques. We solve the **PF** problem adapting optimal-control techniques. For that, we set an infinite-dimensional Lagrangian. The problem with risk is solved through dynamic programming, which involves exploiting some results from functional analysis. These two proofs, allow us to make a transparent connection between the infinite-dimensional Lagrange multipliers and the derivative of the value functional in the dynamic program. The problem with default uses a mix of both techniques.

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[16]The solution can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. The PDE for the bond, has the form of a Hamilton-Jacobi-Bellman equation without a choice variable.

The problem's Lagrangian is:

$$\mathcal{L}\left[\iota, f\right] = \int_0^\infty e^{-\rho t} U\left(y(t) - f(0,t) + \int_0^T \left[q(t,\tau,\iota)\,\iota(\tau,t) - \delta f(\tau,t)\right] d\tau\right) dt$$
$$+ \int_0^\infty \int_0^T e^{-\rho t} j(\tau,t)\left(-\frac{\partial f}{\partial t} + \iota(\tau,t) + \frac{\partial f}{\partial \tau}\right) d\tau dt,$$

where we substituted out consumption in the objective of **PF** using the budget constraint.[17] Since the Lagrange multipliers multiply objects equal to zero, maximizing the Lagrangian amounts to maximizing the objective, just as in standard control. The necessary conditions can be obtained by a classic variational argument: the condition that at the optimum, the optimal issuance and debt paths cannot be improved. Taking an infinitesimal variation over the control ι cannot produce an increase in the Lagrangian, no improvement holds if and only if:

$$\underbrace{U'(c)\left(q(t,\tau,\iota) + \frac{\partial q}{\partial \iota}\iota(\tau,t)\right)}_{\text{Marginal Benefit}} = \underbrace{-j(\tau,t)}_{\text{Marginal Cost}}.$$

This necessary condition is intuitive: the issuance (or buyback) of debt of a given τ and t, produces a marginal cost and a marginal benefit. Both margins must be equated. The marginal benefit is the marginal utility obtained from the increase in expenditures. The marginal increase in expenditures is the price, $q(t,\tau,\iota)$, minus the price impact of the issuance, $\frac{\partial q}{\partial \iota}\iota(\tau,t)$. The marginal cost of the issuance is summarized by the Lagrange multiplier $-j(\tau,t)$. This multiplier is in fact, the present value of the debt payments associated with that maturity and time. Hence, the multiplier captures all forward-looking information.

The forward-looking information encoded in the Lagrange multiplier comes out to surface when we derive the second necessary condition, a second step in the proof. At an optimal path, we should also be unable to improve the Lagrangian with a variation to the stock of debt. Thus, any perturbation around $f(\tau,t)$ must produce a zero value change. We show that the solution cannot be improved as long as the Lagrange multipliers j satisfy the following partial-differential equation (PDE):

$$\rho j(\tau,t) = -U'(c(t))\delta + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \quad \tau \in (0,T], \tag{3.3}$$

with terminal condition: $j(0,t) = -U'(c(t))$.

Each Lagrange multiplier is forward-looking because it takes the form of a continuous-time present-value formula. The first term, is a flow, the dis-utility $-U'(c(t))\delta$. The second term,

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[17] The differences with a standard control problem is that the state variable is a distribution, not a vector. Thus, at each point in time, there is a continuum of Lagrange multipliers (Lagrangians) and not a vector of co-states. We interpret these Lagrangians as having two dimensions: one for time and one for maturity.

$\partial j / \partial t$, captures the change in flow utility. The third term, $\partial j / \partial \tau$ captures that the bond matures with time.[18] For interpretation purposes, it is convenient to convert the multiplier $j(\tau, t)$ from utiles into a cost in consumption units. For that, define:

$$v(\tau, t) \equiv -j(\tau, t) / U'(c(t)).$$

We refer to this object as the internal valuation of the (τ, t) debt.

With this definition, we re-express the first-order condition as

$$\frac{\partial q}{\partial \iota} \iota(\tau, t) + q(t, \tau, \iota) = v(\tau, t), \tag{3.4}$$

and (3.3), into a PDE for the internal valuation:

$$r(t) v(\tau, t) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T), \tag{3.5}$$

with terminal condition $v(0, t) = 1$ and where $r(t)$ is given by:

$$r(t) \equiv \rho - \frac{U''(c(t)) c(t)}{U'(c(t))} \cdot \frac{\dot{c}(t)}{c(t)}. \tag{3.6}$$

Note that $r(t)$ is the classic formula for an instantaneous discount factor. We observe the remarkable connection the internal valuations and the market-price equations. Both the internal valuation and the price of the bond, are a net-present value of the cash-flow of each bond. The only difference in the discounting. The optimal issuances given by (3.4), depend on the spread among both valuations—q is a function of ψ.

Delegation. The characterization above allows us to interpret the optimal issuance policy through a delegation: the Government has a discount factor. Then, the Government designates a continuum of traders, one for each τ, to value its debt with his discount factor and find v. Each trader then issues debt according to (3.4). Of course, the discount factor of the Government must be internally consistent with the consumption path produced his traders issuances. This decentralization makes the interpretations of the solutions transparent, especially when we exploit the functional form for λ of the previous section.

Summary Proposition. The following proposition summarizes the discussion into a full characterization of the problem's solution:

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[18]Note that the PDE for j is the analog of the ODE of the Lagrange multiplier when the Lagrange multiplier is unidimensional. In this case, for example if the government only had access to an instantaneous bond, the PDE for the Lagrange multiplier would be given by $\rho j(t) = -\delta U'(c(t)) + \frac{\partial j}{\partial t}$.

Proposition 2 *(Necessary conditions of the **PF** problem) If a solution to **PF** exists, then:*

$$v\left(\tau,t\right) = e^{-\int_t^{t+\tau} r(u)du} + \delta \int_t^{t+\tau} e^{-\int_t^s r(u)du} ds. \tag{3.7}$$

The optimal issuance $\iota\left(\tau,t\right)$ is given by the condition (3.4). The evolution of the debt mass can be recovered from the law of motion for debt, (2.1), given the initial condition $f\left(\cdot,0\right)$. Finally, $c(t)$ and $r(t)$ must be consistent with the budget constraint (2.2).

Proof. See Appendix C.1. □

The following section uses the same approach to characterize a useful benchmark, a version of the model without liquidity costs.

3.2 Maturity Management without Liquidity Costs

It is insightful to characterize the solution without liquidity frictions. This section verifies the known result that without risk, maturity is indeterminate. Surprisingly, in a subsequent section, we demonstrate that optimal maturity profile is, by the contrary, determinate as liquidity frictions approach zero. Thus, there's a discontinuity at the limit that we study here.

Consider $\lambda\left(\iota,\tau,t\right) = 0$. The necessary conditions are still those of the previous section. Hence, (3.4) still holds but with $\frac{\partial q}{\partial \iota} = 0$. This implies that issuances are unbounded, unless $v\left(\tau,t\right) = \psi\left(t,\tau\right)$. If we combine this information with the fact that (3.7) is also a necessary condition, we conclude that it must be that $\bar{r}(t) = r\left(t\right)$. These simple observations are enough to characterize the solution without liquidity frictions.

Proposition 3 *(Optimal Policy with Liquid Debt) Assume that $\lambda\left(\iota,\tau,t\right) = 0$. Define the aggregate stock debt stock of debt, $b(t)$, by:*

$$b(t) \equiv -\int_0^T \psi\left(\tau,t\right) f\left(\tau,t\right) d\tau, \;\; \forall t \geq 0. \tag{3.8}$$

If a solution exists, then consumption growth satisfies the following ODE:

$$\bar{r}(t) \equiv \rho - \frac{U''\left(c\left(t\right)\right) c\left(t\right)}{U'\left(c\left(t\right)\right)} \frac{\dot{c}(t)}{c(t)},$$

with an integral condition:

$$b(0) = \int_0^\infty \exp\left(-\int_0^s \bar{r}(t)du\right)\left(y(s) - c(s)\right) ds.$$

Any solution $\iota(\tau, t)$ consistent with (2.1), (3.8) and

$$\dot{b}(t) = \bar{r}(t)b(t) + y(t) - c(t), \text{ for } t > 0, \tag{3.9}$$

is an optimal solution.

Proof. See Appendix C.2. □

This solution is also the solution of a standard consumption-savings problem with a single instantaneous bond with initial condition $b(0)$. The result can be anticipated because all bonds are redundant: the yield curve is arbitrage free so there is no way to structure debt to reduce the cash-flow payments given an initial inflow.[19] Next, we characterize the dynamics in presence of liquidity costs and how the portfolio is determinate even as liquidity frictions vanish.

3.3 Characterization

Proposition 2 characterizes the solution to the **PF** problem, but it does not present an explicit formula. For that purpose, we use CRRA utility with coefficient σ and adopt the functional form of the liquidity costs in Prosposition 1. With CRRA utility, $r(t) = \rho + \sigma \dot{c}(t)/c(t)$. The optimal-issuance condition (3.4) translates into:

$$\iota(\tau, t) = \frac{1}{\bar{\lambda}} \frac{\psi(\tau, t) - v(\tau, t)}{\psi(\tau, t)} \tag{3.10}$$

This is the simple rule discussed in the Introduction. As we explained, the spread $\psi(\tau, t) - v(\tau, t)$ is a like an arbitrage available to the fictitious traders. Liquidity costs, contain the desire to issue bonds when the spread is positive. Naturally, issuances fall with a higher $\bar{\lambda}$. The parameter space can be divided into two regions where the dynamics of the solution differ depending on $\bar{\lambda}$. We explain this property next.

Asymptotic Behavior. When the international rate is at steady state, the price function is independent of time:

$$\psi_{ss}(\tau) = \delta \frac{1 - e^{-\bar{r}\tau}}{\bar{r}} + e^{-\bar{r}\tau}, \tag{3.11}$$

where for $\delta = \bar{r}_{ss}$ yields a price $\psi(\tau) = 1$. To set ideas, we let $\delta = \bar{r}_{ss}$.

Proposition 4 *Consider a steady state for the exogenous variables. Then, there exists a steady state in problem **PF** if and only if $\bar{\lambda} > \bar{\lambda}_o$ for some $\bar{\lambda}_o$. If instead, $\bar{\lambda} \leq \bar{\lambda}_o$, there is no steady state, but $r(t) \to r_\infty(\bar{\lambda})$. The assymptotic discount factor $r_\infty(\bar{\lambda})$, is increasing and continuous in $\bar{\lambda}$ with bounds*

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[19]For that reason, we obtain the discount factor of the Government must equal the international rate. Obviously, the discount factor determines consumption growth, and consumption determines the aggregate stock of debt, although not the composition.

$r_\infty\left(\bar{\lambda}_o\right) = \rho$ and $r_\infty\left(0\right) = \bar{r}_{ss}$. Consumption decreases asymptotically at an exponential rate $\bar{r}_{ss} - \rho$. The asymptotic distribution of debt is obtained from $r_\infty\left(\bar{\lambda}\right)$. as the planner's discount factor.

Proof. See Appendix C.3. ☐

A more detailed version of this Proposition 4 is found in Appendix **C.3**. This detailed version includes a formula for $\bar{\lambda}_o$ and for the asymptotic values of the variables in problem **PF**. In the case where liquidity costs are high, i.e. when $\bar{\lambda} > \bar{\lambda}_o$, the steady state has an analytic expression. Let's begin with this case. Proposition 4 establishes that there's a steady state. Consequently, $r_{ss} = \rho$, because consumption doesn't grow in a steady state. With a constant discount factor, steady state valuations satisfy:

$$v_{ss}(\tau) = -\left(\delta\frac{1 - e^{-\rho\tau}}{\rho} + e^{-\rho\tau}\right), \tag{3.12}$$

which is the same formula as (3.11), but using ρ instead of $\bar{r}$. Then, the issuances at steady state, $\iota_{ss}\left(\tau\right)$ follow from (3.10), and the debt outstanding is given by $f_{ss}(\tau) = \int_\tau^T \iota_{ss}(s)ds$. These are all paper and pencil formulas.

Figure 3.1 displays a typical steady state when $\bar{\lambda} > \bar{\lambda}_o$ and $\rho > \bar{r}$. The key object is the distance between $\psi_{ss}(\tau)$ and $v_{ss}(\tau)$ because this spread governs the optimal issuance policy. The figure displays higher issuances at longer maturities. The reason is that the differences in valuations are higher for longer maturities, because the gap in discount rates gets compounded for longer horizons. The distribution of debt, takes the opposite shape. With standard debt, when issuances at steady state are positives for all maturities, there is always a bigger stock of short-term debt, simply because of accounting: maturing long-term debt becomes short term debt but not the other way around. This is obvious from the expression for $f_{ss}(\tau)$, but is a phenomenon, that does not appear if the Government were to issue consols. In addition, as we increase the spread $(\rho - \bar{r})$, the maturity profile shifts towards longer maturities and the overall stock of debt increases.[20]

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[20]We can also conclude that when $\rho = \bar{r}$, steady state debt is zero for all maturities. In a consumption-savings problem with liquidity costs, the stock of debt at steady state is determined by the initial conditions. The result also allow us to There is a threshold value, precisely $\bar{\lambda}_o$, such that the steady-state level of debt consistent with this solution is zero. At the point where $\bar{\lambda}$ crosses the threshold value, the nature of the solution changes.
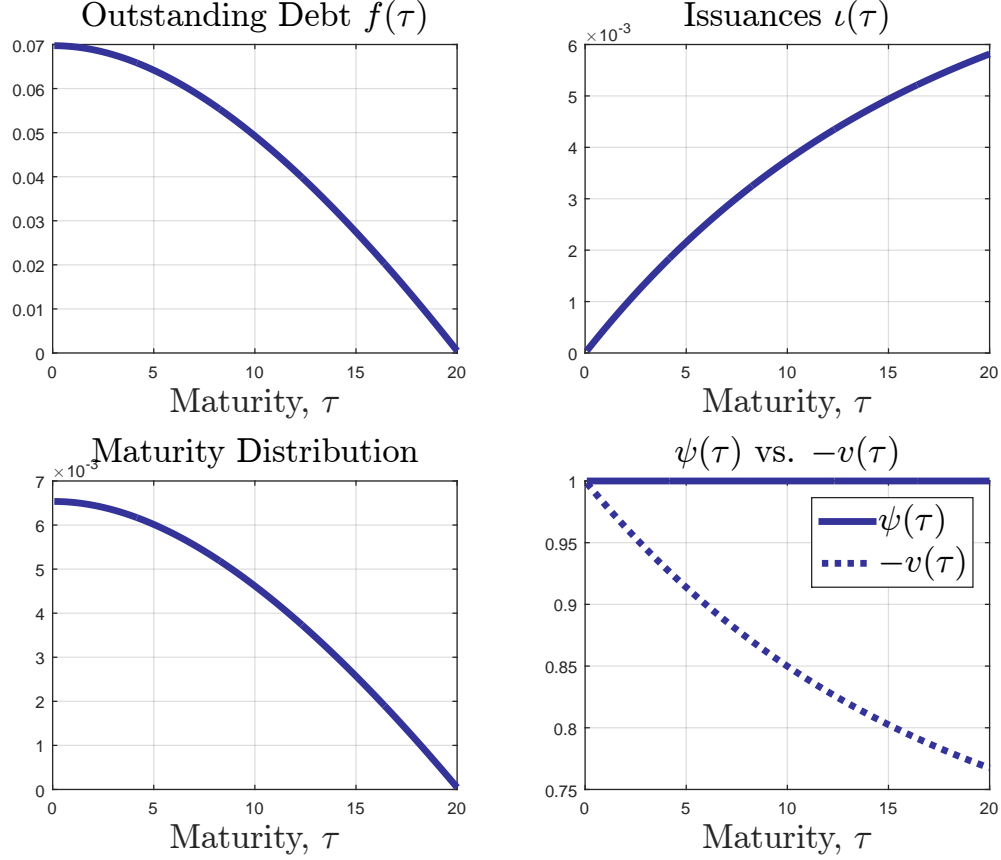
Figure 3.1: The figure describes the steady state values of debt, issuances, maturity distribution and the wedges in valuations.

Let's now turn to the opposite case. Proposition 4 states that when $\bar{\lambda} \leq \bar{\lambda}_o$ and $\rho > \bar{r}$, there is no steady state. However, the asymptotic behavior of the economy can be characterized. As liquidity coefficient $\bar{\lambda}$ crosses the threshold value, the nature of the solution begins to look closer the solution without liquidity costs. Without liquidity costs, $\rho > \bar{r}$, consumption converges to zero at an exponential rate determined by the spread $\rho > \bar{r}$. A similar result holds here, except that an asymptotic discount factor lower than ρ emerges, and is a function of $\bar{\lambda}$. This discount factor $r_\infty(\bar{\lambda})$ decreases once liquidity crosses the threshold $r_\infty(0) = \bar{r}$. Figure 3.2 compares the asymptotic behavior of the solution, as we vary $\bar{\lambda}$. We also observe how consumption and the discount factor converge to zero and $\bar{r}$, as liquidity falls. Naturally, issuances and debt, increase as liquidity costs fall. In fact, we can characterize the solution as $\bar{\lambda} \to 0$.
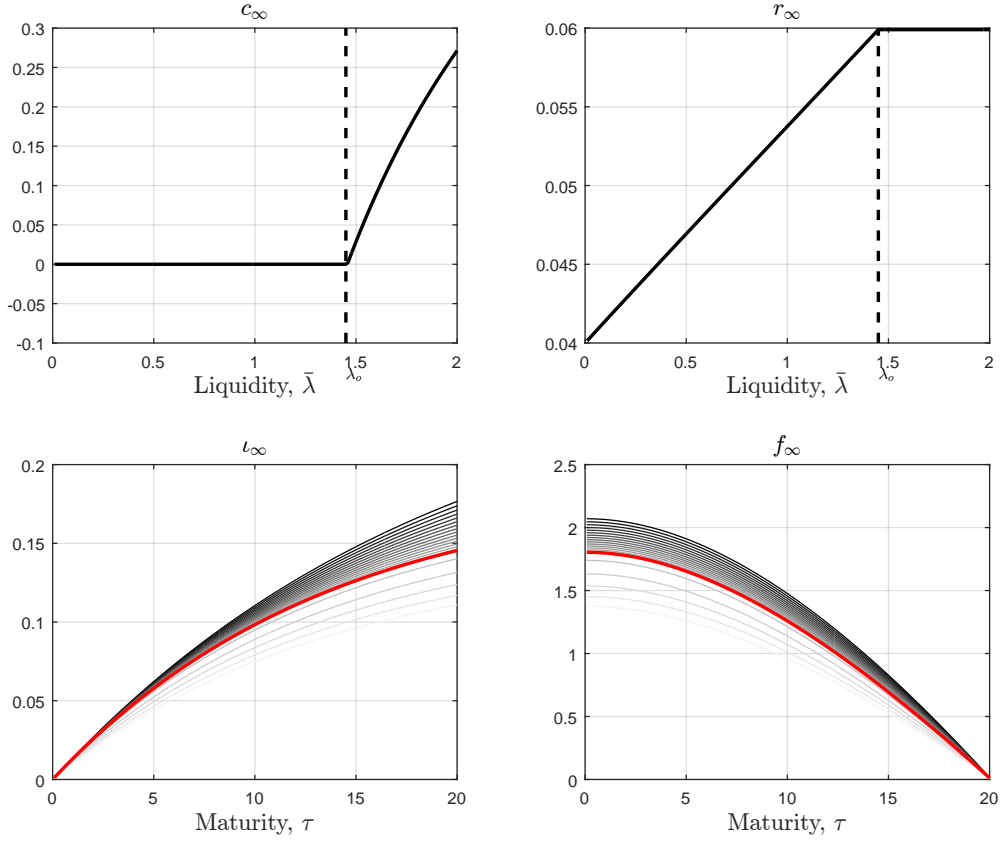
Figure 3.2: The figure describes the asymptotic behavior for multiple $\bar{\lambda}$ values.

Limiting Distribution as Liquidity Costs Vanish. The limiting behavior as liquidity costs vanish is established in the following proposition:

Proposition 5 *In the limit as liquidity costs vanish, $\bar{\lambda} \to 0$, the optimal issuance is*

$$\iota_\infty^*(\tau) = \lim_{\bar{\lambda} \to 0} \iota_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{1 + [-1 + (\bar{r}/\delta - 1)\bar{r}\tau] e^{-\bar{r}\tau}}{1 + [-1 + (\bar{r}/\delta - 1)\bar{r}T] e^{-\bar{r}T}} \frac{\chi(T)}{\chi(\tau)} \kappa,$$

where constant $\kappa > 0$ is such that

$$y - f_\infty^*(0) + \int_0^T [\iota_\infty^*(\tau)\psi(\tau) - \delta f_\infty^*(\tau)] d\tau = 0,$$

and $f_\infty(\tau) = \int_\tau^T \iota_\infty^*(s) ds$.

Proof. See Appendix C.4. □

Proposition 5 gives us the distribution of maturity as the liquidity cost parameter vanishes. As discussed, this result differs from the case where liquidity costs are actually zero because in that case, this distribution is undetermined. Thus, there is a discontinuity at the perfectly liquid limit, since for any arbitrarily small cost, the distribution is determined. Vanishing liquidity costs can be employed as a s*election device* in order to break the indeterminacy problem. Notice how the limiting distribution is only a function of the bond parameters (δ, T), the riskless rate $\bar{r}$ and the income y.

Transitions. A transition to a steady state (or an asymptotic limit) is a fixed point problem in $c(t)$. A given $c(t)$ taken as an input, will a discount factor. The discount factor produces valuations that determine issuance rates $\iota(\tau, t)$. A family of issuance rates produces a family of debt distributions indexed by time, and the budget constraint produces a consumption path, $c(t)$. A transition is a fixed point where the input and output are identical. Appendix E presents the numerical algorithm we use throughout the paper to construct transitions.

The Dual. The solution to the **PF** is also the solution to a cost minimization problem: given a desired consumption path $c(t)$, minimize the net-present value of resources financial expenses. Mathematically, the dual problem (**DP**) is given by:

Problem 3 *(Dual Problem) The **DP** is:*

$$\min_{\{\iota(\tau,t)\}\in\mathcal{I}} \int_0^\infty e^{-\int_0^t r(s)ds} \left(f(0,t) + \int_0^T \delta f(\tau,t)\,d\tau - \int_0^T q(\tau,t,\iota)\iota(\tau,t)d\tau \right) dt$$

where $r(t)$ is given by (3.6), and the minimization is subject to the law of motion of debt (2.1), an initial condition $f(\cdot, 0)$, and debt prices (3.1).

In the problem, the Government's time discount is given by the consumption path. The object in parenthesis are the (net of inflow) financial expenses. We have the following result:

Proposition 6 *Suppose that for a given income path $y(t)$ and initial debt f_0 the solution to **PF** is $\{c^*(t), \iota^*(\tau, t)\}$. Then, $\{\iota^*(\tau, t)\}$ solves **DP** given the path $c^*(t)$ and an initial debt f_0. The solution to **DP** satisfies the budget constraint (2.2) given $y(t)$.*

Proof. See Appendix D. □

The Proposition establishes that the (**DP**) problem can be thought of as a cost minimization problem that includes the price impact. In practice, treasury departments in charge of debt management have the objective in their mandates.

3.4 Applications of the PF Model

Calibration. We provide a calibration in all of the applications. We set xxx. All quantities are expressed in percentage of the steady state output (that is equal to 1). In steady state, the country devotes 6 percent of GDP to debt service; 4.4 percent of GDP to the payment of bond principals and 1.6 percent of GDP to coupon payments. Liquidity costs, that in the current calibration are 0.3 percent of GDP, that is, about 5% of total financial expenses. New debt issuance's are also 4.4. of GDP, since at steady state, they compensate for the payment of the principal. Consumption is 97.6 percent of GDP.

Debt Management after Unexpected Shocks

Coming back to the solution of **P1** in this section we want to illustrate numerically two main forces that drive the solution: consumption smoothing versus the smoothing of adjustment costs. We study a permanent and unexpected shocks to output and the interest rate that revert to steady state These transitional dynamics teach us new lessons: issuance cycles and consumption vs. price smoothing.

Unexpected Output Shock. In Figures 3.3 and 3.4 we analyze the response of issuances, consumption and total debt from a shock to output of 5% that reverts to steady state. The main take out is that to smooth the shock the government increases issuances on impact, and this will generate a wave of payments concentrated in T years. Upon impact, we observe two things. We see an immediate increase of issuances and a pronounced increase on impact of the internal discount factor. The liquidity cost prevents a perfect smoothing of consumption. This is why the internal discount factors jump. Also, there is a cycle of payments. As the initial vintage of borrowings matures, and it is particularly pronounced for long-term bonds, consumption growth slows down, but then accelerates again as the wave passes. This is an interesting phenomenon because it suggests that in presence of liquidity costs, we should expect waves of debt refinancing.

Figure 3.3: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to output of 5%.

Figure 3.4: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to output of 5% when *T*=30 years.

Unexpected Interest Rate Shock. In Figures 3.6 and 3.7 we analyze the response of issuances, consumption and total debt from a shock to the interest rate, that goes to zero, and returns to steady state. We compare the responses when $\sigma = 2$ and $\sigma = 0$. When the IES is not infinite, the model shows that when rates are unusually low, the country increases it's borrowing. This is captured by a spike in consumption beyond it's steady state level. Then, as rates begin to increase, the issuance rate declines. Eventually, there's a period low consumption were debt is being repaid. The reason for this repayment phase is the liquidity cost. As rates return to normality, while the stock is higher due to the past issuances, the country is making higher interest and principal payments, which take consumption to a lower level than at steady state. As the debt is repaid and issues return to steady state, consumption converges back to steady state. Turning on a consumption smoothing motive tampers this effect. There's a trade-off between exploiting the low interest rate environment and smoothing consumption.
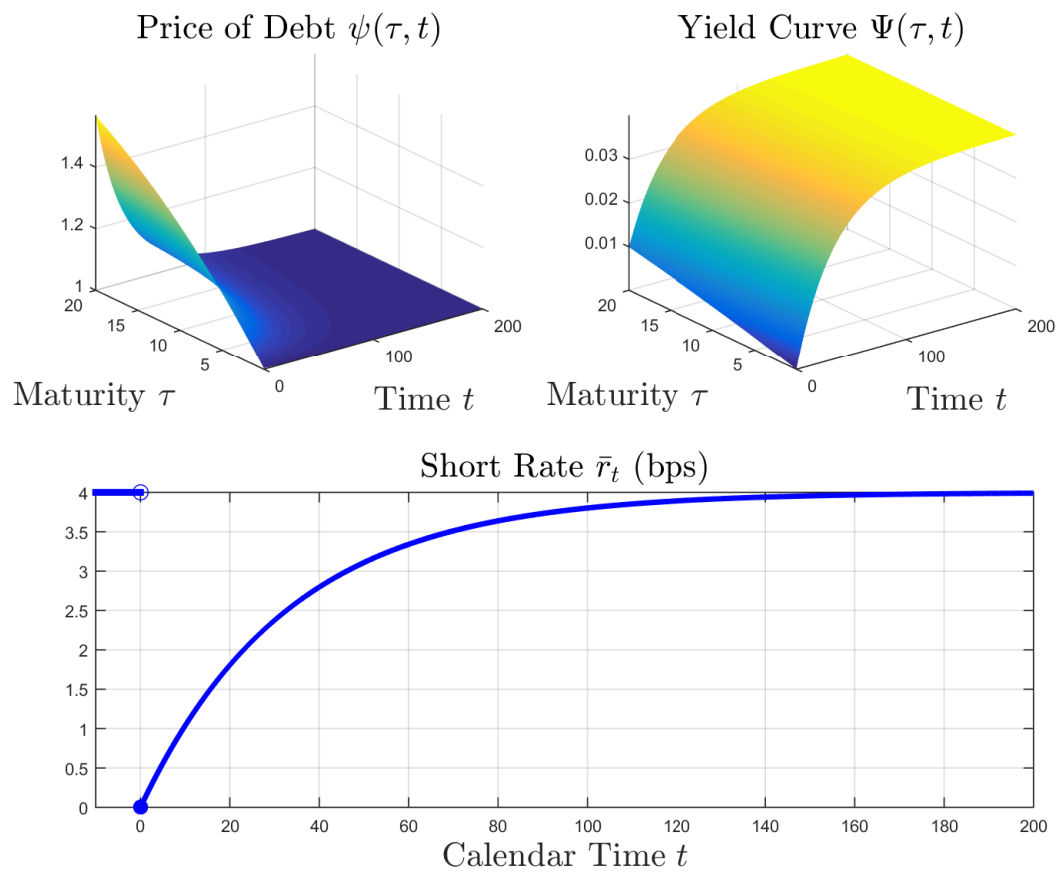
Figure 3.5: The figure describes the response of the yield curve to a shock in the short rate.
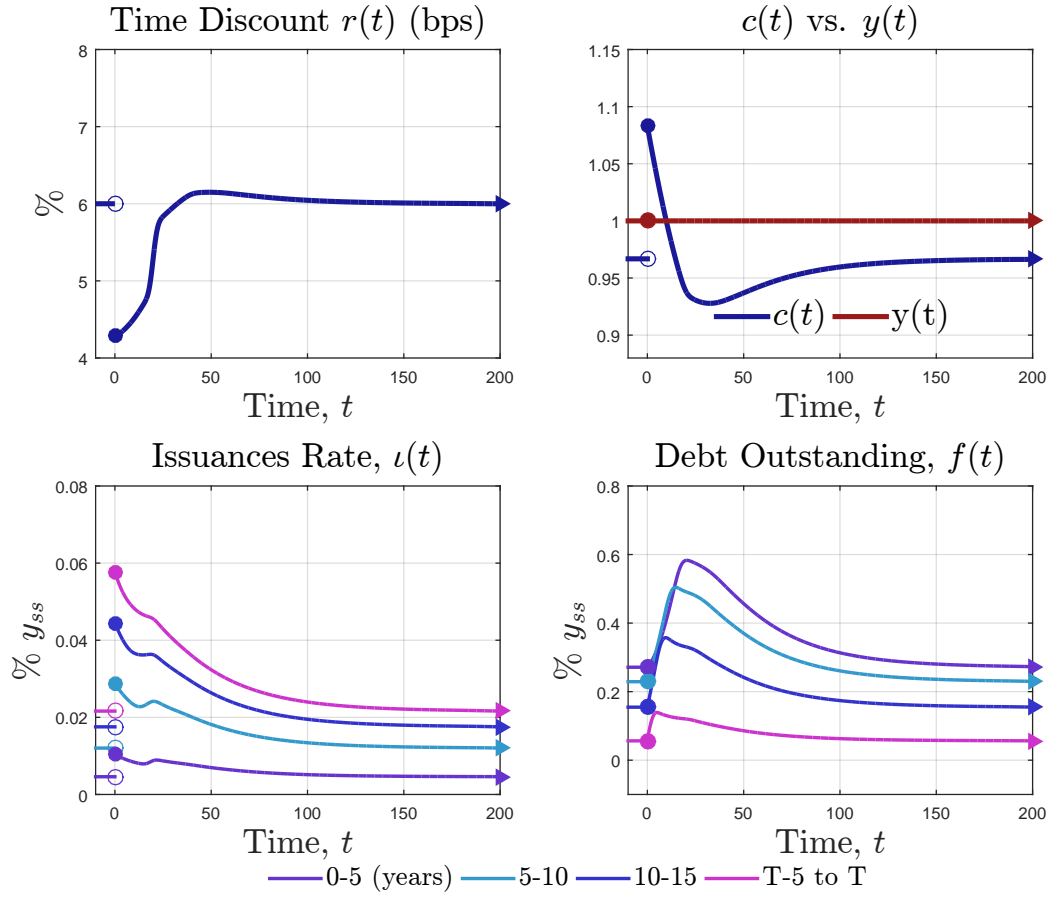
Figure 3.6: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to the short interest rate.
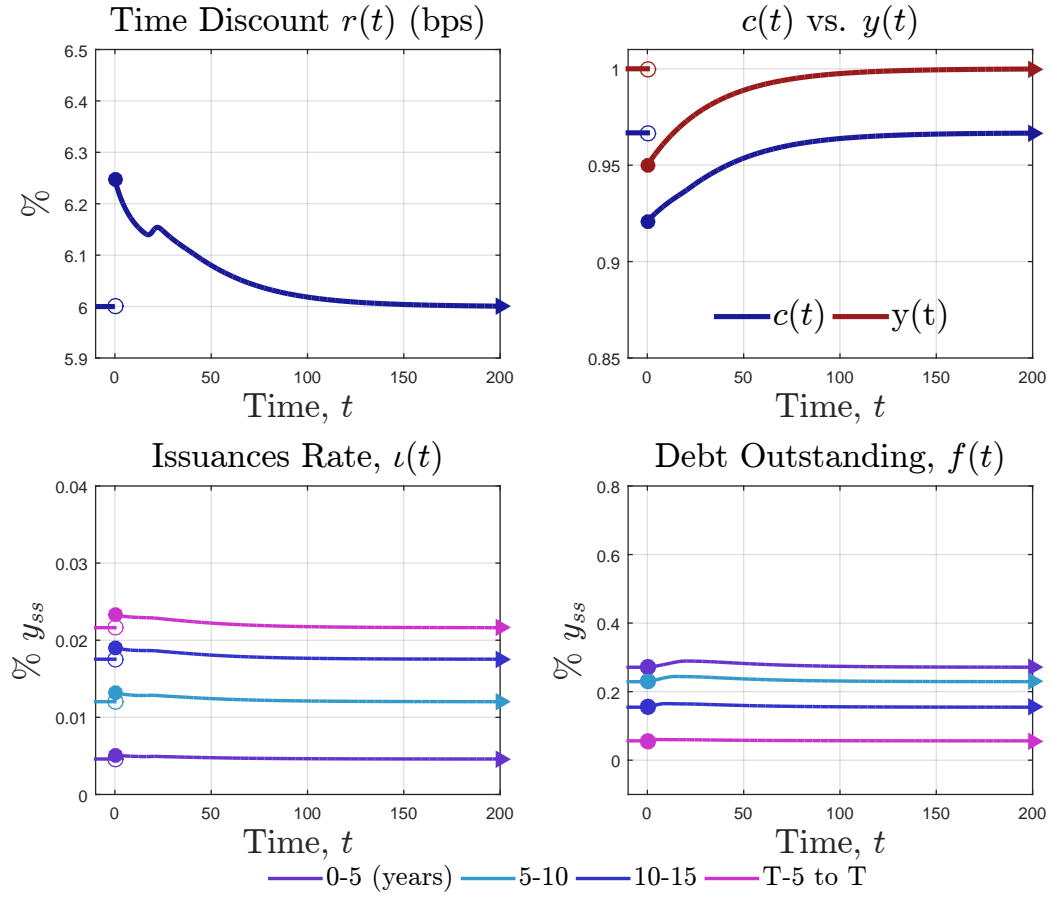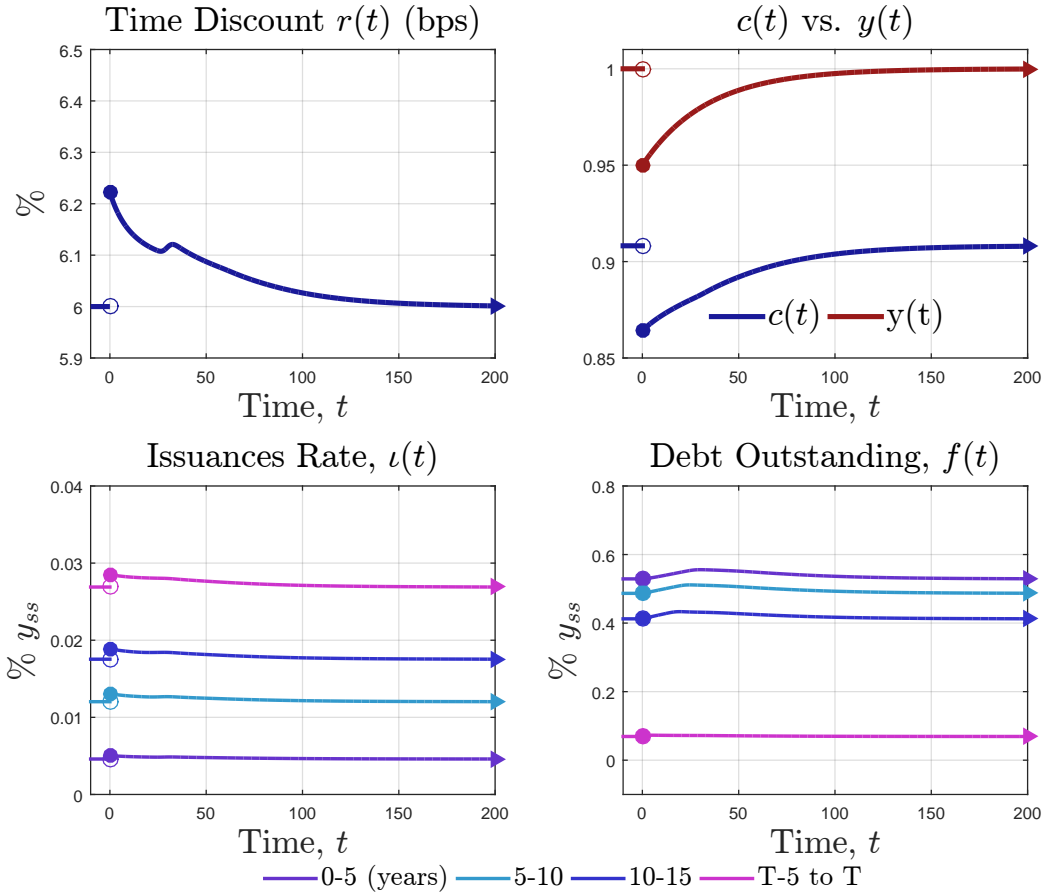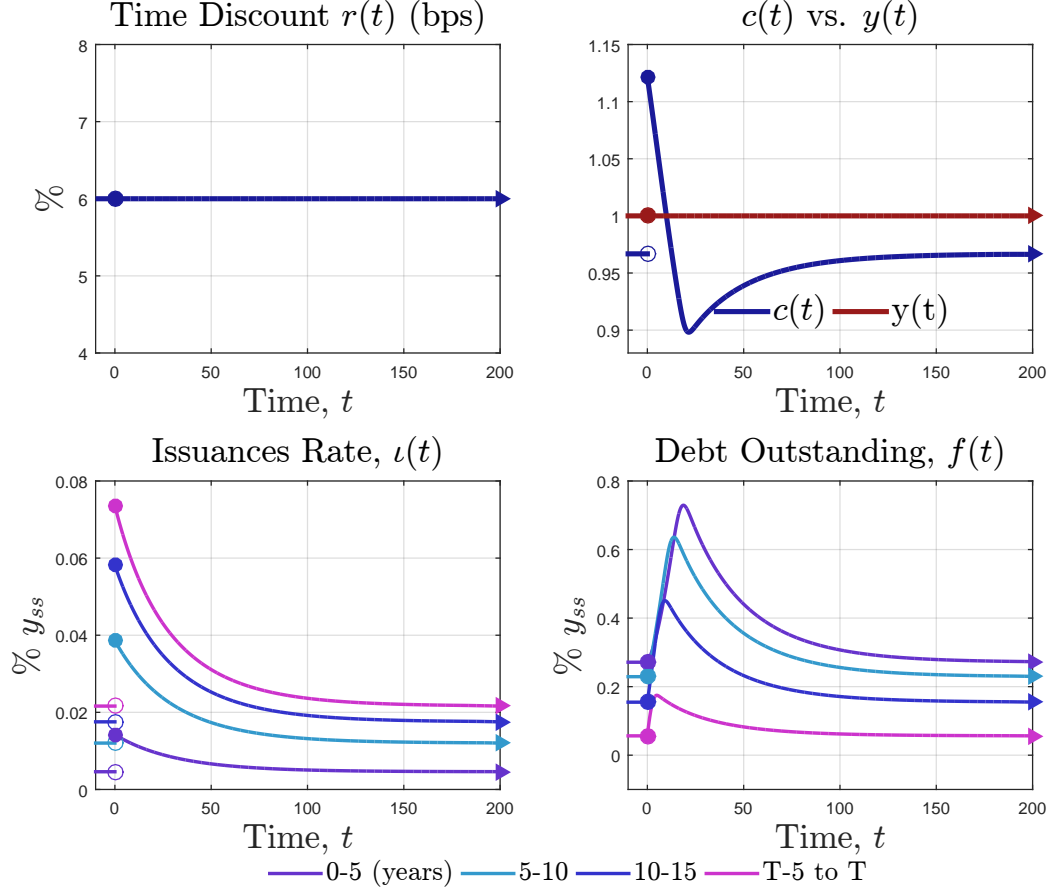
Figure 3.7: The figure describes the response of the government discount factor, issuances, consumption, and total debt, from an unexpected shock to the short interest rate that reverts to the long run mean for the case in which the households are risk neutral.

Consols vs. Standard Debt

Our next application is to compare the value of a Government that issues standard debt and one that only issues consols. In particular, we now study an alternative version of the (**DP**) with consols. Appendix xxx, contains the details. To connect both models, we establish a one-to-one map from each maturity τ to a consol. In particular, we define $m = \frac{1}{\tau}$ to be the decaying rate of a consol associated with a given τ. Thus, we have a continuum of consols $m \in [\frac{1}{T}, \infty)$. Each consol pays a constant coupon rate $z = \bar{r}_{ss}$. Then the mass of consols of maturity m satisfies the following Kolmogorov-Forward equation,

$$\frac{\partial f}{\partial t} = \iota(m, t) - m f(m, t).$$

In addition the budget constraint satisfies:

$$c(t) = y(t) + \int_{1/T}^{\infty} [q(m,t,\iota)\iota(m,t) - (m+z)f(m,t)]\,dm.$$

and the price of debt of debt is given by:

$$\psi(m,t) = (m+z)\int_t^{\infty} \exp\left\{-\int_t^s (r(u)+m)\,du\right\}ds.$$

The solution method is exactly the same, except that now, the valuations:

$$(r(t)+m)v(m,t) = -(m+z) + \frac{\partial v}{\partial t}, \text{ if } m \in [\frac{1}{T},\infty).$$

After we construct the solution to the valuations, we follow the same algorithm to construct a solution. Namely, $\iota(\tau,t)$ is given by the analogue of condition (3.4) where $v(m,t)$ replaces $v(\tau,t)$.

[TBC. Proposition that establishes desirability of Consols.]

4   Maturity-Debt Management under Risk

So far, we studied debt-management under perfect foresight. This section presents a characterization that allows the study of the influence of risk, for a limited form of risk. The goal is to understand how the anticipation of shocks affects the shape of the optimal debt profile. We continue to abstract from default and study only output, liquidity, and interest-rate risks. Towards that goal, we assume that the vector of those exogenous variables, $X(t)$, follows a deterministic path until the arrival of a one-time-and-for-all Poisson shock. The arrival of the Poisson shock has intensity ϕ. Upon the arrival of the shock at a time denoted by t^o, nature draws a new state, $X(t^o)$, that possibly alters one or more of the exogenous variables. This new state is drawn from $X(t^o) \sim F^X(\cdot; X(t^o_-))$, where t^o_- is the left limit.[21] After the jump in the states, $X(t)$ follows a deterministic path back to (a possibly new) steady state.

Obviously, there is risk in this economy before the shock, but after the shock the economy becomes the perfect-foresight economy of the previous section. Hence, we divide the analysis into a pre- and post-shock dynamics. We adopt the following convention to distinguish pre- from the post- shock variables: for any variable w that follows a path $w(t)$, we define by $\hat{w}(t)$, $t < t^o$, the value before the arrival of the shock and simply by $w(t)$, $t \geq t^o$, as the value after the shock arrival, once the economy becomes a perfect foresight economy as in the previous section. We denote by $\mathbb{E}_t^X$ a conditional expectation under the distribution $F^X(\cdot; X(t^o_-))$. Finally, we

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[21] We assume that $X(t)$ is *càdlàg*, that is, everywhere right-continuous and has left limits everywhere.

denote by $\mathbb{E}_t$ as the expectation under the distribution of arrivals of the Poisson event.

Problem with Risk. With the notation in hand, we are ready to study the problem under risk as **PR** and it is given by:

Problem 4 *(Problem with Risk) The **PR** problem is*

$$\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]=\max_{\{\hat{\imath}(\cdot)\}\in\mathcal{I}}\mathbb{E}_t\left[\int_t^{t^o}e^{-\rho(s-t)}\mathcal{U}\left(\hat{c}\left(s\right)\right)ds+e^{-\rho(t^o-t)}\mathbb{E}_{t^o}^X\left[V\left[f\left(\cdot,t^o\right),X\left(t^o\right)\right]\right]\right].$$

The maximization is subject to the law of motion of debt (2.1), the budget constraint (2.2), an initial condition $f\left(\cdot,0\right)=f_0$, and debt prices given by (3.1).

Following our notation, $\mathbb{E}_t^X\left[V\left[f\left(\cdot,t^o\right),X\left(t^o\right)\right]\right]$ is the expected jump in the value functional, conditional on a jump and the current state X.

4.1 Characterization

To characterize the solution, we follow a Dynamic Programming approach and employ results from functional analysis. These results, allow us to derive the analogue of a Hamilton-Jacobi-Bellman (HJB) equation for the case where the state variable is the distribution of debt:

Problem 5 *(Hamilton-Jacobi-Bellman Equation) The **PR** problem has a HJB representation given by:*

$$\rho\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right] \quad = \quad \max_{\{\hat{\imath}(\cdot,t)\}}\mathcal{U}\left(\hat{c}\left(t\right)\right)+\int_0^T\frac{\delta\hat{V}}{\delta f}\frac{\partial f\left(\cdot,t\right)}{\partial t}d\tau\dots \tag{4.1}$$

$$+\phi\left(\mathbb{E}_t^X\left[V\left[f\left(\cdot,t\right),X\left(t\right)\right]\right]-\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]\right) \tag{4.2}$$

subject to (2.1), (2.2), and (3.1).

Other than the integral term, this equation is identical to a standard HJB equation. The integral term is the infinite-dimensional analogue of the sum of the individual derivatives that would appear in an HJB equation with finite states.[22] Here, the value is not a value function, but a value functional, so the term $\frac{\delta\hat{V}}{\delta f}$ represents a Gâteaux derivative, the derivative of the value function with respect to the debt distribution. Results from functional analysis, allow us to obtain this representation and are found in Appendix C.5. As in a standard, HJB equation, this problem features a first-order condition with respect to the issuances. In the case of a continuum of controls, it is obtained by taking a Gâteaux derivative of the value with respect to ι, and point-wise equalizing the result to zero:

$$\mathcal{U}'\left(c\right)\left(\frac{\partial\hat{q}}{\partial\hat{\imath}}\hat{\imath}\left(\tau,t\right)+\hat{q}\left(\tau,t,\iota\right)\right)=-\hat{\jmath}\left(\tau,t\right),$$

---

[22]For example, of in an HJB with two state variables the the sum of two partial derivatives would emerge.

where we define $\hat{j}(\tau,t) \equiv \frac{\delta \hat{V}}{\delta f}$. Note the resemblances with the Lagrange multiplier of the **PF** problem.

Also, akin to a standard HJB equation, this continuous control version has an analogue to an Envelope Theorem. Taking the Gateaux derivative of the value function, we derive a law of motion for $\hat{j}(\tau,t)$:

$$\rho \hat{j}(\tau,t) = -\delta \mathcal{U}'(\hat{c}(t)) + \frac{\partial \hat{j}}{\partial t} - \frac{\partial \hat{j}}{\partial \tau} + \phi \left(\mathbb{E}_t^X [j(\tau,t)] - \hat{j}(\tau,t) \right), \quad \tau \in (0,T] \quad (4.3)$$

$$\hat{j}(0,t) = -\mathcal{U}'(\hat{c}(t)).$$

Here $\mathbb{E}_t^X [j(\tau,t)]$ is the expected value of the Lagrange multiplier associated with the τ-debt of **a PF** problem with initial condition $\{f(\cdot,t^o), X(t^+)\}$. The expectation is taken over the state variable $X(t^+)$ and is conditioned on $X(t^-)$. We can define the debt valuations as we do under perfect foresight:

$$\hat{v}(\tau,t) = -\hat{j}(\tau,t) / \mathcal{U}'(\hat{c}(t)),$$

and thus we have an analogue of the first-order condition (3.4), but that considers risk:

$$\frac{\partial \hat{q}}{\partial \hat{\iota}} \hat{\iota}(\tau,t) + \hat{q}(t,\tau,\iota) = \hat{v}(\tau,t). \quad (4.4)$$

We now establish the following summarizing proposition for the version with risk.

Proposition 7 *If a solution to **PR** exists, then valuations[23] are:*

$$\hat{v}(\tau,t) = e^{-\int_t^{t+\tau} (\hat{r}(u)+\phi) du} + \int_t^{t+\tau} \left(\delta + v(\tau - s, t+s) \frac{\mathcal{U}'(c(t+s))}{\mathcal{U}'(\hat{c}(t+s))} \right) e^{-\int_t^s (\hat{r}(u)+\phi) du} ds. \quad (4.5)$$

The optimal issuance $\hat{\iota}(\tau,t)$ is again given by (4.4). Finally, $\hat{c}(t)$ and $\hat{r}(t)$ must be consistent with the budget constraint (2.2).

Proof. See Appendix C.5. □

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[23]Their PDE representation is:

$$\hat{r}(t)\hat{v}(\tau,t) = \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau} + \phi \mathbb{E}_t^X \left[v(\tau,t) \frac{\mathcal{U}'(c(t))}{\mathcal{U}'(\hat{c}(t))} - \hat{v}(\tau,t) \right], \quad \tau \in (0,T],$$

$$\hat{v}(0,t) = 1, \quad \tau = 0,$$

$$\lim_{t \to \infty} e^{-\rho t} \hat{v}(\tau,t) = 0.$$

The proposition is very similar to the case studied under perfect foresight. However, valuations feature a correction for risk given by:

$$\phi \mathbb{E}_t^X \left[v\left(\tau,t\right) \frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)} - \hat{v}\left(\tau,t\right) \right].$$

This term is akin to the correction in any valuation that features a Poisson jump. The expression simply captures an "exchange rate" between states. The internal valuation, after the shock, $v\left(\tau,t\right)$, is measured in goods after the arrival of the shock. The ratio of marginal utilities tells us how how goods are relatively valued in terms of utilities, before vis-a-vis after the shock. Note that if a shock produces a decline in consumption, this ratio is greater than one. This means that fixing a given maturity, debt will priced more highly. In such a case, not only does the Government value the coupon stream, but increases the valuation of the coupon stream. This extra kick in the valuations affect bonds of different maturity deferentially, as we illustrate next.

Although Proposition 7 characterizes the solution to **PR**, computing its solution involves solving for a fixed point in a a family of distributions. This is because, for each t before the the arrival of the Poisson shock, will be associated with a consumption jump from $\hat{c}\left(t\right)$ to $c\left(t\right)$. That jump is a function of the distribution $f\left(\cdot, t^o\right)$. However, that possible jump affects the choice of ι through its influence in valuations. Thus, a solution is a fixed point in a family of debt distributions. To this date, it is unfeasible without an approximation algorithm. This is a problem to the case with already with absorbent shocks, as in this section, so the problem escalates in the presence of multiple shocks.[24] The subsequent section presents a methodology to deal with this issue: the study of what we label, the computation of a property: the risky steady state. Before we proceed, we describe the solution to the problem with liquidity costs and complete markets.

4.2 Maturity and Risk Management without Liquidity Costs

We now review a version of the model with risk without liquidity costs. This section helps us clarify the the extent of insurance to different shocks that the available sets of maturities allow the Government to achieve. Thus, we consider again the extreme where $\lambda\left(\iota, \tau, t\right) = 0$. The necessary conditions of this fully liquid problem are identical to those when there are liquidity costs. This implies that issuances are unbounded, unless $v\left(\tau, t\right) = \psi\left(t, \tau\right)$. Furthermore, since the PDE (3.7) is also a necessary condition, it must be that the Government's discount factor coincide with the international rate, $\bar{r}(t) = r\left(t\right)$. These observations are enough to characterize

---

[24] An analogous challenge appears in models of incomplete markets which is why the literature uses the approximation in Krusell and Smith (1998). Unfortunately, in our problem, that approximation is not good, because being a linear a approximation, it neglects dealing with risks, precisely the problem we are trying to solve.

the optimal maturity-management at the fully liquid limit.

Proposition 8 *Any solution $\{\iota(\tau,t),\hat{c}(t)\}$ to consistent with (2.1), (3.8)*

$$\left(\rho - \frac{\mathcal{U}''\left(\hat{c}\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\frac{d\hat{c}}{dt}\right)\hat{\psi}\left(\tau,t\right) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \theta\left(\mathbb{E}_t^X\left[\frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\psi\left(\tau,t\right)\right] - \hat{\psi}\left(\tau,t\right)\right) \quad (4.6)$$

which should be satisfied for any (τ,t) in the domain.

Below we discuss some particular cases.

Lack of Insurance. Consider the case of an income shock, that is, the only variable that changes with the shock is $y(t)$. In this case bond prices do not change, $\psi\left(\tau,t\right) = \hat{\psi}\left(\tau,t\right)$, then, the only possible outcome is to have:

$$r(t) = \left(\rho - \frac{\mathcal{U}''\left(\hat{c}\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\frac{d\hat{c}}{dt}\right) = \bar{r}(t) + \theta\left(\mathbb{E}_t^X\left[\frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\right] - 1\right).$$

With lack of insurance, the smalle economy can in fact reach a risky steady state with positive consumption. [TBC]

Complete Markets Allocation. Consider now a shock to interest rate $\bar{r}(t)$. In this case bond prices change with the arrival shock. If we combine again equations (4.6) and (**??**) yields

$$r(t) = \left(\rho - \frac{\mathcal{U}''\left(\hat{c}\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\frac{d\hat{c}}{dt}\right) = \bar{r}(t) + \theta\mathbb{E}_t^X\left[\left(\frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)} - 1\right)\frac{\psi\left(\tau,t\right)}{\hat{\psi}\left(\tau,t\right)}\right]. \quad (4.7)$$

If there is no jump in consumption once the shock arrives, $\hat{c}\left(t\right) = c\left(t\right)$, the solution to this equation yields $r(t) = \bar{r}(t)$. This implies that [TBC].

Imperfect Insurance. Equation (4.7) also has a solution with $\hat{c}\left(t\right) c\left(t\right)$ provided that

$$\mathbb{E}_t^X\left[\left(\frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)} - 1\right)\psi\left(\tau,t\right)\right] = 0.$$

For instance, in the case that $\bar{r}(t)$ remains constant after the shock, this equations results in

$$\int\left(\frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)} - 1\right)\frac{(1 - e^{-x\tau})}{x}dF_x(x) = 0.$$

Notice that this equation should be zero for any time and maturity. Since the solution must be true for any τ and there are τ bonds, there's a one-to-one map form the number of conditions to the number of unknowns. [TBC]

4.3 Risky Steady State

As discussed above, the calculation of a solution requires a fixed point in a family of functions. Instead of taking a numerical approach, we analyze a risky steady state. Our concept is similar to the one used by Coeurdacier et al. (2011). In our context, a risky steady state (RSS) is defined is a point where the economy is expected to converge when shocks are expected permanently, but when they have not yet materialized. We can solve this case because it no longer involves a solution in a fixed

Risky Steady State. We will characterize debt, issuances and consumption, for the case in which the economy is waiting for the shock but the shock has not arrived yet; that is, in histories where the shock has not occurred yet and the economy already converged to a steady state. In this case the solution of the system is composed of the following equations:

Proposition 9 *Valuations at a risky steady state are given by:*

$$\hat{v}_{rss}(\tau) = \int_0^\tau e^{-(r_{rss}+\theta)(\tau-s)} \left(\delta + \theta \mathbb{E}_{rss}^X \left[\frac{\mathcal{U}'(c_{rss})}{\mathcal{U}'(\hat{c}_{rss})} v_{rss}(s) \right] \right) ds, \text{ for } \tau \in (0, T]$$

The optimal issuance $\hat{\imath}(\tau, t)$ is again obtained by (4.4). The evolution of debt is obtained from, (2.1), and given by the initial condition $f(\cdot, 0)$. Finally, $\hat{c}(t)$ and $\hat{r}(t)$ must be consistent with the budget constraint (2.2).

Proof. Immediately from 7. $\qquad\square$

Discussion. The solution takes this form because the presence of risk at the steady state produces a term that alters the valuations:

$$\rho = \delta - \frac{\partial v}{\partial \tau} + \theta \mathbb{E}_{rss}^X \left[v_{rss}(\tau) \frac{\mathcal{U}'(c_{rss})}{\mathcal{U}'(\hat{c}_{rss})} - \hat{v}_{rss}(\tau) \right], \text{ for } \tau \in (0, T]$$

$$\hat{v}_{rss}(0) = 1 \text{ if } \tau = 0.$$

The extra term is $v_{rss}(\tau) \frac{\mathcal{U}'(c_{rss})}{\mathcal{U}'(\hat{c}_{rss})}$ which captures valuations from pre to post shocks. Again, the ratio of marginal utilities, acts like an exchange rate across states. The advantage of the risky steady state is that it provides a closed form solution.

4.4 Applications of the Model with Risk

The Influence of Risk

Next, we present two numerical exercises for the general case. In the first exercises, output is expected to drop by 5% on impact, and then a recovery back to steady state.

Expecting a 5% $y(t)$ drop. First, we compare the risky steady state (red), with the steady state (blue). There are two patterns to be dissected from Figures 4.1 and 4.2. On one hand, the presence of income risk produces an overall decline in issuances and debt outstanding of all maturities. On the other hand, we see a relative decline in shorter maturities. The reason for the decline in overall borrowing is the presence of risk, which captured by the ratio of marginal utilities present in the valuation formulas. This ratio tells us how costly an outflow of payments is once the shock is realized. This penalizes all bonds, and in the valuation equations the effect is analogous to an expected increase in coupon payouts. The second observation is that the decline is more pronounced on short term assets. It's better to explain the logic in the next exercise.

Figure 4.1: Response, in the risky steady state, of the maturity distribution, total debt, issuances, and valuations to an expected 10 percent drop in output that reverts to steady state exponentially.
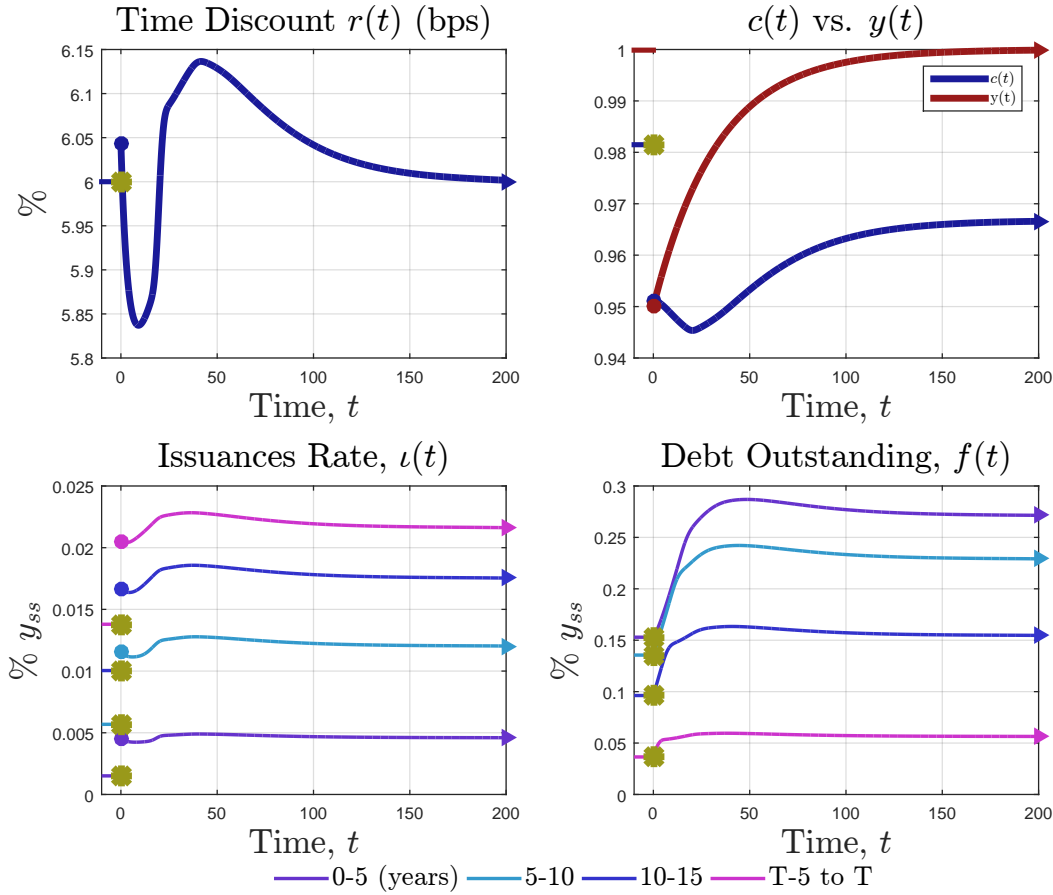
Figure 4.2: Dynamics total debt, issuances, consumption and the internal rate after a 10 percent shock in output that reverts to the long run mean exponentially.

Shock 4% to 8% on impact. In this experiment, depicted in Figures 4.3 and 4.4, we present an expected shock where short term rates are expected to increase suddenly, to 8%. This is a rate above the discount factor of the small economy. As in the previous exercise, the effect of this source of risk is to shrink the issuances at all maturities. However, the exercise results in a more extreme reversal of positions. For very short-term bonds, the country actually begins buying back and eventually accumulating short term bonds. This means that the way the country reacts to risk is by issuing long-term debt but holding short term assets. Again, all of the logic is captured by the valuation equations. A long-term bond will be long lived. With a high probability, it will experience a cycle that begins with a positive spread between $r(t)$ and $\bar{r}(t)$, but then, when the shock is realized, with a converse relationship. When international rates are high, the country is worse off by holding those assets, because paying out coupons when discounts are high is not desirable. However, there are two other phases where the country spread is positive. This makes long-term debt desirable, despite the fact that with a high probability

there will be a period where the debt will be particularly costly because marginal utility is high. The principle, on the other hand, will be likely to be repaid once marginal utility is close to steady state again.

Short-term debt is different. Before the shock is realized, the short-term bond yields a benefit because $r_{rss} - \bar{r}_{ss} > 0$. However, being a short-term bond, it is likely to experience a period of high internal discounting. The chances of entering this period are the same as for the long-term bond. However, the short bond doesn't have the length to cover the reversal of the cycle. Furthermore, it's principle is likely to be paid in periods of high marginal utility. This makes the short-term bond be valuable as an asset.

Both exercises have the same forces behind. However, the calibration of interest-rate risks are strong enough to reverse the positions.
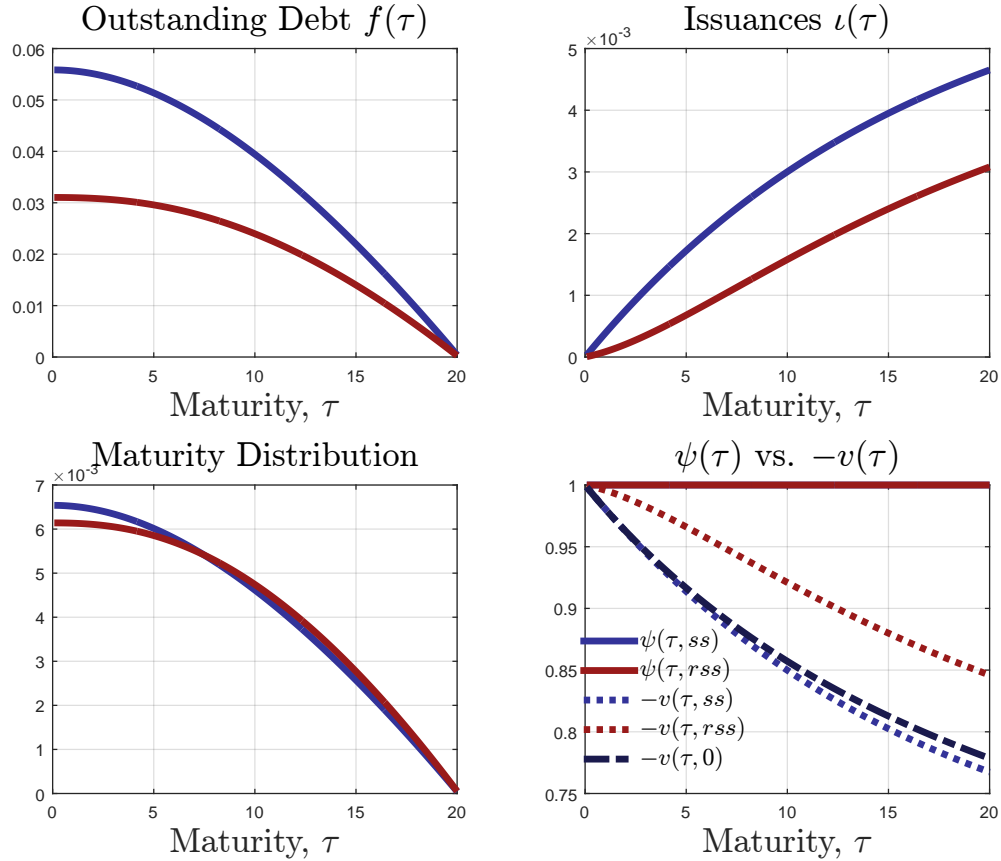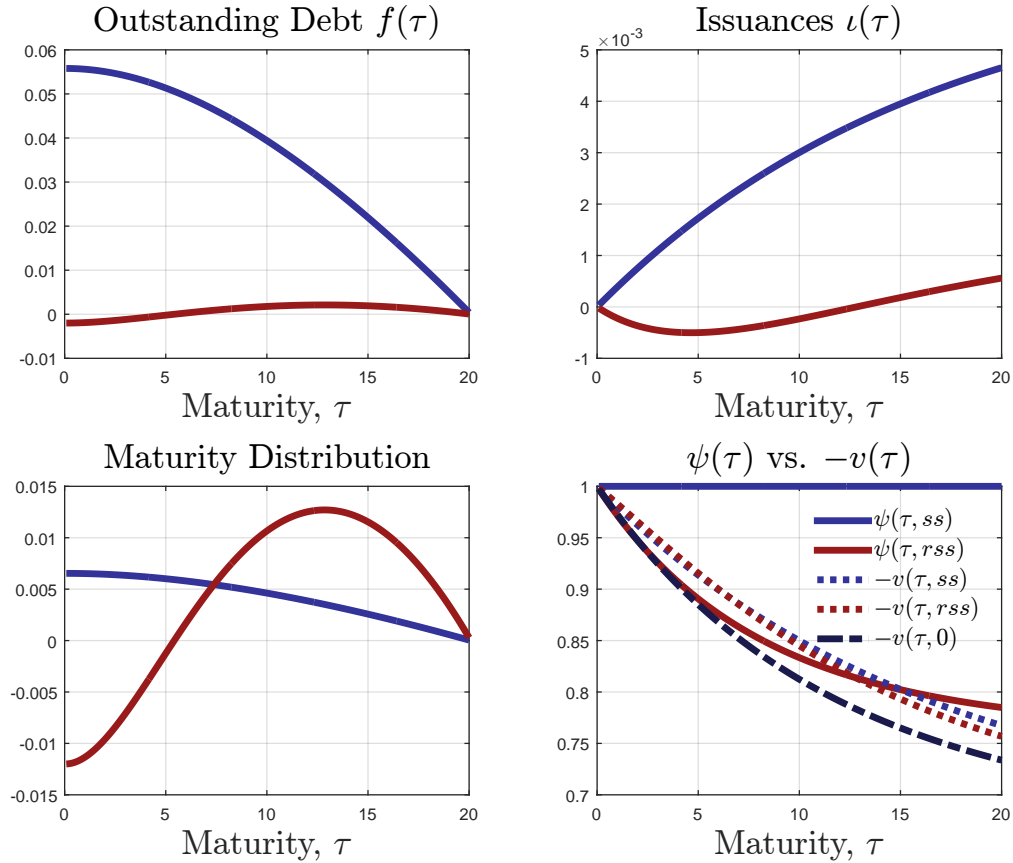

Figure 4.3: Response, in the risky steady state, of the maturity distribution, total debt, issuances, and valuations to an expected permanent increase in the short rate.
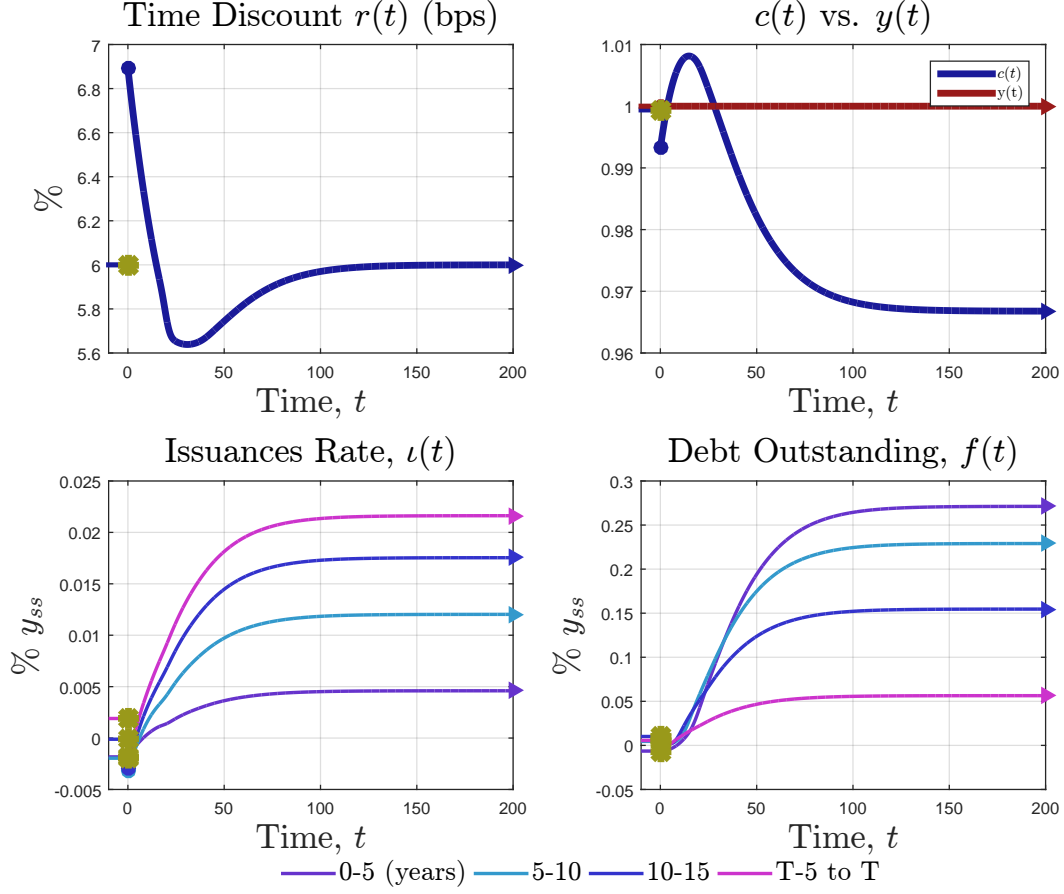
Figure 4.4: Dynamics total debt, issuances, consumption and the internal rate to an expected permanent increase in the short rate.

Limited Issuances

We adapt the model to allow for issuances only at a limited set of maturities. In this case:

$$\iota(\tau,t) = \begin{cases} \frac{\psi(\tau,t)+v(\tau,t)}{\bar{\lambda}} & \textbf{for } \tau \in [\tau_1, \tau_2, ..., \tau_N] \\ 0 & \textbf{for } \tau \notin [\tau_1, \tau_2, ..., \tau_N] \end{cases}.$$

We obtain the following figure plots the solution for this case.
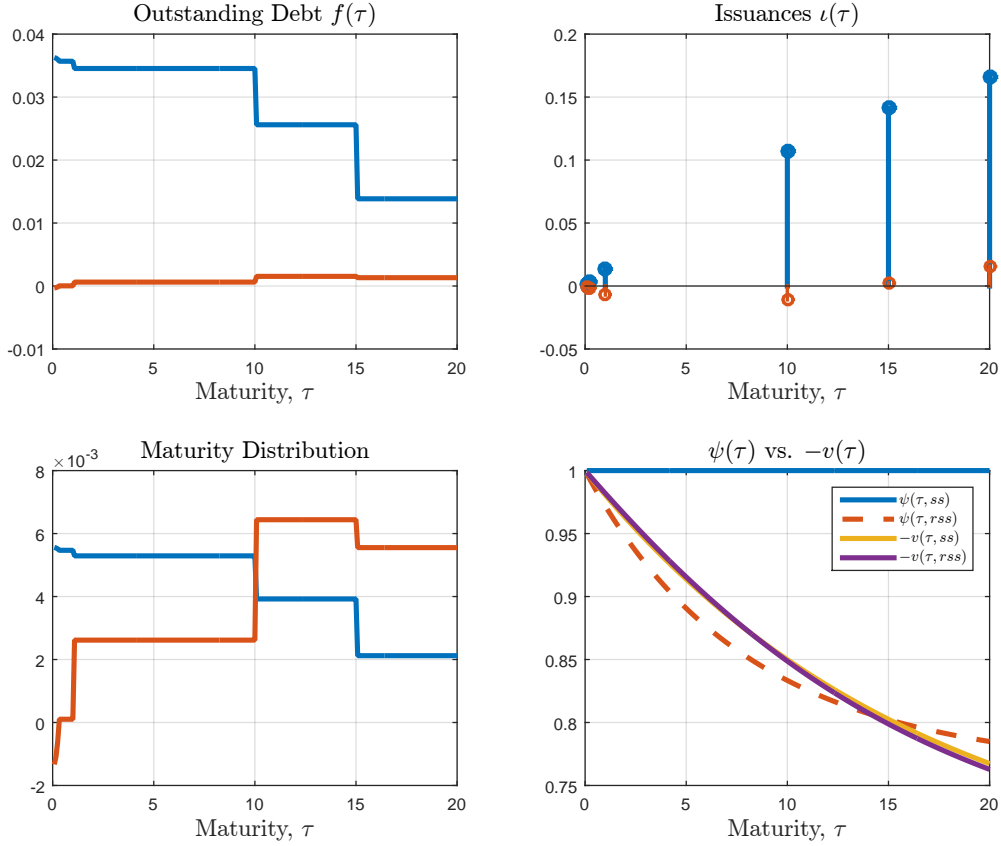
Figure 4.5: The figure describes steady state in a model with limited issuances.

5 The Option to Default

In this section we study how the option to default alters the optimal maturity profile. The nature of the problem changes because, now, bond prices depend on Government actions, and vice versa. Furthermore, it now matters if the Government can commit to a debt issuance program or not. We proceeds as follows. We first describe the solution to the Government's problem when the option to default occurs only once and at known date. This solution is interesting on it's own, but it is instructive to solve the case where the actual date is unknown. We use this intuition to solve the problem with a random arrival date of a default event. In both cases, we solve the problem with and without commitment.

Default. As in the rest of the paper, we maintain the assumption that jumps in the exogenous states occur once. We also maintain the pre- and post-shock notation of the previous

section. For each y^D, we obtain a value-upon default V^D with an induced distribution $\Phi(\cdot)$. Recall from the introductory section that τ^D is the *time of default*. We first consider the case where the only shock that can arrive is the default shock, and keep constant the value of the other parameters: $X(t) = X$. An extension to the general case, which we use in the applications, is immediate, but more intense in notation.

The price is $\psi(\tau, t)$ is given by (2.5). The Government solves::

Problem 6 *The problem with default (**PD**) is*

$$\hat{V}\left[f\left(\cdot, t\right)\right] = \max_{\{\hat{c}(t), \mu(\cdot, t)\}_{t \geq 0}} \mathbb{E}_t \left[\int_t^{t+\tau^D} e^{-\rho(s-t)} \mathcal{U}\left(\hat{c}\left(s\right)\right) ds + e^{-\rho\tau^D} V^D \right], \tag{5.1}$$

subject to the law of motion of debt (2.1), the budget constraint (2.2), and the prices (2.5).

The expectation in the problem is with respect to the $\{\tau^D, V^D\}$. We study four variations of this problem that differ in assumptions about commitment and the occurrence of default events:

For the occurrence of default events, we consider to possible cases:

- Fixed Default Date (**FD**): The default option is known to occur only by date $\tilde{t}$ and occurs with probability θ. In this case, $\tau^D = \tilde{t}$ or ∞.

- Random Default Date (**RD**): the default option occurs with Poisson distribution with coefficient θ.

For the assumption of commitment:

- With Commitment (**WC**): we assume that the Government can commit to an issuance path from time zero. It cannot commit to repay it's debt upon the default option arrives.

- No Commitment (**NC**): we assume that the Government cannot commit to an issuance path neither to repay it's debt upon the default option arrives.

We begin with the problems with a fixed date.

5.1 Characterization

The Problem with Default on a Fixed Date with Commitment (PDFDWC). Consider that at date $\tilde{t}$ the Government has a probability probability θ of a new draw of a default value. If the default event occurs, the new draw is taken from Φ. The Government's default decision is immediate. It defaults if the value of the outside option is $V\left[f\left(\cdot, \tilde{t}\right)\right]$. As in all previous instances, the optimal policy is characterized by solving for the formula for valuations, except that now, valuations have an interesting interaction term.

Proposition 10 (*Necessary conditions of the* **PDFDWC** *problem*) *If a solution to* **PDFDWC** *exists, then:*

$$\hat{v}(\tau, t) = e^{-\int_t^{t+\tau} r(u)du} + \delta \int_t^{t+\tau} e^{-\int_t^s r(u)du} ds \ \text{if} \ t + \tau < \tilde{t}. \tag{5.2}$$

and if $\tau + t \geq \tilde{t}$,

$$
\begin{aligned}
\hat{v}(\tau, t) &= \delta \int_t^{\tilde{t}} e^{-\int_t^s r(u)du} ds + e^{-\int_t^{\tilde{t}} r(u)du} \hat{v}(\tau - (\tilde{t} - t), \tilde{t}) \\
&\quad + e^{-\int_t^{\tilde{t}} r(u)du} (a(\tau, t) + b(\tau, t)) \cdot \frac{\mathcal{U}'(c(\tilde{t}))}{\mathcal{U}'(\hat{c}(\tilde{t}))} v(\tau - (\tilde{t} - t), \tilde{t}),
\end{aligned}
$$

where the terminal value is:

$$\hat{v}(\tau, \tilde{t}) = (1 - \theta(1 - \Phi(V[f(\cdot, \tilde{t})]))) \frac{\mathcal{U}'(c(\tilde{t}))}{\mathcal{U}'(\hat{c}(\tilde{t}))} v(\tau, \tilde{t}), \ \text{for} \ \tau \in (0, T],$$

and the after t price impact is:

$$
\begin{aligned}
a(\tau, t) &= \frac{1}{\bar{\lambda}} \theta \Phi'(V[f(\cdot, \tilde{t})]) \mathcal{U}'(\hat{c}(\tilde{t})) \ldots \\
&\quad \int_t^{\tilde{t}} e^{\int_s^{\tilde{t}} r(u)du} \int_0^T e^{-\int_s^{\tilde{t}} \bar{r}(u)du} \psi(\tau, \tilde{t}) \left(1 - \frac{\hat{v}(\tau + \tilde{t} - s, s)}{\psi(\tau + \tilde{t} - s, s)}\right) d\tau ds,
\end{aligned}
$$

and the before t price impact is:

$$
\begin{aligned}
b(\tau, t) &= \frac{1}{\bar{\lambda}} \theta \Phi'(V[f(\cdot, \tilde{t})]) \mathcal{U}'(\hat{c}(\tilde{t})) \ldots \\
&\quad \int_{max\{\tilde{t}-(T-\tau),0\}}^t e^{\int_s^{\tilde{t}} r(u)du} \int_0^T e^{-\int_s^{\tilde{t}} \bar{r}(u)du} \psi(\tau, \tilde{t}) \left(1 - \frac{\hat{v}(\tau + \tilde{t} - s, s)}{\psi(\tau + \tilde{t} - s, s)}\right) d\tau ds.
\end{aligned}
$$

The optimal issuance $\iota(\tau, t)$ *is given by the condition (3.4). The evolution of the debt mass can be recovered from the law of motion for debt, (2.1), given the initial condition* $f(\cdot, 0)$. *Finally,* $c(t)$ *and* $r(t)$ *must be consistent with the budget constraint (2.2).*

The valuations are broken into two regions. When $t + \tau < \tilde{t}$ a bond vintage will expire before the default period. Thus, the vintage will not influence the probability of default, at least directly. It may affect the probability of default indirectly, but the Envelope Theorem ensures that the indirect effect is not present in the valuations. Furthermore, the valuation is as in the standard case because the bond will be paid in full.

The interesting region is for the vintages such that $\tau + t \geq \tilde{t}$. These are bonds that expire after $\tilde{t}$, have an influence on default decisions, and carry a default-option premium. The valuation of the bond is such that the coupons before $\tilde{t}$ is given by three terms. The term $\delta \int_t^{\tilde{t}} e^{-\int_t^s r(u)du} ds$

is the present value of coupons. The term $e^{-\int_t^{\tilde{t}} r(u)du}\hat{v}\left(\tau-(\tilde{t}-t),\tilde{t}\right)$ is the present value of the valuation of the bond at the date of the default. The valuation term $\hat{v}\left(\tau-(\tilde{t}-t),\tilde{t}\right)$ has the following interpretation. The indexes the remaining maturity of the vintage at the time of the default option. The value itself has several components. The probability that the bond is repaid is:

$$1-\theta+\theta\left(V\left[f\left(\cdot,\tilde{t}\right)\right]\right),$$

Since the bond has no value in the complement probability, this probability multiplies,

$$\frac{\mathcal{U}'\left(c\left(\tilde{t}\right)\right)}{\mathcal{U}'\left(\hat{c}\left(\tilde{t}\right)\right)}v\left(\tau,\tilde{t}\right)$$

which is the value of the bond times the exchange-rate between pre- and post-shock consumption. Note that a similar term also appears in the valuation of the bond by the international investor, but there is no risk-correction given the risk-neutrality assumption.

The more interesting are the terms $a\left(\tau,t\right)$ and $b\left(\tau,t\right)$, which captures the spill-over of default probabilities on other bond vintages. Both terms capture the same effect, and their distinction is only the effects on bonds after t (in the case of $a\left(t\right)$) and before t (in the case of $b\left(t\right)$). The term

$$\theta\Phi'\left(V\left[f\left(\cdot,\tilde{t}\right)\right)\right]\right)\mathcal{U}'\left(\hat{c}\left(\tilde{t}\right)\right),$$

is the marginal change in the probability of default at the terminal date, in terms of goods at time $\tilde{t}$. At any date prior to $\tilde{t}$, the influence on the of default at $\tilde{t}$ on the current price is:

$$\frac{\partial\psi\left(\tau+\tilde{t}-s,\tilde{s}\right)}{\partial\psi\left(\tau,\tilde{t}\right)}=e^{-\int_s^{\tilde{t}}\tilde{r}(u)du}\psi\left(\tau,\tilde{t}\right).$$

This is because the price of each bond will be the present value of dividends plus the present value of the price at the default date. Multiplying this expression by the marginal change in the default probability yields the change in the price at any date. Now, in the Government's problem, the change in the price affects the value of issuances at a given date, which is why each price is multiplied by the corresponding issuance:

$$\frac{1}{\bar{\lambda}}\left(1-\frac{\hat{v}\left(\tau+\tilde{t}-s,s\right)}{\psi\left(\tau+\tilde{t}-s,s\right)}\right)$$

Finally, the terms:

$$\int_t^{\tilde{t}}e^{\int_s^{\tilde{t}}r(u)du}ds\text{ adn }\int_{max\{\tilde{t}-(T-\tau),0\}}^{\tilde{t}}e^{\int_s^{\tilde{t}}r(u)du}ds$$

transform each change in the revenues at any period, into a goods of date $\tilde{t}$. The discount $e^{-\int_t^{\tilde{t}}r(u)du}$ in front of the expression brings it back to time t and multiplication by $\hat{v}\left(\tau-(\tilde{t}-t),\tilde{t}\right)$ measures the change in the value the period of default.

The Problem with Default on a Fixed Date without Commitment (PDFDNC). Without commitment, the Government

Proposition 11 (*Necessary conditions of the* **PDFDNC** *problem*) *If a solution to* **PDFDNC** $\hat{v}(\tau, \tilde{t})$ *is given by the same expression as in the case with commitment* (**PDFDWC**)*. However,* $b(t) = 0$ *whereas the terms* $\hat{v}(\tau, \tilde{t})$ *and* $a(\tau, t)$ *satisfy the same functional equations.*

Valuations at a risky steady state with default are:

The Problem with Default on a Rando Date without Commitment (PDRDNC). Consider now the case of the problem where default opportunities can arrive at any date with Poisson intensity θ.

Proposition 12 (*Necessary conditions of the* **PDRDWC**) *If a solution to* **PDRDWC** *with* **commitment** *exists, then:*

$$\tilde{v}(\tau, t) = \delta \int_t^\tau e^{-\int_t^s (r(u)+\theta)du} ds + e^{-\int_t^{t+\tau}(r(u)+\theta)du} + \pi(\tau, t) + \tilde{a}(\tau, t) + \tilde{b}(\tau, t)$$

where $\tilde{\pi}(\tau, t)$ *is a risk-premium:*

$$\pi(\tau, t) = \theta \int_t^{t+\tau} e^{-\int_t^s (r(u)+\theta)du} \left(1 - \Phi\left(V[f(\cdot, s)]\right)\right) \frac{\mathcal{U}'(c(s))}{\mathcal{U}'(\hat{c}(s))} v(\tau + s - t, s) \, ds, \text{ for } \tau \in (0, T],$$

and $\tilde{b}(\tau, t)$ *are the before-issuance increases in funding costs:*

$$\tilde{b}(\tau, t) = \frac{\theta}{\bar{\lambda}} \mathcal{U}'(\hat{c}(t)) \int_{max\{t-T, 0\}}^t e^{\int_s^t r(u)du} \int_0^T \left(1 - \frac{\tilde{v}(\zeta, s)}{\psi(\zeta, s)}\right) \frac{\partial \psi(\zeta, s)}{\partial f(\tau, t)} d\zeta ds$$

and $\tilde{a}(\tau, t)$ *are the after-issuance increase in funding costs:*

$$\tilde{a}(\tau, t) = \frac{\theta}{\bar{\lambda}} \mathcal{U}'(\hat{c}(t)) \int_t^{t+\tau} e^{\int_s^t r(u)du} \int_0^T \left(1 - \frac{\tilde{v}(\zeta, s)}{\psi(\zeta, s)}\right) \frac{\partial \psi(\zeta, s)}{\partial f(\tau, t)} d\zeta ds$$

where

$$\frac{\partial \tilde{\psi}(\zeta, s)}{\partial f(\tau, t)} = \int_t^{t+\tau} \mathbb{I}_{[s+\zeta>z]} e^{-\int_s^z (\bar{r}(u)+\theta)du} \Phi'\left(V[f(\cdot, z)]\right) \psi(\zeta - z, s + z) \frac{\mathcal{U}'(c(z))}{\mathcal{U}'(\hat{c}(z))} v(\tau - z, t + z) \, dz.$$

The optimal issuance $\iota(\tau, t)$ *is given by the condition* (3.4)*. The evolution of the debt mass can be recovered from the law of motion for debt,* (2.1)*, given the initial condition* $f(\cdot, 0)$*. Finally,* $c(t)$ *and* $r(t)$ *must be consistent with the budget constraint* (2.2)*.*

Proof. See Appendix C.5. □

The Proposition states that in a risky steady state, prices and valuations the same formulas as in all previous cases. Two things change. Both the international rate and the Government's discount factor are now adjusted for by a risk premium. The risk premium captures the chances that a default event occurs and that the bond is repaid. The flows are θ and $\bar{\theta}$ capture how, if a default option occurs but is not triggered, valuations are altered. We now move to two additional applications.

The Problem with Default on a Fixed Date without Commitment (PDRDNC). Without commitment, the Government valuations are given by a similar equation that doesn't look backwards.

Proposition 13 *(Necessary conditions of the **PDRDNC** problem) If a solution to **PDRDNC** $\hat{v}(\tau, \tilde{t})$ is given by the same expression as in the case with commitment (**PDRDWC**). However, $b(t) = 0$ whereas the terms $\hat{v}(\tau, \tilde{t})$ and $a(\tau, t)$ satisfy the same functional equations.*

5.2   Transitions in the models with Fixed Default Dates

5.3   The Risky Steady States of the models with Random Default Dates

5.4   Applications

6   Conclusions

This paper developed the techniques to study a consumption-savings problem when the control variable is a distribution over a set of bonds of different maturity. Central to the environment is the presence of a price impact on each bond. Throughout the paper, we demonstrated how these features, alter common results found in the literature.

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Appendix: Optimal Debt-Maturity Management

A Microfoundation of the Liquidity Costs

Here we describe the microfoundation in more detail...

Consider a bond issuance of $\iota(\tau, t)$ at time t of maturity τ.

A.1 Matching Probabilities

Define s as the time since the beginning of the auction. The time to maturity by s is $\tau' = \tau - s$ the corresponding maturity. The outstanding amount of bonds in hands of the banker by time s after the issuance of the bond is:

$$I(s; \iota(\tau, t)) = \max(\iota(\tau, t) - \mu \cdot s, 0).$$

Clearly, the bond inventory is exhausted by time $\bar{s} = \iota(\tau, t)/\mu$. Per instant of time, the intensity at which bonds are sold is:

$$\gamma(s; \tau, t) = \frac{\mu y_{ss}}{I(s; \iota(\tau, t))} = \frac{1}{\bar{s} - s} \text{ for } s \in [0, \min\{\tau, \bar{s}\})$$

The probability is defined only between $[0, \min\{\tau, \bar{s}\})$ because after the bond matures or after the stock is exhausted, there is no selling probability.

A.2 Valuation for Bankers and Investors

Investor's Valuation. The valuation of the bond by investors is

$$\psi^{(t,\tau)}(\tau', s) = \psi(\tau - s, t + s)$$

identical to the price equation in the body of the paper. Hence, the price equation satisfies the PDE we have studied so far:

$$\bar{r}(t + s)\psi^{(t,\tau)}(\tau', s) = \delta - \frac{\partial \psi^{(t,\tau)}}{\partial \tau'} + \frac{\partial \psi^{(t,\tau)}}{\partial t}.$$

with the terminal condition of $\psi^{(t,\tau)}(0, s) = 1$.

Banker's Valuation. Now consider the valuation of the cash-flows of the bond from the perspective of the banker $q^{(t,\tau)}(\tau', s)$. By analogy, it must satisfy

$$(\bar{r}(t + s) + \eta)q^{(t,\tau)}(\tau', s) = \delta - \frac{\partial q^{(t,\tau)}}{\partial \tau'} + \frac{\partial q^{(t,\tau)}}{\partial t} + \gamma(s)\left(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s)\right). \quad \text{(A.1)}$$

This expression takes this form because the banker extracts surplus when it matches. Before

match, it earns the flow utility, but upon a match, it's value jumps to $\psi^{(t,\tau)} - q^{(t,\tau)}$. This jump arrives with endogenous intensity $\gamma(s)$. The complication with this PDE is its terminal condition. If $\bar{s} \leq \tau$, its terminal condition is given by $q^{(t,\tau)}(\tau', \bar{s}) = \psi^{(t,\tau)}(\tau', \bar{s})$. If $\bar{s} > \tau$, the corresponding terminal condition is $q^{(t,\tau)}(0,s) = 1$, since by the expiration date, it is paid the principal equal to 1.

Competitive Auction Price. The date of the auction $s = 0, \tau' = \tau$ the banker will pay its expected valuation and hence the bond price demand faced by the Government is:

$$q(\iota, \tau, t) \equiv q^{(t,\tau)}(\tau, 0).$$

This is because, there is free entry by investment banks into the auction.

Earlier we expressed $q(\iota, \tau, t)$ as $q(\iota, \tau, t) = \psi(\tau, t) - \lambda(\iota, \tau, t)$. Thus, $\lambda(\iota, \tau, t) = \psi(\tau, t) - q(\iota, \tau, t)$, is the object we are trying to find. To find that expression, we employ the following calculations.

A.3   The Exact Solution

Next, we solve of the liquidity cost $\lambda(\iota, \tau, t)$. To do that, we first solve for the PDE for q. We have the following result:

Problem 7 *The solution to $q(\iota, \tau, t)$ falls in one of two cases:*
   ***Case 1.** If $\bar{s} \leq \tau$, then*

$$q(\iota, \tau, t) = \frac{\int_0^{\bar{s}}(\delta(\bar{s} - v) + \psi(\tau - v, t + v)) \exp\left(- \int_0^v r(t + u) + \eta du\right) dv}{\bar{s}}.$$

   ***Case 2.** $\bar{s} > \tau$*

$$\begin{aligned}
q(\iota, \tau, t) &= \int_0^\tau \left(\frac{\delta(\bar{s} - v) + \psi(\tau - v, t + v)}{\bar{s}}\right) \exp\left(- \int_0^s r(t + u) + \eta du\right) dv \\
&+ \frac{(\bar{s} - \tau)}{\bar{s}} \exp\left(- \int_0^\tau r(t + u) + \eta)du\right).
\end{aligned}$$

Proof. We solve the PDE for q depending on the corresponding case for it's terminal condition.
   Case 1. Consider the first case. The general solution to the PDE equation for $q^{(t,\tau)}(\tau', s)$ is,

$$\int_0^{\bar{s}-s}(\delta + \gamma(s + v)\psi(\tau - v, t + v)) \exp\left(- \int_0^v (r(t + u) + \eta + \gamma(u)) du\right) dv \quad + \quad \text{(A.2)}$$

$$\exp\left(- \int_0^{\bar{s}-s} (r(t + u) + \eta + \gamma(u)) du\right) \psi(\tau' - (\bar{s} - s), t + \bar{s}).$$

This can be checked by taking partial derivatives with respect to time and maturity and applying Leibnitz's rule.[25] Consider the exponentials that appear in both terms. These can be decomposed into:

$$\exp\left(-\int_0^v r(t+u) + \eta du\right) \exp\left(-\int_0^v \gamma(u)du\right).$$

Then, by definition of γ we have:

$$\exp\left(-\int_0^v \gamma(u)du\right) = \exp\left(-\int_0^v \frac{1}{\bar{s}-u}du\right) = \exp\left(\ln(\bar{s}-v) - \ln(\bar{s})\right) = \frac{(\bar{s}-v)}{\bar{s}}.$$

Thus, (A.2) can be expressed as:

$$q^{(t,\tau)}(\tau',s) = \int_0^{\bar{s}-s} \frac{(\bar{s}-v)}{\bar{s}} (\delta + \gamma(s+v)\psi(\tau-v,t+v)) \exp\left(-\int_0^v (r(t+u)+\eta)\,du\right) dv$$

$$+ \exp\left(-\int_0^{\bar{s}-s} (r(t+u)+\eta)\,du\right) \frac{s}{\bar{s}}\psi(\tau' - (\bar{s}-s), t+\bar{s}).$$

When we evaluate this expression at $s = 0$, $\tau' = \tau$ we have that $q(\iota,\tau,t) \equiv q^{(t,\tau)}(\tau,0)$ and thus equals:

$$\int_0^{\bar{s}} \left(\delta + \frac{1}{(\bar{s}-v)}\psi(\tau-v,t+v)\right) \frac{(\bar{s}-v)}{\bar{s}} \exp\left(-\int_0^v r(t+u)+\eta du\right) dv. \qquad \text{(A.3)}$$

After we replace $\gamma(v) = \frac{1}{\bar{s}-v}$ we obtain:

$$q(\iota,\tau,t) = \int_0^{\bar{s}} \left(\delta + \frac{\psi(\tau-v,t+v)}{\bar{s}-v}\right) \frac{(\bar{s}-v)}{\bar{s}} \exp\left(-\int_0^v r(t+u)+\eta du\right) dv.$$

Rearranging terms gives the expression in the Proposition above.

Case 2. The proof in the second case runs parallel. The general solution to the PDE equation in this case is,

$$q^{(t,\tau)}(\tau',s) = \int_0^{\tau'} (\delta + \gamma(s+v)\psi(\tau-v,t+v)) \exp\left(-\int_0^v (r(t+u)+\eta+\gamma(u))\,du\right) dv$$

$$+ \exp\left(-\int_0^{\tau'} (r(t+u)+\eta+\gamma(u))\,du\right),$$

---

[25] Notice that we have directly replaced the value $\psi^{(t,\tau)}(\tau',s) = \psi(\tau-s,t+s)$.

When we evaluate this expression at $s = 0, \tau' = \tau$:

$$
\begin{aligned}
q(\iota, \tau, t) \quad &= \quad \int_0^\tau \left(\frac{\delta(\bar{s} - v) + \psi(\tau - v, t + v)}{\bar{s}} \right) \exp\left(-\int_0^s r(t + u) + \eta du \right) dv \\
&\quad + \frac{(\bar{s} - \tau)}{\bar{s}} \exp\left(-\int_0^\tau r(t + u) + \eta) du \right).
\end{aligned}
$$

$\square$

A.4 Limit Behavior of $q(\iota, \tau, t)$

Price with Zero Issuances. Consider the limit $\iota(\tau, t) \longrightarrow 0$ for any $\tau > 0$. Then, $\bar{s} \longrightarrow 0$. The relevant price equation is

$$
\lim_{\iota(\tau, t) \longrightarrow 0} q(\iota, \tau, t) = \lim_{\bar{s} \longrightarrow 0} \frac{\int_0^{\bar{s}} (\delta(\bar{s} - s) + \psi(\tau - s, t + s)) \exp\left(-\int_0^s r(t + u) + \eta du \right) ds}{\bar{s}}.
$$

Now, both the numerator and the denominator, approach zero as we take the limits. Hence, by L'Hôpital's rule, the limit price is the limit of the ratio of derivatives. The derivative of the numerator is obtained via Leibnitz's rule and thus,

$$
\begin{aligned}
\lim_{\iota(\tau, t) \longrightarrow 0} q(\iota, \tau, t) \quad &= \quad \lim_{\bar{s} \longrightarrow 0} \frac{(\delta(\bar{s} - s) + \psi(\tau - s, t + s)) \exp\left(-\int_0^s r(t + u) + \eta du \right) \big|_{s=\bar{s}}}{1} \\
&= \quad \lim_{\bar{s} \longrightarrow 0} \psi(\tau - \bar{s}, t + \bar{s}) \exp\left(-\int_0^{\bar{s}} r(t + u) + \eta du \right) = \psi(\tau, t).
\end{aligned}
$$

The limit as orders reach infinity are, $\mu \longrightarrow \infty$, then same limit property holds:

$$
\lim_{\mu \longrightarrow \infty} q(\iota, \tau, t) = \psi(\tau, t).
$$

Limit with Zero Orders. Consider the limit with zero orders, $\mu \longrightarrow 0$. Then, for any finite $\tau > 0, \bar{s} \longrightarrow \infty$. Thus,

$$
\begin{aligned}
\lim_{\mu \longrightarrow 0} q(\iota, \tau, t) \quad &= \quad \lim_{\bar{s} \longrightarrow \infty} \int_0^\tau \left(\frac{\delta(\bar{s} - s) + \psi(\tau - s, t + s)}{\bar{s}} \right) \exp\left(-\int_0^s r(t + u) + \eta du \right) ds \\
&\quad + \lim_{\bar{s} \longrightarrow \infty} \frac{(\bar{s} - \tau)}{\bar{s}} \exp\left(-\int_0^\tau r(t + u) + \eta) du \right). \\
&= \quad \int_0^\tau \delta \exp\left(-\int_0^s r(t + u) + \eta du \right) ds + \exp\left(-\int_0^\tau r(t + u) + \eta) du \right).
\end{aligned}
$$

This is the banker's valuation of a bond held until maturity, which we denote $\tilde{\psi}(\tau, t)$.

A.5 Linear Approximation to $q(\iota, \tau, t)$

In this section we proceed towards obtaining a linear approximation of $q(\iota, \tau, t)$. First observe that by definition of $\bar{s}$, we have by the chain rule that:

$$\frac{\partial q(\iota, \tau, t)}{\partial \iota} = \frac{\partial q(\iota, \tau, t)}{\partial \bar{s}} \frac{1}{\mu}.$$

Derivative $q(\iota, \tau, t)$ **with respect to** ι**.** Consider the PDE corresponding to Case 1. The derivative of the price function with respect to $\bar{s}$ is given by

$$
\begin{aligned}
\frac{\partial q(\iota, \tau, t)}{\partial \bar{s}} &= \frac{\partial}{\partial \bar{s}} \left(\frac{\int_0^{\bar{s}} (\delta(\bar{s} - s) + \psi(\tau - s, t + s)) \exp\left(-\int_0^s r(t + u) + \eta du\right) ds}{\bar{s}} \right) \\
&= \frac{\psi(\tau - \bar{s}, t + \bar{s}) \exp\left(-\int_0^{\bar{s}} (r(t + u) + \eta) du\right)}{\bar{s}} - \frac{q(\iota, \tau, t)}{\bar{s}}.
\end{aligned}
$$

Now consider the value of the derivative evaluated at zero $\partial q(\iota, \tau, t)/\partial \iota|_{\iota=0}$. Note the limits of the three terms in the numerator are:

$$\lim_{\bar{s} \to 0} \int_0^{\bar{s}} \delta \exp\left(-\int_0^s r(t + u) + \eta du\right) ds = 0,$$

$$\lim_{\bar{s} \to 0} \psi(\tau - \bar{s}, t + \bar{s})) \exp\left(-\int_0^{\bar{s}} r(t + u) + \eta du\right) = \psi(\tau, t),$$

$$\lim_{\bar{s} \to 0} q(\iota, \tau, t) = -\psi(\tau, t),$$

respectively. Hence, the numerator converges to zero, as does the denominator of $\bar{s}$. We employ L'Hôspital's rule to obtain the derivative of interest. The derivative of the denominator is 1. Thus, we need to obtain the value of the following limit:

$$\lim_{\bar{s} \to 0} \frac{\partial}{\partial \bar{s}} \left(\int_0^{\bar{s}} \delta \exp\left(-\int_0^s r(t + u) + \eta du\right) ds + \psi(\tau - \bar{s}, t + \bar{s})) \exp\left(-\int_0^{\bar{s}} r(t + u) + \eta du\right) \right)$$

$$\dots - q(\iota, \tau, t).$$

Consider the limit of the derivative of the first two terms. Applying Leibnitz's rule and passing

the derivative inside the integral:

$$\lim_{\bar{s}\longrightarrow 0} \delta \exp\left(-\int_0^{\bar{s}} r(t+u) + \eta du\right) \exp\left(-\int_0^{\bar{s}} r(t+u) + \eta du\right) + \dots$$

$$\lim_{\bar{s}\longrightarrow 0}\left(-\frac{\partial}{\partial \tau}\psi(\tau - \bar{s}, t + \bar{s}) + \frac{\partial}{\partial t}\psi(\tau - \bar{s}, t + \bar{s}) - (r(t+\bar{s}) + \eta)\psi(\tau - \bar{s}, t + \bar{s})\right)\dots$$

$$\exp\left(-\int_0^{\bar{s}} r(t+u) + \eta du\right).$$

Taking limits $\bar{s} \longrightarrow 0$ we obtain

$$\delta - \frac{\partial}{\partial \tau}\psi(\tau, t) + \frac{\partial}{\partial t}\psi(\tau, t) - (r(t) + \eta)\psi(\tau, t) \;=\; r(t)\psi(\tau, t) - (r(t) + \eta)\psi(\tau, t)$$

$$=\; -\eta\psi(\tau, t),$$

where the first equality used the PDE of bond prices.

The limit of the second term is given by,

$$\frac{\partial q(\iota, \tau, t)}{\partial \bar{s}}\Big|_{\bar{s}=0} = \mu y_{ss}\frac{\partial q(\iota, \tau, t)}{\partial \iota}\Big|_{\iota=0}.$$

Hence, back into the L'Hôspital limit, we have that,

$$\frac{\partial q(\iota, \tau, t)}{\partial \iota}\Big|_{\iota=0} = \frac{-\eta\psi(\tau, t) - \mu y_{ss}\frac{\partial q(\iota,\tau,t)}{\partial \bar{\iota}}\big|_{\iota=0}}{\mu y_{ss}}.$$

Rearranging terms, we conclude with the term

$$\frac{\partial q(\iota, \tau, t)}{\partial \iota}\Big|_{\iota=0} = -\frac{\eta\psi(\tau, t)}{2\mu y_{ss}}.$$

First-Order Expansion. A first-order Taylor expansion around small emissions yields:

$$q(\iota, \tau, t) \simeq q(\iota, \tau, t)|_{\iota=0} + \frac{\partial q(\iota, \tau, t)}{\partial \iota}\Big|_{\iota=0}\iota.$$

With this, we show that $\bar{\lambda} = -\eta/2\mu y_{ss}$ and $\chi(\tau, t) = \psi(\tau, t)$. This proves Proposition 1.

B Discrete Commitment Problem

Define Δ as an arbitrary time step. A Poisson event that changes $X(t)$—including the option to default— ocurrs at the end $(t, t + \Delta]$ with probability $P(\Delta)$—the discrete analogue of the Poisson event. This implies that the riskless-bond price within that intervale solves:

$$\psi(\tau, t) = \int_t^{t+\min(\Delta,\tau)} e^{-\bar{r}(s-t)} \delta ds + \mathbb{I}_{[\tau \leq \Delta]} e^{-\bar{r}\tau}$$
$$+ \mathbb{I}_{[\Delta < \tau]} e^{-\bar{r}\Delta} \left[P(\Delta) \mathbb{E}_t^X \left[\Phi \left(V \left[f \left(\cdot, t + \Delta, X(t) \right) \right] \right) \psi(\tau, t + \Delta) \right] \right.$$
$$\left. + (1 - P(\Delta)) \psi(\tau, t + \Delta) \right] \tag{B.1}$$

The first two terms are deterministic because there are no shocks in that time frame. Applying the Feynman-Kac formula produces a modified PDE for the bond pricing equation (B.1):

$$\bar{r}\psi(\tau, t) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}, \text{ if } \tau \in (0, T) \tag{B.2}$$
$$\psi(0, t) = 1, \text{ if } \tau = 0,$$
$$\psi(\tau, t + \Delta) = P(\Delta) \mathbb{E}_t^X \left[\Phi \left(\hat{V} \left[f \left(\cdot, t + \Delta, X(t) \right) \right] \right) \psi(\tau, t + \Delta) \right] + (1 - P(\Delta)) \psi(\tau, t + \Delta), \tau \in (0, T)$$

Let the expected value upon default be given by:

$$\Gamma \left(V \left[f \left(\cdot, t + \Delta, X(t) \right) \right] \right) = \mathbb{E}_t^X \left[V^D | V^D > V \left[f \left(\cdot, t + \Delta, X(t) \right) \right] \right].$$

The solution to the **PD** problem is the Markov-Perfect Stackelberg Equilibrium defined as the limit as $\Delta \to 0$ of the family of Δ-**PD** problem's.

With some abuse of notation define $V(t) \equiv V \left[f(\cdot, t), X(t) \right]$ and $\hat{V}(t) \equiv \hat{V} \left[f(\cdot, t), X(t) \right]$ is the aggregate value functional of the problem, which depends on time through its dependence on the distribution.

We now define the Government problem in that interval:

Problem 8 *The Δ-**PD** problem is given by:*

$$\hat{V} \left[f(\cdot, t), X(t) \right] = \max_{\{\iota(\cdot)\} \in \mathcal{I}} \int_t^{t+\Delta} e^{-\rho(s-t)} \mathcal{U}(c(s)) ds$$
$$+ e^{-\rho\Delta} \left[P(\Delta) \left[\Gamma \left(V(t + \Delta) \right) + V(t + \Delta) \Phi \left(V(t + \Delta) \right) \right] \right.$$
$$\left. + (1 - P(\Delta)) \hat{V}(t + \Delta) \right] \tag{B.3}$$

subject to the law of motion of debt (2.1), the budget constraint (2.2) and the bond pricing equation (B.2).

The Δ-**PD** problem is defined over arbitrary finite-length interval $(t, t + \Delta]$. Default may only happen at the end of the interval provided that Poisson jump arrived $P(\Delta)$. If the option indeed arrives, the continuation value is $\Gamma\left(V\left(t + \Delta\right)\right) + V\left(t + \Delta\right)\left(t + \Delta\right)\Phi\left(V\left(t + \Delta\right)\right)$. The first term is $\Gamma\left(V\left(t + \Delta\right)\right)$ is the expected value of a default option, which is of course, execcuted only if the values of default exceed $V(t + \Delta)$. In turn, $V\left(t + \Delta\right)\left(t + \Delta\right)\Phi\left(V\left(t + \Delta\right)\right)$ is the probability that the option is not executed times the value of the deterministic problem. If the option has not arrived, the value is $\hat{V}\left(t + \Delta\right)$.

C Proofs

C.1 Proof of Proposition 2

First we construct a Lagrangian on the space of functions g such that are square-intergrable, $\left\| e^{-\rho t/2} g(\tau, t) \right\|^2 < \infty$. The Lagrangian, after replacing $c(t)$ from the budget constraint, is:

$$
\mathcal{L}[\iota, f] = \int_0^\infty e^{-\rho t} U\left(y(t) - f(0, t) + \int_0^T \left[q(t, \tau, \iota) \iota(\tau, t) - \delta f(\tau, t) \right] d\tau \right) dt
$$
$$
+ \int_0^\infty \int_0^T e^{-\rho t} j(\tau, t) \left(-\frac{\partial f}{\partial t} + \iota(\tau, t) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
$$

where $j(\tau, t)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h(\tau, t)$, $e^{-\rho t} h \in L^2([0, T] \times [0, \infty))$, around the optimal solution. Since the initial distribution f_0 is given, any feasible perturbation must have $h(\tau, 0) = 0$. In addition, we know that $f(T, t) = 0$ because $f(T^+, t) = 0$ (by construction) and issuances are infinitesimal. Thus, any admissible variation must feature $h(T, t) = 0$.

At an optimal solution f, the Lagrangian must satisfy $\mathcal{L}[\iota, f] \geq \mathcal{L}[\iota, f + \alpha h]$ for any perturbation $h(\tau, t)$. Taking derivative with respecto to α —i.e., computing the Gâteaux derivative, for any suitable $h(\tau, t)$ we obtain:

$$
\left. \frac{\partial}{\partial \oslash} \mathcal{L}[\iota, f + \alpha h] \right|_{\alpha=0} = \int_0^\infty e^{-\rho t} U'(c) \left[-h(0, t) - \int_0^T \delta h(\tau, t) d\tau \right] dt
$$
$$
- \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j(\tau, t) d\tau dt
$$
$$
+ \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j(\tau, t) d\tau dt.
$$

We employ intergration by parts to show that:to note that:

$$
\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j(\tau, t) d\tau dt = \int_0^\infty e^{-\rho t} [h(T, t) j(T, t) - h(0, t) j(0, t) dt - \int_0^T h(\tau, t) \frac{\partial j}{\partial \tau} d\tau] dt,
$$

and

$$\int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j\left(\tau, t\right) d\tau dt = \int_0^T \int_0^\infty e^{-\rho t} \frac{\partial h}{\partial t} j\left(\tau, t\right) dt d\tau$$

$$= \int_0^T \left(\lim_{s \to \infty} e^{-\rho s} [h\left(\tau, s\right) j\left(\tau, s\right)] - h\left(\tau, 0\right) j\left(\tau, 0\right) \right) d\tau$$

$$- \int_0^T \int_0^\infty e^{-\rho t} \left(\frac{\partial j\left(\tau, t\right)}{\partial t} - \rho j\left(\tau, t\right) \right) h(\tau, t) dt d\tau$$

$$= \int_0^T \left(\lim_{s \to \infty} e^{-\rho s} [h\left(\tau, s\right) j\left(\tau, s\right)] - h\left(\tau, 0\right) j\left(\tau, 0\right) \right) d\tau -$$

$$\int_0^\infty e^{-\rho t} \int_0^T \left(\frac{\partial j\left(\tau, t\right)}{\partial t} - \rho j\left(\tau, t\right) \right) h(\tau, t) d\tau dt.$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$0 = \int_0^\infty e^{-\rho t} U'\left(c\right) \left[-h\left(0, t\right) - \int_0^T \delta h\left(\tau, t\right) d\tau \right] dt$$

$$+ \int_0^\infty \int_0^T e^{-\rho t} \left(-\rho j - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h\left(\tau, t\right) d\tau dt$$

$$+ \int_0^\infty e^{-\rho t} \left(h\left(T, t\right) j\left(T, t\right) - h\left(0, t\right) j\left(0, t\right) \right) dt$$

$$- \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h\left(\tau, s\right) j\left(\tau, s\right) d\tau.$$

We rearrange terms to obtain:

$$0 = - \int_0^\infty e^{-\rho t} \left[U'\left(c\right) - j\left(0, t\right) \right] h\left(0, t\right) dt$$

$$+ \int_0^\infty \int_0^T e^{-\rho t} \left(-\rho j - U'\left(c\right) \delta - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h\left(\tau, t\right) d\tau dt$$

$$- \int_0^\infty e^{-\rho t} \left(h\left(T, t\right) j\left(T, t\right) \right) dt$$

$$- \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h\left(\tau, s\right) j\left(\tau, s\right) d\tau.$$

Since, $h\left(T, t\right) = 0$ is a condition for an admisible control, then, the term is zero. Since the condition is required for any control, then all terms multiplying the non-zero terms in $h(\tau, t) = 0$. This yields a system of necessary conditions for the Lagrangian multipliers.

$$\rho j\left(\tau, t\right) = -\delta U'\left(c\left(t\right)\right) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0, T] \quad \text{(C.1)}$$

$$j\left(0, t\right) = -U'\left(c\left(t\right)\right), \text{ if } \tau = 0,$$

$$\lim_{t \to \infty} e^{-\rho t} j\left(\tau, t\right) = 0.$$

Next, we perturb the control. We proceed in a similar fashion :

$$\frac{\partial}{\partial \alpha} \mathcal{L} \left[\iota + \alpha h, , f \right] \Big|_{\alpha = 0} = \int_0^\infty e^{-\rho t} U'(c) \left[\int_0^T \left(\frac{\partial q}{\partial \iota} \iota (\tau, t) + q (t, \tau, \iota, f) \right) h (\tau, t) \, d\tau \right] dt$$
$$+ \int_0^\infty \int_0^T e^{-\rho t} h (\tau, t) j (\tau, t) \, d\tau dt.$$

Collecting terms and setting the Lagrangian to zero, we obtain:

$$\int_0^\infty \int_0^T e^{-\rho t} \left(j (\tau, t) + U'(c) \left[\left(\frac{\partial q}{\partial \iota} \iota (\tau, t) + q (t, \tau, \iota, f) \right) h (\tau, t) \, d\tau \right] \right) dt.$$

Thus, setting the term in parenthesis to zero, amounts to setting:

$$U'(c) \left(\frac{\partial q}{\partial \iota} \iota (\tau, t) + q (t, \tau, \iota, f) \right) = -j (\tau, t).$$

Next, we define the Lagrangian multiplier in terms of goods:

$$v (\tau, t) = -j (\tau, t) / U'(c(t)). \tag{C.2}$$

Taking the derivative of $v (\tau, t)$ with respect to t and τ we can express the necessary conditions in terms of v. In particular, we transform the PDE equation (D.1) into the summary equations in the Propositions. That is:

$$\left(\rho - \frac{U''(c(t))}{U'(c(t))} c(t) \frac{\dot{c}(t)}{c(t)} \right) v (\tau, t) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, \infty),$$
$$v (0, t) = 1, \text{ if } \tau = 0,$$
$$\lim_{t \to \infty} e^{-\rho t} v (\tau, t) = 0,$$

and the first-order condition, (C.2), is

$$\frac{\partial q}{\partial \iota} \iota (\tau, t) + q (t, \tau, \iota, f) = v (\tau, t).$$

C.2 Proof of Proposition 3

The first part of the proof is just a direct consequence of the first-order condition $v (\tau, t) = -\psi (\tau, t)$. Bond prices are given by (3.1) while the Government valuations still given by (3.7). Since both equations must be equal in a bounded solution, we conclude that

$$\bar{r}(t) = r(t) = \rho - \frac{\mathcal{U}''(c(t))}{\mathcal{U}'(c(t))} \frac{dc}{dt},$$

must describe the dynamics of consumption.

The second part of the proof derives the law of motion of $b(t)$. First we take the derivative with respect to time at both sides of definition (3.8):

$$\dot{b}(t) = -\int_0^T (\psi_t(\tau,t)f(\tau,t) + \psi(\tau,t)f_t(\tau,t))d\tau.$$

Next, we substitute out for $\iota(\tau,t)$ from (??) using (??) to express the budget constraint in terms of f.

$$c(t) = y(t) - f(0,t) + \int_0^T \psi(\tau,t)\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \tau}\right) - \delta f(\tau,t)\,d\tau. \qquad (C.3)$$

Next, apply integration by parts to the following expression:

$$\int_0^T \psi(\tau,t)\frac{\partial f}{\partial \tau}d\tau = \psi(T,t)f(T,t) - \psi(0,t)f(0,t) - \int_0^T \psi_\tau(\tau,t)f(\tau,t)\,d\tau.$$

Observe that $f(T,t) = 0$ as long as the solution is smooth. Also, recall that by construction $\psi(0,t) = 1$. Hence:

$$\int_0^T \psi(\tau,t)\frac{\partial f}{\partial \tau}d\tau = -f(0,t) - \int_0^T \psi_\tau(\tau,t)f(\tau,t)\,d\tau.$$

The next step uses (??) and substitutes $\psi_\tau(\tau,t)$ out of the expression above. We obtain:

$$\int_0^T \psi(\tau,t)\frac{\partial f}{\partial \tau}d\tau = -f(0,t) - \int_0^T (\delta + \psi_t(\tau,t) - r(t)\psi(\tau,t))f(\tau,t)\,d\tau.$$

We substitute this expression into (C.3), and thus:

$$\begin{aligned}
c(t) =& y(t) - f(0,t) + \int_0^T \psi(\tau,t)\frac{\partial f}{\partial t} - \delta f(\tau,t)\,d\tau... \\
& - \left(-f(0,t) - \int_0^T (\delta + \psi_t(\tau,t) - r(t)\psi(\tau,t))f(\tau,t)\,d\tau\right) \\
=& y(t) + \int_0^T \psi(\tau,t)f_t(\tau,t) + \psi_t(\tau,t)f(\tau,t)\,d\tau - \int_0^T \bar{r}(t)\psi(\tau,t)f(\tau,t).
\end{aligned}$$

Rearraging terms and employing the defintions above, we obtain:

$$\dot{b}(t) = y(t) - c(t) + \bar{r}(t)b(t),$$

as desired.

C.3 Proof of Proposition 3

Here we proof a more detailed version of Propositions 3. The full proposition is:

Proposition 14 *Consider a steady state for the exogenous variables. Then, there exists a steady state in problem* **PF** *if and only if* $\bar{\lambda} > \bar{\lambda}_o$ *for some* $\bar{\lambda}_o$. *If instead,* $\bar{\lambda} \leq \bar{\lambda}_o$, *there is no steady state but consumption evolves asymptotically. In particular, the assymptotic behavior of the* **PF** *problem is:*

*Case 1 (**High Liquidity Costs**). For liquidity costs above the threshold value* $\bar{\lambda} > \bar{\lambda}_o$, *variables converge to a steady state characterized by the following system:*

$$\frac{\dot{c}_{ss}}{c_{ss}} = 0$$

$$r_{ss} = 0$$

$$v_{ss}(\tau) = \frac{\delta}{\rho}(1 - e^{-\rho\tau}) + e^{-\rho\tau}$$

$$\iota_{ss}(\tau) = \frac{\psi(\tau) - v_{ss}(\tau)}{\bar{\lambda}\chi(\tau)}$$

$$f_{ss}(\tau) = \int_{\tau}^{T} \iota_{ss}(s)ds$$

$$c_{ss} = y - f_{ss}(0) + \int_{0}^{T}\left[\iota_{ss}(\tau)\psi(\tau) - \frac{\bar{\lambda}\chi(\tau)}{2}\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau)\right]d\tau. \tag{C.4}$$

*Case 2 (**Low Liquidity Costs**). For liquidity costs below the threshold value* $0 < \bar{\lambda} \leq \bar{\lambda}_o$, *variables converge assymptotically to:*

$$\lim_{T\to\infty}\frac{c(T)}{c(t)} = e^{-\frac{(\rho - r_\infty(\bar{\lambda}))(T-t)}{\sigma}}$$

$$v_\infty(\tau, r_\infty(\bar{\lambda})) = -\delta(1 - e^{-r_\infty(\bar{\lambda})\tau}) + e^{-r_\infty(\bar{\lambda})\tau}$$

$$\iota_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{\psi(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\chi(\tau)}$$

$$f_\infty(\tau,, r_\infty(\bar{\lambda})) = \int_{\tau}^{T} \iota_\infty(s, r_\infty(\bar{\lambda}))ds$$

and $r_\infty(\bar{\lambda})$ *satisfies* $\bar{r} \leq r_\infty(\bar{\lambda}) < \rho$ *and solves:*

$$
\begin{aligned}
c_\infty &= 0 \\
&= y_{ss} - f_\infty(0,, r_\infty(\bar{\lambda})) + \int_{0}^{T}\left[\iota_\infty(\tau, r_\infty(\bar{\lambda}))\psi(\tau) - \frac{\bar{\lambda}\chi(\tau, t)}{2}\iota_\infty(\tau, r_\infty(\bar{\lambda}))^2 - \delta f_\infty(\tau, r_\infty(\bar{\lambda}))\right]d\tau.
\end{aligned}
$$

Threshold. *The threshold* $\bar{\lambda}_o$ *solves* $|c_{ss}|_{\bar{\lambda}=\bar{\lambda}_o} = 0$ *in (C.4) and* $\lim_{\bar{\lambda}\to\bar{\lambda}_0} r_\infty(\bar{\lambda}) = \rho$.

First observe that as $\bar{\lambda} \to \infty$, the optimal issuance policy (3.10) approaches $\iota(\tau, t) = 0$. Thus,

for that limit, $c_{ss} = y > 0$ and $f_{ss}(\tau) = 0$. Next, consider the system in Case 1 of **??**.

$$\iota_{ss}(\tau) = \frac{\psi(\tau) - v_{ss}(\tau)}{\bar{\lambda}\chi(\tau)}, \tag{C.5}$$

$$v_{ss}(\tau) = \frac{\delta}{\rho}(1 - e^{-\rho\tau}) + e^{-\rho\tau} \tag{C.6}$$

$$f_{ss}(\tau) = \int_{\tau}^{T} \iota_{ss}(s)ds \tag{C.7}$$

$$c_{ss} = y_{ss} - f_{ss}(0) + \int_{0}^{T} \left[\psi_{ss}(\tau)\iota_{ss}(\tau) - \frac{\bar{\lambda}\chi(\tau)}{2}\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau) \right]. \tag{C.8}$$

The system is conitunuous in $\bar{\lambda}$ and by continuity there exists a solution to $c_{ss} > 0$, $\iota_{ss} > 0$ and $f_{ss} > 0$ — as long as $\rho > r$ then $v_{ss}(\tau) > \psi(\tau)$, and then thus $\dot{c}(t) = 0$ we have

$$r_{ss} \equiv r(t) = \rho.$$

Next, we proof that c_{ss} decreases with $\bar{\lambda}$ decreases. Observe that, steady state internal valuations $v_{ss}(\tau)$ in (C.6) and bond prices $\psi(\tau)$ are independent of $\bar{\lambda}$. As steady state debt issuances $\iota_{ss}(\tau)$ (C.5) are a monotonously decreasing function of $\bar{\lambda}$:

$$\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}\iota_{ss}(\tau) < 0,$$

then the total amount of debt at each maturity $f_{ss}(\tau)$ in (C.7) is also decreasing with $\bar{\lambda}$:

$$\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} == -\frac{1}{\bar{\lambda}}f_{ss}(\tau) < 0.$$

If we take derivatives with respect to $\bar{\lambda}$ in the budget constraint (C.8) to obtain

$$\begin{aligned}
\frac{\partial c_{ss}}{\partial \bar{\lambda}} &= \frac{\partial f_{ss}(0)}{\partial \bar{\lambda}} + \int_{0}^{T} \left[\psi(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \frac{\chi(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\chi(\tau)\iota_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \delta\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} \right] d\tau \\
&= -\frac{1}{\bar{\lambda}}f_{ss}(0) - \frac{1}{\bar{\lambda}}\int_{0}^{T} \left[\psi(\tau)\iota_{ss}(\tau) + \bar{\lambda}\frac{\chi(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\chi(\tau)\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau) \right] d\tau \\
&= -\frac{1}{\bar{\lambda}}c_{ss} < 0.
\end{aligned}$$

Third, observe that $\iota_{ss}(\tau)$ can be made arbitrarilly large by increasing $\bar{\lambda}$. Thus, there exist a value of $\bar{\lambda} \geq 0$ such that $c_{ss} = 0$ in the system above. We denote this value by $\bar{\lambda}_0$. For $\bar{\lambda} \leq \bar{\lambda}_0$, if a steady state exists, it would imply $c_{ss} < 0$, outside of the range of admissible values. Therefore, there is no steady state in this case.

Assume the economy grows asymptotically at rate $g_\infty(\bar{\lambda}) \equiv \lim_{t\to\infty} \frac{1}{c(t)}\frac{dc}{dt}$. If $g_\infty(\bar{\lambda}) > 0$ then consumption would grow to infinity, which violates the budget constraint. Thus, if

there exists an assymptotic the growth rate, it is negative: $g_\infty(\bar{\lambda}) < 0$. If we define $r_\infty(\bar{\lambda}) \equiv (\rho + \gamma g_\infty(\bar{\lambda})) < \rho$, the growth rate of the economy can be expressed as

$$g_\infty(\bar{\lambda}) = -\frac{(\rho - r_\infty(\bar{\lambda}))}{\gamma}.$$

When this is the case, the assymptotic valuation is

$$v_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{\delta\left(1 - e^{-r_\infty(\bar{\lambda})\tau}\right)}{r_\infty(\bar{\lambda})} + e^{-r_\infty(\bar{\lambda})\tau}.$$

To obtain the discount factor bounts, observe that if $|v_\infty(\tau, r_\infty(\bar{\lambda}))| \le \psi(\tau)$ the optimal issuance is non-negative. Otherwise issuances would be negative at all maturities and the country would be an asymptotic net debt holder. This cannot an optimal solution as this implies that consumption can be increased just by reducing the amount of foreign assets. Therefore, $r_\infty(\bar{\lambda}) \ge \bar{r}$.

C.4 Proof of Proposition 5

Consider the following limit:

$$
\begin{aligned}
\iota_\infty^*(\tau) &= \lim_{\bar{\lambda}\to 0} \iota_\infty(\tau, r_\infty(\bar{\lambda})) = \lim_{\bar{\lambda}\to 0} \frac{\psi(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\chi(\tau)} \\
&= \lim_{\bar{\lambda}\to 0} \frac{1}{\bar{\lambda}\chi(\tau)}\left[\frac{\delta(1 - e^{-\bar{r}\tau})}{\bar{r}} - \frac{\delta\left(1 - e^{-r_\infty(\bar{\lambda})\tau}\right)}{r} + e^{-\bar{r}\tau} - e^{-r_\infty(\bar{\lambda})\tau}\right].
\end{aligned}
$$

This is a limit of the form $\frac{0}{0}$ as $\lim_{\bar{\lambda}\to 0} r_\infty(\bar{\lambda}) = \bar{r}$. We do not have an expression for $r_\infty(\bar{\lambda})$, so we cannot apply L'Hôpital's rule directly. Instead, we compute the following limit:

$$
\lim_{\bar{\lambda}\to 0} \frac{\iota_\infty(\tau, r_\infty(\bar{\lambda}))}{\iota_\infty(T, r_\infty(\bar{\lambda}))} = \lim_{r_\infty(\bar{\lambda})\to\bar{r}} \frac{\frac{\delta(1-e^{-\bar{r}\tau})}{\bar{r}} - \frac{\delta(1-e^{-r_\infty(\bar{\lambda})\tau})}{r} + e^{-\bar{r}\tau} - e^{-r_\infty(\bar{\lambda})\tau}}{\frac{\delta(1-e^{-\bar{r}T})}{\bar{r}} - \frac{\delta(1-e^{-r_\infty(\bar{\lambda})T})}{r} + e^{-\bar{r}T} - e^{-r_\infty(\bar{\lambda})T}} \frac{\chi(T)}{\chi(\tau)},
$$

which also has a limit of the form $\frac{0}{0}$. Now we can apply L'Hôpital's. We obtain:

$$
\lim_{\bar{\lambda}\to 0} \frac{\iota_\infty(\tau, r_\infty(\bar{\lambda}))}{\iota_\infty(T, r_\infty(\bar{\lambda}))} = \frac{\frac{-\delta\bar{r}\tau e^{-\bar{r}\tau} + \delta(1-e^{-\bar{r}\tau})}{\bar{r}^2} + \tau e^{-\bar{r}\tau}}{\frac{-\delta\bar{r}T e^{-\bar{r}T} + \delta(1-e^{-\bar{r}T})}{\bar{r}^2} + Te^{-rT}} \frac{\chi(T)}{\chi(\tau)} = \frac{1 + [-1 + (\bar{r}/\delta - 1)\bar{r}\tau]e^{-\bar{r}\tau}}{1 + [-1 + (\bar{r}/\delta - 1)\bar{r}T]e^{-\bar{r}T}} \frac{\chi(T)}{\chi(\tau)}.
$$

If we define $\kappa \equiv \lim_{\bar{\lambda}\to 0} \iota_\infty(T, r_\infty(\bar{\lambda}))$ and thus$\lim_{\bar{\lambda}\to 0} \iota_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{1 + [-1 + (\bar{r}/\delta-1)\bar{r}\tau]e^{-\bar{r}\tau}}{1 + [-1 + (\bar{r}/\delta-1)\bar{r}T]e^{-\bar{r}T}} \frac{\chi(T)}{\chi(\tau)}\kappa.$

The value of κ then must be consistent with zero consumption:

$$y - f_\infty^*(0) + \int_0^T \left[\iota_\infty^*(\tau)\,\psi(\tau) - \delta f_\infty^*(\tau)\right] d\tau = 0,$$

for $f_\infty(\tau) = \int_\tau^T \iota_\infty^*(s)\,ds$.

C.5   Proof of Proposition 7

To compute the dynamics of $\iota\left[f\left(\cdot,t\right),X\left(t\right)\right]$ we employ dynamic programming. The value functional before the shock arrives is

$$\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]=\max_{\{\hat{\iota}(\cdot)\}\in\mathcal{I}}\mathbb{E}_{t}\left[\int_{t}^{t^{o}}e^{-\rho(s-t)}\mathcal{U}\left(\hat{c}\left(s\right)\right)ds+e^{-\rho(t^{o}-t)}\mathbb{E}_{t^{o}}^{X}\left[V\left[f\left(\cdot,t^{o}\right),X\left(t^{o}\right)\right]\right]\right].$$

If we are explicit about the arrival time distribution of the shock, the value function can be expressed as:

$$\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]=\max_{\{\hat{\iota}(\cdot)\}\in\mathcal{I}}\int_{t}^{\infty}e^{-\rho(s-t)}\left[e^{-\phi(s-t)}\mathcal{U}\left(\hat{c}\left(s\right)\right)+\left(1-e^{-\phi(s-t)}\right)\mathbb{E}_{s}^{X}\left[V\left[f\left(\cdot,s\right),X\left(s\right)\right]\right]\right]ds.$$

Note that $e^{-\phi(s-t)}$ is the probability that the shock arrives later that times conditional on not having arrived at time t. If we apply Bellman's Principle of Optimality, for an arbitrary $t'>t$, the value functional can be expressed recursively through:

$$
\begin{aligned}
\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right] \quad &= \quad \max_{\{\hat{\iota}(\cdot)\}\in\mathcal{I}}\int_{t}^{t'}e^{-\rho(s-t)}\left[e^{-\phi(s-t)}\mathcal{U}\left(\hat{c}\left(s\right)\right)+\left(1-e^{-\phi(s-t)}\right)\mathbb{E}_{s}^{X}\left[V\left[f\left(\cdot,s\right),X\left(s\right)\right]\right]\right]ds \\
&+e^{-\rho(t'-t)}\left[e^{-\phi(t'-t)}\hat{V}\left[f\left(\cdot,t'\right),X\left(t'\right)\right]+\left(1-e^{-\phi(t'-t)}\right)\mathbb{E}_{t'}^{X}\left[V\left[f\left(\cdot,t'\right),X\left(t'\right)\right]\right]\right].
\end{aligned}
$$

Then we take the derivative with respect to t' on both sides, and then the limit $t'\to t$:

$$
\begin{aligned}
0 \quad &= \quad \max_{\{\iota(\cdot)\}\in\mathcal{I}}\mathcal{U}\left(\hat{c}\left(t\right)\right)-\left(\rho+\phi\right)\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right] \\
&+ \quad \frac{1}{dt}d\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]+\left(-\rho+\left(\rho+\phi\right)\right)\mathbb{E}_{t}^{X}\left[V\left[f\left(\cdot,t\right),X\left(t\right)\right]\right].
\end{aligned}
$$

Rearranging terms we arrive at:

$$
\begin{aligned}
\rho\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right] \quad &= \quad \max_{\{\iota(\cdot)\}\in\mathcal{I}}\mathcal{U}\left(\hat{c}\left(t\right)\right)+\frac{1}{dt}d\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right] & \text{(C.9)} \\
&+ \quad \phi\left[\mathbb{E}_{t}^{X}\left[V\left[f\left(\cdot,t\right),X\left(t\right)\right]\right]-\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]\right]. & \text{(C.10)}
\end{aligned}
$$

We work with the HJB (C.9) in the space of functions $L^{2}\left(\left[0,T\right)\right.$. In order to compute the derivative $\frac{1}{dt}d\hat{V}\left[f\left(\cdot,t\right),X\left(t\right)\right]$ we need to be able to compute the Gâteaux derivative of V with respect to f in an arbitrary direction $h\left(\tau,t\right)\in L^{2}\left(\left[0,T\right]\right)$. Notice the abuse of notation as t is now an index identifying the point in time at which the HJB (C.9) is evaluated. If $\hat{V}$ is Fréchet derivable with respect to f then the Gâteaux derivative is a linear functional on h. The Riesz Representation Theorem allows us to represent any linear functional as an inner product and hence the

Gâteaux derivative with respect to f:

$$\frac{\partial}{\partial \alpha} \hat{V} \left[f\left(\cdot, t\right) + \alpha h, X(t) \right] \bigg|_{\alpha=0} = \int_0^T \frac{\delta \hat{V}}{\delta f} h\left(\tau, t\right) d\tau \tag{C.11}$$

where $\frac{\delta \hat{V}}{\delta f}\left(\tau, t, , X\left(t\right)\right) \in L^2\left([0,T] \times [0,\infty) \times \mathbb{R}\right)$ is a function. Applying the definition of derivative and the chain rule, the derivative $\frac{1}{dt} d\hat{V}\left[f\left(\cdot, t\right), X\left(t\right)\right]$ can be expressed as

$$\frac{1}{dt} d\hat{V} \left[f\left(\cdot, t\right), X(t)\right] = \frac{\partial}{\partial \alpha} \hat{V} \left[f\left(\cdot, t\right) + \alpha \frac{\partial f\left(\cdot, t\right)}{\partial t}, X(t) \right]\bigg|_{\alpha=0} + \frac{\partial \hat{V}}{\partial X} \frac{dX}{dt},$$

Combining this expression with the previous result, we obtain

$$\frac{1}{dt} d\hat{V} \left[f\left(\cdot, t\right), X(t)\right] = \int_0^T \frac{\delta \hat{V}}{\delta f} \frac{\partial f\left(\cdot, t\right)}{\partial t} d\tau + \frac{\partial \hat{V}}{\partial X} \frac{dX}{dt}.$$

The HJB (C.9) can thus be expressed as

$$\rho \hat{V} \left[f\left(\cdot, t\right), X\left(t\right)\right] = \max_{\{\iota(\cdot)\} \in \mathcal{I}} \mathcal{U}\left(\hat{c}\left(t\right)\right) + \int_0^T \frac{\delta \hat{V}}{\delta f} \frac{\partial f\left(\cdot, t\right)}{\partial t} d\tau + \frac{\partial \hat{V}}{\partial X} \frac{dX}{dt} \tag{C.12}$$

$$+ \phi \left[\mathbb{E}_t^X \left[V\left[f\left(\cdot, t\right), X\left(t\right)\right]\right] - \hat{V}\left[f\left(\cdot, t\right), X\left(t\right)\right] \right] \tag{C.13}$$

$$= \max_{\iota(\cdot,t)} \mathcal{U}\left(y\left(t\right) - f\left(0, t\right) + \int_0^T \left[\hat{q}\left(\tau, t, \hat{\iota}\right)\hat{\iota}\left(\tau, t\right) - \delta f\left(\tau, t\right)\right] d\tau\right)$$

$$+ \int_0^T \frac{\delta \hat{V}}{\delta f} \left(\hat{\iota}\left(\tau, t\right) + \frac{\partial f}{\partial \tau}\right) d\tau + \frac{\partial \hat{V}}{\partial X} \frac{dX}{dt}$$

$$+ \phi \left[\mathbb{E}_t^X \left[V\left[f\left(\cdot, t\right), X\left(t\right)\right]\right] - \hat{V}\left[f\left(\cdot, t\right), X\left(t\right)\right]\right]. \tag{C.14}$$

where we substituted consumption out using the budget constraint and used the KFE to substitute out $\partial f / \partial t$. The first-order condition with respect to $\hat{\iota}(\tau, t)$ can be obtained by computing the Gâteaux derivative in (C.12) with respect to the control:

$$0 = \frac{\partial}{\partial \alpha} \mathcal{U} \left(y\left(t\right) - f\left(0, t\right) + \int_0^T \left[\hat{q}\left(\tau, t, \hat{\iota}(\tau, t) + \alpha h(\tau, t)\right)\left(\hat{\iota}(\tau, t) + \alpha h(\tau, t)\right) - \delta f(\tau, t)\right] d\tau\right)\bigg|_{\alpha=0}$$

$$+ \frac{\partial}{\partial \alpha} \int_0^T \frac{\delta \hat{V}}{\delta f} \left(\hat{\iota}(\tau, t) + \alpha h(\tau, t) + \frac{\partial f}{\partial \tau}\right) d\tau \bigg|_{\alpha=0},$$

and

$$0 = \mathcal{U}'\left(\hat{c}\left(t\right)\right) \left[\int_0^T \left(\frac{\partial \hat{q}}{\partial \hat{\iota}} \hat{\iota}\left(\tau, t\right) + \hat{q}\left(\tau, t, \iota\right)\right) h\left(\tau, t\right) d\tau\right] dt$$

$$+ \int_0^T \frac{\delta \hat{V}}{\delta f} h\left(\tau, t\right) d\tau.$$

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Because, the Gâteaux derivative should be zero for any direction $h(\tau, t) \in L^2([0, T])$, the first-order condition is equivalent to:

$$\mathcal{U}'(\hat{c}(t)) \left(\frac{\partial \hat{q}}{\partial \hat{\iota}} \hat{\iota}(\tau, t) + \hat{q}(\tau, t, \iota) \right) = -\frac{\delta \hat{V}}{\delta f}(\tau, t, X(t)).$$

If we define

$$\hat{j}(\tau, t) \equiv \frac{\delta \hat{V}}{\delta f}(\tau, t, X(t)). \tag{C.15}$$

Then the first-order condition results

$$\mathcal{U}'(\hat{c}(t)) \left(\frac{\partial \hat{q}}{\partial \iota} \hat{\iota}(\tau, t) + \hat{q}(\tau, t, \iota) \right) = -\hat{j}(\tau, t). \tag{C.16}$$

Next, we compute Gâteaux derivatives with respect to f in the HJB equation (C.12) applying the Envelope Theorem:

$$\rho \frac{\partial}{\partial \alpha} \hat{V}[f(\cdot, t) + \alpha h(\cdot, t), X(t)] \Big|_{\alpha=0} =$$

$$+ \frac{\partial}{\partial \alpha} \mathcal{U} \left(y(t) - (f(0, t) + \alpha h(0, t)) + \int_0^T [q\hat{\iota} - \delta(f + \alpha h)] d\tau \right) \Big|_{\alpha=0}$$

$$+ \frac{\partial}{\partial \alpha} \int_0^T \frac{\delta \hat{V}}{\delta(f + \alpha h)} \left(\hat{\iota} + \frac{\partial(f + \alpha h)}{\partial \tau} \right) d\tau \Big|_{\alpha=0}$$

$$+ \frac{\partial}{\partial \alpha} \frac{\partial \hat{V}}{\partial X} [f(\cdot, t) + \alpha h(\cdot, t), X(t)] \Big|_{\alpha=0} \frac{dX}{dt}$$

$$+ \phi \frac{\partial}{\partial \alpha} \left[\mathbb{E}_t^X [V[f + \alpha h, X(t)]] - \hat{V}[f + \alpha h, X(t)] \right] \Big|_{\alpha=0}.$$

Applying the Riesz Representation Theorem, we can express the Gâteaux derivatives as

$$\frac{\partial}{\partial \alpha} \hat{V}[f(\cdot, t) + \alpha h, X(t)] \Big|_{\alpha=0} = \int_0^T \frac{\delta \hat{V}}{\delta f} h(\tau, t) d\tau,$$

$$\int_0^T \frac{\partial}{\partial \alpha} \frac{\delta \hat{V}[f(\tau', t) + \alpha h(\tau', t), X(t)]}{\delta f(\tau, t)} \Big|_{\alpha=0} \left(\hat{\iota} + \frac{\partial f}{\partial \tau} \right) d\tau =$$

$$\int_0^T \left[\int_0^T \frac{\delta^2 \hat{V}}{\delta f^2}(\tau, \tau', t, X(t)) h(\tau', t) d\tau' \right] \left(\hat{\iota}(\tau, t) + \frac{\partial f}{\partial \tau}(\tau, t) \right) d\tau,$$

$$\int_0^T \frac{\delta \hat{V}}{\delta f} \frac{\partial}{\partial \alpha} \left(\hat{\iota} + \frac{\partial(f + \alpha h)}{\partial \tau} \right) \Big|_{\alpha=0} d\tau = \int_0^T \frac{\delta \hat{V}}{\delta f} \frac{\partial h}{\partial \tau}(\tau, t) d\tau,$$

$$\frac{\partial}{\partial \alpha} \frac{\partial \hat{V}}{\partial X} \left[f\left(\cdot, t\right) + \alpha h\left(\cdot, t\right), X\left(t\right) \right] \Bigg|_{\alpha=0} = \int_0^T \frac{\partial}{\partial X} \left(\frac{\delta \hat{V}}{\delta f} \right) h\left(\tau, t\right) d\tau,$$

$$\frac{\partial}{\partial \alpha} \left[\mathbb{E}_t^X \left[V\left[f + \alpha h, X\left(t\right) \right] \right] - \hat{V} \left[f + \alpha h, X\left(t\right) \right] \right] \Bigg|_{\alpha=0} = \int_0^T \left(\mathbb{E}_t^X \left[\frac{\delta V}{\delta f} \right] - \frac{\delta \hat{V}}{\delta f} \right) h\left(\tau, t\right) d\tau.$$

The term $\int_0^T \frac{\delta^2 V}{\delta f^2} \left(\tau, \tau', t, X(t)\right) h\left(\tau', t\right) d\tau'$ is the second Gâteaux derivative of V. There is a mapping between this term and $\frac{\partial \hat{j}}{\partial t}$, where $\hat{j}$ has been defined in (C.15):

$$\begin{aligned}
\frac{\partial \hat{j}}{\partial t} &= \frac{\partial}{\partial t} \frac{\delta \hat{V}}{\delta f} \\
&= \frac{\partial}{\partial \alpha} \frac{\delta \hat{V}}{\delta f} \left[f\left(\cdot, t\right) + \alpha \frac{\partial f\left(\cdot, t\right)}{\partial t}, X(t) \right] \Bigg|_{\alpha=0} + \frac{\partial}{\partial X} \left(\frac{\delta \hat{V}}{\delta f} \right) \frac{dX}{dt} \\
&= \int_0^T \frac{\delta^2 \hat{V}}{\delta f^2} \frac{\partial f}{\partial t} d\tau + + \frac{\partial \hat{j}}{\partial X} \frac{dX}{dt} \\
&= \int_0^T \frac{\delta^2 \hat{V}}{\delta f^2} \left(\hat{\imath}\left(\tau, t\right) + \frac{\partial f}{\partial \tau}\left(\tau, t\right) \right) d\tau + \frac{\partial \hat{j}}{\partial X} \frac{dX}{dt}.
\end{aligned}$$

Collecting terms and replacing $\frac{\delta \hat{V}}{\delta f}$ by $\hat{j}$, the derivative with respect to f results in:

$$\begin{aligned}
\rho \int_0^T \hat{j}\left(\tau, t\right) h\left(\tau, t\right) d\tau &= \mathcal{U}'\left(\hat{c}\left(t\right)\right) \left[-h\left(0, t\right) - \int_0^T \delta h\left(\tau, t\right) d\tau \right] \\
&+ \int_0^T \left(\frac{\partial \hat{j}}{\partial t} + \hat{j}\left(\tau, t\right) \frac{\partial h}{\partial \tau} \right) d\tau \\
&+ \phi \int_0^T \left(\mathbb{E}_t^X \left[\frac{\delta V}{\delta f} \right] - \hat{j}\left(\tau, t\right) \right) h\left(\tau, t\right) d\tau.
\end{aligned}$$

Defining $\hat{j}\left(\tau, t, X(t)\right) \equiv \frac{\delta V}{\delta f}\left(\tau, t, X(t)\right)$ and integrating by parts we obtain

$$\begin{aligned}
\rho \int_0^T \hat{j}\left(\tau, t\right) h\left(\tau, t\right) d\tau &= \mathcal{U}'\left(\hat{c}\left(t\right)\right) \left[-h\left(0, t\right) - \int_0^T \delta h\left(\tau, t\right) d\tau \right] \\
&+ \int_0^T \left(\frac{\partial \hat{j}}{\partial t} - \frac{\partial \hat{j}}{\partial \tau} \right) h\left(\tau, t\right) d\tau + \hat{j}\left(T, t\right) h\left(T, t\right) - \hat{j}\left(0, t\right)) h\left(0, t\right) \\
&+ \phi \int_0^T \left(\mathbb{E}_t^X \left[j\left(\tau, t\right) \right] - \hat{j}\left(\tau, t\right) \right) h\left(\tau, t\right) d\tau.
\end{aligned}$$

Because, $f\left(T, t\right) = 0$ then the directions h are forced to have $h\left(T, t\right) = 0$. As the Gâteaux derivative should be zero for any direction $h\left(\tau, t\right)$:

$$\rho \hat{j}(\tau,t) \quad = -\delta \ \mathcal{U}'\left(\hat{c}\left(t\right)\right) + \frac{\partial \hat{j}}{\partial t} - \frac{\partial \hat{j}}{\partial \tau} + \phi \left(\mathbb{E}_t^X\left[j\left(\tau,t\right)\right] - \hat{j}\left(\tau,t\right)\right), \text{ if } \tau \in (0,T] \quad \text{(C.17)}$$

$$\hat{j}\left(0,t\right) \quad = \quad -\mathcal{U}'\left(\hat{c}\left(t\right)\right), \ \text{ if } \tau = 0.$$

Finally, if we define the variables

$$v\left(\tau,t\right) \quad = \quad -j\left(\tau,t\right)/\mathcal{U}'\left(c\left(t\right)\right),$$

$$\hat{v}\left(\tau,t\right) \quad = \quad -\hat{j}\left(\tau,t\right)/\mathcal{U}'\left(\hat{c}\left(t\right)\right),$$

the PDE equation (C.17) results in

$$\left(\rho - \frac{\mathcal{U}''\left(\hat{c}\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\frac{d\hat{c}}{dt}\right)\hat{v}\left(\tau,t\right) = \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau} + \phi \left(\frac{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\mathbb{E}_t^X\left[v\left(\tau,t\right)\right] - \hat{v}\left(\tau,t\right)\right), \text{ if } \tau \in (0,\infty),$$

$$\hat{v}\left(0,t\right) = 1, \ \text{ if } \tau = 0,$$

and the first-order condition is

$$\frac{\partial \hat{q}}{\partial \hat{\imath}}\hat{\imath}\left(\tau,t\right) + \hat{q}\left(\tau,t,\hat{\imath}\right) = \hat{v}\left(\tau,t\right).$$

A. Proof of Proposition ??

The proof has three steps. We first solve the problem where the Government commits to a debt schedule and cannot default within a Δ time interval. In that sense, we solve a problem with commitment within an arbitrary interval $(t, t+\Delta]$. Then we take the limit as $\Delta \to 0$. Finally we employ dynamic programming to obtain the value of the value functional $V[f(\cdot, t)]$.

Before any further steps, we recall that the Gâteaux derivative of the value functions before and after the default event arrives, can be expressed as we did in equation (C.11) in the proof of the model with risk,

$$\frac{\partial}{\partial \alpha} V[f(\cdot, t) + \alpha h(\cdot, t)]|_{\alpha=0} = \int_0^T \frac{\delta V}{\delta f}(\tau, t) h(\tau, t) d\tau, \tag{C.18}$$

$$\frac{\partial}{\partial \alpha} \hat{V}[f(\cdot, t) + \alpha h(\cdot, t)]|_{\alpha=0} = \int_0^T \frac{\delta \hat{V}}{\delta f}(\tau, t) h(\tau, t) d\tau, \tag{C.19}$$

where $\left\{ \frac{\delta \hat{V}}{\delta f}(\tau, t), \frac{\delta \hat{V}}{\delta f}(\tau, t) \right\} \in L^2([0, T] \times [0, \infty))$. We employ these expressions multiple times in the steps that follow.

Step 1. Withing Commitment Interval Problem

Assume first that the time interval $[0, \infty)$ is divided in subintervals $[0, \Delta] \cup (\Delta, 2\Delta] \cup (2\Delta, 3\Delta] \cup$ ... Shocks arrive at the beginning of each interval —except at time $t = 0$. Default can be triggered only at the instant when the shock arrives.

The probability of receiving the shock within an interval $(t, t+\Delta]$ conditional on not having received it in the past is $(1 - e^{-\phi\Delta})$. The expected value functional is $\Gamma(\hat{V}[f(\cdot, t+\Delta)]) + V[f(\cdot, t+\Delta)] \Phi(\hat{V}[f(\cdot, t+\Delta)])$. The first term is the portion corresponding to default states $\Gamma(\hat{V}[f(\cdot, t+\Delta)])$. The second, the portion corresponding to no default, $V[f(\cdot, t+\Delta)]$.

The Government solves a commitment problem within each subinterval. The solution procedure to the discrete commitment problem follows the same lines as in the perfect foresight case, because there are no shocks within the interval. The Lagrangian is:

$$\begin{aligned}
\mathcal{L}[\iota, f, \hat{\psi}] = & \int_t^{t+\Delta} e^{-\rho(s-t)} \mathcal{U}(c(t)) ds \\
& + \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} \hat{j}(\tau, s) \left(-\frac{\partial f}{\partial s} + \iota(\tau, s) + \frac{\partial f}{\partial \tau} \right) d\tau ds \\
& + \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu(\tau, s) \left(-\bar{r}\hat{\psi}(\tau, s) + \delta + \frac{\partial \hat{\psi}}{\partial s} - \frac{\partial \hat{\psi}}{\partial \tau} \right) d\tau ds \\
& + e^{-\rho\Delta} \left(1 - e^{-\phi\Delta} \right) \left[\Gamma(\hat{V}[f(\cdot, t+\Delta)]) + V[f(\cdot, t+\Delta)] \Phi(\hat{V}[f(\cdot, t+\Delta)]) \right] \\
& + e^{-\rho\Delta} e^{-\phi\Delta} \hat{V}[f(\cdot, t+\Delta)].
\end{aligned}$$

Here $\hat{j}(\tau, t)$ and $\mu(\tau, t)$ are the Lagrange multipliers associated with the law of motion of debt

(2.1), as before, and with the bond price equation (??). Note that the only difference is that the Government carries the law of motion for prices as a constraint. The reason is that the Government knows that by controlling the state, it influences the chances of default, and thus, influences prices. This happens through the terminal condition,

$$\hat{\psi}\left(\tau, t+\Delta\right) = \left[\left(1 - e^{-\theta\Delta}\right)\Phi\left(\hat{V}\left[f\left(\cdot, t+\Delta\right)\right]\right) + e^{-\theta\Delta}\right]\psi\left(\tau, t+\Delta\right),\qquad\text{(C.20)}$$
$$\hat{\psi}\left(0, s\right) = 1.$$

The terminal condition reflects that after a the arrival of a default option the bond price is zero if default occurs or the perfect foresight price $\psi\left(\tau, t\right)$, if default does not occur.

As an intermediate step, integrate by parts the terms that involve time or maturity derivatives of f and $\hat{\psi}$. We otain the following terms:

$$-\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho(s-t)}\frac{\partial f}{\partial s}\hat{j}\left(\tau, s\right) d\tau ds = -\int_{0}^{T} e^{-\rho\Delta}f\left(\tau, t+\Delta\right)\hat{j}\left(\tau, t+\Delta\right) d\tau \qquad\text{(C.21)}$$
$$+\int_{0}^{T} f\left(\tau, t\right)\hat{j}\left(\tau, t\right) d\tau$$
$$+\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho(s-t)}f\left(\tau, s\right)\left(\frac{\partial\hat{j}(\tau, s)}{\partial s} - \rho\hat{j}(\tau, s)\right) d\tau ds.$$

For this we exchanged the limits of integration, integrated by parts with respect to time, and reversed the order of limits again. We proceed in the same way with the following term:

$$\int_{t}^{t+\Delta}\int_{0}^{T}\frac{\partial f}{\partial\tau}\hat{j}\left(\tau, s\right) d\tau ds = \int_{t}^{t+\Delta} e^{-\rho(s-t)}f\left(T, s\right)\hat{j}\left(T, s\right) ds \qquad\text{(C.22)}$$
$$-\int_{t}^{t+\Delta} e^{-\rho(s-t)}f\left(0, s\right)\hat{j}\left(0, s\right) ds$$
$$-\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho(s-t)}f\left(\tau, s\right)\frac{\partial\hat{j}(\tau, s)}{\partial\tau} d\tau ds.$$

For the price Lagrange multipliers we have:

$$\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho(t-s)}\mu\left(\tau, s\right)\frac{\partial\hat{\psi}}{\partial s} d\tau ds = \int_{0}^{T}\left[e^{-\rho\Delta}\mu\left(\tau, t+\Delta\right)\hat{\psi}\left(\tau, t+\Delta\right)\right] d\tau \qquad\text{(C.23)}$$
$$-\int_{0}^{T}\mu\left(\tau, t\right)\hat{\psi}\left(\tau, t\right) d\tau$$
$$-\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho t}\hat{\psi}\left(\tau, s\right)\left(\frac{\partial\mu\left(\tau, s\right)}{\partial s} - \rho\mu\left(\tau, s\right)\right) d\tau ds,$$

and

$$-\int_t^{t+\Delta}\int_0^T e^{-\rho(t-s)}\mu\left(\tau,s\right)\left(\frac{\partial\hat{\psi}}{\partial\tau}\right)d\tau ds \quad = \quad -\int_t^{t+\Delta}e^{-\rho(s-t)}\mu\left(T,s\right)\hat{\psi}\left(T,s\right) \qquad\text{(C.24)}$$

$$+\quad e^{-\rho t}\mu\left(0,s\right)\hat{\psi}\left(0,s\right)$$

$$+\quad \int_t^{t+\Delta}\int_0^T e^{-\rho t}\hat{\psi}\left(\tau,s\right)\frac{\partial\mu}{\partial\tau}d\tau ds.$$

These 12 terms appear in two of the integrals. Hence, we expand the Langrangean and use the expressions to later on connect terms. We also, use the terminal conditions for prices and mass of debt are $f\left(T,s\right)=0$.

We substitute both (C.21-C.24) and the terminal conditions into the Lagrangian :

$$
\begin{aligned}
\mathcal{L}\left[\iota, f, \hat{\psi}\right] \;=\;& \int_t^{t+\Delta} e^{-\rho(s-t)} \mathcal{U}\left(c\left(t\right)\right) ds \\
&+ \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} \hat{j}\left(\tau, s\right) \iota\left(\tau, s\right) d\tau ds \\
&+ \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu\left(\tau, s\right)\left(\delta - \bar{r}\psi(\tau, s)\right) d\tau ds \\
&- \int_0^T e^{-\rho\Delta} f\left(\tau, t+\Delta\right) \hat{j}\left(\tau, t+\Delta\right) d\tau \\
&+ \int_0^T f\left(\tau, t\right) \hat{j}\left(\tau, t\right) d\tau \\
&+ \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} f(\tau, s)\left(\frac{\partial \hat{j}(\tau, s)}{\partial s} - \rho \hat{j}(\tau, s)\right) ds d\tau \\
&+ \int_t^{t+\Delta} e^{-\rho(s-t)} f\left(T, s\right) \hat{j}\left(T, s\right) ds \\
&- \int_t^{t+\Delta} e^{-\rho(s-t)} f\left(0, s\right) \hat{j}\left(0, s\right) ds \\
&- \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} f(\tau, s)\frac{\partial \hat{j}(\tau, s)}{\partial \tau} d\tau ds \\
&+ \int_0^T e^{-\rho\Delta} \mu\left(\tau, t+\Delta\right)\left[\left(1 - e^{-\phi\Delta}\right) \Phi\left(\hat{V}\left[f\left(\cdot, t+\Delta\right)\right]\right) + e^{-\phi\Delta}\right] \psi\left(\tau, t+\Delta\right) d\tau \\
&- \int_0^T \mu\left(\tau, t\right) \hat{\psi}\left(\tau, t\right) d\tau \\
&- \int_t^{t+\Delta} \int_0^T e^{-\rho t} \hat{\psi}\left(\tau, s\right)\left(\frac{\partial \mu\left(\tau, s\right)}{\partial s} - \rho \mu\left(\tau, s\right)\right) d\tau ds \\
&- \int_t^{t+\Delta} e^{-\rho(s-t)} \mu\left(T, s\right) \hat{\psi}\left(T, s\right) ds \\
&+ \int_t^{t+\Delta} e^{-\rho t} \mu\left(0, s\right) ds \\
&+ \int_t^{t+\Delta} \int_0^T e^{-\rho t} \hat{\psi}\left(\tau, s\right) \frac{\partial \mu}{\partial \tau} d\tau ds. \\
&+ e^{-\rho\Delta}\left(1 - e^{-\phi\Delta}\right)\left[\Gamma\left(\hat{V}\left[f\left(\cdot, s+\Delta\right)\right]\right) + V\left[f\left(\cdot, t+\Delta\right)\right] \Phi\left(\hat{V}\left[f\left(\cdot, t+\Delta\right)\right]\right)\right] \\
&+ e^{-\rho\Delta} e^{-\phi\Delta} \hat{V}\left[f\left(\cdot, t+\Delta\right)\right].
\end{aligned}
$$

Next, we compute the Gâteaux derivatives with respect to each of the three arguments of the value function at a time.

Gâteaux derivative of the issuances. First, we consider a perturbation around issuances:

$$\frac{\partial}{\partial \alpha} \mathcal{L} \left[\iota(\tau, t) + \alpha h(\tau, s), f, \hat{\psi} \right] \Big|_{\alpha=0}$$

$$= \frac{\partial}{\partial \alpha} \left[\int_t^{t+\Delta} e^{-\rho(s-t)} \mathcal{U} \left(\hat{c}(t) + \int_0^T \left(\hat{\psi}(\tau, s) + \lambda(\tau, \iota) \right) \alpha h(\tau, s)) d\tau \right) ds \right.$$

$$\left. + \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} \hat{j}(\tau, s) \left(\iota(\tau, s) + \alpha h(\tau, s) \right) d\tau ds \right] \Bigg|_{\alpha=0}.$$

Note that we ommited the terms in the Lagrangean where the perturbation does not appear. Passing limits inside the integrals to compute derivatives we obtain:

$$\int_t^{t+\Delta} e^{-\rho(s-t)} \int_0^T \left(\mathcal{U}'(\hat{c}(t)) \left(\hat{\psi}(\tau, s) + \lambda(\tau, \iota) \right) + \hat{j}(\tau, s) \right) h(\tau, s)) d\tau ds.$$

The Gâteaux derivative should be zero for any suitable $h(\tau, t)$. Thus, we obtain the first-order condition, that we obtain in the previous problems:

$$\mathcal{U}'(\hat{c}(t)) \left(\frac{\partial q}{\partial \iota} \iota(\tau, t) + q(\tau, \iota, f) \right) = -\hat{j}(\tau, t). \tag{C.25}$$

or more conveniently:

$$\iota(\tau, t) = \frac{-\left(q(\tau, \iota, f) + \hat{j}(\tau, t) \right) / \mathcal{U}'(\hat{c}(t))}{\frac{\partial q}{\partial \iota}}$$

Gâteaux derivative of the debt density. Since the distribution at the beginning of the interval $f(\tau, t)$ is given, any feasible perturbation must feature $h(\tau, t) = 0$ for any $\tau \in [0, T]$. In addition, we know that $h(T, s) = 0$, and $s \in (t, t + \Delta]$, because $f(T, s) = 0$. The Gâteaux derivative of the Lagrangian with respect to the debt density is:

$$\frac{\partial}{\partial \alpha} \mathcal{L} \left[\iota, f(\tau, t) + \alpha h(\tau, t), \hat{\psi} \right]\big|_{\alpha=0}$$

$$= \frac{\partial}{\partial \alpha} \left[\int_t^{t+\Delta} e^{-\rho(s-t)} \mathcal{U} \left(\hat{c}(t) - h(0, s) - \delta \int_0^T (\hat{\psi}(\tau, s) + \lambda(\tau, \iota)) \alpha h(\tau, s) d\tau \right) ds \right.$$

$$- \int_0^T e^{-\rho \Delta} (f(\tau, t+\Delta) + \alpha h(\tau, t+\Delta)) \hat{j}(\tau, t+\Delta) d\tau$$

$$+ \int_0^T (f(\tau, t) + \alpha h(\tau, t)) \hat{j}(\tau, t) d\tau$$

$$+ \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} (f(\tau, s) + \alpha h(\tau, s)) \left(\frac{\partial \hat{j}(\tau, s)}{\partial s} - \rho \hat{j}(\tau, s) \right) ds d\tau$$

$$+ \int_t^{t+\Delta} e^{-\rho(s-t)} (f(T, s) + \alpha h(T, s)) \hat{j}(T, s) ds$$

$$- \int_t^{t+\Delta} e^{-\rho(s-t)} (f(0, s) + \alpha h(0, s)) \hat{j}(0, s) ds$$

$$- \int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} (f(\tau, s) + \alpha h(\tau, s)) \frac{\partial \hat{j}(\tau, s)}{\partial \tau} d\tau ds$$

$$+ \int_0^T e^{-\rho \Delta} \mu(\tau, t+\Delta) \left[\left(1 - e^{-\phi \Delta} \right) \Phi \left(\hat{V} \left[f(\cdot, t+\Delta) + \alpha h(\tau, t+\Delta) \right] \right) + e^{-\phi \Delta} \right] \psi(\tau, t) d\tau$$

$$+ e^{-\rho \Delta} \left(1 - e^{-\phi \Delta} \right) \Gamma \left(\hat{V} \left[f(\cdot, s+\Delta) + \alpha h(\tau, t+\Delta) \right] \right)$$

$$+ e^{-\rho \Delta} \left(1 - e^{-\phi \Delta} \right) V \left[f(\cdot, t+\Delta) + \alpha h(\tau, t+\Delta) \right] \Phi \left(\hat{V} \left[f(\cdot, t+\Delta) + \alpha h(\tau, t+\Delta) \right] \right)$$

$$+ e^{-\rho \Delta} e^{-\phi \Delta} \hat{V} \left[f(\cdot, t+\Delta) + \alpha h(\tau, t+\Delta) \right] \bigg|_{\alpha=0}.$$

If we evaluate the functional derivatives, the right-hands side becomes:

$$-\int_{t}^{t+\Delta} e^{-\rho(s-t)}\mathcal{U}\left(\hat{c}\left(s\right)\right)\left(h\left(0,s\right)+\delta\int_{0}^{T}h\left(\tau,s\right)d\tau\right)ds$$

$$-\int_{0}^{T} e^{-\rho\Delta}h\left(\tau,t+\Delta\right)\hat{j}\left(\tau,t+\Delta\right)d\tau$$

$$+\int_{0}^{T} h\left(\tau,t\right)\hat{j}\left(\tau,t\right)d\tau$$

$$+\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho(s-t)}h(\tau,s)\left(\frac{\partial\hat{j}(\tau,s)}{\partial s}-\rho\hat{j}(\tau,s)\right)dsd\tau$$

$$+\int_{t}^{t+\Delta} e^{-\rho(s-t)}h\left(T,s\right)\hat{j}\left(T,s\right)ds$$

$$-\int_{t}^{t+\Delta} e^{-\rho(s-t)}h\left(0,s\right)\hat{j}\left(0,s\right)ds$$

$$-\int_{t}^{t+\Delta}\int_{0}^{T} e^{-\rho(s-t)}h(\tau,s)\frac{\partial\hat{j}(\tau,s)}{\partial\tau}d\tau ds$$

$$+A+B+C+D.$$

Where,

$$A=\int_{0}^{T} e^{-\rho\Delta}\mu\left(\tau,t+\Delta\right)\left[\left(1-e^{-\phi\Delta}\right)\frac{\partial}{\partial\alpha}\Phi\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)+\alpha h(\tau,t+\Delta)\right]\right)\Bigg|_{\alpha=0}\right]\psi\left(\tau,t+\Delta\right)d\tau$$

$$B=e^{-\rho\Delta}\left(1-e^{-\phi\Delta}\right)\frac{\partial}{\partial\alpha}\left[\Gamma\left(\hat{V}\left[f\left(\cdot,s+\Delta\right)+\alpha h(\tau,t+\Delta)\right]\right)\right]\Bigg|_{\alpha=0}$$

$$C=e^{-\rho\Delta}\left(1-e^{-\phi\Delta}\right)\frac{\partial}{\partial\alpha}\left[V\left[f\left(\cdot,t+\Delta\right)+\alpha h(\tau,t+\Delta)\right]\Phi\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)+\alpha h(\tau,t+\Delta)\right]\right)\right]\Bigg|_{\alpha=0}$$

$$D=\frac{\partial}{\partial\alpha}e^{-\rho\Delta}e^{-\phi\Delta}\hat{V}\left[f\left(\cdot,t+\Delta\right)+\alpha h(\tau,t+\Delta)\right]\Bigg|_{\alpha=0}.$$

If we collact terms, and use $h\left(\tau,t\right)=h\left(T,s\right)=0$, and the derivative of the The Gâteaux derivative of the Lagrangian is then:

$$-\int_t^{t+\Delta} e^{-\rho(s-t)}\left(\mathcal{U}\left(\hat{c}\left(s\right)\right)+\hat{j}\left(0,s\right)\right)h\left(0,s\right)$$

$$+\int_t^{t+\Delta}\int_0^T e^{-\rho(s-t)}h(\tau,s)\left(\frac{\partial\hat{j}(\tau,s)}{\partial s}-\frac{\partial\hat{j}(\tau,s)}{\partial\tau}-\rho\hat{j}(\tau,s)-\delta\mathcal{U}'\left(\hat{c}\left(s\right)\right)\right)dsd\tau \qquad \text{(C.26)}$$

$$-\int_0^T e^{-\rho\Delta}\hat{j}\left(\tau,t+\Delta\right)h\left(\tau,t+\Delta\right)d\tau$$

$$+A+B+C+D.$$

Next, we calculate the terms $\{A,B,C,D\}$ using the expressions for the Gâteaux derivatives, equations (C.18,C.19). We begin with the term A:

$$\int_0^T e^{-\rho\Delta}\mu\left(\tau,t+\Delta\right)\left[\left(1-e^{-\phi\Delta}\right)\Phi'\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)\right]\right)\int_0^T\frac{\delta\hat{V}}{\delta f}\left(\tau',t+\Delta\right)h\left(\tau',t+\Delta\right)d\tau'\right]\psi\left(\tau,t+\Delta\right)d\tau,$$

and after we exchange the order of integration, we obtain:

$$A=\left[\int_0^T e^{-\rho\Delta}a\left(t,\Delta\right)\frac{\delta\hat{V}}{\delta f}\left(\tau',t+\Delta\right)h\left(\tau',t+\Delta\right)d\tau'\right].$$

for

$$a\left(t,\Delta\right)=\left(1-e^{-\phi\Delta}\right)\Phi'\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)\right]\right)\int_0^T\mu\left(\tau,t+\Delta\right)\psi\left(\tau,t+\Delta\right)d\tau.$$

For the second term, B, we have:

$$e^{-\rho\Delta}\left(1-e^{-\phi\Delta}\right)\left[\Gamma'\left(\hat{V}\left[f\left(\cdot,s+\Delta\right)\right]\right)\int_0^T\frac{\delta\hat{V}}{\delta f}\left(\tau',t+\Delta\right)h\left(\tau',t+\Delta\right)d\tau'\right]$$

and observing that $\Gamma'\left(\hat{V}\left[f\left(\cdot,s+\Delta\right)\right]\right)=-\hat{V}\left[f\left(\cdot,s+\Delta\right)\right]\Phi'\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)\right]\right)$, we end with:

$$B=-e^{-\rho\Delta}\left(1-e^{-\theta\Delta}\right)\left[\hat{V}\left[f\left(\cdot,s+\Delta\right)\right]\Phi'\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)\right]\right)\frac{\delta\hat{V}}{\delta f}\left(\tau',t+\Delta\right)h\left(\tau',t+\Delta\right)d\tau'\right]$$

For the third term, C, we have

$$C=\int_0^T\frac{\delta V}{\delta f}\left(\tau',t\right)h\left(\tau',t+\Delta\right)d\tau'\Phi\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)\right]\right)+V\left[f\left(\cdot,t+\Delta\right)\right]\ldots$$

$$\Phi'\left(\hat{V}\left[f\left(\cdot,t+\Delta\right)\right]\right)\int_0^T\frac{\delta\hat{V}}{\delta f}\left(\tau',t+\Delta\right)h\left(\tau',t+\Delta\right)d\tau'.$$

We combine both conditions to obtain:

$$B + C = \int_0^T e^{-\rho\Delta} b_1(t, \Delta) \frac{\delta \hat{V}}{\delta f}(\tau, t) h(\tau, t + \Delta) d\tau \ldots$$

$$+ \int_0^T e^{-\rho\Delta} b_2(t, \Delta) \frac{\delta V}{\delta f}(\tau, t) h(\tau, t + \Delta) d\tau$$

where

$$b_1(t, \Delta) = \left(1 - e^{-\phi\Delta}\right) \left(\left(V[f(\cdot, t + \Delta)] - \hat{V}[f(\cdot, s + \Delta)]\right) \Phi'\left(\hat{V}[f(\cdot, t + \Delta)]\right)\right)$$

and

$$b_2(t, \Delta) = \left(1 - e^{-\phi\Delta}\right) \Phi\left(\hat{V}[f(\cdot, t + \Delta)]\right).$$

The final term in the construction, D, is given by:

$$\int_0^T e^{-\rho\Delta} d(t, \Delta) \frac{\delta \hat{V}}{\delta f}(\tau', t + \Delta) h(\tau', t + \Delta) d\tau',$$

where

$$d(t, \Delta) = e^{-\phi\Delta}.$$

Notice that the terms that multiply $h(\tau, t + \Delta)$ in the Gâteaux derivative of the Lagrangian, (C.26), is:

$$-\int_0^T e^{-\rho\Delta} \hat{j}(\tau, t + \Delta) h(\tau, t + \Delta) d\tau + A + B + C + D =$$

$$-e^{-\rho\Delta} \int_0^T \left(\hat{j}(\tau, t + \Delta) + a(t, \Delta) \frac{\delta \hat{V}}{\delta f}(\tau', t + \Delta) + b_1(t, \Delta) \frac{\delta \hat{V}}{\delta f}(\tau', t) + b_2(t, \Delta) \frac{\delta V}{\delta f}(\tau', t) + e^{-\theta\Delta}\right) h(\tau, t +$$

The value of the Gâteaux derivative of the Lagrangian, (C.26), for any perturbation, must be zero. Thus, again a necessary condition is to have all terms that multiply any entry of $h(\tau, t)$ by zero.

Thus, we summarize the necessary conditions into:

$$\rho \hat{j}(\tau, s) = -\delta \mathcal{U}'(\hat{c}(s)) + \frac{\partial \hat{j}}{\partial s} - \frac{\partial \hat{j}}{\partial \tau}, \text{ for } \tau \in (0, T] \tag{C.27}$$

$$\hat{j}(0, s) = -\mathcal{U}'(\hat{c}(s)), \text{ for } \tau = 0,$$

$$\hat{j}(\tau, t + \Delta) = (a(t, \Delta) + b_1(t, \Delta) + d(t, \Delta)) \frac{\delta \hat{V}}{\delta f}(\tau, t + \Delta) + b_2(t, \Delta) \frac{\delta V}{\delta f}(\tau, t) = \hat{j}(\tau, t + \Delta), \text{ for } \tau \in (0,T] \tag{C.28}$$

where

$$a(t, \Delta) = \left(1 - e^{-\phi\Delta}\right) \Phi'\left(\hat{V}\left[f(\cdot, t + \Delta))\right]\right) \int_0^T \mu(\tau, t + \Delta) \psi(\tau, t + \Delta) d\tau.$$

$$b_1(t, \Delta) = \left(1 - e^{-\phi\Delta}\right) \left(\left(V\left[f(\cdot, t + \Delta)\right] - \hat{V}\left[f(\cdot, s + \Delta)\right]\right) \Phi'\left(\hat{V}\left[f(\cdot, t + \Delta)\right]\right)\right)$$

$$b_2(t, \Delta) = \left(1 - e^{-\phi\Delta}\right) \Phi\left(\hat{V}\left[f(\cdot, t + \Delta)\right]\right)$$

$$d(t, \Delta) = e^{-\phi\Delta}.$$

Gâteaux derivative of the bond price. A final calculation, requires us to compute the Gâteaux derivatives with respect to the evolution of the price ψ :

$$\begin{aligned}
&\frac{\partial}{\partial \alpha} \mathcal{L}\left[\iota, f, \hat{\psi}(\tau, t) + \alpha h(\tau, t)\right]\Big|_{\alpha=0} \\
=\ &\frac{\partial}{\partial \alpha} \Bigg[\int_t^{t+\Delta} e^{-\rho(s-t)} \mathcal{U}\left(\hat{c}(t) + \alpha \int_0^T \iota(\tau, s) h(\tau, s) d\tau\right) ds \\
&+ \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu(\tau, s) \left(\delta - \bar{r}\left(\hat{\psi}(\tau, s) + \alpha h(\tau, s)\right)\right) d\tau ds \\
&- \int_0^T \mu(\tau, t) \left(\hat{\psi}(\tau, t) + \alpha h(\tau, t)\right) d\tau \\
&- \int_t^{t+\Delta} \int_0^T e^{-\rho t} \left(\hat{\psi}(\tau, s) + \alpha h(\tau, s)\right) \left(\frac{\partial \mu(\tau, s)}{\partial s} - \rho \mu(\tau, s)\right) d\tau ds \\
&- \int_t^{t+\Delta} e^{-\rho(s-t)} \mu(T, s) \left(\hat{\psi}(T, s) + \alpha h(T, s)\right) ds \\
&+ \int_t^{t+\Delta} e^{-\rho t} \mu(0, s) ds \\
&+ \int_t^{t+\Delta} \int_0^T e^{-\rho t} \left(\hat{\psi}(\tau, s) + \alpha h(t, s)\right) \frac{\partial \mu}{\partial \tau} d\tau ds \Bigg]\Bigg|_{\alpha=0}.
\end{aligned}$$

Note that the perturbation is only around $\hat{\psi}(\tau, s)$ and not $\psi(\tau, t + \Delta)$, the terminal price after default, which is given. Since at maturity, bonds have a value of 1, $\mu(0, s) = 0$ for the Lagrangian to reprsent the value.

Then, we evaluate the Gâteaux derivatives and obtain:

$$\frac{\partial}{\partial \alpha} \mathcal{L} \left[\iota, f, \hat{\psi}(\tau, t) + \alpha h(\tau, t) \right] \Big|_{\alpha=0}$$

$$= \left[\int_t^{t+\Delta} \int_0^T e^{-\rho(s-t)} \mathcal{U}' \left(\hat{c}(t) \right) \iota(\tau, s) h(\tau, s) \, d\tau ds \right.$$

$$- \int_t^{t+\Delta} \int_0^T e^{-\rho(t-s)} \mu(\tau, s) \bar{r} h(\tau, s) \, d\tau ds$$

$$- \int_0^T \mu(\tau, t) h(\tau, t) \, d\tau$$

$$- \int_t^{t+\Delta} \int_0^T e^{-\rho t} \left(\frac{\partial \mu(\tau, s)}{\partial s} - \rho \mu(\tau, s) \right) h(\tau, s) \, d\tau ds$$

$$- \int_t^{t+\Delta} e^{-\rho(s-t)} \mu(T, s) h(T, s) \, ds$$

$$+ \left. \int_t^{t+\Delta} \int_0^T e^{-\rho t} \left(h(\tau, s) \right) \frac{\partial \mu}{\partial \tau} \, d\tau ds \right|_{\alpha=0}.$$

Again, as the Gâteaux derivative should be zero for any suitable $h(\tau, t)$, the optimality condition is:

$$\begin{aligned}
(\bar{r} - \rho) \mu(\tau, s) &= \mathcal{U}'(\hat{c}(t)) \iota(\tau, s) + \frac{\partial \mu}{\partial \tau} - \frac{\partial \mu}{\partial s}, \text{ if } \tau \in (0, T), \ s \in (t, t+\Delta) \\
\mu(T, s) &= 0, \text{ if } \tau = T, \ s \in (t, t+\Delta) \\
\mu(\tau, t) &= 0, \text{ if } \tau \in (0, T).
\end{aligned} \tag{C.29}$$

Step 2. $\Delta \longrightarrow 0$ Limit

We express the HJB equation (**??**) as

$$\begin{aligned}
\hat{\jmath}(\tau, t) &= \int_t^{t+\Delta} e^{-\rho(t-s)} \left\{ \mathcal{U}'(\hat{c}(s)) (-\delta) \right\} ds && \text{(C.30)} \\
&\quad + e^{-\rho \Delta} \mathbb{E}_t \left[\frac{\delta \hat{V}}{\delta f}(\tau, t+\Delta) \right]. && \text{(C.31)} \\
j(0, s) &= -\mathcal{U}'(c(s)), \text{ if } \tau = 0,
\end{aligned}$$

where the term $\frac{\delta \hat{V}}{\delta f}(\tau, t+\Delta)$ depends on the arrival of a shock with probability $(1 - e^{-\theta \Delta})$ that decreases the value by

$$\left[\left(V[f(\tau, t+\Delta)] - \hat{V}[f(\tau, t+\Delta)] \right) \Phi' \left(\hat{V}[f(\tau, t+\Delta)] \right) + \Phi \left(\hat{V}[f(\cdot, t+\Delta)] \right) \frac{\frac{\delta V}{\delta f}}{\frac{\delta \hat{V}}{\delta f}} + \phi \left(\hat{V}[f(\cdot, t+\Delta)] \right) \int_0^T \right.$$

We take the limit as $\Delta \to 0$. In this case, the Lagrange multiplier $\mu(\tau, t)$ collapses to zero

given (**??**):

$$\mu\left(\tau,t\right)=0,\ \text{for all}\ \tau\in\left(0,T\right],\ t\in\left(0,\infty\right),\tag{C.32}$$

reflecting the lack of commitment over finite intervals.

Conditional on no shock arrival, the limit case $\Delta\to 0$ in equation (**??**) results in

$$\hat{j}\left(\tau,t\right)=\frac{\delta\hat{V}}{\delta f}\left(\tau,t\right),\ \text{for all}\ \tau\in\left(0,T\right],\ t\in\left(0,\infty\right).\tag{C.33}$$

Taking into account (C.32) and (C.33), the limit as $\Delta\to 0$ of equation (C.30) can be expressed as an HJB of the form

$$\rho\hat{j}\left(\tau,t\right)=\mathcal{U}'\left(\hat{c}\left(t\right)\right)\left(-\delta\right)+\frac{\partial\hat{j}}{\partial t}-\frac{\partial\hat{j}}{\partial\tau}+\theta\left[\Phi\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right)\frac{j\left(\tau,t\right)}{\hat{j}\left(\tau,t\right)}+\left(V\left[f\left(\cdot,t\right)\right]-\hat{V}\left[f\left(\cdot,t\right)\right]\right)\phi\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right)-$$

$$\hat{j}\left(0,t\right)=-\mathcal{U}'\left(c\left(t\right)\right),\ \text{if}\ \tau=0.$$

Defining the variable

$$\hat{v}\left(\tau,t\right)=-\hat{j}\left(\tau,t\right)/\mathcal{U}'\left(\hat{c}\left(t\right)\right),$$

the HJB results in

$$\left(\rho-\frac{\mathcal{U}''\left(\hat{c}\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}\frac{d\hat{c}}{dt}\right)\hat{v}\left(\tau,t\right)\ =\ \delta+\frac{\partial\hat{v}}{\partial t}-\frac{\partial\hat{v}}{\partial\tau}-\phi\left[\Phi\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right)\frac{v\left(\tau,t\right)}{\hat{v}\left(\tau,t\right)}\frac{\mathcal{U}'\left(c\left(t\right)\right)}{\mathcal{U}'\left(\hat{c}\left(t\right)\right)}+\left(V\left[f\left(\cdot,t\right)\right]-\hat{V}\left[f\left(\cdot,t\right)\right]\right.$$

$$\hat{v}\left(0,t\right)\ =\ 1,\ \text{if}\ \tau=0.$$

and the first order condition is

$$\frac{\partial q}{\partial\iota}\iota\left(\tau,t\right)+q\left(t,\tau,\iota\right)=\hat{v}\left(\tau,t\right).$$

Step 3. HJB Equation

The last step is to construct the aggregate HJB in order to obtain the value of $\hat{V}\left[f\left(\cdot,t\right)\right]$. The idea is to compute the derivative with respect to $t+\Delta$ in the dynamic programming equation and then to take the limit as $\Delta\to 0$:

$$\rho\hat{V}\left[f\left(\cdot,t\right)\right]\ =\ \mathcal{U}\left(\hat{c}\left(t\right)\right)+\int_{0}^{T}\frac{\delta\hat{V}}{\delta f}\frac{\partial f}{\partial t}d\tau$$

$$+\phi\left\{\Gamma\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right)+\Phi\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right)V\left[f\left(\cdot,t\right)\right]-\hat{V}\left[f\left(\cdot,t\right)\right]\right\},$$

or equivalently

$$
\begin{aligned}
\rho \hat{V}\left[f\left(\cdot,t\right)\right] \;=\; & \mathcal{U}\left(\hat{c}\left(t\right)\right) + \int_0^T \mathcal{U}'\left(\hat{c}\left(t\right)\right) \hat{v}\left(\tau,t\right) \frac{\partial f}{\partial t} d\tau \\
& + \phi\left\{\Gamma\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right) + \Phi\left(\hat{V}\left[f\left(\cdot,t\right)\right]\right) V\left[f\left(\cdot,t\right)\right] - \hat{V}\left[f\left(\cdot,t\right)\right]\right\}.
\end{aligned}
$$

D Appendix: Duality

The Primal. Given a path of resources $y(t)$, the problem is:

$$V\left[f\left(\cdot,0\right)\right] = \max_{\{\iota(\tau,t),c(t)\}_{t\in[0,\infty),\tau\in[0,T]}} \int_t^\infty e^{-\rho(s-t)} u(c(s)) ds \text{ s.t.}$$

$$c\left(t\right) = y\left(t\right) - f\left(0,t\right) + \int_0^T \left[q\left(\tau,t,\iota\right)\iota(\tau,t) - \delta f\left(\tau,t\right)\right] d\tau$$

$$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}; f(\tau,0) = f_0(\tau)$$

Definition 1. Given a path of income $\{y(t)\}$ and initial debt f_0 a solution to **P1** is a path of consumption $c(t)$ and debt issuances $\iota(\tau,t)$ such that 1) the budget constraint holds for every t 2) the evolution of debt satisfies the KFE 3) the no ponzi condition holds, there is not other path of consumption and debt issuances $\{\tilde{c},\tilde{\iota}\}$ that is feasible and yields strictly higher utility at zero.

Let $j(\tau,t)$ be the lagrange multiplier associated with the KFE. It measures the change maginal utility of issuing more debt.

Proposition 1. *If a solution to* **P1** *with* $e^{-\rho t}f$, $e^{-\rho t}\iota \in L^2\left([0,T]\times[0,\infty)\right), e^{-\rho t}c \in L^2[0,\infty)$, *given by* $\{\iota(\tau,t),c(t)\}_{t=0}^\infty$ *exists, it satisfies the PDE*

$$\rho j\left(\tau,t\right) = \frac{\partial q}{\partial f}\iota - \delta + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0,T]$$

$$j\left(0,t\right) = -u'(c(t)),$$

$$\lim_{t\to\infty} e^{-\rho t}j\left(\tau,t\right) = 0$$

where $v\left(\tau,t\right)$ *is the marginal value of a unit of debt with time-to-maturity* τ, *the interest rate* $r\left(t\right)$ *is given by* $r\left(t\right) = \rho - \frac{U''(c(t))}{U'(c(t))}\frac{\dot{c}(t)}{c(t)}$ *and* $e^{-\rho t}v \in L^2\left([0,T]\times[0,\infty)\right)$; *the optimal issuance* $\iota\left(\tau,t\right)$ *is given by*

$$\left(\frac{\partial q}{\partial \iota}\iota\left(\tau,t\right) + q\left(t,\tau,\iota\right)\right) u'(c(t)) = -j\left(\tau,t\right)$$

The Dual. This problem finds the lowest cost of achieving a particular path of consumption with the lowest amount of resources. Given a desired path of consumption $c(t)$ the objective is to minimize the resources needed to achieve that path. More precisely, **P2** is given by:

$$D\left[f\left(\cdot,0\right)\right] = \min_{\{\iota(\tau,t),y_t\}_{t=\tau\in[0,\infty),\tau\in[0,T]}} \int_0^\infty e^{-\int_0^t r(s)ds} y_t dt \text{ s.t.}$$

$$c\left(t\right) = y\left(t\right) - f\left(0,t\right) + \int_0^T \left[q(\tau,t,\iota)\iota(\tau,t) - \delta f\left(\tau,t\right)\right] d\tau$$

$$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}; f(\tau,0) = f_0(\tau)$$

$$r(t) = \rho + \sigma \frac{\dot{c}(t)}{c(t)}$$

Definition 2. Given a path of consumption $\{c(t)\}$ and initial debt f_0 a solution to **P2** is a path of income $y(t)$ and debt issuances $\iota(\tau,t)$ such that 1) the budget constraint holds for every t 2) the evolution of debt satisfies the KFE 3) the no ponzi condition holds, there is not other path of consumption and debt issuances $\{\tilde{y}, \tilde{\iota}\}$ that is feasible and has lower resources associated.

Let $v(\tau,t)$ be the Lagrange multiplier associated with KFE. it measure marginal resources needed if a unit of debt is issued. The necessary conditions are the following:

Proposition 2. *If a solution to **P2** with $e^{-\rho t}f$, $e^{-\rho t}\iota \in L^2([0,T] \times [0,\infty))$, $e^{-\rho t}c \in L^2[0,\infty)$, given by $\{\iota(\tau,t),c(t)\}_{t=0}^{\infty}$ exists, it satisfies the PDE*

$$r(t)\, v(\tau,t) = \frac{\partial q}{\partial f}\iota - \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0,T]$$

$$v(0,t) = -1,$$

$$\lim_{t \to \infty} e^{-\rho t} v(\tau,t) = 0$$

where $v(\tau,t)$ is the marginal value of a unit of debt with time-to-maturity τ, the interest rate $r(t)$ is given by $r(t) = \rho - \frac{U''(c(t))}{U'(c(t))}\frac{\dot{c}(t)}{c(t)}$ and $e^{-\rho t}v \in L^2([0,T] \times [0,\infty))$; the optimal issuance $\iota(\tau,t)$ is given by

$$\frac{\partial q}{\partial \iota}\iota(\tau,t) + q(t,\tau,\iota) = -v(\tau,t)$$

Proof. See below. . $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The Connection.

Corollary 1. *Suppose that for a given income path $y(t)$ and initial debt f_0 the solution to **P1** is $c^*(t), \iota^*(\tau,t), j^*(\tau,t)$. Then, $y(t), \iota^*(\tau,t), \frac{j^*(\tau,t)}{u'(c(t))}$ solves **P2** given the path $c^*(t)$.*

D.1 Proof of Proposition

First we construct a Lagrangian in the space of functions g such that $\left\| e^{-\rho t/2}g(\tau,t) \right\|_{L^2} < \infty$. The Lagrangian is

$$\begin{aligned}
\mathcal{L}(\iota,f) &= \int_0^{\infty} e^{-r(t)t}\left(c(t) + f(0,t) - \int_0^T [q(t,\tau,\iota,f)\,\iota(\tau,t) - \delta f(\tau,t)]\,d\tau \right) dt \\
&+ \int_0^{\infty}\int_0^T e^{-r(t)t} v(\tau,t)\left(-\frac{\partial f}{\partial t} + \iota(\tau,t) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
\end{aligned}$$

where $j(\tau, t)$ is the Lagrange multiplier associated to the law of motion of debt. Taking Gateaux derivatives, for any suitable $h(\tau, t)$ such that $e^{-\rho t}h \in L^2([0,T] \times [0,\infty))$:

$$\lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \mathcal{L}(\iota, f + \alpha h) = \int_0^\infty e^{-r(t)t} \left[h(0,t) - \int_0^T \left(\frac{\partial q}{\partial f} \iota(\tau, t) - \delta \right) h(\tau, t) d\tau \right] dt$$
$$+ \int_0^\infty \int_0^T e^{-r(t)t} \frac{\partial h}{\partial t} v(\tau, t) d\tau dt$$
$$- \int_0^\infty \int_0^T e^{-r(t)t} \frac{\partial h}{\partial \tau} v(\tau, t) d\tau dt,$$

The last two terms can be integrated by parts

$$- \int_0^T \frac{\partial h}{\partial \tau} v(\tau, t) d\tau = - \quad h(T,t) v(T,t) + h(0,t) v(0,t) + \int_0^T h \frac{\partial v}{\partial \tau} d\tau,$$
$$+ \int_0^\infty e^{-r(t)t} \frac{\partial h}{\partial t} v(\tau, t) dt = \quad \lim_{s \to \infty} e^{-r(t)s} h(\tau, s) v(\tau, s) - h(\tau, 0) v(\tau, 0) - \int_0^\infty e^{-r(t)t} h(\tau, t) \left(\frac{\partial v}{\partial t} - r(t)v \right) d\tau.$$

As the initial distribution f_0 is given the value of $h(\tau, 0) = 0$. The Gateaux derivative should be zero for any suitable $h(\tau, t)$

$$0 = \int_0^\infty e^{-r(t)t} \left[+h(0,t) - \int \left(\frac{\partial q}{\partial \tau} \iota(\tau, t) - \delta \right) h(\tau, t) d\tau \right] d\tau$$
$$- \int_0^\infty \int_0^T e^{-r(t)t} \left(-r(t)v - \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t} \right) h(\tau, t) d\tau dt$$
$$- \int_0^\infty e^{-\rho t} \left(h(T,t) v(T,t) - h(0,t) v(0,t) \right) dt$$
$$+ \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h(\tau, s) v(\tau, s) d\tau.$$

Therefore, as $f(T^+, t) = 0$ then $h(T, t) = 0$ and we have

$$r(t)v(\tau, t) = \left(\frac{\partial q}{\partial f} \iota - \delta \right) + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T] \tag{D.1}$$
$$v(0, t) = -1, \text{ if } \tau = 0,$$
$$\lim_{t \to \infty} e^{-r(t)t} v(\tau, t) = 0.$$

Proceeding similarly in the case of ι

$$\lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \mathcal{L}(\iota + \alpha h, f) = \int_0^\infty e^{-r(t)t} \left[- \int_0^T \left(\frac{\partial q}{\partial \iota} \iota(\tau, t) + q(t, \tau, \iota, f) \right) h(\tau, t) d\tau \right] dt$$
$$- \int_0^\infty \int_0^T e^{-r(t)t} h(\tau, t) v(\tau, t) d\tau dt,$$

and hence

$$\left(\frac{\partial q}{\partial \iota}\iota\left(\tau,t\right) + q\left(t,\tau,\iota,f\right)\right) = -v\left(\tau,t\right).$$

The PDE equation (D.1) results in

$$
\begin{aligned}
r(t)v\left(\tau,t\right) &= \frac{\partial q}{\partial f}\iota - \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0,\infty),\\
v\left(0,t\right) &= -1, \text{ if } \tau = 0,\\
\lim_{t\to\infty} e^{-r(t)t}v\left(\tau,t\right) &= 0,
\end{aligned}
$$

and the first order condition is

$$\frac{\partial q}{\partial \iota}\iota\left(\tau,t\right) + q\left(t,\tau,\iota,f\right) = -v\left(\tau,t\right).$$

E   Computational Method Deterministic Dynamics

We describe the numerical algorithm used to jointly solve for the equilibrium value function, $v(\tau, t)$, bond price, $q(t, \tau, \iota)$, consumption $c(t)$, issuance $\iota(\tau, t)$ and density $f(\tau, t)$. The equilibrium is characterized by the HJB equation

$$r(t) v(\tau, t) = -\delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0, T] \tag{E.1}$$

$$v(0, t) = -1, \text{ if } \tau = 0, \tag{E.2}$$

where the interest rate $r(t)$ is given by:

$$r(t) = \rho + \frac{\gamma}{c(t)} \frac{dc}{dt}, \tag{E.3}$$

where γ is the risk coefficient in $U(c) := \frac{c^{1-\gamma}-1}{1-\gamma}$. The optimal issuance $\iota(\tau, t)$ is given by

$$\iota = \frac{1}{\bar{\lambda}} \left(\frac{\delta(1 - e^{-\bar{r}\tau})}{\bar{r}} + e^{-\bar{r}\tau} + v(\tau, t) \right). \tag{E.4}$$

The law of motion of the density of maturities is given by the Kolmogorov Forward equation

$$\frac{\partial f}{\partial t} = \iota(\tau, t) + \frac{\partial f}{\partial \tau}, \tag{E.5}$$

and consumption by the budget constraint

$$c(t) = \bar{y} - f(0, t) + \int_0^T \left[\left(\frac{\delta(1 - e^{-\bar{r}\tau})}{\bar{r}} + e^{-\bar{r}\tau} - \frac{1}{2}\bar{\lambda}\iota(\tau, t) \right) \iota(\tau, t) - \delta f(\tau, t) \right] d\tau. \tag{E.6}$$

The parameters are T, δ, $\bar{y}$, γ, $\bar{\lambda}$, ρ and $\bar{r} = \rho$. The initial distribution is $f(\tau, 0) = f_0(\tau)$. The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the Hamilton-Jacobi-Bellman equation   The HJB equation (E.1) is solved using an *upwind finite difference* scheme similar to Achdou et al. (2014). We approximate the value function $v(\tau)$ on a finite grid with step $\Delta\tau : \tau \in \{\tau_1, ..., \tau_I\}$, where $\tau_i = \tau_{i-1} + \Delta\tau = \tau_1 + (i-1)\Delta\tau$ for $2 \leq i \leq I$. The bounds are $\tau_1 = \Delta\tau$ and $\tau_I = T$, such that $\Delta\tau = T/I$. We use the notation $v_i := v(\tau_i)$, and similarly for the policy function ι_i. Notice first that the HJB equation involves first derivatives of the value function. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable. As in our case, the drift is always negative, we employ a backward

approximation in state:

$$\frac{\partial v(\tau_i)}{\partial \tau} \approx \frac{v_i - v_{i-1}}{\Delta \tau}. \tag{E.7}$$

The HJB equation is approximated by the following upwind scheme,

$$\rho v_i = -\delta + \frac{v_{i-1}}{\Delta \tau} - \frac{v_i}{\Delta \tau},$$

with terminal condition $v_0 = v(0) = -1$. This can be written in matrix notation as

$$\rho \mathbf{v} = \mathbf{u} + \mathbf{A} \mathbf{v},$$

where

$$\mathbf{A} = \frac{1}{\Delta \tau}\begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{I-1} \\ v_I \end{bmatrix}, \tag{E.8}$$

$$\mathbf{u} = \begin{bmatrix} -\delta - 1/\Delta \tau \\ -\delta \\ -\delta \\ \vdots \\ -\delta \\ -\delta \end{bmatrix}.$$

The solution is given by

$$\mathbf{v} = (\rho \mathbf{I} - \mathbf{A})^{-1} \mathbf{u}. \tag{E.9}$$

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}$.

To analyze the transitional dynamics, define $t^{\max}$ as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and time is discretized as $t_n = t_{n-1} + \Delta t$, in intervals of length

$$\Delta t = \frac{t^{\max}}{N - 1},$$

where N is a constant. We use now the notation $v_i^n := v(\tau_i, t_n)$. The value function at $t^{\max}$ is the stationary solution computed in (E.9) that we denote as $\mathbf{v}^N$.[26] We choose a forward approxima-

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[26]You may begin directly by employing the analytical solution from equation (??) as $\mathbf{v}^N$.

tion in time. The dynamic HJB (E.1) can thus be expressed

$$r^n \mathbf{v}^n = \mathbf{u} + \mathbf{A}\mathbf{v}^n + \frac{\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right)}{\Delta t},$$

where $r^n := r(t_n)$. By defining $\mathbf{B}^n = \left(\frac{1}{\Delta t} + r^n\right)\mathbf{I} - \mathbf{A}$ and $\mathbf{d}^{n+1} = \mathbf{u} + \frac{\mathbf{v}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = (\mathbf{B}^n)^{-1}\,\mathbf{d}^{n+1}, \tag{E.10}$$

which can be solved backwards from $n = N - 1$ until $n = 1$.

The optimal issuance is given by

$$\iota_i^n = \frac{1}{\bar{\lambda}}\left(\psi_i + v_i^n\right),$$

where

$$\psi_i = \frac{\delta\left(1 - e^{-\rho\tau_i}\right)}{\rho} + e^{-\rho\tau_i}.$$

Step 2: Solution to the Kolmogorov Forward equation   Analogously, the KFE equation (2.1) can be approximated as

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} = \iota_i^n + \frac{f_{i+1}^n - f_i^n}{\Delta\tau},$$

where we have employed the notation $f_i^n := f(\tau_i, t_n)$. This can be written in matrix notation as:

$$\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t} = \iota^n + \mathbf{A}^{\mathbf{T}}\mathbf{f}^n, \tag{E.11}$$

where $\mathbf{A}^{\mathbf{T}}$ is the transpose of $\mathbf{A}$ and

$$\mathbf{f}^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \\ f_I^n \end{bmatrix}.$$

Given $\mathbf{f}_0$, the discretized approximation to the initial distribution $f_0(\tau)$, we can solve the KF equation forward as

$$\mathbf{f}_n = \left(\mathbf{I} - \Delta t \mathbf{A}^{\mathbf{T}}\right)^{-1}\left(\iota^n \Delta t + \mathbf{f}_{n-1}\right), \quad n = 1, .., N. \tag{E.12}$$

Step 3: Computation of consumption   The discretized budget constraint (**??**) can be expressed as

$$c^n = \bar{y} - f_1^{n-1}\Delta\tau + \sum_{i=1}^{I} \left[\left(\psi_i - \frac{1}{2}\bar{\lambda}\iota_i^n \right) \iota_i^n - \delta f_i^n \right] \Delta\tau, \quad n = 1, .., N.$$

Compute

$$r^n = \rho + \frac{\gamma}{c^n}\frac{c^{n+1} - c^n}{\Delta t}, n = 1, .., N - 1.$$

Complete algorithm   The algorithm proceeds as follows. First guess an initial path for consumption, for example $c^n = \bar{y}$, for $n = 1, .., N$. Set $k = 1$;

Step 1: HJB. Given c_{k-1} solve the HJB and obtain ι.

Step 2: KF. Given ι solve the KF equation with initial distribution f_0 and obtain the distribution f.

Step 3: Consumption. Given ι and f compute consumption c. If $\|c - c_{k-1}\| = \sum_{n=1}^{N} \left| c^n - c_{k-1}^n \right| < \varepsilon$ then stop. Otherwise compute

$$c_k = \omega c + (1 - \omega) c_{k-1}, \ \lambda \in (0, 1),$$

set $k := k + 1$ and return to step 1.