# Collateral Constraints and Asset Prices* 

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#### Abstract

We consider an economy populated by investors with heterogeneous preferences and beliefs who receive non-pledgeable labor incomes. We study the effects of collateral constraints that require investors to maintain sufficient pledgeable capital to cover their liabilities. We show that these constraints inflate stock prices, give rise to clusters of stock return volatilities, and produce spikes and crashes in price-dividend ratios and volatilities. Furthermore, the mere possibility of a crisis significantly decreases interest rates and increases Sharpe ratios. The stock price has a large collateral premium over non-pledgeable incomes. Asset prices are in closed form, and investors survive in the long run.


Journal of Economic Literature Classification Numbers: D52, G12.
Keywords: collateral, non-pledgeable labor income, heterogeneous preferences, disagreement, asset prices, stationary equilibrium.

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## 1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors' incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are backed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of financial markets at a cost of restricting risk sharing among investors. In this paper, we develop a parsimonious model which sheds light on the economic effects of such restrictions and show that they give rise to rich dynamics of asset prices. In particular, we show how collateralization inflates asset prices, generates repeated booms and busts in the stock market, and leads to spikes, crashes, and clustering of stock return volatilities, as well as cycles of high and low leverage. Our analysis is facilitated by closed-form solutions and the stationarity of equilibrium processes.

We consider a pure exchange economy with one consumption good produced by a tree with i.i.d. shocks, similar to Lucas (1978). The economy is populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the output growth rate. Each investor receives a fraction of the tree's output as labor income and invests total wealth in financial assets such as bonds and stocks. The investors have limited liability and can re-enter the financial market following defaults on risky positions in financial assets. In the event of default the financial assets can be seized by counterparties but labor income cannot be expropriated. The arising moral hazard problem is resolved by requiring risky positions to be backed by collateral in such a way that each investor's total financial wealth stays positive at all times, and hence, investors can always pay back to counterparties. We also allow the aggregate consumption to experience rare large negative shocks, which help us explore how mere anxiety about the possibility of a production crisis affects the economy by tightening collateral constraints. Our closed-form solutions allow us to prove some of the results for general model parameters rather than for particular calibrations.

First, we show that collateral constraints increase the prices of all tradable assets with positive cash flows relative to a frictionless economy. Moreover, these increases in prices are larger when investors are closer to their default boundaries. In particular, the stock price-dividend ratio is a U-shaped function of one of the investor's share of the aggregate
consumption. Consequently, it spikes upwards in response to small economic shocks near default boundaries giving rise to repeated periods of high and low stock prices.

The intuition for the latter results is as follows. In a frictionless economy, the investors' consumption shares gradually approach zero or one, and hence the economic impact of one of the investors vanishes in the long-run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015). The collateral requirements restrict financial losses and protect investors from losing their consumption shares. The result is that the consumption shares are bounded away from zero and one. Moreover, the constraints never bind simultaneously for both investors, and at each moment one of the investors is unconstrained. The unconstrained investor's marginal utility of consumption is proportional to the prices of Arrow-Debreu securities. This marginal utility is expected to be higher in the economy with constraints because the unconstrained investor's consumption is expected to be lower than in the unconstrained economy due to the upper bound on the consumption share, discussed above. Consequently, the prices of Arrow-Debreu securities, and hence, the prices of all assets with positive cash flows, are higher in the constrained economy.

The dynamics of the price-dividend ratio determines the effect of constraints on volatilities. We show that collateral constraints dampen volatilities in bad times, when aggregate consumption is low, and amplify them in good times, when aggregate consumption is high. The latter effect makes collateral requirements a useful tool for curbing excessive volatility in bad times. The explanation is that the U-shaped price-dividend ratio is procyclical in good and countercyclical in bad times. As a result, the price-dividend ratio and the dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the price-dividend ratio and the dividend, the stock return volatility increases in good times and decreases in bad times. The volatility experiences spikes and crashes due to the sensitivity of price-dividend ratios to small shocks when investors are close to hitting their constraints. Moreover, the periods of high and low volatility are persistent because of the persistence of periods when constraints are likely to bind, as discussed below, which gives rise to the clustering of volatilities.

We also derive the distributions of investors' consumption shares in analytic form and show that they are stationary and non-degenerate (i.e., their support is a closed interval rather than a single point). The analysis of these distributions yields three economic insights. First, there is non-trivial time-variation of asset prices in the long run. Second, periods of binding collateral requirements are persistent. That is, the economy stays close
to default boundaries for some time because hitting a constraint makes likely hitting it again in the near future due to slow accumulation of wealth over time. Third, we show that all investors, including those with incorrect beliefs, survive in the long run and can have a large economic impact in equilibrium because the constraints prevent investors from losing their consumption shares, similar to the related literature (e.g., Blume and Easley, 2006; Cao, 2018). We note that the non-degeneracy of consumption share distributions and the persistence of the periods of binding constraints are more difficult to demonstrate than survival, and, to our best knowledge, these results are new to the literature.

Next, we show that mere possibility of a large (albeit unpredictable) drop in the aggregate output next period decreases interest rates and increases Sharpe ratios in the current period when the irrational optimist is close to hitting the collateral constraint. The latter effect only occurs when production crises and collateral requirements are jointly present in the economy. Hence, the collateral requirements amplify the spillover of the production crisis to the financial market. The amplification effect arises because investors "fly to quality" by buying riskless bonds when there is a possibility of hitting the collateral constraint next period. We note that lower interest rates and higher Sharpe ratios can be generated by alternative mechanisms and constraints, discussed in the literature review below. However, the amplification mechanism, to our best knowledge, has not been studied before. We also show that investor heterogeneity and the stationarity of equilibrium give rise to cycles of high and low leverage. In particular, the leverage is high when investors are far away from their default boundaries, and drops to zero when investors hit their constraints.

Finally, we measure the collateral liquidity premium of the stock versus labor income. This premium arises because dividends and labor incomes are collinear but incomes are non-pledgeable. First, we derive shadow prices of claims to labor incomes such that exchanging marginal units of these claims for the consumption good at shadow prices does not affect investors' welfare. Then, we construct portfolios of stocks that replicate labor incomes. We define the collateral liquidity premium as the percentage difference in the value of the replicating portfolio and the shadow price. The premium from the view of a particular investor widens close to that investor's default boundary and ranges from $0 \%$ to $35 \%$ in our calibration, which demonstrates the economic importance of collateralization. We also show that the non-tradability of labor income does not contribute to this premium. This is because in economies with pledgeable labor income investors circumvent non-tradability by taking short positions in the stock, and hence, the liquidity premium
in such economies is zero.
The paper develops a new methodology for studying the effects of collateral requirements. This new methodology allows us to obtain closed-form equilibrium processes and prove their properties which previously could only be studied numerically. For example, we prove that our constraints increase price-dividend ratios and generate spikes in asset prices, and lead to non-degeneracy and stationarity of consumption share distributions. Hence, collateralization emerges as a tractable way of inducing the stationary of equilibrium. Finally, the paper introduces a tractable discrete-time framework that makes exposition less technical and permits taking continuous-time limits. The tractability and stationarity make our model a convenient benchmark for asset pricing research that can be extended in various directions.

Related Literature. Closest to us are papers that study economies where investors have limited liability and face solvency constraints. Deaton (1990) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and face a non-negativity constraint on their financial wealth. Detemple and Serrat (2003) also study the non-negative wealth constraint in a model where investors have heterogeneous beliefs and identical risk aversions. They show that this constraint introduces a singularity component into interest rates when the constraint binds while stock risk premia have the same structure as in unconstrained economies. They do not compute price-dividend ratios, volatilities, and consumption share distributions as we do in this paper. They also do not study the effects of rare production crises and heterogeneity in preferences. We show that the production crises give rise to new effects. They amplify the effects of constraints on interest rates and Sharpe ratios and have an impact on them not only at the boundaries (as in Detemple and Serrat, 2003) but also in the internal area of the state-space.

Chien and Lustig (2010) study a similar constraint in an economy with a continuum of ex-ante identical investors that receive non-pledgeable labor incomes affected by idiosyncratic shocks. Lustig and Van Nieuwerburgh (2005) study the role of housing collateral when labor income is non-pledgeable. The main difference of our paper from the latter two papers is that our investors are ex-ante heterogeneous and are not affected by idiosyncratic shocks to labor income. The economic effects of heterogeneity in preferences and beliefs are different from the effects of ex-post heterogeneity in realized idiosyncratic labor income shocks in the above literature. For example, Krueger and Lustig (2010) show the
irrelevance of market incompleteness induced by these income shocks for the risk premia.
Cao (2018) proves that investors with incorrect beliefs have strictly positive shares of consumption in the long run (i.e., survive in the long-run) in economies with general collateral constraints and stationary endowment processes bounded away from zero. Similar results are also shown numerically in an example with non-stationary endowments. Blume et al (2015) explore potential benefits from imposing trading restrictions, such as natural borrowing constraints, in economies with bounded endowments and investors with heterogeneous beliefs. In contrast to these works, our results do not rely on bounded endowments. Moreover, in addition to showing the survival of investors, we derive consumption share distributions in closed form, and establish their bimodality, stationarity, and non-degeneracy (i.e., their support is a closed interval rather than a single point), and derive new equilibrium effects. Kubler and Schmedders (2013) prove the existence of stationary equilibria in dynamic economies with general collateral constraints. Rampini and Viswanathan (2018) study household insurance in an economy with collateral constraints with limited enforcement and deep-pocket risk-neutral lenders, who provide statecontingent claims to households at zero risk premium. Our model is different from the latter paper in that all investors in our economy are risk averse, and risk premia are endogenous and time-varying. Gromb and Vayanos (2002, 2010, 2018) and Brunnermeier and Pedersen (2009) study economies with CARA investors subject to margin constraints, which have similarities with our constraints. In contrast to their models, in our model all investors have CRRA preferences and interest rates are endogenous.

Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008, 2014), and Geanakoplos and Zame (2014) develop the theory of collateral constraints in two- and three-period multinomial general equilibrium economies. Similar to the baseline analysis in Geanakoplos (2003, 2009), our constraint also prevents investors from defaulting in the worst-case scenario. Simsek (2013) studies a two-period economy with a continuum of states and shows that collateral constraints have asymmetric disciplining effects, depending on investor's beliefs, and also shows how defaultable debt endogenously emerges in equilibrium. Biais, Hombert, and Weill (2018) study a two-period economy with multiple trees and imperfect collateral pledgeability. In contrast to this literature, we focus on the non-pledgeability of labor income rather than imperfect pledgeability of assets. Hence, our model generates a different set of predictions. Our constraint is also more tractable and allows us to study multiperiod economies where investors have different preferences and beliefs and the
output follows a geometric Brownian motion with jumps.
Kehoe and Levine (1993), Kocherlakota (1996), Tsyrennikov (2012), and Osambela (2015) study economies where investors are weakly better off not defaulting and are permanently excluded from securities markets if they default. Alvarez and Jermann (2000) show that such constraints can be implemented by imposing certain "not too tight" solvency portfolio constraints. Alvarez and Jermann (2001) find that such constraints help explain equity premia in the U.S. economy. They solve a simple example in closed form and develop a numerical method for the general case. In contrast to this literature, our investors have limited liability and can re-enter the market after a default.

Our paper is related to the literature on the economic effects of borrowing, margin, short-sale and position limit constraints (e.g., Harrison and Kreps, 1978; Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Pavlova and Rigobon, 2008; Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014; Brumm et al, 2015; Buss et al, 2016), portfolio insurance (e.g., Basak, 1995) and VaR constraints (e.g., Basak and Shapiro, 2001). Our economic results are different from the results in this literature. First, the latter constraints can increase or decrease stock prices depending on whether the investors' risk aversions are greater or less than one (e.g., Chabakauri, 2015), whereas our collateral requirements always increase stock prices irrespective of risk aversions and beliefs. Second, these constraints typically dampen stock return volatility whereas our collateral constraints amplify them in some states of the economy. We also uncover new effects such as spikes and crashes of volatilities and stock prices, and clusters of volatility.

We note that our collateral requirements are different from the margin and borrowing constraints in some of the related models discussed above (e.g., Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014). The latter models focus on partial nonpledgeability of stocks and do not incorporate labor incomes. The economic effects of constraints in these models are driven by reduced risk-sharing. In contrast to these works, we explore the non-pledgeability of labor incomes in a setting with fully pledgeable financial assets serving as collateral. The effects of constraints in our model are driven by increased marginal utilities of investors and collateral premia, which inflate asset prices.

Our methodology is also different from the approaches in the related literature. The equilibrium in models with constraints is often constructed using fictitious complete market economies (Cvitanić and Karatzas, 1992; Cuoco, 1997; Detemple and Murthy, 1997;

Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Pavlova and Rigobon, 2008; Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015). Moreover, when investors have non-logarithmic utilities the equilibrium is characterized in terms of non-linear differential equations that can only be solved numerically (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014). In contrast to these works, we do not employ fictitious markets and derive the equilibrium using the envelope theorem. To our best knowledge, this paper is the first to derive analytical prices, distributions of consumption shares, and conditions for the constraints to be binding.

The paper is also related to macro-finance, financial intermediation, and banking literatures that study economies with frictions (Kiyotaki and Moore, 1997; Krusell and Smith, 1998; Brunnermeier and Sannikov, 2014; Klimenko, Pfeil, Rochet, and De Nicolo, 2016; Kondor and Vayanos, 2018) and to the literature on frictionless economies with heterogeneous investors (e.g., Chan and Kogan, 2002; Basak, 2005; Yan, 2008; Bhamra and Uppal, 2014; Atmaz and Basak, 2018; Borovička, 2018; Massari, 2018, among others).

## 2. Economic setup

We consider a pure-exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree. The economy is populated by two representative heterogeneous investors $A$ and $B$ that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start with a discrete-time economy with dates $t=0, \Delta t, 2 \Delta t, \ldots$, and later take a continuous-time limit.

At each point of time $t=0, \Delta t, 2 \Delta t, \ldots$ the economy is in one of the three states: $\omega_{1}$, $\omega_{2}$, and $\omega_{3}$. With probability $1-\lambda \Delta t$ the economy is either in state $\omega_{1}$ or state $\omega_{2}$, which we call normal states, and with probability $\lambda \Delta t$ in state $\omega_{3}$, which we call the crisis state. Parameter $\lambda>0$ is the crisis intensity. States $\omega_{1}$ and $\omega_{2}$ have probabilities $1 / 2$ conditional on the economy being in a normal state. Figure 1 depicts the structure of uncertainty.

### 2.1. Output, financial markets, and investor heterogeneity

At date $t$ the tree produces $D_{t} \Delta t$ units of aggregate output, where $D_{t}$ follows a process

$$
\begin{equation*}
\Delta D_{t}=D_{t}\left[\mu_{D} \Delta t+\sigma_{D} \Delta w_{t}+J_{D} \Delta j_{t}\right] \tag{1}
\end{equation*}
$$



## Figure 1

## States of the Economy

After time $t$ the economy moves to a normal state with probability $1-\lambda \Delta t$ and to a crisis state with probability $\lambda \Delta t$. Conditional on being in a normal state the economy moves to either $\omega_{1}$ or $\omega_{2}$ with equal probabilities.
where $\mu_{D} \geq 0, \sigma_{D}>0$, and $J_{D} \leq 0$ are output growth mean, volatility, and drop during a crisis, respectively, and $\Delta D_{t}=D_{t+\Delta t}-D_{t}$ is the change in output. Processes $w_{t}$ and $j_{t}$ are discrete-time analogues of a Brownian motion and Poisson processes, respectively. ${ }^{1}$ These processes follow dynamics $w_{t+\Delta t}=w_{t}+\Delta w_{t}$ and $j_{t+\Delta t}=j_{t}+\Delta j_{t}$, where increments $\Delta w_{t}$ and $\Delta j_{t}$ are i.i.d. random variables given by:

$$
\Delta w_{t}=\left\{\begin{array}{rl}
+\sqrt{\Delta t}, & \text { in state } \omega_{1},  \tag{2}\\
-\sqrt{\Delta t}, & \text { in state } \omega_{2}, \\
0, & \text { in state } \omega_{3}
\end{array} \quad \Delta j_{t}= \begin{cases}0, & \text { in state } \omega_{1} \\
0, & \text { in state } \omega_{2} \\
1, & \text { in state } \omega_{3}\end{cases}\right.
$$

It can be easily verified that $\mathbb{E}_{t}\left[\Delta w_{t} \mid\right.$ normal $]=0$ and $\operatorname{var}_{t}\left[\Delta w_{t} \mid\right.$ normal $]=\Delta t$, similar to a Brownian motion, where $\mathbb{E}_{t}[\cdot]$ and $\operatorname{var}_{t}[\cdot]$ are expectation and variance conditional on time- $t$ information. Parameters $\mu_{D}, \sigma_{D}$, and $J_{D}$ are such that $D_{t}>0$ for all $t$.

The economy is populated by two representative investors $A$ and $B$. Each investor stands for a continuum of identical investors of unit mass. Fractions $l_{A}$ and $l_{B}$ of the aggregate output $D_{t} \Delta t$ are paid to investors $A$ and $B$ as their labor incomes, respectively. Labor incomes are non-tradable. Fractions $l_{A}$ and $l_{B}$ can also be interpreted as non-tradable shares in the aggregate output such as holdings of illiquid assets rather than shares of labor income. The remaining fraction $1-l_{A}-l_{B}$ is paid as a dividend to the shareholders.

The investors can trade three securities at each date $t$ : 1) a riskless bond in zero net supply, which pays one unit of consumption at date $t+\Delta t ; 2)$ one stock in net supply of one unit, which is a claim to the stream of dividends $\left.\left(1-l_{A}-l_{B}\right) D_{t} \Delta t ; 3\right)$ a one-period

[^1]insurance contract in zero net supply, which pays one unit of consumption in the crisis state $\omega_{3}$ and zero otherwise. Absent any frictions the market is complete. Market completeness and the absence of idiosyncratic shocks to labor income are required for tractability, and allow us to solve the model in closed form. Bond, stock, and insurance prices $B_{t}, S_{t}$, and $P_{t}$, respectively, are determined in equilibrium.

### 2.2. Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

$$
u_{i}(c)=\left\{\begin{array}{lll}
\frac{c^{1-\gamma_{i}}}{1-\gamma_{i}}, & \text { if } & \gamma_{i} \neq 1  \tag{3}\\
\ln (c), & \text { if } & \gamma_{i}=1
\end{array}\right.
$$

where $i=A, B$. The investors agree on time- $t$ asset prices and the aggregate output but disagree on the probabilities of states. Investor $A$ is rational and has correct probabilities

$$
\begin{equation*}
\pi_{A}\left(\omega_{1}\right)=\frac{1-\lambda \Delta t}{2}, \quad \pi_{A}\left(\omega_{2}\right)=\frac{1-\lambda \Delta t}{2}, \quad \pi_{A}\left(\omega_{3}\right)=\lambda \Delta t \tag{4}
\end{equation*}
$$

where $\lambda$ is such that probabilities (4) are positive. Investor $B$ has biased probabilities

$$
\begin{equation*}
\pi_{B}\left(\omega_{1}\right)=\frac{1-\lambda_{B} \Delta t}{2}(1+\delta \sqrt{\Delta t}), \quad \pi_{B}\left(\omega_{2}\right)=\frac{1-\lambda_{B} \Delta t}{2}(1-\delta \sqrt{\Delta t}), \quad \pi_{B}\left(\omega_{3}\right)=\lambda_{B} \Delta t, \tag{5}
\end{equation*}
$$

where crisis intensity $\lambda_{B}$ and disagreement parameter $\delta$ are such that probabilities (5) are positive. It is immediate to verify that $\pi_{B}\left(\omega_{1}\right)+\pi_{B}\left(\omega_{2}\right)+\pi_{B}\left(\omega_{3}\right)=1$, and hence, $\pi_{B}(\omega)$ is a probability measure. Throughout the paper, by $\mathbb{E}_{t}^{i}[\cdot]$ and $\operatorname{var}_{t}^{i}[\cdot]$ we denote conditional expectations and variances under the probability measure of investor $i$.

It can be easily verified that time- $t$ conditional expected output growth rate in normal times under the beliefs of investor $B$ is given by:

$$
\begin{equation*}
\mathbb{E}_{t}^{B}\left[\left.\frac{\Delta D_{t}}{D_{t}} \right\rvert\, \text { normal }\right]=\left(\mu_{D}+\delta \sigma_{D}\right) \Delta t \tag{6}
\end{equation*}
$$

Therefore, parameter $\delta$ measures the extent of the investor disagreement about the expected output growth during normal times. For tractability, we assume that investor $B$ does not update probabilities over time. We also assume that investor $B$ is weakly less risk averse and more optimistic than investor $A: \gamma_{A} \geq \gamma_{B}, \lambda \geq \lambda_{B}$ and $\delta \geq 0$. The assumption that the less risk averse investor is also more optimistic is imposed to simplify
the exposition and does not affect the qualitative results in the paper. ${ }^{2}$ We allow for the heterogeneity in both risk aversions and beliefs for generality. Main qualitative results do not change if we keep only one source of heterogeneity.

At date 0 the investors have certain endowments of financial assets. The total time- $t$ disposable wealth of investor $i$ is given by $W_{i t}+l_{i} D_{t} \Delta t$, where $W_{i t}$ is the financial wealth, defined as the time- $t$ value of all positions in financial assets acquired at the previous date, and $l_{i} D_{t} \Delta t$ is the labor income. At date $t$, investor $i$ allocates wealth to $c_{i t} \Delta t$ units of consumption, $b_{i t}$ units of bond, and a portfolio of risky assets $n_{i t}=\left(n_{i, S t}, n_{i, p t}\right)$, where $n_{i, s t}$ and $n_{i, P t}$ are units of stock and insurance, respectively.

In a frictionless economy, the financial wealth $W_{i t}$ can become negative when investors take risky positions backed by their future labor income. However, in our economy only financial assets are pledgeable whereas labor incomes are not. Moreover, the investors have limited liability. That is, they can default when their financial wealth becomes negative and then re-enter the market, which gives rise to a moral hazard problem, similar to the related literature (e.g., Chien and Lustig, 2010; Geanakoplos, 2009). This problem is addressed here by requiring the investors to keep their next-period financial wealth $W_{i, t+\Delta t}$ positive at all times, so that their pledgeable capital is sufficient to cover all liabilities such as debt and short positions. Intuitively, constraint $W_{i, t+\Delta t} \geq 0$ requires investors to crosscollateralize their pledgeable financial assets in such a way that losses on one position are always offset by gains on the other positions.

Investor $i=A, B$ maximizes expected discounted utility with time discount $\rho$

$$
\begin{equation*}
\max _{c_{i t}, b_{i t}, n_{i t}} \mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{\infty} e^{-\rho \tau} u_{i}\left(c_{i \tau}\right) \Delta t\right], \tag{7}
\end{equation*}
$$

subject to the self-financing budget constraints, given by

$$
\begin{align*}
W_{i t}+l_{i} D_{t} \Delta t & =c_{i t} \Delta t+b_{i t} B_{t}+n_{i t}\left(S_{t}, P_{t}\right)^{\top}  \tag{8}\\
W_{i, t+\Delta t} & =b_{i t}+n_{i t}\left(S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t, \mathbf{1}_{\left\{\omega_{t+\Delta t}=\omega_{3}\right\}}\right)^{\top} \tag{9}
\end{align*}
$$

and the collateral constraint:

$$
\begin{equation*}
W_{i, t+\Delta t} \geq 0 \tag{10}
\end{equation*}
$$

[^2]where $W_{i, t+\Delta t}$ is the financial wealth at date $t+\Delta t$ given by equation (9).
To provide further intuition for the constraint (10), following Gromb and Vayanos (2018), we observe that it is equivalent to the following collateral constraint: ${ }^{3}$
$W_{i t}+\left(l_{i} D_{t}-c_{t}\right) \Delta t \geq \max _{\omega_{t+\Delta t}}\left\{n_{i, s t}\left(S_{t}-\frac{S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t}{1+r_{t} \Delta t}\right)+n_{i, P t}\left(P_{t}-\frac{1_{\left\{\omega_{t+\Delta t}=\omega_{3}\right\}}}{1+r_{t} \Delta t}\right)\right\}$.
The expression on the right-hand side of the constraint (11) represents the largest possible loss of a risky position evaluated in present value terms. Therefore, this constraint indicates that the investors are allowed to invest in portfolios of assets using these portfolios as collateral, but are required to put up sufficient amount of their own capital to cover the losses in the worst-case scenario. The coefficients multiplying asset holdings $n_{i, S t}$ and $n_{i, P t}$ in (11) and evaluated at the worst-case state $\omega_{t+\Delta t}$ are endogenous margin requirements that show the investors' own capital invested per unit of asset.

The constraint (11) is similar to collateral constraints in Brunnermeier and Pedersen (2008) and Gromb and Vayanos (2018) with the difference being that we allow investors to "cross-margin" their positions so that one risky asset can be used to cover margins on the other. Brunnermeier and Pedersen (2008) discuss the institutional features of such constraints and point out that it is increasingly possible to "cross-margin".

Remark 1 (Partially pledgeable labor income). Our model can be easily extended to economies where fraction $k_{i} \in[0,1]$ of investor $i$ 's labor income can be pledged. The requirement to keep next-period pledgeable wealth is then given by:

$$
\begin{equation*}
W_{i, t+\Delta t}+\underbrace{\frac{k_{i} l_{i}}{1-l_{A}-l_{B}}\left(S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t\right)}_{\text {measure of pledgeable labor income }} \geq 0 . \tag{12}
\end{equation*}
$$

The second term in constraint (12) measures the value of the pledgeable income. Let $k_{i} l_{i} D_{t} \Delta t$ be the pledgeable income of investor $i$. This income is proportional to stock dividends $\left(1-l_{A}-l_{B}\right) D_{t} \Delta t$, and hence, can be replicated by a portfolio of $\widehat{n}_{i}=k_{i} l_{i} /(1-$ $\left.l_{A}-l_{B}\right)$ units of stock with cum-dividend value $\widehat{n}_{i}\left(S_{t}+\left(1-l_{A}-l_{B}\right) D_{t} \Delta t\right)$. The investors can circumvent the non-tradability of pledgeable income by shorting stocks against this income. Hence, the claims to pledgeable income are, effectively, tradable and have the same value as the replicating portfolio. The requirement to have positive pledgeable wealth then

[^3]becomes $W_{i, t+\Delta t}+\widehat{n}_{i}\left(S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t\right) \geq 0$, which is equivalent to constraint (12). Lemma A. 1 in the Appendix shows that models with $k_{i} \neq 0$ reduce to models with $k_{i}=0$ by a change of variable. Hence, the economic implications of our baseline model with constraint (10) and the model with a more general constraint (12) are the same.

### 2.3. Equilibrium

Definition. An equilibrium is a set of asset prices $\left\{B_{t}, S_{t}, P_{t}\right\}$ and of consumption and portfolio policies $\left\{c_{i t}^{*}, b_{i t}^{*}, n_{i t}^{*}\right\}_{i \in\{A, B\}}$ that solve optimization problem (7) for each investor, given processes $\left\{B_{t}, S_{t}, P_{t}\right\}$, and consumption and securities markets clear:

$$
\begin{equation*}
c_{A t}^{*}+c_{B t}^{*}=D_{t}, \quad b_{A t}^{*}+b_{B t}^{*}=0, \quad n_{A, S t}^{*}+n_{B, S t}^{*}=1, \quad n_{A, P t}^{*}+n_{B, P t}^{*}=0 . \tag{13}
\end{equation*}
$$

In addition to asset prices, we derive price-dividend and wealth-output ratios $\Psi=$ $S /\left(\left(1-l_{A}-l_{B}\right) D\right)$ and $\Phi_{i}=W_{i}^{*} / D$, respectively. We also derive annualized $\Delta t$-period riskless interest rates $r_{t}$, stock mean-returns $\mu_{t}$ and volatilities $\sigma_{t}$ in normal times, and the percentage change of the stock price in the crisis state, denoted by $J_{t}$.

We derive the equilibrium in terms of state variable $v_{t}$ given by the log-ratio of marginal utilities of investors evaluated at their shares of the aggregate consumption $c_{i t}^{*} / D_{t}$ :

$$
\begin{equation*}
v_{t}=\ln \left(\frac{\left(c_{A t}^{*} / D_{t}\right)^{-\gamma_{A}}}{\left(c_{B t}^{*} / D_{t}\right)^{-\gamma_{B}}}\right) . \tag{14}
\end{equation*}
$$

Substituting consumption shares of investors $A$ and $B$, denoted by $s_{t}=c_{A t}^{*} / D_{t}$ and $1-s_{t}=$ $c_{B t}^{*} / D_{t}$, into equation (14), we express $v_{t}$ as a function of $s_{t}$ :

$$
\begin{equation*}
v_{t}=\gamma_{B} \ln \left(1-s_{t}\right)-\gamma_{A} \ln \left(s_{t}\right) . \tag{15}
\end{equation*}
$$

Variable $v_{t}$ is a decreasing function $s_{t}$, and hence, $s_{t}$ is an alternative state variable.
We assume that the exogenous model parameters are such that

$$
\begin{equation*}
\mathbb{E}^{i}\left[e^{-\rho \Delta t}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{i}}\right]<1, \quad i=A, B . \tag{16}
\end{equation*}
$$

Condition (16) is necessary and sufficient for the existence of equilibrium in homogeneousagent economies populated only by investor $A$ or investor $B$.

## 3. Characterization of equilibrium

First, we derive the investors' state price densities (SPD) $\xi_{A t}$ and $\xi_{B t}$ defined as processes such that asset prices can be expressed as follows (e.g., Duffie (2001, p.23)):

$$
\begin{align*}
B_{t} & =\mathbb{E}_{t}^{i}\left[\frac{\xi_{i, t+\Delta t}}{\xi_{i t}}\right]  \tag{17}\\
S_{t} & =\mathbb{E}_{t}^{i}\left[\frac{\xi_{i, t+\Delta t}}{\xi_{i t}}\left(S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t\right)\right],  \tag{18}\\
P_{t} & =\mathbb{E}_{t}^{i}\left[\frac{\xi_{i, t+\Delta t}}{\xi_{i t}} \mathbf{1}_{\left\{\omega_{t+\Delta t}=\omega_{3}\right\}}\right] \tag{19}
\end{align*}
$$

where $i=A, B$. The state price density $\xi_{i t}$ exists for each investor $i$ due to the absence of arbitrage opportunities in our economy. ${ }^{4}$ The investors can eliminate arbitrage because strategies with zero investment and non-negative payoffs are feasible given constraints (8)(10). The SPDs $\xi_{A t}$ and $\xi_{B t}$ differ due to heterogeneity in beliefs and are linked by the change of measure equation ${ }^{5}$

$$
\begin{equation*}
\frac{\xi_{B, t+\Delta t}}{\xi_{B t}}=\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \frac{\pi_{A}\left(\omega_{t+\Delta t}\right)}{\pi_{B}\left(\omega_{t+\Delta t}\right)} \tag{20}
\end{equation*}
$$

We find the SPDs from the first order conditions in terms of investors' marginal utilities of consumption and Lagrange multipliers for collateral requirements (10). First, we rewrite the budget equations (8)-(9) in a static form that expresses the current wealth in terms of current consumption and the expected discounted future wealth (e.g., Cox and Huang, 1989). Then, we solve investor optimizations by dynamic programming and the method of Lagrange multipliers. Lemma 1 below reports the results.

## Lemma 1 (Dynamic programming and the first order condition).

1) Let $V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)$ denote the value function of investor $i$, where $v_{t}$ is the state variable. Then, the value function solves the following equation of dynamic programminng:

$$
\begin{equation*}
V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)=\max _{c_{i t}}\left\{u_{i}\left(c_{i t}\right) \Delta t+e^{-\rho \Delta t} \mathbb{E}_{t}^{i}\left[V_{i}\left(W_{i, t+\Delta t}, v_{t+\Delta t} ; l_{i}\right)\right]\right\} \tag{21}
\end{equation*}
$$

subject to the static budget and collateral constraints:

$$
\begin{align*}
W_{i t}+l_{i} D_{t} \Delta t & =c_{i t} \Delta t+\mathbb{E}_{t}^{i}\left[\frac{\xi_{i, t+\Delta t}}{\xi_{i t}} W_{i, t+\Delta t}\right]  \tag{22}\\
W_{i, t+\Delta t} & \geq 0 \tag{23}
\end{align*}
$$

[^4]2) Value function $V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)$ is a concave function of wealth $W_{i t}$.
3) The SPDs $\xi_{i t}$ and optimal consumptions $c_{i t}^{*}$ satisfy the first order conditions
\[

$$
\begin{equation*}
\frac{\xi_{i, t+\Delta t}}{\xi_{i t}}=e^{-\rho \Delta t} \frac{\left(c_{i, t+\Delta t}^{*}\right)^{-\gamma_{i}}+\ell_{i, t+\Delta t}}{\left(c_{i t}^{*}\right)^{-\gamma_{i}}} \tag{24}
\end{equation*}
$$

\]

where $\ell_{i, t+\Delta t} \geq 0$ is the Lagrange multiplier for collateral requirement (23) satisfying the complementary slackness condition $\ell_{i, t+\Delta t} W_{i, t+\Delta t}^{*}=0$.

We use Lemma 1 to derive the dynamics of state variable $v_{t}$. First, suppose the constraints do not bind. Hence, Lagrange multipliers $\ell_{i, t+\Delta t}$ vanish and the first order conditions (24) are the same as in a frictionless economy. The dynamics of the state variable $v_{t}$ in the unconstrained region of the state-space is then the same as in the frictionless economy, and is found in closed form below. Next, let $\bar{v}$ and $\underline{v}$ be the values of the state variable $v_{t}$ when constraints (10) of investors $A$ and $B$ bind, respectively. We show that state variable $v_{t}$ stays within boundaries $\underline{v} \leq v_{t} \leq \bar{v}$. Intuitively, binding collateral constraints restrict the investors' losses of wealth and consumption, which traps the state variable in the interval $[\underline{v}, \bar{v}]$. The boundaries $\bar{v}$ and $\underline{v}$ are found from the condition that the constraints bind: $W_{i, t+\Delta t}=0$. Dividing these constraints by $D_{t+\Delta t}$, we obtain equations

$$
\begin{equation*}
\Phi_{A}(\bar{v})=0, \quad \Phi_{B}(\underline{v})=0, \tag{25}
\end{equation*}
$$

where $\Phi_{i}\left(v_{t}\right)$ are wealth-output ratios given by equations (A19) and (A20) in the Appendix. Proposition 1 below reports the dynamics of $v_{t}$.

Proposition 1 (Closed-form dynamics of state variable $v_{t}$ ).
Given the boundaries $\bar{v}$ and $\underline{v}$, the equilibrium dynamics of state variable $v_{t}$ is given by:

$$
\begin{equation*}
v_{t+\Delta t}=\max \left\{\underline{v} ; \min \left\{\bar{v} ; v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}\right\}\right\}, \tag{26}
\end{equation*}
$$

where drift $\mu_{v}$, volatility $\sigma_{v}$, and jump $J_{v}$ are given in closed form by:

$$
\begin{align*}
& \mu_{v}=\frac{1}{2 \Delta t}\left(\left(\gamma_{A}-\gamma_{B}\right) \ln \left[\left(1+\mu_{D} \Delta t\right)^{2}-\sigma_{D}^{2} \Delta t\right]+\ln \left(\frac{1-\lambda_{B} \Delta t}{1-\lambda \Delta t}\right)^{2}+\ln \left(1-\delta^{2} \Delta t\right)\right)  \tag{27}\\
& \sigma_{v}=\frac{1}{2 \sqrt{\Delta t}}\left(\left(\gamma_{A}-\gamma_{B}\right) \ln \left(\frac{1+\mu_{D} \Delta t+\sigma_{D} \sqrt{\Delta t}}{1+\mu_{D} \Delta t-\sigma_{D} \sqrt{\Delta t}}\right)+\ln \left(\frac{1+\delta \sqrt{\Delta t}}{1-\delta \sqrt{\Delta t}}\right)\right)  \tag{28}\\
& J_{v}=\left(\gamma_{A}-\gamma_{B}\right) \ln \left(1+\mu_{D} \Delta t+J_{D}\right)+\ln \left(\frac{\lambda_{B}}{\lambda}\right)-\mu_{v} \Delta t \tag{29}
\end{align*}
$$

Boundaries $\bar{v}$ and $\underline{v}$ are reflecting when $\Delta t$ is sufficiently small; that is, $v_{t}$ does not stay at the boundaries forever: $\operatorname{Prob}\left(\bar{v}>v_{t+\Delta t} \mid v_{t}=\bar{v}\right)>0$ and $\operatorname{Prob}\left(v_{t+\Delta t}>\underline{v} \mid v_{t}=\underline{v}\right)>0$.

Equation (26) reveals the exact structure of the state variable and sheds light on the equilibrium effects of the collateral requirement. The equation demonstrates that the constraint does not alter the dynamics of the state variable when the constraint does not bind, and all its effects are due to imposing bounds on process $v_{t}$. This property of state variable $v_{t}$ plays important role in establishing the clustering of volatilities and other results in Section 4 below, and it is difficult to see using numerical methods instead of a closed-form dynamics.

Proposition B. 1 in technical appendix B proves the existence of time-independent bounds finite $\bar{v}$ and $\underline{v}$ satisfying equations (25). The intuition for the existence of these bounds is as follows. Suppose, for example, that $\bar{v}=+\infty$. Then, equation (15) for the consumption share $s\left(v_{t}\right)$ of investor $A$ implies that $s\left(v_{t}\right) \approx 0$ when $v_{t}$ is sufficiently large, and hence, investor $A$ 's consumption net of labor income $\left(s\left(v_{t}\right)-l_{A}\right) D_{t}$ can be negative for a long period. As a result, investor $A$ 's wealth is negative for a sufficiently large $v_{t}$ because this wealth equals the present value of net consumptions. However, negative wealth contradicts the constraint $W_{A, t+\Delta t} \geq 0$, and hence $\bar{v}$ should be finite.

The closed-form dynamics (26) helps us build a theory of collateral constraints. In particular, we use this dynamics to prove the existence of equilibrium and stationarity of equilibrium processes, derive asset prices, and to study the effects of collateralization on asset prices. Proposition 2 below reports the SPD and the stock price.

## Proposition 2 (State price density and the effects on asset prices).

1) The state price density under the beliefs of investor $A$ is given by:

$$
\begin{equation*}
\frac{\xi_{A, t+\Delta t}}{\xi_{A t}}=e^{-\rho \Delta t}\left(\frac{s\left(v_{t+\Delta t}\right)}{s\left(v_{t}\right)} \frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}} \exp \left(\max \left\{0 ; v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}-\bar{v}\right\}\right), \tag{30}
\end{equation*}
$$

where investor $A$ 's time- $t$ consumption share $s\left(v_{t}\right)$ solves equation (15).
2) The price-dividend ratio $\Psi\left(v_{t}\right)$ is uniformly bounded, the stock price $S_{t}$ is given by

$$
\begin{equation*}
S_{t}=\left(1-l_{A}-l_{B}\right) D_{t} \mathbb{E}_{t}^{A}\left[\sum_{\tau=t+\Delta t}^{+\infty} \frac{\xi_{A \tau}}{\xi_{A t}} \frac{D_{\tau}}{D_{t}}\right], \tag{31}
\end{equation*}
$$

and the prices of the bond and the insurance contract are given by $B_{t}=\mathbb{E}_{t}^{A}\left[\xi_{A, t+\Delta t}\right] / \xi_{A t}$ and $P_{t}=\mathbb{E}_{t}^{A}\left[\xi_{A, t+\Delta t} 1_{\left\{\omega_{t+\Delta t}=\omega_{3}\right\}}\right] / \xi_{A t}$, respectively.
3) The prices of bond, stock, and the insurance contract are higher in the economy with collateral constraints than in the frictionless economy, conditional on two economies having the same current output $D_{t}$ and the state variable $v_{t}$.

Equation (30) captures the effect of collateralization on the SPD in our economy. It shows that the change in the $\mathrm{SPD}, \xi_{t+\Delta t} / \xi_{t}$, can be decomposed into two terms. The first term, $e^{-\rho \Delta t}\left(s\left(v_{t+\Delta t}\right) D_{t+\Delta t}\right)^{-\gamma_{A}} /\left(s\left(v_{t}\right) D_{t}\right)^{-\gamma_{A}}$, given by the ratio of marginal utilities of investor $A$ at times $t+\Delta t$ and $t$, is the change in SPD in the frictionless economy. The second term captures the effect of the friction on the SPD, and is only activated when the constraint of investor $A$ is binding. An equivalent representation of SPD can be obtained in terms of the marginal utilities of investor $B$.

Proposition 2 demonstrates that collateralization inflates asset prices. This is because the SPD in the constrained economy exceeds its counterpart in the frictionless economy due to the positive Lagrange multiplier $\ell_{i, t+\Delta t}$ in the first order condition (24). This result is in contrast to the effects of borrowing, margin, and restricted participation constraints in the related literature (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014), which increase or decrease the stock prices depending on the investors' elasticities of intertemporal substitution. Moreover, this literature evaluates the effects of frictions numerically, whereas we provide proves aided by the closed-form dynamics of the state variable (26) and the SPD (30). We discuss the intuition and further economic differences between our constraint and the constraints in the literature in Section 4.1.

Proposition B. 2 in the technical Appendix B provides the verification theorem for the optimality of investors' optimal strategies, and is not reported here for brevity. In particular, this proposition shows that in the economy where the state price density is given by equation (30) the dynamic programming problem (21)-(23) has a unique solution $V_{i t}$ and this solution is the indirect utility function of investor $i$.

### 3.1. Closed-form solution in a continuous-time limit

Next, we take continuous-time limit $\Delta t \rightarrow 0$ and derive the equilibrium in closed form. Taking the limit allows rewriting equations (A32) and (A33) for the price-dividend and wealth-consumption ratios, $\Psi_{t}$ and $\Phi_{i t}$, as differential-difference equations. For tractability, we derive ratios $\Psi_{t}$ and $\Phi_{i t}$ in terms of a transformed ratio $\widehat{\Psi}(v ; \theta)$, which satisfies a simpler equation reported in Lemma 2 below.

Lemma 2 (Differential-difference equation). In the limit $\Delta t \rightarrow 0$, the price-dividend ratio $\Psi$ and wealth-output ratios $\Phi_{i}$ are given by:

$$
\begin{equation*}
\Psi(v)=\widehat{\Psi}\left(v ;-\gamma_{A}\right) s(v)^{\gamma_{A}} \tag{32}
\end{equation*}
$$

$$
\begin{equation*}
\Phi_{i}(v)=\left(\left(\mathbf{1}_{\{i=A\}}-\mathbf{1}_{\{i=B\}}\right) \widehat{\Psi}\left(v ; 1-\gamma_{A}\right)+\left(\mathbf{1}_{\{i=B\}}-l_{i}\right) \widehat{\Psi}\left(v ;-\gamma_{A}\right)\right) s(v)^{\gamma_{A}} \tag{33}
\end{equation*}
$$

where $s(v)$ solves equation (15) and $\widehat{\Psi}(v ; \theta)$ satisfies a differential-difference equation

$$
\begin{align*}
\frac{\widehat{\sigma}_{v}^{2}}{2} \widehat{\Psi}^{\prime \prime}(v ; \theta) & +\left(\widehat{\mu}_{v}+\left(1-\gamma_{A}\right) \sigma_{D} \widehat{\sigma}_{v}\right) \widehat{\Psi}^{\prime}(v ; \theta) \\
& -\left(\lambda+\rho-\left(1-\gamma_{A}\right) \mu_{D}+\frac{\left(1-\gamma_{A}\right) \gamma_{A}}{2} \sigma_{D}^{2}\right) \widehat{\Psi}(v ; \theta)  \tag{34}\\
& +\lambda\left(1+J_{D}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(\max \left\{\underline{v} ; v+\widehat{J}_{v}\right\} ; \theta\right)+s(v)^{\theta}=0
\end{align*}
$$

subject to the reflecting boundary conditions

$$
\begin{equation*}
\widehat{\Psi}^{\prime}(\underline{v} ; \theta)=0, \quad \widehat{\Psi}^{\prime}(\bar{v} ; \theta)-\widehat{\Psi}(\bar{v} ; \theta)=0 \tag{35}
\end{equation*}
$$

where $\widehat{\mu}_{v}, \widehat{\sigma}_{v} \geq 0$, and $\widehat{J}_{v} \leq 0$ are constants given by:

$$
\begin{align*}
& \widehat{\mu}_{v}=\left(\gamma_{A}-\gamma_{B}\right)\left(\mu_{D}-\frac{\sigma_{D}^{2}}{2}\right)+\lambda-\lambda_{B}-\frac{\delta^{2}}{2}  \tag{36}\\
& \widehat{\sigma}_{v}=\left(\gamma_{A}-\gamma_{B}\right) \sigma_{D}+\delta  \tag{37}\\
& \widehat{J}_{v}=\left(\gamma_{A}-\gamma_{B}\right) \ln \left(1+J_{D}\right)+\ln \left(\frac{\lambda_{B}}{\lambda}\right) \tag{38}
\end{align*}
$$

The boundaries $\bar{v}$ and $\underline{v}$ solve the following equations:

$$
\begin{equation*}
\frac{\widehat{\Psi}\left(\bar{v} ; 1-\gamma_{A}\right)}{\widehat{\Psi}\left(\bar{v} ;-\gamma_{A}\right)}=l_{A}, \quad \frac{\widehat{\Psi}\left(\underline{v} ; 1-\gamma_{A}\right)}{\widehat{\Psi}\left(\underline{v} ;-\gamma_{A}\right)}=1-l_{B} . \tag{39}
\end{equation*}
$$

We observe that equation (34) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2012; Chabakauri, 2013, 2015; Rytchkov, 2014). This equation is a differential-difference equation with a "delayed" argument in the fourth term on the left-hand side of the equation because $\widehat{J}_{v} \leq 0$. This term is further complicated by the fact that the delayed argument is restricted to stay above the lower boundary $\underline{v}$, which gives rise to the dependence of the fourth term on a peculiar argument $\max \left\{\underline{v} ; v+\widehat{J}_{v}\right\}$. This term captures investors' decisions in anticipation of hitting their collateral constraint.

Before deriving the equilibrium in the general case, in Corollary 1 below, we provide analytical price-dividend ratios when there is no crisis and investors have log preferences.

Corollary 1 (Analytical asset prices with log preferences). Suppose, investors $A$ and $B$ have logarithmic preferences and there is no production crisis, that is, $\lambda=\lambda_{B}=0$. Then, price-dividend ratio $\Psi(v)$ is given by:

$$
\begin{equation*}
\Psi(v)=\frac{1}{\rho}+\frac{C_{1} e^{\varphi+v}+C_{2} e^{\varphi-v}}{1+e^{v}} \tag{40}
\end{equation*}
$$

where $\varphi_{ \pm}=0.5\left(1 \pm \sqrt{1+8 \rho / \delta^{2}}\right)$, and constants $C_{1}$ and $C_{2}$ are given by equations (A46) and (A47) in the Appendix, respectively.

In Section 4 below, we argue that the analytical price-dividend ratio (40) captures some important properties of price-dividend ratio that hold in the general case with arbitrary risk aversions and crises. Hence, this special case can be used as a tractable benchmark in asset pricing research. Nevertheless, we undertake a comprehensive investigation of equilibrium in the general case.

Proposition B. 3 in Appendix B presents the closed-form price-dividend ratio for general CRRA risk aversions and beliefs. These solutions are in terms of exogenous model parameters and do not require solving equations (34)-(35). Although the closed-form solution in Proposition B. 3 is complex, it provides a constructive proof for the existence of price-dividend ratios. We also solve equations (34)-(35) using the method of finite differences, and double-check that the numerical solution coincides with the exact one reported in Proposition B. 3 in Appendix B.

We call the interval $v \in\left[\underline{v}, \underline{v}-\widehat{J}_{v}\right]$ in the state-space a period of anxious economy, similar to Fostel and Geanakoplos (2008). ${ }^{6}$ When the economy falls into this state, even a small possibility of a crisis renders the collateral requirement binding and leads to deleveraging in the economy. To explore the economic effects of the anxious economy, we provide closedform expressions for the interest rates $r_{t}$ and risk premia in normal times $\mu_{t}-r_{t}$, which can be easily obtained using equations for asset prices and the state price density derived in Proposition 2. Proposition 3 below reports the results.

Proposition 3 (Interest rates and risk premia in the limit). For a sufficiently small

[^5]interval $\Delta t$, the interest rate $r_{t}$ and the risk premium $\mu_{t}-r_{t}$ in normal times are given by:
\[

$$
\begin{align*}
& r_{t}=\left\{\begin{array}{l}
\tilde{r}_{t}-\lambda\left(1+J_{D}\right)^{-\gamma_{A}}\left(\frac{s\left(\max \left\{\underline{v} ; v_{t}+\widehat{J}_{v}\right\}\right)}{s_{t}}\right)^{-\gamma_{A}}+O(\Delta t), \text { for } \underline{v}<v_{t}<\bar{v}, \\
\frac{\left(1-s_{t}\right) \Gamma_{t}\left(\mathbf{1}_{\{v=\underline{v}\}}-\mathbf{1}_{\{v=\bar{v}\}}\right)-\gamma_{B}}{2 \gamma_{B} \sqrt{\Delta t}} \widehat{\sigma}_{v}+O(1), \text { for } v=\underline{v} \text { or } v=\bar{v},
\end{array}\right.  \tag{41}\\
& \mu_{t}-r_{t}=\left(\gamma_{A} \sigma_{D}-\frac{\left(1-s_{t}\right) \Gamma_{t} \widehat{\sigma}_{v}}{\gamma_{B}}+\frac{\left(1-s_{t}\right) \Gamma_{t} \widehat{\sigma}_{v}\left(\mathbf{1}_{\{v=\underline{v}\}}+\mathbf{1}_{\{v=\bar{v}\}}\right)-\gamma_{B} \widehat{\sigma}_{v} \mathbf{1}_{\{v=\bar{v}\}}}{2 \gamma_{B}}\right) \sigma_{t} \\
&-\lambda\left(1+J_{D}\right)^{-\gamma_{A}} J_{t}\left(\frac{s\left(\max \left\{\underline{v} ; v_{t}+\widehat{J}_{v}\right\}\right)}{s_{t}}\right)^{-\gamma_{A}}+O(\sqrt{\Delta t}), \tag{42}
\end{align*}
$$
\]

where $\tilde{r}_{t}$ is the interest rate in the unconstrained economy without crisis risk, given by:

$$
\begin{align*}
\tilde{r}_{t}=\lambda & +\rho+\gamma_{A} \mu_{D}-\frac{\gamma_{A}\left(1+\gamma_{A}\right)}{2} \sigma_{D}^{2}+\left(\frac{\gamma_{A} \sigma_{D} \widehat{\sigma}_{v}-\widehat{\mu}_{v}}{\gamma_{B}}\right)\left(1-s_{t}\right) \Gamma_{t} \\
& -\widehat{\sigma}_{v}^{2}\left(\frac{1}{2 \gamma_{B}^{2}}\left(1-s_{t}\right)^{2} \Gamma_{t}^{2}+\frac{1}{2 \gamma_{A}^{2} \gamma_{B}^{2}} s_{t}\left(1-s_{t}\right) \Gamma_{t}^{3}\right), \tag{43}
\end{align*}
$$

drift $\widehat{\mu}_{v}$, volatility $\widehat{\sigma}_{v}$, and $\widehat{J}_{v}$ of the state variable $v$ are given by equations (36)-(38), volatility $\sigma_{t}$ and jump size $J_{t}$ are given by equations (B27)-(B28), respectively, and $\Gamma_{t} \equiv$ $\gamma_{A} \gamma_{B} /\left(\gamma_{A}\left(1-s_{t}\right)+\gamma_{B} s_{t}\right)$ is the risk aversion of a representative investor.

The effects of collateral requirements on interest rates and risk premia arise due to the investors' concern that a potential crisis may render the constraint binding next period when the economy is close to boundary $\underline{v}$. The last term in the first equation in (41) for the interest rate quantifies the impact of collateralization on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Equations (41) and (42) also feature terms with indicator functions $\mathbf{1}_{\{v=v\}}$ and $\mathbf{1}_{\{v=\bar{v}\}}$, which are non-zero only at the boundaries $\underline{v}$ and $\bar{v}$. For the interest rate $r_{t}$ these terms have the order of magnitude proportional to $1 / \sqrt{\Delta t}$, and hence, the interest rate has singularities at the boundaries $\underline{v}$ and $\bar{v}$ when $\Delta t \rightarrow 0$. The intuition is that near the boundaries $\underline{v}$ and $\bar{v}$ even a small shock $\Delta w_{t}$ may lead to a default. Thus, when the investor's constraint binds at time $t$, this investor allocates a larger fraction of income to bond than in the interior region $\underline{v}<v_{t}<\bar{v}$ and requires a higher risk premium. Therefore, the interest rate decreases and Sharpe ratio increases at the boundaries.

Similar singularities arise in a continuous-time model of Detemple and Serrat (2003). Our discrete-time analysis sheds new light on these singularities by uncovering their order of magnitude $1 / \sqrt{\Delta t}$. Consequently, the per-period rate $r_{t} \Delta t$ is finite and has an order of magnitude $O(\sqrt{\Delta t})$. Moreover, in contrast to the latter paper, due to production crises, the collateralization in our model affects the interest rates and Sharpe ratios not only at the boundaries but also in the internal area of the state-space interval, which we call the period of anxious economy. We also note that the interest rate (41) and the risk-premium (42) are significantly different from those in the literature on borrowing and margin constraints (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014) where they feature Lagrange multipliers for the constraints that bind in an interval of a state space and do not have singularities.

### 3.2. Stationary distribution of consumption share

Absent any frictions, state variable $v$ follows an arithmetic Brownian motion with a jump. This process is non-stationary and induces non-stationarity in the unconstrained equilibrium where one of the investor's consumption share gradually converges to zero. As a result, with the exception of some knife-edge parameter combinations, only one of the investors has a significant impact on asset prices in the frictionless economy in the long run (e.g., Blume and Easley, 2001; Yan, 2008; Chabakauri, 2015).

Imposing collateral constraints (10) helps both investors survive and have an impact on equilibrium in the long-run because these constraints protect investors against losing their shares of aggregate consumption beyond certain limits. This intuition for the survival of investors has also been discussed in the previous literature (e.g., Blume and Easley, 2001; Cao, 2018, among others). However, this intuition does not reveal the shape of the distribution of consumption share $s$, whether this distribution is well-defined or degenerate (e.g., fully concentrated at boundaries $\underline{s}$ or $\bar{s}$ ), and which parameters determine the relative dominance of investors in the economy. Our main contribution in this section is that armed with the closed-form dynamics of the state variable $v_{t}$ in (26), we derive the probability density function (PDF) of consumption share $s$ in closed form, show that this PDF is stationary and non-degenerate, and find parameters that determine its shape. The latter result is important because it implies non-trivial time-variation of asset prices in the long run. For simplicity, we assume that there is no crisis risk so that $\lambda=\lambda_{B}=0$. Proposition 4 reports the results.

Proposition 4 (Stationary distribution of consumption share). Suppose, $\lambda=$ $\lambda_{B}=0$. Then, the PDF $f\left(s, \tau ; s_{t} ; \tau\right)$ of consumption share $s$ at time $\tau$ conditional on observing share $s_{t}$ at time $t$ is given in closed form by expression (A67) in the Appendix. Furthermore, the stationary PDF of consumption share s is given by:

$$
\begin{equation*}
f(s)=\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}}\left(\frac{\gamma_{A}}{s}+\frac{\gamma_{B}}{1-s}\right) \frac{\left((1-s)^{\gamma_{B}} / s^{\gamma_{A}}\right)^{2 \widehat{\mu}_{v} / \widehat{\sigma}_{v}^{2}}}{\left((1-\underline{s})^{\gamma_{B}} / \underline{s}^{\gamma_{A}}\right)^{2 \widehat{\mu}_{v} / \widehat{\sigma}_{v}^{2}}-\left((1-\bar{s})^{\gamma_{B}} / \bar{s}^{\gamma_{A}}\right)^{2 \widehat{\mu}_{v} / \widehat{\sigma}_{v}^{2}}} \mathbf{1}_{\{\underline{s} \leq s \leq \bar{s}\}}, \tag{44}
\end{equation*}
$$

where $\widehat{\mu}_{v}=\left(\gamma_{A}-\gamma_{B}\right)\left(\mu_{D}-\sigma_{D}^{2} / 2\right)-\delta^{2} / 2, \widehat{\sigma}_{v}=\left(\gamma_{A}-\gamma_{B}\right) \sigma_{D}+\delta, \mathbf{1}_{\{\underline{s} \leq s \leq \bar{s}\}}$ is an indicator function, and $\underline{s}$ and $\bar{s}$ are the bounds on the consumption share $s$, which solve equation (15) for $\bar{v}$ and $\underline{v}$, respectively.

Proposition 4 confirms that both investors survive in the long run, and that consumption share $s$ has a well-defined stationary distribution. The beliefs enter PDF (44) via the ratio of the drift and variance of process $v_{t}$, given by $\widehat{\mu}_{v} / \widehat{\sigma}_{v}^{2}$. This ratio determines the relative dominance of investors in the economy. In particular, for bounds $\underline{s}$ and $\bar{s}$ that are symmetric around 0.5 , the PDF is concentrated around $\underline{s}$ if $\widehat{\mu}_{v}>0$ and around $\bar{s}$ if $\widehat{\mu}_{v}<0$.

Figure 2 plots the stationary PDF (44) and transition densities $f\left(s, t ; s_{0}, 0\right)$, for parameters described in the legend and explained in Section 4 below. The stationary PDF has a larger mass around $s=0.1$ because the labor share $l_{B}=0.14$ of investor $B$ exceeds the labor share $l_{A}=0.123$ of investor $A$ in this example in order to get boundary values $\underline{s}=0.1$ and $\bar{s}=0.9$ symmetric around 0.5 . From Figure 2 we observe that both rational and irrational investors can occasionally have large consumption shares.

Another notable feature of PDF (44) is that it is bimodal, with a large mass of the distribution concentrated around boundaries $\underline{s}$ and $\bar{s}$. The economic implication of this bimodality is that the periods of binding constraints are likely to be persistent. The closedform dynamics (26) for the state variable $v$ helps explain the bimodality of the PDF. From this dynamics, we observe that after hitting a boundary the process $v_{t}$ remains in its vicinity for some time. Hence, because variable $v$ follows an arithmetic Brownian motion in the interval $(\underline{v}, \bar{v})$, the probability of hitting the same boundary again is high.

Proposition 4 implies that the PDF of consumption share $s$ is always stationary when investors have positive labor incomes $l_{B}>0$ and $l_{A}>0$ because in this case $1>\bar{s}>\underline{s}>0$, and hence, the $\operatorname{PDF}(44)$ is well-defined. The $\operatorname{PDF}$ of $s$ is also stationary when $l_{A}=0$, $l_{B}>0$, and $\widehat{\mu}_{v}<0$, or $l_{A}>0, l_{B}=0$, and $\widehat{\mu}_{v}>0$. In the latter cases, $\underline{s}=0$ or $\bar{s}=1$, respectively. Then, we observe that the $\operatorname{PDF}(44)$ is well-defined when $\underline{s}=0$ and $\widehat{\mu}_{v}<0$,


Figure 2
Convergence to stationary distribution of consumption share $s_{t}=c_{A, t}^{*} / D_{t}$
The Figure shows transition densities $f\left(s, t ; s_{0}, 0\right)$ for the starting point $s_{0}=0.2$ and the stationary distribution $f(s)$ (i.e., density for $t=\infty$ ). We set $\gamma_{A}=2, \gamma_{B}=1.5, \mu_{D}=0.018, \sigma_{D}=0.032$, $\lambda=\lambda_{B}=0, \rho=0.02, \delta=0.1125, \underline{s}=0.1, \bar{s}=0.9, l_{A}=0.123$, and $l_{B}=0.14$.
and when $\bar{s}=1$ and $\widehat{\mu}_{v}>0$, and hence is stationary. The PDF of $s$ is non-stationary when $l_{A}=0$ and $l_{B}=0$, and is derived in closed form in Chabakauri (2015). In the latter case only investor $A$ survives if $\widehat{\mu}_{v}<0$, and only investor $B$ survives when $\widehat{\mu}_{v}>0$.

## 4. Analysis of Equilibrium

In this section, we demonstrate the economic implications of our model. In Section 4.1, we show that capital requirements amplify the effect of rare crises on generating lower interest rates and higher Sharpe ratios, lead to spikes and crashes of stock prices and stock return volatilities, amplify volatility in good times and decrease it in bad times, and generate volatility clusters. Section 4.2 measures the economic significance of collateralization by quantifying the collateral premium of the stock.

We study the equilibrium for calibrated parameters. We set the parameters of the aggregate consumption process to $\mu_{D}=0.018, \sigma_{D}=0.032, J_{D}=-0.2$, and the crisis intensities of investors $A$ and $B$ to $\lambda=0.017$ and $\lambda_{B}=0.01$, respectively. ${ }^{7}$ The risk aversions are $\gamma_{A}=2$ and $\gamma_{B}=1.5$, and the time discount is $\rho=0.02$. The disagreement parameter is $\delta=0.1125$, which corresponds to the mean growth rate (6) under investor

[^6]$B$ 's beliefs equal to $1.2 \mu_{D}$. The shares of labor income $l_{A}=0.123$ and $l_{B}=0.14$ are chosen to generate symmetric bounds on investor $A$ 's consumption share: $\underline{s}=0.1$ and $\bar{s}=0.9 .{ }^{8}$

We plot the equilibrium distributions and processes as functions of consumption share $s_{t}=c_{A t}^{*} / D_{t}$ because $s$ lies in the interval $(0,1)$ and is more intuitive than variable $v$. We observe that consumption share $s$ is countercyclical in the sense that $\operatorname{corr}_{t}\left(d s_{t}, d D_{t}\right)<$ 0 . Intuitively, the aggregate wealth and consumption shift to (away from) investor $A$ following negative (positive) shocks to output because this investor is more risk averse and pessimistic than investor B. We call a process procyclical (countercyclical) if that process is a decreasing (increasing) function of $s$. We interpret periods of low (high) $s_{t}$ as good (bad) times in the economy, because during these periods the output $D_{t}$ is high (low).

### 4.1. Equilibrium processes

Figure 3 depicts investor $B$ 's leverage/market ratio $L_{t} / S_{t}$ and stock holdings $n_{B t}$ in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) demonstrates the cyclicality of leverage. The leverage is lowest when either investor $A$ or investor $B$ bind on their constraints. Intuitively, when $s=\bar{s}$, investor $B$ 's financial wealth is zero, and hence, $B$ lacks collateral and cannot borrow. When $s=\underline{s}$, investor $A$ 's financial wealth is zero and the labor income $l_{A} D_{t} \Delta t$ is infinitesimally small in the continuous-time limit. The liquidity dries up because investor $A$ cannot supply credit. The leverage cycles are present only in the constrained economy. They do not occur in the unconstrained economy where the state variable $s$ is non-stationary and gradually converges to 0 or 1 .

Panel (b) presents the number of stocks held by investor $B$. Consider first the unconstrained economy where the labor income is pledgeable. From panel (b) we observe that in this economy investor $B$ shorts stocks despite being more optimistic than investor $A$ when consumption share $s$ is close to 1 . The intuition is that in bad times, following a sequence of negative shocks to output, investor $B$ shorts stocks to finance consumption and backs short positions by the pledgeable labor income. The stream of labor income $l_{B} D_{t} \Delta t$ is equivalent to dividends from holding $\widehat{n}_{B}=l_{B} /\left(1-l_{A}-l_{B}\right)$ units of non-tradable shares in the Lucas tree. Short-selling allows the investor to circumvent the non-tradability of labor income and freely adjust the effective share $\hat{n}_{B}+n_{B, S t}$ in the Lucas tree. Overcoming the

[^7]

## Figure 3

Leverage and stock holdings of optimistic and less risk averse investor $B$
Panels (a) and (b) depict optimistic and less risk averse investor $B$ 's leverage/market price ratio $L_{t} / S_{t}$ and the number of shares $n_{S t}$, respectively, as functions of consumption share $s_{t}=c_{A t}^{*} / D_{t}$. The solid and dashed lines correspond to constrained and unconstrained economies, respectively.
non-tradability of labor incomes makes this economy similar to the non-stationary unconstrained economy where investors can freely trade shares in the Lucas tree. The financial wealth can then become negative. The collateral requirement imposes non-negative wealth constraint, which precludes investor $B$ from shorting. The trading strategy of investor $A$ equals $1-n_{B t}^{*}$ in equilibrium and can be analyzed similarly. Investor $A$ also has an additional motive to short stocks due to being more pessimistic than investor $B$.

Figure 4 depicts the interest rate $r_{t}$, Sharpe ratio $\left(\mu_{t}-r_{t}\right) / \sigma_{t}$, price-dividend ratio $\Psi$, and excess stock return volatility $\left(\sigma_{t}-\sigma_{D}\right) / \sigma_{D}$ in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) shows the interest rates $r_{t} .{ }^{9}$ The interest rate declines sharply when the economy enters into an anxious state close to the boundary $\bar{s}$ where even a small possibility of a crisis next period makes the constraint of investor $B$ binding. The intuition is as follows. In the unconstrained economy, a crisis around state $\bar{s}$ generates wealth transfer to the pessimistic and more risk averse investor $A$ and increases her consumption share $s$ above $\bar{s}$. In the constrained economy, consumption share $s$ is capped by $\bar{s}$. Consequently, following a crisis, investor $A$ 's marginal utility $\left(c_{A}^{*}\right)^{-\gamma_{A}}$ is higher in the constrained than in the unconstrained economy. As a result, investor $A$ is more willing to smooth consumption in the constrained economy, and hence, the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down. Panel (b) of Figure 4 shows that the

[^8]

Figure 4

## Equilibrium processes

Panels (a)-(d) show interest rate $r_{t}$, Sharpe ratio $\left(\mu_{t}-r_{t}\right) / \sigma_{t}$, price-dividend ratio $\Psi_{t}$, and excess volatility $\left(\sigma_{t}-\sigma_{D}\right) / \sigma_{D}$ as functions of $s_{t}=c_{A t}^{*} / D_{t}$ for the constrained (solid lines) and unconstrained (dashed lines) economies.

Sharpe ratio increases to compensate investor $A$ for buying risky assets from investor $B$.
Our results on interest rates and Sharpe ratios indicate that the rare crises and collateral requirements reinforce the effects of each other. In particular, the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints (10) are simultaneously present. Removing the constraint but keeping the crisis risk increases the interest rate and decreases the Sharpe ratio. Equation (41) for the interest rate and equation (42) for the risk premium show that removing the crisis risk (i.e., setting $\lambda=\lambda_{B}=0$ ) but keeping the constraint leads to $r_{t}$ and $\mu_{t}-r_{t}$ which are the same as in the frictionless economy when $\underline{v}<v_{t}<\bar{v}$, consistent with findings in Detemple and Serrat (2003). Absent any crises, the constraints affect $r_{t}$ and $\mu_{t}-r_{t}$ only at the boundaries of the state-space, as shown in Section 3.1.

From panel (c), we observe that the collateral requirements give rise to higher pricedividend ratio $\Psi$ than in the unconstrained economy, $\Psi_{t}^{\text {constr }}-\Psi_{t}^{\text {unc }}>0$, as proven in

Proposition 2. The increases in ratio $\Psi$ are larger around the boundaries $\underline{s}$ and $\bar{s}$, which makes ratio $\Psi$ a U-shaped function of $s$ sensitive to small shocks close to boundaries. The U-shape is a robust phenomenon that does not require rare crises or investors that differ both in risk aversions and beliefs. When both investors have identical risk aversions $\gamma_{A}=\gamma_{B}=1$ but different beliefs and there is no crisis risk (i.e., $\lambda_{A}=\lambda_{B}=0$ ), the U-shape is an analytical result that follows from the closed-form expression (40) for ratio $\Psi$. This ratio remains U-shaped when investors have different risk aversions but identical beliefs.

The intuition for the U-shape is as follows. Suppose, consumption share $s$ is close to the boundary $\bar{s}$, where investor $B$ 's constraint is likely to bind but investor $A$ is unconstrained. Because investor $A$ 's constraint is loose the state price density $\xi_{A t}$ is proportional to investor $A$ 's marginal utility $\left(c_{A t}^{*}\right)^{-\gamma_{A}}$. In the constrained economy the consumption share of investor $A$ is capped by $\bar{s}<1$ whereas in the unconstrained economy it can increase above $\bar{s}$. Therefore, the marginal utility of investor $A$ and, hence, the state price density are expected to be higher in the constrained than in the unconstrained economy. Consequently, stocks are more valuable in the constrained economy around the boundary $\bar{s}$. The intuition around $\underline{s}$ can be analyzed in a similar way. An additional economic force contributing to higher stock price is that the stock can be used as collateral that helps relax the constraint, which gives rise to a premium. This force is explored in Section 4.2.

The results on panel (d) demonstrate that the constraint makes volatility more procyclical, reducing it in bad times (around $\bar{s}$ ) and increasing it in good times (around $\underline{s}$ ). This is because U-shaped price-dividend ratio in the constrained economy is more procyclical in good times (i.e., around $\underline{s}$ ) and more countercyclical in bad times (i.e., around $\bar{s})$ than in the unconstrained economy. Stock price $S_{t}=\Psi_{t} D_{t}$ is more volatile in good times (around $\underline{s}$ ) because both $\Psi$ and $D_{t}$ change in the same direction, and is less volatile in bad times (around $\bar{s}$ ) because $\Psi$ and $D_{t}$ change in opposite directions and partially offset the effects of each other. Lower volatility in bad times is in line with the previous literature on the effects of portfolio constraints on asset prices (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others). The empirical literature finds that the volatility tends to be higher in bad times (e.g., Schwert, 1989). However, high volatility can be explained by high uncertainty about the economic growth and learning effects in bad times (e.g., Veronesi, 1999), which we do not study in this paper to focus on the effects of collateral constraints which are not confounded by other effects.

Boundary conditions (35) allow us to explore volatility $\sigma_{t}$ near the boundaries $\underline{s}$ and $\bar{s}$


Figure 5
Simulated $\mathbf{P} / \mathbf{D}$ ratio $\Psi$ and stock return volatility $\sigma$ over time
Panels (a) and (b) show the spikes and crashes of simulated $\mathrm{P} / \mathrm{D}$ ratio and volatility $\sigma$, and clustering of volatility $\sigma$ over the period of 50 years.
using closed form expressions in Corollary 2 below.
Corollary 2 (Stock return volatility at the boundaries). Stock return volatility in normal times $\sigma_{t}$ satisfies the following boundary conditions:

$$
\begin{equation*}
\sigma(\underline{s})=\sigma_{D}+\frac{\gamma_{B} \underline{s} \widehat{\sigma}_{v}}{\gamma_{A}(1-\underline{s})+\gamma_{B} \underline{s}}>\sigma_{D}, \quad \sigma(\bar{s})=\sigma_{D}-\frac{\gamma_{A}(1-\bar{s}) \widehat{\sigma}_{v}}{\gamma_{A}(1-\bar{s})+\gamma_{B} \bar{s}}<\sigma_{D} . \tag{45}
\end{equation*}
$$

By continuity, inequalities (45) also hold in a vicinity of the boundaries. Panel (d) shows that volatility $\sigma_{t}$ is very steep at the boundaries: it spikes close to $\underline{s}$ and crashes close to $\bar{s}$, consistent with Corollary 2. It also evolves in three regimes of low, medium, and high volatility, which resembles volatility clustering documented in the empirical literature (e.g., Bollerslev, 1987). The distribution of consumption share $s$ on Figure 2 implies that the economy persists in these clusters for some time.

Figure 5 plots the simulated dynamics of $\mathrm{P} / \mathrm{D}$ ratio and stock return volatility over a period of 50 years. Consistent with our discussion above, the dynamics of $\mathrm{P} / \mathrm{D}$ ratio on panel (a) exhibits intervals of booms and busts around the times when the collateral requirements become binding. These intervals resemble periods of inflating and deflating bubbles in the economy. The volatility $\sigma$ on panel (b) evolves in clusters of high and low volatility, as explained above.

The economic effects of collateral requirements are different from the effects of margin and borrowing constraints in the related literature discussed in the introduction. In par-
ticular, those constraints increase or decrease price-dividend ratios and make them pro- or counter-cyclical depending on investors' risk aversions (e.g., Chabakauri, 2015). They also shrink volatility towards the output volatility $\sigma_{D}$ by reducing the risk-sharing. The main mechanism in this paper is different, and is driven by the increased marginal utilities due to endogenously arising bounds on the consumption share $s$. As a result, in our model price-dividend ratios increase irrespective of the beliefs and risk aversions of investors, and the volatility deviates further away from the output volatility (Figure 4). Other new effects relative to this literature include the cyclicality of leverage, mutual amplification of effects of rare crises and collateral constraints, and spikes in price-dividend ratios and volatilities.

### 4.2. Collateral liquidity premium

In this section, we measure the liquidity premium of stocks over labor income arising because stocks can be used as collateral. We consider a marginal representative investor $i$ that does not affect asset prices and characterize this investor's shadow indifference price $\widehat{S}_{i t}$ of labor income. We define $\widehat{S}_{i t}$ as the price such that exchanging marginal $\Delta l_{i}$ units of labor income for $\widehat{S}_{i t} \Delta l_{i}$ units of wealth leaves the investor's utility unchanged. Consider the investor's value function $V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)$ satisfying the dynamic programming equation (21) subject to constraints (22) and (23). Price $\widehat{S}_{i t}$ is the solution of equation $V_{i}\left(W_{i t}^{*}, v_{t} ; l_{i}\right)=V_{i}\left(W_{i t}^{*}+\widehat{S}_{i t} \Delta l_{i}, v_{t} ; l_{i}-\Delta l_{i}\right)$ when $\Delta l_{i} \rightarrow 0$. In the limit, we find:

$$
\begin{equation*}
\widehat{S}_{i t}=\frac{\partial V_{i}\left(W_{i t}^{*}, v_{t} ; l_{i}\right) / \partial l_{i}}{\partial V_{i}\left(W_{i t}^{*}, v_{t} ; l_{i}\right) / \partial W_{i t}} . \tag{46}
\end{equation*}
$$

The definition of shadow indifference price $\widehat{S}_{i t}$ comes from the literature on the valuation of derivative securities in incomplete markets (e.g., Davis, 1997).

The labor incomes $l_{i} D_{t} \Delta t$ are proportional to dividends $\left(1-l_{A}-l_{B}\right) D_{t} \Delta t$. Therefore, if claims on labor incomes were tradable and pledgeable, shadow price $\widehat{S}_{i t}$ would have been equal to $S_{t} /\left(1-l_{A}-l_{B}\right)$. However, labor incomes are non-tradable and non-pledgeable. Hence, from the view of investor $i$, the stock enjoys liquidity premium, which we define as

$$
\begin{equation*}
\Lambda_{i t}=\frac{S_{t} /\left(1-l_{A}-l_{B}\right)-\widehat{S}_{i t}}{S_{t} /\left(1-l_{A}-l_{B}\right)} . \tag{47}
\end{equation*}
$$

We find derivatives in equation (46) using the envelope theorem. Then, we derive prices $\widehat{S}_{i t}$ and show that premia (47) are positive and large. Proposition 5 reports our results.


Figure 6
Collateral liquidity premia from the view of investors $A$ and $B$
The Figure shows the collateral liquidity premia (47) of stocks over non-pledgeable labor incomes from the view of investors $A$ and $B$.

Proposition 5 (Shadow prices and the liquidity premium). In the limit $\Delta t \rightarrow 0$, investor $i^{\prime}$ s shadow price of a unit of labor income is given by:

$$
\begin{equation*}
\widehat{S}_{i t}=\widehat{\Psi}_{i}\left(v ;-\gamma_{A}\right) s(v)^{\gamma_{A}} D_{t}, \quad i=A, B \tag{48}
\end{equation*}
$$

where $\widehat{\Psi}_{i}(v ; \theta)$ satisfies differential-difference equation (34) subject to the following boundary conditions for investors $A$ and $B$

$$
\begin{array}{ll}
\widehat{\Psi}_{A}^{\prime}(\underline{v} ; \theta)=0, & \widehat{\Psi}_{A}^{\prime}(\bar{v} ; \theta)=0 \\
\widehat{\Psi}_{B}^{\prime}(\underline{v} ; \theta)=\widehat{\Psi}_{B}(\underline{v} ; \theta), & \widehat{\Psi}_{B}^{\prime}(\bar{v} ; \theta)=\widehat{\Psi}_{B}(\bar{v} ; \theta) \tag{50}
\end{array}
$$

The investors' liquidity premia for stocks $\Lambda_{A}$ and $\Lambda_{B}$ are positive, and hence,

$$
\begin{equation*}
S_{t} /\left(1-l_{A}-l_{B}\right)>\widehat{S}_{A t}, \quad S_{t} /\left(1-l_{A}-l_{B}\right)>\widehat{S}_{B t} \tag{51}
\end{equation*}
$$

The premium $\Lambda_{i t}>0$ arises because the stock can be used as a collateral whereas the labor income cannot. We note that this premium is zero in the unconstrained economy, and hence the non-tradability of labor income and the possibility of shorting stocks do not contribute to the premium. This is because, as discussed in Section 4.1, in an unconstrained economy with fully pledgeable labor income the investors can circumvent the non-tradability of labor income by shorting stocks. We further remark that the shadow prices and liquidity premia can be found in closed form, similar to stock prices in Section 3 , but we do not present them for brevity.

Figure 6 plots the liquidity premia (47) for the same calibrated parameters as in Section 4.1. We observe that investors $A$ and $B$ have different valuations of their labor incomes due to differences in preferences and beliefs. Their premia $\Lambda_{i}$ are close to zero when the investors are far away from the boundaries where their respective collateral requirements become binding. The premia increase up to $35 \%$ close to the boundaries where the stock is more valuable for the purposes of relaxing the constraints. Large premia $\Lambda_{i t}$ imply the economic significance of stock pledgeability.

## 5. Conclusion

We develop a parsimonious and tractable theory of asset pricing under collateral requirements. We show that requiring investors to collateralize their trades has significant effects on asset prices and their moments. The constraints decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries. They also increase price-dividend ratios, amplify volatilities in good states and dampen them in bad states. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.

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## Appendix A: Proofs

Lemma A. 1 (Change of variable). Let $\widehat{n}_{i}=k_{i} l_{i} /\left(1-l_{A}-l_{B}\right)$. Maximization of expected discounted utility (7) subject to budget constraints (8) and (9), and constraint (12) is equivalent to maximizing (7) with respect to $c_{i t}, b_{i t}$ and $\tilde{n}_{i t}$ subject to the following set of constraints:

$$
\begin{align*}
\widetilde{W}_{i t}+l_{i} D_{t} \Delta t & =c_{i t} \Delta t+b_{i t} B_{t}+\widetilde{n}_{i t}\left(S_{t}, P_{t}\right)^{\top}  \tag{A1}\\
\widetilde{W}_{i, t+\Delta t} & =b_{i t}+\widetilde{n}_{i t}\left(S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t, \mathbf{1}_{\left\{\omega_{t+\Delta t}=\omega_{3}\right\}}\right)^{\top},  \tag{A2}\\
\widetilde{W}_{i, t+\Delta t} & \geq 0 \tag{A3}
\end{align*}
$$

where $\widetilde{W}_{i t}=W_{i t}+\widehat{n}_{i} S_{t}$ and $\widetilde{W}_{i, t+\Delta t}=W_{i, t+\Delta t}+\widehat{n}_{i}\left(S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t}\right)$.
Proof of Lemma A.1. Substituting $n_{i t}=\widetilde{n}_{i t}-\left(\widehat{n}_{i}, 0\right)$ into (8) and (9), we obtain constraints (A1) and (A2). Rewriting constraint (12) in terms of variable $\widetilde{W}_{i, t+\Delta t}$, we obtain (A3). Finally, we note that $\widetilde{W}_{i t}=W_{i t}+\widehat{n}_{i} S_{t}$ is worth $\widetilde{W}_{i, t+\Delta t}$ next period. Hence, (A1) and (A2) can be seen as self-financing budget constraints.

## Proof of Lemma 1.

1) We start by demonstrating the equivalence of the dynamic (8)-(9) and static budget constraints (22). Multiplying equation (9) by $\xi_{i, t+\Delta t} / \xi_{i t}$, taking expectation operator $\mathbb{E}_{t}^{i}[\cdot]$ on both sides, and using equations (17)-(19) for asset prices, we obtain:

$$
\begin{equation*}
\mathbb{E}_{t}^{i}\left[\frac{\xi_{i, t+\Delta t}}{\xi_{i t}} W_{i, t+\Delta t}\right]=b_{i t} B_{t}+n_{i t}\left(S_{t}, P_{t}\right)^{\top} \tag{A4}
\end{equation*}
$$

From the budget constraint equation (8), we observe that the right-hand side of (A4) equals $W_{i t}+l_{i} D_{t} \Delta t$, and hence, we obtain the static budget constraint (22). Conversely, if there exists $W_{i, t+\Delta t}$ satisfying constraints (22) and (23) there exist trading strategies $b_{i t}$ and $n_{i t}$ that replicate $W_{i, t+\Delta t}$ because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Then, rewriting the optimization problem (7) in a recursive form, we obtain the dynamic programming equation (21) for the value function. 2) Consider wealth levels $W_{i t}$ and $\widehat{W}_{i t}$. Let $\left\{c_{i t}^{*}, b_{i t}^{*}, n_{i t}^{*}\right\}$ and $\left\{\widehat{c}_{i t}^{*}, \widehat{b}_{i t}^{*}, \widehat{n}_{i t}^{*}\right\}$ be optimal consumptions and portfolios that correspond to $W_{i t}$ and $\widehat{W}_{i t}$, respectively, and satisfy constraints (8)-(10). For any $\alpha \in[0,1]$, policies $\left\{\alpha \widehat{c}_{i t}^{*}+(1-\alpha) c_{i t}^{*}, \alpha \widehat{b}_{i t}^{*}+(1-\alpha) b_{i t}^{*}, \alpha \widehat{n}_{i t}^{*}+(1-\right.$
$\left.\alpha) n_{i t}^{*}\right\}$ are admissible for wealth $\alpha W_{i t}+(1-\alpha) \widehat{W}_{i t}$. By concavity of CRRA utilities:

$$
\begin{align*}
V_{i}\left(\alpha W_{i t}+(1-\alpha) \widehat{W}_{i t}, v_{t} ; l_{i}\right) & \geq \sum_{\tau=t}^{\infty} u_{i}\left(\alpha \widehat{c}_{i t}^{*}+(1-\alpha) c_{i t}^{*}\right) \\
& \geq \sum_{\tau=t}^{\infty}\left(\alpha u_{i}\left(\widehat{c}_{i t}^{*}\right)+(1-\alpha) u_{i}\left(c_{i t}^{*}\right)\right)  \tag{A5}\\
& =\alpha V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)+(1-\alpha) V_{i}\left(\widehat{W}_{i t}, v_{t} ; l_{i}\right)
\end{align*}
$$

Therefore, $V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)$ is a concave function of wealth.
3) Consider the following Lagrangian:

$$
\begin{align*}
\mathscr{L} & =u_{i}\left(c_{i t}\right) \Delta t+e^{-\rho \Delta t} \mathbb{E}_{t}^{i}\left[V_{i}\left(W_{i, t+\Delta t}, v_{t+\Delta t} ; l_{i}\right)\right] \\
& \left.+\eta_{i t}\left(W_{i t}+l_{i} D_{t} \Delta t-c_{i t} \Delta t-\mathbb{E}_{t}^{i}\left[\frac{\xi_{i, t+\Delta t}}{\xi_{i t}} W_{i, t+\Delta t}\right]\right)+\mathbb{E}_{t}^{i}\left[e^{-\rho \Delta t} \ell_{i, t+\Delta t} W_{i, t+\Delta t}\right)\right], \tag{A6}
\end{align*}
$$

where multiplier $\ell_{i, t+\Delta t}$ satisfies the complementary slackness condition $\ell_{i, t+\Delta t} W_{i, t+\Delta t}=0$. Differentiating the Lagrangian (A6) with respect to $c_{i t}$ and $W_{i, t+\Delta t}$, we obtain:

$$
\begin{align*}
u_{i}^{\prime}\left(c_{i t}^{*}\right) & =\eta_{i t}  \tag{A7}\\
e^{-\rho \Delta t}\left(\frac{\partial V_{i}\left(W_{t+\Delta t}, v_{t+\Delta t} ; l_{i}\right)}{\partial W}+\ell_{i, t+\Delta t}\right) & =\eta_{i t} \frac{\xi_{i, t+\Delta t}}{\xi_{i t}} \tag{A8}
\end{align*}
$$

By the envelope theorem (e.g, Back (2010, p.162)):

$$
\begin{equation*}
\frac{\partial V_{i}\left(W_{i, t+\Delta t}, v_{t+\Delta t} ; l_{i}\right)}{\partial W}=u_{i}^{\prime}\left(c_{i, t+\Delta t}^{*}\right) \tag{A9}
\end{equation*}
$$

Substituting the partial derivative of the value function (A9) and the marginal utility (A7) into equation (A8), and then dividing both sides of the equation by $u_{i}^{\prime}\left(c_{i t}^{*}\right)$, we obtain the expression for the SPD (24).

## Proof of Proposition 1.

Step 1. Consider the case when constraints do not bind, and hence, $\ell_{i, t+\Delta t}=0$. Then, using equation (14) for the state variable $v_{t}$ and the first order conditions (24), we obtain:

$$
v_{t+\Delta t}-v_{t}=\ln \left(\frac{\left(c_{A, t+\Delta t}^{*} / c_{A t}^{*}\right)^{-\gamma_{A}}}{\left(c_{B, t+\Delta t}^{*} / c_{B t}^{*}\right)^{-\gamma_{B}}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right)=\ln \left(\frac{\xi_{A, t+\Delta t} / \xi_{A t}}{\xi_{B, t+\Delta t} / \xi_{B t}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right) .
$$

From the above equation and the change of measure equation (20), which relates SPDs $\xi_{A, t+\Delta t}$ and $\xi_{B, t+\Delta t}$, we obtain the dynamics of $v_{t}$ when constraints do not bind:

$$
\begin{equation*}
v_{t+\Delta t}-v_{t}=\ln \left(\frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right) . \tag{A10}
\end{equation*}
$$

Let $\bar{v}$ and $\underline{v}$ be the boundaries satisfying Equations (25), at which the constraints of investors $A$ and $B$ bind, respectively. Let investor $A$ 's constraint be binding so that $v_{t+\Delta t}=\bar{v}$, and hence, $\ell_{A, t+\Delta t} \geq 0$. Using Equation (14) for $v_{t}$, first order conditions (24), and $\ell_{A, t+\Delta t} \geq 0$, we obtain:

$$
\begin{align*}
\bar{v}-v_{t} & \leq \ln \left(\frac{\left(\left(c_{A, t+\Delta t}^{*}\right)^{-\gamma_{A}}+\ell_{A, t+\Delta t}\right) /\left(c_{A t}^{*}\right)^{-\gamma_{A}}}{\left(c_{B, t+\Delta t}^{*} / c_{B t}^{*}\right)^{-\gamma_{B}}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right)  \tag{A11}\\
& =\ln \left(\frac{\xi_{A, t+\Delta t} / \xi_{A t}}{\xi_{B, t+\Delta t} / \xi_{B t}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right)=\ln \left(\frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right) .
\end{align*}
$$

Similarly, for $v_{t+\Delta t}=\underline{v}$ we obtain that $\underline{v}-v_{t} \geq \ln \left(\pi_{B}\left(\omega_{t+\Delta t}\right) / \pi_{A}\left(\omega_{t+\Delta t}\right)\left(D_{t+\Delta t} / D_{t}\right)^{\gamma_{A}-\gamma_{B}}\right)$. The latter two inequalities imply that when the constraint binds $v_{t+\Delta t}$ is given by:

$$
\begin{equation*}
v_{t+\Delta t}=\max \left\{\underline{v} ; \min \left\{\bar{v} ; v_{t}+\ln \left(\frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right)\right\}\right\} . \tag{A12}
\end{equation*}
$$

We observe that (A12) is also satisfied in the unconstrained case where $\underline{v}<v_{t+\Delta t}<\bar{v}$. It remains to prove that $v_{t}$ does not escape $[\underline{v}, \bar{v}]$ interval. Consider a marginal investor of type $A$. We guess that $v_{t}$ follows dynamics (A12) and verify that the consumption choice of investor $A$ indeed implies this dynamics. The analysis for investor $B$ is similar.

We have shown above that $v_{t}$ satisfies inequality (A11) when investor $A$ is constrained. Now, we show the opposite: investor $A$ is constrained when $v_{t}$ satisfies (A11). Hence, $v_{t+\Delta t}$ cannot exceed $\bar{v}$. Consider $v_{t}$ such that $v_{t}+\ln \left(\pi_{B}\left(\omega_{t+\Delta t}\right) / \pi_{A}\left(\omega_{t+\Delta t}\right)\left(D_{t+\Delta t} / D_{t}\right)^{\gamma_{A}-\gamma_{B}}\right)>\bar{v}$ for some $\omega_{t+\Delta t}$ and $v_{t} \in(\underline{v}, \bar{v})$. Because $\underline{v}<v_{t}<\bar{v}$, investor $A$ consumes $c_{A t}^{*}=s\left(v_{t}\right) D_{t}$, as shown above. We show that the constraint of investor $A$ binds and $c_{A, t+\Delta t}^{*}=s(\bar{v}) D_{t+\Delta t}$. This consumption level confirms that $v_{t+\Delta t}=\bar{v}$ is indeed an equilibrium outcome.

Consider the constraint of investor $A$ at date $t$ in the state $\omega_{t+\Delta t}$ where $v_{t+\Delta t}=\bar{v}$ :

$$
\begin{equation*}
W_{A, t+\Delta t} \geq 0 \equiv \Phi_{A}(\bar{v}) D_{t+\Delta t} \tag{A13}
\end{equation*}
$$

where the last equality holds by the definition of $\bar{v}$. Using the concavity of the value function, proven in Lemma 1, and condition (A9) from the envelope theorem, we obtain:

$$
\begin{equation*}
u_{A}^{\prime}\left(c_{A, t+\Delta t}^{*}\right)=\frac{\partial V_{A}\left(W_{A, t+\Delta t}, \bar{v} ; l_{A}\right)}{\partial W} \leq \frac{\partial V_{A}\left(\Phi_{A}(\bar{v}) D_{t+\Delta t}, \bar{v} ; l_{A}\right)}{\partial W}=u_{A}^{\prime}\left(s(\bar{v}) D_{t+\Delta t}\right) . \tag{A14}
\end{equation*}
$$

Because $u_{i}^{\prime}(c)$ is a decreasing function, we find that $c_{A, t+\Delta t}^{*} / D_{t+\Delta t} \geq s(\bar{v})$.
Investor $B$ is unconstrained when $v_{t+\Delta t}=\bar{v}$, and hence, has SPD

$$
\begin{equation*}
\frac{\xi_{B, t+\Delta t}}{\xi_{B t}}=e^{-\rho \Delta t}\left(\frac{c_{B, t+\Delta t}^{*}}{c_{B t}^{*}}\right)^{-\gamma_{B}}=e^{-\rho \Delta t}\left(\frac{(1-s(\bar{v})) D_{t+\Delta t}}{\left(1-s\left(v_{t}\right)\right) D_{t}}\right)^{-\gamma_{B}} \tag{A15}
\end{equation*}
$$

From the change of measure equation (20) and the FOC (24), the SPD of investor $A$ is

$$
\begin{align*}
\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} & =\frac{\xi_{B, t+\Delta t}}{\xi_{B t}} \frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)} \\
& =e^{-\rho \Delta t} \frac{\left(c_{A, t+\Delta t}^{*}\right)^{-\gamma_{A}}+\ell_{A, t+\Delta t}}{\left(c_{A t}^{*}\right)^{-\gamma_{A}}} . \tag{A16}
\end{align*}
$$

From (A16) and (A15), we find the Lagrange multiplier:

$$
\begin{aligned}
\frac{l_{A, t+\Delta t}}{\left(c_{A, t+\Delta t}^{*}\right)^{-\gamma_{A}}} & =\left(\frac{c_{A, t+\Delta t}^{*}}{c_{A t}^{*}}\right)^{\gamma_{A}}\left(\frac{(1-s(\bar{v})) D_{t+\Delta t}}{\left(1-s\left(v_{t}\right)\right) D_{t}}\right)^{-\gamma_{B}} \frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}-1 \\
& \geq\left(\frac{s(\bar{v}) D_{t+\Delta t}}{s\left(v_{t}\right) D_{t}}\right)^{\gamma_{A}}\left(\frac{(1-s(\bar{v})) D_{t+\Delta t}}{\left(1-s\left(v_{t}\right)\right) D_{t}}\right)^{-\gamma_{B}} \frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}-1 \\
& =\left(\frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right) e^{v_{t}-\bar{v}}-1>0 .
\end{aligned}
$$

The first inequality follows from the fact that $c_{A, t+\Delta t}^{*} \geq s(\bar{v}) D_{t+\Delta t}$ we proved above. The second equality holds by the definition of state variable (14). The second inequality comes from the assumption that $v_{t}+\ln \left(\pi_{B}\left(\omega_{t+\Delta t}\right) / \pi_{A}\left(\omega_{t+\Delta t}\right)\left(D_{t+\Delta t} / D_{t}\right)^{\gamma_{A}-\gamma_{B}}\right)>\bar{v}$. Hence, the Lagrange multiplier $l_{A, t+\Delta t}$ is strictly positive. From the complementary slackness condition, the constraint (A13) must be binding. Therefore, inequality (A14) becomes an equality, and hence, $c_{A, t+\Delta t}^{*}=s(\bar{v}) D_{t+\Delta t}$.
Step 2. We now look for coefficients $\mu_{v}, \sigma_{v}$ and $J_{v}$ such that:

$$
\begin{align*}
\mu_{v} \Delta t & +\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}=\ln \left(\frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}\right) \\
& =\ln \left(\frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\right)+\left(\gamma_{A}-\gamma_{B}\right) \ln \left(1+\mu_{D} \Delta t+\sigma_{D} \Delta w_{t}+J_{v} \Delta j_{t}\right) \tag{A17}
\end{align*}
$$

We write identity (A17) in each of the states $\omega_{t+\Delta t} \in\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$ and obtain the following system of three linear equations with three unknowns $\mu_{v}, \sigma_{v}$ and $J_{v}$ :

$$
\begin{align*}
& \mu_{v} \Delta t+\sigma_{v} \sqrt{\Delta t}=\ln \left(\frac{\left(1-\lambda_{B} \Delta t\right)(1+\delta \Delta t)}{1-\lambda \Delta t}\right)+\left(\gamma_{A}-\gamma_{B}\right) \ln \left(1+\mu_{D} \Delta t+\sigma_{D} \sqrt{\Delta t}\right), \\
& \mu_{v} \Delta t-\sigma_{v} \sqrt{\Delta t}=\ln \left(\frac{\left(1-\lambda_{B} \Delta t\right)(1-\delta \Delta t)}{1-\lambda \Delta t}\right)+\left(\gamma_{A}-\gamma_{B}\right) \ln \left(1+\mu_{D} \Delta t-\sigma_{D} \sqrt{\Delta t}\right),  \tag{A18}\\
& \mu_{v} \Delta t+J_{v} \quad=\ln \left(\frac{\lambda_{B}}{\lambda}\right)+\left(\gamma_{A}-\gamma_{B}\right) \ln \left(1+\mu_{D} \Delta t+J_{D}\right) .
\end{align*}
$$

Solving the above system, we obtain $\mu_{v}, \sigma_{v}$ and $J_{v}$ reported in Proposition 1.

Step 3. Finally, we show that the boundaries are reflecting for a sufficiently small $\Delta t$. Suppose, two conditions are satisfied: $\mu_{v} \Delta t-\sigma_{v} \sqrt{\Delta t}<0$ and $\mu_{v} \Delta t+\sigma_{v} \sqrt{\Delta t}>0$. Then, the boundaries are reflecting: 1) if $v_{t}=\bar{v}$, then $v_{t+\Delta t}=\bar{v}+\mu_{v} \Delta t-\sigma_{v} \sqrt{\Delta t}<\bar{v}$ with positive probability; 2) if $v_{t}=\underline{v}$, then $v_{t+\Delta t}=\underline{v}+\mu_{v} \Delta t+\sigma_{v} \sqrt{\Delta t}>\underline{v}$ with positive probability. It can be easily verified that as $\Delta t \rightarrow 0, \mu_{v} \rightarrow \widehat{\mu}_{v}$ and $\sigma_{v} \rightarrow \widehat{\sigma}_{v}$, where $\widehat{\mu}_{v}$ and $\widehat{\sigma}_{v}$ are constants given by equations (36) and (37), respectively. Because $\sigma_{v}>0$ and $\sqrt{\Delta t}$-terms dominate $\Delta t$-terms for small $\Delta t$, we find that $\mu_{v} \Delta t-\sigma_{v} \sqrt{\Delta t}<0$ and $\mu_{v} \Delta t+\sigma_{v} \sqrt{\Delta t}>0$ for all sufficiently small $\Delta t$. Hence, the boundaries are reflecting.

Lemma A. 2 (Wealth-output ratios). The investors' wealth-output ratios $\Phi_{i}$ are uniformly bounded and given by:

$$
\begin{align*}
& \Phi_{A}\left(v_{t}\right)=\mathbb{E}_{t}^{A}\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{A}}\left(\frac{s\left(v_{\tau}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}}\left(s\left(v_{\tau}\right)-l_{A}\right) \Delta t\right]  \tag{A19}\\
& \Phi_{B}\left(v_{t}\right)=\mathbb{E}_{t}^{B}\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{B}}\left(\frac{1-s\left(v_{\tau}\right)}{1-s\left(v_{t}\right)}\right)^{-\gamma_{B}}\left(l_{B}-s\left(v_{\tau}\right)\right) \Delta t\right] . \tag{A20}
\end{align*}
$$

Proof of Lemma A.2. Substituting FOC (24) into the budget constraint (22) and using the complementary slackness condition $\ell_{i, t+\Delta t} W_{i, t+\Delta t}^{*}=0$, we obtain:

$$
\begin{equation*}
W_{A t}^{*}=\mathbb{E}_{t}^{A}\left[e^{-\rho \Delta t}\left(\frac{c_{A, t+\Delta t}^{*}}{c_{A t}^{*}}\right)^{-\gamma_{A}} W_{A, t+\Delta t}^{*}\right]+\left(c_{A t}^{*}-l_{A} D_{t}\right) \Delta t . \tag{A21}
\end{equation*}
$$

Substituting $W_{A t}^{*}=\Phi_{A t} D_{t}$ and $c_{A t}^{*}=s\left(v_{t}\right) D_{t}$ into equation (A21) and iterating, we obtain equation (A19). Let $\bar{s}=s(\underline{v}) \underline{s}=s(\bar{v})$, where $s(v)$ is given by equation (15). Then, $\bar{s} \geq s \geq \underline{s}>0$. Using the bounds on $s_{t}$, we obtain the following uniform bound on $\Phi_{A}$ :

$$
\Phi_{A}\left(v_{t}\right) \leq \text { Const } \times \mathbb{E}_{t}^{A}\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{A}} \Delta t\right] .
$$

The series on the right-hand side of the latter inequality is convergent due to condition (16) on model parameters. Equation (A20) is obtained along the same lines.

Proof of Proposition 2. 1) First, we derive the $\mathrm{SPD} \xi_{A t}$ under the correct beliefs of investor $A$. When investor $A$ 's constraint does not bind, substituting $c_{A t}^{*}=s\left(v_{t}\right) D_{t}$ into the first order condition (24) we find that

$$
\begin{equation*}
\frac{\xi_{A, t+\Delta t}}{\xi_{A t}}=e^{-\rho \Delta t}\left(\frac{s\left(v_{t+\Delta t}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}} . \tag{A22}
\end{equation*}
$$

Equation (A22) is consistent with SPD (30) because when the constraint does not bind $v_{t+\Delta t}=v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}<\bar{v}$, and hence the exponential term in (30) vanishes.

When the constraint of investor $A$ binds, the constraint of investor $B$ is loose: the constraints cannot bind simultaneously because stock market would not clear otherwise. Therefore, the ratio $\xi_{B, t+\Delta t} / \xi_{B t}$ is given by FOC (24) for investor $B$ with $\ell_{B}=0$. Using equation (20), we rewrite the latter SPD under the correct beliefs of investor $A$ :

$$
\begin{equation*}
\frac{\xi_{A, t+\Delta t}}{\xi_{A t}}=e^{-\rho \Delta t}\left(\frac{1-s\left(v_{t+\Delta t}\right)}{1-s\left(v_{t}\right)}\right)^{-\gamma_{B}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{B}} \frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)} . \tag{A23}
\end{equation*}
$$

Next, from equation (15) for consumption share $s$ we find that $\left(1-s_{t}\right)^{-\gamma_{B}}=e^{-v_{t}} s_{t}^{-\gamma_{A}}$. Substituting the latter equality into equation (A23), and also using equation (A17) for the increment $v_{t+\Delta t}-v_{t}$, we obtain:

$$
\begin{align*}
& \frac{\xi_{A, t+\Delta t}}{\xi_{A t}}=e^{-\rho \Delta t}\left(\frac{s\left(v_{t+\Delta t}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}} e^{v_{t}-v_{t+\Delta t}} \frac{\pi_{B}\left(\omega_{t+\Delta t}\right)}{\pi_{A}\left(\omega_{t+\Delta t}\right)}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{\gamma_{A}-\gamma_{B}}  \tag{A24}\\
& \quad=e^{-\rho \Delta t}\left(\frac{s\left(v_{t+\Delta t}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}} \exp \left\{v_{t}-v_{t+\Delta t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}\right\} .
\end{align*}
$$

The fact that the constraint of investor $A$ is binding means that $v_{t+\Delta t}=\bar{v}$ and $v_{t}+\mu_{v} \Delta t+$ $\sigma_{v} \Delta w_{t}+J_{v} \geq \bar{v}$ (because otherwise $v_{t+\Delta t}<\bar{v}$, and hence, the constraint does not bind). Therefore, the exponential term $\exp \left(v_{t}-v_{t+\Delta t}\right)$ in equation (A24) can be replaced with $\exp \left(\max \left\{0, v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}-\bar{v}\right\}\right)$. When the constraint of investor $A$ does not bind the latter term vanishes and we obtain equation (A22). Therefore, both equations (A22) and (A24) are summarized by equation (30) for $\xi_{A, t+\Delta t} / \xi_{A t}$.
2) Lemma A. 2 derives the wealth-output ratios $\Phi_{i}\left(v_{t}\right)$ and shows that they are uniformly bounded. From the market clearing condition $S_{t}=W_{A t}+W_{B t}$. Dividing by $D_{t}$, we obtain that $\Psi\left(v_{t}\right)=\Phi_{A}\left(v_{t}\right)+\Phi_{B}\left(v_{t}\right)$. Hence, $\Psi\left(v_{t}\right)$ is uniformly bounded. The fact that stock price $S_{t}$ is given by (31) can be verified by substituting $S_{t}$ into the recursive equation (18).
3) In the unconstrained economy, the state variable $v_{t}^{u n c}$ follows dynamics:

$$
\begin{equation*}
v_{t}^{u n c}=\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t} . \tag{A25}
\end{equation*}
$$

Define processes $U_{t+\Delta t}=U_{t}+\Delta U_{t}$ and $V_{t+\Delta t}=V_{t}+\Delta V_{t}$, where increments are given by:
$\Delta U_{t}=\max \left\{0 ; v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}-\bar{v}\right\}, \quad \Delta V_{t}=\max \left\{0 ; \underline{v}-v_{t}-\mu_{v} \Delta t-\sigma_{v} \Delta w_{t}-J_{v} \Delta j_{t}\right\}$.

The process for the state variable in the constrained economy can be rewritten as

$$
\begin{equation*}
v_{t+\Delta t}=v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}+\Delta V_{t}-\Delta U_{t} . \tag{A27}
\end{equation*}
$$

If the state variables have the same value at time 0 , i.e., $v_{0}=v_{0}^{u n c}$, we obtain:

$$
\begin{equation*}
v_{t}=v_{t}^{u n c}+V_{t}-U_{t} \tag{A28}
\end{equation*}
$$

Next, we prove that the SPD is higher in the constrained economy.

$$
\begin{align*}
& \frac{\xi_{A, t+\Delta t}}{\xi_{A t}}=e^{-\rho \Delta t}\left(\frac{s\left(v_{t+\Delta t}\right)}{s\left(v_{t}\right)} \frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}} \exp \left(\Delta U_{t}\right)  \tag{A29}\\
& \frac{\xi_{A, t+\Delta t}^{u n c}}{\xi_{A t}^{u n c}}=e^{-\rho \Delta t}\left(\frac{s\left(v_{t+\Delta t}^{u n c}\right)}{s\left(v_{t}^{u n c}\right)} \frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}} . \tag{A30}
\end{align*}
$$

Iterating the above equations, we obtain:

$$
\begin{aligned}
\frac{\xi_{A t}}{\xi_{A 0}} & =e^{-\rho t}\left(\frac{s\left(v_{t}\right)}{s\left(v_{0}\right)} \frac{D_{t}}{D_{0}}\right)^{-\gamma_{A}} \exp \left(U_{t}\right) \\
\frac{\xi_{A t}^{u n c}}{\xi_{A 0}^{u n c}} & =e^{-\rho t}\left(\frac{s\left(v_{t}^{u n c}\right)}{s\left(v_{0}\right)} \frac{D_{t}}{D_{0}}\right)^{-\gamma_{A}}
\end{aligned}
$$

By the definition of $s(v)$ in equation (15), we have $e^{v}=(1-s(v))^{\gamma_{B}} \cdot s(v)^{-\gamma_{A}}$. Hence,

$$
\begin{align*}
\frac{\xi_{A t} / \xi_{A 0}}{\xi_{A t}^{u n c}} / \xi_{A 0}^{u n c} & =\left(\frac{s\left(v_{t}\right)}{s\left(v_{t}^{u n c}\right)}\right)^{-\gamma_{A}} \exp \left(U_{t}\right)=\left(\frac{s\left(v_{t}^{u n c}+V_{t}-U_{t}\right)}{s\left(v_{t}^{u n c}\right)}\right)^{-\gamma_{A}} e^{v_{t}^{u n c}} e^{-\left(v_{t}^{u n c}-U_{t}\right)} \\
& \geq s\left(v_{t}^{u n c}-U_{t}\right)^{-\gamma_{A}} e^{-\left(v_{t}^{u n c}-U_{t}\right)} \cdot s\left(v_{t}^{u n c}\right)^{\gamma_{A}} e^{u_{t}^{u n c}}  \tag{A31}\\
& =\left(1-s\left(v_{t}^{u n c}-U_{t}\right)\right)^{-\gamma_{B}} \cdot\left(1-s\left(v^{u n c}\right)\right)^{\gamma_{B}} \geq 1
\end{align*}
$$

Therefore, we conclude that $\xi_{A t} / \xi_{A 0} \geq \xi_{A t}^{u n c} / \xi_{A 0}^{u n c}$. The latter inequality and the equations for asset prices (17)-(19) imply that prices are higher in the constrained economy.

Proof of Lemma 2. The price-dividend ratio $\Psi$ and wealth-aggregate consumption ratios $\Phi_{i}$ are functions of the state variable $v$, and satisfy equations:

$$
\begin{align*}
\Psi\left(v_{t}\right) & =\mathbb{E}_{t}^{A}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \frac{D_{t+\Delta t}}{D_{t}}\left(\Psi\left(v_{t+\Delta t}\right)+\Delta t\right)\right]  \tag{A32}\\
\Phi_{i}\left(v_{t}\right) & =\mathbb{E}_{t}^{A}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \frac{D_{t+\Delta t}}{D_{t}} \Phi_{i}\left(v_{t+\Delta t}\right)\right]+\left(\mathbf{1}_{\{i=A\}} s\left(v_{t}\right)+\mathbf{1}_{\{i=B\}}\left(1-s\left(v_{t}\right)\right)-l_{i}\right) \Delta t . \tag{A33}
\end{align*}
$$

These equations are obtained by substituting $S_{t}=\left(1-l_{A}-l_{B}\right) D_{t} \Psi\left(v_{t}\right)$ into equation (18) for the stock price, and $\Psi_{i}=D_{t} W_{i t}$ into static budget constraints (22). Define the following function in discrete time:

$$
\begin{equation*}
\widehat{\Psi}\left(v_{t} ; \theta\right)=\mathbb{E}_{t}^{A}\left[e^{-\rho \Delta t+\Delta U_{t}}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right)\right]+s\left(v_{t}\right)^{\theta} \Delta t, \tag{A34}
\end{equation*}
$$

where $\Delta U_{t}$ is given by equation

$$
\begin{equation*}
\Delta U_{t}=\max \left\{0 ; v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}-\bar{v}\right\} \tag{A35}
\end{equation*}
$$

Comparing equation (A34) with equations (A32) and (A33) for $\Psi$ and $\Phi_{i}$ and using the linearity of equation (A34), it easy to observe that $\Psi\left(v_{t}\right)$ and $\Phi_{i}\left(v_{t}\right)$ are given by the following equations in terms of $\widehat{\Psi}\left(v_{t} ; \theta\right)$ :

$$
\begin{aligned}
& \Psi\left(v_{t}\right)=\widehat{\Psi}\left(v_{t},-\gamma_{A}\right) s\left(v_{t}\right)^{\gamma_{A}}-\Delta t \\
& \Phi\left(v_{t}\right)=\left(\left(\mathbf{1}_{\{i=A\}}-\mathbf{1}_{\{i=B\}}\right) \widehat{\Psi}\left(v ; 1-\gamma_{A}\right)+\left(\mathbf{1}_{\{i=B\}}-l_{i}\right) \widehat{\Psi}\left(v ;-\gamma_{A}\right)\right) s(v)^{\gamma_{A}} .
\end{aligned}
$$

Taking limit $\Delta t \rightarrow 0$, we obtain equations (32) and (33) for $\Psi\left(v_{t}\right)$ and $\Phi_{i}\left(v_{t}\right)$.
First, we derive the equation for $\widehat{\Psi}\left(v_{t} ; \theta\right)$ when $v_{t}$ belongs to the interior $(\underline{v}, \bar{v})$. For a sufficiently small $\Delta t$ we have $\Delta U_{t}=0$, where $\Delta U_{t}$ is given by (A35). Then, we rewrite the expectation of $\left.\left(D_{t+\Delta t}\right) / D_{t}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t} ; \theta\right)$ as follows:

$$
\begin{align*}
\mathbb{E}_{t}^{A}\left[\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right)\right]= & (1-\lambda \Delta t) \mathbb{E}_{t}^{A}\left[\left.\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right) \right\rvert\, \text { normal }\right] \\
& +\lambda \Delta t \mathbb{E}_{t}^{A}\left[\left.\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right) \right\rvert\, \text { crisis }\right] \tag{A36}
\end{align*}
$$

Noting that in the crisis $D_{t+\Delta t} / D_{t}=1+\mu_{v} \Delta t+J_{D}$ and $v_{t+\Delta t}=\max \left\{\underline{v} ; v_{t}+\mu_{v} \Delta t+J_{v}\right\}$ and in the normal state $D_{t+\Delta t} / D_{t}=1+\mu_{D} \Delta t+\sigma_{D} \Delta w_{t}$ and $v_{t+\Delta t}=v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}$, using Taylor expansions for $\left(D_{t+\Delta t} / D_{t}\right)^{1-\gamma_{A}}$ and $\widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right)$, we find:

$$
\begin{align*}
\mathbb{E}_{t}^{A} & {\left[\left.\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right) \right\rvert\, \text { crisis }\right] }
\end{aligned}=\left(1+J_{D}\right)^{1-\gamma_{A} \widehat{\Psi}\left(\max \left\{\underline{v} ; v_{t}+J_{v}\right\} ; \theta\right) .} \begin{aligned}
\mathbb{E}_{t}^{A}\left[\left.\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right) \right\rvert\, \text { normal }\right] & =\left[1+\left(\left(1-\gamma_{A}\right) \mu_{D}+\frac{\left(1-\gamma_{A}\right) \gamma_{A} \sigma_{D}^{2}}{2}\right) \Delta t\right] \widehat{\Psi}\left(v_{t} ; \theta\right)  \tag{A37}\\
+\left(\mu_{v}+\left(1-\gamma_{A}\right) \sigma_{D} \sigma_{v}\right) \widehat{\Psi}^{\prime}\left(v_{t} ; \theta\right) \Delta t & +\frac{\sigma_{v}^{2}}{2} \widehat{\Psi}^{\prime \prime}\left(v_{t} ; \theta\right) \Delta t+o(\Delta t) .
\end{align*}
$$

Substituting (A37)-(A38) into (A34), we obtain:

$$
\begin{align*}
\widehat{\Psi}\left(v_{t} ; \theta\right) & =\left[1-\left(\lambda+\rho-\left(1-\gamma_{A}\right) \mu_{D}+\frac{\left(1-\gamma_{A}\right) \gamma_{A}}{2} \sigma_{D}^{2}\right) \Delta t\right] \widehat{\Psi}\left(v_{t} ; \theta\right) \\
& +\left(\mu_{v}+\left(1-\gamma_{A}\right) \sigma_{D} \sigma_{v}\right) \widehat{\Psi}^{\prime}(v ; \theta) \Delta t+\frac{\sigma_{v}^{2}}{2} \widehat{\Psi}^{\prime \prime}(v ; \theta) \Delta t  \tag{A39}\\
& +\lambda\left(1+J_{D}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(\max \left\{\underline{v} ; v_{t}+J_{v}\right\} ; \theta\right) \Delta t+s(v)^{\theta} \Delta t+o(\Delta t) .
\end{align*}
$$

Canceling similar terms, diving by $\Delta t$, taking limit $\Delta t \rightarrow 0$, and noting that $\mu_{v}, \sigma_{v}$ and $J_{v}$ converge to $\widehat{\mu}_{v}, \widehat{\sigma}_{v}$ and $\widehat{J}_{v}$ given by (36)-(38), we obtain equation (34) for $\widehat{\Psi}\left(v_{t} ; \theta\right)$.

Next, we derive the boundary conditions for $\widehat{\Psi}\left(v_{t} ; \theta\right)$. From equation (26), the state variable dynamics at lower bound is $v_{t+\Delta t}=\underline{v}+\max \left\{0, \mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}\right\}$. We use $\Delta v_{t}$ to denote the difference of $v_{t+\Delta t}$ and $v_{t}$. In this case,

$$
\begin{equation*}
\Delta v_{t}=\max \left\{0, \mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}\right\} . \tag{A40}
\end{equation*}
$$

For sufficiently small $\Delta t$ increment $\Delta v_{t}$ is positive only in state $\omega_{1}$ and is zero otherwise. In state $\omega_{1}, \Delta v_{t}=\mu_{v} \Delta t+\sigma_{v} \sqrt{\Delta t}$. Therefore, the order of $\mathbb{E}_{t}^{A}\left[\Delta v_{t}\right]$ is $\sqrt{\Delta t}$, but second order terms involving $\Delta v_{t}$ have lower order:

$$
\begin{equation*}
\lim _{\Delta t \rightarrow 0} \frac{\mathbb{E}_{t}^{A}\left[\Delta v_{t}\right]}{\sqrt{\Delta t}}=\frac{\widehat{\sigma}_{v}}{2} \tag{A41}
\end{equation*}
$$

$\lim _{\Delta t \rightarrow 0} \frac{\mathbb{E}_{t}^{A}\left[\left(\Delta v_{t}\right)^{2}\right]}{\sqrt{\Delta t}}=\lim _{\Delta t \rightarrow 0} \frac{\mathbb{E}_{t}^{A}\left[\Delta v_{t} \Delta t\right]}{\sqrt{\Delta t}}=\lim _{\Delta t \rightarrow 0} \frac{\mathbb{E}_{t}^{A}\left[\Delta v_{t} \Delta w_{t}\right]}{\sqrt{\Delta t}}=\lim _{\Delta t \rightarrow 0} \frac{\mathbb{E}_{t}^{A}\left[\Delta v_{t} \Delta j_{t}\right]}{\sqrt{\Delta t}}=0$.
Taylor expansion of $\widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right)$ at $v_{t}=\underline{v}$ is given by

$$
\begin{equation*}
\widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right)=\widehat{\Psi}(\underline{v} ; \theta)+\widehat{\Psi}^{\prime}(\underline{v} ; \theta) \Delta v_{t}+\frac{1}{2} \widehat{\Psi}^{\prime \prime}(\underline{v} ; \theta) \Delta v_{t}^{2}+o(\sqrt{\Delta t}) . \tag{A42}
\end{equation*}
$$

In subsequent calculations we keep terms with order of $\sqrt{\Delta t}$. Using the above results, we obtain the following expansion:

$$
\begin{align*}
& \mathbb{E}_{t}^{A}\left[\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{1-\gamma_{A}} \widehat{\Psi}\left(v_{t+\Delta t} ; \theta\right)\right] \\
& =\mathbb{E}_{t}^{A}\left[\left(1+\mu_{D} \Delta t+\sigma_{D} \Delta w_{t}+J_{v} \Delta j_{t}\right)^{1-\gamma_{A}}\left(\widehat{\Psi}(\underline{v} ; \theta)+\widehat{\Psi}^{\prime}(\underline{v} ; \theta) \Delta v_{t}+\frac{1}{2} \widehat{\Psi}^{\prime \prime}(\underline{v} ; \theta) \Delta v_{t}^{2}\right)\right]  \tag{A43}\\
& =\widehat{\Psi}(\underline{v} ; \theta)+\widehat{\Psi}^{\prime}(\underline{v} ; \theta) \mathbb{E}_{t}^{A}\left[\Delta v_{t}\right]+o(\sqrt{\Delta t}) .
\end{align*}
$$

Substituting (A43) into (A34), taking into account that $\Delta U_{t}=0$ at $v_{t}=\underline{v}$, and canceling $\widehat{\Psi}(\underline{v} ; \theta)$ on both sides, we obtain the first boundary condition $\widehat{\Psi}^{\prime}(\underline{v} ; \theta)=0$.

At the upper bound $v_{t}=\bar{v}$ investor $A$ is constrained, and hence, $\Delta U_{t}$ in (A35) is positive. From (26) the state variable at the upper bound is

$$
\begin{equation*}
v_{t+\Delta t}=\min \left\{\bar{v}, v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}\right\}=v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}-\Delta U_{t} . \tag{A44}
\end{equation*}
$$

The order of $\mathbb{E}_{t}^{A}\left[\Delta U_{t}\right]$ is $\sqrt{\Delta t}$, but second order terms involving $\Delta U_{t}$ have order $o(\sqrt{\Delta t})$. Proceeding in the same way as (A41)-(A43), we arrive at

$$
\widehat{\Psi}(\bar{v} ; \theta)=\widehat{\Psi}(\bar{v} ; \theta)+\left[\widehat{\Psi}(\bar{v} ; \theta)-\widehat{\Psi}^{\prime}(\bar{v} ; \theta)\right] \mathbb{E}_{t}^{A}\left[\Delta U_{t}\right]+o(\sqrt{\Delta t}) .
$$

Canceling similar terms, taking limit $\Delta t \rightarrow 0$, we obtain condition $\widehat{\Psi}(\bar{v} ; \theta)-\widehat{\Psi}^{\prime}(\bar{v} ; \theta)=0$.
Finally, we derive the equations for $\bar{v}$ and $\underline{v}$. Taking limit $\Delta t \rightarrow 0$ in equations (25), we find that these equations become: $\Phi_{A}(\bar{v})=0, \Phi_{B}(\underline{v})=0$. Substituting $\Phi_{i}(v)$ and $\Psi(v)$ in terms of $\widehat{\Psi}(v ; \theta)$ from equations (33) into the latter equations for the boundaries, after some algebra, we obtain equations (39).

Proof of Corollary 1. Consider the case $\lambda=\lambda_{B}=0$ and $\gamma_{A}=\gamma_{B}=1$. Then, $s(v)$ solving equation (26) is given by $s(v)=1 /\left(1+e^{v}\right), \Psi(v)=\widehat{\Psi}(v) s(v)$, where $\widehat{\Psi}(v)$ solves a special case of equation (34) given by:

$$
\begin{equation*}
\frac{\delta^{2}}{2} \widehat{\Psi}^{\prime \prime}(v)-\frac{\delta^{2}}{2} \widehat{\Psi}^{\prime}(v)-\rho \widehat{\Psi}(v)+1+e^{v}=0 \tag{A45}
\end{equation*}
$$

subject to boundary conditions (35). It can be easily verified that $\widehat{\Psi}(v)=C_{1} e^{\varphi-v}+$ $C_{2} e^{\varphi+v}+\left(1+e^{v}\right) / \rho$ satisfies (A45). Substituting $\widehat{\Psi}(v)$ into boundary conditions (35), we obtain the following system for coefficients $C_{1}$ and $C_{2}$ :

$$
C_{1} \varphi_{-} e^{\varphi_{-} \underline{v}}+C_{2} \varphi_{+} e^{\varphi_{+} \underline{v}}+e^{\underline{v}} / \rho=0 ; \quad C_{1}\left(\varphi_{-}-1\right) e^{\varphi-\bar{v}}+C_{2}\left(\varphi_{+}-1\right) e^{\varphi_{+} \bar{v}}-1 / \rho=0 .
$$

Solving these equations, we obtain:

$$
\begin{align*}
C_{1} & =\frac{1}{\rho} \frac{\left(\varphi_{+}-1\right) e^{\underline{v}+\varphi_{+} \bar{v}}+\varphi_{+} e^{\varphi_{+}} \underline{v}}{\varphi_{+}\left(\varphi_{-}-1\right) e^{\varphi_{-\bar{v}}+\varphi_{+} \underline{v}}-\varphi_{-}\left(\varphi_{+}-1\right) e^{\varphi_{+}+\bar{v}+\varphi_{-}} \underline{v}},  \tag{A46}\\
C_{2} & =-\frac{1}{\rho} \frac{\left(\varphi_{-}-1\right) e^{\underline{v}+\varphi_{-}}+\varphi_{-} e^{\varphi-\underline{v}}}{\varphi_{+}\left(\varphi_{-}-1\right) e^{\varphi_{+}+\underline{v}+\bar{v}}-\varphi_{-}\left(\varphi_{+}-1\right) e^{\varphi+\bar{v}+\varphi_{-}}} . \tag{A47}
\end{align*}
$$

Proof of Proposition 3. From equation (17) for the bond price and the fact that $1=B_{t}\left(1+r_{t} \Delta t\right)$ we find that the riskless interest rate $r_{t}$ is given by:

$$
\begin{align*}
r_{t} & =\frac{1-\mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t}\right]}{\mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t}\right] \Delta t} \\
& =\frac{1-(1-\lambda \Delta t) \mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t} \mid \text { normal }\right]-\lambda \Delta t \mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t} \mid \text { crisis }\right]}{\mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t}\right] \Delta t} \tag{A48}
\end{align*}
$$

where $\xi_{A, t+\Delta t} / \xi_{A t}$ is given by equation (30). We separately calculate $\mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t} \mid\right.$ normal $]$ and $\mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t} \mid\right.$ crisis $]$, and then take the limit $\Delta t \rightarrow 0$.

We start with the derivation of $\mathbb{E}_{t}\left[\xi_{A, t+\Delta t} / \xi_{A t} \mid\right.$ normal $]$ when $\underline{v}<v_{t}<\bar{v}$, and hence, by continuity, for a sufficiently small $\Delta t$ the economy is unconstrained next period, so that $\underline{v}<v_{t+\Delta t}<\bar{v}$. In the unconstrained region $\Delta v_{t}=\widehat{\mu}_{v} \Delta t+\widehat{\sigma}_{v} \Delta w_{t}$ and the SPD is given by (A22). From the expression for the SPD, using expansions (A57) and (A59), we obtain:

$$
\begin{align*}
& \mathbb{E}_{t}\left[\left.\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \right\rvert\, \text { normal }\right]=\mathbb{E}_{t}\left[\left(\left(1+a_{t} \Delta v_{t}+b_{t}\left(\Delta v_{t}\right)^{2}\right)\left(1-r_{A} \Delta t-\kappa_{A} \Delta w_{t}\right) \mid \text { normal }\right]+o(\Delta t)\right. \\
& \quad=\mathbb{E}_{t}\left[1+a_{t} \Delta v_{t}+b_{t}\left(\Delta v_{t}\right)^{2}-r_{A} \Delta t-\kappa_{A} \Delta w_{t}-\kappa_{A} a_{t} \Delta v_{t} \Delta w_{t} \mid \text { normal }\right]+o(\Delta t) \\
& \quad=1+a_{t} \widehat{\mu}_{v} \Delta t+b_{t} \widehat{\sigma}_{v}^{2} \Delta t-r_{A} \Delta t-\kappa_{A} a_{t} \widehat{\sigma}_{v} \Delta t+o(\Delta t) \tag{A49}
\end{align*}
$$

Conditioning on the crisis state, we have:

$$
\begin{align*}
\mathbb{E}_{t}\left[\left.\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \right\rvert\, \text { crisis }\right] & =(1-\rho \Delta t)\left(1+\mu_{D} \Delta t+J_{D}\right)^{-\gamma_{A}}\left(\frac{s\left(\max \left\{\underline{v}, v_{t}+\mu_{v} \Delta t+J_{v}\right\}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}} \\
& =\left(1+J_{D}\right)^{-\gamma_{A}}\left(\frac{s\left(\max \left\{\underline{v}, v_{t}+\widehat{J}_{v}\right\}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}}+o(\Delta t) \tag{A50}
\end{align*}
$$

Substituting $a_{t}$ and $b_{t}$ from (A58) into equation (A49), and then substituting (A49) and (A50) into equation (A48), after simple algebra, we obtain $r_{t}$ in (41) for the case $\underline{v}<v_{t}<\bar{v}$.

Now, we derive $r_{t}$ at the boundaries $\underline{v}$ and $\bar{v}$. The SPD is given by (30). Using expansions (A57) and (A59), we obtain the following expansion:

$$
\begin{align*}
\mathbb{E}_{t}\left[\left.\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \right\rvert\, \text { normal }\right]= & \mathbb{E}_{t}\left[\left(\left(1+a_{t} \Delta v_{t}+b_{t}\left(\Delta v_{t}\right)^{2}\right)\left(1-r_{A} \Delta t-\kappa_{A} \Delta w_{t}\right)\right.\right. \\
& \left.\times\left(1+\Delta U_{t}+0.5\left(\Delta U_{t}\right)^{2}\right) \mid \text { normal }\right]+o(\Delta t) \\
= & \mathbb{E}_{t}\left[1+a_{t} \Delta v_{t}+b_{t}\left(\Delta v_{t}\right)^{2}-r_{A} \Delta t-\kappa_{A} \Delta w_{t}-\kappa_{A} a_{t} \Delta v_{t} \Delta w_{t}\right. \\
+ & \left.\Delta U_{t}-\kappa_{A} \Delta w_{t} \Delta U_{t}+a_{t} \Delta U_{t} \Delta v_{t}+0.5\left(\Delta U_{t}\right)^{2} \mid \text { normal }\right]+O(\Delta t) \tag{A51}
\end{align*}
$$

where $\Delta U_{t}$ is given by equation (A35). Using equation (26) for the process $v_{t}$ and equation (A35) for $\Delta U_{t}$, for a fixed $v_{t}$ and sufficiently small $\Delta t$, we find that $\Delta v_{t}$ and $\Delta U_{t}$ at the boundaries are given by:

$$
\Delta v_{t}= \begin{cases}\min \left(0, \mu_{v} \Delta t+\sigma_{v} \Delta w_{t}\right), & \text { if } v_{t}=\bar{v}  \tag{A52}\\ \max \left(0, \mu_{v} \Delta t+\sigma_{v} \Delta w_{t}\right), & \text { if } v_{t}=\underline{v}\end{cases}
$$

$$
\Delta U_{t}=\left\{\begin{array}{cl}
0, & \text { if } v_{t}<\bar{v},  \tag{A53}\\
\max \left(0, \mu_{v} \Delta t+\sigma_{v} \Delta w_{t}\right), & \text { if } v_{t}=\bar{v},
\end{array}\right.
$$

We note that for a sufficiently small $\Delta t$ the sign of $\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}$ is solely determined by the second term $\sigma_{v} \Delta w_{t}$ because it has the order of magnitude $\sqrt{\Delta t}$. Volatility $\sigma_{v}$ is positive because under our assumptions investor $A$ is more risk averse and more pessimistic. Using the latter observation, substituting equations (A52) and (A53) into equation (A51) and computing the expectation, we obtain:
$\mathbb{E}_{t}\left[\left.\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \right\rvert\,\right.$ normal $]=1+\left\{\begin{array}{l}\left(\frac{a_{t}\left(\mu_{v}-\kappa_{A} \sigma_{v}\right)}{2}+\frac{b_{t} \sigma_{v}^{2}}{2}+\frac{\mu_{v}+\kappa_{A} \sigma_{v}+\sigma_{v}^{2}}{2}-r_{A}\right) \Delta t \\ +\frac{\sigma_{v}\left(1-a_{t}\right)}{2} \sqrt{\Delta t}+O(\Delta t), \quad \text { if } \quad v_{t}=\underline{v}, \\ \left(\frac{a_{t} \mu_{v}-a_{t} \kappa_{A} \sigma_{v}+b_{t} \sigma_{v}^{2}}{2}\right) \Delta t+\frac{a_{t} \sigma_{v}}{2} \sqrt{\Delta t}+O(\Delta t), \quad \text { if } \quad v_{t}=\bar{v} .\end{array}\right.$

Substituting (A54) and (A50) into equation (A48) for the interest rate $r_{t}$, we obtain $r_{t}$ in (41) for the case when $v_{t}$ is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$
\begin{equation*}
\frac{\Delta S_{t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t}{S_{t}}=\mu_{t} \Delta t+\sigma_{t} \Delta w_{t}+J_{t} \Delta j_{t} . \tag{A55}
\end{equation*}
$$

Multiplying both sides of (A55) by $\xi_{A, t+\Delta t} / \xi_{A t}$ and taking expectations, we obtain:

$$
\mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \frac{\Delta S_{t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t}{S_{t}}\right]=\mu_{t} \Delta t \mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}}\right]+\sigma_{t} \mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \Delta w_{t}\right]+J_{t} \mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \Delta j_{t}\right]
$$

On the other hand, from the equation for stock price (18) we find that:

$$
\mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \frac{\Delta S_{t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t}{S_{t}}\right]=1-\mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}}\right]
$$

Combining the last two equations and the equation (A48) for the interest rate, we obtain:

$$
\begin{equation*}
\mu_{t}-r_{t}=-\left(\sigma_{t} \mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \Delta w_{t}\right]+J_{t} \mathbb{E}_{t}\left[\frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \Delta j_{t}\right]\right) \frac{1+r_{t} \Delta t}{\Delta t} \tag{A56}
\end{equation*}
$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain equation (42) for the risk premium.

Lemma A. 3 (Useful expansions).

1) For small increment $\Delta v_{t}=v_{t+\Delta t}-v_{t}$ the ratio $\left(s\left(v_{t+\Delta t}\right) / s\left(v_{t}\right)\right)^{-\gamma_{A}}$ has expansion:

$$
\begin{equation*}
\left(\frac{s\left(v_{t+\Delta t}\right)}{s\left(v_{t}\right)}\right)^{-\gamma_{A}}=1+a_{t} \Delta v_{t}+b_{t}\left(\Delta v_{t}\right)^{2}+o(\Delta t) \tag{A57}
\end{equation*}
$$

where coefficients $a_{t}$ and $b_{t}$ are given by:

$$
\begin{equation*}
a_{t}=\frac{\left(1-s_{t}\right) \Gamma_{t}}{\gamma_{B}}, \quad b_{t}=\frac{1}{2 \gamma_{B}^{2}}\left(1-s_{t}\right)^{2} \Gamma_{t}^{2}+\frac{1}{2 \gamma_{A}^{2} \gamma_{B}^{2}} s_{t}\left(1-s_{t}\right) \Gamma_{t}^{3}, \tag{A58}
\end{equation*}
$$

$\Gamma_{t}=\gamma_{A} \gamma_{B} /\left(\gamma_{A}(1-s)+\gamma_{B} s\right)$ is the risk aversion of the representative investor and $s_{t}$ is consumption share of investor $A$ that solves equation (15).
2) For the case $J_{D}=0$, the SPD in a one-investor economy can be expanded as follows:

$$
\begin{equation*}
e^{-\rho \Delta t}\left(\frac{D_{t+\Delta t}}{D_{t}}\right)^{-\gamma_{A}}=1-r_{A} \Delta t-\kappa_{A} \Delta w_{t}+o(\Delta t) \tag{A59}
\end{equation*}
$$

where $r_{A}$ and $\kappa_{A}$ are the riskless rate and the Sharpe ratio in an economy populated only by investor $A$, given by:

$$
\begin{equation*}
r_{A}=\rho+\gamma_{A} \mu_{D}-\frac{\gamma_{A}\left(1+\gamma_{A}\right)}{2} \sigma_{D}^{2}, \quad \kappa_{A}=\gamma_{A} \sigma_{D} \tag{A60}
\end{equation*}
$$

Proof of Lemma A.3. 1) We expand the ratio on the left-hand side of (A57) using Taylor's formula, and observe that $a_{t}=\left(s\left(v_{t}\right)^{-\gamma_{A}}\right)^{\prime} / s\left(v_{t}\right)^{-\gamma_{A}}$ and $b_{t}=0.5\left(s\left(v_{t}\right)^{-\gamma_{A}}\right)^{\prime \prime} / s\left(v_{t}\right)^{-\gamma_{A}}$. Differentiating, we obtain the following expressions for $a_{t}$ and $b_{t}$ :

$$
\begin{equation*}
a_{t}=-\gamma_{A} \frac{s^{\prime}\left(v_{t}\right)}{s\left(v_{t}\right)}, \quad b_{t}=\frac{\gamma_{A}\left(1+\gamma_{A}\right)}{2}\left(\frac{s^{\prime}\left(v_{t}\right)}{s\left(v_{t}\right)}\right)^{2}-\frac{\gamma_{A}}{2} \frac{s^{\prime \prime}(v)}{s(v)} . \tag{A61}
\end{equation*}
$$

To find derivatives $s^{\prime}(v)$ and $s^{\prime \prime}(v)$, we differentiate equation (15) twice with respect to $v$, and obtain two equations for the derivatives:

$$
\begin{align*}
& 1=-\left(\frac{\gamma_{A}}{s_{t}}+\frac{\gamma_{B}}{1-s_{t}}\right) s^{\prime}\left(v_{t}\right),  \tag{A62}\\
& 0=\left(\frac{\gamma_{A}}{s_{t}^{2}}-\frac{\gamma_{B}}{\left(1-s_{t}\right)^{2}}\right)\left(s^{\prime}\left(v_{t}\right)\right)^{2}-\left(\frac{\gamma_{A}}{s_{t}}+\frac{\gamma_{B}}{1-s_{t}}\right) s^{\prime \prime}\left(v_{t}\right) . \tag{A63}
\end{align*}
$$

Finding $s^{\prime}(v)$ and $s^{\prime \prime}(v)$ from the system (A62)-(A63) and substituting them into expressions (A61) for coefficients $a_{t}$ and $b_{t}$, after some algebra, we obtain expressions (A58).
2) Substituting $D_{t+\Delta t} / D_{t}$ from (1) into equation (A59), after some algebra, we obtain:

$$
\begin{align*}
e^{-\rho \Delta t}\left(\frac{D_{t+\Delta t}}{D_{t}}\right) & =e^{-\rho \Delta t}\left(1+\mu_{D} \Delta t+\sigma_{D} \Delta w_{t}\right)^{-\gamma_{A}} \\
& =(1-\rho \Delta t)\left(1-\left(\gamma_{A} \mu_{D}-\frac{\gamma_{A}\left(1+\gamma_{A}\right)}{2} \sigma_{D}^{2}\right) \Delta t-\gamma_{A} \sigma_{D}\right)+o(\Delta t)  \tag{A64}\\
& =1-r_{A} \Delta t-\kappa_{A} \Delta w_{t}+o(\Delta t)
\end{align*}
$$

Proof of Proposition 4. Consider a reflected arithmetic Brownian motion with boundaries $\underline{v}$ and $\bar{v}$ and dynamics $d v_{t}=\widehat{\mu}_{v} d t+\widehat{\sigma}_{v} d w_{t}$ when $\underline{v}<v_{t}<\bar{v}$, where $w_{t}$ is a Brownian motion. The transition density for this process is given by (see Veestraeten, 2004):

$$
\begin{align*}
& f_{v}\left(v, \tau ; v_{t}, t\right)=\frac{1}{\sqrt{2 \pi \widehat{\sigma}_{v}^{2}(\tau-t)}} \sum_{n=-\infty}^{+\infty}\left[\exp \left(-\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}} n(\bar{v}-\underline{v})-\frac{\left(v-v_{t}-\widehat{\mu}_{v}(\tau-t)+2 n(\bar{v}-\underline{v})\right)^{2}}{2 \widehat{\sigma}_{v}^{2}(\tau-t)}\right)\right. \\
& \left.+\exp \left(-\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}}\left(v_{t}-\underline{v}+n(\bar{v}-\underline{v})\right)-\frac{\left(v-v_{t}-\widehat{\mu}_{v}(\tau-t)+2\left(v_{t}-\underline{v}+n[\bar{v}-\underline{v}]\right)\right)^{2}}{2 \widehat{\sigma}_{v}^{2}(\tau-t)}\right)\right] \\
& +\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}} \sum_{n=0}^{+\infty}\left[\exp \left(-\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}}(\bar{v}-v+n[\bar{v}-\underline{v}])\right) \mathcal{N}\left(\frac{v_{t}+\widehat{\mu}_{v}(\tau-t)-v-2(\bar{v}-v+n[\bar{v}-\underline{v}])}{\widehat{\sigma}_{v} \sqrt{\tau-t}}\right)\right. \\
& \left.-\exp \left(\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}}(v-\underline{v}+n[\bar{v}-\underline{v}])\right)\left(1-\mathcal{N}\left(\frac{v_{t}+\widehat{\mu}_{v}(\tau-t)-v+2(v-\underline{v}+n[\bar{v}-\underline{v}])}{\widehat{\sigma}_{v} \sqrt{\tau-t}}\right)\right)\right], \tag{Ā65}
\end{align*}
$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution of a standard normal distribution. By $F_{v}\left(v, \tau ; v_{t}, t\right)$ $=\operatorname{Prob}\left\{v_{\tau} \leq v \mid v_{t}\right\}$ we denote the corresponding cumulative distribution function of $v$ conditional on observing $v_{t}$ at time $t$. We observe that $s_{t}=s\left(v_{t}\right)$ is a decreasing function of $v_{t}$ implicitly defined by equation (15). From the latter equation we also find that $s^{-1}(x)=\gamma_{B} \ln (1-s)-\gamma_{A} \ln (s)$. The cumulative distribution function of consumption share $s_{\tau}$ at time $\tau$ conditional on observing $s_{t}$ at time $t$ can then be found as follows:

$$
\begin{align*}
F\left(x, \tau ; s_{t}, t\right) & =\operatorname{Prob}\left\{s_{\tau} \leq x \mid s_{t}\right\} \equiv \operatorname{Prob}\left\{s\left(v_{\tau}\right) \leq x \mid s_{t}\right\} \\
& =1-\operatorname{Prob}\left\{v_{\tau} \leq s^{-1}(x) \mid v_{t}\right\}  \tag{A66}\\
& =1-\operatorname{Prob}\left\{v_{\tau} \leq \gamma_{B} \ln (1-x)-\gamma_{A} \ln (x) \mid v_{t}\right\} \\
& =1-F_{v}\left(\gamma_{B} \ln (1-x)-\gamma_{A} \ln (x), \tau ; v_{t}, t\right) .
\end{align*}
$$

Substituting $v_{t}=\gamma_{B} \ln \left(1-s_{t}\right)-\gamma_{A} \ln \left(s_{t}\right)$ into (A66), differentiating CDF $F\left(x, \tau ; s_{t}, t\right)$ with respect to $x$ and setting $x=s$, we find that the transition PDF for $s$ is given by:

$$
\begin{equation*}
f\left(s, \tau ; s_{t}, t\right)=\left(\frac{\gamma_{A}}{s}+\frac{\gamma_{B}}{1-s}\right) f_{v}\left(\gamma_{B} \ln (1-s)-\gamma_{A} \ln (s), \tau ; \gamma_{B} \ln \left(1-s_{t}\right)-\gamma_{A} \ln \left(s_{t}\right), t\right), \tag{A67}
\end{equation*}
$$

where transition density $f_{v}\left(v, \tau ; v_{t}, t\right)$ is given by equation (A65).
The stationary distribution of variable $v$, calculated in Veestraeten (2004), is given by:

$$
\begin{equation*}
f_{v}(v)=\frac{2 \widehat{\mu}_{v}}{\widehat{\sigma}_{v}^{2}} \frac{\exp \left(\left(2 \widehat{\mu}_{v} / \widehat{\sigma}_{v}^{2}\right) v\right)}{\exp \left(\left(2 \widehat{\mu}_{v} / \widehat{\sigma}_{v}^{2}\right) \bar{v}\right)-\exp \left(\left(2 \widehat{\mu}_{v} / / \hat{\sigma}_{v}^{2}\right) \underline{v}\right)} \tag{A68}
\end{equation*}
$$

Proceeding in the same way as for the derivation of transition PDF (A67), we obtain stationary PDF (44) for consumption share $s$.

Proof of Corollary 2. The proof easily follows by substituting boundary conditions (35) into the equation (B27) for volatility $\sigma_{t}$ at the boundary values $\underline{v}$ and $\bar{v}$.

Proof of Proposition 5. Consider Lagrangian (A6) for the dynamic optimization of investor $i$. Differentiating this Lagrangian with respect to $l_{i}$ and $c_{i t}$, we obtain:

$$
\begin{align*}
\frac{\partial V_{i}\left(W_{i t}^{*}, v_{t} ; l_{i}\right)}{\partial l_{i}} & =\eta_{i t} D_{t} \Delta t+e^{-\rho \Delta t} \mathbb{E}_{t}^{i}\left[\frac{\partial V_{i}\left(W_{i, t+\Delta t}^{*}, v_{t+\Delta t} ; l_{i}\right)}{\partial l_{i}}\right],  \tag{A69}\\
u^{\prime}\left(c_{i t}^{*}\right) & =\eta_{i t} . \tag{A70}
\end{align*}
$$

By the envelope theorem (e.g., Back (2010, p.162)):

$$
\begin{gather*}
\frac{\partial V_{i}\left(W_{i t}, v_{t} ; l_{i}\right)}{\partial W}=u_{i}^{\prime}\left(c_{i t}^{*}\right),  \tag{A71}\\
\frac{\partial V_{i}\left(W_{i, t+\Delta t}, v_{t+\Delta t} ; l_{i}\right)}{\partial W}=u_{i}^{\prime}\left(c_{i, t+\Delta t}^{*}\right) . \tag{A72}
\end{gather*}
$$

Substituting (46), (A70), (A71), and (A72) into equation (A69), and simplifying, we find:

$$
\begin{equation*}
\widehat{S}_{i t}=D_{t} \Delta t+\mathbb{E}_{t}^{i}\left[e^{-\rho \Delta t} \frac{u_{i}^{\prime}\left(c_{i, t+\Delta t}^{*}\right)}{u_{i}^{\prime}\left(c_{i t}^{*}\right)} \widehat{S}_{i, t+\Delta t}\right] . \tag{A73}
\end{equation*}
$$

From equation (30), we recall that the SPD of investor $A$ is given by

$$
\begin{equation*}
\frac{\xi_{A, t+\Delta t}}{\xi_{A t}}=e^{-\rho \Delta t+\Delta U_{t}} \frac{\left(c_{A, t+\Delta t}^{*}\right)^{-\gamma_{A}}}{\left(c_{A t}^{*}\right)^{-\gamma_{A}}} \frac{D_{t+\Delta t}}{D_{t}} \tag{A74}
\end{equation*}
$$

where $\Delta U_{t}=\max \left\{0 ; v_{t}+\mu_{v} \Delta t+\sigma_{v} \Delta w_{t}+J_{v} \Delta j_{t}-\bar{v}\right\}$. Rewriting equation (A73) for investor $A$ in terms of SPD (A74), we obtain:

$$
\begin{equation*}
\widehat{S}_{A t}=D_{t} \Delta t+\mathbb{E}_{t}^{A}\left[e^{-\Delta U_{t}} \frac{\xi_{A, t+\Delta t}}{\xi_{A t}} \widehat{S}_{A, t+\Delta t}\right] \tag{A75}
\end{equation*}
$$

Following the same steps as in the proof of Lemma 2, we find that $\widehat{S}_{A t}=\widehat{\Psi}_{i}\left(v_{t} ;-\gamma_{A}\right) s\left(v_{t}\right)^{\gamma_{A}} D_{t}$, where $\widehat{\Psi}_{i}(v ; \theta)$ satisfies differential-difference equation (34) with boundary conditions (49).

Iterating equation (18) for stock and equation (A75) for shadow prices, we obtain:

$$
\begin{align*}
S_{t}+\left(1-l_{A}-l_{B}\right) D_{t} \Delta t & =\frac{1}{\xi_{t}} \mathbb{E}_{t}^{A}\left[\sum_{\tau=t}^{\infty} \xi_{\tau}\left(1-l_{A}-l_{B}\right) D_{\tau} \Delta t\right],  \tag{A76}\\
\widehat{S}_{A t} & =\frac{1}{\xi_{t}} \mathbb{E}_{t}^{A}\left[\sum_{\tau=t}^{\infty} e^{-\left(U_{\tau}-U_{t}\right)} \xi_{\tau} D_{\tau} \Delta t\right] . \tag{A77}
\end{align*}
$$

Inequality $\left(S_{t}+\left(1-l_{A}-l_{B}\right) D_{t} \Delta t\right) /\left(1-l_{A}-l_{B}\right)>\widehat{S}_{A t}$ follows from the fact that $U_{t}=$ $\sum_{\tau=0}^{t} \Delta U_{\tau}$ is a non-decreasing processes. In the continuous-time limit, we obtain that $S_{t} /\left(1-l_{A}-l_{B}\right)>\widehat{S}_{A t}$. Hence, the liquidity premium $\Lambda_{A t}$ is positive. The derivation of the shadow price of investor $B$ is analogous and available upon request.

## Appendix B: Technical results.

Proposition B. 1 (Existence of boundaries $\underline{v}$ and $\bar{v}$ ). There exist constant boundaries $\underline{v}$ and $\bar{v}$ for the state variable $v_{t}$ process (26) that solve equations (25).

Proof of Proposition B.1. We here show the existence of $\bar{v}$ that solves $\Phi_{A}(\bar{v})=0$, where $\Phi_{A}(v)$ is given by equation (A19) in Appendix A. The proof for $\underline{v}$ is analogous.

We note that $\Phi_{A}\left(v_{t}\right) \geq 0$ because of the constraint $W_{A t} \geq 0$. Suppose, $\bar{v}$ does not exist, and hence $\Phi_{A}\left(v_{t}\right)>0$ for all $v_{t}$. From equation (15) for consumption share $s$ we observe that $s\left(v_{t}\right) \rightarrow 0$ when $v_{t} \rightarrow+\infty$. For arbitrary $\varepsilon \in\left(0, l_{A}\right)$ choose $v_{t}$ sufficiently large, so that $s\left(v_{t}\right)-l_{A}<-\varepsilon$. Let $T\left(v_{t}\right)$ be the stopping time, defined as

$$
\begin{equation*}
T\left(v_{t}\right)=\inf \left\{\tau: s\left(v_{\tau}\right)-l_{A} \geq-\varepsilon\right\} . \tag{B1}
\end{equation*}
$$

From equation (A19) for $\Phi_{A}\left(v_{t}\right)$ we obtain the following inequality:

$$
\begin{align*}
\Phi_{A}\left(v_{t}\right) s\left(v_{t}\right)^{-\gamma_{A}} \leq & -\varepsilon \mathbb{E}_{t}^{A}\left[\sum_{\tau=t}^{T\left(v_{t}\right)} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{A}} s\left(v_{\tau}\right)^{-\gamma_{A}} \Delta t\right] \\
+ & \mathbb{E}_{t}^{A}\left[\sum_{\tau=T\left(v_{t}\right)+\Delta t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{A}} s\left(v_{\tau}\right)^{-\gamma_{A}}\left(s\left(v_{\tau}\right)-\varepsilon\right) 1_{\left\{s\left(v_{\tau}\right) \geq \varepsilon\right\}} \Delta t\right] \\
\leq & -\varepsilon\left(l_{A}-\varepsilon\right)^{-\gamma_{A}} \mathbb{E}_{t}^{A}\left[\sum_{\tau=t}^{T\left(v_{t}\right)} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{A}} \Delta t\right]  \tag{B2}\\
& +\max \left(1 ; \varepsilon^{1-\gamma_{A}}\right) \mathbb{E}_{t}^{A}\left[\sum_{\tau=T\left(v_{t}\right)+\Delta t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{A}} \Delta t\right] .
\end{align*}
$$

Next, we show that $T\left(v_{t}\right) \rightarrow+\infty$ as $v_{t} \rightarrow+\infty$. Let $\hat{v}$ be such that $s(\widehat{v})=l_{A}-\varepsilon$. Then, because $s\left(v_{t}\right)$ is a decreasing function, $v_{t} \geq \widehat{v}$ and the stopping time (B1) can be rewritten as $T\left(v_{t}\right)=\inf \left\{\tau: v_{\tau} \leq \widehat{v}\right\}$. We note that $T\left(v_{t}\right) \geq \widehat{T}$, where $\widehat{T}$ is the minimal time required to get from $v_{t}$ to $\widehat{v}$, which is the time when $\Delta w_{t}=-\sqrt{\Delta t}$ and $\Delta j_{t}=1$ along the path. Time $\widehat{T}$ is found from the condition $v_{t}+(\widehat{T} / \Delta t)\left(\mu_{v} \Delta t-\sigma_{v} \sqrt{\Delta t}+J_{v}\right)=\widehat{v}$, where $J_{v}<0$. We observe that $\widehat{T} \rightarrow+\infty$ as $v_{t} \rightarrow+\infty$, and hence $T\left(v_{t}\right) \rightarrow+\infty$. We also note that $\mathbb{E}_{t}\left[\sum_{\tau=t}^{\infty} e^{-r(\tau-t)} D_{\tau}^{1-\gamma_{A}} \Delta t\right]<+\infty$ by condition (16). Therefore, for a sufficiently large $v_{t}$ we obtain from inequality (B2) that $\Phi_{A}\left(v_{t}\right)<0$, which contradicts initial assumption that $\Phi_{A}\left(v_{t}\right)>0$ for all $v_{t}$. Hence, there exists $\bar{v}$ such that $\Phi_{A}(\bar{v})=0$.

Lemma B. 1 (Unconstrained optimization). Consider an infinitesimal unconstrained investor with risk aversion $\gamma_{i}$ and labor income $l_{i} D_{t}, i=A, B$, who lives in the economy where the state price density is given by (30). The investor's value function is given by

$$
\begin{equation*}
V_{i}^{u n c}\left(W_{t}, v_{t}\right)=\frac{\left(W_{t}+l_{i} /\left(1-l_{A}-l_{B}\right) S_{t}\right)^{1-\gamma_{i}}}{1-\gamma_{i}} h_{i}\left(v_{t}\right)^{\gamma_{i}}, \tag{B3}
\end{equation*}
$$

where $h\left(v_{t}\right)$ is a uniformly bounded wealth-consumption ratio, given by:

$$
\begin{equation*}
h_{i}\left(v_{t}\right)=\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{+\infty}\left(\frac{\xi_{i \tau}}{\xi_{i t}}\right)^{1-1 / \gamma_{i}} e^{-\rho(\tau-t) / \gamma_{i}} \Delta t\right] . \tag{B4}
\end{equation*}
$$

The investor's optimal consumption is given by $c_{i \tau}^{*}=\ell\left(\xi_{i \tau} e^{\rho(\tau-t)}\right)^{-1 / \gamma_{i}}$, where $\ell$ is a constant. Moreover, for all feasible consumptions $c_{t}$ the following inequalities are satisfied:

$$
\begin{gather*}
\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_{i}\left(c_{\tau}\right) \Delta t \leq \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_{i}\left(c_{\tau}^{*}\right) \Delta t=V_{i}^{u n c}\left(W_{t}, v_{t}\right),  \tag{B5}\\
\lim _{T \rightarrow \infty} \sup e^{-\rho T} \mathbb{E}_{t}\left[V_{i}^{u n c}\left(W_{T}, v_{T}\right)\right] \leq 0 \tag{B6}
\end{gather*}
$$

Proof of Lemma B.1. We solve the problem using the martingale method. The static budget constraint is given by:

$$
\begin{equation*}
\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{+\infty} \frac{\xi_{i \tau}}{\xi_{i t}} c_{\tau}^{*}\right]=W_{t}+\frac{l_{i} S_{t}}{1-l_{A}-l_{B}} \tag{B7}
\end{equation*}
$$

where the last term is the value of the labor income. Because the dividends and labor incomes are collinear, the value of the labor income is given by:

$$
\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{+\infty} \frac{\xi_{i \tau}}{\xi_{i t}}\left(l_{i} D_{\tau}\right)\right]=\frac{l_{i} S_{t}}{1-l_{A}-l_{B}}
$$

The first order condition gives the optimal consumption $c_{\tau}^{*}=\ell\left(\xi_{i \tau} / \xi_{i t} e^{\rho(\tau-t)}\right)^{-1 / \gamma_{i}}$, where $\ell$ is the Lagrange multiplier that can be found by substituting $c_{\tau}^{*}$ into (B7). Finding the multiplier $\ell$ and substituting $c_{\tau}^{*}$ into the objective function, we obtain the value function (B3), where $h\left(v_{t}\right)$ is given by (B4).

Next, we show that $h\left(v_{t}\right)$ is uniformly bounded. First, we consider the case $\gamma_{i} \geq 1$. Using equation (B4) and Hölder's inequality, we obtain:
$h_{i}\left(v_{t}\right)=\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{\infty}\left(\frac{\xi_{i \tau}}{\xi_{i t}}\right)^{1-1 / \gamma_{i}} e^{-\rho(\tau-t) / \gamma_{i}}\right] \leq\left(\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{\infty} \frac{\xi_{i \tau} D_{\tau}}{\xi_{i t} D_{t}}\right]\right)^{1-1 / \gamma_{i}}\left(\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{i}}\right]\right)^{1 / \gamma_{i}}$.

We note that both multipliers on the right-hand side of the latter inequality are bounded. The first multiplier equals the price-dividend ratio and is bounded by Proposition 2. The second multiplier is bounded due to condition (16) on the model parameters. Consider now the case $\gamma_{i} \leq 1$. From the FOCs (24) and the fact that $\underline{s} \leq s \leq \bar{s}$, we obtain:

$$
\frac{\xi_{i T}}{\xi_{i t}} \geq e^{-\rho(T-t)}\left(\frac{c_{T}^{*}}{c_{t}^{*}}\right)^{-\gamma_{i}} \geq e^{-\rho(T-t)}\left(\frac{D_{T}}{D_{t}}\right)^{-\gamma_{i}}\left(\frac{\bar{s}}{\underline{s}}\right)^{-\gamma_{i}}
$$

From the latter inequality it follows that

$$
\begin{equation*}
\mathbb{E}_{t}^{i}\left[\left(\frac{\xi_{i \tau}}{\xi_{i t}}\right)^{1-1 / \gamma_{i}} e^{-\rho(\tau-t) / \gamma_{i}}\right] \leq\left(\frac{\bar{s}}{\underline{s}}\right)^{1-\gamma_{i}} \mathbb{E}_{t}^{i}\left[e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{i}}\right] \tag{B8}
\end{equation*}
$$

The inequality (B8) and condition (16) imply that the infinite series in (B4) converges and function $h_{i}(v)$ is uniformly bounded. We also observe that $h_{i}(v) \geq \Delta t>0$.

Now, we prove inequality (B5). We consider feasible consumption streams satisfying condition $W_{t}+l_{i} /\left(1-l_{A}-l_{B}\right) S_{t} \geq 0$ for all $t$, which means that investor's aggregate wealth is non-negative at all times so that investor does not go bankrupt. From the investor's budget constraint and the latter inequality for all feasible consumptions we obtain:

$$
\begin{equation*}
W_{t}+\frac{l_{i} S_{t}}{1-l_{A}-l_{B}} \geq \mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T} \frac{\xi_{i \tau}}{\xi_{i t}} c_{\tau} \Delta t\right]+\mathbb{E}_{t}^{i}\left[\frac{\xi_{i T}}{\xi_{i t}}\left(W_{T}+\frac{l_{i} S_{T}}{1-l_{A}-l_{B}}\right)\right] \geq \mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T} \frac{\xi_{i \tau}}{\xi_{i t}} c_{\tau} \Delta t\right] \tag{B9}
\end{equation*}
$$

Consider the weighting function $w_{t}$ given by

$$
\begin{equation*}
w_{\tau}=\frac{\left(\frac{\xi_{i \tau}}{\xi_{i t}}\right)^{1-1 / \gamma_{i}} e^{-\rho(\tau-t) / \gamma_{i}}}{\widehat{h}_{i T}\left(v_{t}\right)}, \quad \text { where } \widehat{h}_{i T}\left(v_{t}\right)=\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T}\left(\frac{\xi_{i \tau}}{\xi_{i t}}\right)^{1-1 / \gamma_{i}} e^{-\rho(\tau-t) / \gamma_{i}} \Delta t\right] \tag{B10}
\end{equation*}
$$

We note that $\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T} w_{\tau} \Delta t\right]=1$. Using Jensen's inequality and inequality (B9), we obtain:

$$
\begin{align*}
\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T} \frac{e^{-\rho(\tau-t)} c_{\tau}^{1-\gamma_{i}}}{1-\gamma_{i}} \Delta t\right] & =\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T} \frac{\left(\left(\xi_{i \tau} / \xi_{i t}\right)^{1 / \gamma_{i}} e^{\rho(\tau-t) / \gamma_{i}} c_{\tau}\right)^{1-\gamma_{i}} w_{\tau} \Delta t}{1-\gamma_{i}}\right] \widehat{h}_{i T}\left(v_{t}\right) \\
& \leq \frac{\left(\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T}\left(\xi_{i \tau} / \xi_{i t}\right)^{1 / \gamma_{i}} e^{\rho(\tau-t) / \gamma_{i}} c_{\tau} w_{\tau} \Delta t\right]\right)^{1-\gamma_{i}}}{1-\gamma_{i}} \widehat{h}_{i T}\left(v_{t}\right)  \tag{B11}\\
& =\frac{\left(\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{T}\left(\xi_{i \tau} / \xi_{i t}\right) c_{\tau} \Delta t\right]\right)^{1-\gamma_{i}}}{1-\gamma_{i}} \widehat{h}_{i T}\left(v_{t}\right)^{\gamma_{i}} \\
& \leq \frac{\left(W_{t}+\frac{l_{i} S_{t}}{1-l l_{A}-l_{B}}\right)^{1-\gamma_{i}}}{1-\gamma_{i}} \widehat{h}_{i T}\left(v_{t}\right)^{\gamma_{i}} .
\end{align*}
$$

Taking limit $T \rightarrow \infty$ in (B11), and noting that $\widehat{h}_{i T}\left(v_{t}\right) \rightarrow h_{i}\left(v_{t}\right)$, we obtain (B5).
Finally, we prove inequality (B6). Because $c_{\tau} \geq 0$, from inequality (B9), we obtain:

$$
\begin{equation*}
\mathbb{E}_{t}^{i}\left[\frac{\xi_{i T}}{\xi_{i t}}\left(W_{T}+\frac{l_{i} S_{T}}{1-l_{A}-l_{B}}\right)\right] \leq W_{t}+\frac{l_{i} S_{t}}{1-l_{A}-l_{B}} . \tag{B12}
\end{equation*}
$$

Using Jensen's inequality following the same steps as in inequality (B11), we obtain:

$$
\begin{aligned}
\frac{\mathbb{E}_{t}^{i}\left[\left(W_{T}+\frac{l_{i} S_{T}}{1-l_{-}-l_{B}}\right)^{1-\gamma_{i}}\right]}{1-\gamma_{i}} & \leq \frac{\left(\mathbb{E}_{t}^{i}\left[\frac{\xi_{i T}}{\xi_{i t}}\left(W_{T}+\frac{l_{i} S_{T}}{1-l_{A}-l_{B}}\right)\right]\right)^{1-\gamma_{i}}}{1-\gamma_{i}}\left(\mathbb{E}_{t}^{i}\left[\left(\frac{\xi_{i T}}{\xi_{i t}}\right)^{-\frac{1-\gamma_{i}}{\gamma_{i}}}\right]\right)^{\gamma_{i}} \\
& \leq \frac{\left(W_{t}+\frac{l_{i} S_{t}}{1-l_{A}-l_{B}}\right)^{1-\gamma_{i}}}{1-\gamma_{i}}\left(\mathbb{E}_{t}^{i}\left[\left(\frac{\xi_{i T}}{\xi_{i t}}\right)^{-\frac{1-\gamma_{i}}{\gamma_{i}}}\right]\right)^{\gamma_{i}} .
\end{aligned}
$$

The above inequality and the boundedness of $h_{i}\left(v_{t}\right)$ then imply the following inequality:

$$
\begin{equation*}
e^{-\rho(\tau-t)} \mathbb{E}_{t}^{i}\left[V_{i T}^{u n c}\right] \leq \text { Const } \times V_{i t}^{u n c}\left(\mathbb{E}_{t}^{i}\left[\left(\frac{\xi_{i T}}{\xi_{i t}}\right)^{-\frac{1-\gamma_{i}}{\gamma_{i}}} e^{-\rho(\tau-t) / \gamma_{i}}\right]\right)^{\gamma_{i}} \tag{B13}
\end{equation*}
$$

Inequality (B13) also holds for $\gamma_{i}=1$ if CRRA preferences are replaced with logarithmic preferences. Suppose, $\gamma_{i}>1$. Then, inequality (B6) is satisfied because $V_{i}^{u n c}<0$. Suppose, $\gamma_{i} \leq 1$. Then, using inequalities (B8), (B13), and condition (16), we obtain:

$$
e^{-\rho(\tau-t)} \mathbb{E}_{t}^{i}\left[V_{i T}^{u n c}\right] \leq \text { Const } \times\left(\mathbb{E}_{t}^{i}\left[e^{-\rho(\tau-t)}\left(\frac{D_{\tau}}{D_{t}}\right)^{1-\gamma_{i}}\right]\right)^{\gamma_{i}} \rightarrow 0, \text { as } T \rightarrow \infty
$$

Lemma B.2. Let $\mathcal{P}(V)$ be a point-wise monotone operator such that for all point-wise bounded functions $V_{1}$ and $V_{2}$ such that $V_{1} \leq V_{2} \Rightarrow \mathcal{P}\left(V_{1}\right) \leq \mathcal{P}\left(V_{2}\right)$. Suppose further there exist point-wise bounded functions $\underline{V}$ and $\bar{V}$ such that $\underline{V} \leq \bar{V}, \mathcal{P}(\underline{V}) \geq \underline{V}$, and $\mathcal{P}(\bar{V}) \leq \bar{V}$. Then, there exists a point-wise bounded function $V^{*}$ such that: 1) $\left.\underline{V} \leq V^{*} \leq \bar{V} ; 2\right)$ $\left.V^{*} \leq \mathcal{P}\left(V^{*}\right) ; 3\right) \mathcal{P}^{n}(\underline{V}) \rightarrow V^{*}$ point-wise as $n \rightarrow \infty$.

Proof of Lemma B.2. From the monotonicity of the operator $\mathcal{P}(V)$ and the definitions of $V$ and $\bar{V}$, we obtain:

$$
\begin{equation*}
\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}(\bar{V}) \leq \bar{V} . \tag{B14}
\end{equation*}
$$

Applying operator $\mathcal{P}$ to inequalities (B14), and then using the definitions of $V$ and $\bar{V}$, we obtain: $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^{2}(\underline{V}) \leq \bar{V}$. Proceeding in the same way $n$ times we obtain $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^{2}(\underline{V}) \leq \ldots \leq \mathcal{P}^{n}(\underline{V}) \leq \bar{V}$. Consequently, $\mathcal{P}^{n}(\underline{V})$ is point-wise increasing and bounded, and hence converges to some function $V^{*}$ such that $\underline{V} \leq V^{*} \leq \bar{V}$ and $\mathcal{P}^{n}(\underline{V}) \leq V^{*}$. Applying operator to both sides of the latter inequality, we find that $\mathcal{P}^{n+1}(\underline{V}) \leq \mathcal{P}\left(V^{*}\right)$. Taking limit, we find that $V^{*} \leq \mathcal{P}\left(V^{*}\right)$.

Proposition B. 2 (Verification of optimality). Consider an infinitesimal investor $i$ who lives in an economy where the state price density is given by equation (30). Suppose, this investor maximizes expected discounted utility (7) subject to a self-financing budget constraint and the collateral constraint (10). Then, there exists unique bounded value function $V_{i}^{*}$ satisfying the dynamic programming equation (21) and the transversality condition, such that for all feasible consumptions

$$
\begin{equation*}
V_{i t}^{*} \geq \mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{+\infty} u\left(c_{i \tau}\right) \Delta t\right] \tag{B15}
\end{equation*}
$$

and, moreover,

$$
\begin{equation*}
V_{i t}^{*}=\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{+\infty} u\left(c_{i \tau}^{*}\right) \Delta t\right], \tag{B16}
\end{equation*}
$$

for the optimal consumptions given by FOCs (24).
Proof of Proposition B.2. Consider the following operator:

$$
\begin{equation*}
\left.\mathcal{P}_{i}(V)=\max _{c_{t}}\left\{u_{i}\left(c_{t}\right) \Delta t+e^{-\rho \Delta t} \mathbb{E}_{t}^{i}\left[V_{i, t+\Delta t}\right)\right]\right\}, \quad i=A, B \tag{B17}
\end{equation*}
$$

where maximization is subject to budget constraint (22) and collateral constraint (23). Consider the following functions:

$$
\underline{V}_{i t}=\left\{\begin{array}{ll}
0, & \gamma_{i}<1,  \tag{B18}\\
\mathbb{E}_{t}^{i}\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_{i}\left(l_{i} D_{\tau}\right) \Delta t\right], & \gamma_{i} \geq 1,
\end{array} \quad \bar{V}_{i t}= \begin{cases}V_{i t}^{u n c}, & \gamma_{i} \leq 1, \\
0, & \gamma_{i}>1,\end{cases}\right.
$$

where $V_{t}^{u n c}$ is given by (B3).
We observe that for $\gamma_{i} \geq 1$ function $\underline{V}_{i}$ is bounded due to condition (16) imposed on model parameters. Because $c_{t}=l_{i} D_{t}$ is feasible, we obtain that

$$
\mathcal{P}\left(\underline{V}_{i}\right) \geq u_{i}\left(l_{i} D_{t}\right)+e^{-\rho \Delta t} \mathbb{E}_{t}^{i}\left[\sum_{\tau=t+\Delta t}^{+\infty} e^{-\rho(\tau-t)} u_{i}\left(l_{i} D_{t}\right) \Delta t\right]=\underline{V}_{i} .
$$

For $\gamma_{i}<1$ it is easy to see that $\mathcal{P}\left(\underline{V}_{i}\right) \geq \underline{V}_{i}$ because $u_{i}(c)>0$. Next, we prove that $\mathcal{P}_{i}\left(\bar{V}_{i}\right) \leq \bar{V}_{i}$. The latter inequality is straightforward for $\gamma_{i}>1$ because $\mathcal{P}_{i}(0) \leq 0$. Suppose now, $\gamma_{i} \leq 1$. Consider operator $\widetilde{\mathcal{P}}_{i}\left(V_{i}\right)$ given by equation (B17), where the maximization is subject to the budget constraint (22), but without the collateral constraint (23). Hence, $\mathcal{P}_{i}\left(V_{i}\right) \leq \widetilde{\mathcal{P}}_{i}\left(V_{i}\right)$. By Lemma B.1, $\bar{V}_{i}^{\text {unc }}$ is the solution of the unconstrained optimization, and hence $\bar{V}_{i}=\widetilde{\mathcal{P}}_{i}\left(\bar{V}_{i}\right)$. Therefore, $\mathcal{P}_{i}\left(\bar{V}_{i}\right) \leq \widetilde{\mathcal{P}}_{i}\left(\bar{V}_{i}\right)=\bar{V}_{i}$.

We drop subscript and superscript $i$ for convenience. Consider the sequence $V_{n+1}=$ $\mathcal{P}\left(V_{n}\right)$, with $V_{0}=\underline{V}$, where $\underline{V}$ is given in (B18). Then, by Lemma B.2, $V_{n} \rightarrow V^{*}$ pointwise as $n \rightarrow \infty$. Next, we show that $V^{*}$ is the value function and $\mathcal{P}\left(V^{*}\right)=V^{*}$. By the definition operator $\mathcal{P}(V)$ in (B17), for all feasible consumption streams

$$
\begin{align*}
V_{n+1} & \geq u\left(c_{t}\right) \Delta t+e^{-\rho \Delta t} \mathbb{E}_{t}\left[V_{n}\left(W_{t+\Delta t} ; v_{t+\Delta t}\right)\right] \\
& \geq \mathbb{E}_{t}\left[\sum_{\tau=t}^{n \Delta t} e^{-\rho(\tau-t)} u\left(c_{\tau}\right) \Delta t\right]+e^{-\rho n \Delta t} \mathbb{E}_{t}[\underline{V}] . \tag{B19}
\end{align*}
$$

Taking point-wise limit $n \rightarrow \infty$ in (B19) and taking into account that $\mathbb{E}_{t}[\underline{V}]$ is point-wise bounded, we obtain inequality (B15).

By Lemma B. $2, V^{*} \leq \mathcal{P}\left(V^{*}\right)$ and $V^{*} \leq \bar{V}$, where $\bar{V}$ is given in (B18), and hence

$$
\begin{align*}
V^{*}\left(W_{t}, v_{t}\right) & \leq u\left(c_{t}^{*}\right) \Delta t+e^{-\rho \Delta t} \mathbb{E}_{t}\left[V^{*}\left(W_{t+\Delta t} ; v_{t+\Delta t}\right)\right] \\
& \leq \mathbb{E}_{t}\left[\sum_{\tau=t}^{T} u\left(c_{\tau}^{*}\right) \Delta t\right]+e^{-\rho T} \mathbb{E}_{t}\left[V^{*}\left(W_{T}, v_{T}\right)\right]  \tag{B20}\\
& \leq \mathbb{E}_{t}\left[\sum_{\tau=t}^{T} u\left(c_{\tau}^{*}\right) \Delta t\right]+e^{-\rho T} \mathbb{E}_{t}\left[\bar{V}\left(W_{T}, v_{T}\right)\right]
\end{align*}
$$

where $c^{*}$ is the optimal consumption that solves optmization in equation (B17).
We note that $\bar{V}=0$ for $\gamma>1$ and $\lim \sup e^{-\rho T} \mathbb{E}_{t}\left[\bar{V}\left(W_{T}, v_{T}\right)\right] \leq 0$ as $T \rightarrow \infty$ for $\gamma \leq 1$, by Lemma B.1. Taking limit $T \rightarrow \infty$ in (B20) we find that $V^{*} \leq \mathbb{E}_{t}\left[\sum_{\tau=t}^{+\infty} u\left(c_{\tau}^{*}\right) \Delta t\right]$, which along with inequality (B15) yields (B16). Equation (B16) along with inequality (B20) also imply that $V^{*}=\mathcal{P}\left(V^{*}\right)$. Moreover, $V^{*}$ is point-wise bounded because $\underline{V} \leq V^{*} \leq$ $\bar{V}$. Then, given the existence of the value function, the optimal consumptions are given by (24). Finally, we show that $V^{*}$ satisfies the transversality condition. We note that $e^{-\rho(T-t)} \mathbb{E}_{t}\left[\underline{V}_{T}\right] \leq e^{-\rho(T-t)} \mathbb{E}_{t}\left[V_{T}^{*}\right] \leq e^{-\rho(T-t)} \mathbb{E}_{t}\left[\bar{V}_{T}\right]$. Taking limit $T \rightarrow 0$ we find that the upper and lower bound in the latter equation converge to 0 , and hence the transversality condition is satisfied for $V^{*}$.

## Proposition B. 3 (Closed-form solutions).

1) In the limit $\Delta t \rightarrow 0$ the price-dividend ratio $\Psi$ and wealth-consumption ratios $\Phi_{i}$ are given by equations (32) and (33), where function $\widehat{\Psi}(v ; \theta)$ is given by:

$$
\begin{equation*}
\widehat{\Psi}(v ; \theta)=\int_{\underline{v}}^{v} s(y)^{\theta} \widehat{\psi}(v-y) d y+\frac{\int_{\underline{v}}^{\bar{v}} s(y)^{\theta}\left[\widehat{\psi}^{\prime}(\bar{v}-y)-\widehat{\psi}(\bar{v}-y)\right] d y}{1+H\left(\widehat{\psi}(\bar{v}-\underline{v})-\int_{0}^{\bar{v}-\underline{v}} \widehat{\psi}(y) d y\right)}\left(1-H \int_{0}^{v-\underline{v}} \widehat{\psi}(y) d y\right), \tag{B21}
\end{equation*}
$$

where $s(y)$ solves equation ${ }^{10}(15)$, and $\widehat{\psi}(x), H$ and some auxiliary variables are given by:

$$
\begin{align*}
& \widehat{\psi}(x)=\frac{2}{\widehat{\sigma}_{v}^{2}} \sum_{n=0}^{\infty}\left[\left(\frac{2 \lambda\left(1+J_{D}\right)^{1-\gamma_{A}}}{\widehat{\sigma}_{v}^{2}}\right)^{n} \frac{\exp \left(\left(\zeta_{+}+\zeta_{-}\right)\left(x+n \widehat{J}_{v}\right) / 2\right)}{\left(\zeta_{+}-\zeta_{-}\right)^{2 n+1} n!}\right.  \tag{B22}\\
& \left.\times Q_{n}\left(\frac{\left(\zeta_{+}-\zeta_{-}\right)\left(x+n \widehat{J}_{v}\right)}{2}\right) \mathbf{1}_{\left\{x+n \widehat{J}_{v} \geq 0\right\}}\right],  \tag{B23}\\
& Q_{n}(x)=\exp (-x) \sum_{m=0}^{n}(2 x)^{n-m} \frac{(n+m)!}{m!(n-m)!}-\exp (x) \sum_{m=0}^{n}(-2 x)^{n-m} \frac{(n+m)!}{m!(n-m)!},  \tag{B24}\\
& H=\lambda+\rho-\left(1-\gamma_{A}\right) \mu_{D}+\frac{\left(1-\gamma_{A}\right) \gamma_{A}}{2} \sigma_{D}^{2}-\lambda\left(1+J_{D}\right)^{1-\gamma_{A}},  \tag{B25}\\
& \zeta_{ \pm}=-\frac{\widehat{\mu}_{v}+\left(1-\gamma_{A}\right) \widehat{\sigma}_{v} \sigma_{D} \mp \sqrt{\left(\widehat{\mu}_{v}+\left(1-\gamma_{A}\right) \widehat{\sigma}_{v} \sigma_{D}\right)^{2}+2 \widehat{\sigma}_{v}^{2}\left(\lambda+\rho-\left(1-\gamma_{A}\right) \mu_{D}+\frac{\left(1-\gamma_{A}\right) \gamma_{A}}{2} \sigma_{D}^{2}\right)}}{\widehat{\sigma}_{v}^{2}} . \tag{B26}
\end{align*}
$$

2) Stock return volatility in normal times and the jump size $J_{t}$ are given by:

$$
\begin{align*}
\sigma_{t} & =\sigma_{D}+\left(\frac{\widehat{\Psi}^{\prime}\left(v_{t} ;-\gamma_{A}\right)}{\widehat{\Psi}\left(v_{t} ;-\gamma_{A}\right)}-\frac{\gamma_{A}\left(1-s\left(v_{t}\right)\right)}{\gamma_{A}\left(1-s\left(v_{t}\right)\right)+\gamma_{B} s\left(v_{t}\right)}\right) \widehat{\sigma}_{v},  \tag{B27}\\
J_{t} & =\frac{\left(1+J_{D}\right) \widehat{\Psi}\left(\max \left\{\underline{v} ; v_{t}+\widehat{J}_{v}\right\} ;-\gamma_{A}\right) s\left(\max \left\{\underline{v} ; v_{t}+\widehat{J}_{v}\right\}\right)^{\gamma_{A}}}{\widehat{\Psi}\left(v_{t} ;-\gamma_{A}\right) s\left(v_{t}\right)^{\gamma_{A}}}-1 . \tag{B28}
\end{align*}
$$

Numbers of shares $n_{i, s t}^{*}$ and leverage $L_{i t}=-b_{i t} B_{i t}$ to market price $S_{t}$ ratio are given by:

$$
\begin{equation*}
n_{i, S t}^{*}=\frac{\Phi_{i}\left(v_{t}\right) \sigma_{D}+\Phi_{i}^{\prime}\left(v_{t}\right) \widehat{\sigma}_{v}}{\Psi\left(v_{t}\right) \sigma_{t}}, \quad \frac{L_{i t}}{S_{t}}=n_{i, S t}-\frac{\Phi_{i}\left(v_{t}\right)}{\Psi\left(v_{t}\right)\left(1-l_{A}-l_{B}\right)} . \tag{B29}
\end{equation*}
$$

Proof of Proposition B.3. 1) First, we solve the differential-difference equation in Lemma 2. We denote $g(x)=\widehat{\Psi}(x+\underline{v} ; \theta)$ and apply the following changes of variables:

$$
\begin{align*}
& x=v-\underline{v}, \quad \tilde{\sigma}=\widehat{\sigma}_{v}, \quad \tilde{\mu}=\widehat{\mu}_{v}+\left(1-\gamma_{A}\right) \sigma_{D} \widehat{\sigma}_{v}, \quad \tilde{J}=-\widehat{J}_{v}, \quad \tilde{\lambda}=\lambda\left(1+J_{D}\right)^{1-\gamma_{A}}, \\
& \tilde{\rho}=\lambda+\rho-\left(1-\gamma_{A}\right) \mu_{D}+\frac{\left(1-\gamma_{A}\right) \gamma_{A}}{2} \sigma_{D}^{2} \tag{B30}
\end{align*}
$$

Equations (34) and (35) with new variables now become:

$$
\begin{equation*}
\frac{\tilde{\sigma}^{2}}{2} g^{\prime \prime}(x)+\tilde{\mu} g^{\prime}(x)-\tilde{\rho} g(x)+\tilde{\lambda} g(\max \{x-\tilde{J}, 0\})+s(x+\underline{v})^{\theta}=0 \tag{B31}
\end{equation*}
$$

[^9]\[

$$
\begin{equation*}
g^{\prime}(0)=0, \quad g(\bar{v}-\underline{v})-g^{\prime}(\bar{v}-\underline{v})=0 . \tag{B32}
\end{equation*}
$$

\]

Let $\mathcal{L}[g(x)]=\int_{0}^{\infty} e^{-z x} g(x) d x$ be the Laplace transform of $g(x)$, and similarly for other functions. The Laplace transforms of $g^{\prime}(x), g^{\prime \prime}(x)$ and $g(\max \{x-\tilde{J}, 0\})$ are given by:

$$
\begin{align*}
\mathcal{L}\left[g^{\prime}(x)\right] & =z \mathcal{L}[g(x)]-g(0), \\
\mathcal{L}\left[g^{\prime \prime}(x)\right] & =z^{2} \mathcal{L}[g(x)]-z g(0)-g^{\prime}(0), \\
\mathcal{L}[g(\max \{x-\tilde{J}, 0\})] & =\int_{0}^{\infty} e^{-z x} g(\max \{x-\tilde{J}, 0\}) d x  \tag{B33}\\
& =\int_{0}^{\tilde{J}} e^{-z x} g(0) d x+\int_{\tilde{J}}^{\infty} e^{-z x} g(x-\tilde{J}) d x \\
& =\frac{1}{z}\left(1-e^{-\tilde{J} z}\right) g(0)+e^{-\tilde{J} z} \mathcal{L}[g(x)] .
\end{align*}
$$

Applying the transform to equation (B31), we arrive at the following equation:

$$
\begin{align*}
\frac{\tilde{\sigma}^{2}}{2}\left(z^{2} \mathcal{L}[g(x)]-z g(0)-g^{\prime}(0)\right) & +\tilde{\mu}(z \mathcal{L}[g(x)]-g(0))-\tilde{\rho} \mathcal{L}[g(x)] \\
& +\tilde{\lambda}\left(e^{-\tilde{J} z} \mathcal{L}[g(x)]+\frac{1}{z}\left(1-e^{-\tilde{J}_{z}}\right) g(0)\right)+\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]=0 \tag{B34}
\end{align*}
$$

Applying boundary condition $g^{\prime}(0)=0$ and solving for $\mathcal{L}[g(x)]$, we obtain:

$$
\begin{equation*}
\mathcal{L}[g(x)]=\frac{\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}-\tilde{\lambda} e^{-\tilde{J} z}}+g(0)\left(\frac{1}{z}-\frac{\tilde{\rho}-\tilde{\lambda}}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}-\tilde{\lambda} e^{-\tilde{J} z}} \cdot \frac{1}{z}\right) . \tag{B35}
\end{equation*}
$$

Now define a new function $\widehat{\psi}(x)$ through inverse Laplace transform

$$
\begin{equation*}
\widehat{\psi}(x)=\mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}-\tilde{\lambda} e^{-\tilde{J} z}}\right] \tag{B36}
\end{equation*}
$$

Next, we apply inverse transform to each term in (B35). Noting that $\mathcal{L}^{-1}[1 / z]=1$ and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

$$
\begin{gather*}
\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}-\tilde{\lambda} e^{-\tilde{J} z}}\right]=\int_{0}^{x} s(y+\underline{v})^{\theta} \cdot \widehat{\psi}(x-y) d y  \tag{B37}\\
\mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}-\tilde{\lambda} e^{-\tilde{J} z}} \cdot \frac{1}{z}\right]=\int_{0}^{x} 1_{\{y \geq 0\}} \cdot \widehat{\psi}(x-y) d y=\int_{0}^{x} \widehat{\psi}(y) d y .
\end{gather*}
$$

The linearity of the Laplace transform gives the following equation:

$$
\begin{equation*}
g(x)=\mathcal{L}^{-1}[\mathcal{L}[g(x)]]=\int_{0}^{x} s(y+\underline{v})^{\theta} \cdot \widehat{\psi}(x-y) d y+g(0)\left[1-(\tilde{\rho}-\tilde{\lambda}) \int_{0}^{x} \widehat{\psi}(y) d y\right] . \tag{B38}
\end{equation*}
$$

We calculate $g(0)$ below, and then after changing the variable back from $x$ to $v=x+\underline{v}$, substituting in expressions for $\tilde{\rho}$ and $\tilde{\lambda}$ from (B30), we obtain (B21).

Next, we solve for $\widehat{\psi}(x)$ in closed form. We expand $\mathcal{L}[\widehat{\psi}(x)]$ as series, and sum up the inverse transforms of each term in the summation to get $\widehat{\psi}(x)$.

$$
\begin{align*}
\mathcal{L}[\widehat{\psi}(x)] & =\frac{1}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}-\tilde{\lambda} e^{-\tilde{J} z}} \\
& =\left(\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}\right)^{-1} \cdot\left(1-\frac{\tilde{\lambda} e^{-\tilde{J} z}}{\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}}\right)^{-1}  \tag{B39}\\
& =\sum_{n=0}^{\infty} \frac{\tilde{\lambda}^{n} e^{-n \tilde{J} z}}{\left(\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}\right)^{n+1}} .
\end{align*}
$$

The above series converges for $z$ such that $\left|\tilde{\rho}-\tilde{\mu} z-\left(\tilde{\sigma}^{2} / 2\right) z^{2}\right|>|\tilde{\lambda} \exp (-\tilde{J} z)|$. This holds if the real part of $z$ is sufficiently large, e.g., $\Re(z)>4|\tilde{\mu}| / \tilde{\sigma}^{2}+(2 / \tilde{\sigma}) \sqrt{\tilde{\rho}+\tilde{\lambda}}$. The inverse Laplace transform can then be calculated along the line $(\bar{z}-i \infty, \bar{z}+i \infty)$ in the complex domain where $\bar{z}>4|\tilde{\mu}| / \tilde{\sigma}^{2}+(2 / \tilde{\sigma}) \sqrt{\tilde{\rho}+\tilde{\lambda}}$, and hence, the inequality for $\Re(z)$ is satisfied.

Let $\zeta_{-}<\zeta_{+}$be roots of $\tilde{\rho}-\tilde{\mu} z-\tilde{\sigma}^{2} z^{2} / 2=0$, given by (B26). We use the following inversion formula for $1 /\left[\left(z-\zeta_{+}\right)\left(z-\zeta_{-}\right)\right]^{n+1}$ from Gradshteyn and Ryzhik (2007, p. 1117):

$$
\begin{equation*}
\mathcal{L}^{-1}\left[\frac{1}{\left[\left(z-\zeta_{+}\right)\left(z-\zeta_{-}\right)\right]^{n+1}}\right]=\frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{x^{n+\frac{1}{2}}}{\left(\zeta_{+}-\zeta_{-}\right)^{n+\frac{1}{2}}} e^{\frac{\zeta_{+}+\zeta_{-} x}{2} x} I_{n+\frac{1}{2}}\left(\frac{\zeta_{+}-\zeta_{-}}{2} x\right) \tag{B40}
\end{equation*}
$$

Function $e^{-n \tilde{J} z}$ in the complex domain corresponds to a shift from $x$ to $x-n \tilde{J}$. Therefore,

$$
\begin{align*}
& \mathcal{L}^{-1}\left[\frac{\tilde{\lambda}^{n} e^{-n \tilde{J} z}}{\left(\tilde{\rho}-\tilde{\mu} z-\frac{\tilde{\sigma}^{2}}{2} z^{2}\right)^{n+1}}\right]=\tilde{\lambda}^{n}\left(-\frac{\tilde{\sigma}^{2}}{2}\right)^{-n-1} \mathbf{1}_{x \geq n \tilde{J}}  \tag{B41}\\
& \quad \times \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{(x-n \tilde{J})^{n+\frac{1}{2}}}{\left(\zeta_{+}-\zeta_{-}\right)^{n+\frac{1}{2}}} e^{\zeta_{+}+\zeta_{-}(x-n \tilde{J})} I_{n+\frac{1}{2}}^{2}\left(\frac{\left(\zeta_{+}-\zeta_{-}\right)(x-n \tilde{J})}{2}\right) .
\end{align*}
$$

Consequently, the explicit expression for $\widehat{\psi}(x)$ is given by:

$$
\begin{equation*}
\widehat{\psi}(x)=\sum_{n=0}^{\infty} \tilde{\lambda}^{n}\left(-\frac{\tilde{\sigma}^{2}}{2}\right)^{-n-1} \frac{\mathbf{1}_{\{x \geq n \tilde{j}\}} \sqrt{\pi}}{\Gamma(n+1)} \frac{(x-n \tilde{J})^{n+\frac{1}{2}}}{\left(\zeta_{+}-\zeta_{-}\right)^{n+\frac{1}{2}}} e^{\frac{\zeta_{+}+\zeta_{-}}{2}(x-n \tilde{J})} I_{n+\frac{1}{2}}\left(\frac{\left(\zeta_{+}-\zeta_{-}\right)(x-n \tilde{J})}{2}\right), \tag{B42}
\end{equation*}
$$

where function $I_{n+\frac{1}{2}}(\cdot)$ is a modified Bessel function of the first kind, $\zeta_{-}<\zeta_{+}$are given by (B26) and $\tilde{\rho}, \tilde{\mu}, \tilde{\sigma}, \tilde{\lambda}$, and $\tilde{J}$ are defined in (B30). Bessel function $I_{n+\frac{1}{2}}(\cdot)$ is given by (see
equation 8.467 in Gradshteyn and Ryzhik (2007)):

$$
\begin{equation*}
I_{n+\frac{1}{2}}(z)=\frac{1}{\sqrt{2 \pi z}}\left[e^{z} \sum_{m=0}^{n} \frac{(-1)^{m}(n+m)!}{m!(n-m)!(2 z)^{m}}+(-1)^{n+1} e^{-z} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!(2 z)^{m}}\right] . \tag{B43}
\end{equation*}
$$

Substituting (B43) into (B42), after minor algebra, we obtain expression (B23) for $\widehat{\psi}(x)$. The infinite series (B42) has only finite number of non-zero terms because for a fixed $x$ indicators $1_{\{x \geq n \tilde{J}\}}$ vanish for sufficiently large $n$, and hence, (B42) is well-defined.

To find $g(0)$ in equation (B38), we first evaluate $\widehat{\psi}(0)$. From the above formula (B42), because $1_{\{0 \geq n \tilde{J}\}}=0$ for all $n>0$, we obtain

$$
\begin{equation*}
\widehat{\psi}(0)=-\frac{2}{\tilde{\sigma}^{2}} \cdot \frac{e^{\zeta_{+} \cdot 0}-e^{\zeta_{-} \cdot 0}}{\zeta_{+}-\zeta_{-}}=0 \tag{B44}
\end{equation*}
$$

Differentiating (B38) and using $\widehat{\psi}(0)=0$, we find:

$$
\begin{equation*}
g^{\prime}(x)=\int_{0}^{x} s(y+\underline{v})^{\theta} \cdot \widehat{\psi}^{\prime}(x-y) d y-g(0) \cdot(\tilde{\rho}-\tilde{\lambda}) \widehat{\psi}(x), \tag{B45}
\end{equation*}
$$

We solve for $g(0)$ from the boundary condition $g(\bar{v}-\underline{v})-g^{\prime}(\bar{v}-\underline{v})=0$ and obtain:

$$
\begin{equation*}
g(0)=\frac{\int_{0}^{\bar{v}-\underline{v}} s(y+\underline{v})^{\theta} \cdot\left[\widehat{\psi}^{\prime}(\bar{v}-\underline{v}-y)-\widehat{\psi}(\bar{v}-\underline{v}-y)\right] d y}{1-(\tilde{\rho}-\tilde{\lambda}) \int_{0}^{\bar{v}-\underline{v}} \widehat{\psi}(y) d y+(\tilde{\rho}-\tilde{\lambda}) \widehat{\psi}(\bar{v}-\underline{v})} . \tag{B46}
\end{equation*}
$$

Substituting (B46) into (B38), we derive equation (B21) for $\widehat{\Psi}(v ; \theta)$.
2) Next we solve for stock volatility and jump size. In the unconstrained region $\underline{v}<v_{t}<\bar{v}$, stock price $S_{t}$, dividend $D_{t}$ and state variable $v_{t}$ follow processes:

$$
\begin{align*}
d S_{t} & =S_{t}\left[\mu_{t} d_{t}+\sigma_{t} d w_{t}+J_{t} d j_{t}\right] \\
d D_{t} & =D_{t}\left[\mu_{D} d_{t}+\sigma_{D} d w_{t}+J_{D} d j_{t}\right]  \tag{B47}\\
d v_{t} & =\widehat{\mu}_{v} d t+\widehat{\sigma}_{v} d w_{t}+\left(\max \left\{\underline{v} ; v_{t}+\widehat{J}_{v}\right\}-v_{t}\right) d j_{t}
\end{align*}
$$

Applying Ito's lemma to $S_{t}=\left(1-l_{A}-l_{B}\right) \widehat{\Psi}\left(v_{t} ;-\gamma_{A}\right) s\left(v_{t}\right)^{\gamma_{A}} D_{t}$, and matching $d w_{t}$ and $d j_{t}$ terms, after some algebra, we obtain $\sigma_{t}$ and $J_{t}$ in Proposition B.3.

Equation equation (9) for $W_{i, t+\Delta t}$, implies the following expressions for $n_{i, s t}^{*}$ and $b_{i t}^{*}$ :

$$
\begin{aligned}
n_{i, s t}^{*} & =\sqrt{\frac{\operatorname{var}_{t}\left[W_{i, t+\Delta t}-W_{i t} \mid \text { normal }\right]}{\operatorname{var}_{t}\left[\Delta S_{t}+\left(1-l_{A}-l_{B}\right) D_{t} \Delta t \mid \text { normal }\right]}} \\
b_{i t}^{*} & =\mathbb{E}_{t}\left[W_{i, t+\Delta t \mid} \text { normal }\right]-n_{i t} \mathbb{E}_{t}\left[S_{t+\Delta t}+\left(1-l_{A}-l_{B}\right) D_{t+\Delta t} \Delta t \mid \text { normal }\right] .
\end{aligned}
$$

Taking limit $\Delta t \rightarrow 0$ in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in equation (B29).


[^0]:    *Contacts: G.Chabakauri@lse.ac.uk, Y.Han7@lse.ac.uk. We are grateful to Ulf Axelson, Jaroslav Borovička, Bernard Dumas, David Easley, Peter Kondor, Tao Li, Hanno Lustig, Igor Makarov, Ian Martin, Kjell Nyborg, Jean-Charles Rochet, Christoph Roling, Andres Schneider, Raman Uppal, Dimitri Vayanos, Pietro Veronesi, Grigory Vilkov, Mindy Zhang, Alexandre Ziegler and seminar participants at the Adam Smith Workshop in Asset Pricing, China International Conference in Finance (Hangzhou), Copenhagen Business School, European Finance Association (Oslo), European Winter Finance Summit (Davos), FIRS (Lisbon), Frankfurt School of Finance and Management, IE Business School, London School of Economics, Paris December Finance Meeting, SFS Cavalcade (Toronto), University of New South Wales, University of Sydney, University of Technology Sydney, University of Zurich, and Western Finance Association (San Diego) for helpful comments. All errors are our responsibility. We are grateful to Paul Woolley Centre at the LSE for financial support. This paper was previously entitled "Capital Requirements and Asset Prices".

[^1]:    ${ }^{1}$ Chabakauri (2014) shows that process (1) converges to a continuous-time Lévy process as $\Delta t \rightarrow 0$.

[^2]:    ${ }^{2}$ Assuming that the less risk averse investor is more optimistic makes our main state variable $s_{t}=$ $c_{A t}^{*} / D_{t}$ (introduced in Section 2.3 below) countercyclical, which facilitates the analysis of the results. If this assumption is relaxed, the qualitative results remain the same, but additional analysis is required to determine whether the state variable $s$ is counter- or pro-cyclical.

[^3]:    ${ }^{3}$ The constraint (11) is obtained by substituting bond holding $b_{i t}=\left(W_{i t}+\left(l_{i} D_{t}-c_{i t}\right) \Delta t-\right.$ $\left.n_{i t}\left(S_{t}, P_{t}\right)^{\top}\right) / B_{t}$ from equation (8) into equation (9) for wealth $W_{i, t+\Delta t}$, and then rearranging term in the inequality $W_{i, t+\Delta t} \geq 0$.

[^4]:    ${ }^{4}$ The proof of existence of the SPD in arbitrage-free economies can be found in Duffie (2001, p.4).
    ${ }^{5}$ Three equations (17)-(19) can be rewritten as equations for three unknowns $\pi_{i}\left(\omega_{k}\right) \xi_{i, t+\Delta t}\left(\omega_{k}\right) / \xi_{i t}$, where $k=1,2,3$ and $i$ is set to either $A$ or $B$. The solution of these equations is unique when the matrix of asset payoffs is invertible, and hence, $\pi_{B}\left(\omega_{t+\Delta t}\right) \xi_{B, t+\Delta t} / \xi_{B t}=\pi_{A}\left(\omega_{t+\Delta t}\right) \xi_{A, t+\Delta t} / \xi_{A t}$ for all states.

[^5]:    ${ }^{6}$ However, in contrast to Fostel and Geanakoplos (2008), the disagreement about the consumption growth dynamics in our economy does not increase during these periods.

[^6]:    ${ }^{7}$ Drift $\mu_{D}$ and volatility $\sigma_{D}$ are within the ranges considered in the literature (e.g., Basak and Cuoco, 1998; Chan and Kogan, 2002; Rytchkov, 2014), intensity $\lambda=0.017$ is from Barro (2009).

[^7]:    ${ }^{8}$ To avoid finding bounds $\underline{s}$ and $\bar{s}$ numerically, we set them exogenously to $\underline{s}=0.1$ and $\bar{s}=0.9$ and then recover the shares of labor incomes $l_{A}=0.123$ and $l_{B}=0.14$ that imply these bounds in equilibrium. First, we find $\underline{v}$ and $\bar{v}$ from equation (15) for $v$, and then find $l_{A}$ and $l_{B}$ from equations (39).

[^8]:    ${ }^{9}$ We exclude the singularities in the dynamics of $r_{t}$ and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.

[^9]:    ${ }^{10}$ Although $s(y)$ is not in closed form, we observe from equation (15) that its inverse is given by $s^{-1}(x)=$ $\gamma_{B} \ln (x)-\gamma_{A} \ln (1-x)$. The change of variable $x=s(y)$ eliminates implicit functions, similar to Chabakauri (2015). We keep all integrals in terms of $s(y)$ because $s(y)$ is intuitive and easily computable from (15).

