The Analytic Theory of
a Monetary Shock*

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Abstract

We propose a new method to analyze the propagation of a once and for all shock in a broad class of sticky price models. The method is based on the eigenvalue-eigenfunction representation of the cross-sectional process for price adjustments and provides a thorough characterization of the entire impulse response function, for any moment of interest, in response to both small as well as large shocks. We use the method to discuss (i) a general analytic characterization of the "selection effect" in sticky-price models, (ii) a parsimonious representation of the impulse response function and its shape, (iii) the propagation of monetary shocks after a change in volatility, and (iv) multiproduct models. We conclude by extending the method to models featuring asymmetric return functions and asymmetric law of motion of the state.

JEL Classification Numbers: E3, E5

Key Words: Menu costs, Impulse response, Dominant Eigenvalue, Selection, Volatility.

1 Introduction

Economists are often faced with the analysis of dynamic high dimensional objects, such as cross sectional distributions of incomes, assets, markups, and other economic variables of interest. This is the case for instance when studying impulse response functions, resulting from the dynamics of selected moments computed on the distribution of interest. We present a powerful method for such analyses, which typically require solving the partial differential equation that characterizes the evolution of the distribution of interest. The method is the eigenvalue-eigenfunction decomposition that allows to solve the partial differential equation through a neat separation of the time-dimension from the state-dimension, providing a tractable solution to a non-trivial problem. Few recent papers in economics have applied such method to study the transition of the cross-section distribution of incomes, Gabaix, Lasry, Lions, and Moll (2016), and to analyze asset pricing for long term risk, Hansen and Scheinkman (2009).[1]

We apply this method to analytically compute the entire impulse response function to a once and for all monetary shock in a broad class of sticky price models including versions of Taylor (1980), Calvo (1983), Golosov and Lucas (2007), a version of the "CalvoPlus" model by Nakamura and Steinsson (2010), as well as the multi-product models of Midrigan (2011), Bhattarai and Schoenle (2014) and Alvarez and Lippi (2014), and the model with "price-plans" as in Eichenbaum, Jaimovich, and Rebelo (2011) and Alvarez and Lippi (2019). In these models firms are hit by idiosyncratic shocks and face a price setting problem featuring (possibly random) menu costs, as well as "price plans" (i.e. the possibility of choosing 2 prices instead of a single one upon resetting). As in most of the general equilibrium literature on the topic we abstract from strategic complementarities to retain tractability. These problems are typically computationally intensive and numerical solutions may hinder a clear understanding of the mechanism at work. The approach we propose greatly facilitates the solution of such

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[1] See also Caballero (1993) for an early use eigenvalue-eigenfunctions to analyze the dynamics of a cross-section distribution and Krieger (2002) for a related early attempt to represent the dynamics of a high-dimensional state in macroeconomic models as *eigenstates*.

models, which in many cases has a simple-to-derive analytic form, while at the same time unveiling the key forces and deep parameters behind the results.

Our method delivers analytical representation of the *whole profile* of the impulse response function, as opposed to previous analytic results on the impact effect of shocks, such as Caballero and Engel (2007), or analytic results on the cumulated impulse response to shocks, such as Alvarez, Le Bihan, and Lippi (2016) and Baley and Blanco (2019a). The results also provide straightforward characterizations for several features of interest for a large class of sticky price models, such as duration analysis and the dynamics of *any* moment of interest after an aggregate shock. Moreover the analysis applies to *shocks of any size* as well as to *shocks to higher moments*, such as uncertainty shocks, differently from previous analytic investigations focusing mostly on approximations for small monetary shocks, as in e.g. Gertler and Leahy (2008); Alvarez, Le Bihan, and Lippi (2016); Baley and Blanco (2019b). After presenting the setup of the analysis and our main result in Section 3, we illustrate the power of the method by discussing four substantive economic applications.

First, in Section 4, we provide an analytic characterization of the "selection effect", which is one of the main reason why different sticky price models yield different real effects. The selection effect, first discussed by Golosov and Lucas, refers to the fact that the prices that adjust following a monetary shock are those of a selected group of firms. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Calvo model. We present an analytic result showing how the selection effect creates a wedge between the duration of price spells and the duration of the aggregate output response. The two durations coincide when there is no selection. We show that such a wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of aggregate output.

Second, Section 5 discusses the possibility to obtain a parsimonious approximate characterization of the impulse response function by using selected eigenvalues. The question arises since in Hansen and Scheinkman (2009) and Gabaix et al. (2016) the dominant eigenvalue is a convenient and accurate description of an otherwise complicated infinitely dimensional object. It is thus natural to ask whether a single eigenvalue might be found to represent an approximate impulse response function. We present several results. We show that the dominant eigenvalue, which characterizes the asymptotic behaviour of the income distribution in Gabaix et al. (2016), gives the asymptotic hazard rate of price changes in our class of sticky price models. Yet the impulse response of output, including its asymptotic behavior, is unrelated to the dominant eigenvalue. This is because the output's IRF depends on the difference between price increases and decreases, as opposed to the hazard rate of any price change, i.e. the hazard rate of either increases or decreases. Indeed, as we consider models with "less selection" on the price changes –say moving from the menu cost model of Golosov and Lucas towards Calvo pricing, or increasing the number of products in the multi product model–, the dominant and the second eigenvalues get closer to each other. We show that in general it is not possible to summarize the impulse response function, nor its asymptotic behavior, with a single eigenvalue-eigenfunction pair. As a concrete example of a case where no single eigenvalue can be used to characterize the IRF we discuss a class of sticky price models that gives rise to a hump-shaped impulse response function.[2] We provide analytic conditions for the hump-shaped impulse response to arise. Similar results are established for multiproduct models. We show, however, that an interesting special case exists: the canonical menu cost model with 1 good and no price plans can be effectively summarized by a single eigenfunction.

Third, in Section 6, we study how the propagation of monetary shocks is affected by the volatility of shocks faced by firms. Bloom (2009) analyzed the macroeconomic impact of uncertainty shocks, and other scholars have shown that recessions are times in which the

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[2]This hump-shaped behavior emerges in economies where the price setting technology features "price plans" and some randomness in the cost of price changes (weak selection).

volatility of shocks is higher. We use our method to analyze whether monetary policy is more or less powerful in recessions, interpreted as times in which volatility is high. The question relates to the recent quantitative investigation by Vavra (2014). We show that the answer crucially depends on the time elapsed since the volatility shock occurred: the propagation of a monetary shock that occurs together with the volatility shock differs substantively from the propagation of a shock that occurs a long time after the new volatility is in place. The reason is that in the first case, which we label the "short run", the new volatility immediately affects the firm's decision rule, but the cross sectional distribution of firms is still the old invariant distribution. For instance, an increase in uncertainty widens the inaction region, so that no firm is initially located close to the boundary and the number of adjustments is lower (so that monetary policy is more powerful). This effect reverses in the long run: as the new invariant distribution gets settled, the higher uncertainty will trigger more adjustments and thus a less powerful monetary policy. The model also allows us to quantify the time that it takes for the long-run effect to settle in.

Fourth, we apply our method to study an economy with multiproduct firms in Section 7. Such model assumes a firm produces n different products and that it faces increasing returns in the price adjustment: if a pays a fixed cost it can adjust simultaneously the n prices. Variations on this model have been studied by Midrigan (2011) and Bhattarai and Schoenle (2014). These models are appealing because they match several empirical regularities: synchronization among price changes within a store, and coexistence of both small and large price changes. Their economic analysis is of interest because in an economy populated by multiproduct firms the monetary shocks have more persistent real effects. In Alvarez and Lippi (2014) we derived results for impulse responses to this multidimensional setup and explore the sense in which such a model is realistic. Here we show that the characterization of the selection effect, as the difference between the survival function and the output IRF holds in this model, with the number of products n serving as the parameter that controls selection.

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Section 8 concludes the paper by discussing how our method can be used in setups that, unlike the workhorse sticky-price models described in this paper, allow for an asymmetric return function as well as an asymmetric law of motion for the state (e.g. high inflation), or both. We show that, although analytical results are now harder to obtain, the method retains tractability. Such a framework is a promising avenue to study, among others, the role of higher-order shocks of the kind discussed by Fernandez-Villaverde et al. (2011) and Fernandez-Villaverde et al. (2015) in contexts where decisions rules and the state of the economy feature various kinds of asymmetries. Section 9 discusses avenues for future work.

2   Set up

This section introduces the main objects of our analysis. First we set up a standard mathematical definition of the impulse response. Second, we present a simple baseline sticky-price model that is used to illustrate several applications of interest.

The standard set up is made by the following objects: the law of motion of the Markov process $\{x(t)\}$ for each individual firm, the function of interest $f(x)$, the cross-sectional initial distribution of x, denoted by $P(x; 0)$. At this general level the set-up and definition of an impulse response is closely related to the one in Borovicka, Hansen, and Scheinkman (2014). The law of motion for the process $f(x)$, with $x \in X \equiv [\underline{x}, \bar{x}]$, is also Markov and is described using

$$\mathcal{H}(f)(x, t) = \mathbb{E}\left[f\left(x\left(t\right)\right) | x(0) = x\right] \tag{1}$$

where the operator $\mathcal{H}$ computes the t period ahead expected value of the function $f : X \to \mathbb{R}$ conditional on the state $x = x(0)$. Next we describe the initial distribution of x, which we denote by $P(\cdot; 0) : X \to \mathbb{R}$. This represents the measure of firms that start with value smaller or equal than x at time $t = 0$, each of them following the stochastic process described in $\mathcal{H}(f)$, with independent realizations. We allow the distribution $P(x; 0)$ to have mass points. In particular P has a piecewise continuous derivative (density) which we extend

to the entire domain, so that $p(\cdot, 0) : [\underline{x}, \bar{x}] \to \mathbb{R}$, where P can have countably many jump discontinuities (mass points), denoting the difference between the right and left limits by $p_m(\cdot; 0) : \{x_k\}_{k=1}^\infty \to \mathbb{R}$, so that x_k is the location of the mass points.[3] While the possibility of handling a distribution with mass point is of theoretical interest, most of the economic applications that we discuss have no mass points and hence the density p will suffice. Summarizing, we assume that the "shocks" are idiosyncratic, and that the initial condition is given by a cross sectional distribution $P(\cdot, 0)$.

We are interested in the standard impulse response function H defined for each $t > 0$ as:

$$H\left(t; f, P - \bar{P}\right) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x, t) \left[dP(x; 0) - d\bar{P}(x)\right] \tag{2}$$

where $\bar{P}$ is the invariant distribution of x, the distribution that x will converge to in the long run.

We note that the impulse response can also be written in terms of the evolution of the cross sectional distribution $P(\cdot, t)$, namely: $H\left(t; f, P - \bar{P}\right) = \int_{\underline{x}}^{\bar{x}} f(x) \left[dP(x; t) - d\bar{P}(x)\right]$. The two definitions are equivalent. To characterize equation (2) requires solving a Kolmogorov backward (KB) equation, while the latter requires solving a Kolmogorov forward (KF) equation. In spite of the equivalence, there are two reasons why equation (2) is sometimes preferred. The first reason depends on how well behaved are $\hat{P}$ vs f. If we start with a distribution with mass points, as will be the case after a large shock, then initial condition will not be a density. The dynamics of the distribution will then involve Dirac functions. Instead, using the KB we do not need to deal with this. Notice the asymmetry: f are functions, where $\hat{P}$ is a generalized function, i.e. can include Dirac functions. Second, even if we do not deal with generalized functions, so that both f and $\hat{P}$ are functions, the boundary

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[3]Given our assumption on P we can write the expectations of any function g as:

$$\int_{\underline{x}}^{\bar{x}} g(x)\, dP(x; 0) = \int_{\underline{x}}^{\bar{x}} g(x)\, p(x; 0) dx + \sum_{k=1}^\infty g(x_k)\, p_m(x_k; 0)\,.$$

conditions can be trickier in the KF than in the KB. The boundary conditions for the KF are stated in terms of right and left derivatives, and indeed the corresponding eigenfunctions are not differentiable at the reinjection point. While using the KB the boundary conditions are standard -just as the ones in a Bellman equation- and the eigenfunctions are differentiable at the reinjection point.

The ergodicity of $\{x\}$ implies that we can also write

$$H\left(t; f, P - \bar{P}\right) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x, t) \, dP(x; 0) - \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x) \, .$$

The interpretation of $H(t)$ is the expected value of the cross-sectional distribution of f *in deviation from its steady state value*, where each $x(t)$ has followed the Markov process associated with $\mathcal{H}(\cdot)$ and whose cross sectional distribution at time zero is given by $P(\cdot; 0)$. In other words, for ergodic processes we are forcing the impulse response to go to zero as t diverges. Since we evaluate H only for the difference between two measures, i.e. only for signed measures, when it is convenient we introduce the notation $\hat{P} \equiv P - \bar{P}$ and likewise for the densities $\hat{p} = p - \bar{p}$. Thus

$$\hat{P}(x, t) \equiv P(x, t) - \bar{P}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t \geq 0 \, .$$

We also define the discounted cumulative response, given by the (discounted) area under the impulse response function $H(t; f, \hat{P})$:

$$\mathbf{H}(r; f, \hat{P}) = \int_0^\infty e^{-rt} \, H(t; f, \hat{P}) \, dt \, . \tag{3}$$

For many problems, such as the sufficient statistics for monetary policy discussed in Alvarez, Le Bihan, and Lippi (2016), the cumulative response $\mathbf{H}$ is a convenient summary for the effects of a shock, but of course such statistics is not informative about *the shape* of the impulse response function.

We give an alternative definition of the impulse response, which uses a stopping time τ, and a modified expectation operator $\mathcal{G}$ defined as:

$$\mathcal{G}(f)(x,t) = \mathbb{E}\left[1_{\{t \leq \tau\}} f\left(x\left(t\right)\right) | x(0) = x\right] \tag{4}$$

The operator $\mathcal{G}$ computes the t period ahead expected value of the function $f : X \to \mathbb{R}$ starting from the value of the state $x = x(0)$, conditional on x surviving. The indicator function $1_{\{t \leq \tau\}}$ becomes zero when the first adjustment following the shock occurs at the stopping time τ.

In the context of the price setting models with sS rules we refer to the operator $\mathcal{H}$ as the one for the problem with "reinjection", i.e. one in which the operator keeps following the firm until an adjustment occurs (at time τ). In contrast, we refer to the operator $\mathcal{G}$ as one for the problem without "reinjection", i.e. not tracking the firm after the first adjustment. Note that $\mathcal{G}$ is losing measure with time, i.e. $\mathcal{G}(1)(x,t) \leq 1$, and the inequality can be strict for most x's. To be concrete, in the sticky price models discussed below the stopping time τ will be given by the occurrence of a price adjustment.

We define the impulse response function G for each $t > 0$ as:

$$G(t; f, P) = \int_{\underline{x}}^{\bar{x}} \mathcal{G}(f)(x,t) \, dP(x; 0). \tag{5}$$

The interpretation of $G(t)$ is the expected value of the cross-sectional distribution of f, conditional on surviving, where each $x(t)$ follows the Markov process, and the cross-sectional distribution at time zero is given by $P(\cdot; 0)$. Note the difference with $\mathcal{H}$, which was defined for measures relative to their steady state value. In the case without "reinjection" we don't need to subtract the invariant, since as t diverges the measure of surviving units converges to zero, and so does the impulse response.

As done above for H, we define the cumulative effect as the area under the impulse

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response produced by G, formally:

$$\mathbf{G}(r; f, P) = \int_0^\infty e^{-rt} G(t; f, P)\, dt\,. \tag{6}$$

While H is the impulse response as commonly defined, it turns out that G is simpler to characterize and that in a large class of problems an equivalence holds so that the cumulative impulse response $\mathbf{G}$ coincides with the cumulative response $\mathbf{H}$ (see Proposition 1 below). Moreover, Proposition 2 below will show that under slightly more stringent conditions the impulse response $G(t; f, P)$ will also coincide with $H(t; f, \hat{P})$ for all t. Given the simpler nature of G we first develop our main results for setups where the $G = H$ equivalence holds, and present more general results for the cases where it does not hold in Section 8.

The initial condition.   The primal impulse in the setup is encoded in the initial condition, $P(x, 0)$, which denotes the distribution of the state variable $x \in (\underline{x}, \bar{x})$ at time zero. In particular $P(\cdot; 0)$ describes the cross-sectional distribution of the state immediately after the shock. As time elapses the initial distribution will converge to the invariant distribution $\bar{P}(x)$, tracing out the impulse response for the function of interest $f(x)$. We discussed above that our method allows the initial distribution to have mass points. This can be useful, for instance, if the initial shock is large enough to displace a non-negligible mass of agents onto the return point x^*. Also notice that our formulation allows us to handle a variety of shocks. Several papers have focussed on a small uniform displacement $\delta > 0$ of the whole distribution relative to the invariant $\bar{P}(x)$, what Borovicka, Hansen, and Scheinkman (2014) label the "marginal response function". In this case the initial condition is $P(x, 0) = \bar{P}(x + \delta)$ and it is straighforward that we can rewrite the signed measure $\hat{P}(x, 0) \equiv P(x, 0) - \bar{P}(x)$ using a first order expansion as $\hat{P}(x, 0) = \delta \bar{P}'(x)$. We will sometime focus on such marginal shocks for convenience and to relate to the literature. We stress, however, that our method can handle any type of initial condition, such as one triggered by a large shock, or the one triggered by a higher-order shock.

2.1 A baseline price setting problem

This section lays out the price setting problem solved by a firm in the "Calvo plus" model. In this model, which can be seen as one with random menu costs, the firm is allowed to change prices either by paying a fixed menu cost or upon receiving a random free adjustment opportunity (a menu cost equal to zero). The setup nests several models of interest, from the canonical menu cost problem to the pure Calvo model. Analogous results to the ones described in this section can be derived for models with Price-plans as well as for the multi-product price-setting problem with slight changes in the math of the problem (see Section 5 and Section 7 for such extensions).

The firm problem in the Calvo plus model. We describe the price setting problem for a firm in steady state. The firm cost follows a Brownian motion with variance σ^2 and drift μ, where the latter is due to inflation. The firm can change its price at any time paying a fixed cost $\psi > 0$. At exogenously given times, which occur with a Poisson rate ζ, the firm faces a zero menu cost. The price gap x is defined as the price currently charged by the firm relative to the price that will maximize current profits, which is proportional to the firm cost (measured as the log of the ratio between these prices). The optimal policy is to change the price when the gap x reaches either of two barriers, $\underline{x} < \bar{x}$, or when the menu cost is zero. In either case, at the time of a price change, the firm sets a new price which determines a price gap x^*, which is the optimal return point after the adjustment.

The flow cost of the firm is given by $R(x)$. An example is $R(x) = Bx^2$, where the coefficient B measures the curvature of the profit function measured around the static profit maximizing point. The firm maximizes expected discounted profits, using a discount rate $r > 0$. Thus, the optimal policy can be describe by three numbers: $\underline{x} < x^* < \bar{x}$, whose values can be found by solving the value function for the cost in the inaction region:

$$(r + \zeta)v(x) = R(x) + \mu v'(x) + \frac{\sigma^2}{2}v''(x) + \zeta v(x^*) \text{ for } x \in [\underline{x}, \bar{x}] \tag{7}$$

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and imposing the relevant boundary conditions: value matching, smooth pasting, and opti-
mality of the return point:

$$v(\bar{x}) = v(\underline{x}) = v(x^*) + \psi, \text{ and } v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0 \ . \tag{8}$$

The density $\bar{p}$ of the invariant distribution for price gaps generated by the policy $\{\underline{x}, x^*, \bar{x}\}$
and the law of motion of x is the solution to the Kolmogorov forward equation:

$$\zeta \bar{p}(x) = -\mu \bar{p}'(x) + \frac{\sigma^2}{2} \bar{p}''(x) \text{ for } x \in [\underline{x}, \bar{x}], x \neq x^* \tag{9}$$

with boundary conditions at the exit points, unit mass, and continuity requirements:

$$\bar{p}(\underline{x}) = \bar{p}(\bar{x}) = 0, \int_{\underline{x}}^{\bar{x}} \bar{p}(x)dx = 1, \text{ and } \bar{p} \text{ continuous at } x = x^* \ .$$

The boundary conditions at the exit points are immediate in sS models with fixed costs since
no mass can accumulate at the boundary of the inaction region as long as $\sigma > 0$.

2.2 Cumulative impulse response

We now discuss a useful result concerning the cumulated impulse response function that holds
in all models described in the previous section. The result states that if the function of interest
is $f(x) = R'(x)$, then the cumulated impulse response can be readily computed using the
derivative of the value function v, given any arbitrary initial distribution. The function $R'(x)$
is of interest because in several problems the derivative of the firm's function is proportional to
the firm's contribution to aggregate output. Recall that the initial condition for the problem
is the signed measure $\hat{P}(\cdot, 0) \equiv P(\cdot, 0) - \bar{P}(\cdot)$ where $\bar{P}$ is the invariant distribution. We have
the following result (see Appendix A for the proof of all propositions):

PROPOSITION 1. Consider the problem described by $\mathbb{P} \equiv \{\mu, \sigma, r, \psi, \zeta, R(\cdot)\}$. Let $\{\underline{x}, x^*, \bar{x}\}$
and $v(\cdot)$ be the optimal policy and value functions solving the problem $\mathbb{P}$, so the thresholds

and value function solve equation (7) and equation (8). Let R' be the derivative of the return function R and $P(\cdot, 0)$ be the distribution of x right after the shock. Then:

$$\mathbf{H}(r, R', \hat{P}) = \int_{\underline{x}}^{\bar{x}} v'(x) d\hat{P}(x; 0) = \mathbf{G}(r, R', \hat{P}) \ .$$

The result is surprising to us. The problem features asymmetric sS bands and the state has a drift, yet we can compute the cumulative IRF as in the symmetric case with no drift. Moreover, when computed this way the value of the optimal return point x^* is irrelevant –literally it does not enter in the computations.[4] We make several comments to this result. First, notice that the proposition holds for $f = R'$, not for an arbitrary function f. The first equality says that if one has solved for the value function in equation (7)-(8), then it has the cumulative IRF for the function $f = R'$. Note that for $R(x) = Bx^2$, i.e. for a second order approximation to the profit function, the derivative $R'(x) = 2Bx$, and hence it is proportional to the firm's contribution of the IRF of output. In particular, the cumulative IRF of output after a monetary shock is obtained by setting $p(x, 0) = \bar{p}(x + \delta)$ and $f(x) = -R'(x)/(2B) = -x$. Second, we stress that the proposition holds for any initial condition $\hat{P}$. This allows us to study either small shocks, such as a marginal displacement of the invariant distribution $\bar{P}$, as well as large shocks that give rise to any type of initial signed mass $\hat{P} = P(0) - \bar{P}$.

Proposition 1 yields a straightforward analysis of an otherwise computationally intensive question. The last equality states that we can obtain the cumulative IRF by simply keeping track of each firm until the time it makes the first adjustment following the shock. This is convenient both for analytical computation as well as for simulations. Figure 1 illustrates this point by exploring how the the cumulated output response, following a small monetary shock, changes with the inflation rate. To this end we use a simple Golosov-Lucas menu cost model calibrated to produce 1 price adjustment per year, on average, at zero inflation. We then vary

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[4] An identical result holds for the model with price plans. We relegate this result to an appendix since the math is more cumbersome in that case.

the inflation rate and compute the cumulated area as $\mathbf{H}(r, R', \hat{P}) = \int_{\underline{x}}^{\bar{x}} v'(x)\bar{p}'(x; 0)dx$ where the value function $v(x)$ solves equation (7) and equation (8), and the density function $\bar{p}(x)$ solves equation (9). The small shock assumption is used by postulating that the distribution right after the shock is equal to a small displacement of the invariant, namely that $p(x, 0) = \bar{p}(x+\delta)$, so that $\hat{p}(x) = p(x, 0) - \bar{p}(x) \approx \bar{p}'(x)\delta$, where δ is the aggregate monetary shock. The figure shows that the cumulated output effect is not responsive to inflation when inflation is low (it is easy to prove that the function has a zero derivative at $\pi = 0$). As inflation increases however the real (cumulated) effect of policy vanishes fast. At an inflation rate equal to 50% per year the effect is about 1/5th of the effect at low inflation.

Figure 1: Cumulative output response at different inflation rates
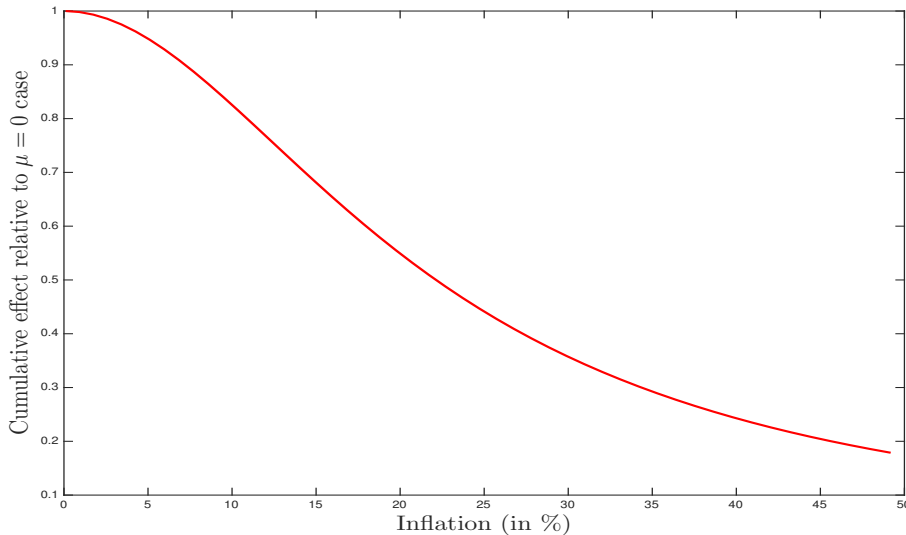
Note: cumulated output response triggered by a small monetary shock, at different inflation rates, computed using Proposition 1 with a return function $R(x) = x^2$. The cumulated output, measured on the vertical axis, is normalized relative to the one at zero inflation rate ($\mu = 0$). The underlying menu cost model is calibrated to produce 1 price adjustment per year at zero inflation.

2.3 Impulse Response Functions for symmetric sS problems

We define a problem and its associated sS rule to be symmetric if the state variable has no drift, so that $\mu = 0$, and if the return point x^* is equidistant from the upper and lower barriers: $x^* - \underline{x} = \bar{x} - x^*$, so that $x^* = (\underline{x} + \bar{x})/2$. Indeed, when $\mu = 0$ and $R(x)$ is

symmetric the *optimal* decision rule is symmetric. Let $\{x(t)\}$ be the value of the state for a firm following the optimal policy. Let $g(x; t, x^*)$ be the density of distribution of $x(t)$, conditional on $x(0) = x^*$. This distribution is symmetric with respect to x around x^* for all $t > 0$, i.e. its density is given by $g(z + x^*; t, x^*) = g(-z + x^*; t, x^*)$ for all $z \in [0, \bar{x} - x^*]$. This symmetry comes from the combination of the symmetry of the distribution of a BM without drift, and the symmetry of the boundaries relative to the optimal return point.

Next we present a proposition, for the symmetric case, under which the standard IRF $H(t)$ coincides with $G(t)$, i.e. the simple IRF computed stopping after the first adjustment (see Appendix A for the proof). The conditions for this to happen are that either the function of interest f, or that the initial distribution $P(\cdot, 0)$ are antisymmetric, where a function $\nu(x)$ is antisymmetric about x^* if it satisfies $\nu(x - x^*) = -\nu(x^* - x)$ for all $x \in [\underline{x}, \bar{x}]$.

PROPOSITION 2. Assume the problem is symmetric. Then if either

(i) the function of interest $f : [\underline{x}, \bar{x}] \to \mathbb{R}$ is antisymmetric and $P(\cdot, 0)$ is arbitrary

(ii) the signed measure (initial condition) $p(\cdot, 0) - \bar{p}(\cdot) : [\underline{x}, \bar{x}] \to \mathbb{R}$ as well as its mass points $p_m(\cdot, 0) - \bar{p}_m(\cdot) : \{x_k\}_{k=1}^{\infty} \to \mathbb{R}$ are antisymmetric and $f(\cdot)$ is arbitrary

we have that $G(t; f, P) = H\left(t; f, P - \bar{P}\right)$ for all t.

The proposition's requirement that either the function of interest f, or the initial distribution $P - \bar{P}$, is anti-symmetric is not that restrictive for our applications. The main function of interest for the paper, used to compute the IRF for output, is anti-symmetric in the class of models we analyse. For example $f(x) = -x$ in the Calvo$^+$ model. Also, our benchmark case in this class of models is that the density $p - \bar{p}$ is anti-symmetric. This is because in our setup what matters is $p - \bar{p}$, namely the deviation from a symmetric steady after a monetary shock, which can be shown to be antisymmetric –we will state and prove this later. We believe that the strongest assumption is the symmetry on the sS decision rules, which is appropriate for monetary models in the neighborhood of price stability, but it may

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be inappropriate for other set-ups. An extension to analyse problems with asymmetries is studied in Section 8.

3 Analytic Impulse Response Functions

This section presents a fundamental decomposition that allows us to develop an analytic solution for the operator $\mathcal{G}(f)(x,t)$ defined in equation (4). The main assumption of this section is that the problem is symmetric, i.e. that x has no drift ($\mu = 0$), and that the stopping barriers $\underline{x}, \bar{x}$ are symmetric around the optimal return point x^* (see Section 2.3). As shown in Proposition 2 we know that in this case the impulse response is equivalent to the one of a problem without reinjections and is given by $G(t, f, P)$. We see the symmetric setup as a convenient starting point for the analysis and one which is relevant to analyze most state of the art sticky price models. In Section 8 and Appendix B we show how to extend the analysis presented here to problems with asymmetries and drift, which will involve reinjections. While such problems are in principle more involved we show that the method retains tractability and remains useful.

Assume the process for the firm's price gap is given by a drift-less brownian motion, with instantaneous variance per unit of time σ^2. The stopping time, i.e. the rule at which prices are change, is given by the first time at which $x(t)$ hits either $\underline{x}$ or $\bar{x}$, or that a Poisson counter, with instantaneous rate ζ, changes its value. The definition of $\mathcal{G}(f)(x,t)$ as an expected value implies that this function must satisfy the following partial differential equation:

$$\partial_t \mathcal{G}(f)(x,t) = \frac{\sigma^2}{2}\partial_{xx}\mathcal{G}(f)(x,t) - \zeta\mathcal{G}(f)(x,t) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and } t > 0 \tag{10}$$

with boundary conditions

$$\mathcal{G}(f)(\underline{x}, t) = \mathcal{G}(f)(\bar{x}, t) = 0 \text{ for all } t > 0 \text{ and} \tag{11}$$

$$\mathcal{G}(f)(x, 0) = f(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{12}$$

where boundary conditions for $t > 0$ are an implication from $\bar{x}$ and $\underline{x}$ being exit points, and hence close to them the survival rate goes to zero. The boundary condition at $t = 0$ follows directly from the definition of $\mathcal{G}(f)$.

Two key steps, based on the properties of eigenvalues and eigenfunctions proved below, allow us to solve for $\mathcal{G}(f)$. First, that the function f can be represented by a linear combination of eigenfunctions, with typical member φ_j, as follows $f(x) = \sum_{j=1}^{\infty} b_j[f]\,\varphi_j(x)$, where $b_j[f]$ are coefficients. Second, that the solution for $\mathcal{G}(\varphi_j)$ for each eigenfunction is multiplicatively separable in (x, t):

$$\mathcal{G}(\varphi_j)(x, t) = e^{\lambda_j t}\,\varphi_j(x)$$

for some constant λ_j. Thus the partial differential equation in (10) becomes the following ordinary differential equation:

$$\lambda_j \varphi_j(x) = \varphi_j''(x)\frac{\sigma^2}{2} - \zeta \varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ with boundary conditions } \varphi_j(\bar{x}) = \varphi_j(\underline{x}) = 0 .$$

Note that the boundary condition for $\mathcal{G}(\varphi_j)$ at $t = 0$, in equation (12), holds by construction. We will denote the φ_j as eigenfunctions and the corresponding λ_j as eigenvalues. The eigenvalues and eigenfunctions are:

$$\lambda_j = -\left[\zeta + \frac{\sigma^2}{2}\left(\frac{j\pi}{\bar{x} - \underline{x}}\right)^2\right] \text{ and } \varphi_j(x) = \frac{1}{\sqrt{(\bar{x} - \underline{x})/2}} \sin\left(\frac{[x - \underline{x}]}{[\bar{x} - \underline{x}]} j\pi\right) \text{ for } j = 1, 2, 3, ...$$

$$(13)$$

This is the solution to a well known problem which has been studied extensively.[5] A key property of the set of eigenfunctions $\{\varphi_j\}$ is that they form an orthonormal base for the functions $f : [\underline{x}, \bar{x}] \to \mathbb{R}$ and for which $\int_{\underline{x}}^{\bar{x}}[f(x)]^2 dx < \infty$ so that we can write:

$$\hat{f}(x) \equiv \sum_{j=1}^{\infty} b_j[f]\,\varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}]$$

---

[5]That this is the solution of the o.d.e. follows since $\sin(0) = \sin(j\pi) = 0$ for all $j \geq 1$ and also because $\sin''(x) = -\sin(x)$ for all x. Matching coefficients in the o.d.e. we obtain the eigenvalues.

where $\int_{\underline{x}}^{\bar{x}} (\hat{f}(x) - f(x))^2 dx = 0$ and the projection coefficients $b_j[f]$ are:

$$b_j[f] = \int_{\underline{x}}^{\bar{x}} f(x)\, \varphi_j(x) dx \text{ for all } j \geq 1\,,$$

which uses that $\{\varphi_j\}$ is an orthonormal base. Next we present a Lemma for the representation of the projections (expectations) of the function of interest f:

LEMMA 1. Assume that $f : [\underline{x},\, \bar{x}] \to \mathbb{R}$ is piece-wise differentiable, with countably many discontinuities, and $\int_{\underline{x}}^{\bar{x}} [f(x)]^2\, dx < \infty$ then:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}\, b_j[f]\, \varphi_j(x) \ \text{ for all } x \in [\underline{x}, \bar{x}]$$

Now we turn to the impulse response. We define the projection coefficients for the initial distribution $P(\cdot, 0)$ as follows:

$$b_j[P(\cdot, 0)] = \int_{\underline{x}}^{\bar{x}} \varphi_j(x) dP(x; 0) \equiv \int_{\underline{x}}^{\bar{x}} \varphi_j(x) p(x; 0) dx + \sum_{k=1}^{\infty} \varphi_j(x) p_m(x_k; 0)$$

so that if P has no mass points, it coincides with the definition for a function f. Using Lemma 1 and the definition of the projection coefficients $b_j[P]$ we can write the impulse response function in equation (5) as follows:

PROPOSITION 3. Assume that $f : [\underline{x},\, \bar{x}] \to \mathbb{R}$ is piece-wise differentiable, with countably many discontinuities, and $\int_{\underline{x}}^{\bar{x}} [f(x)]^2\, dx < \infty$. Furthermore assume that P has a piecewise continuous density and at most countably many mass points, then:

$$G(t; f, P) = \sum_{j=1}^{\infty} e^{\lambda_j t}\, \beta_j \qquad \text{where} \qquad \beta_j \equiv b_j[f]\, b_j[P(\cdot, 0)]\,. \tag{14}$$

For the cumulative impulse response defined in equation (6) we then have:

$$\mathbf{G} \equiv \int_0^\infty G(t)dt = \sum_{j=1}^\infty \frac{\beta_j}{-\lambda_j} \tag{15}$$

For instance, the impulse response of output can be obtained using $f(x) = -x$, since the contribution of output of each firm is proportional to their price gap. It is also straightforward to analyze the slope of the impulse response at $t = 0$, a model feature we discuss later in the application to the Calvo$^+$ model.

3.1   Application to the canonical menu cost model.

We illustrate a concrete application of the above results using the menu-cost model, obtained by setting $\mu = \zeta = 0$ in the problem of Section 2.1, which yields the symmetric inaction region $\underline{x} = -\bar{x}$ with optimal return $x^* = 0$. To compute the impulse response of output we use $f(x) = -x$ since the contribution of a firm to the deviation of output (relative to steady state) is inversely proportional to its price gap. Integrating $f(x)$ against $\varphi_j(x)$ we find the projection coefficients $b_j[f]$ in equation (14):

$$b_j[f] = \frac{4\bar{x}^{3/2}}{j\,\pi} \text{ for } j = 2, 4, 6, \ldots, \text{ and } b_j[f] = 0 \text{ otherwise.} \tag{16}$$

In this example we consider a *small* monetary shock so that the initial condition is given by $p(x, 0) - \bar{p}(x) = \delta \bar{p}'(x)$, as discussed above. The invariant distribution for this model is readily derived from equation (9) and the associated boundary conditions, which gives the triangular density $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$ for $x \in (-\bar{x}, \bar{x})$. It is apparent that $\bar{p}'(x)$ is a step function, equal to $1/\bar{x}^2$ for $x \in [-\bar{x}, 0)$ and equal to $-1/\bar{x}^2$ for $x \in (0, \bar{x}]$. We thus construct the projection coefficients $b_j[\bar{p}']$ by integrating $\bar{p}'(x)$ against $\varphi_j(x)$. This gives

$$b_j[\bar{p}'] = \frac{8}{j\,\pi\bar{x}^{3/2}} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[\bar{p}'] = 0 \text{ otherwise.} \tag{17}$$

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Thus the impulse response $b_j[\bar{p}']b_j[f]$ coefficients for equation (14) are:

$$b_j[\bar{p}']b_j[f] = \frac{32}{(j\,\pi)^2} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[\bar{p}']b_j[f] = 0 \text{ otherwise.}$$

and the output impulse response, which we denote by $Y(t) = G(t)/\delta$ is:

$$Y(t) = \sum_{i=0}^{\infty} \frac{32}{((2+4i)\,\pi)^2} e^{-N\frac{((2+4i)\pi)^2}{8}t} \tag{18}$$

where $N = \sigma^2/\bar{x}^2$ is the average number of price changes per period, the only parameter in this expression.

4   Characterization of the selection effect

This section uses the results of Section 3 to provide an analytic illustration of why different models display different degrees of "selection". To ensure the results are applicable we focus on a symmetric problem (symmetric return function and law of motion). The term selection, coined by Golosov and Lucas, refers to the fact that the prices that adjust following a monetary shock are those of a selected group of firms. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Calvo model. It is known that different amounts of selection critically affect the propagation of monetary shocks. Our application casts light on the mechanism behind this result.

We present an analytic result showing how the selection effect creates a wedge between the duration of price changes and that of output. The two durations coincide when there is no selection. In this case the frequency of price changes is a sufficient statistic for the output effect of monetary policy. We show that such wedge is visible in the magnitude of

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the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of output. Next we illustrate this result using the Calvo-plus model, a model that nests several special cases featuring different degrees of selection, from Golosov-Lucas to the pure Calvo model. The result also holds in several other models, featuring multiproduct firms or price plans.

Application to the Calvo$^+$ model. Next we use the decision problem defined in Section 2.1, and assume zero inflation ($\mu = 0$) and a quadratic profit function $R = x^2$. It is straightforward that $\bar{x} = -\underline{x} > 0$ and that the optimal return is $x^* = 0$. Given the policy parameters $\{-\bar{x}, \bar{x}\}$ and the law of motion of the state $dx = \sigma dW$ it is immediate that the eigenvalues-eigenfunctions of the problem are those computed in equation (13). Since the eigenvalues depend on the speed at which prices are changed, we find it convenient to rewrite them in terms of the average number of price changes per unit of time, N and ϕ.

To this end we compute the expected number of adjustments per unit of time, the reciprocal of the expected time until an adjustment,

$$
N = \frac{\zeta}{1 - \text{sech}(\sqrt{2\phi})} \quad \text{where} \quad \phi \equiv \frac{\zeta \bar{x}^2}{\sigma^2} \quad .
$$

Note that as $\bar{x} \to \infty$ then $N \to \zeta$, which is the Calvo model where all adjustment occur after an exogenous poisson shock. As $\zeta \to 0$ then $N \to \sigma^2/\bar{x}^2$ so that the model is Golosov and Lucas. This single parameter $\phi \in (0, \infty)$ controls the degree to which the model varies between Golosov-Lucas and Calvo. Note that with this parameterization we can distinguish between N and the importance of the randomness in the menu cost ζ vs the width of the barriers, $\bar{x}^2/\sigma^2$. Indeed ζ/N, the share of adjustment due to random free-adjustments, depends only on ϕ. We let:

$$
\frac{\zeta}{N} = \ell(\phi) \text{ where this function is defined as } \ell(\phi) = 1 - \text{sech}\left(\sqrt{2\phi}\right) \quad .
$$

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The function $\ell(\cdot)$ is increasing in ϕ, and ranges from 0 to 1 as ϕ goes from 0 to ∞. Using the formula for N and equation (13) we have:

$$\lambda_j = -\zeta - \frac{\sigma^2}{\bar{x}^2}\frac{(j\pi)^2}{8} = -\zeta\left[1 + \frac{(j\pi)^2}{8\phi}\right] = -N\,\ell(\phi)\left[1 + \frac{(j\pi)^2}{8\phi}\right] \quad . \tag{19}$$

Interpretation of the Dominant Eigenvalue. The dominant eigenvalue has the interpretation of the asymptotic hazard rate of price changes. In particular, let $h(t)$ be the hazard rate of price spells as a function of the duration of the price spell t. Let τ be the stopping time for prices, i.e. τ is the first time at which $\sigma W(t)$, which started at $W(0) = 0$, either hits $\bar{x}$ or $\underline{x} = -\bar{x}$, or that the Poisson process changes. Let $S(t)$ be the survival function, i.e.: $S(t) = \Pr\{\tau \geq t\}$. Notice that the function of interest to compute the survival function is the indicator $f(x) = 1$ for all $t < \tau$. The hazard rate is defined as $h(t) = -S'(t)/S(t)$. Application of Proposition 3 gives the following:

COROLLARY 1. The Survival function $S(t)$ depends only on the odd-indexed eigenvalues-eigenfunctions, i.e. (λ_i, φ_i) for $i = 1, 3, 5, \ldots$. Let $h(t)$ be the hazard rate of price changes. Then, the dominant eigenvalue λ_1 is equal to the asymptotic hazard rate, i.e.

$$S(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j-1}t}\beta_{2j-1} \quad\text{and}\quad -\lambda_1 = \lim_{t\to\infty} h(t)$$

where $\beta_j \equiv b_j[1]\, b_j[\delta_0]$ where δ_0 is the Dirac delta function.

Average survival function. Next we derive a result, closely related to the previous one, showing that the even-index eigenvalue-eigenfunctions $(j = 2, 4, \ldots)$ are irrelevant (literally they do not appear) in the equation for the average number of price changes following a monetary shock.

Define the average survival function $Q(t)$ to be the number of firms that survive up to period t among those that started at time zero with a distribution of x given by $p(x; 0)$. In

terms of our notation this corresponds to $f(x) = 1$, as in the survival function described above. The difference with the survival function $S(\cdot)$ defined above, is that the initial condition is $p(\cdot; 0)$ instead of the dirac function. Specifically, $S(t)$ is the probability of a firm surviving up to time t conditional on starting with $x = 0$ at time zero. Instead $Q(t)$ is the fraction of firms surviving up to t for a cohort that starts at zero with distribution $p(\cdot; 0)$. The function $Q(\cdot)$ is important because our objective is to characterize the impulse response after an aggregate shock, which is modeled as a particular initial distribution $p(\cdot; 0)$. Indeed, together with the output impulse response, it allows us to decompose the effect of a monetary shock into the effect on the probabilities of price changes vs the effect on the average price level at each horizon.

The definition above gives us:

$$Q(t) = \int_{\underline{x}}^{\bar{x}} q(x, t)\, p(x; 0)\, dx \quad \text{where} \quad q(x, t) = \mathcal{G}(1)(x, t)$$

so $q(x, t)$ is the probability that a firm with x at time zero will survive until time t. Before setting the next result we define as $\bar{Q}(\cdot)$ the average survival function at steady state, i.e. Q when $p(x, 0) = \bar{p}(x)$, where $\bar{p}$ is the density of the invariant distribution. Application of Proposition 3 gives:

COROLLARY 2. The average survival function $Q(t)$ for the Calvo$^+$ model after a small monetary shock is equal to its steady state value, i.e. $Q(t) = \bar{Q}(t)$ for all $t \geq 0$. The function $\bar{Q}$, depends only on the odd-indexed eigenfunctions-eigenvalues $(j = 1, 3, ...)$ and $-\lambda_1$ is the asymptotic hazard rate of Q, i.e.

$$Q(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j-1} t} \beta_{2j-1} \quad \text{and} \quad -\lambda_1 = \lim_{t \to \infty} h(t)$$

where $\beta_j \equiv b_j[1]\, b_j[\bar{p}]$.

Corollary 2 means one should not expected to detect difference in the average hazard

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rates for price changes before and after a (small) monetary shock. This is because what is important for the output IRF is the impact of the monetary shock on the average price level. For example, an equal number of increases and decreases on prices do not contribute to output IRF.

Irrelevance of dominant eigenvalue for output IRF. Next we show that the dominant eigenvalue λ_1, as well as all other odd-indexed eigenvalue-eigenfunction pairs, play no role in the output impulse response. Consider the output coefficients in the impulse response, given by equation (16). It is apparent that the coefficients $b_j[f]$ for all the odd-indexed eigenvalues-eigenfunctions $(j = 1, 3, \dots)$ are zero, i.e. the loading of these terms are zero. This implies that the coefficient corresponding to the dominant eigenvalue λ_1 is zero. The first non-zero term, which we call the "leading" eigenvalue, involves λ_2. This is because $\varphi_j(\cdot)$ is symmetric around $x = 0$ for j odd, and antisymmetric for j even. Thus:

$$\int_{\underline{x}}^{\bar{x}} \varphi_j(x) f(x) dx = 0 \implies b_j[f] = 0 \text{ for } j = 1, 3, \dots$$

This happens since all the odd-indexed eigenfunctions φ_j $(j = 1, 3, \dots)$ are symmetric functions, and thus the projection onto them of an asymmetric function, such as $f(x) = -x$, yields a zero b_j coefficient. We summarize this result in the next corollary:

COROLLARY 3. The output impulse response function for the Calvo$^+$ model depends only on the even-indexed eigenvalue-eigenfunctions (λ_j, φ_j), and has zero loadings on the odd-indexed ones, such as the dominant eigenvalue. Thus the first leading term corresponds to the second eigenvalue:

$$G(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} \beta_{2j} \text{ and } \lambda_2 = \lim_{t \to \infty} \frac{\log Y(t)}{t}$$

where $\beta_j \equiv b_j[f] \, b_j[p(\cdot, 0)]$.

Figure 2: Selection effect in Calvo$^+$ model
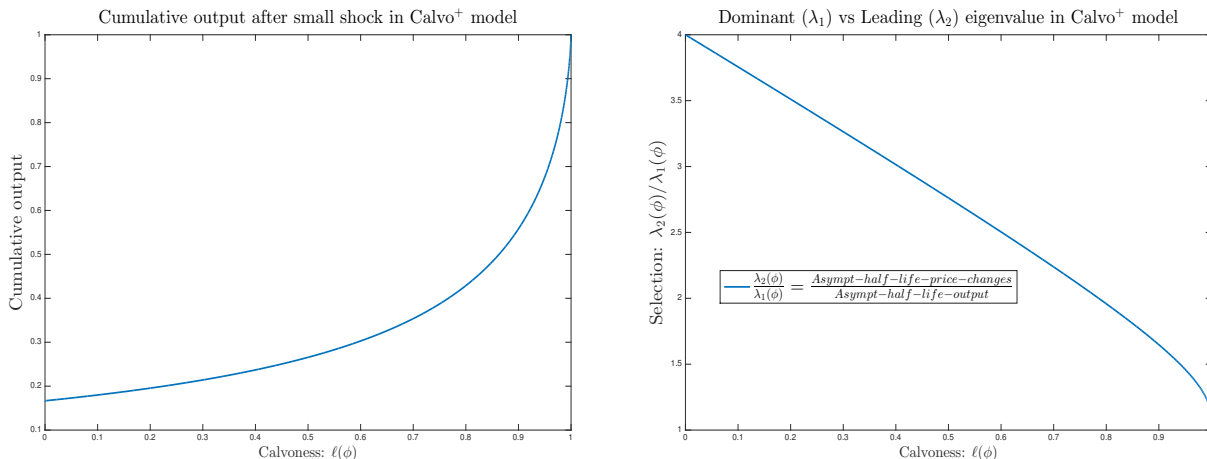
The corollary states that only half of the eigenvalues (those with an even index) show up in the output impulse response function. The largest eigenvalue is λ_2, which we call the "leading" eigenvalue of the output response function. It is interesting to notice that the dominant eigenvalue λ_1 does not appear in the impulse response for output. Notice the difference with the survival function where the only eigenvalues that appear are those with an odd-index. The right panel of Figure 2 plots the ratio between the leading eigenvalue for output λ_2 and the dominant eigenvalue λ_1. It is straightforward to see that the ratio is $\frac{\lambda_2}{\lambda_1} = \frac{8\phi + 4\pi^2}{8\phi + \pi^2}$ depends only on ϕ, so that it can be immediately mapped into the "Calvoness" of the problem $\ell(\phi) \in (0, 1)$. It appears that the ratio, which can also be interpreted as the ratio between the asymptotic duration of price changes over the asymptotic duration of the output impulse response, is monotonically decreasing in ℓ, and converges to 1 as $\ell \to 1$. The economics of this result is that the shape of the impulse response of output depends on the differential impact of the aggregate shock on price *increases* and price *increases*. Instead the dominant eigenvalue controls the asymptotic behavior of price *changes*, both increases and decreases. As $\ell \to 1$ selection disappears from the model and the two durations coincide. The left panel of the figure uses the particular case of a small monetary shock (developed in detail in the next subsection) to illustrate that as ℓ increases the cumulated output effect

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becomes larger due to a muted selection effect.

4.1 How do monetary shocks affect the dispersion of prices?

We conclude this section with a discussion of a relevant topic in monetary models that concerns the welfare effects of shocks. In many monetary models the presence of price stickiness implies that welfare of the representative consumer depends on the dispersion of prices, or the dispersion of markups (i.e. prices relative to a flexible price benchmark where dispersion is nil). It is therefore of interest to analyze how the dispersion of markups behaves following a small monetary shock of size δ.[6] Let $\mathcal{D}(t, \delta) \equiv var(x, t; \delta)$ denote the cross sectional variance of markups t periods after the monetary shock δ hits an economy at the steady state. More in general let $M_k(t) \equiv \mathbb{E}\left[x(t) - \mathbb{E}(x(t))\right]^k$, for $k = 2, 4, ...$, denote the k-th even centered moment of x. We have the following result:

PROPOSITION 4. Assume the initial condition $\hat{p}$, the signed mass right after the aggregate shock, is odd. Then the initial impulse $\hat{p}$ does not have a first-order effect on any even centered moment $f(x) = x^k$ with k =2,4,.....

The proposition shows that a small (marginal) monetary shock does not have a first order impact on the dispersion of markups at all $t > 0$ after the monetary shock. An identical logic (see the proof) shows that a zero first-order effect is predicted for all even centered moments of the distribution of markups (such as Kurtosis). Instead, uneven moments, such as the the mean markup (proportional to total output) or the the skewness of the distribution display a non-zero first order effect following a small monetary shock.

5 On the shape of the impulse response function

This section discusses whether it is possible to approximate the impulse response function in a parsimonious way, a question that is naturally related to the shape of the impulse

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[6] We are thankful to Nobu Kiyotaki for posing this question to us.

response. A natural candidate would be to analyze the impulse response associated to the leading eigenvalue as defined in Section 4, namely the largest eigenvalue associated with non-zero projection coefficient b_j in equation (14), for a case in which the IRF is close to exponential. We analyze this question by focusing on a small monetary shock that causes a marginal displacement of the invariant distribution. We assume a symmetric problem and present results for the baseline Calvo$^+$ model as well as for a model with price plans.

Initial Condition $p(\cdot, 0)$ for the impulse response to a monetary shock. The invariant density function $\bar{p}$ solves the Kolmogorov forward $\zeta \bar{p}(x) = \sigma^2/2 \bar{p}''(x)$ in the support, except at $x = 0$, integrates to one and it is zero at $\pm \bar{x}$. This gives:

$$\bar{p}(x) = \frac{\theta \left[e^{\theta(2\bar{x} - |x|)} e - e^{\theta|x|} \right]}{2 \left(1 - e^{\theta \bar{x}} \right)^2} \text{ for } x \in [-\bar{x}, \bar{x}] \quad \text{where} \quad \theta \equiv \sqrt{2\zeta/\sigma^2} \quad . \tag{20}$$

(note that to simplify notation we define a new parameter θ). The invariant distribution $\bar{p}(\cdot)$ is symmetric with $\bar{p}(x) = \bar{p}(-x)$ and $\bar{p}'(x) = -\bar{p}'(-x)$ for all $x \in [-\bar{x}, 0)$. The density of the distribution is continuous but non-differentiable at the injection point $x = 0$. For concreteness we focus below on the response to a small monetary shock δ, starting at the steady state. For a small shock we can disregard the fraction of firms that change prices on impact, i.e. this effect is of order δ^2. Thus the initial condition is

$$p(x, 0) = \bar{p}(x + \delta) = \bar{p}(x) + \bar{p}'(x)\delta + o(\delta) \text{ for all } x \in [-\bar{x}, 0) \text{ and } x \in (0, \bar{x}]$$
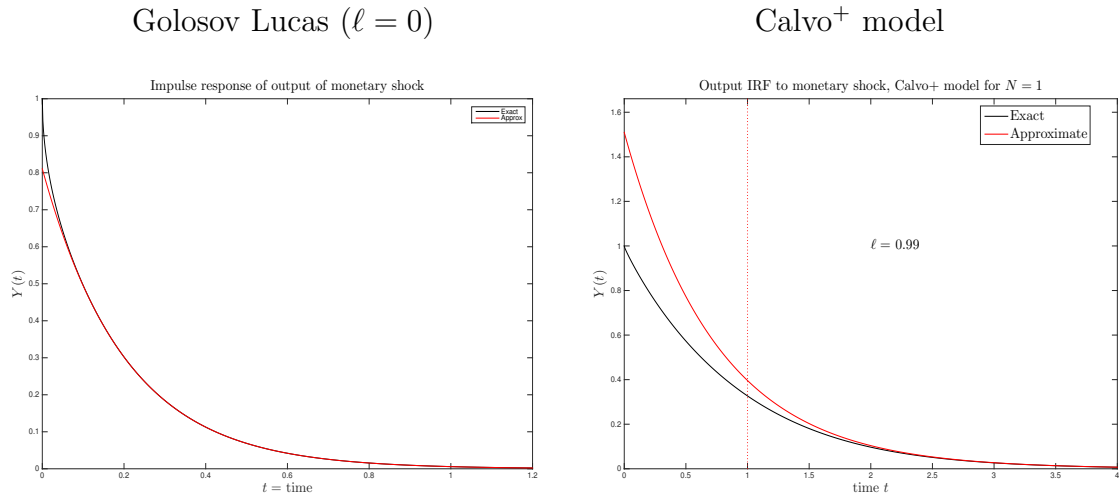
Notice that we can write:

$$G(t, \delta) = \frac{\partial}{\partial \delta} G(t, \delta) \big|_{\delta=0} \delta + o(\delta)$$

From now on for the Impulse response function of output we simply write

$$Y(t) \equiv \frac{\partial}{\partial \delta} G(t, \delta)\big|_{\delta=0} = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} b_{2j}[f] b_{2j}[\bar{p}']$$

which is the output impulse response per unit of the monetary shock. Application of Proposition 3 gives the following:

Figure 3: Exact vs approximate IRF

Golosov Lucas ($\ell = 0$)   Calvo$^+$ model

PROPOSITION 5.   The coefficients for the impulse response to a small monetary shock in the Calvo$^+$ model are given by:

$$\beta_j(\phi) \equiv b_j[\bar{p}'] b_j[f] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2 \left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{1-\cosh\left(\sqrt{2\phi}\right)}\right] \left[\frac{1}{1+\frac{j^2 \pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2 \left[\frac{1}{1+\frac{j^2 \pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases} \qquad (21)$$

Recall that $\phi \equiv \sigma^2 \zeta / \bar{x}^2 \in (0, \infty)$ is a single parameter that locates the Calvo$^+$ model between the Golosov-Lucas ($\phi = 0$) and the Calvo ($\phi \to \infty$) model. Each of these cases can

be easily computed by simple calculus while keeping N constant. The solution for the first case, $\phi \to 0$, was given in equation (16) and equation (17). The second case, $\phi \to \infty$, is peculiar because in this limit the spectrum is no longer discrete. To see this notice that each of the eigenvalues $\lambda_j \to -\zeta$ and:

$$\lim_{\phi \to \infty} \beta_j(\phi) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even and } \frac{i}{2} \text{ is odd} \\ -2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

Note that in this case, using just the "leading" eigenvalue, i.e. the term with the first non-zero weight, gives an impulse response $G(t)$ that is twice as large than the true one, for each t. This is in stark contrast with the case when $\phi \to 0$ where the approximation is extremely accurate. The difference is less than 1.5%, to see this note that when $\phi \to 0$ then $\mathbf{G} \to 1/(6N) \approx 0.1677/N$. On the other hand, using only the term corresponding to the second eigenvalue we obtain $b_2/(-\lambda_2) = 16/(\pi^4)/N \approx 0.1643/N$.

The next proposition gives a characterization of the ratio between the true area under the impulse response and the approximate one, computed using only the leading eigenvalue:

PROPOSITION 6. Define the ratio of the approximate cumulative impulse response based on the second eigenvalue to the area under the impulse response as

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty} \beta_j(\phi)/\lambda_j(\phi)} = 2\frac{\left[1 + \cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi}) - 1 - \phi\right]\left[1 + \frac{\pi^2}{2\phi}\right]^2}$$

We note that $m_2(0) = \frac{16}{\pi^4}6 \approx 0.98$, $m_2'(\phi) > 0$ and $m_2(\phi) \to 2$ as $\phi \to \infty$.

We can also use the expression for the coefficients of the impulse response to show that the slope of Y at $t = 0$ is minus infinity. This is intuitive since, after the shock, there are firms that are just on the boundary of the inaction region where they will increase prices, but there are no firms at the boundary at which they want to decrease prices.

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Figure 4: Calvo$^+$ model

Ratio of approx to exact cumulative IRF $\mathcal{M}$ for Calvo+ (cumulative IRF)
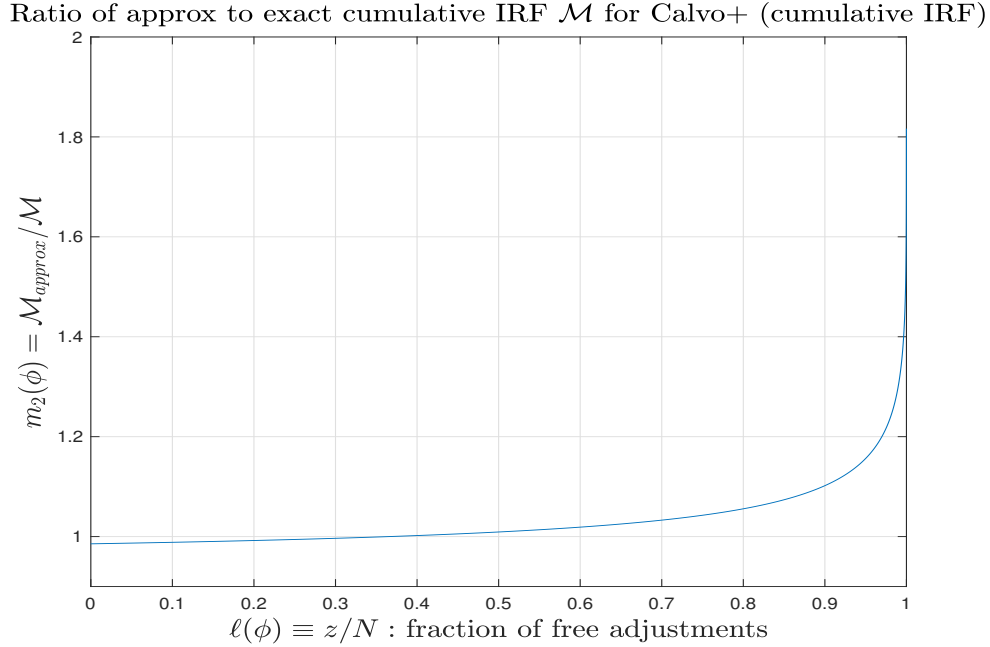
PROPOSITION 7. The derivative of the IRF with respect to t at $t = 0$ is given by:

$$\frac{\partial}{\partial t}Y(t)\bigg|_{t=0} = -\infty \quad \text{for} \quad 0 \leq \phi < \infty.$$

Note that when $\phi \to \infty$, so we get the pure Calvo model, so that the impulse response becomes $Y(t) = \exp(-Nt)$, and thus $Y'(0)$ is finite.[7]

Price Plans and the hump-shaped Output IRF. The model with price plans assumes that upon paying the menu cost the firm can choose two, instead of one price. At any point in time the firm is free to charge either price within the current plan, but changing to a new plan (another pair of prices) is costly. The idea was first proposed by Eichenbaum, Jaimovich, and Rebelo (2011) to model the phenomenon of temporary price changes (prices that move from a reference value for a short period of time and then return to it). In Alvarez and Lippi

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[7] See the web Appendix F analyzes the the output IRF for the pure Calvo model.

(2019) we provide an analytic solution to this problem and characterize the determinants of $\bar{x}$, the threshold where a new plan is chosen, as well as the optimal prices within the plan, named $\tilde{x}$ and $-\tilde{x}$. When $x \in [-\bar{x}, 0]$ the firm charges $-\tilde{x}$ and when $x \in (0, \bar{x}]$ it charges $\tilde{x}$. The invariant density of price gaps is still given by equation (20). For a given threshold $\bar{x}$ the value of $\tilde{x}$ is given by:

$$\tilde{x} = \bar{x} \left[\frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi} \left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2 \right)} \right] \equiv \bar{x} \, \rho(\phi) > 0 \quad \text{where} \quad \phi = \bar{x}^2 \zeta / \sigma^2$$

and the function $\rho(\phi)$ gives the optimal price within the plan as a function of the adjustment threshold, namely $\tilde{x} = \rho(\phi)\bar{x}$, as a function of ϕ. Simple analysis shows that the images of the function $\rho(\phi)$ lie in the interval $(0, 1/3)$, that it is decreasing, and that it converges to $1/3$ as $\phi \to 0$ (see Alvarez and Lippi (2019)).

In the model with plans the contribution to the aggregate of a firm with output price gap x is, instead of $f(x) = -x$, the following function:

$$\tilde{f}(x) = \begin{cases} -x - \tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ -x + \tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

By the linearity of Fourier series we can add to the coefficients of the function $f(x) = -x$, shown in equation (16) the ones of the step function:

$$f_0(x) = \begin{cases} -\tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ +\tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

Importantly, we note the function $\tilde{f}(x) = f(x) + f_0(x)$ is still an asymmetric function. The function f_0 has Fourier sine coefficients equal to:

$$b_j[f_0] = -\frac{8\bar{x}^{3/2}\rho(\phi)}{j\,\pi} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[f_0] = 0 \text{ otherwise}$$

From here we conclude that:

$$\beta_j^0(\phi) = b_j[f_0]b_j[\bar{p}'] = \begin{cases} 0 & \text{if } j \text{ is odd} \\[2ex] -4\,\rho(\phi)\left[\dfrac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\dfrac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\[2ex] 0 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases} \quad (22)$$

Thus the impulse response is given by:

$$Y_{\text{Plan}}(t) = Y_{\text{Calvo}^+}(t) + \sum_{j=1}^{\infty} \beta_j^0(\phi)e^{\lambda_j(\phi)t}$$

While the impulse response is monotone decreasing in the Calvo$^+$ model, in the price plan model the impulse response can be hump shaped. Indeed, as the ϕ increases, the impulse response goes from decreasing to humped shaped. The reason for this difference is that in the price plan model there is a non-negligible impact effect, due to the non-negligible set of firms that within the plan adjust from one price to the other. The difference across models, as we increase ϕ, is that invariant distribution has more firms with price gaps close to zero, and hence more firms that can change from one price to the other within the plan. This is because the higher is the impact effect on prices, the smaller is the effect on output. The next proposition indeed shows that when $\phi = 1$, so that $\zeta = \sigma^2/\bar{x}^2$, i.e. the number of plan changes in a pure-barrier model equals the one in a pure-random-plan model, are equal. For $\phi = 1$ the impulse response has a "hump", but it is infinitesimal.

PROPOSITION 8. For $0 \leq \phi \leq 1$, the impulse response $Y(t)$ is decreasing. For $\phi > 1$, the impulse response is hump shaped. The value for $t_0 N$ at which the maximum is reached, $Y'(t_0 N) = 0$, increases relative to expected time to adjustment $1/N$.

Given that the impulse response is hump-shaped any single eigenvalue cannot approximate the impulse response. Even for low values of ϕ, where the impulse response is monotone, using λ_2 gives a very bad approximation.

Figure 5: Price Plan model

Output IRF to monetary shock, Price Plan Model
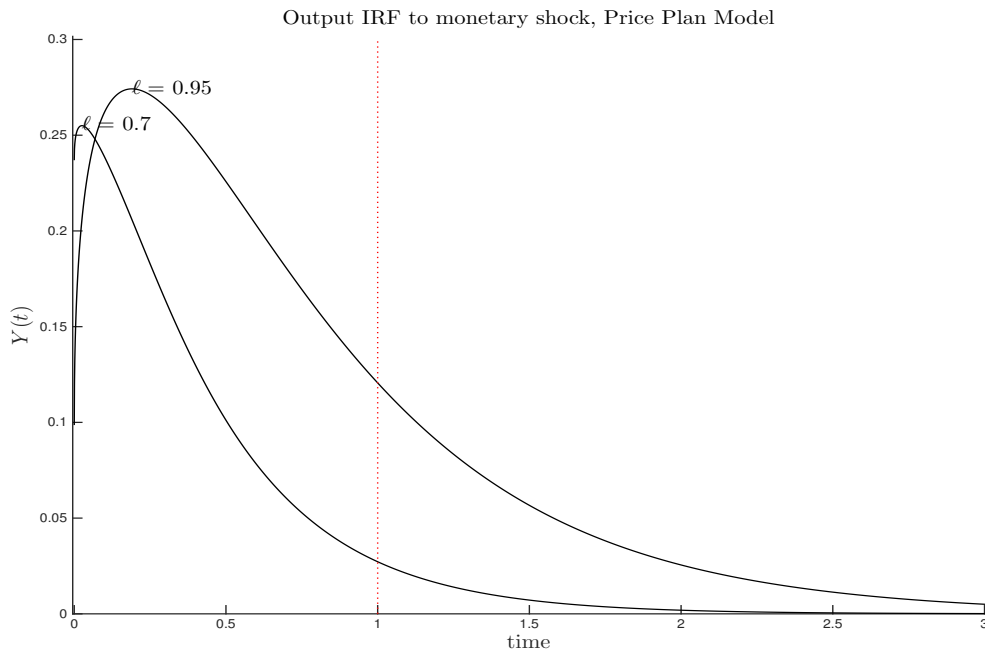
6 Volatility shocks and the propagation of monetary impulses

This section discusses the effect that changes to the volatility of shocks exert on the propagation of monetary shocks. The issue matters to e.g. the effectiveness of monetary policy in recessions vs boom, when the state of the economy is assumed to feature, respectively, high vs low volatility of shocks as in Vavra (2014). Our method provides a sharp analytic answer to this question.

For concreteness we illustrate the problem by using the pure menu cost model (without Calvo adjustment i.e. $\zeta = 0$ so that $\phi = \ell = 0$), whose output response to a small monetary shock was given in equation (18). We conduct comparative statics exercise to analyze how the propagation is affected by an innovation of the "volatility shocks", namely a permanent change in the common value of the idiosyncratic volatility σ.[8]

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[8] For simplicity and clarity of the results we consider here once and for all shocks to volatility. It is simple to modify the setup to consider a two-state Markov switching volatility process and to solve the associated

We start with a steady state for the model with idiosyncratic volatility σ. We characterize the effect of a small monetary shock, $\delta > 0$, which occurs $\tau \geq 0$ periods after a change in idiosyncratic volatility from σ to $\tilde{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$. In particular we let $Y(t; \tau, d\sigma/\sigma)\delta$ denote the output's IRF $t \geq 0$ periods after of an unexpected monetary shock of size δ starting with a cross sectional distribution that has evolved τ periods since the change in σ.[9]

While we characterize Y for all t and τ's, two interesting cases are worthwhile to mention separately: the long-run and short-run effect of volatility. The long-run effect is $Y(t; \infty, d\sigma/\sigma)$, or $\tau \to \infty$. It is equivalent to computing the effect of a monetary shock δ for a new steady state with volatility $\tilde{\sigma}$. We refer to this as the *long run* effect of volatility on the output IRF after a monetary shock, since it is the effect of an unanticipated monetary shock once the distribution of price gaps has achieved its new invariant distribution. In this case the firm's decision rule corresponds to the new volatility $\tilde{\sigma}$ and the economy is described by the new invariant distribution of price gaps.

The other case is the short-term effect, defined as $Y(t; 0, d\sigma/\sigma)$, or $\tau = 0$. This case consists of starting with the original volatility σ and considering a simultaneous permanent change of both σ (to $\tilde{\sigma}$) and $\delta > 0$. As in the previous case, the forward looking firm's decision rules adjusts immediately to the new volatility $\tilde{\sigma}$. The difference with the long-run case is that the initial distribution of price gaps corresponds to the stationary distribution produced by the old decision rule, i.e the decision rules implied by volatility σ.

The general case characterizes an IRF whose coefficients are indexed by the parameter $0 < \tau < \infty$. The key feature of this case is that the monetary shock δ occurs τ periods after the volatility shock, thus displacing a cross-section distribution of price gaps that is in a transition towards the new invariant distribution. Our analytic method allows us to exactly compute the evolution of this distribution and hence the effect of a monetary shock.

The next proposition uses the notation introduced above, which means that $Y(t; 0, 0)$ is

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firm's decision rules.

[9] We will keep using the notation of Y as the output's IRF per unit of monetary shock, and then omit the δ in the expressions below.

the impulse before any change in volatility occurs, which we use as a benchmark. Also, the difference $Y\left(t;\tau,\frac{d\sigma}{\sigma}\right) - Y\left(t;\infty,\frac{d\sigma}{\sigma}\right)$ is the correction to the long run effect of a volatility shock $d\sigma/\sigma$ due to a finite duration τ.

PROPOSITION 9. Let $Y(t)$ be the output impulse response to a small monetary shock for the idiosyncratic volatility σ, and let the new volatility of shocks be $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$. The *long run* effect of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right);0,0\right) \quad \text{for all } t \geq 0 . \tag{23}$$

The *short run* effect of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$Y\left(t;0,\frac{d\sigma}{\sigma}\right) = \left(1 + \frac{d\sigma}{\sigma}\right) Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right);0,0\right) \quad \text{for all } t \geq 0 . \tag{24}$$

The deviation from the long run response as a function of τ is given by:

$$Y\left(t;\tau,\frac{d\sigma}{\sigma}\right) - Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k}t} b_{2k}[f] \, b_{2k}\left[\hat{p}'(\cdot,\tau)\right] \quad \text{for all } t,\tau \geq 0 . \tag{25}$$

where $\hat{p}'(\cdot,\tau)$ is the initial condition (i.e. a displaced cross section) at the time of the monetary shock, τ periods after the change in volatility, whose projection coefficients are given by:

$$b_{2k}\left[\hat{p}'(\cdot,\tau)\right] = \frac{d\sigma}{\sigma} \sum_{j=1,3,5,\dots}^{\infty} e^{\lambda_j \tau} \left(2\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\pi)^2}\right) \left(\frac{4kj}{(4k^2 - j^2))}\right) \quad , \quad k = 1,2,3,\dots \tag{26}$$

A few comments are in order.

(i) Figure 6 illustrates the difference between the short run and long run effect of an increase in volatility on the output's response to a monetary shock. The left panel compares the IRF with no change in volatility, $Y(t;0,0)$ to the one where the volatility increase has occurred $\tau = \infty$ periods ago, i.e. $Y(t;\infty,d\sigma/\sigma)$ the long run effect. The right panel compares

the IRF with no change in volatility, $Y(t; 0, 0)$ to the one where the volatility increase has occurred at the same time as the monetary shock $\tau = 0$ periods ago, i.e. $Y(t; 0, d\sigma/\sigma)$ the short run effect.

Figure 6: Short-run and long-run IRF vs. IRF before volatility increases

<div align="center">

Long run $(\tau \to \infty)$ Short run $(\tau = 0)$

</div>
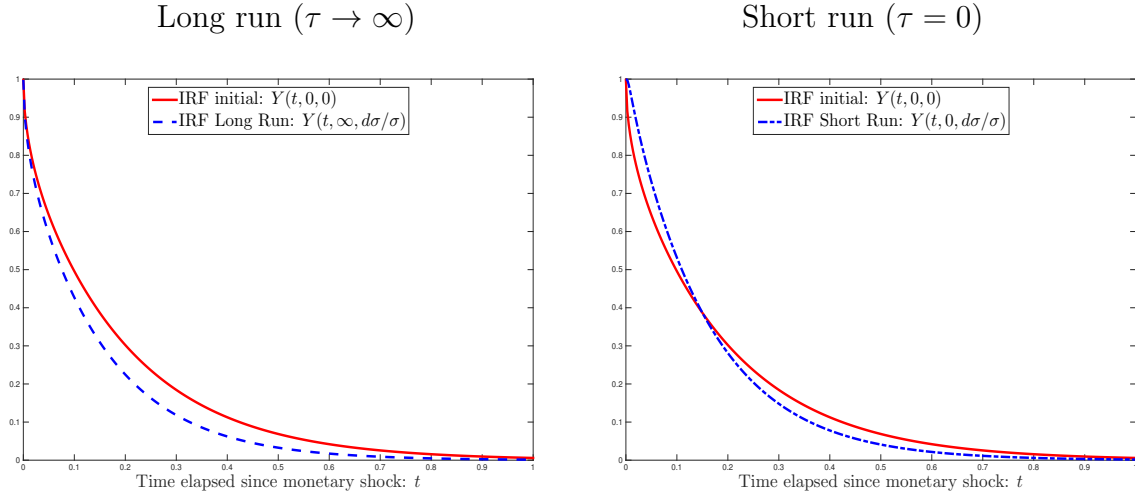
Note: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

(ii) For this proposition we use the form of the decision rules for the threshold $\bar{x}$, which as the discount rate goes to zero is $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$ where ψ is the fixed cost –as fraction of the frictionless profit and B is the curvature of the profit function around the frictionless profit. This implies that the elasticity of $\bar{x}$ to σ is $1/2$. This elasticity is the so called "option value" effect on the optimal decision rules.

(iii) The rescaling of time in $Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right); 0, 0\right)$ in the expressions for the long and short run effect of volatility reflects the change in the eigenvalues, which depend on the value of N, the implied average number of price changes per unit of time, as $\lambda_j = -N(\pi j)^2/8$ (see equation (19) for $\zeta = 0$). Recall that $N = (\sigma/\bar{x})^2$, and hence all the eigenvalues change proportionally with σ.

(iv) For the case of the impact effect and in which $\tilde{\sigma} > \sigma$, the invariant distribution just before the monetary shock is narrower than the range of inaction that corresponds to the new wider barriers. This explains the extra multiplicative term level $\left(1 + \frac{d\sigma}{\sigma}\right)$ in the impact

effect in equation (24): since firms have price gaps that are discretely away from the inaction bands, then prices react more slowly, generating the extra effect on output. The logic for the case where $\tilde{\sigma} < \sigma$ is similar.

(v) We have found that Y is differentiable with respect to $\tilde{\sigma}$, when evaluated at $\tilde{\sigma} < \sigma$. It may be surprising that the right and left derivatives (corresponding to the cases of increases and decreases on σ) are the same, because in the case of a decrease in σ there is a positive mass of firms that change prices on impact. Nevertheless, this effect is of smaller order of magnitude than the change on σ.

(vi) In equation (25) we use only the even terms for the projections, i.e. the index for the projection $b_{2k}[\cdot]$ for runs on $2k$ because f is antisymmetric. This means that, as in the case without volatility shocks, the eigenvalues that control the effect of the horizon t in the IRF are the even ones, i.e. $\lambda_2, \lambda_4, \ldots$, starting with the leading one λ_2.

(vii) The expressions in equation (25) and equation (26) shows that what governs the difference between the long run and the short run volatility effects are the odd eigenvalues, i.e. $\lambda_1, \lambda_2, \ldots$, since these are the only elements where τ affect the expressions. In particular, λ_1 is the dominant eigenvalue.

(viii) We note that the expression for correction term in equation (25) for the general case of $0 < \tau < \infty$ involves no parameter for the model with the exception of N, which enters only in the eigenvalues $\lambda_j = -N(j\pi)^2/8$. This gives a meaning to the units of t and τ, which are measured relative to the (new) steady state duration of price changes $1/N$.

(ix) To illustrate the general case of $0 < \tau < \infty$ in Figure 7 we display two type of plots. First, the left panel of Figure 7 plots equation (25), evaluating at 4 values of τ for an interval of times t. It is apparent that as τ becomes bigger monetary policy becomes less effective and gradually converges to the long run value. This can be seen comparing the correction for any given t across the four values of τ. Second, in the right panel, we plot the cumulated IRF of a monetary shock τ periods after the volatility shock relative to the cumulative IRF

Figure 7: The propagation of monetary shocks as τ grows

$$Y(t; \tau, d\sigma/\sigma) - Y(t; \infty, d\sigma/\sigma) \qquad \text{Change in Cumulated output: } \mathcal{C}(\tau, d\sigma/\sigma)$$
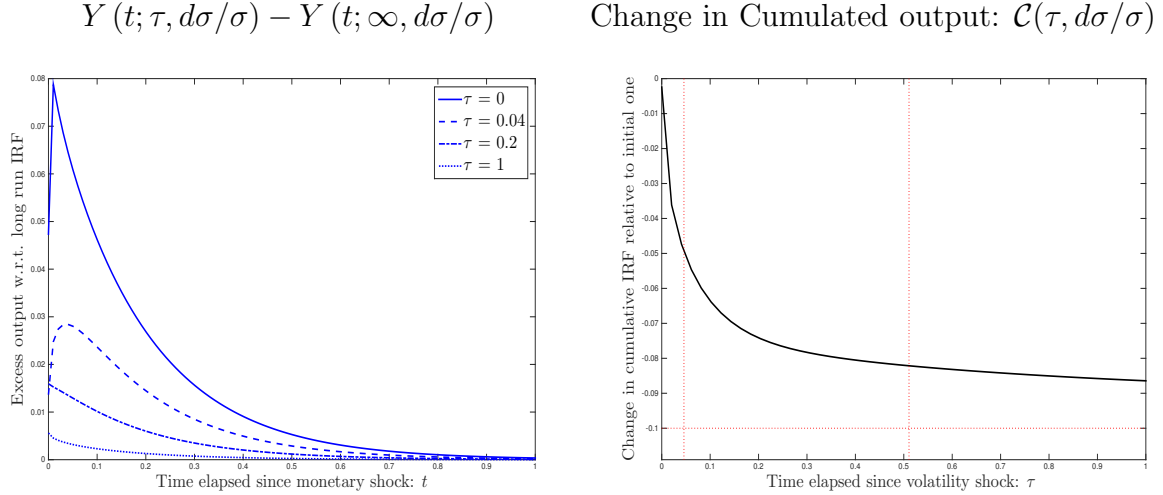
Note: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

of a monetary shock when there is no volatility shock. In particular we plot:

$$\mathcal{C}(\tau, d\sigma/\sigma) \equiv \frac{\int_0^\infty Y(t, \tau, d\sigma/\sigma)dt}{\int_0^\infty Y(t, 0, 0)dt} - 1$$

We use the cumulated IRF to obtain a simple one-dimensional summary of this effect across all times t. Notice the following properties of $\mathcal{C}$: for all τ we have $\mathcal{C}(\tau, d\sigma/\sigma) = (d\sigma/\sigma)\mathcal{C}(\tau, 1)$, since it is based on a derivative, and for extreme values of τ we have $\mathcal{C}(\infty, d\sigma/\sigma) = -d\sigma/\sigma$, and $\mathcal{C}(0, d\sigma/\sigma) = 0$. From Figure 7 it is clear that the transition to the higher volatility occurs very fast, a cumulative effect of $\mathcal{C}$ half as large as half of the one in $\tau = \infty$ will occur when $\tau_{1/2} \approx 0.05$, a half-life indicated by a vertical bar in the right panel. More precisely, $\tau_{1/2}$ is defined as $\mathcal{C}(\tau_{1/2}, d\sigma/\sigma) = -(1/2)(d\sigma/\sigma)$. This effect is much faster than the half-life corresponding to the dominant eigenvalue $\lambda_1 = -N\pi^2/8$, which is given by $t_{1/2} \equiv -8\log(0.5)/(N\pi^2) \approx 0.56$, and it is indicated by another vertical bar in the right panel. The ratio of the two times is very large: $t_{1/2}/\tau_{1/2} \approx 12$, and it is independent of any parameter of the model.[10] From this comparison we conclude that for this particular

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[10] The vertical distance on the correction between $\tau =$ and $\tau = \infty$ plotted on the right panel is $d\sigma/\sigma$, which is 0.1 for this example. For other values, the vertical axis scales proportionally.

model using exclusively the dominant eigenvalue λ_1 to approximate the time it takes for the distribution to converge after the change in volatility will be misleading. Summarizing, in the Golosov-Lucas model the short run effect of the volatility change is only relevant when the monetary shock occurs almost immediately after the volatility change.

7 Sticky Price Multiproduct firms

Multiproduct models consider a firm that produces n different products and that faces increasing returns in the price adjustment: if a pays a fixed cost it can adjust simultaneously the n prices. Variations on this model have been studied by Midrigan (2011) and Bhattarai and Schoenle (2014). These models are appealing because they match several empirical regularities: synchronization among price changes within a store and the coexistence of both small and large price changes. Their economic analysis is of interest because in an economy populated by multiproduct firms the monetary shocks have more persistent real effects. In Alvarez and Lippi (2014) we derived results for impulse responses to this multidimensional setup and explore the sense in which such a model is realistic. Here we show that the characterization of the selection effect, as the difference between the survival function and the output IRF holds in this model, with the number of products n serving as the parameter that control selection. We also show that in this case a single eigenvalue gives a poor characterization of the output IRF.

In the multiproduct model the price gap is given by a vector of n price gaps, each of them given by an independently standard BM's $(p_1, p_2, \ldots p_n)$, driftless and with innovation variance σ^2. We are interested only on two functions of this vector, the sum of its squares and it sum:

$$y = \sum_{i=1}^{n} p_i^2 \text{ and } z = \sum_{i=1}^{n} p_i$$

It is interesting to notice that while the original state is n dimensional, (y, z) can be described

as a two dimensional diffusion –see Alvarez and Lippi (2014) and Appendix C for details.

We are interested in the sum of its squares y because in Alvarez and Lippi (2014) under the assumption of symmetric demand the optimal decision rule is to adjust the firm time that y hits a critical value $\bar{y}$. We are interested in z, the sum of the price gaps, because this give the contribution of firm to the deviation of the price level relative to the steady state value, and hence $-z$ is proportional to its contribution to output. Note that the domain of (y, z) is $0 \leq y \leq \bar{y}$ and $-\sqrt{ny} \leq z \leq \sqrt{n\,y}$. In Alvarez and Lippi (2014) we show that the expected number of adjustments per unit of time is given by $N = \frac{n\sigma^2}{\bar{y}}$ and also give a characterization of $\bar{y}$ in terms of the parameters for the firm's problem. For the purpose in this paper we find it convenient to rewrite the state as (x, w) defined as

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}.$$

In Lemma 2 in Appendix C we analyze the behavior of the $(x, w) \in [0, \bar{x}] \times [-1, 1]$ process with $\bar{x} \equiv \sqrt{\bar{y}}$. Clearly we can recover (y, z) from (x, w). For instance, $z = w\sqrt{n}x$. Yet with this change on variables, even though the original problem is n dimensional, we define a two dimensional process for which we can analytically find its associated eigenfunctions and eigenvalues for the operator:

$$\mathcal{G}(f)\left((x, w), t\right) = \mathbb{E}\left[f\left(x(t), w(t)\right) 1_{y \geq \bar{y}} \,\Big|\, (x(0), w(0)) = (x, w) \right]$$

where $f : [0, \bar{x}] \times [-1, 1] \rightarrow \mathbb{R}$. The relevant p.d.e. is defined and its solution via eigenfunctions and eigenvalues, is characterized in Proposition 15 in Appendix C. Moreover the eigenfunctions and eigenvalues are indexed by a countably double infinity indices $\{m, k\}$.

Eigenfunctions. The eigenfunctions φ have a multiplicative nature, so $\varphi_{m,k}(x, w) = h_m(w)g_{m,k}(x)$ where for each number of products n then h_m and $g_{m,k}$ are known analytic functions indexed by k and by (k, m) respectively. Indeed h_m are scaled Gegenbauer polynomials, and $g_{m,k}$ are scaled Bessel functions –see Proposition C for the exact expressions and

definition.[11]

Eigenvalues. For each n the eigenvalues can be also indexed by a countably double-infinity $\{\lambda_{m,k}\}$. As in the baseline case, the eigenvalues are proportional to N, the expected number of price changes per unit of time:

$$\lambda_{m,k} = -N \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \quad \text{for } m = 0, 1, \ldots, \text{ and } k = 1, 2, \ldots$$

$j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

The second sub-index k in the root of the Bessel function denote their ordering, so $k = 1$ is the smallest positive root. Also fixing k the roots $j_{m+\frac{n}{2}-1,k}$ are increasing in m. Thus, the *dominant* eigenvalue is given by $\lambda_{0,1}$. We will argue below that the smallest (in absolute value) eigenvalue that is featured in the (marginal) output IRF is $\lambda_{1,1}$. A very accurate approximation of the eigenvalues consists on using the first three leading terms in its expansion, as is given by: $j_{\nu,k} \approx \nu + \nu^{1/3}2^{-1/3}a_k + (3/20)(a_k)^2 2^{1/3}\nu^{-1/3}$ where a_k are the zeros of the Airy function. [12] Using this approximation into the expression for the eigenvalues, one can see that keeping fixed N, the absolute value both $\lambda_{0,1}$ and $\lambda_{1,1}$ go to infinity, and that the difference between the two decreases and converges to $N/2$. Figure 8 displays the difference between these two eigenvalues.

Impulse response. As before, we want to compute $G(t)$, the conditional expectation of $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$ for (x, w) following equation (40)-equation (41), integrated with respect to $p(w, x; 0)$. We are interested in functions $f : [0, \bar{x}] \times [-1, 1]$ that can be written as:

$$f(x, w) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{m,k}[f]\varphi_{m,k}(x, w)$$

---

[11] The Gegenbauer polynomials are orthogonal to each other, and so are the Bessel functions when using an appropriately weighted inner product, as defined in Appendix C.

[12] In our case, we are interested in $k = 1$ which is about $a_1 = -2.33811$. See Figure 10 in the APP where we plots both eigenvalues, as well as its approximation for several n.

Using the same logic as in the one dimensional case:[13]

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k} t} b_{m,k}[f] \, b_{m,k} \left[p(\cdot, 0)/\omega \right] \, \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

where the term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ appears because the have, as it is customary in this case, use an orthogonal, but not orthonormal base, and where $\omega(w, x)$ is a weighing function appropriately defined – see Appendix C. So that $b_{m,k} \left[p(\cdot, 0)/\omega \right]$ are the projections of the ratio of the functions $p(\cdot, 0)$ and ω.

Functions of interest. We analyze two important functions of interest f. The first one a constant, $f(w, x) = 1$ which is used to compute the measure of firms that have not adjusted, or the survival function $S(t)$. The second one is the one that gives the average price gap among the n product of the firm, i.e. $f(w, z) = -z/n = -wx/\sqrt{n}$. This is, as before, the negative of the average across the n products of the price gaps. This is the function f used for the impulse response of output to a monetary shock. An important property of the Gegenbauer polynomials is that the $m = 0$ equals a constant, for $m = 1$ is proportional to w, and in general for m odd are antisymmetric on w and symmetric for even m. Thus for $f = 1$ we can use just the Gegenbauer polynomial with $m = 0$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$. Instead for $f(w, x) = wx/sqrtn = z$ we can use just the Gegenbauer polynomial with $m = 1$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$.

Initial shifted distribution for a small shock. We have derived the invariant distribution of (z, y) in Alvarez and Lippi (2014). Using the change in variables (y, z) to $y = x^2$ and $z = \sqrt{yn} \, w =$ we can define the steady state density as $\bar{p}(w, x) = \bar{h}(w)\bar{g}(x)$ – see Appendix C for the expressions. We perturb this density with a shock of size δ in each of the n price gaps. We want to subtract δ to each component of $(p_1, ..., p_n)$. This means that the density for each $x = ||p||$ just after the shock becomes the density of $x(\delta) = ||(p_1 + \delta, \ldots, p_n + \delta)||$ just before. Likewise the density corresponding to each w becomes th one for $w(\delta) = (z + n\delta)/(\sqrt{n} \, x(\delta))$.

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[13]See Appendix C for a derivation

We consider the initial condition given by density $p_0(w, x; \delta) = \bar{h}(w(\delta))\bar{g}(x(\delta))$. We will use the first order terms, which are appropriate for the case of a small shock δ. The expressions can be found in

Interpretation of dominant eigenvalue, and irrelevance for the marginal IRF. We are now ready to generalize our interpretation of the dominant eigenvalue (as well as those corresponding to symmetric functions of z), as well as its irrelevance for the marginal output IRF.

PROPOSITION 10.   The coefficient of the marginal impulse response of output for a monetary shock are a function of the $\{\lambda_{1,k}, \varphi_{1,k}\}_{k=1}^{\infty}$ eigenvalue-eigenfunctions pairs, so that:

$$Y(t) = \sum_{k=1}^{\infty} \beta_{1,k}\, e^{\lambda_{1,k} t} \;\; \text{and} \;\; -\lambda_{1,1} = \lim_{t\to\infty} \frac{\log|Y(t)|}{t}$$

where $\beta_{1,k} = b_{1,k}\left[wx/\sqrt{n}\right] b_{1,k}\left[\bar{p}'(w, x)\right]$. In particular, the dominant eigenvalue $\lambda_{0,1}$ does not characterize the limiting behavior of the impulse response. Instead the survival function for price changes $S(t)$, can be written in terms of $\{\lambda_{0,k}, \varphi_{0,k}\}_{k=1}^{\infty}$, and hence the asymptotic hazard rate is equal to the dominant eigenvalue $\lambda_{0,1}$, i.e.
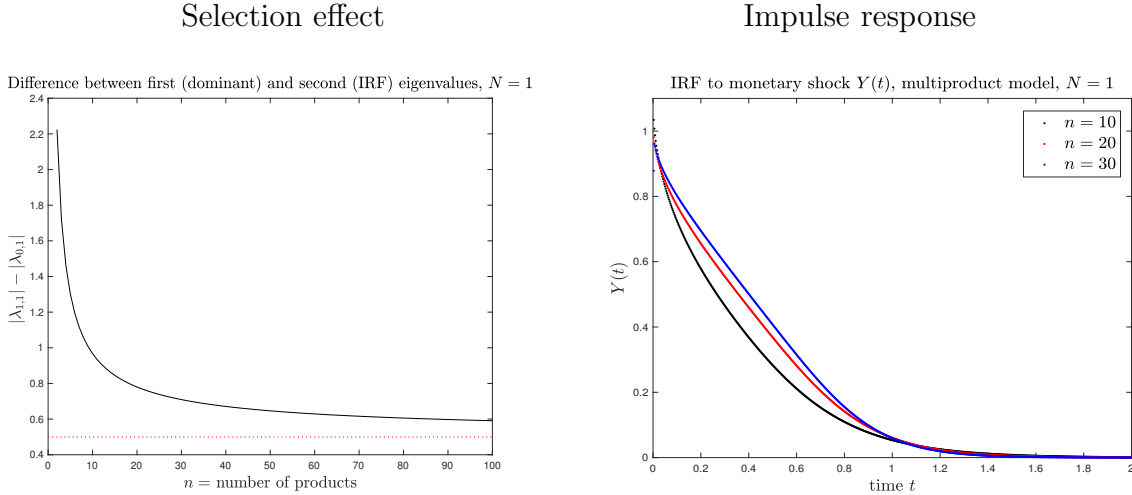
$$S(t) = \sum_{k=1}^{\infty} \beta_{0,k}\, e^{\lambda_{0,k} t} \;\; \text{and} \;\; -\lambda_{0,1} = \lim_{t\to\infty} \frac{\log S(t)}{t}$$

where $\beta_{0,k} = b_{0,k}\left[1\right] b_{0,k}\left[\delta_0\right]$ where δ_0 is the Dirac delta function for $(p_1, \ldots, p_n)$ transformed to the (x, w) coordinates. Recall that $0 > \lambda_{0,1} > \lambda_{1,1}$.

Given the importance of the difference between the eigenvalues $\lambda_{1,k}$ and $\lambda_{0,k}$ we show that for a fixed k they both increase with n, but it difference decreases to asymptote to $1/2$.

PROPOSITION 11.   Fixing $k \geq 1$, the kth eigenvalue for the IRF $Y(\cdot)$ given by $\lambda_{1,k}$ and the kth eigenvalue for the survival function $S(\cdot)$ given by $\lambda_{1,k}$ both increase with the number of products n, diverging towards $-\infty$ as $n \to \infty$. The difference $\lambda_{0,k} - \lambda_{1,k} > 0$ decreases with n, converging to $1/2$ as $n \to \infty$.

Figure 8: Shock propagation in Multiproduct models

Selection effect · Impulse response

Difference between first (dominant) and second (IRF) eigenvalues, $N = 1$

IRF to monetary shock $Y(t)$, multiproduct model, $N = 1$

Keeping fixed $N = 1$ for all n

Figure 8 illustrates Proposition 11 for the case of $k = 1$, i.e. the eigenvalue that dominates the long run behaviour of the survival and IRF functions. Proposition 11 extends the result for all k. Increasing the number of products n in the multi product model decreases the selection effect at the time of a price change. As n goes to infinity, the eigenvalues that control the duration of the price changes (S) and those that control the marginal output IRF (Y) converge. This result shows that the characterization of selection effect in terms of dynamics controlled by two different types of eigenvalues is present not only in the Calvo$^+$ model, but also in this setup.

In Appendix C we include Proposition 16 which gives an closed form solution for $\bar{p}'(w, x; 0)$ and for the coefficients for $b_{1,k}$ of the output impulse response function. All these expressions depends only of the number of products n. Instead we include a figure of the impulse responses for three values of n. It is clear both the output IRF and the survival function cannot be well described using one eigenfunction-eigenvalue for large n. For instance, as $n \to \infty$ the output's IRF Y becomes a linearly declining function until it hits zero at $t = 1/N$, and the survival function S is zero until it becomes infinite at $t = 1/N$.

8 Asymmetric problems: dealing with reinjection

In this section we study the impulse response function for problems where the symmetry assumptions of Proposition 2 do not hold. In such a case the computation of the impulse response function requires keeping track of firms after their first adjustment, so that the impulse response function $H(t)$ cannot be computed by means of the simpler operator $G(t)$. The solution to this problem is to compute the law of motion of the cross-sectional distribution using the Kolmogorov forward equation, keeping track of the reinjenctions that occur after the adjustments.

The nature of reinjection in our set up differs from the one in Gabaix et al. (2016) and hence we cannot use their results for the ergodic case. The added complexity of our case originates because the exit points (of our pricing problem) are not independent of where x is, as in the case of poisson adjustments. Rather, prices are changed when either barrier $\underline{x}$ or $\bar{x}$ is hit, and then the measure of products whose prices are changed are all reinjected at single value, the optimal return point x^*.

The set-up consists of an unregulated BM $dx = \mu dt + \sigma dB$, which returns to a single point x^* the first time that x hits either of the barriers $\underline{x}$ or $\bar{x}$ or that a Poisson counter with intensity ζ changes. As implied by Proposition 2 we cannot ignore the reinjenctions at x^* if either $x^* \neq (\bar{x} + \underline{x})/2$ or $\mu \neq 0$. For simplicity consider the case with no drift, so we set $\mu = 0$. We can use the method in Appendix B to modify the result accordingly. This set up can be used to study the problem of a firm with a non-symmetric period return function in an economy without inflation ($\mu = 0$). In this case the optimal decision rule implies $x^* \neq (\bar{x} + \underline{x})/2$, i.e. the reinjection point (after adjustment) is not located in the middle of the inaction region. Note that the number of price adjustment per unit of time is given by $N = \frac{\sigma^2}{(\bar{x}-\underline{x})^2 \alpha(1-\alpha)}$.

Let $\hat{p}(x)$ denote the initial condition for the density of firms relative the invariant distribution $\bar{p}(x)$, i.e. $\hat{p}(x) = p(x) - \bar{p}(x)$ for some density p, where $\bar{p}$ is the asymmetric (steady state) tent map. Notice that to analyze small shocks δ, i.e. an initial condition $p(x) = \bar{p}(x + \delta)$ the

signed measure is $\hat{p}(x) = \delta\bar{p}'(x)$ by a simple Taylor expansion and mass preservation requires that $\int_{\underline{x}}^{\bar{x}} \hat{p}(x)dx = 0$, so that

$$\hat{p}(x)/\delta = \begin{cases} \frac{2}{(\bar{x}-\underline{x})^2\alpha} & \text{if } x \in [\underline{x}, x^*) \\ \frac{-2}{(\bar{x}-\underline{x})^2(1-\alpha)} & \text{if } x \in (x^*, \bar{x}]. \end{cases} \tag{27}$$

We define the Kolmogorov forward operator $\mathcal{H}^*(\hat{p}) : [\underline{x}, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}$ for the process with reinjection, where $\mathcal{H}^*(\hat{p})(x, t)$ denotes the cross-sectional density of the firms t periods after the shock. In this case we have that for all $x \in [\underline{x}, x^*) \cup (x^*, \bar{x}]$ and for all $t > 0$:

$$\partial_t \mathcal{H}^*(\hat{p})(x, t) = \frac{\sigma^2}{2}\partial_{xx}\mathcal{H}^*(\hat{p})(x, t) - \zeta\mathcal{H}^*(\hat{p})(x, t) \tag{28}$$

with boundary conditions:

$$\mathcal{H}^*(\hat{p})(\underline{x}, t) = \mathcal{H}^*(\hat{p})(\bar{x}, t) = 0 \quad, \quad \lim_{x\uparrow x^*}\mathcal{H}^*(\hat{p})(x, t) = \lim_{x\downarrow x^*}\mathcal{H}^*(\hat{p})(x, t) \tag{29}$$

$$\partial_x^-\mathcal{H}^*(\hat{p})(\bar{x}, t) - \partial_x^+\mathcal{H}^*(\hat{p})(x^*, t) + \partial_x^-\mathcal{H}^*(\hat{p})(x^*, t) - \partial_x^+\mathcal{H}^*(\hat{p})(\underline{x}, t) = \frac{2\zeta}{\sigma^2} \tag{30}$$

$$\mathcal{H}^*(\hat{p})(x, 0) = \hat{p}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{31}$$

The p.d.e. in equation (28) is standard, we just note that it does not need to hold at the reinjection point x^*. The boundary conditions in equation (29) are also standard, given that $\underline{x}$ and $\bar{x}$ are exit points, and that with $\sigma^2 > 0$, the density must be continuous everywhere. The condition in equation (30) ensures that the measure is preserved, or equivalently that there is no change in total mass across time: $\int_{\underline{x}}^{\bar{x}} \mathcal{H}^*(\hat{p})(x, t)dx = \int_{\underline{x}}^{\bar{x}} \hat{p}(x)dx$ for all t. This is a small extension of Proposition 1 in Caballero (1993).

We can use $\mathcal{H}^*$ to compute the Impulse response function defined above as follows:

$$H(t, f, \hat{p}) = \int_{\underline{x}}^{\bar{x}} f(x)\,\mathcal{H}^*(\hat{p})(x, t)\,dx. \tag{32}$$

If $\zeta > 0$ and $\bar{x} \to \infty$ as well as $\underline{x} = \infty$, we will have the pure Calvo case, and we can use the ideas in Gabaix et al. (2016), and thus the case without reinjection and with reinjection are quite similar. To highlight the difference, we consider the opposite case, and set $\zeta = 0$ and use an eigenvalue decomposition of $\mathcal{H}^*$.

PROPOSITION 12. Assume that $\zeta = 0$ and that α is not a rational number. The orthonormal eigenfunctions of $\mathcal{H}^*$ are:

$$\varphi_j^m(x) = \sqrt{\frac{2}{(\bar{x} - \underline{x})}} \sin\left(\left[\frac{x - \underline{x}}{\bar{x} - \underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, \bar{x}] \tag{33}$$

$$\varphi_j^l(x) = \sqrt{\frac{2}{(x^* - \underline{x})}} \sin\left(\left[\frac{x - \underline{x}}{x^* - \underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, x^*] \text{ and } 0 \text{ otherwise} \tag{34}$$

$$\varphi_j^h(x) = \sqrt{\frac{2}{(\bar{x} - x^*)}} \sin\left(\left[\frac{x - x^*}{\bar{x} - x^*}\right] 2\pi j\right) \text{ if } x \in [x^*, \bar{x}] \text{ and } 0 \text{ otherwise} , \tag{35}$$

with corresponding eigenvalues:

$$\lambda_j^m = -\frac{\sigma^2}{2} \frac{(2\pi j)^2}{(\bar{x} - \underline{x})^2} , \ \lambda_j^l = -\frac{\sigma^2}{2} \frac{(2\pi j)^2}{(x^* - \underline{x})^2}, \text{ and } \lambda_j^h = -\frac{\sigma^2}{2} \frac{(2\pi j)^2}{(\bar{x} - x^*)^2} , \tag{36}$$

for all $j = 1, 2, \ldots$. The eigenfunctions in the set $\{\varphi_j^m\}_{j=1}^\infty$ are orthogonal to each other, and so are those in the set $\{\varphi_j^l, \varphi_j^h\}_{j=1}^\infty$. The eigenfunctions $\{\varphi_j^m, \varphi_j^l, \varphi_j^h\}_{j=1}^\infty$ span the set of functions $g : [\underline{x}, \bar{x}] \to \mathbb{R}$, piecewise differentiable, with countably many discontinuities, and with $\int_{\underline{x}}^{\bar{x}} g(x)dx = 0$.

By defining $\mathcal{H}^*$ for initial conditions given by the differences of a density relative to the density of the invariant distribution, we are excluding the zero eigenvalue and its corresponding eigenfunction, the invariant distribution $\bar{p}$ from the its representation. From the proposition we see what are the first two non-zero eigenvalues.

$$\lambda_1 = -\frac{\sigma^2}{2} \left(\frac{2\pi}{\bar{x} - \underline{x}}\right)^2 > \lambda_2 = -\frac{\sigma^2}{2} \left(\frac{2\pi}{\max\{(\bar{x} - x^*), (x^* - \underline{x})\}}\right)^2 \tag{37}$$

Notice that the difference between the first and the second eigenvalues depends on the asymmetry of the bands.

The proposition also proves that the eigenfunctions φ_j^k with $k = \{l, h, m\}$ form a base, so that projecting the initial condition onto them is possible. It is however more involved than in the symmetric case since the eigenfunctions are not all orthogonal with each other, e.g. $\langle \varphi_j^m, \varphi_j^h \rangle \neq 0$. Note however that $\{\varphi_j^m, \lambda_j^m\}$ coincide withe the antisymmetric eigenfunction and eigenvalues for the case without reinjection and $\mu = \zeta = 0$. Because of this, any piecewise differentiable function $g : [\underline{x}, \bar{x}] :\rightarrow \mathbb{R}$ that is antisymmetric around $(\underline{x} + \bar{x})/2$ can be represented, in a L^2 sense, as a Fourier series using $\{\varphi_j^m\}$.

In spite of the lack of orthogonality, the general logic for constructing the impulse response function is the same. Given the projection of the initial condition on the eigenfunctions

$$\hat{p}(x, 0) = \sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k(x)$$

where $k = \{l, h, m\}$. We use the linearity of $\mathcal{H}^*$ to write the operator in equation (32) as

$$\mathcal{H}^*(\hat{p})(x, t) = \mathcal{H}^* \left(\sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k \right)(x, t) = \sum_k \sum_{j=1}^{\infty} a_j^k \mathcal{H}^*(\varphi_j^k)(x, t) = \sum_k \sum_{j=1}^{\infty} a_j^k e^{\lambda_j^k t} \varphi_j^k(x)$$

where the last equality uses that the $\varphi_j^k(x)$ are eigenfunctions. Thus, given the a_j^k coefficients (whose computation is discussed below), we can write the impulse response in equation (32) as

$$H(t, f, \hat{p}) = \sum_{\{k=l, h, m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t} a_j^k \int_{\underline{x}}^{\bar{x}} f(x) \varphi_j^k(x) \, dx \, .$$

or, computing the inner products $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x) \varphi_j^k(x) \, dx$

$$H(t, f, \hat{p}) = \sum_{k=\{l, h, m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t} a_j^k b_j^k[f] \, . \tag{38}$$

A straightforward numerical approach to finding the projection coefficients a_j^k requires

running a simple linear regression of $\hat{p}(x, 0)$ on the basis $\{\varphi_j^h(x), \varphi_j^l(x), \varphi_j^m(x)\}_{j=1}^J$ (up to some order frequency J). For the output impulse response, given the function of interest $f(x) = -x$, the projection coefficients are also readily computed $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j^k(x)dx$ for $k = \{m, l, h\}$, and $j = 1, 2, 3....$ which gives

$$b_j^m[f] = \frac{(\bar{x} - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^l[f] = \frac{(x^* - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^h[f] = \frac{(\bar{x} - x^*)^{3/2}}{\sqrt{2}\pi j} \quad . \tag{39}$$

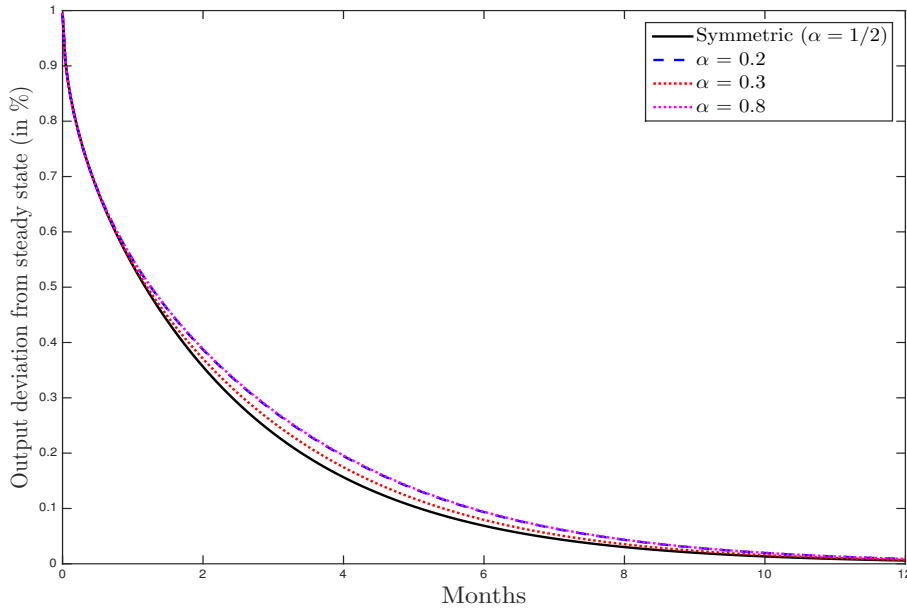Figure 9: Response to monetary shock for asymmetric problem

Figure 9 displays some impulse response generated by asymmetric problems where $\alpha \neq 1/2$ and contrasts them to the one produced by the symmetric problem where $\alpha = 1/2$. Two remarks are in order: first, modest degrees of asymmetry do not have a major effect on the impulse response: the impulse response function for $\alpha = 0.4$ would be barely distinguishable from the symmetric impulse response. Second, once quantitatively large asymmetries are considered, such as the small values of α considered in the figure, the impulse response becomes more persistent than the symmetric one. The presence of the asymmetry makes the convergence to the mean x value of the invariant distribution slower; this is intuitive since for

symmetric problems the mean of the distribution is obtained right after the first adjustment, while this is not anymore true.

Irrelevance of the sign of the reinjection point. We discuss here the irrelevance of whether the optimal return point x^* is to the left of the interval's midpoint, as when $x^* < 0$, or to the right of it hence with $x^* > 0$. Formally we consider two problems: the first one has $\alpha = 1/2 - z$ where $z \in (0, 1/2)$ and the second problem has $\tilde{\alpha} = 1/2 + z$. We will show that, somewhat surprisingly to us, the sign of the optimal return point x^* is irrelevant for the impulse response which is the same one for the problem with α and for the one with $\tilde{\alpha}$. We have the following result

PROPOSITION 13. Consider the inaction region for x defined by the interval $(-\bar{x}, \bar{x})$, let $z \in (0, 1/2)$ be a non-rational number. Consider a problem with reinjection point $\alpha = 1/2 - z$ and another problem with reinjection point $\tilde{\alpha} = 1/2 + z$. Then the impulse response function is the same for both problems.

Figure 9 illustrates the results of the proposition by showing that the impulse response for $\alpha = 0.2$ coincides with the one for $\alpha = 0.8$. An important implication of this property is that the derivative of the impulse response function with respect to α evaluated at $alpha = 1/2$ must be zero, which explains why small deviations from the symmetric benchmark produce results that are essentially almost indistinguishable from those produced by the symmetric case. Overall this result suggests that the symmetric benchmark is an accurate approximation of problems with modest degrees of asymmetry.

9 Conclusion and Future Work

(TBD)

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A   Proofs

Proof. of Proposition 1. Using the definitions of **H**, H and $\mathcal{H}$ note that for any f:

$$
\begin{aligned}
\mathbf{H}(r, f, p) &\equiv \int_0^\infty e^{-rt} H(t; f, p) dt \\
&= \int_0^\infty e^{-rt} \left[\int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x, t)\, p(x, 0)\, dx \right] dt \\
&= \int_0^\infty e^{-rt} \int_{\underline{x}}^{\bar{x}} \mathbb{E}\left[f(x(t)) \mid x(0) = x \right] p(x, 0)\, dx\, dt \\
&= \int_{\underline{x}}^{\bar{x}} \int_0^\infty e^{-rt} \mathbb{E}\left[f(x(t)) \mid x(0) = x \right] p(x, 0)\, dt\, dx \\
&= \int_{\underline{x}}^{\bar{x}} \mathbb{E}\left[\int_0^\infty e^{-rt} f(x(t)) dt \mid x(0) = x \right] p(x, 0)\, dx
\end{aligned}
$$

where the first equality is the definition of **H**, the second and third lines use the definition of H and $\mathcal{H}$, the fourth line exchanges the order of integration, the fifth line uses the linearity of the expectations operator (where p stands for a generic signed measure).

The rest of the proof consists of two steps. The first is to show that if $f(x) = R'(x)$, then

$$
v'(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} f(x(t)) dt \mid x(0) = x \right] \quad \text{for all } x.
$$

This establishes the first equality in the proposition. The second step is to show that this expression implies that its expectation over x using the signed measure $p(\cdot, 0)$ gives $\mathbf{G}(r, R', p)$, which establishes the second equality in the proposition.

For the first step we take the value function in inaction for the Calvo$^+$ problem and differentiate it with respect to x obtaining:

$$
(r + \zeta) v'(x) = R'(x) + \mu v''(x) + \frac{\sigma^2}{2} v'''(x) \text{ for } x \in [\underline{x}, \bar{x}]
$$

Moreover the boundary conditions of the Calvo$^+$ problem (smooth pasting and optimal return) give:

$$
v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0
$$

Thus, given the ode stated above for v' and $v'(\underline{x}) = v'(\bar{x}) = v'(x^*)$, which act as value matching, then:

$$
v'(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} R'(x(t)) dt \mid x(0) = x \right]
$$

which is the sequence problem corresponding to the Hamilton Jacobi Bellman equation written above.

For the second step we notice that $v'(x^*) = 0$ and hence:

$$
v'(x) = \mathbb{E}\left[\int_0^\tau e^{-rt} R'(x(t)) dt + e^{-r\tau} v'(x(\tau)) \mid x(0) = x \right]
$$

where τ is the stopping time denoting the first time that x hits either $\underline{x}$ or $\bar{x}$ or that the free adjustment opportunity occurs. Since $x(\tau) = x^*$ and $v'(x^*) = 0$, then:

$$v'(x) = \mathbb{E}\left[\int_0^\tau e^{-rt} R'(x(t))\, dt\,|\, x(0) = x\right]$$

Thus to compute the cumulative impulse response it suffices to compute the stochastic integral up to the first price change. $\square$

Proof.   (of Proposition 2). Using the definitions of $\mathcal{H}$, $\mathcal{G}$ and τ we have the following recursion:

$$\mathcal{H}(f)(x,t) = \mathcal{G}(f)(x,t) + \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)|x\right] \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t > 0.$$

Let us begin by defining the following object: $D(x,t) \equiv \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)|x\right]$. We first consider case (i) and show that $D(x,t) = 0$ for all x and all t. This follows since $\mathcal{H}(f)(x^*, s) = 0$ for all s. This in turn follows because f is antisymmetric, thus we have $\mathbb{E}\left[f(x(t))|x(\tau) = x^*\right] = 0$, which follows immediately by the symmetry of the distribution $g(x,t)$ and the antisymmetric property of f. It follows that $\mathbb{E}\left[1_{\{t\geq\tau\}}f(x(t))|x(0) = x\right] = 0$. Hence, since $\mathcal{H} = \mathcal{G}$, this implies that $G(t) = H(t)$ for any $p(\cdot, t)$.

Now we turn to case (ii). We note that $D(x,t)$ is symmetric in x around $x^* = (\underline{x} + \bar{x})/2$. This follows since the law of motion of x is symmetric so $g(x,t)$ is symmetric around x^*. This in turn implies that the probability of hitting either barrier at time s, starting with $x(0) = x$, is symmetric in x, which directly implies the symmetry of $D(x,t)$. Now we use that $D(x,t)$ is symmetric and that

$$H(t, f, p) - G(t, f, p) = \int_{\underline{x}}^{\bar{x}} D(x,t)\left(p(x,0) - \bar{p}(x)\right)\, dx \ .$$

Since $D(x,t)$ is symmetric and $p(x,0) - \bar{p}(x)$ is antisymmetric we have that the right hand side is zero so that $H(t) = G(t)$. $\square$

Proof.   of Lemma 1. The proof uses the linearity of $\mathcal{G}$ to write $\mathcal{G}(\hat{f} + f - \hat{f}) = \mathcal{G}(\hat{f}) + \mathcal{G}(f - \hat{f})$. The projection $\hat{f}$ converges pointwise to f at any point at which f is differentiable. Additionally, by hypothesis, f is not differentiable (at most) at countably many points. Finally, we have defined $\mathcal{G}(f)(x,t) = \mathbb{E}\left[1_{\{t<\tau\}}f(x(t))|x(0) = x\right]$. We note that this expected value is given by the integral that uses a continuous density, i.e. the density of BM starting at $x(0) = x$ and reaching $x(t) = y$ at t, which is continuous on y for $t > 0$. Hence the function $f - \hat{f}$ is non-zero only at countably many points, and thus its integral with respect to continuous density is zero, i.e. $\mathcal{G}(f - \hat{f})(x,t) = 0$ for all x and $t > 0$. Then we have

$$\mathcal{G}(f)(x,t) = \mathcal{G}\left(\hat{f}\right)(x,t) + \mathcal{G}(f - \hat{f})(x,t) = \mathcal{G}\left(\hat{f}\right)(x,t)$$

$$= \mathcal{G}\left(\sum_{j=1}^\infty b_j[f]\varphi_j\right)(x,t) = \sum_{j=1}^\infty b_j[f]\mathcal{G}\left(\varphi_j\right)(x,t) = \sum_{j=1}^\infty b_j[f]e^{\lambda_j t}\varphi_j(x)$$

where we have used the linearity of $\mathcal{G}$, the definition of $\hat{f}$, and the form of the solution for $\mathcal{G}\left(\varphi_j\right)(x,t)$. $\square$

Proof. (of Proposition 3). The result follows by Lemma 1, the definition of the projection coefficients $\{b_j[P]\}$ and the definition of the response function in equation (5).

Proof. (of Proposition 4) Let us define the centered even k-th moment for the variable x: $M_k(t,\delta) \equiv \mathbb{E}_\delta\left(x(t) - \mathbb{E}(x(t))\right)^k$, where $k = 2, 4....$ and the subscript δ denotes that probabilities are those of an impulse response following a marginal shock δ to the invariant distribution of gaps at zero inflation.

The objective is to show that the $\left.\frac{\partial}{\partial\delta} M_k(t,\delta)\right|_{\delta=0} = 0$ for all t, i.e. that a marginal shock δ has no first-order effect on the even centered moments at every t. The proof follows two steps. First, to show that the impulse response of any *even* moment is flat at zero. Second, to show that the impulse response of any *centered* moment is well approximated, up to second order terms, by the impulse response of the corresponding non-centered moment.

The first step is readily established since a marginal shock triggers an antisymmetric displaced distribution $\hat{p}(x,0) = \bar{p}'(x)\delta$, whose projection coefficients on all even-indexed eigenfunctions $j = 2, 4, ...$ are zero (since such eigenfunctions are symmetric). Note next that even (non-centered) moments $k = 2, 4, ...$ are symmetric by definition, which immediately implies that their projection coefficients on all odd-indexed eigenfunctions $j = 1, 3, ...$ are zero. It follows that none of the eigenfunctions will have a non-zero coefficient. This proves the first step.

To prove the second step write in terms of the non-centered moments

$$M_k(t,\delta) = B_0\mathbb{E}_\delta\left(x(t)^k\right) + B_1\mathbb{E}_\delta\left(x(t)^{k-1}\right)\mathbb{E}_\delta\left(x(t)\right) + ... + B_{k-1}\mathbb{E}_\delta\left(x(t)\right)\left(\mathbb{E}_\delta\left(x(t)\right)\right)^{k-1} + B_k\left(\mathbb{E}_\delta\left(x(t)\right)\right)^k$$

where the B_j are the binomial coefficients. Next, lets us replace each of the moments with its first order expansion in δ, namely let $\mathbb{E}_\delta\left(x(t)^k\right) = a_k\delta + o(\delta)$ where a_k is moment-k first derivative. We get

$$M_k(t,\delta) = B_0(a_k\delta + o(\delta)) + B_1(a_{k-1}\delta + o(\delta))(a_1\delta + o(\delta)) + ... + B_k(a_1\delta + o(\delta))^k$$

It is apparent that the only first order term in δ is a_k, i.e. the coefficient of the non-centered moment. This concludes the proof. $\square$

Proof. (of Proposition 5) Straightforward differentiation of the density function $\bar{p}(x)$ gives

$$\bar{p}'(x) = \begin{cases} -\frac{\theta^2\left[-e^{-\theta x} - e^{2\theta\bar{x}}e^{\theta x}\right]}{2\left[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}\right]} & \text{if } x \in [-\bar{x}, 0] \\ -\frac{\theta^2\left[e^{\theta x} + e^{2\theta\bar{x}}e^{-\theta x}\right]}{2\left[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}\right]} & \text{if } x \in [0, \bar{x}] \end{cases}$$

where $\theta \equiv \bar{x}^2 \zeta / \sigma^2$. The linear projection of $\bar{p}'(x)$ onto φ_j gives the projection coefficients:

$$b_j[\bar{p}'] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -\left[\frac{2\theta^2 j\pi}{4\theta^2 \bar{x}^2 + j^2\pi^2}\right]\left[\frac{1+\cosh(\bar{x}\theta)}{1-\cosh(\bar{x}\theta)}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -\left[\frac{2\theta^2 j\pi}{4\theta^2 \bar{x}^2 + j^2\pi^2}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

To see this compute: $\int_{-\bar{x}}^{\bar{x}} \bar{p}'(x)\varphi_j(x)dx = 2\int_{-\bar{x}}^0 \bar{p}'(x)\varphi_j(x)dx$ for $j = 2, 4, 6, \ldots$. The function $\bar{p}'$ is antisymmetric and φ_j is antisymmetric for j even, with respect to $x = 0$. For $j = 1, 3, 5, \ldots$ this integral is zero, since φ_j is symmetric, see equation (13). For $j = 2, 4, \ldots$ we thus have:

$$b_j[\bar{p}'] = 2\int_{-\bar{x}}^0 \bar{p}'(x)\varphi_j(x)dx = \frac{\theta^2}{[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}]}\int_{-\bar{x}}^0 \left[e^{-\theta x} + e^{2\theta\bar{x}}e^{\theta x}\right]\frac{1}{\sqrt{\bar{x}}}\sin\left(\frac{(x+\bar{x})}{2\bar{x}}j\pi\right)dx$$

$$= \frac{e^{\bar{x}\theta}4\theta^2\bar{x}}{\sqrt{\bar{x}}[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}]}\frac{\left[j\pi\left(1 - \cosh(\bar{x}\theta)(-1)^{j/2}\right)\right]}{4\theta^2\bar{x}^2 + j^2\pi^2}$$

$$= \frac{8\phi\,e^{\sqrt{2\phi}}}{\bar{x}^{3/2}\left[1 - 2e^{\sqrt{2\phi}} + e^{2\sqrt{2\phi}}\right]}\frac{\left[j\pi\left(1 - \cosh\left(\sqrt{2\phi}\right)(-1)^{j/2}\right)\right]}{8\phi + \pi^2 j^2}$$

$$= \frac{j\pi}{4\bar{x}^{3/2}}\frac{(-2)}{\left(1 + \frac{j^2\pi^2}{8\phi}\right)}\frac{1 - \cosh\left(\sqrt{2\phi}\right)(-1)^{j/2}}{1 - \cosh\left(\sqrt{2\phi}\right)}$$

where we used that $\theta\bar{x} = \sqrt{2\phi}$ and that $\cosh(x) = (1 + e^x)/(2e^x)$. Combining it with the expression for $b_j[f]$ in equation (16) gives the desired result. $\square$

Proof. (of Proposition 6) Rewriting the expression for m_2:

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^\infty \beta_j(\phi)/\lambda_j(\phi)} = \frac{\beta_2(\phi)/\lambda_2(\phi)}{Kurt(\phi)/(6\,N)}$$

$$= \frac{\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh\left(\sqrt{2\phi}\right)-1}\right]\left[\frac{8(2\phi)}{4(2\phi)+4\pi^2}\right]}{N\,\ell\left(\sqrt{2\phi}\right)\left[1 + \frac{\pi^2}{2\phi}\right]}\frac{N\left(\exp(\sqrt{2\phi}) + \exp(-\sqrt{2\phi}) - 2\right)^2}{(\exp(\sqrt{2\phi}) + \exp(-\sqrt{2\phi}))(\exp(\sqrt{2\phi}) + \exp(-\sqrt{2\phi}) - 2 - 2\phi)}$$

$$= 2\frac{\left[1 + \cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi}) - 1 - \phi\right]\left[1 + \frac{\pi^2}{2\phi}\right]^2}$$

where the first line follows from the definition, and the first equality from the sufficient statistic result in Alvarez, Le Bihan, and Lippi (2016). The second line uses the expression for β_2, λ_2 derived above, as well as the expression for the Kurtosis derived in Alvarez, Le Bihan, and Lippi (2016). The third line uses the expression for ℓ. The remaining lines are simplifications. $\square$

Proof. (of Proposition 7) First use Proposition 5 to write

$$\frac{\partial}{\partial t}Y(t)|_{t=0} = \lim_{M\to\infty}\sum_{j=1}^{M}\beta_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty}\sum_{i=0}^{M}[\beta_{2+4i}\lambda_{2+4i} + \beta_{4+4i}\lambda_{4+4i}]$$

Using the coefficients for β_j in Proposition 5 and the expression for the eigenvalues in equation (19) we write

$$\frac{\partial}{\partial t}Y(t)|_{t=0} = -N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M}2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]-1\right) = -2N\ell(\phi)\lim_{M\to\infty}M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]-1\right)$$

which diverges towards minus infinity for any $0 \le \phi < \infty$. □

Proof. (of Proposition 8) We let $\bar\phi$ to be the value of ϕ for which the impulse response, as a function of t, has a local maximum at $t = 0$. For larger values of ϕ the IRF will have an interior maximum at some $t > 0$, and hence the IRF will be hump shaped. For lower values of ϕ it will be monotonically decreasing in t. Thus we characterize the critical value of ϕ for which the IRF has a local maximum, and also verify that the second derivative with respect to time is negative at $t = 0$ for that critical value of ϕ. The slope of the impulse response function at $t = 0$ is given by:

$$\frac{\partial}{\partial t}Y(t)\Big|_{t=0} = \lim_{M\to\infty}\sum_{j=1}^{M}\left(\beta_j(\phi) + \beta_j^0(\phi)\right)\lambda_j(\phi) = \lim_{M\to\infty}\sum_{i=0}^{M}\left[\left(\beta_{2+4i}+\beta_{2+4i}^0\right)\lambda_{2+4i} + \left(\beta_{4+4i}+\beta_{4+4i}^0\right)\lambda_{4+4i}\right]$$

Using the coefficients for b_j in Proposition 5, the expression for $b_j^0(\phi)$ given in equation (22) and the expression for the eigenvalues λ_j in equation (19)

$$\frac{\partial}{\partial t}Y(t)\Big|_{t=0} = -N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M}2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1\right)$$

$$= -N\ell(\phi)\lim_{M\to\infty}2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1\right) \ .$$

Replacing the value of the value of the optimal decision rule for the plans model $\rho(\phi)$ in the previous expression we have

$$\frac{\partial}{\partial t}Y(t)\Big|_{t=0} = -N\ell(\phi)\lim_{M\to\infty}2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[1-2\left[\frac{\exp\left(\sqrt{2\phi}\right)-\exp\left(-\sqrt{2\phi}\right)-2\sqrt{2\phi}}{\sqrt{2\phi}\left(\exp\left(\sqrt{2\phi}\right)+\exp\left(\sqrt{-2\phi}\right)-2\right)}\right]\right]-1\right)$$

$$= -N\ell(\phi)\lim_{M\to\infty}2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{\sqrt{2\phi}\cosh\left(\sqrt{2\phi}\right)-2\sinh\left(\sqrt{2\phi}\right)+\sqrt{2\phi}}{\sqrt{2\phi}\left(\cosh\left(\sqrt{2\phi}\right)-1\right)}\right]-1\right) \ .$$

Thus, this expression is equal to zero for $\bar\phi$ solving:

$$\sqrt{2\phi}\left(\cosh\left(\sqrt{2\phi}\right)-1\right)^2 = \left[1+\cosh\left(\sqrt{2\phi}\right)\right]\left[\sqrt{2\phi}\cosh\left(\sqrt{2\phi}\right)-2\sinh\left(\sqrt{2\phi}\right)+\sqrt{2\phi}\right] \ .$$

Analysis of this function shows that $\bar\phi = 1$. Next we check the value of the second derivative

at $\phi = 1$; we have:

$$\frac{\partial^2}{\partial t^2} Y(t)|_{t=0} = \left[\left(\beta_{2+4i} + \beta_{2+4i}^0 \right) \lambda_{2+4i}^2 + \left(\beta_{4+4i} + \beta_{4+4i}^0 \right) \lambda_{4+4i}^2 \right]$$

$$= N\ell(\phi) \lim_{M \to \infty} \sum_{i=0}^{M} 2 \left(\left[\frac{1 + \cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi}) - 1} \right] (1 - 2\rho(\phi)) |\lambda_{2+4i}| - |\lambda_{4+4i}| \right) = -\infty$$

since $|\lambda_{2+4i}| > |\lambda_{4+4i}|$ and since: $\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1} \right] (1 - 2\rho(\phi)) - 1 = 0$ at $\phi = 1$. To summarize: at $\phi = 1$ the slope of the impulse response at $t = 0$ is zero and the second derivative is negative, thus it is a maximum. $\square$

Proof. (of Proposition 9) First we consider case (i), i.e. the *long run* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right) \sigma$ on the impulse response of output to a monetary shock. We note that the expression for $Y(t)$ for the Golosov Lucas model does not feature $\bar{x}$, which is a function of σ (see equation (18)). Indeed the only place where σ enters in the expression for $Y(t)$ is in the eigenvalues (the parameter $N(j\pi)^2/8$ in equation (18)). Since, $N = \sigma^2/\bar{x}^2$ and $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$, then $d\log\bar{x} = 1/2 d\log\sigma$ and $d\log N = 2\left(d\log\bar{x} - d\log\sigma\right)$, hence $d\log N = d\log\sigma$. Substituting this into the eigenvalue $\lambda_j = -\tilde{N}(j\pi)^2/8 = -(1 + \frac{d\sigma}{\sigma})N(j\pi)^2/8$ were N is the average number of price changes before the volatility shock. Using the expression for the impulse response in terms of the post-shock objects we have:

$$\tilde{Y}(t) = \sum_{j=1}^{\infty} b_j[f] b_j[\bar{p}'] e^{-\left(1+\frac{d\sigma}{\sigma}\right) N \frac{(j\pi)^2}{8} t} = Y\left(t\left(1 + \frac{d\sigma}{\sigma}\right)\right)$$

and we obtain the desired result.

Now we consider case (ii), i.e. the *impact* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right) \sigma$ on the impulse response of output to a monetary shock. As in the previous case the eigenvalues can be written as functions of the shock and the old value of the expected number of price changes. Also as the previous case we have $f(x) = -x$. The difference is on the initial distribution $p(x, 0)$. The initial condition is given by $p(x, 0) = \bar{p}(x + \delta; \bar{x}(\sigma))$ where we write $\bar{x}(\sigma)$ to indicate that the distribution depends on σ. Indeed, since we are using the expression for $\tilde{Y}(t)$ in terms of the value of $\bar{x}$ that corresponds to the post-shock value of σ, we need to consider the effect on $\bar{x}$ of a decrease of σ in the proportion $d\sigma/\sigma$. To do this we take a second order expansion of $p(x, 0) = \bar{p}(x + \delta; \bar{x}(\sigma))$ with respect to δ and σ evaluated

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at $\delta = 0$ and $d\sigma = 0$.

$$p(x;0) \equiv \bar{p}(x+\delta;\bar{x}(\sigma)) = \bar{p}(x) + \frac{\partial}{\partial\delta}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta - \frac{\partial}{\partial\bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial\delta^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta^2$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial\bar{x}^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\left(\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\right)^2 d\sigma^2 + \frac{1}{2}\frac{\partial}{\partial\bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial^2\bar{x}(\sigma)}{\partial\sigma^2}d\sigma^2$$

$$- \frac{\partial^2}{\partial\bar{x}\partial\delta}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma\delta + o\left(||(\delta,d\sigma)||^2\right)$$

for $x \in [\underline{x},\bar{x}]$ and $x \neq 0$. Recall that the invariant distribution for this model is the triangular density $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$ for $x \in (-\bar{x},\bar{x})$. Using this functional form we have:

$$\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x}) = \begin{cases} +\frac{1}{\bar{x}^2} & \text{if } x \in [-\bar{x},0) \\ -\frac{1}{\bar{x}^2} & \text{if } x \in (0,\bar{x}] \end{cases} \quad , \quad \frac{\partial}{\partial\bar{x}}\bar{p}(\delta+x;\bar{x})\big|_{\delta=0} = \begin{cases} -\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x},0) \\ +\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0,\bar{x}] \end{cases}$$

$$\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x}) = \begin{cases} -\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x},0) \\ +\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0,\bar{x}] \end{cases} \quad , \quad \frac{\partial^2}{\partial\bar{x}^2}\bar{p}(\delta+x;\bar{x}) = \begin{cases} +\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in [-\bar{x},0) \\ -\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in (0,\bar{x}] \end{cases}$$

Notice that the first order derivatives with respect to δ as well as the cross partial derivative are antisymmetric functions of x around $x = 0$, while the derivatives with respect to $\bar{x}$ are symmetric functions of x. Finally we have $\frac{\partial^2}{\partial\delta^2}\bar{p}(x+\delta;\bar{x}) = 0$.

Now we use the expansion and compute the impulse response coefficients $b_j \equiv b_j[f]b_j[p(\cdot,0)]$. The first order term for $d\sigma$ is zero because f is antisymmetric (so that $b_j[f] = 0$ for $j = 2,4,6,...$) and the first derivative with respect to $\bar{x}$ is symmetric (so that $b_j[p(\cdot,0)] = 0$ for $j = 1,3,5,...$) hence the $\beta_j = 0$ for $j = 1,2,3,4,...$. Likewise the second order terms for $d\sigma^2$ are zero since f is antisymmetric and the first and second derivative with respect to $\bar{x}$ are symmetric. The second order term δ^2 is zero because the second derivative with respect to δ is zero. This leaves us with two non-zero terms. The first order term on δ, which is the term for the IRF with respect to a monetary shock, and the second order term corresponding to the cross-derivative. For the cross-partial term we note that, using that $\bar{x}$ has elasticity $1/2$ with respect to σ, we can write

$$-\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x})\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma\delta = -\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x})\bar{x}(\sigma)\left[\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\frac{\sigma}{\bar{x}(\sigma)}\right]\frac{d\sigma}{\sigma}\delta$$

$$= -\frac{2}{\bar{x}(\sigma)}\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x})\bar{x}(\sigma)\frac{1}{2}\frac{d\sigma}{\sigma}\delta \quad = -\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x})\frac{d\sigma}{\sigma}\delta$$

Thus we have that each b_j term is given by the sum of the (non-zero) terms corresponding to the first order term on δ and the second-order term corresponding to the cross-derivative:

$$b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta + b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta\frac{d\sigma}{\sigma} = b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta\left(1 + \frac{d\sigma}{\sigma}\right)$$

This gives the second adjustment for $\tilde{Y}(t)$ in terms of the expression for $Y(t)$. $\square$

Proof. (of Proposition 12) The proof consists on checking that the functions described are the only ones that satisfy the sufficient conditions equation (28), equation (29), equation (30) and equation (31) for $\mathcal{H}^*(\varphi_j^k(x)e^{\lambda_j^k t})$ for $k \in l, m, j$ and $j = 1, 2, \ldots$.

First, let's consider the case of eigenvalues $\lambda \neq 0$. In this case the only non-constant real function that satisfy the o.d.e.: $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$ for $\lambda < 0$ is $\varphi(x) = \sin(\phi + \omega x)$ for some ϕ and for $\lambda = -\frac{\sigma^2}{2}\omega^2$. The cases below use this characterization when the function is not constant, to determine the values of ϕ and ω.

Second, consider the case of functions that are differentiable in the entire domain $[\underline{x}, \bar{x}]$. The continuity at $x = x^*$ is satisfied immediately. In this case, the o.d.e. : $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$, with boundaries equation (29) and equation (30) is satisfied only by $\varphi(x) = \varphi_j^m(x)$ for all $j = 1, 2, \ldots$. This gives the particular value of ϕ and ω, for $j = 1, 2, \ldots$, and hence no other differentiable function different from zero satisfy all the conditions.

Third, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[\underline{x}, x^*]$. Then $\varphi(x) = 0$, to satisfy the boundary condition equation (29) at $x = \underline{x}$. Then $\varphi(x) = 0$ for all $x \in [\underline{x}, x^*)$, since φ can only be non-differentiable at $x = x^*$. Then, for $x \in (x^*, \bar{x}]$ it has to be differentiable, non-identically equal to zero, satisfy $\varphi(x^*) = 0$ so that it is continuous at $x = x^*$, and also $\varphi(\bar{x}) = 0$, to satisfy the boundary condition equation (29) at $x = \bar{x}$. Finally, to satisfy the measure preserving condition equation (30), it has to be of the form of $\varphi_j^l(x)$ for $j = 1, 2, \ldots$.

Fourth, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[x^*, \bar{x}]$. Following the same steps as in the previous case, we obtain that $\varphi(x) = \varphi_j^h(x)$ for $j = 1, 2, \ldots$ for this case.

For the fifth and remaining case, we consider the case of functions $\varphi(x)$ which are non-constant for all intervals included in $[\underline{x}, \bar{]}$, and that $\varphi(x)$ is not differentiable at $x = x^*$. To satisfy the o.d.e. in each segment $[\underline{x}, x^*)$ and $(x^*, \bar{x}]$ then we must have $\varphi(x) = \sin(\underline{\phi} + \underline{\omega}x)$ and $\varphi(x) = \sin(\bar{\phi} + \bar{\omega}x)$ in each of the respective segments. Since the eigenvalue has to be the same for all segments, then we have that $\underline{\omega} = \bar{\omega} \equiv \omega$. The eigenfunction φ must be measure preserving, so that

$$0 = \cos(\underline{\phi} + \omega x^*) - \cos(\underline{\phi} + \omega\underline{x}) + \cos(\bar{\phi} + \omega\bar{x}) - \cos(\bar{\phi} + \omega x^*)$$

To satisfy the boundary conditions equation (29) we require $\sin(\underline{\phi} + \omega\underline{x}) = \sin(\bar{\phi} + \omega\bar{x}) = 0$. Thus $\cos(\underline{\phi} + \omega\underline{x}) = \pm 1$ and $\cos(\bar{\phi} + \omega\bar{x}) = \pm 1$. Hence, we have that:

$$\text{either } 0 = \cos(\underline{\phi} + \omega x^*) - \cos(\bar{\phi} + \omega x^*) \text{ or } \pm 2 = \cos(\underline{\phi} + \omega x^*) - \cos(\bar{\phi} + \omega x^*)$$

In the first case we have:

$$0 = \cos(\underline{\phi} + \omega x^*) - \cos(\bar{\phi} + \omega x^*) \text{ and } 0 = \sin(\underline{\phi} + \omega x^*) - \sin(\bar{\phi} + \omega x^*)$$

so the function is differentiable at $x = x^*$, which is a contradiction. So we must have the second case, and because eigenfunctions are defined up to sign, must have:

$$2 = \cos(\underline{\phi} + \omega x^*) - \cos(\bar{\phi} + \omega x^*) \text{ and } 0 = \sin(\underline{\phi} + \omega x^*) - \sin(\bar{\phi} + \omega x^*)$$

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Using the properties of cos it must be the case that $\sin\left(\underline{\phi} + \omega x^*\right) = \sin\left(\bar{\phi} + \omega x^*\right) = 0$. Then, φ must be zero in the extremes of each of the two following segment $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$. This requires that $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$ be an in an multiple integer of each other, since in each of the segments φ is a sine function with the same frequency ω which is zero at the two extremes. But this violate that $[\underline{x}, x^*]/[x^*, \bar{x}]$ is not rational.

Now we show that the eigenfunctions span the densities for the signed measures. It suffices to show that if $g : [\underline{x}, \bar{x}] \to \mathbb{R}$ is in the domain of $\mathcal{H}^*$, and $\langle g, \varphi_j^m \rangle = \langle g, \varphi_j^l \rangle = \langle g, \varphi_j^h \rangle = 0$ for all $j = 1, 2, \ldots$, then it must be that $g = 0$.

As a way of contradiction, suppose we have a function $g \neq 0$ that g is orthogonal to all the eigenfunctions. Given that the eigenfunctions can span antisymmetric functions defined in different domains as explained above, it must be that g is a symmetric function as defined in $[\underline{x}, \bar{x}]$, so that it is orthogonal to $\{\varphi_j^m\}_{j=1}^{\infty}$, and also a symmetric in the following restricted domains $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$, so that g is also orthogonal each of eigenfunctions $\{\varphi_j^l\}_{j=1}^{\infty}$ and $\{\varphi_j^h\}_{j=1}^{\infty}$ when defined in the restricted domains.

Now, without loss of generality, assume that $x^* < (\underline{x} + \bar{x})/2$. Below we sketch a proof that for a function g to be even (or symmetric) in these three domains, it must be the case that $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, which contradicts the assumption that $[x^* - \underline{x}]/[\bar{x} - x^*]$ is not a rational number.

Let $L = x^* - \underline{x}$. To arrive to this conclusion we first notice that since g must be symmetric in the entire domain $[\underline{x}, \bar{x}]$, then it must be the case that g has identical symmetric shape in the segment $[\underline{x}, \underline{x} + L]$ than in the segment $[\bar{x} - L, \bar{x}]$. Then using that g is symmetric in the restricted domain $[x^*, \bar{x}]$, it must be that it also has the same symmetric shape in the interval $[x^*, x^* + L]$ than in both intervals $[\underline{x}, \underline{x} + L]$ and $[\bar{x} - L, \bar{x}]$. If it is the case that $x^* + L = \bar{x} - L$, then $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, and find a contradiction. If this is not the case, i.e. if $x^* + L < \bar{x} - L$, we use the g is symmetric in the entire domain, to say that again g must take the same symmetric shape in the interval $[\bar{x} - 2L, \bar{x} - L]$. Now either $x* + L = \bar{x} - 2L$, which gives a contradiction, or we continue using the symmetry of g in either the entire domain $[\underline{x}, \bar{x}]$ or in the restricted domain $[x^*, \bar{x}]$ until we get that $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, which is a contradiction with our assumption. Formally, this can be set up as an induction step, but it requires to develop enough notation, which we skip to shorten. $\square$

Proof. of Proposition 13. Consider the first problem with $\alpha < 1/2$. Normalize (WLOG) the interval width to $2\bar{x} = 1$ and rewrite the initial condition as $\hat{p} = \hat{p}^s + \hat{p}^a$, respectively the symmetric and antisymmetric component as

$$
\hat{p}^s(x) = \begin{cases} \frac{1-2\alpha}{\alpha(1-\alpha)} \\ \frac{-1}{(1-\alpha)} \end{cases} \quad , \quad \hat{p}^a(x) = \begin{cases} \frac{1}{\alpha(1-\alpha)} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ 0 & \text{for } x \in (-z, z) \end{cases}
$$

Notice that for $\alpha = 1/2 - z$ the slope of the antisymmetric part is either zero or $\frac{1}{1/4 - z^2}$. The same obtains for $\tilde{\alpha} = 1/2 + z$. Thus the asymmetric component of the initial condition $\hat{p}^a(x)$ is the same for α and for $\tilde{\alpha}$. The symmetric component $\hat{p}^s(x)$ is as follows

$$
\hat{p}^s(x, \alpha) = \begin{cases} \frac{2z}{1/4 - z^2} \\ \frac{-1}{1/2 + z} \end{cases} \quad , \quad \hat{p}^s(x, \tilde{\alpha}) = \begin{cases} \frac{-2z}{1/4 - z^2} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ \frac{1}{1/2 + z} & \text{for } x \in (-z, z) \end{cases}
$$

which reveals that the symmetric component of the initial condition for the problem with $\tilde{\alpha}$ is given by -1 times the symmetric component of the initial condition for the problem with α.

Now let's consider the consequences for the output impulse response as defined in equation (32). For the problem with α we use the decomposition $\hat{p} = \hat{p}^s + \hat{p}^a$ and the linearity of $\mathcal{H}^*$ to write the IRF as

$$H_\alpha(t, f, \hat{p}(\alpha)) = H_\alpha(t, f, \hat{p}^a(\alpha)) + H_\alpha(t, f, \hat{p}^s(\alpha))$$

where we use the subscript to emphasize that this is the impulse response for the problem with reinjection point α. Using the properties for the initial condition associated to the problem with $\tilde{\alpha}$ discussed above we can the write its impulse response as

$$H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha})) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha)) + H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$$

where we used that $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and that $\hat{p}^s(\alpha) = -\hat{p}^s(\tilde{\alpha})$.

It is immediate to see that $H_\alpha(t, f, \hat{p}^a(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha))$, i.e. that the IRF component triggered by the asymmetric part of the initial condition, is the same in both problems. This follows since $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and because both problems share the same identical base for asymmetric functions, given by the eigenfunctions φ_j^m.

Finally, we argue that $H_\alpha(t, f, \hat{p}^s(\alpha)) = H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$. To see this notice that the symmetric part of the impulse response function is obtained by projecting the initial condition on the orthogonalized symmetric eigenfunctions v_j^k, where $k = \{l, h\}$, produced by e.g. the Gram-Schmidt algorithm. The key is to notice that the symmetrized eigenfunction for the problem with $\tilde{\alpha}$, equals -1 times the eigenfunctions for the problem with α, formally $v_j^k(\alpha) = -v_j^k(\tilde{\alpha})$. Inspection if the eigenfunctions φ_j^h and φ_j^l reveals that, for all $x \in (-\bar{x}, \bar{x})$ they obey $\varphi_1^h(x; x^* = -z) = -\varphi_1^l(-x; x^* = z)$. It therefore follows that $H_\alpha(t, f, \hat{p}(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha}))$. $\square$

B Symmetric problem with drift (inflation)

In this section we introduce a non-zero drift μ to the process for x and solve for the eigenfunctions and eigenvalues for $\mathcal{G}(f)(x, t)$, i.e. for the process without reinjection. To lighten the notation we use $g(x, t) = \mathcal{G}(f)(x, t)$.

PROPOSITION 14.   Assume that the process has a drift μ, and variance σ^2, so the Kolmogorov backward equation is:

$$\partial_t \mathcal{G}(f)(x, t) = \partial_x \mathcal{G}(f)(x, t)\mu + \partial_{xx}\mathcal{G}(f)(x, t)\frac{\sigma^2}{2} \text{ for all } x \in [\underline{x}, \bar{x}]$$

with boundary conditions $\mathcal{G}(f)(\underline{x}, t) = \mathcal{G}(f)(\bar{x}, t) = 0$ for all $t > 0$ and $\mathcal{G}(f)(x, 0) = f(x)$ for

all x. The eigenvalues, eigenfunctions, projections, and inner product are given by:

$$\lambda_j = -\left[\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x}-\underline{x}}\right)^2\right] \text{ for all } j = 1, 2, \ldots$$

$$\varphi_j(x) = \sqrt{\frac{2}{\bar{x}-\underline{x}}}\,\sin\left(\left[\frac{x-\underline{x}}{\bar{x}-\underline{x}}\right]j\pi\right)e^{-\frac{\mu}{\sigma^2}x} \text{ for all } x \in [\underline{x}, \bar{x}] \text{ where}$$

$$b_j[f] = \frac{\langle f, \varphi_j\rangle}{\langle\varphi_j, \varphi_j\rangle} \text{ where } \langle a, b\rangle \equiv \int_{\underline{x}}^{\bar{x}} a(x)b(x)e^{2\frac{\mu}{\sigma^2}x}dx$$

Thus the solution for $\mathcal{G}(f)$ is:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}b_j[f]\,\varphi_j(x) \text{ for all } t \geq 0 \text{ and } x \in [\underline{x}, \bar{x}].$$

Proof. (of Proposition 14) To lighten the notation, denote $g(x,t) = \mathcal{G}(f)(x,t)$. We start by rewriting g as

$$g(x,t) = h(x,t)s(x)$$

for some function s, so that the Kolmogorov backward equation is:

$$s(x)\partial_t h(x,t) = \left[s'(x)h(x,t) + s(x)\partial_x h(x,t)\right]\mu + \left[s''(x)h(x,t) + 2s'(x)\partial_x h(x,t) + s(x)\partial_{xx}h(x,t)\right]\frac{\sigma^2}{2}$$

We will take $s(x) = \exp(ax)$ for some constant a to be determined. Replacing this function and its derivatives, and cancelling we get

$$\partial_t h(x,t) = \left[ah(x,t) + \partial_x h(x,t)\right]\mu + \left[a^2 h(x,t) + 2a\partial_x h(x,t) + \partial_{xx}h(x,t)\right]\frac{\sigma^2}{2}$$

$$= h(x,t)\left[a\mu + a^2\frac{\sigma^2}{2}\right] + \partial_x h(x,t)\left[\mu + a\sigma^2\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2}$$

Setting $a = -\frac{\mu}{\sigma^2}$ into the p.d.e. for h we have:

$$\partial_t h(x,t) = h(x,t)\left[-\frac{\mu^2}{\sigma^2} + \frac{\mu^2}{\sigma^4}\frac{\sigma^2}{2}\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2}$$

$$= h(x,t)\left[-\frac{\mu^2}{\sigma^2} + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2}$$

$$= -h(x,t)\frac{1}{2}\frac{\mu^2}{\sigma^2} + \partial_{xx}h(x,t)\frac{\sigma^2}{2}$$

Since $s(x) \neq 0$, the boundary conditions for h are the same as for g, namely $h(\underline{x}, t) = h(\bar{x}, t) =$

0. The equation for the eigenvalues-eigenfunctions for h is the same as in the Calvo+ model

$$\left(\lambda_j + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right)\phi_j(x) = \partial_{xx}\phi_j(x)\frac{\sigma^2}{2}$$

so that we have the same expression for the eigenvalues-eigenfuctions:

$$\lambda_j = -\left[\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x}-\underline{x}}\right)^2\right] \text{ for all } j = 1, 2, \ldots$$

$$\phi_j(x) = \sin\left(\left[\frac{x-\underline{x}}{\bar{x}-\underline{x}}\right]j\pi\right) \text{ for all } x \in [\underline{x}, \bar{x}]$$

Thus we can write the solution for h as:

$$h(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}\frac{\int_{\underline{x}}^{\bar{x}} h(x',0)\phi_j(x')dx'}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x'))^2dx'}\phi_j(x)$$

Multiplying both sides by $s(x)$:

$$h(x,t)s(x) = \sum_{j=1}^{\infty} e^{\lambda_j t}\frac{\int_{\underline{x}}^{\bar{x}} h(x',0)\phi_j(x')dx'}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x'))^2dx'}\phi_j(x)s(x)$$

Thus we define

$$\langle a, b\rangle = \int_{\underline{x}}^{\bar{x}} a(x)b(x)\frac{1}{s(x)^2}dx \text{ and }$$

$$\varphi_j(x) = \phi_j(x)s(x) \text{ so that}$$

$$\langle \varphi_j, \varphi_i\rangle = 0 \text{ if } i \neq j \text{ since } \int_{\underline{x}}^{\bar{x}} \phi_j(x)\phi_i(x)dx = 0 \text{ for } i \neq i$$

Note that we can write the solution for g as follows:

$$\frac{\langle g(x,0), \varphi_j\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\int_{\underline{x}}^{\bar{x}} g(x,0)\varphi_j(x)\frac{dx}{s(x)^2}}{\int_{\underline{x}}^{\bar{x}}(\varphi_j(x))^2\frac{dx}{s(x)^2}} = \frac{\int_{\underline{x}}^{\bar{x}} \frac{g(x,0)}{s(x)}\phi_j(x)dx}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x))^2dx} = \frac{\int_{\underline{x}}^{\bar{x}} h(x,0)\phi_j(x)dx}{\int_{\underline{x}}^{\bar{x}}(\phi_j(x))^2dx}$$

where we use that $g(x,0) = s(x)h(x,0)$. $\square$

C   Details of the multiproduct model

Law of motion for y, z.

$$dy = \sigma^2 n\, dt + 2\sigma\sqrt{y}\, dW^a$$

$$dz = \sigma\sqrt{n}\left[\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b\right]$$

where W^a, W^b are independent standard BM's.

LEMMA 2.   Define

$$x = \sqrt{y}\ \text{ and }\ w = \frac{z}{\sqrt{ny}}$$

so that the domain is $0 \le x \le \bar{x} \equiv \sqrt{\bar{y}}$ and $-1 \le w \le 1$. They satisfy:

$$dx = \sigma^2 \frac{n-1}{2\,x} dt + \sigma dW^a \tag{40}$$

$$dw = \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \frac{\sqrt{1-w^2}}{x}\, dW^b \tag{41}$$

We look for a solution to the eigenvalue-eigenfunction problem (λ, φ) given by equation (40) and equation (41). They must satisfy

$$\lambda\varphi(w,x) = \varphi_x(w,x)\sigma^2\left(\frac{n-1}{2x}\right) + \varphi_w(w,x)\frac{w}{x^2}\left(\frac{1-n}{2}\right)$$

$$+ \frac{1}{2}\varphi_{ww}(w,x)\frac{(1-w^2)}{x^2} + \frac{1}{2}\sigma^2\varphi_{xx}(w,x)$$

for all $(x,w) \in [0, \bar{x}] \times [-1, 1]$, with $\varphi(\bar{x}, w) = 0$, all w and φ^2 integrable.

PROPOSITION 15.   The eigenfunctions-eigenvalues of (w, x) satisfying equation (40)-equation (41) denoted by $\{\varphi_{m,k}(\cdot), \lambda_{m,k}\}$ for $k = 1, 2, \dots$ and $m = 0, 1, \dots$ are given by:

$$\varphi_{m,k}(x,w) = h_m(w)\, g_{m,k}(x)\ \text{ where}$$

$$h_m(w) = C_m^{\frac{n}{2}-1}(w)\ \text{ for } m = 0, 1, 2, \dots \text{ and}$$

$$g_{m,k}(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(j_{\frac{n}{2}-1+m,k}\frac{x}{\bar{x}}\right)\ \text{ for } k = 1, 2, \dots \text{ and}$$

$$\lambda_{m,k} = -N\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n}\ \text{ for } m = 0, 1, \dots, \text{ and } k = 1, 2, \dots$$

where $C_m^{\frac{n}{2}-1}(\cdot)$ denote the Gegenbauer polynomials, and where $J_{\frac{n}{2}-1+m}(\cdot)$ denote the Bessel function of the first kind, $j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

Note that the expressions for the eigenfunctions are only valid only for $n > 2$. For $n = 2$ the expression take a different special form, which we skip to save space. The expressions for the eigenvalues are valid for $n \geq 2$.

We remind the reader how the Gegenbauer polynomial and Bessel function, which form an orthogonal base, are defined. The Gegenbauer polynomial $C_m^{\frac{n}{2}-1}(w)$ is given by:

$$C_m^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{\Gamma(m-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(m-2k)!}(2w)^{m-2k} \tag{42}$$

For a fixed n, the polynomials are orthogonal on with respect to the weighting function $(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}$ so that:[14]

$$\int_{-1}^{1} C_m^{(\frac{n}{2}-1)}(w) \, C_j^{(\frac{n}{2}-1)}(w) \, (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}} \, dw = 0 \text{ for } m \neq j \tag{43}$$

and for $m = j$ we get

$$\int_{-1}^{1} \left[C_m^{(\frac{n}{2}-1)}(w)\right]^2 (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}} \, dw = \frac{\pi \, 2^{1-2(\frac{n}{2}-1)} \Gamma(m+2(\frac{n}{2}-1))}{m!(m+\frac{n}{2}-1)[\Gamma(\frac{n}{2}-1)]^2} \tag{44}$$

The Bessel function of the first kind is given by :

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{45}$$

For a given ν, the following functions are orthogonal, using the weighting function x^{n-1} so that:[15]

$$\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right] \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right)\right] x^{n-1} \, dx$$

$$= \int_0^{\bar{x}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right) J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right) x \, dx = 0 \text{ if } k \neq s \in \{1,2,3,\dots\} \text{ and}$$

$$\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 x^{n-1} \, dx = \bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} \tag{46}$$

$$= \frac{1}{2} \left(\bar{x} \, J_{\nu+1}(j_{\nu,k})\right)^2 \text{ for all } k \in \{1,2,3\dots\} \tag{47}$$

where $j_{\nu,k}$ and $j_{\nu,s}$ are two zeros of $J_\nu(\cdot)$.

---

[14]By this we mean that we define the inner product between functions a,b from $[-1,1]$ to $\mathbb{R}$ as : $\langle a,b \rangle = \int_{-1}^{1} a(w)b(w) \left(1-w^2\right)^{\frac{n}{2}-1-\frac{1}{2}} \, dw$.

[15]By this we mean that we define the inner product between functions a,b from $[0,\bar{x}]$ to $\mathbb{R}$ as: $\langle a,b \rangle = \int_0^{\bar{x}} a(x)b(x)x^{n-1}dx$.

Figure 10: Eigenvalues for multiproduct model

Abs. value of First (dominant) & Second (IRF) eigenvalues, for $N = 1$
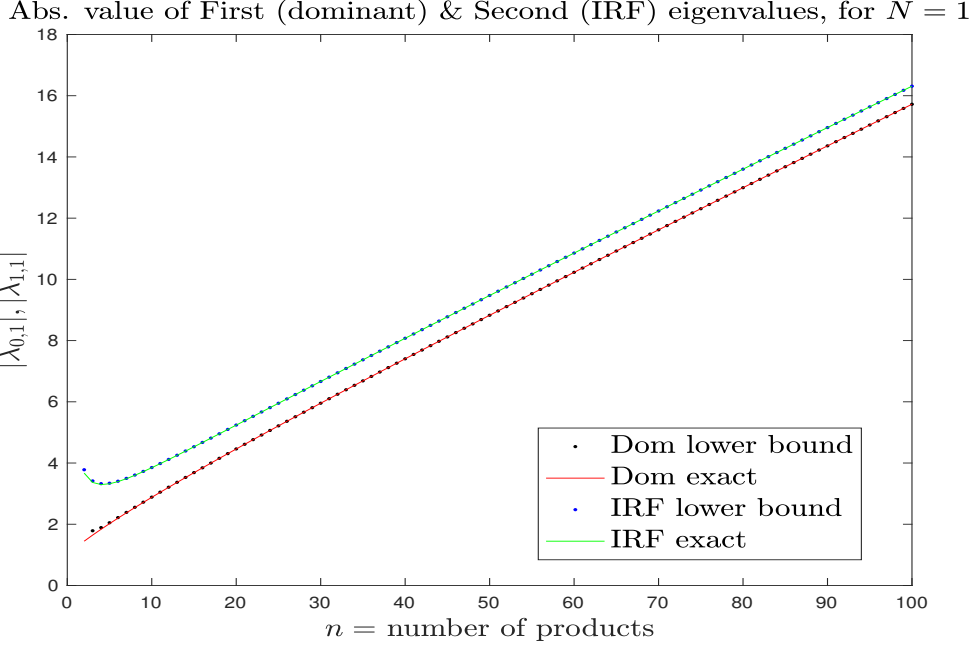
Kepping fixed $N = 1$ for all n

Derivation of IRF. Thus we have

$$G(t) \equiv \int_0^{\bar{x}} \int_{-1}^1 \mathcal{G}(f)(x, w, t)\, p(x, w; 0) dw\, dx$$

As in Section 3, we can write this expected value as:

$$
\begin{aligned}
Y(t) &= \int_0^{\bar{x}} \int_{-1}^1 \mathcal{G}\left(\sum_{k=1}^\infty \sum_{m=0}^\infty b_{m,k}[f]\, \varphi_{k,m}\right)(x, w, t)\, p(x, w; 0) dw\, dx \\
&= \int_0^{\bar{x}} \int_{-1}^1 \sum_{k=1}^\infty \sum_{m=0}^\infty b_{m,k}[f]\, \mathcal{G}\left(\varphi_{k,m}\right)(x, w, t)\, p(x, w; 0) dw\, dx \\
&= \int_0^{\bar{x}} \int_{-1}^1 \sum_{k=1}^\infty \sum_{m=0}^\infty b_{m,k}[f]\, e^{\lambda_{m,k} t}\, \varphi_{m,k}(x, w)\, p(x, w; 0) dw\, dx \\
&= \sum_{k=1}^\infty \sum_{m=0}^\infty e^{\lambda_{m,k} t}\, b_{m,k}[f] \int_0^{\bar{x}} \int_{-1}^1 \varphi_{m,k}(x, w)\, p(x, w; 0) dw\, dx
\end{aligned}
$$

Then we get:

$$G(t) = \sum_{k=1}^\infty \sum_{m=0}^\infty e^{\lambda_{m,k} t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot, 0)/\omega\right] \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

66

Inner product. We let $\omega(w, x) = x^{1-n} (1 - w^2)^{\frac{n-3}{2}}$. The inner product of functions a, b from $[0, \bar{x}] \times [-1, 1]$ to $\mathbb{R}$ is defied as

$$\langle a, b \rangle = \int_0^{\bar{x}} \int_{-1}^1 a(x, w) \, b(x, w) \, x^{1-n} (1 - w^2)^{\frac{n-3}{2}} \, dw \, dx$$

The term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ is given by the product of equation (44) and equation (47) found above. Indeed since the polynomials are orthogonal we have:

$$b_{m,k}[f] = \frac{\langle f, \varphi_{m,k} \rangle}{\langle \varphi_{m,k}, \varphi_{m,k} \rangle} = \frac{\int_0^{\bar{x}} \left[\int_{-1}^1 f(x, w) h_m(w) (1 - w^2)^{\frac{n-3}{2}} dw \right] g_{m,k}(x) \, x^{n-1} \, dx}{\left[\int_{-1}^1 (h_m(w))^2 (1 - w^2)^{\frac{n-3}{2}} dw \right] \left[\int_0^{\bar{x}} (g_{m,k}(x))^2 \, x^{n-1} \, dx \right]}$$

$$= \frac{\int_0^{\bar{x}} \left[\int_{-1}^1 f(x, w) C_m^{\frac{n}{2}-1}(w) (1 - w^2)^{\frac{n-3}{2}} dw \right] J_{m+\frac{n}{2}-1} \left(j_{m+\frac{n}{2}-1, k} \frac{x}{\bar{x}} \right) x^{\frac{n}{2}} \, dx}{\left[\int_{-1}^1 \left(C_m^{\frac{n}{2}-1}(w) \right)^2 (1 - w^2)^{\frac{n-3}{2}} dw \right] \left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1} \left(j_{m+\frac{n}{2}-1, k} \frac{x}{\bar{x}} \right) \right)^2 x \, dx \right]}$$

Invariant Distribution. After the change in variables we have:

$$\bar{h}(w) = \frac{1}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \left(1 - w^2 \right)^{(n-3)/2} \quad \text{for} \quad w \in (-1, 1) \tag{48}$$

$$\bar{g}(x) = x \, (\bar{x})^{-n} \left(\frac{2n}{n-2} \right) \left[\bar{x}^{n-2} - x^{n-2} \right] \qquad \text{for } x \in [0, \bar{x}] \tag{49}$$

Initial distribution after a small monetary shock.

$$p(w, x; 0) = \bar{h}(w(\delta)) \bar{g}(x(\delta)) = \bar{h}(w) \bar{g}(x) + \bar{p}'(w, x; 0) \delta + o(\delta) \text{ with}$$
$$\bar{p}'(w, x; 0) = \bar{g}(x) \bar{h}'(w) w'(0) + \bar{h}(w) \bar{g}'(x) x'(0)$$

where:

$$\frac{\partial}{\partial \delta} x(\delta)|_{\delta=0} = x'(0) = \sqrt{n} \, w \quad \text{and} \quad \frac{\partial}{\partial \delta} \bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w) w'(0)$$

$$\frac{\partial}{\partial \delta} w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n} \, (1 - w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial \delta} \bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x) x'(0)$$

PROPOSITION 16. The expressions for $\bar{p}'(x, w; 0)$ and the coefficients $b_{1,k}(n)$ for the impulse response of output are given by:

$$\bar{p}'(w, x; 0) = \bar{g}(x) \bar{h}'(w) w'(0) + \bar{h}(w) \bar{g}'(x) x'(0)$$

$$= \frac{w \, (1 - w^2)^{(n-3)/2}}{Beta \left(\frac{n-1}{2}, \frac{1}{2} \right)} \sqrt{n} \left(\frac{2n}{n-2} \right) \frac{[(4-n) \bar{x}^{n-2} - (4+n) x^{n-2}]}{\bar{x}^n}$$

and the coefficients for the impulse response $b_{1,k}(n) = b_{1,k}[f] \, b_{1,k}[\bar{p}'(\cdot, 0)/\omega] \, \langle \varphi_{1,k}, \varphi_{1,k} \rangle$ are

given by

$$b_{1,k}(n) = -\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} - \frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}} \right)\right.$$

$$\left. - (4+n)2^{-1-\frac{n}{2}}\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right) \right]$$

where ${}_1\tilde{F}_2(a_1; b_1, b_2; z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1; b_1, b_2; z) = {}_1F_2(a_1; b_1, b_2; z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function and $j_{\frac{n}{2},k}$ is the kth ordered zero of the Bessel function $J_{\frac{n}{2}}(\cdot)$.

Note that, as our notation emphasizes, the coefficients $b_j(n)$ depends only on the number of products.