Fiscal Origins of Monetary Paradoxes

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Abstract

We revisit the monetary paradoxes of standard monetary models in a liquidity trap and study the channels through which they occur. We focus on two paradoxes: the Forward Guidance Puzzle and the Paradox of Flexibility. First, we propose a decomposition of consumption into *substitution* and *wealth effects*, both of which take into account the general equilibrium effects on output and inflation, and we show that the substitution effect cannot account for the puzzles. Instead, monetary paradoxes are the result of strong wealth effects which, generically, are solely determined by the expected fiscal response to the monetary shocks. We estimate the fiscal response to monetary policy shocks with US data and find responses with the opposite sign to the ones implied by the standard equilibrium. Finally, we introduce the estimated fiscal responses into a medium-size DSGE model. We find that the impulse-response of consumption and inflation do not match the data, suggesting that wealth effects induced by fiscal policy may be important even outside of the liquidity trap. We show that models with constrained agents can produce strong wealth effects if gross private debt is different than zero.

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1 INTRODUCTION

In the aftermath of the financial crisis that started in 2008, many central banks reached the effective lower bound on nominal interest rates, leading them to search for alternative, unconventional, instruments. An alternative that has received considerable attention is ”forward guidance”, i.e., promises of future interest rate changes in an attempt to affect current macroeconomic conditions. However, when central banks and academics turned to the standard model of monetary analysis to evaluate the impact of such policies, it was found that forward guidance produces counterfactual responses, to the point many of the results of New Keynesian models in a liquidity trap are deemed as “paradoxes” or “puzzles”.[1] In particular, the prediction that changes in interest rates in the far distant future have arbitrarily large effects on current output is called the “The Forward Guidance Puzzle”.[2]

The extreme sensitivity of current conditions to events in a distant future led to a surge in work related to attenuating the forward looking nature of the New Keynesian model. A burgeoning literature has explored several mechanisms to dampen these forward looking effects, and intertemporal substitution effects in particular, through OLG models (Del Negro, Giannoni and Patterson (2015)), heterogeneity and incomplete markets (McKay, Nakamura and Steinsson (2016)), deviations from common knowledge (Angeletos and Lian (2016)), behavioral agents (Gabaix (2016), Farhi and Werning (2017)), or adaptive expectations (Gertler (2017)), to cite a few recent exemples. Recognizing the general relevance of these mechanisms to analyze a range of macroeconomic questions, we propose an alternative diagnosis to the paradoxical results found in a liquidity trap scenario: powerful wealth

[1]Despite the prominance of the effects of forward guidance, several other puzzles have been identified, such as the “Paradox of Flexibility” (Eggertsson and Krugman (2012), Werning (2011)), the backloading of fiscal multipliers (Farhi and Werning (2016)), or the ”Paradox of Toil” (Eggertsson (2010), Wieland (2014)).

[2]The term was coined by Del Negro, Giannoni and Patterson (2015) who also found that the model predictions were at odds with estimates of the effect of forward guidance even for changes in a relatively short horizon.

effects typically caused by strong fiscal responses. Our analysis suggests that weakening intertemporal substitution effects is not necessary, and in many occasions not sufficient, to eliminate the counterfactual predictions of the model.

We start by showing how to decompose the consumption response of *any equilibrium* of the standard New Keynesian model into *substitution* and *wealth effects*. It is well understood that the Forward Guidance Puzzle is the result of general equillibrium effects.[3] Hence, it is important to go beyond the standard decomposition in partial equilibrium and extend it to a general equilibrium setting, where, for instance, the substitution effect is consistent with the inflationary consequences of substituting consumption intertemporally. Our first main result is that, in the absence of wealth effects, the equilibrium does not present any of the paradoxical results: even after fully taking into account general equilibrium effects on output and inflation, the effect of changes in interest rates is *reduced* with the horizon of the intervention and the equilibrium is continuous in the price flexibility parameter.

The importance of wealth effects remains even if we take into account some of the mechanisms proposed to reduce the role of forward looking aspects of the model. In particular, many of the proposed formulations boil down to a modified version of the New Keynesian model with a *discounted Euler equation*, as in McKay, Nakamura and Steinsson (2017). As long the discounting is not too large, such that the equilibrium remains indeterminate under an interest rate peg, the puzzles are *attenuated*, but not eliminated.[4] We extend our decomposition to an economy with a discounted Euler equation and show that again, in the absence of wealth effects, the puzzles are eliminated.

Given the relevance of wealth effects in the counterfactual predictions of the model, it is important to understand how wealth effects are determined. In the knife-edge case where government debt and proportional taxation are equal to zero, such that monetary actions have no fiscal conse-

[3] See, for instance, Angeletos and Lian (2016) and Kaplan, Moll and Violante (2017).

[4] Diba and Loisel (2017) also stress this point.

quence, wealth effects are purely self-fulfilling. Expectations for a path of output consistent with a higher value for (physical and human) wealth will increase demand, increasing output, and confirming the initial expectation. In the case where either government debt or proportional taxes are different from zero, however small, the response will depend on the expectation of the fiscal response.

Generically, wealth effects are determined by a mechanism we call the *Intermporal Keynesian Cross*, given that its logic is reminiscent of undergraduate textbook analysis. The effect of autonomous movements in wealth, components that are not directly a function of the expected path of output, as expected fiscal transfers or the revaluation of bonds, is amplified in a similar way as autonomous variations in income are amplified in the standard Keynesian cross. We find that the powerful wealth effects required by the liquidity trap equilibrium cannot be generated by the revaluations of bonds, but it is the result of expected fiscal transfers.

Our results are consistent with the findings of Cochrane (2017*b*) and Cochrane (2017*a*) who argue that the puzzles are eliminated by introducing the Fiscal Theory of the Price Level. In his case, the only wealth effects are the ones generated by bond revaluations, which are not enough to create the puzzling outcomes. In contrast, we emphasize that whether we are in a monetary or fiscal dominance regime is immaterial, as it is the expected equilibrium behavior of the fiscal authority that matters, not the off-equilibrium interactions of the fiscal and monetary authority. Hence, we view our contribution as complementary to Cochrane's work.

Finally, we analyze whether the channels emphasized in our analysis are a feature of the zero lower bound or they are relevant to understand the monetary policy transmission mechanism in normal times as well. In particular, we study the role of wealth effects and fiscal responses to monetary shocks in a medium-scale DSGE model, as in Smets and Wouters (2007). First, we show that the implicit transfers necessary to sustain the standard equilibrium play an important role in the quantitative predictions of the model. Absent the transfers, the impulse response functions for output and inflation tend to display the opposite signs than with transfers. Next, we ask what is the actual fiscal response to monetary shocks in the data. We use the high-frequency identification approach adopted in Gertler and Karadi (2015), augmented to account for fiscal variables. We find that not only the size, but the sign of the estimated fiscal response go against the ones implied by the standard equilibrium. As a final step, we feed the medium-scale DSGE model with the estimated monetary and fiscal impulse response functions, and compute the resulting dynamics of output and inflation. It is worth emphasizing that this exercise is a test of the New Keynesian model that does not rely on policy rules but instead imposes the observed path of monetary and fiscal variables as restrictions, and hence is independent of the debate about fiscal and monetary dominance. We find that by adding the additional constraint on fiscal policy, the model has difficulty in generating reasonable impulse responses of consumption and inflation to monetary shocks. This finding differs from the result in Kaplan, Moll and Violante (2017) that the main transmission mechanism of monetary policy in the representative agent New Keynesian model is the intertemporal Euler equation.[5] We conclude that it may be useful to explore models with a richer portfolio structure in order to generate richer wealth effects in the absence of strong fiscal response. In particular, we show that a medium-scale DSGE model augmented by a heterogeneous agents model that produces indebted hand-to-mouth improves the quantitative predictions of the New Keynesian model.

The rest of the paper is organized as follows. Section 2 presents the monetary paradoxes in the context of a continuous-time New Keynesian model. Section 3 presents the equilibrium decomposition into substitution and wealth effects, first with rigid prices, then with sticky prices. Section 4 discuss the determination of wealth effects and the fiscal origins of the monetary paradoxes. Section 5 presents the analysis of these channels outside

[5] In footnote 11, Kaplan, Moll and Violante (2017) quote John Cochrane and suggest that the standard New Keynesian model could be renamed as ”sticky-price intertemporal substitution model”.

of the zero lower bound. Section 6 concludes.

2 The Standard Liquidity Trap Equilibrium

We develop a simple New Keynesian model in continuous time in the spirit of Werning (2011) and Cochrane (2017*b*), augmented to incorporate fiscal variables and explicitly account for the households' budget constraint. The objective of this section is to present the monetary paradoxes that characterize the standard Liquidity Trap equilibrium, i.e., the Forward Guidance Puzzle and the Paradox of Flexibility, and show how fiscal variables, in particular lump-sum transfers, adjust in the background.

Time is continuous and denoted by $t \in \mathbb{R}_+$. There are two types of agents in the economy: a large number of identical, infinitely-lived households, and a infinitely-lived government. There is also a continuum of mass one of firms that produce a differentiated good. Households' preferences are such that final consumption is a CES aggregator of the purchases of each of the differentiated goods. In the benchmark model, we assume that the government doesn't consume, but it raises some proportional sales taxes, issues short-term nominal debt (which is in positive net supply in steady-state) and distributes lump-sum transfers (which can be negative).

The focus of this paper is to understand the presence of paradoxes when the economy is in a Liquidity Trap. As is standard in the literature, we will log-linearize the model around it's steady-state equilibrium to study the first-order approximation of the equilibrium response of the economy to exogenous shocks. Since the model is linear, we do not need to introduce the shock that takes the economy to a Liquidity Trap, but just study the economy as if the nominal rate was fixed.[6] The joint dynamics of a shock that takes the economy to a Liquidity Trap and our exercises would be just the sum of the two.[7]

[6] See Angeletos and Lian (2016) for a similar strategy.

[7] For a detailed exposition of the dynamics of the economy after a shock that leads to a Liquidity Trap, see Werning (2011) and Cochrane (2017*b*).

The log-linearized solution to the model can be characterized by four equations: an intertemporal Euler equation

$$\dot{c}_t = \sigma^{-1}(i_t - \pi_t - \rho), \tag{1}$$

a New Keynesian Phillips Curve

$$\dot{\pi}_t = \rho\pi_t - \kappa c_t, \tag{2}$$

the household's intertemporal budget constraint

$$\int_0^\infty e^{-\rho t} c_t dt = \int_0^\infty e^{-\rho t} \left[(1-\overline{\tau})y_t + \overline{b}(i_t - \pi_t - \rho) + T_t\right] dt, \tag{3}$$

and a resource constraint

$$c_t = y_t, \tag{4}$$

plus a policy rule for monetary and fiscal policy. Here, c_t denotes the percentage difference between actual consumption and the level of consumption in a steady state that features a constant path for the policy variables and zero inflation; π_t denotes inflation; i_t denotes the nominal, short-term, risk-free interest rate; $\overline{\tau}_t$ is the steady state level level of proportional sales taxes; $\overline{b}$ is the steady state level of short-term government debt; T_t is a lump-sum transfer expressed as a fraction of output; σ denotes the inverse of the intertemporal elasticity of substitution; ρ denotes the subjective discount factor of the households; and κ is the slope of the Phillips curve.

Since our analysis emphasizes the role of the household's budget constraint in the dynamic behavior of consumption, it is useful to briefly describe its components. The left hand side of the household's budget constraint is the present value of consumption. The right hand side are the sources of income: the after tax wage and profits, the interest from financial assets, and government's lump-sum transfers. In the standard analysis, equation (3) is dropped because transfers T_t are assumed to automatically adjust so that the government's budget constraint is always satisfied for any path of the endogenous and exogenous variables. Since lump-sum transfers do not affect any of the other equilibrium equations, they provide a free variable that guarantees that the solution to the system given by (1) and (2) (plus a boundary condition) is consistent with the equilibrium of the economy. Given the focus of our analysis, we will explicitly account for the presence of the budget constraint and explore the role of each component of income in the dynamic behavior of consumption.

The system of differential equations (1)-(2) can be written as

$$\begin{bmatrix} \dot{c}_t \\ \dot{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 & -\sigma^{-1} \\ -\kappa & \rho \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma^{-1}(i_t - \rho) \\ 0 \end{bmatrix}.$$

The eigenvalues of the system above are given by

$$\overline{\omega} = \frac{\rho + \sqrt{\rho^2 + 4\kappa\sigma^{-1}}}{2}, \quad \underline{\omega} = \frac{\rho - \sqrt{\rho^2 + 4\kappa\sigma^{-1}}}{2}.$$

Notice that the system has a positive and a negative eigenvalue. Focusing on bounded solutions, we need one additional condition to determine equilibrium. The liquidity trap literature has used $c_{T^*} = 0$ for some T^*, potentially in the far future, as such condition. In this section, we stick to this selection. The next lemma characterizes the solution and shows some of its properties.

Lemma 1 (Standard Liquidity Trap Equilibrium)**.** *Consider the New Keynesian model with the standard Liquidity Trap equilibrium, $c_{T^*} = 0$. The response of initial consumption to a monetary shock is given by*

$$c_0^{NK} = -\frac{\kappa\sigma^{-1}}{\sigma(\overline{\omega} - \underline{\omega})} \int_0^{T^*} \left(\frac{e^{-\overline{\omega}t}}{\overline{\omega}} - \frac{e^{-\underline{\omega}t}}{\underline{\omega}} \right) (i_t - \rho) dt,$$

while the response of initial inflation is given by

$$\pi_0^{NK} = -\frac{\kappa\sigma^{-1}}{\overline{\omega} - \underline{\omega}} \int_0^{T^*} (e^{-\underline{\omega}t} - e^{-\overline{\omega}t})(i_t - \rho) dt.$$

Moreover, initial consumption and inflation are decreasing in the nominal interest rate at all horizons,

$$\frac{\partial c_0^{NK}}{\partial i_t} < 0, \qquad \frac{\partial \pi_0^{NK}}{\partial i_t} \leq 0, \qquad \forall t \geq 0.$$

Two characteristics of the solution are worth mentioning. First, in the standard equilibrium, a monetary shock that increases the nominal interest rate has a contractionary effect in the economy, reducing initial consumption and generating negative inflation. Second, fiscal policy, given by $\overline{\tau}$, $\overline{b}$ and $\{T_t\}_{t=0}^{\infty}$, is irrelevant in the standard Liquidity Trap equilibrium, as long as the government's intertemporal budget constraint is satisfied. However, as we show below, this does not imply that they do not play a role in determining the channels through which monetary policy operates.

Next, we present the monetary puzzles and paradoxes the literature has identified and formally show that they are present in the standard liquidity trap equilibrium. Suppose that the central bank reduces the short-term nominal interest rate in some period t. As a result, consumption and inflation increase at the time of the news. How does the response of consumption depend on the time of the intervention and the degree of price rigidity in the economy? The Forward Guidance Puzzle refers to the theoretical result that the promise to hold interest rates lower in the future becomes more powerful the further in the future the actual intervention takes place. This term was coined by Del Negro, Giannoni and Patterson (2015), who find that the estimated response of the US economy to forward guidance shocks are significantly smaller than the ones predicted by the standard New Keynesian model. In the extreme, the response of consumption becomes unboundedly large as the horizon of the policy intervention goes to infinity. Another counter-intuitive result attributed to the New Keynesian model is that the effect of monetary policy shocks become stronger as price flexibility increases. As we approach the flexible price limit, monetary policy becomes arbitrarily strong. Since shocks to the nominal interest rate have no impact on real variables in a flexible price economy, this result is known as the Para-

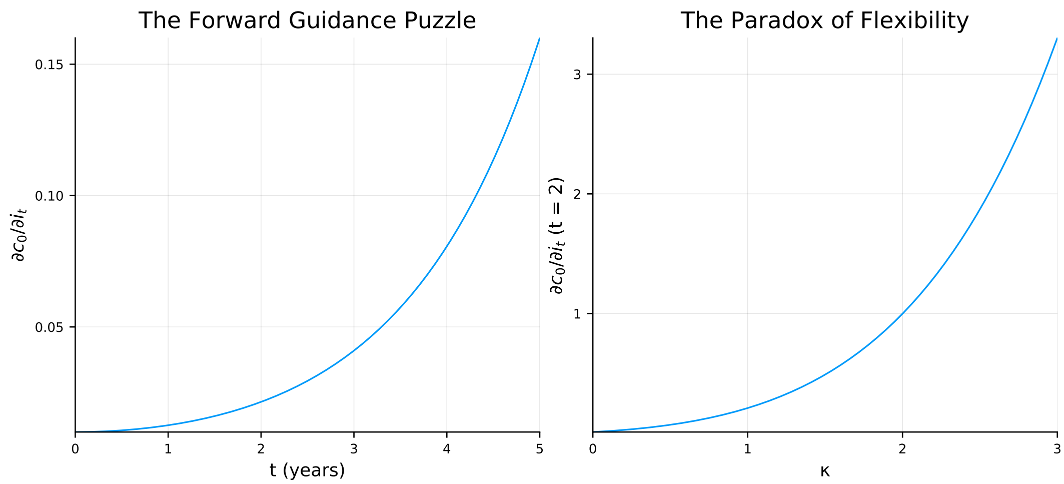

FIGURE 1: The Forward Guidance Puzzle and The Paradox of Flexibility

dox of Flexibility. Proposition 1 formally shows that the Forward Guidance Puzzle and the Paradox of Flexibility are present in the standard liquidity trap equilibrium selection.

Proposition 1 (The Standard Liquidity Trap Equilibrium and its Paradoxes)**.** *Consider the standard Liquidity Trap equilibrium and let $t < T^*$. The economy presents the following dynamics:*

i) Forward Guidance Puzzle

$$\frac{\partial^2 c_0^{NK}}{\partial t \partial i_t} < 0, \qquad \lim_{t\to\infty} \frac{\partial c_0^{NK}}{\partial i_t} = -\infty,$$
$$\frac{\partial^2 \pi_0^{NK}}{\partial t \partial i_t} < 0, \qquad \lim_{t\to\infty} \frac{\partial \pi_0^{NK}}{\partial i_t} = -\infty,$$

ii) Paradox of Flexibility

$$\lim_{\kappa\to\infty} \frac{\partial c_0^{NK}}{\partial i_t} = -\infty, \qquad \lim_{\kappa\to\infty} \frac{\partial \pi_0^{NK}}{\partial i_t} = -\infty.$$

Figure 1 shows graphically these results. The standard narrative is as follows. A reduction in the interest rate translates into a reduction in the real interest rate of the economy due to nominal rigidities. This reduction in the real rate generates a boom in consumption in that period, which pushes

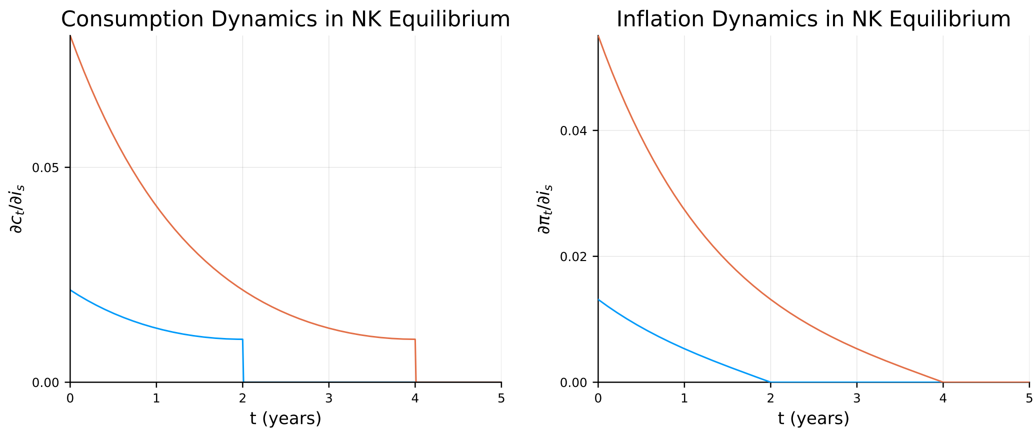

FIGURE 2: Consumption and Inflation Dynamics after a Monetary Shock

prices up, increasing inflation. Since inflation is forward looking, a higher *expected* inflation increases inflation in all of the previous periods. Solving the problem backwards, this implies that the feedback loop force achieves its maximum in period 0. Thus, the furthest in the future the policy promise is, the more inflation can increase. And since nominal interest rates in previous periods are kept fixed due to the liquidity trap, the bigger the reduction in real rates and the bigger the consumption boom today. Moreover, since nominal rigidities are preventing some prices to increase, the more flexible prices are, the more inflation the monetary shocks generates, and hence the more powerful the effect is. Figure 2 shows the dynamics of consumption and inflation for monetary shocks that happen in year 2 and year 4.

Given this logic, it is natural that many solutions to these puzzles focused on reducing the forward looking nature of consumption and inflation. However, as we show in the next section, most of the solutions can at most attenuate the effects, but do not eliminate them. In contrast, we show that the nature of the puzzles is generically fiscal rather than monetary, meaning that it is the response of the fiscal authority the one that is generating the paradoxical results. Moreover, in section **??** we argue that these responses are unlikely to hold in the data.

3 Consumption Decomposition: Substitution and Wealth Effects

Next, we dig deeper into the channels through which monetary policy operates. To better understand Proposition 1, we decompose c_t into two components: a substitution effect and a wealth effect. To this end, we first present two objects that will be important in this characterization.

First, for a given path of the nominal interest rate and inflation, $\{i_t, \pi_t\}_{t=0}^{\infty}$, we define c_t^S as the Hicksian demand, which is given by

$$c_t^S \equiv \sigma^{-1} \int_0^t (i_s - \pi_s - \rho) ds - \sigma^{-1} \int_0^\infty e^{-\rho s} (i_s - \pi_s - \rho) ds. \tag{5}$$

Equation (5) is the log-linear approximation of the solution to the problem of minimizing the household's expenditures subject to achieving a reservation utility. In this setting, the different goods are given by consumption at different dates, and the price of one unit of consumption at date t is $e^{-\int_0^t (i_s - \pi_s - \rho) ds}$. Importantly, the total cost of the Hicksian demand at steady state prices is zero, that is

$$\int_0^\infty e^{-\rho t} c_t^S dt = 0.$$

The Hicksian demand will be tightly connected to the substitution effect in general equilibrium.

The second object we define here is the average consumption, C,

$$C \equiv \rho \int_0^\infty e^{-\rho t} \left[(1 - \overline{\tau}) y_t + \overline{b}(i_t - \pi_t - \rho) + T_t \right] dt. \tag{6}$$

Average consumption is the consumption path that would prevail if the household was forced to consume the same amount every period while still satisfying their budget constraint. Average consumption will be related to the wealth effect.

In the rest of this section, we analyze the role of the substitution and wealth effects in generating the dynamics of consumption. We do this in two steps. First, we study an economy with fixed prices. This economy provides a useful benchmark by shutting down an important general equilibrium feedback effect that works through inflation. Next, we allow for prices to move and show that the insights of the fixed prices case are amplified in general equilibrium.

3.1 Rigid Prices

Suppose prices are fixed, i.e.. $\kappa = 0$. From the Euler equation, we have that in the standard Liquidity Trap equilibrium

$$c_t = -\sigma^{-1} \int_t^{\infty} (i_s - \rho) ds. \tag{7}$$

Consider the effect of a one time monetary shock at time s. Since prices are fixed, an increase in the nominal interest rate translates into a one-to-one increase in the real interest rate. Thus, an increase in the interest rate in period s implies an increase in the relative price of consumption in all periods $t \leq s$ and a reduction in the relative price of consumption in all periods $t > s$. Equation (7) tells us that consumption decreases by $\sigma^{-1} \Delta i_s$ in all periods $t \leq s$, and goes back to zero afterwards.

Moreover, the effect on initial consumption is independent of the time of the shock. That is

$$\frac{\partial c_0^{NK}}{\partial i_s} = -\sigma^{-1} \qquad \forall s. \tag{8}$$

Thus, the fixed price case has a attenuated form of the forward guidance puzzle: the time of the intervention is irrelevant for its effect on consumption in period zero. Note that this does not mean that the time of intervention is irrelevant for the whole *path* of consumption. In fact, the further in the future the intervention takes place, the longer its cumulative effect on consumption.

However, even though the timing of the intervention has no effect on initial consumption, the channel through which the effect takes place does vary with the horizon of the intervention. To see this, let's first decompose consumption into substitution and wealth effect. Under fixed prices, the substitution effect is the Hicksian demand evaluated at $\pi_t = 0 \; \forall t$, while the wealth effect is given by average consumption, C. The next lemma formalizes this.

Lemma 2 (Substitution and Wealth Effects with Fixed Prices)**.** *Suppose $\kappa = 0$. In the standard Liquidity Trap equilibrium, consumption can be decomposed as*

$$c_t^{NK} = \underbrace{\sigma^{-1}\int_0^t (i_s - \rho)ds - \sigma^{-1}\int_0^\infty e^{-\rho s}(i_s - \rho)ds}_{\text{substitution effect}} + \underbrace{\sigma^{-1}\int_0^\infty (e^{-\rho s} - 1)(i_s - \rho)ds}_{\text{wealth effect}},$$

where the substitution effect equals the Hicksian demand defined in (5) *evaluated at $\pi_t = 0 \; \forall t$, and the wealth effect equals the average consumption defined in* (6)*.*

Now that we have the decomposition of consumption between substitution and wealth effect, we can study the channels through which a monetary shock works. Consider again a shock in period s. We know from (8) that the effect on initial consumption is independent of s. Let's analyze the substitution and wealth effect. The substitution effect of a monetary shock in s is given by

$$\frac{\partial c_t^S}{\partial i_s} = \begin{cases} -\sigma^{-1}e^{-\rho s} < 0 & \text{if } t < s, \\ \sigma^{-1}(1 - e^{-\rho s}) > 0 & \text{if } t \geq s. \end{cases}$$

We can see two things. First, the substitution effect of an increase in the interest rate in s is negative for $t < s$ and positive afterwards. This is the standard result from consumer theory: the substitution effect is negative for goods that suffer a relative price increases and vice-versa. This effect is depicted in Figure 3 Panel (a) for a negative interest rate shock in $s = 2$. Second, the substitution effect on initial consumption *decreases* with the horizon of the intervention and vanishes in the limit.

$$\frac{\partial^2 c_0^S}{\partial s \partial i_s} = \rho\sigma^{-1}e^{-\rho s} > 0, \qquad \lim_{s\to\infty} \frac{\partial c_0^S}{\partial i_s} = 0.$$

The intuition for this result is the following. While the size of the change in consumption at date s depends only on the elasticity of intertemporal substitution, how much of the adjustment will fall on current versus future consumption depends on the marginal rate of substitution (MRS) between consumption in these two dates. Since the marginal utility of future consumption declines with $e^{-\rho t}$, the indifference curves get flatter over time. Hence, a smaller change in initial consumption will be necessary to keep the same utility level.[8]

That is, the *intertemporal substitution* channel gets smaller as the time of the intervention increases. Thus, absent wealth effects, this model predicts a negative relation between the time of the shock and the size of the initial effect on consumption and, therefore, does not feature a Forward Guidance Puzzle.

However, since we know that the initial consumption does not change with the time of the intervention, it must be that the wealth effect picks up the slack. How does this happen? Figure 3 Panel (b) shows a graphical representation of the channel. The reduction in the interest rate in period $s = 2$ generate a positive substitution effect in all periods before $s = 2$ and a negative substitution effect after. Since consumption has to be at its steady state level at $T^* > s$ because of the Liquidity Trap equilibrium (and since consumption is constant after $s = 2$), the wealth effect is exactly the negative of the substitution effect after $s = 2$. As the time of the intervention s moves into the future, the substitution effect in 0 decreases, but the negative substitution effect after the shock increases. Therefore, the wealth effect has to be larger in order to compensate for the negative wealth effect and allow consumption to go back to the steady state after the shock, moving

[8]This can be easily seen in a two good example. If σ^{-1} is the elasticity of substitution between goods x and y, and we start at a point where $x = y$, then $\frac{\partial \log x}{\partial p_y/p_x} = \frac{MRS}{1+MRS}\sigma^{-1}$.

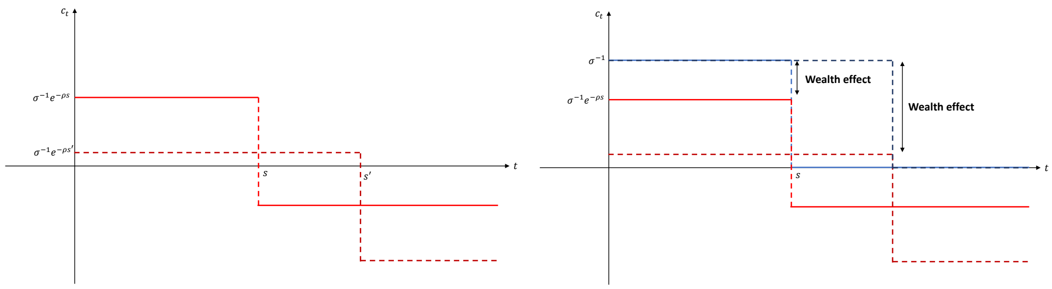

FIGURE 3: Substitution and Wealth Effects with Fixed Prices

the whole consumption path upwards.

To fix ideas, consider the following example.[9] Suppose there is an unexpected monetary shock in period 0 that takes the following form:

$$i_t = \begin{cases} \rho & \text{for } t < z, \\ \rho + e^{-\eta(t-z)}(r-\rho) & \text{for } t \geq z, \end{cases}$$

where $r > \rho$ and z is known for the agent. Note that if $z = 0$ this is a contemporaneous shock with persistence governed by η. As we increase z, it becomes a forward guidance shock. Replacing this path for the interest rate in (2) at $t = 0$, we get

$$c_0^{NK} = -\left[\underbrace{\frac{\sigma^{-1}}{\eta} \frac{\eta e^{-\rho z}}{\rho + \eta}}_{\text{substitution effect}} + \underbrace{\frac{\sigma^{-1}}{\eta} \frac{\rho + \eta(1 - e^{-\rho z})}{\rho + \eta}}_{\text{wealth effect}} \right] (r - \rho).$$

For $z = 0$, the relative importance of the substitution and the wealth effect is governed by the relative magnitudes of η and ρ. Kaplan, Moll and Violante (2017) calibrate this case and argue that for reasonable values of η and ρ, contemporaneous shocks are mostly driven by the substitution effect. However, we can see that as z increases, the relative importance of the substitution effect decreases. In particular, as $z \to \infty$, the effect of a forward

[9]This is an extension of an example in Kaplan, Moll and Violante (2017).

guidance shock is exclusively explained by the wealth effect.

Interestingly, the results in this subsection imply that even though the response of initial consumption to a monetary shock is independent of the timing of the intervention, the channel through which this happens varies with it. In particular, the substitution effect is more relevant for contemporaneous shocks while the wealth effect is the most important channel for forward guidance shocks. We showed that it is the wealth effect the one that explains the non-decreasing response of initial consumption to the horizon of the policy intervention and not a strong intertemporal substitution effect or the intertemporal Euler equation. Since our main interest is in the equilibrium dynamics in a liquidity trap where the monetary authority cannot control prices, we next present the general case with $\kappa > 0$ and show that an even starker result arises when prices are allowed to move.

3.2 Sticky Prices

Decomposing consumption into substitution and wealth effect was relatively easy when prices were fixed. Since there were no general equilibrium feedback effects through inflation, the substitution effect was just given by the exogenous shocks to the nominal interest rate. However, in general equilibrium, the "right" decomposition is less obvious. The next example clarifies the conceptual difficulty of the task and justifies the choices we make in the rest of the paper.

Consider an economy that is in steady state with zero inflation and all policy variables are at their zero inflation steady state level. Suppose that the household receives an unexpected endowment in period zero, $e_0 > 0$. This represents a positive *wealth shock* in period 0. As a consequence, the household will try to increase their consumption in all periods. Without explicitly solving for it, it should be clear that this behavioral response will generate some inflationary consequences if nominal rates remain fixed. Now, let's decompose the response of consumption into substitution and wealth effects. If we were to use the equilibrium inflation to calculate the substitution effect, we would end up attributing a fraction of the response to it. However, we know that the shock was a purely wealth effect shock. Since we don't want to have this type of results, we define the substitution effect in this economy as the Hicksian demand for the observed path of nominal rates, $\{i_t\}_{t=0}^{\infty}$ and the inflation *induced* by the substitution effect, $\{\pi_t^S\}_{t=0}^{\infty}$. That is, $\{c_t^S, \pi_t^S\}_{t=0}^{\infty}$ is the solution to the following system of equations

$$\begin{aligned} c_t^S &= \sigma^{-1}\int_0^t (i_s - \pi_s^S - \rho)ds - \sigma^{-1}\int_0^{\infty} e^{-\rho s}(i_s - \pi_s^S - \rho)ds, \\ \pi_t^S &= \kappa \int_t^{\infty} e^{-\rho(s-t)} c_s^S ds. \end{aligned} \tag{9}$$

In the example above, since $i_t = \rho \ \forall t$, the only bounded solution to the system above has $c_t^S = 0 \ \forall t$.

Using this definition of the substitution effect, the next proposition decomposes consumption into substitution and wealth effects.

Proposition 2 (Substitution and Wealth Effects with Sticky Prices)**.** *Suppose $\kappa > 0$. Consumption can be decomposed as*

$$c_t = \underbrace{c_t^S}_{\text{substitution effect}} + \underbrace{\underbrace{C}_{\text{direct wealth effect}} + \underbrace{\left(\frac{\overline{\omega}}{\rho} e^{\underline{\omega} t} - 1\right) C}_{\text{GE amplification of wealth effect}}}_{\text{wealth effect}},$$

where c_t^S is the solution to (9) *and C is the average consumption as defined in* (6)*. The GE amplification of the wealth effect is always positive in $t = 0$.*

As it was evident in the example with the endowment shock, the wealth effect affects the economy through two different channels. The first one is the analogous to the one we found in the fixed prices case. When the household's wealth increases, they respond by increasing their consumption. But now there is an extra term, which takes into account that this direct effect has an impact on prices which in turn generate indirect effects on consumption. In particular, a positive wealth shock generates inflation which translates into a reduction in the real interest rate when nominal rates are fixed. This generates a positive effect in period zero consumption.

The main result of this section is that when prices are sticky, the substitution effect is a force *against* the monetary paradoxes of Proposition 1.

Proposition 3 (Substitution Effect with Sticky Prices). *Suppose $\kappa > 0$. Then, the substitution effect satisfies*

$$\frac{\partial c_0^S}{\partial s \partial i_s} > 0, \quad \lim_{s \to \infty} \frac{\partial c_0^S}{\partial i_s} = 0, \quad \lim_{\kappa \to \infty} \frac{\partial c_0^S}{\partial i_s} = 0.$$

Proposition 3 implies that the paradoxes are not a result of general equilibrium amplification of substitution effects. In the absence of wealth effects, an increase in interest rates shifts consumption from the present to the future. However, the effect becomes weaker as the date of the shock moves further into the future. Moreover, the substitution effect is continuous in the price flexibility parameter κ, in sharp contrast with the standard liquidity trap equilibrium.

Thus, both the fixed prices and sticky prices exercise identify the wealth effect, through the average consumption, as the channel that generate the monetary paradoxes. In the next section, we open the average consumption and study the dynamics of each one of its components, with a special interest in the fiscal variables.

In order to highlight the central role of the average consumption in the equilibrium of the New Keynesian model, we present a characterization of *all* the solutions to the system of equations (1)-(3), indexing them by the value of average consumption, C.

Lemma 3. *In the bounded solutions to the system* (1)-(3)*, consumption is given by*

$$c_t = c_t^S + \frac{\overline{\omega}}{\rho} e^{\underline{\omega} t} C,$$

where c_t^S is defined as in (9) *and C is average consumption. Moreover, all equilibria* *satisfy monetary neutrality in the long run.*

This lemma says that all solutions to the New Keynesian model have a common component given by the substitution effect, $\{c_t^S\}_{t=0}^{\infty}$, and differ only in their average consumption, and hence the wealth effect. The standard Liquidity Trap equilibrium selects a particular wealth effect. However, as it is well known, that is just one of the many solutions to the system given by (1)-(2). A prominent example of the characterization of these solutions in the context of a Liquidity Trap exercise is Cochrane (2017*b*). Note that our characterization is not inconsistent with the characterization in that paper. The technical reason for the multiplicity of solutions is that the system (1)-(2) needs a an extra boundary condition. This condition could take many forms, including initial inflation or initial consumption. Since the focus of our paper is on average consumption, and, in particular, on the fiscal variables that determine the average consumption, it is convenient to work with the characterization from Lemma 3.

Moreover, Lemma 3 provides a different justification for our decomposition of Proposition 2. Our definition of the substitution effect coincides with the equilibrium of the New Keynesian model characterized by the system (1)-(3) when wealth effects are zero.

Corollary 3.1. *Consider the equilibrium of the New Keynesian model characterized by equations* (1)-(3)*, with $C = 0$. Then, the equilibrium consumption is given by the substitution effect, i.e., by the solution of the system given by* (9)*.*

The analysis in this section suggests that the responsible for the monetary paradoxes is not the substitution effect, and hence the Euler equation, but the wealth effect, through average consumption. In the rest of this section we show that, in fact, the solutions proposed in the literature that are based on the "discounted Euler equation" principle cannot, in general, get rid of the paradoxes but they just attenuate them.

3.3 The Discounted Euler Equation

A range of different economic environments have been proposed attempting to solve the Forward Guidance Puzzle and related paradoxes. For instance, McKay, Nakamura and Steinsson (2017) proposed a heterogenous agent model with incomplete markets, Angeletos and Lian (2016) relaxed the assumption of common knowledge in New Keynesian models, Gabaix (2016) introduced (behavioral) inattention, and Gertler (2017) adaptive expectations. Perhaps surprisingly, despite the vastly different microfoundations, all these proposed solutions essentially boil down to changes in the aggregate Euler equation, where the direct impact of future interest rate changes is attenuated.

In order to illustrate the mechanism, we will consider a version of the heterogeneous agent model in McKay, Nakamura and Steinsson (2017), henceforth MNS, but our results apply to any environment that generates an aggregate discounted Euler equation. Households now face uninsurable idiosyncratic risk and they cannot borrow. Low productivity households cannot produce and receive a government transfer, while all the labor supply is provided by the high-productivity workers. A household of type $j \in \{H, L\}$ switches type with Poisson intensity $\lambda_j \geq 0$. There is no liquidity in this economy, i.e., government bonds are equal to zero at all dates $B_t = 0$. In equilibrium, the high-productivity household will be unconstrained and his Euler equation will be given by

$$\frac{\dot{C}_{H,t}}{C_{H,t}} = \frac{r_t - \rho}{\sigma} + \frac{\lambda_H}{\sigma} \left(\frac{C_{L,t}^{-\sigma} - C_{H,t}^{-\sigma}}{C_{H,t}^{-\sigma}} \right)$$

The second-term in the expression above captures the *self-insurance* motive and it will imply that consumption reacts less strongly to future real interest rate changes. By linearizing the Euler equation around a symmetric steady state and assuming that the transfer to low productivity households stays at the steady state level as in MNS[10], we can derive the *discounted Euler*

[10] Bilbiie (2017*a*) shows that to obtain a discounted Euler equation the consumption of

equation:

$$\dot{c}_t = \delta c_t + \zeta \sigma^{-1}(r_t - \rho) \tag{10}$$

where $\zeta = \frac{\lambda_L}{\lambda_L + \lambda_H}$, $\delta = \zeta \lambda_H$, and c_t denote deviations of aggregate consumption to steady state.

The parameter δ controls the amount of discounting and it is zero in the special case where $\lambda_H = 0$, when we recover the representative agent case. Integrating the discounted Euler equation forward, we obtain

$$c_t = -\zeta \sigma^{-1} \int_t^{\infty} e^{-\delta(s-t)}(i_s - \pi_s - \rho) ds$$

Despite of the extra discounting, moderate values of δ are not enough to get rid of the puzzles, even though it attenuates its impact. The following proposition shows that Forward Guidance Puzzle and the Paradox of Flexibility are still present even with a discounted Euler equation.

Proposition 4. *Suppose* $0 < \delta < \frac{\kappa \zeta \sigma^{-1}}{\rho}$.[11] *Let* c_t^{DE} *denote consumption in an equilibrium with the discounted Euler equation* (10) *and the standard selection. Then,*

$$\frac{\partial c_0^{DE}}{\partial i_t} < 0, \qquad \lim_{t \to \infty} \frac{\partial c_0^{DE}}{\partial i_t} = -\infty, \qquad \lim_{\kappa \to \infty} \frac{\partial c_0^{DE}}{\partial i_t} = -\infty, \qquad \lim_{t \to \infty} \frac{\frac{\partial c_0^{DE}}{\partial i_t}}{\frac{\partial c_0^{NK}}{\partial i_t}} = 0$$

Changes in nominal interest rates in the far distant future still have arbitrarily large effects on consumption, but this effect grows at a smaller rate than in the case with the standard Euler equation. Decomposing the consumption allocation between a wealth effect $C^{DE} = \rho \int_0^{\infty} e^{-\rho t} c_t^{DE} dt$ and a substitution effect may help provide some intuition for this result.

Proposition 5. *Suppose* $0 < \delta < \frac{\kappa \zeta \sigma^{-1}}{\rho}$. *Let* c_t^{DE} *denote consumption in an*

low productivity households must react less than one-to-one to aggregate income. See also Werning (2015) for the role of the cyclicality of income on the aggregate Euler equation.

[11] The upper bound on δ guarantees that there is multiplicity of equilibrium under a interest rate peg, as in the standard New Keynesian model. For a standard calibration, the value of δ is more than one order of magnitude smaller than the bound.

equilibrium with the discounted Euler equation (10) *and arbitrary equilibrium selection. Then,*

1. *Consumption decomposition:*

$$c_t^{DE} = c_t^{DE,S} + \frac{\overline{\omega}_d - \delta}{\rho} e^{\underline{\omega}_d t} C^{DE} \tag{11}$$

   where $\overline{\omega}_d > \delta$ and $\underline{\omega}_d < 0$ are parameters defined in the appendix.

2. *Substitution effect:*

$$\frac{\partial c_0^{DE,S}}{\partial i_t} < 0, \quad \lim_{t\to\infty} \frac{\partial c_0^{DE,S}}{\partial i_t} = 0, \quad \lim_{\kappa\to\infty} \frac{\partial c_0^{DE,S}}{\partial i_t} = 0, \quad \int_0^\infty e^{-\rho t} c_t^{DE,S} dt = 0$$

As before, any equilibrium in the economy with a discounted Euler equation can be decomposed into a wealth effect C^{DE} and a substitution effect $c_t^{DE,S}$. As before, under the substitution effect equilibrium, average consumption is equal to zero: intertemporal substitution only reallocates consumption over time. Since average consumption is negative under the standard equilibrium, this requires a negative wealth effect, which needs to be larger the further into the future is the interest rate change.

Hence, wealth effects are important to understand the Forward Guidance Puzzle or the Paradox of Flexibility even after allowing for a discounted Euler equation. In the next section, we show how fiscal policy is typically an important determinant of the wealth effects in the standard New Keynesian model.

4 The Fiscal Origins of Monetary Paradoxes

Recall that average consumption is given by

$$C = \rho \int_0^\infty e^{-\rho t} \left[(1 - \overline{\tau}) y_t + \overline{b}(i_t - \pi_t - \rho) + T_t \right] dt.$$

Thus, it is immediate to ask the relative importance of each term in generating the monetary paradoxes.

In order to do this, we start with the knife-edge case in which $\overline{\tau} = \overline{b} = 0$. Using the resource constraint $c_t = y_t$, average consumption is given by

$$C = \rho \int_0^\infty e^{-\rho t}(c_t + T_t)dt = C + \rho \int_0^\infty e^{-\rho t} T_t dt,$$

which implies that the equilibrium requires $\int_0^\infty e^{-\rho t} T_t dt = 0$, and, therefore, the budget constraint of the household provides no restriction on what average consumption is. In particular, the level of average consumption, and hence the wealth effect, has a self-fulfilling nature. If agents expect to receive higher income, $\int_0^\infty e^{-\rho t} y_t dt$, then they increase their consumption accordingly, and since output is demand determined in this model, output increases to satisfy that demand. But since the household's income equals the present value of output, the increase in consumption becomes self-fulfilling. Thus, when $\overline{\tau} = \overline{b} = 0$, the monetary paradoxes are the result of the exacerbated self-fulfilling nature of the wealth effect, and not the intertemporal substitution effect.

However, we now show that this self-fulfilling nature of average consumption is limited to the knife-edge case. As we move away from this case, the average consumption is exactly determined by the policy variables. It is important to highlight that this decomposition has no implication about the discussion of passive-active fiscal policy. All our analysis is consistent with the so-called Ricardian fiscal policy. It just brings to the forefront the role that the policy variables have on the determination of equilibrium and the channels through which they operate.

Suppose $\overline{\tau} > 0$ and $\overline{b} > 0$. In this case, there are three terms that determine the value of C: the *spending-income spiral*, the *spending-inflation spiral*, and the *present-value of government transfers*, given, respectively, by

$$y_t = c_t, \qquad \pi_t = \pi_t^* - \sigma \underline{\omega} c_t, \qquad T \equiv \int_0^\infty e^{-\rho t} T_t dt,$$

where π_t^* is a function of $\{i_t\}_{t=0}^{\infty}$.

The next proposition shows how to determine the average consumption.

Proposition 6. *Suppose $\overline{\tau} > 0$ and $\overline{b} > 0$. The average consumption, C, solves*

$$C = [1 - (\overline{\tau} - \sigma\underline{\omega}\overline{b})]Y + A, \tag{12}$$

where $A \equiv \rho\overline{b}\int_0^{\infty} e^{-\rho s}(i_s - \pi_s^* - \rho)ds + \rho T$. Thus, average consumption is given by

$$C = \frac{A}{\overline{\tau} - \sigma\underline{\omega}\overline{b}}.$$

Moreover, for a given sequence of $\{i_t, T_t\}_{t=0}^{\infty}$,

$$\frac{\partial C}{\partial i_t} > 0, \quad \frac{\partial^2 C}{\partial t \partial i_t} < 0, \quad \lim_{t\to\infty} \frac{\partial C}{\partial i_t} = 0, \quad \lim_{\kappa\to\infty} \frac{\partial C}{\partial i_t} = 0.$$

Interestingly, equation (12) shows that the average consumption is determined according to an *Intertemporal Keyensian Cross*, in the spirit of the old Keynesian logic found in many introductory textbooks. To grasp the intuition in this model, consider the impact of a shock that increases the value of autonomous spending by Δ. If we were to keep inflation and output constant, this would generate an increase of consumption of Δ. But the higher consumption raises demand, increases the households' income by $1 - \overline{\tau}$, and generates inflation, reducing the real return on the household's assets by $\sigma\underline{\omega}\overline{b}$ (remember $\underline{\omega} < 0$). As a result, there is a (first-round) net increase in wealth of $1 - \left(\overline{\tau} - \sigma\underline{\omega}\overline{b}\right)$. This additional income further increases consumption, which increases net income again, in the following way

$$\Delta + \left(1 - \left(\overline{\tau} - \sigma\underline{\omega}\overline{b}\right)\right)\Delta + \left(1 - \left(\overline{\tau} - \sigma\underline{\omega}\overline{b}\right)\right)^2 \Delta + \ldots = \frac{\Delta}{\overline{\tau} - \sigma\underline{\omega}\overline{b}}$$

Thus, an intuition analogous to the standard Keynesian cross is useful to think about wealth effects in the New Keynesian model.

Therefore, the response of average consumption to monetary shocks depends on the response of the autonomous component of consumption to shocks. In particular, in this settings it is the financial wealth shocks, given by $\rho\bar{b}\int_0^\infty e^{-\rho s}(i_s - \pi_s^* - \rho)ds$, and the present value of government transfers, T. This reduces the candidates of generating the monetary paradoxes to two: general equilibrium effects of financial wealth and government transfers.

In order to get a sense of the magnitude of the transfers necessary to sustain the standard Liquidity Trap equilibrium, let's compute how they adjust to monetary shocks. The budget constraint of the household's implies that

$$c_0 = \int_0^\infty \chi_{i,t}^c (i_t - \rho)dt + \chi_T^c T,$$

where $T \equiv \int_0^\infty e^{-\rho t} T_t dt$ is the present value of government transfers, and

$$\chi_{i,t}^c \equiv -\sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d} e^{-\overline{\omega}t}.$$

The next proposition summarizes the behavior of lump-sum transfers to shocks.

Proposition 7 (Fiscal backing in standard equilibrium selection)**.** *Consider the standard liquidity trap equilibrium selection and let $t < T^*$. Then*

$$\frac{\partial T}{\partial i_t} < 0,$$

$$\frac{\partial^2 T}{\partial t \partial i_t} < 0, \qquad \lim_{t\to\infty} \frac{\partial T}{\partial i_t} = -\infty, \qquad \lim_{\kappa\to\infty} \frac{\partial T}{\partial i_t} = -\infty.$$

Proposition 7 shows that the forward guidance puzzle and the paradox of flexibility imply a fiscal response with transfers that increases unboundedly (in absolute terms) as the horizon and the price flexibility go to infinity. Figure 4 depict these results.

We are ready to present the main result of the paper. The next proposition identifies the transfers as the only source responsible for the monetary paradoxes.

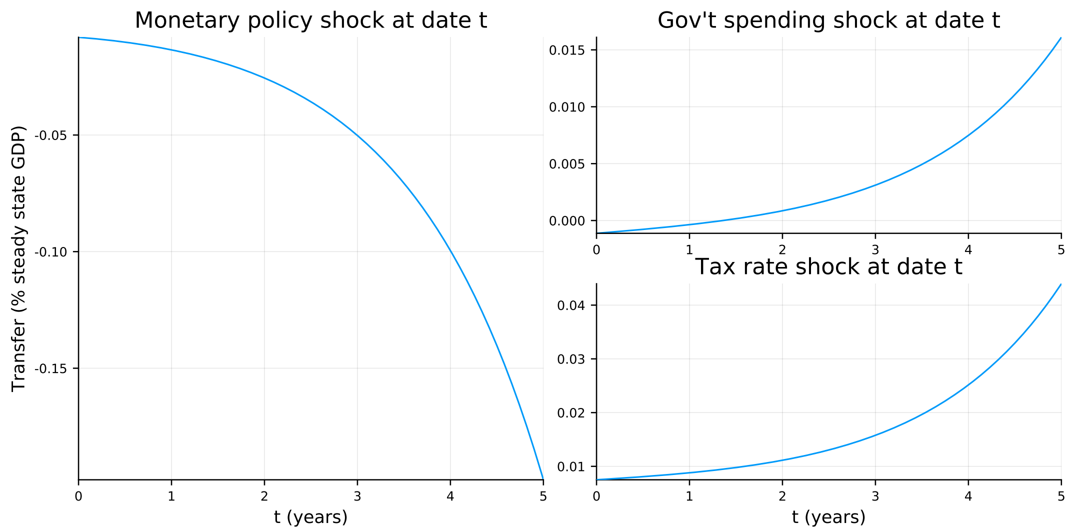

FIGURE 4: Fiscal Backing to Shocks in the Standard Liquidity Trap Equilibrium

Proposition 8. *Suppose $\overline{\tau} > 0$ and $\overline{b} > 0$. In the standard Liquidity Trap equilibrium, the Forward Guidance Puzzle and the Paradox of Flexibility are the consequence of the fiscal response to monetary shocks. In particular,*

$$\lim_{s\to\infty} \frac{\partial c_0}{\partial i_s} = -\infty \iff \lim_{s\to\infty} \frac{\partial T}{\partial i_s} = -\infty$$

where $T = \rho \int_0^\infty e^{-\rho t} T_t dt$ and

$$\lim_{\kappa\to\infty} \frac{\partial c_0}{\partial i_t} = 0 \iff \lim_{\kappa\to\infty} \frac{\partial T}{\partial i_s} = 0.$$

This is a powerful result. It states that if government transfers don't grow too fast as we postpone the time of the monetary shock or as we increase price flexibility, the New Keynesian model does not exhibit any of the monetary paradoxes emphasized in the literature. Thus, monetary paradoxes are not the result of a fiscal response to monetary shocks but to a sufficiently large one.

5 Outside the ZLB: Monetary Policy in Normal Times

Until now we have focused on the paradoxical results of the New Keynesian model in a Liquidity Trap, and we showed that they are generically the consequence of extreme fiscal policy reactions implied in the standard model. Although these findings are important to understand the predictions of the model when the economy is against the zero lower bound, it is less clear whether these analysis is also relevant to study the monetary policy transmission mechanism of the New Keynesian model in normal times. In this section we tackle this question. In particular, we show that the implicit fiscal policy implied in the standard equilibrium selection plays a key role in the quantitative success of the New Keynesian model emphasized in the literature (e.g., Christiano, Eichenbaum and Evans (2005)).

The main exercise of this section is the following. We feed a medium-sized DSGE model with the impulse response functions of the monetary and fiscal policies estimated from the data in the previous section, and compare the predicted response of output and inflation with the ones estimated in the data. To do this, we augment the New Keynesian model from Smets and Wouters (2007) to incorporate fiscal variables. This exercise is a test of the New Keynesian model that does not rely in the specification of the policy rules. We can do this because of the following result: for a given path of monetary and fiscal variables, the New Keynesian model has a unique equilibrium path for all the other variables, independently of the policy rules that the monetary and fiscal authorities follow. Importantly, this is true independently of the policy regime (monetary or fiscal dominance). Thus, we are able to separate the discussion of the policy regime (and the determination of equilibrium) from the analysis of the usefulness of the New-Keynesian model as an approximation of reality.

The rest of this section is organized as follows. First, we briefly describe the model we use in our quantitative exercise. The model is the natural extension of Smets and Wouters (2007) to account for fiscal variables. We show that the implicit transfers necessary to sustain the standard equilibrium play an important role in the quantitative predictions of the model. Next, we use the strategy in Gertler and Karadi (2015) to estimate the fiscal response to a monetary shock in the US data. We find impulse response functions for transfers that have the opposite sign than the implied in the standard New Keynesian equilibrium. Finally, we feed the model with the impulse response functions for monetary and fiscal variables we estimated from the data, and compare the predicted impulse response functions for output and inflation with the ones obtained in the data. Our results suggest that the workhorse New Keynesian model is not consistent with the empirical evidence.

5.1 The Model

Time is discrete and denoted by $t = 0, 1, 2, \ldots, \infty$. The economy is populated by a continuum of mass one of infinitely-lived households. Households derive utility from the consumption of a final good and leisure. Their preference for consumption exhibits an external habit variable. Labor supply is differentiated across households. It is assumed that the wages of each type of labor is negotiated by a union, which chooses the wage but is subject to nominal rigidities à la Calvo. Households are the owners of the capital of the economy. They rent capital services to the firms, which is a function of the capital stock they hold and the utilization level they choose, which comes at the cost of higher depreciation. Households also decide how much capital to accumulate given the adjustment costs they face.

There are two types of firms in the economy. There is a continuum of intermediate goods producer firms, which transform labor and capital services into a differentiated good and set prices subject to the Calvo friction. Those wages and prices that cannot be re-optimized in a given period, are partially indexed to past inflation. The second type of firm is a representative firm that produces the final consumption good using the intermediate goods as inputs and sells the output in competitive markets. Finally, there is a government that chooses a path for the nominal interest rate, government spending, proportional sales taxes, lump-sum transfers, and debt.

In order to keep the exposition short, below we present the household's budget constraint, followed by a description of the differences that incorporating the fiscal variables introduce to the model. The reader can refer to the appendix for a detailed derivation. The budget constraint of the household is given by

$$
\begin{aligned}
c_y c_t + i_y i_t + q^L b_y (b_t^L + q_t^L) = (1-\tau^*)(y_t - \tau_t) - z_y z_t + \frac{\rho_L q^L b_y}{1+\pi^*} q_t^L \\
- \frac{(1+\rho_L q^L) b_y}{1+\pi^*} \pi_t + \frac{(1+\rho_L q^L) b_y}{1+\pi^*} b_{t-1}^L + T_t, \quad (13)
\end{aligned}
$$

where q_L is the price of long-term government bonds

$$
q_t^L = \frac{\rho_L}{1+\bar{r}} q_{t+1}^L - r_t, \tag{14}
$$

y_t is output, c_t is consumption, i_t is investment, g_t is government spending, z_t is the capital utilization rate, l_t is hours worked, r_t is the nominal interest rate (set by the monetary authority), π_t is the inflation rate, q_t is the Tobin's Q, r_t^k is the rental rate of capital services, k_t^s is capital services, k_t is the stock of capital, w_t is the real wage, μ_t^p is the price mark-up, μ_t^w is the wage mark-up, τ_t is the proportional sales tax, b_t^L is government bonds, q_t^L is the price of the long-term bond, and T_t are government lump-sum transfers. The rest are positive constants defined in the appendix.

The model follows very closely Smets and Wouters (2007), with thee differences. First, because we introduce sales taxes, the NK Phillips curve has an extra term and the coefficients depend on $1-\tau^*$. Second, we drop the Taylor rule for monetary policy. Third, we explicitly introduce the household's budget constraint. The first point is not important for our analysis, and introducing this in Smets and Wouters (2007) exercise has no impact on the results. The second and third point are important and related to our test of the model. Since we want to determine the performance of the New Keynesian model when we feed it with the observed impulse responses of the monetary and fiscal variables, we do not need to specify any policy rules. A similar result can be found in Werning (2011) and Cochrane (2017*b*) in a different context. Here, we formalize the result and use it to show that any equilibrium of this model can be the outcome of an economy under either a monetary or fiscal dominance regime.

Lemma 4. *Suppose* $\Xi^* \equiv (y_t^*, c_t^*, l_t^*, i_t^*, z_t*, q_t*, r_t^{k*}, \pi_t^*, k_t^*, k_t^{s*}, \mu_t^{p*}, \mu_t^{w*}, w_t^*, b_t^*)$ *satisfies the system of equations given by* (16)-(30) *given a sequence of monetary and fiscal variables* (r_t, g_t, τ_t, T_t)*. Then, there exists a Taylor rule that implements* Ξ^* *as an equilibrium of an economy characterized by equations* (16)-(28)*, given a sequence of fiscal variables* (g_t, τ_t)*. Moreover, suppose* Ξ^* *satisfies the system of equations given by* (16)-(28) *and a Taylor rule, given a sequence of fiscal variables* (g_t, τ_t)*. Then, there exists a sequence* $(\tilde{r}_t, \tilde{T}_t)$ *that implements* Ξ^* *as a solution to* $(16) - (30)$*.*

Lemma 4 implies that we can test the predictions of the model independently of the policy regime if we feed the system of equations $(16) - (30)$ with a path for (r_t, g_t, τ_t, T_t). If by doing this we reject the model, then assuming monetary or fiscal dominance will not change the results. In this sense, the policy regime discussion is one about out-of-sample performance of the model rather than in-sample evaluation.

The first question we ask is the following: how large are the necessary transfers in a medium-sized New Keynesian model to sustain the standard equilibrium after a monetary shock? We answer this question by simulating the model above calibrated using Smets and Wouters (2007) and calculate the implied transfers from the budget constraint.

Figure 5 shows the results. The solid line in the graphs depicts the impulse response function of inflation and GDP to a monetary shock, as well as the transfers that come out from the budget constraint. The transfers are negative and of a similar order of magnitude of the change in output. From a "permanent income" view, these transitory transfers should not have a big

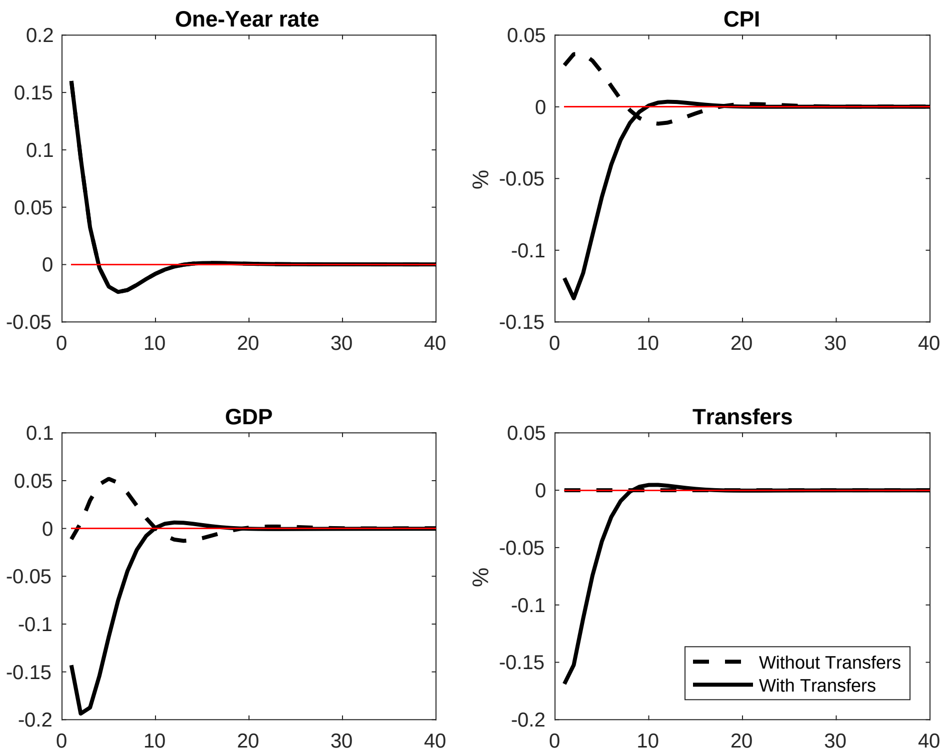

FIGURE 5: SW with and without transfers

effect in the economy. The dashed line in Figure 5 shows the equilibrium for the same path of interest rates but no reaction of transfers. The impulse response of inflation and GDP are dramatically different. First, the monetary shock generates inflation rather than deflation. Second, the shock generates a small recession on impact and later a sizable boom. How can this be explained? The answer is the decomposition from Section 3. The big effect of transfers in this economy is not coming from a ″permanent income″ logic, but from the interaction of transfers and GE effects through inflation. As Figure 5 shows, the transfers have a big impact on inflation, and the effect feeds itself. Thus, transfers have an important role even in a medium-sized New Keynesian model after a transitory monetary policy shock and outside of a Liquidity Trap.

5.2 Empirical Evidence of Fiscal Response to Monetary Shocks

Now we want to evaluate the empirical plausibility of the transfers derived from the theory. There is an extensive literature studying the response of the main macroeconomic variables (output, consumption, investment, inflation, wages) to monetary policy shocks. However, there is still not a consensus with respect to the best set of assumptions that identify exogenous monetary shocks. Our objective in this section is not to contribute to this debate, but to use some of the results in the literature to determine whether the data supports the fiscal response to monetary shocks implied by the standard New Keynesian equilibrium selection.

To this end, we extend the exercise in Gertler and Karadi (2015) to account for the dynamics of fiscal policy after an exogenous monetary shock identified using the change of three-month ahead fed funds rate future in a 30-minute window around FOMC announcements. The strategy is a combination of a VAR estimation and an external instruments approach. First, we estimate a VAR in seven variables. The first four variables are the ones in Gertler and Karadi (2015): the one-year government bond rate, log industrial production, log consumer price index and the Gilchrist and Zakrajšek

(2012) excess bond premium as a measure of credit spread. We also include three fiscal variables: government spending, total revenues over GDP and government transfers. From this VAR, we obtain the reduced form shocks, which are a linear combination of the structural shocks, in particular, the monetary policy shock.

The second stage is to estimate the sensitivity of the VAR variables to a monetary policy shock. A standard approach consists of putting the so called "timing restrictions" on the relationship between reduced form and structural shocks.[12] Instead, Gertler and Karadi (2015) use an external instrument approach. This is achieved by regressing the estimated reduced-form residuals to the change of the three-months ahead fed funds rate futures in a 30-minute window around FOMC announcements, where the fed-fund futures act as an instrument of the actual monetary shocks.

We use quarterly data over the period 1979:3 to 2012:2, due to the limitations imposed by the fiscal variables. Our results suggest that there are no important differences in the estimates of the impulse response functions, though we get wider confidence intervals.

Figure 6 shows the results. As in Gertler and Karadi (2015), we find that a positive monetary shock reduces output and prices, though only the effect on output is significant at 90% confidence level. With respect to the dynamics of fiscal variables, a monetary shock has a statistically zero effect on government purchases, while revenues over output (a proxy of proportional taxes) decrease, and transfers increase. The effect on proportional taxes and transfers is likely to come from the automatic stabilizer mechanisms embedded in the government accounts. Since a monetary shock is contractionary, households' income and employment decrease. This has two effects. First, since income taxes are progressive, the average income tax in the economy decreases. Second, since a large fraction of government transfers are unemployment benefits, it is natural that it increases in a recession.

These results show that a monetary policy shock does not trigger the fis-

[12]For a detailed explanation see Christiano, Eichenbaum and Evans (1999).

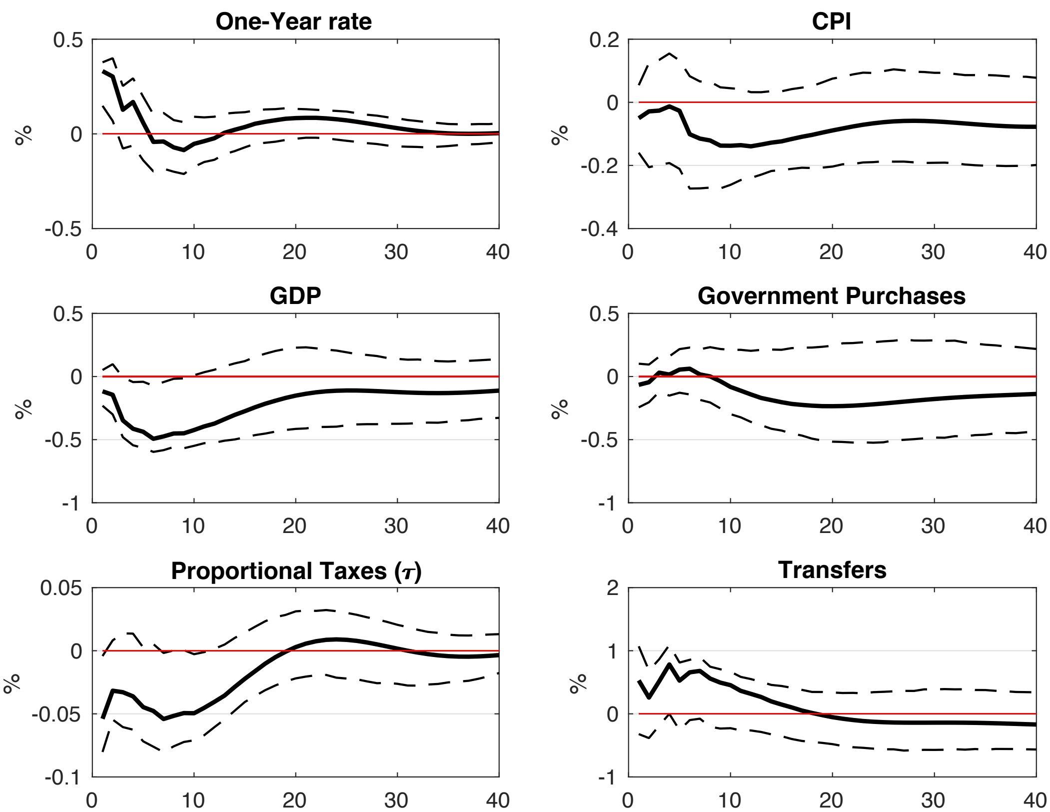

FIGURE 6: Impulse Response Function to a monetary shock

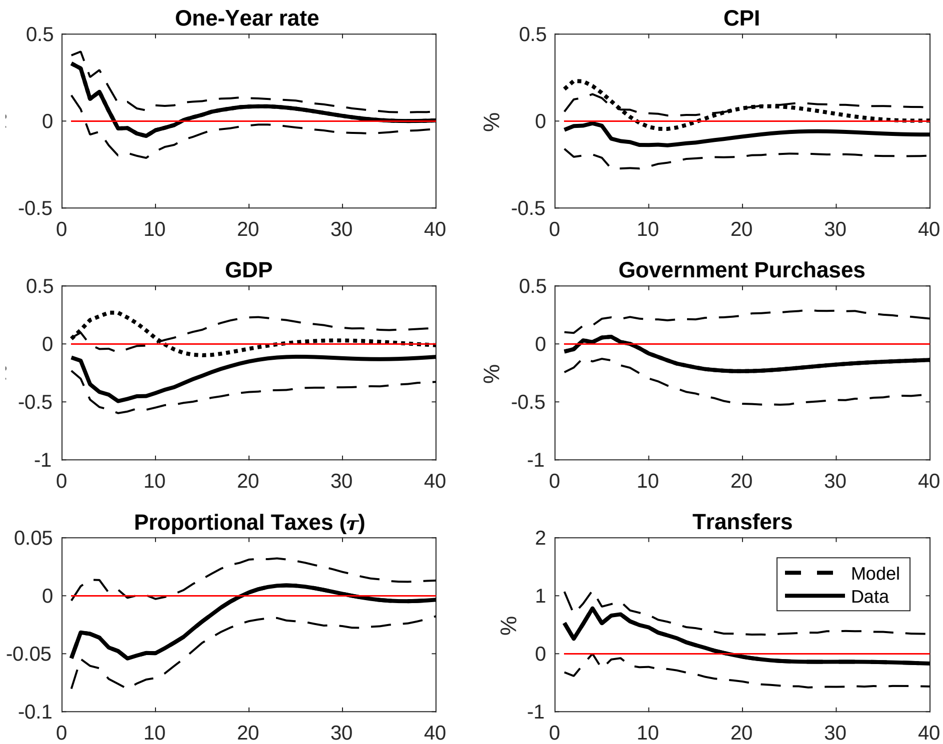

FIGURE 7: Model impulse response functions to a monetary shock. Interest rates and fiscal variables match the data.

cal response implied by the standard New Keynesian equilibrium selection. Government transfers increase rather than decrease and proportional taxes decrease. The point estimate of government spending is negative, which contributes to the standard equilibrium mechanism, but statistically zero. Thus, the evidence does not seem to support the standard equilibrium selection. Next, we feed the monetary and fiscal policy impulse response functions in the New Keynesian model to evaluate the quantitative success of the model when both monetary and fiscal policy follow the observed paths in the data.

5.3 Testing the New Keynesian model

Next, we present the main exercise of this section. We feed the model with the impulse responses for the interest rate, government purchases, proportional taxes and transfers estimated from the data, and compare the resulting impulse response functions for inflation and output with the ones obtained from the data. Figure 7 depicts the results. The solid line is the impulse response estimated from the data, with the dashed lines being the 90% confidence intervals. The pointed line is the impulse response of the model. We can see that the model predicts higher inflation than the data, though mostly inside the confidence bands. However, the data rejects the impulse response for output. While the data implies that a positive monetary shock generates a recession in the short run, the model predicts a boom in that same period. Figure 8 performs a similar exercise but imposing that government spending does not react to a monetary shock, given that the data cannot reject that the changes are zero. The difference with the data exacerbates. Since the point estimate of the impulse response of government purchases is negative, their contribution in the model is a force towards lower inflation and output. When we set government spending to zero, the output boom exacerbates, and now we have statistically significant increase in inflation.

Thus, we conclude that the standard New Keynesian model cannot produce impulse response functions on output and inflation that resemble the ones obtained in the data. This result contrasts some of the findings in the literature, as for example Christiano, Eichenbaum and Evans (2005). The difference relies on our use of fiscal policy as an extra restriction in the model. When fiscal transfers are set to match their empirical counterpart, the standard New Keynesian model is dominated by substitution effects that do not match the data. Importantly, we want to emphasize that this result does not depend on equilibrium selection. The test in this section can be interpreted as being the answer to the following question: is there *any* equilibrium in the standard New Keynesian model that can produce impulse

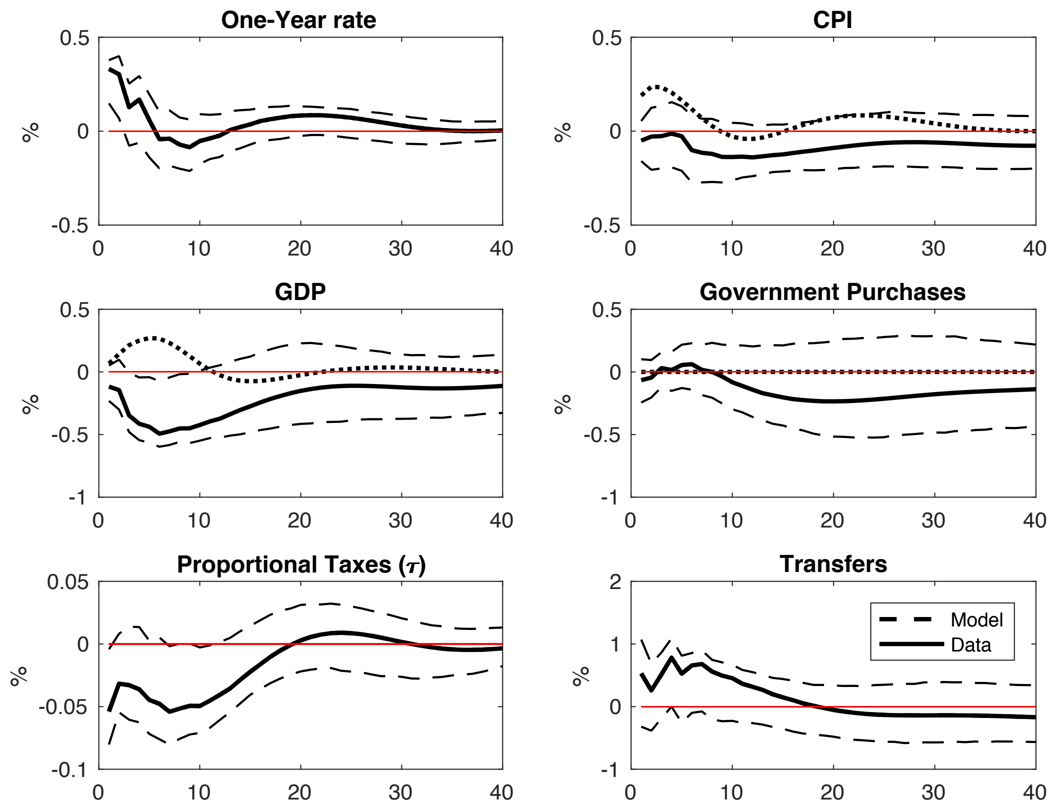

FIGURE 8: Model impulse response functions to a monetary shock. Government spending is set to zero.

response functions similar to the ones in the data, *once we impose observed paths for monetary and fiscal variables after a monetary shock*? Our answer is no.

In light of these results, next we briefly explore an extension of the standard model that relies on endogenous strong *private* wealth effects in response to monetary shocks.

5.4 Private Wealth Effects: Indebted Hand-to-Mouth

The previous analysis showed two things: first, implied government transfers are important for the quantitative success of the standard New Keynesian model; second, once transfers are disciplined with the data, the standard New Keynesian model cannot produce wealth effects strong enough to match the impulse response functions of output and inflation in the data. With this in mind, we study a very simple extension of the New Keynesian model to account for private wealth effects, and show that it produces dynamics similar to the ones found in the data.

The model is a standard borrower-saver model. There are two types of agents in the economy: borrowers and savers. The main difference between these two groups is their discount factor: borrowers are more impatient than savers. Agents in this economy are subject to a borrowing constraint of the form:

$$\frac{B_t}{P_t} \leq \overline{D},$$

where B_t is the level of the agent's nominal debt, P_t is the price level, and $\overline{D}$ is a the real borrowing limit. The rest of the economy follows Smets and Wouters (2007).

In the steady state of the economy, the borrowers are against their borrowing constraint due to their impatience. Therefore, their consumption in period t is given by

$$C_t^b = w_t l_t^b - \frac{r_t}{1+r_t}\overline{D} + T_t,$$

where C_t^b is consumption of borrowers, w_t is the real wage, l_t^b is labor of the borrowers, r_t is the real interest rate, and T_t is lump-sum transfers.

It is immediate to see that this model has the potential of producing strong wealth effects from monetary policy shocks. The borrower is effectively a hand-to-mouth consumer: every period they consume all their income net of interest payments. If the interest rate goes up, borrowers will not be able to smooth consumption due to the binding borrowing constraint. As a consequence, their consumption level will adjust one-to-one to the higher interest payments, introducing a channel that, as we show below, greatly amplifies the effect of monetary shocks.

To understand the importance of having positive debt rather than just hand-to-mouth agents, the next Lemma states that if $\overline{D} = 0$, aggregate variables would behave as if it was a representative agent model.

Lemma 5. *Suppose $\overline{D} = 0$. Then, aggregate variables in the heterogenous agents economy behave as if they were the result of a representative agent model.*

This Lemma implies that the results we obtain below are not just coming from having hand-to-mouth agents, but from the fact that these agents are indebted and, as a consequence, their reaction to interest rate changes has a direct effect on their consumption decisions.

Now, suppose that $\overline{D} > 0$. We evaluate the quantitative performance of this model in the context of Smets and Wouters (2007) augmented by fiscal variables as studied before. The only difference with respect to the representative agent model is that we now have that aggregate consumption is given by the sum of the consumption of borrowers and savers. In the calibration, there are two important new parameters: the fraction of borrowers and the fraction of private debt-to-GDP. As a benchmark, we set the fraction of borrowers to $1/3$, in line to the findings of Kaplan, Violante and Weidner (2014), and private debt-to-GDP to 50% (total private debt in the US is close to one GDP). It is important to emphasize that this exercise should be seen as a proof-of-concept and not a serious quantitative evaluation. The objective of this subsection is to show that this direction of research seems promising in producing dynamics that match the data. A serious evaluation would consider a richer model and a more careful calibration.

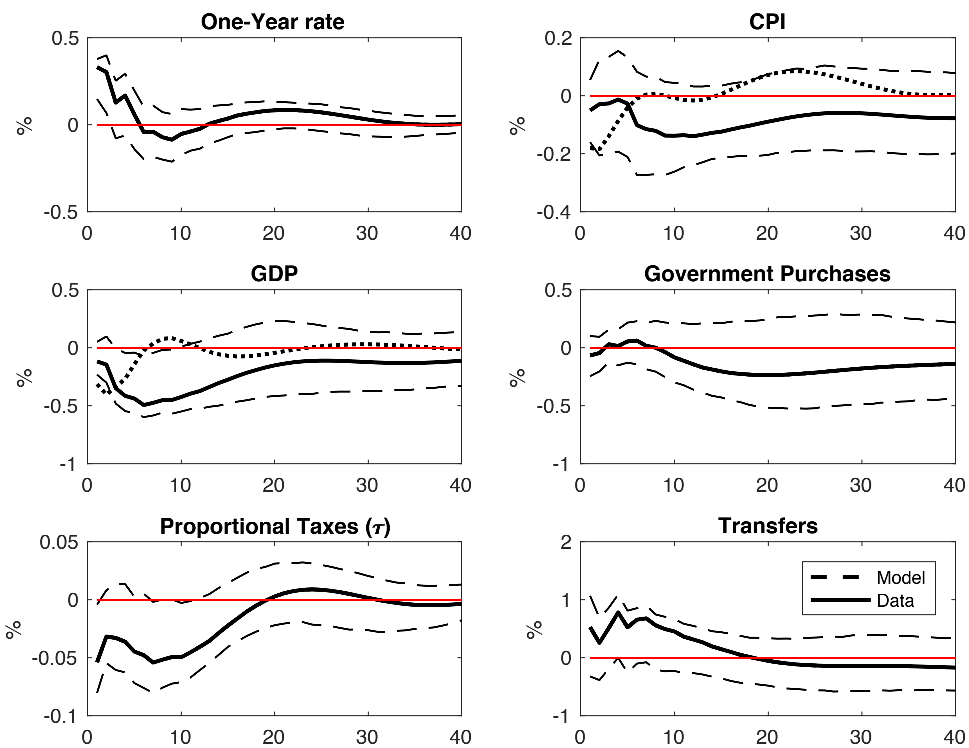

FIGURE 9: Heterogeneous Agent Model impulse response functions to a monetary shock.

Figure 9 shows the results. The fit of the model improves considerably by adding the indebted hand-to-mouth. The impulse response functions for output and inflation match the effect of a monetary shock on impact, though there are some discrepancies on the effects after one year. The change in the interest rate has a strong affect on the borrowers' consumption, which cannot be smoothed because they are against their constraint. This partially offsets the absence of wealth effects coming from government transfers, improving the quantitative performance of the model when fiscal variables are set to match the data. These results show that, in order to understand the monetary policy transmission mechanism, the direction of research should be to better understand the determinants of wealth effects and its interactions with monetary policy, rather than mitigating the forward looking nature of rational households.

6 Conclusion

In this paper we revisit the monetary paradoxes in the standard New Keynesian model and study the channels through which they occur. We focus on two paradoxes: the Forward Guidance Puzzle and the Paradox of Flexibility. Our main finding is that monetary paradoxes are mainly driven by strong wealth effects embedded in the standard equilibrium selection, rather than on the intertemporal substitution effect. In particular, we show that, generically, the paradoxes arise exclusively due to a counterfactual fiscal response to monetary shocks.

To do this, we propose a decomposition of consumption into substitution and wealth effects, both of which take into account the general equilibrium effects of output and inflation. Our first main result is that, in the absence of wealth effects, the equilibrium does not present any of the paradoxical results: even after fully taking into account general equilibrium effects on output and inflation, the effect of changes in interest rates is reduced with the horizon of the intervention and the equilibrium is continuous in the price flexibility parameter. Instead, monetary paradoxes are the result of strong wealth effects which, in the presence of fiscal consequences of monetary shocks, are solely determined by the expected fiscal response to the shock. Moreover, we show that this result holds in models that boil down to a discounted Euler equation, in which case the paradoxes are attenuated but do not disappear.

Moreover, we find that the prescribed fiscal response in the standard equilibrium does not hold in the data. We estimate the fiscal response to monetary shocks using the high-frequency data approach in Gertler and Karadi (2015), which combines a VAR estimation and an external instruments approach, where the instruments are the changes of the three-month ahead fed funds rate in a 30-minute window around FOMC announcements. We find empirical fiscal responses to monetary shocks with the opposite sign to the ones implied by the standard equilibrium.

Finally, we introduce the estimated fiscal responses into a medium-size DSGE model. We find that the impulse-response of consumption and inflation do not match the data, suggesting that wealth effects induced by fiscal policy may be important even outside of the liquidity trap. This result suggests that the New Keynesian model should be augmented with mechanisms that generate stronger wealth effects in order to get impulse responses of output and inflation consistent with the data. In particular, we show that introducing an indebted hand-to-mouth improves the quantitative performance of the model.

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Appendix

Proof of Lemma 1. Define the following rotation of the system:

$$Z_t = \begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} = -\frac{\kappa\omega_c}{\overline{\omega} - \underline{\omega}} \begin{bmatrix} 1 & (\sigma\underline{\omega})^{-1} \\ -1 & -(\sigma\overline{\omega})^{-1} \end{bmatrix} \begin{bmatrix} c_t \\ \pi_t \end{bmatrix}$$

$$\eta_t = \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix} = \begin{bmatrix} -\frac{\kappa\omega_c}{\overline{\omega}-\underline{\omega}} m_t - \frac{\overline{\omega}}{\overline{\omega}-\underline{\omega}} f_t \\ \frac{\kappa\omega_c}{\overline{\omega}-\underline{\omega}} m_t + \frac{\underline{\omega}}{\overline{\omega}-\underline{\omega}} f_t \end{bmatrix}$$

where $f_t \equiv \kappa\left(\omega_g g_t + \tau_t\right)$ and $m_t \equiv \sigma^{-1}(i_t - \rho)$.

The system in the new coordinates can be written as

$$\begin{bmatrix} \dot{Z}_{1t} \\ \dot{Z}_{2t} \end{bmatrix} = \begin{bmatrix} \overline{\omega} & 0 \\ 0 & \underline{\omega} \end{bmatrix} \begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} + \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}$$

Integrating the system above:

$$e^{-\overline{\omega}T} Z_{1T} - e^{-\overline{\omega}t} Z_{1t} = \int_t^T e^{-\overline{\omega}s} \eta_{1s} ds$$

$$e^{-\underline{\omega}T} Z_{2T} - e^{-\underline{\omega}t} Z_{2t} = \int_t^T e^{-\underline{\omega}s} \eta_{2s} ds$$

Solving the first equation forward and the second backwards, we obtain

$$Z_{1t} = -\int_t^\infty e^{-\overline{\omega}(s-t)} \eta_{1s} ds$$

$$Z_{2t} = e^{\underline{\omega}t} Z_{20} + \int_0^t e^{\underline{\omega}(t-s)} \eta_{2s} ds$$

Rotating the system back to the original coordinates, we obtain

$$c_t = -(\sigma\underline{\omega})^{-1} e^{\underline{\omega}t} Z_{20} - (\sigma\underline{\omega})^{-1} \int_0^t e^{\underline{\omega}(t-s)} \eta_{2s} ds + (\sigma\overline{\omega})^{-1} \int_t^\infty e^{-\overline{\omega}(s-t)} \eta_{1s} ds$$

$$\pi_t = e^{\underline{\omega}t} Z_{20} + \int_0^t e^{\underline{\omega}(t-s)} \eta_{2s} ds - \int_t^\infty e^{-\overline{\omega}(s-t)} \eta_{1s} ds.$$

Evaluating these expressions at $t=0$, we get

$$c_0 = -\left(\sigma\underline{\omega}\right)^{-1} Z_{20} + \left(\sigma\overline{\omega}\right)^{-1} \int_0^\infty e^{-\overline{\omega}s}\eta_{1s}ds$$
$$\pi_0 = Z_{20} - \int_0^\infty e^{-\overline{\omega}s}\eta_{1s}ds.$$

Thus, the relationship between c_0 and π_0 is given by

$$\pi_0 = \sigma\underline{\omega}\left[\frac{\overline{\omega}-\underline{\omega}}{\sigma\hat{\kappa}}\int_0^\infty e^{-\overline{\omega}s}\eta_{1s}ds - c_0\right]$$

Replacing back in c_t and π_t, we get

$$c_t = e^{\underline{\omega}t}c_0 - \sigma^{-1}e^{\underline{\omega}t}\int_0^t\left(\frac{e^{-\overline{\omega}s}}{\overline{\omega}}\eta_{1s} + \frac{e^{-\underline{\omega}s}}{\underline{\omega}}\eta_{2s}\right)ds + \sigma^{-1}\frac{e^{\overline{\omega}t}-e^{\underline{\omega}t}}{\overline{\omega}}\int_t^\infty e^{-\overline{\omega}s}\eta_{1s}ds$$
$$\pi_t = e^{\underline{\omega}t}\pi_0 + e^{\underline{\omega}t}\int_0^t\left(e^{-\overline{\omega}s}\eta_{1s} + e^{-\underline{\omega}s}\eta_{2s}\right)ds - \left(e^{\overline{\omega}t}-e^{\underline{\omega}t}\right)\int_t^\infty e^{-\overline{\omega}s}\eta_{1s}ds$$

Finally, using the expression for η_t we get

$$c_t = e^{\underline{\omega}t}c_0 + c_t^m + c_t^f$$
$$\pi_t = e^{\underline{\omega}t}\pi_0 + \pi_t^m + \pi_t^f$$

where

$$c_t^m \equiv \frac{\hat{\kappa}\sigma^{-1}}{\overline{\omega}-\underline{\omega}}\left[e^{\underline{\omega}t}\int_0^t\left(\frac{e^{-\overline{\omega}s}}{\overline{\omega}} - \frac{e^{-\underline{\omega}s}}{\underline{\omega}}\right)(i_s-\rho)ds - \frac{e^{\overline{\omega}t}-e^{\underline{\omega}t}}{\overline{\omega}}\int_t^\infty e^{-\overline{\omega}s}(i_s-\rho)ds\right]$$
$$c_t^f \equiv \frac{\kappa\sigma^{-1}}{\overline{\omega}-\underline{\omega}}\left[e^{\underline{\omega}t}\int_0^t\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)(\omega_g g_s+\tau_s)ds - \left(e^{\overline{\omega}t}-e^{\underline{\omega}t}\right)\int_t^\infty e^{-\overline{\omega}s}(\omega_g g_s+\tau_s)ds\right]$$
$$\pi_t^m \equiv \frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\left[-e^{\underline{\omega}t}\int_0^t\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)(i_s-\rho)ds + \left(e^{\overline{\omega}t}-e^{\underline{\omega}t}\right)\int_t^\infty e^{-\overline{\omega}s}(i_s-\rho)ds\right]$$
$$\pi_t^f \equiv \frac{\kappa}{\overline{\omega}-\underline{\omega}}\left[-e^{\underline{\omega}t}\int_0^t\left(\overline{\omega}e^{-\overline{\omega}s}-\underline{\omega}e^{-\underline{\omega}s}\right)(\omega_g g_s+\tau_s)ds + \overline{\omega}\left(e^{\overline{\omega}t}-e^{\underline{\omega}t}\right)\int_t^\infty e^{-\overline{\omega}s}(\omega_g g_s+\tau_s)ds\right]$$

The standard liquidity trap equilibrium selection sets $c_T = 0$ and assumes no shocks after T. Thus, initial consumption is given by

$$c_0^{NK} = -e^{-\underline{\omega}T}(c_T^m + c_T^f),$$
$$= -\frac{\kappa}{\sigma(\overline{\omega} - \underline{\omega})}\left[\sigma^{-1}\omega_c \int_0^T \left(\frac{e^{-\overline{\omega}s}}{\overline{\omega}} - \frac{e^{-\underline{\omega}s}}{\underline{\omega}}\right)(i_s - \rho)ds + \int_0^T \left(e^{-\overline{\omega}s} - e^{-\underline{\omega}s}\right)(\omega_g g_s + \tau_s)ds\right]$$

and initial inflation is given by

$$\pi_0^{NK} = \sigma\underline{\omega}\left[\int_0^T e^{-\overline{\omega}s}\left[-\sigma^{-1}(i_s - \rho) + \frac{\kappa}{\sigma\underline{\omega}}(\omega_g g_s + \tau_s)ds\right] - c_0^{NK}\right].$$

Finally, the comparative statics for consumption are

$$\frac{\partial c_0^{NK}}{\partial i_s} = -\frac{\hat{\kappa}}{\sigma(\overline{\omega} - \underline{\omega})}\left(\frac{e^{-\overline{\omega}s}}{\overline{\omega}} - \frac{e^{-\underline{\omega}s}}{\underline{\omega}}\right) < 0,$$
$$\frac{\partial c_0^{NK}}{\partial g_s} = -\frac{\kappa\omega_g}{\sigma(\overline{\omega} - \underline{\omega})}\left(e^{-\overline{\omega}s} - e^{-\underline{\omega}s}\right) > 0,$$
$$\frac{\partial c_0^{NK}}{\partial \tau_s} = -\frac{\kappa}{\sigma(\overline{\omega} - \underline{\omega})}\left(e^{-\overline{\omega}s} - e^{-\underline{\omega}s}\right) > 0,$$

and for inflation

$$\frac{\partial \pi_0^{NK}}{\partial i_s} = \sigma\underline{\omega}\left(-\sigma^{-1}e^{-\overline{\omega}s} - \frac{\partial c_0^{NK}}{\partial i_s}\right) = \frac{\hat{\kappa}}{\overline{\omega} - \underline{\omega}}\left(e^{-\overline{\omega}s} - e^{-\underline{\omega}s}\right) < 0,$$
$$\frac{\partial \pi_0^{NK}}{\partial g_s} = \sigma\underline{\omega}\left(\frac{\kappa}{\sigma\underline{\omega}}\omega_g e^{-\overline{\omega}s} - \frac{\partial c_0^{NK}}{\partial g_s}\right) = \frac{\kappa\omega_g}{\overline{\omega} - \underline{\omega}}\left(\overline{\omega}e^{-\overline{\omega}s} - \underline{\omega}e^{-\underline{\omega}s}\right) > 0,$$
$$\frac{\partial \pi_0^{NK}}{\partial \tau_s} = \sigma\underline{\omega}\left(\frac{\kappa}{\sigma\underline{\omega}}e^{-\overline{\omega}s} - \frac{\partial c_0^{NK}}{\partial \tau_s}\right) = \frac{\kappa}{\overline{\omega} - \underline{\omega}}\left(\overline{\omega}e^{-\overline{\omega}s} - \underline{\omega}e^{-\underline{\omega}s}\right) > 0,,$$

■

Proof of Proposition 1. For the Forward Guidance Puzzle we have

$$\frac{\partial^2 c_0^{NK}}{\partial t \partial i_s} = -\frac{\hat{\kappa}}{\sigma(\overline{\omega}-\underline{\omega})}\left(-e^{-\overline{\omega}s}+e^{-\underline{\omega}s}\right) < 0,$$
$$\frac{\partial^2 \pi_0^{NK}}{\partial t \partial i_s} = \frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\left(-\overline{\omega}e^{-\overline{\omega}s}+\underline{\omega}e^{-\underline{\omega}s}\right) < 0,$$

and the limits are straightforward from Lemma 1. For the Paradox of Flexibility note that

$$\lim_{\kappa\to\infty}\hat{\kappa}=\infty, \quad \lim_{\kappa\to\infty}\overline{\omega}=\infty, \quad \lim_{\kappa\to\infty}\underline{\omega}=-\infty, \quad \lim_{\kappa\to\infty}\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}=\infty.$$

Then, the result is straightforward after some manipulation of the expressions. ■

Proof of Proposition 7. From the households' budget constraint we get that

$$c_0 = \int_0^\infty \left[\chi_{i,t}^c(i_t-\rho)+\chi_{g,t}^c g_t+\chi_{\tau,t}^c\hat{\tau}_t\right]dt+\chi_T^c T,$$

where $T \equiv \int_0^\infty e^{-\rho t}T_t dt$ is the present value of government transfers, and

$$\chi_{i,t}^c \equiv -\sigma^{-1}\frac{\overline{\tau}\varsigma_c-\sigma\rho\varsigma_d}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}e^{-\overline{\omega}t},$$
$$\chi_{g,t}^c \equiv \overline{\omega}\frac{\overline{\tau}\varsigma_c-\sigma\rho\varsigma_d}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}\frac{\omega_g}{\omega_c}(e^{-\rho t}-e^{-\overline{\omega}t})+\overline{\omega}\frac{1-\overline{\tau}}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}\varsigma_g e^{-\rho t},$$
$$\chi_{\tau,t}^c \equiv \overline{\omega}\frac{\overline{\tau}\varsigma_c-\sigma\rho\varsigma_d}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}\frac{1}{\omega_c}(e^{-\rho t}-e^{-\overline{\omega}t})-\overline{\omega}\frac{1-\overline{\tau}}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}e^{-\rho t},$$
$$\chi_T^c \equiv \frac{\overline{\omega}}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}.$$

On the other hand, we know that in the standard equilibrium selection, initial consumption is given by

$$c_0^{NK} = -\frac{\kappa}{\sigma(\overline{\omega}-\underline{\omega})}\left[\sigma^{-1}\omega_c\int_0^T\left(\frac{e^{-\overline{\omega}s}}{\overline{\omega}}-\frac{e^{-\underline{\omega}s}}{\underline{\omega}}\right)(i_s-\rho)ds+\int_0^T\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)(\omega_g g_s+\tau_s)ds\right].$$

Equalizing $c_0^{NK} = c_0$, we can isolate T and get

$$T^{NK} = \int_0^T \left[\chi_{i,t}^T (i_t - \rho) + \chi_{g,t}^T g_t + \chi_{\tau,t}^T \tau_t \right] dt,$$

where

$$\chi_{i,t}^T = -\frac{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}{\overline{\omega}} \left[\frac{\hat{\kappa}}{\sigma(\overline{\omega} - \underline{\omega})} \left(\frac{e^{-\overline{\omega}t}}{\overline{\omega}} - \frac{e^{-\underline{\omega}t}}{\underline{\omega}} \right) - \sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\overline{\tau}}\varsigma_c - \sigma\rho\varsigma_d} e^{-\overline{\omega}t} \right],$$
$$\chi_{g,t}^T = -\frac{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}{\overline{\omega}} \left[\frac{\kappa\omega_g}{\sigma(\overline{\omega} - \underline{\omega})} (e^{-\overline{\omega}t} - e^{-\underline{\omega}t}) + \overline{\omega} \frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\overline{\tau}}\varsigma_c - \sigma\underline{\omega}\varsigma_d} \frac{\omega_g}{\omega_c} (e^{-\rho t} - e^{-\overline{\omega}t}) + \overline{\omega} \frac{1 - \overline{\tau}}{\overline{\overline{\tau}}\varsigma_c - \sigma\underline{\omega}\varsigma_d} \varsigma\right.$$
$$\chi_{\tau,t}^T = -\frac{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}{\overline{\omega}} \left[\frac{\kappa}{\sigma(\overline{\omega} - \underline{\omega})} (e^{-\overline{\omega}t} - e^{-\underline{\omega}t}) + \overline{\omega} \frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\overline{\tau}}\varsigma_c - \sigma\underline{\omega}\varsigma_d} \frac{1}{\omega_c} (e^{-\rho t} - e^{-\overline{\omega}t}) - \overline{\omega} \frac{1 - \overline{\tau}}{\overline{\overline{\tau}}\varsigma_c - \sigma\underline{\omega}\varsigma_d} e\right.$$

Note that $\chi_{i,0}^T = -\frac{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}{\overline{\omega}} \left[\frac{\hat{\kappa}}{\sigma(\overline{\omega} - \underline{\omega})} \frac{\underline{\omega} - \overline{\omega}}{\overline{\omega}\underline{\omega}} - \sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\overline{\tau}}\varsigma_c - \sigma\rho\varsigma_d} e^{-\overline{\omega}t} \right]$ and since $\overline{\omega}\underline{\omega} = -\hat{\kappa}$, and $\frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\overline{\tau}}\varsigma_c - \sigma\rho\varsigma_d} > 1$, then $\chi_{i,0}^T < 0$. Moreover,

$$\frac{\partial \chi_{i,t}^T}{\partial t} < 0.$$

Hence

$$\frac{\partial T}{\partial i_t} < 0.$$

Taking the partial derivative with respect to time, we get

$$\frac{\partial^2 T}{\partial t \partial i_t} = -\frac{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}{\overline{\omega}} \left[\frac{\hat{\kappa}}{\sigma(\overline{\omega} - \underline{\omega})} \left(-e^{-\overline{\omega}t} + e^{-\underline{\omega}t} \right) + \overline{\omega}\sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}{\overline{\overline{\tau}}\varsigma_c - \sigma\rho\varsigma_d} e^{-\overline{\omega}t} \right] < 0.$$

It is straightforward to see that as $t \to \infty$, $\chi_{i,t}^T \to -\infty$. ■

Proof of Proposition ??. The effect of government spending on initial consumption is given by

$$\frac{\partial c_0^{NK}}{\partial g_t} = -\frac{\kappa\omega_g}{\sigma(\overline{\omega} - \underline{\omega})} (e^{-\overline{\omega}s} - e^{-\underline{\omega}s}) > 0,$$

$$\frac{\partial^2 c_0^{NK}}{\partial t \partial g_t} = -\frac{\kappa\omega_g}{\sigma(\overline{\omega} - \underline{\omega})} (-\overline{\omega}e^{-\overline{\omega}s} + \underline{\omega}e^{-\underline{\omega}s}) > 0,$$

$$\lim_{t\to\infty} \frac{\partial c_0^{NK}}{\partial g_t} = \lim_{t\to\infty} -\frac{\kappa\omega_g}{\sigma(\overline{\omega}-\underline{\omega})}(\underbrace{e^{-\overline{\omega}s}}_{\to 0} - \underbrace{e^{-\underline{\omega}s}}_{\to\infty}) = \infty$$

$$\lim_{\kappa\to\infty} \frac{\partial c_0^{NK}}{\partial g_t} = \lim_{\kappa\to\infty} -\underbrace{\frac{\kappa\omega_g}{\sigma(\overline{\omega}-\underline{\omega})}}_{\to\infty}(\underbrace{e^{-\overline{\omega}s}}_{\to 0} - \underbrace{e^{-\underline{\omega}s}}_{\to\infty}) = \infty,$$

and on inflation, by

$$\frac{\partial \pi_0^{NK}}{\partial g_t} = \frac{\kappa\omega_g}{\overline{\omega}-\underline{\omega}}\left(\overline{\omega}e^{-\overline{\omega}s} - \underline{\omega}e^{-\underline{\omega}s}\right) > 0,$$

$$\lim_{t\to\infty} \frac{\partial \pi_0^{NK}}{\partial g_t} = \lim_{t\to\infty} \frac{\kappa\omega_g}{\overline{\omega}-\underline{\omega}}\left(\underbrace{\overline{\omega}e^{-\overline{\omega}s}}_{\to 0} - \underbrace{\underline{\omega}e^{-\underline{\omega}s}}_{\to -\infty}\right) = \infty,$$

$$\lim_{\kappa\to\infty} \frac{\partial \pi_0^{NK}}{\partial g_t} = \lim_{\kappa\to\infty} \underbrace{\frac{\kappa\omega_g}{\overline{\omega}-\underline{\omega}}}_{\to\infty}\left(\underbrace{\overline{\omega}e^{-\overline{\omega}s}}_{\to 0} - \underbrace{\underline{\omega}e^{-\underline{\omega}s}}_{\to -\infty}\right) = \infty.$$

Since $\frac{\partial c_0^N K}{\partial g_t} = \omega_g \frac{\partial c_0^{NK}}{\partial \tau_t}$, all the results go through for τ_t as well. ■

Proof of Lemma ??. From the proof of Lemma 1 we know that any equilibrium of the New Keynesian model can be written as

$$c_t = e^{\underline{\omega}t}c_0 + c_t^m + c_t^f$$
$$\pi_t = e^{\underline{\omega}t}\pi_0 + \pi_t^m + \pi_t^f$$

for some $c_t^m, c_t^f, \pi_t^m, \pi_t^f$, functions of $\{i_t, g_t, \tau_t\}_{t=0}^{\infty}$,

$$\pi_0 = \sigma\underline{\omega}\left[\frac{\overline{\omega}-\underline{\omega}}{\sigma\hat{\kappa}}\int_0^\infty e^{-\overline{\omega}s}\eta_{1s}ds - c_0\right],$$

and

$$c_0 = \int_0^\infty \left[\chi_{i,t}^c(i_t-\rho) + \chi_{g,t}^c g_t + \chi_{\tau,t}^c\hat{\tau}_t\right]dt + \chi_T^c T,$$

for some $\chi_{i,t}^c, \chi_{g,t}^c, \chi_{\tau,t}^c, \chi_{T,t}^c$, independent of the shocks. Thus, we can index all equilibria of the New Keynesian model by the level of lump-sum transfers, T. By choosing T, the equations above completely characterize the equilibrium path of consumption and inflation. In particular, we have

$$\tilde{c}_0 \equiv \int_0^\infty \left[\chi^c_{i,t}(i_t - \rho) + \chi^c_{g,t} g_t + \chi^c_{\tau,t} \hat{\tau}_t\right] dt,$$

$$\tilde{\pi}_0 = \sigma \underline{\omega} \left[\frac{\overline{\omega} - \underline{\omega}}{\sigma \hat{\kappa}} \int_0^\infty e^{-\overline{\omega} s} \eta_{1s} ds - \tilde{c}_0\right],$$

and

$$\begin{aligned}\tilde{c}_t &= e^{\underline{\omega} t} \tilde{c}_0 + c^m_t + c^f_t \\ \tilde{\pi}_t &= e^{\underline{\omega} t} \tilde{\pi}_0 + \pi^m_t + \pi^f_t.\end{aligned}$$

It is straightforward to see that

$$c_t = \tilde{c}_t + \frac{\overline{\omega}}{\overline{\tau}\varsigma_c - \sigma \underline{\omega} \varsigma_d} e^{\underline{\omega} t} T,$$

and

$$\pi_t = \tilde{\pi}_t + \frac{\kappa \omega_c}{\overline{\tau}\varsigma_c - \sigma \underline{\omega} \varsigma_d} e^{\underline{\omega} t} T.$$

■

Proof of Proposition ??. We have

$$\frac{\partial \tilde{c}_0}{\partial i_t} = \chi^c_{i,t} = -\sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma \rho \varsigma_d}{\overline{\tau}\varsigma_c - \sigma \underline{\omega} \varsigma_d} e^{-\overline{\omega} t} < 0,$$

$$\begin{aligned}\frac{\partial^2 \tilde{c}_0}{\partial t \partial i_t} &= \sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma \rho \varsigma_d}{\overline{\tau}\varsigma_c - \sigma \underline{\omega} \varsigma_d} \overline{\omega} e^{-\overline{\omega} t} > 0, \\ \lim_{t\to\infty} \frac{\partial \tilde{c}_0}{\partial i_t} &= -\sigma^{-1} \frac{\overline{\tau}\varsigma_c - \sigma \rho \varsigma_d}{\overline{\tau}\varsigma_c - \sigma \underline{\omega} \varsigma_d} \lim_{t\to\infty} e^{-\overline{\omega} t} = 0, \\ \lim_{\kappa\to\infty} \frac{\partial \tilde{c}_0}{\partial i_t} &= -\lim_{\kappa\to\infty} \sigma^{-1} \underbrace{\frac{\overline{\tau}\varsigma_c - \sigma \rho \varsigma_d}{\overline{\tau}\varsigma_c - \sigma \underline{\omega} \varsigma_d}}_{\to 0} \underbrace{e^{-\overline{\omega} t}}_{\to 0} = 0.\end{aligned}$$

■

Proof Lemma ??. First, let's compute the substitution effect. The Hicksian demand of the non-linear model is obtained as the solution to the following problem

$$\min_{\{C_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\int_0^t (i_s - \pi_s) ds} C_t dt$$

$$st \quad \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} dt \geq \overline{U},$$

for some $\overline{U} \in \mathbb{R}$. The FOCs of this problem are given by

$$e^{-\int_0^t (i_s - \pi_s) ds} = \lambda e^{-\rho t} C_t^{-\sigma} \quad \forall t,$$

where λ is the Lagrange multiplier associated to the constraint. This implies that

$$C_t = e^{\frac{1}{\sigma} \int_0^t (i_s - \pi_s) ds} \lambda^{\frac{1}{\sigma}} e^{-\frac{\rho}{\sigma} t} \implies e^{-\rho t} C_t^{1-\sigma} = e^{-\frac{\rho}{\sigma} t} e^{-\frac{\sigma-1}{\sigma} \int_0^t (i_s - \pi_s) ds} \lambda^{\frac{1-\sigma}{\sigma}},$$

and hence

$$\lambda = \left[\frac{(1-\sigma)\overline{U}}{\int_0^{\infty} e^{-\frac{\rho}{\sigma} t} e^{-\frac{\sigma-1}{\sigma} \int_0^t (i_s - \pi_s) ds} dt} \right]^{\frac{\sigma}{1-\sigma}}.$$

Replacing in the FOC for C_t, we get

$$C_t = \frac{e^{-\frac{\rho}{\sigma} t} e^{\frac{1}{\sigma} \int_0^t (i_s - \pi_s) ds}}{\left[\int_0^{\infty} e^{-\frac{\rho}{\sigma} t} e^{-\frac{\sigma-1}{\sigma} \int_0^t (i_s - \pi_s) ds} dt \right]^{\frac{1}{1-\sigma}}} \left[(1-\sigma)\overline{U} \right]^{\frac{1}{1-\sigma}}.$$

Log-linearizing around the zero inflation steady state we get,

$$c_t^S = \frac{1}{\sigma} \int_0^t (i_s - \pi_s - \rho) ds - \frac{1}{\sigma} \int_0^{\infty} e^{-\rho t} (i_s - \pi_s - \rho) dt.$$

The present discounted value of the substitution effect is given by

$$\begin{aligned}
\int_0^\infty e^{-\rho t} c_t^S dt &= \frac{1}{\sigma}\int_0^\infty e^{-\rho t}\int_0^t (i_s-\pi_s-\rho) ds dt - \frac{1}{\sigma}\int_0^\infty e^{-\rho t}\int_0^\infty e^{-\rho s}(i_s-\pi_s-\rho)ds, \\
&= \frac{1}{\rho\sigma}\int_0^\infty e^{-\rho t}(i_s-\pi_s-\rho)ds - \frac{1}{\rho\sigma}\int_0^\infty e^{-\rho s}(i_s-\pi_s-\rho)ds, \\
&= 0.
\end{aligned}$$

Now, let's calculate the wealth effect. From the Euler equation we have

$$c_t = c_0 + \frac{1}{\sigma}\int_0^t e^{-\rho s}(i_s-\pi_s-\rho)ds,$$

hence

$$c_t^W = c_t - c_t^S = c_0 + \frac{1}{\sigma}\int_0^\infty e^{-\rho s}(i_s-\pi_s-\rho)ds,$$

which is constant over time. Let $C \equiv c_t^W$. Then net present value of consumption is given by

$$\int_0^\infty e^{-\rho t}\varsigma_c c_t dt = \int_0^\infty e^{-\rho t}\varsigma_c(c_t^S + C)dt = \frac{\varsigma_c}{\rho}C,$$

where the last equality follows from the fact that the present value of the substitution effect is zero. Introducing this in the budget constraint, we get

$$c_t^W = C = \frac{\rho}{\varsigma_c}\int_0^\infty e^{-\rho t}[(1-\overline{\tau})(y_t-\tau_t) + \varsigma_d(i_t-\pi_t-\rho)]dt.$$

■

Proof of Lemma ??. Immediate from the expression for c_t^S. ■

Proof of Proposition ??. The substitution effect is given by

$$c_t^S = \sigma^{-1}\int_0^t (i_s-\pi_s-\rho)ds - \sigma^{-1}\int_0^\infty e^{-\rho s}(i_s-\pi_s-\rho)ds.$$

Consider a monetary shock at date s. The substitution effect of consumption at dates $t < s$ is given by

$$-\sigma^{-1}e^{-\rho s} + \sigma^{-1}\int_0^\infty e^{-\rho z}\frac{\partial \pi_z}{\partial i_s}dz.$$

Moreover,

$$\frac{\partial \pi_z}{\partial i_s} = e^{\underline{\omega} z}\frac{\partial \pi_0}{\partial i_s} + \frac{\partial \pi_z^m}{\partial i_s},$$

where

$$\frac{\partial \pi_0}{\partial i_s} = -\underline{\omega}\frac{\sigma\overline{\omega}\varsigma_d}{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}e^{-\overline{\omega}s}$$

and

$$\frac{\partial \pi_z^m}{\partial i_s} = \begin{cases} \frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}(e^{\overline{\omega}z} - e^{\underline{\omega}z})e^{-\overline{\omega}s}, & \text{if } z < s \\ -\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}e^{\underline{\omega}z}(e^{-\overline{\omega}s} - e^{-\underline{\omega}s}), & \text{if } z > s \end{cases}$$

Putting all the pieces together, and after some algebra, we get

$$\left.\frac{\partial c_t^S}{\partial i_s}\right|_{t<s} = -\sigma^{-1}\frac{\overline{\tau}\varsigma_c}{\overline{\tau}\varsigma_c - \sigma\rho\varsigma_d}e^{-\overline{\omega}s} < 0.$$

It is straightforward that $\lim_{s\to\infty}\frac{\partial c_0^S}{\partial i_s} = 0$. Moreover,

$$\lim_{\kappa\to\infty}\frac{\partial c_0^S}{\partial i_s} = \left(\frac{1}{\omega_c} - \sigma^{-1}\right)e^{-\rho s}.$$

Now consider the substitution effect at dates $t > s$

$$\sigma^{-1}(1 - e^{-\rho s}) - \sigma^{-1}\int_0^t \frac{\partial \pi_z}{\partial i_s}dz + \sigma^{-1}\int_0^\infty e^{-\rho z}\frac{\partial \pi_z}{\partial i_s}dz$$

Consider the third term

$$\int_0^\infty e^{-\rho z}\left(e^{\underline{\omega}z}\frac{\partial \pi_0}{\partial i_s} + \frac{\partial \pi_z^m}{\partial i_s}\right)dz = \frac{1}{\overline{\omega}}\frac{\partial \pi_0}{\partial i_s} + \int_0^\infty e^{-\rho z}\frac{\partial \pi_z^m}{\partial i_s}dz$$

where

$$\frac{\partial \pi_0}{\partial i_s} = -\underline{\omega}\frac{\sigma\overline{\omega}\varsigma_d}{\overline{\tau}\varsigma_c - \sigma\underline{\omega}\varsigma_d}e^{-\overline{\omega}s}$$

$$\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\int_0^s e^{-\rho z}\left(e^{\overline{\omega}z}-e^{\underline{\omega}z}\right)e^{-\overline{\omega}s}dz=\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\left(-\frac{e^{-\rho s}}{\underline{\omega}}+\frac{e^{-\overline{\omega}s}}{\underline{\omega}}+\frac{e^{-2\overline{\omega}s}}{\overline{\omega}}-\frac{e^{-\overline{\omega}s}}{\overline{\omega}}\right)$$

$$-\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\int_s^\infty e^{-\rho z}e^{\underline{\omega}z}\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)dz=-\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\left(\frac{e^{-2\overline{\omega}s}}{\overline{\omega}}-\frac{e^{-\rho s}}{\overline{\omega}}\right)$$

$$\int_0^\infty e^{-\rho z}\frac{\partial \pi_z^m}{\partial i_s}dz=e^{-\rho s}-e^{-\overline{\omega}s}$$

Hence

$$\sigma^{-1}\int_0^\infty e^{-\rho z}\frac{\partial \pi_z}{\partial i_s}dz=-\sigma^{-1}\frac{\sigma\underline{\omega}\varsigma_d}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}e^{-\overline{\omega}s}+\sigma^{-1}\left(e^{-\rho s}-e^{-\overline{\omega}s}\right).$$

Now consider the first term

$$\int_0^t\left(e^{\underline{\omega}z}\frac{\partial \pi_0}{\partial i_s}+\frac{\partial \pi_z^m}{\partial i_s}\right)dz=\frac{e^{\underline{\omega}t}-1}{\underline{\omega}}\frac{\partial \pi_0}{\partial i_s}+\int_0^t\frac{\partial \pi_z^m}{\partial i_s}dz$$

where

$$\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\int_0^s\left(e^{\overline{\omega}z}-e^{\underline{\omega}z}\right)e^{-\overline{\omega}s}dz=\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\left(\frac{1-e^{-\overline{\omega}s}}{\overline{\omega}}-\frac{e^{(\underline{\omega}-\overline{\omega})s}-e^{-\overline{\omega}s}}{\underline{\omega}}\right)$$

$$-\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\int_s^t e^{\underline{\omega}z}\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)dz=-\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\frac{e^{\underline{\omega}t}\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)-e^{(\underline{\omega}-\overline{\omega})s}+1}{\underline{\omega}}$$

$$\int_0^t\frac{\partial \pi_z^m}{\partial i_s}dz=\left(1-e^{-\overline{\omega}s}\right)-\frac{\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\frac{e^{\underline{\omega}t}\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)}{\underline{\omega}}$$

Hence

$$-\sigma^{-1}\int_0^t\frac{\partial \pi_z^m}{\partial i_s}dz=-\sigma^{-1}\left(1-e^{-\overline{\omega}s}\right)+\frac{\sigma^{-1}\hat{\kappa}}{\overline{\omega}-\underline{\omega}}\frac{e^{\underline{\omega}t}\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)}{\underline{\omega}}$$

Therefore

$$\sigma^{-1}\frac{\sigma\overline{\omega}\varsigma_d}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}e^{-\overline{\omega}s}e^{\underline{\omega}t}-\sigma^{-1}\frac{\sigma\rho\varsigma_d}{\overline{\tau}\varsigma_c-\sigma\underline{\omega}\varsigma_d}e^{-\overline{\omega}s}-\sigma^{-1}\frac{\overline{\omega}}{\overline{\omega}-\underline{\omega}}e^{\underline{\omega}t}\left(e^{-\overline{\omega}s}-e^{-\underline{\omega}s}\right)$$

■

Proof of Proposition ??. Then

$$\rho \int_0^\infty e^{-\rho s}(1-\overline{\tau})(\tilde{y}_s - \tau_s)ds = \underbrace{\rho \int_0^\infty e^{-\rho s}(1-\overline{\tau})\tilde{c}_s ds}_{=(1-\overline{\tau})C} + \rho \int_0^\infty e^{-\rho s}(1-\overline{\tau})(g_s - \tau_s)ds$$

$$= (1-\overline{\tau})C + \rho \int_0^\infty e^{-\rho s}(1-\overline{\tau})(g_s - \tau_s)ds,$$

and

$$\rho\overline{b} \int_0^\infty e^{-\rho s}(i_s - \tilde{\pi}_s - \rho)ds = \rho\overline{b} \int_0^\infty e^{-\rho s}(i_s - \pi_t^* + \sigma\underline{\omega}\tilde{c}_t - \rho)ds$$

$$= \rho\overline{b} \int_0^\infty e^{-\rho s}(i_s - \pi_t^* - \rho)ds + \sigma\underline{\omega}\overline{b}C.$$

Introducing these two results into (**??**), we get

$$C = [1 - (\overline{\tau} - \sigma\underline{\omega}\overline{b})]C + A, \tag{15}$$

or

$$C = \frac{A}{\overline{\tau} - \sigma\underline{\omega}\overline{b}},$$

where $A \equiv \rho\overline{b} \int_0^\infty e^{-\rho s}(i_s - \pi_t^* - \rho)ds + \rho \int_0^\infty e^{-\rho s}(1-\overline{\tau})(g_s - \tau_s)ds$ is the autonomous component of consumption. ■

Proof of Lemma ??. Consider firm's j problem in a flexible price economy

$$\max_{P_t(j)} (1-\tau_t)P_t(j)Y_t(j) - W_t \left(\frac{Y_t(j)}{A_t}\right)^{\frac{1}{\alpha}}$$

$$s.t. \quad Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t.$$

Plugging in the demand in the objective function and taking firs order conditions gives

$$(\epsilon - 1)(1-\tau_t)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t = \frac{\epsilon}{\alpha}\frac{W_t}{P_t(j)}\left(\frac{P_t(j)}{P_t}\right)^{-\frac{\epsilon}{\alpha}}\left(\frac{Y_t}{A_t}\right)^{\frac{1}{\alpha}}.$$

Rearranging,

$$P_t = \frac{\epsilon}{\epsilon-1}\frac{1}{1-\tau_t}\frac{W_t}{\alpha}\frac{Y_t^{\frac{1-\alpha}{\alpha}}}{A_t^{\frac{1}{\alpha}}} = \frac{\epsilon}{\epsilon-1}\frac{1}{1-\tau_t}\frac{W_t}{\alpha}\frac{N_t^{1-\alpha}}{A_t} \implies w_t - p_t = -\tau_t - (1-\alpha)n_t.$$

Labor supply is given by

$$\sigma c_t + \phi n_t = w_t - p_t.$$

The resource constraint is given by

$$y_t = c_t + g_t.$$

The production function gives

$$y_t = \alpha n_t.$$

Combining labor demand and labor supply, we obtain

$$n_t = -\frac{\sigma c_t + \tau_t}{\phi + 1 - \alpha}.$$

Plugging the production function and the expression for labor into the resource constraint

$$-\frac{\alpha\sigma}{\phi+1-\alpha}c_t - \frac{\tau_t}{\phi+1-\alpha} = c_t + g_t \implies c_t = -\underbrace{\frac{\phi+1-\alpha}{\phi+1-\alpha+\alpha\sigma}}_{\omega_g} g_t - \underbrace{\frac{1}{\phi+1-\alpha+\alpha\sigma}}_{\omega_\tau} \tau_t.$$

Hence, the present discounted value of consumption is given by

$$C^{fp} = -\rho \int_0^\infty e^{-\rho t}[\omega_g g_t + \omega_\tau \tau_t]dt,$$

which gives us the derivatives $\frac{\partial C^{fp}}{\partial g_t} = -\rho e^{-\rho t}\omega_g$ and $\frac{\partial C^{fb}}{\partial \tau_t} = -\rho e^{-\rho t}\omega_\tau$.

Consider the limit involving T

$$\lim_{\kappa\to\infty} \frac{T}{\overline{\tau} - \underline{\omega}\sigma\varsigma_d} = \lim_{\kappa\to\infty} \frac{\sqrt{\kappa}}{\overline{\tau} - \underline{\omega}\sigma\varsigma_d} \frac{T}{\sqrt{\kappa}} = \lim_{\kappa\to\infty} \sqrt{-\sigma\frac{\overline{\omega}}{\underline{\omega}}} \frac{-\underline{\omega}}{\overline{\tau} - \underline{\omega}\sigma\varsigma_d} \frac{T}{\sqrt{\kappa}} \propto \lim_{\kappa\to\infty} \frac{T}{\sqrt{\kappa}},$$

since $\lim_{\kappa\to\infty} \sqrt{-\sigma\frac{\overline{\omega}}{\underline{\omega}}} \frac{-\underline{\omega}}{\overline{\tau} - \underline{\omega}\sigma\varsigma_d} = \frac{1}{\sqrt{\sigma\varsigma_d}}$. Hence, the wealth effect converges to its value in the flexible price equilibrium if and only if $\lim_{\kappa\to\infty} \frac{T}{\sqrt{\kappa}} = 0$. ∎

.1 Smets & Wouters (2007) Revisited

In order to keep the exposition short, we present the log-linearized version of the model, followed by a description of the differences that incorporating the fiscal variables introduce to the model.

The equilibrium of the economy is characterized by the following system of linear rational expectations equations:

- the aggregate resource constraint

$$y_t = c_y c_t + i_y i_t + g_y g_t + z_y z_t, \tag{16}$$

- the consumption Euler equation

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (l_t - E_t l_{t+1}) - c_3 (r_t - E_t \pi_{t+1}), \tag{17}$$

- the investment Euler equation

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t, \tag{18}$$

- the Tobin's Q

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) E_t r^k_{t+1} - (r_t - E_t \pi_{t+1}), \tag{19}$$

- the production function

$$y_t = \phi_p \left(\alpha k_t^s + (1-\alpha) l_t \right), \tag{20}$$

- the capital services equation

$$k_t^s = k_{t-1} + z_t, \tag{21}$$

- the capital utilization rate

$$z_t = z_1 r_t^k, \tag{22}$$

- the capital accumulation equation

$$k_t = k_1 k_{t-1} + (1-k_1) i_t, \tag{23}$$

- price mark-up

$$\mu_t^p = \alpha(k_t^s - l_t) - w_t, \tag{24}$$

- the NK Phillips curve

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \pi_3 (1-\tau^*) \tau_t, \tag{25}$$

- firms' cost minimization equation

$$r_t^k = -(k_t^s - l_t) + w_t, \tag{26}$$

- wage mark-up

$$\mu_t^w = w_t - \left[\sigma_l l_t + \frac{1}{1-\frac{h}{\gamma}} \left(c_t - \frac{h}{\gamma} c_{t-1} \right) \right], \tag{27}$$

- the aggregate wage index

$$w_t = w_1 w_{t-1} + (1 - w_1)(E_t w_{t+1} + E_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w, \tag{28}$$

- long-term bond price

$$q_t^L = \frac{\rho_L}{1 + \bar{r}} * q_{t+1}^L - r_t; \tag{29}$$

- the household's budget constraint

$$c_y c_t + i_y i_t + q^L b_y (b_t^L + q_t^L) = (1 - \tau^*)(y_t - \tau_t) - z_y z_t + \frac{\rho_L q^L b_y}{1 + \pi^*} q_t^L - \frac{(1 + \rho_L q^L) b_y}{1 + \pi^*} \pi_t + \frac{(1 + \rho_L q^L) b_y}{1 + \pi^*} b_{t-1}^L + T_t, \tag{30}$$

where y_t is output, c_t is consumption, i_t is investment, g_t is government spending, z_t is the capital utilization rate, l_t is hours worked, r_t is the nominal interest rate (set by the monetary authority), π_t is the inflation rate, q_t is the Tobin's Q, r_t^k is the rental rate of capital services, k_t^s is capital services, k_t is the stock of capital, w_t is the real wage, μ_t^p is the price mark-up, μ_t^w is the wage mark-up, τ_t is the proportional sales tax, b_t^L is government bonds, q_t^L is the price of the long-term bond, and T_t are government lump-sum transfers.