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The Analytic Theory of a Monetary Shock

by

Fernando Alvarez

(University of Chicago and NBER)

Francesco Lippi

(Luiss University and EIEF)

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Fernando Alvarez
University of Chicago and NBER

Francesco Lippi
LUISS University and EIEF

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Abstract

We propose a new method to analyze the propagation of a once and for all shock in a broad class of sticky price models. The method is based on the eigenvalue-eigenfunction representation of the cross-sectional process for price adjustments and provides a thorough characterization of the *entire* impulse response function of *any moment* or function of interest, in response to a once-and-for-all aggregate shock (*any* displacement of the initial distribution). We use the method (i) to discuss a general analytic characterization of the "selection effect" in sticky-price models, (ii) to show that the response of the cross-sectional dispersion of prices to a small shock is zero at all horizons, (iii) to derive a parsimonious representation of the output response to monetary shocks, and the key parameters determining its shape, (iv) to study the propagation of monetary shocks after a change in volatility.

JEL Classification Numbers: E3, E5

Key Words: Menu costs, Impulse response, Dominant Eigenvalue, Selection, Volatility.

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1 Introduction

Economists are often faced with the analysis of dynamic high dimensional objects, such as cross sectional distributions of incomes, assets, markups, and other economic variables of interest. This is the case for instance when studying impulse response functions, resulting from the dynamics of selected moments computed on the distribution of interest. We present a powerful method for such analyses, which typically require solving the partial differential equation that characterizes the evolution of the distribution of interest. The method is the eigenvalue-eigenfunction decomposition that allows to solve the partial differential equation through a neat separation of the time-dimension from the state-dimension, providing a tractable solution to a non-trivial problem. Two examples of use of eigenfunction-eigenvalues in economics are Gabaix, Lasry, Lions, and Moll (2016), who use it to study the transition of the cross-section distribution of incomes, and Hansen and Scheinkman (2009) who use it to define and study long run risk in asset pricing.[1]

We apply this method to analytically compute the entire impulse response function to a once and for all monetary shock in a broad class of sticky price models including versions of Taylor (1980), Calvo (1983), Golosov and Lucas (2007), a version of the "CalvoPlus" model by Nakamura and Steinsson (2010), a generalization of the Calvo plus model to arbitrary random menu costs as in Dotsey, King, and Wolman (1999), multi-product models as in Midrigan (2011), Bhattarai and Schoenle (2014) and Alvarez and Lippi (2014), and the model with "price-plans" as in Eichenbaum, Jaimovich, and Rebelo (2011) and Alvarez and Lippi (2019). In these models firms are hit by idiosyncratic shocks and face a price setting problem featuring (possibly random) menu costs, as well as "price plans" (i.e. the possibility of choosing 2 prices instead of a single one upon resetting). As in most of the general equilibrium literature on the topic we abstract from strategic complementarities to retain tractability. These problems are typically computationally intensive and numerical solutions

[1] See also Caballero (1993) for an early use eigenvalue-eigenfunctions to analyze the dynamics of a cross-section distribution and Krieger (2002) for an early exploration of the dynamics of a high-dimensional state in macroeconomic models using *eigenstates*.

may hinder a clear understanding of the mechanism at work. The approach we propose greatly facilitates the solution of such models, which in many cases has a simple-to-derive analytic form, while at the same time unveiling the key forces and deep parameters behind the results.

Our method delivers an analytic representation of the *whole profile* of the impulse response function, as opposed to previous analytic results on the impact effect of shocks, such as Caballero and Engel (2007), or analytic results on the cumulated impulse response to shocks such as Alvarez, Le Bihan, and Lippi (2016) and the extension developed by Baley and Blanco (2019a) for problems with asymmetries. The results also provide straightforward characterizations for several features of interest for a large class of sticky price models, such as duration analysis and the dynamics of *any* moment of interest after an aggregate shock. Moreover the analysis applies to *shocks of any size* as well as to *shocks to higher moments*, such as uncertainty shocks, differently from previous analytic investigations focusing mostly on approximations for small monetary shocks, as in e.g. Gertler and Leahy (2008); Alvarez, Le Bihan, and Lippi (2016); Baley and Blanco (2019b).

The paper is organized as follows: Section 2 presents the setup of the analysis, the mathematical definition of impulse response and the benchmark economy. Section 3 establishes some preliminary results that are useful to clarify the scope of the analytic results that follow. We show that while our main results are derived under a benchmark model with zero inflation and a symmetric return function, this benchmark provides an accurate description of economies where inflation is low and the firm's return function features a moderate asymmetry, i.e. we show a local insensitivity of the impulse response function to both drift and asymmetry. Section 4 presents our main result, namely the analytic representation of the impulse response function which neatly separates the effect of the shock (shape of the initial impulse), the effect of the shape of the function or moment of interest, and the effect of the time horizon. The rest of the paper illustrates the power of the method by discussing three substantive economic applications that we briefly describe next.

First we provide an analytic characterization of the "selection effect", which is one of the main reason why different sticky price models yield different real effects (Section 5). The selection effect, first discussed by Golosov and Lucas, refers to the fact that the prices that adjust following a monetary shock are those of a selected group of firms. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Calvo model. We present an analytic result showing how the selection effect creates a wedge between the duration of price spells and the duration of the aggregate output response. The two durations coincide when there is no selection. We show that such a wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of aggregate output. In the same section we present a surprising result showing that, following a small monetary shock, the response of even centered moments is zero at all horizons. This applies for instance to the variance of price gaps, to the cross-sectional price dispersion, to the kurtosis of the price changes, as well as to the survival function. Since the cross-sectional price dispersion is the main measure of inefficiency in models with price stickiness, this result implies that in these models such a cost should be measured in an "average" sense, and not as a consequence of a particular shock. The analytic approach clearly identifies the forces behind this surprising result.

Second, we analyze the shape of the impulse response function, and discuss the accuracy of an approximate characterization obtained by using "selected" eigenvalues (Section 6). The latter question arises in, for example, Gabaix et al. (2016), where a single eigenvalue, namely the "dominant eigenvalue", is used to provide a convenient description of the speed of the dynamic adjustment. It is thus interesting to ask whether a single eigenvalue can be found to approximate the impulse response function. We present several results. We show that an interesting special case exists: the canonical menu cost model can be effectively summarized by a single eigenvalue-eigenfunction pair. We explore the accuracy of approximations that use a single "leading" eigenvalue in the Calvo-plus model. We show that in general it is not possible to summarize the impulse response function, nor its asymptotic behavior, with a single eigenvalue-eigenfunction pair. As a concrete example of a case where no single eigenvalue can be used to characterize the IRF we discuss a class of sticky price models that gives rise to a hump-shaped impulse response function.[2] We provide analytic conditions for the hump-shaped impulse response to arise.

Third we study how the propagation of monetary shocks is affected by the aggregate shocks to the volatility faced by firms (Section 7). Bloom (2009) analyzed the macroeconomic impact of volatility shocks, and other scholars have shown that recessions are times in which the volatility of shocks is higher. We use our method to analyze whether monetary shocks propagate more quickly in times when volatility is high. The question relates to the recent quantitative investigation by Vavra (2014) and the empirical studies on volatility and the propagation of monetary shocks by Tenreyro and Thwaites (2016); Eickmeier, Metiu, and Prieto (2019). We show that the answer crucially depends on the time elapsed since the volatility shock occurred: we show that if the monetary shock and the volatility shock occur nearly simultaneously, a case that we label the "short run", then the cumulative effect of a monetary shock is, to first order, unaffected. Indeed this is the combination of two forces, faster propagation coming from higher volatility, but lower initial effect of the monetary shock because sS bands have become wider due to the option value effect. However, if the monetary shock occurs long enough after the increase in volatility, so that the cross-sectional distribution of firms has settled at the new invariant distribution, a case that we call the "long-run", then the result changes. In the long run the shock propagation is faster and the cumulative effect of a monetary shock is smaller by a factor proportional to the increase of volatility. This is because after a long period of high volatility, the cross sectional distribution settled to have a wider support consistent with the wider sS bands. Our method allows us

[2]This hump-shaped behavior emerges in economies where the price setting technology features "price plans" and some randomness in the cost of price changes (weak selection).

to quantify the time that it takes for the long-run effect to settle in. We show that this time is very short, i.e. a very small fraction of the typical length between price changes, a finding that suggests that higher volatility does led to "less powerful" monetary policy in most realistic cases. The result of the short high life if, at least for the pure menu cost model, independent of any parameter values.

Section 8 concludes by discussing developments and avenues for future work.

2 Set up

This section introduces the main objects of the analysis. First we set up a standard mathematical definition of the impulse response. Second, we present a benchmark sticky-price model that is used to illustrate several applications of interest.

The standard set up is made by the following objects: the law of motion of the Markov process $\{x(t)\}$ for each individual firm, the function of interest $f(x)$, the cross-sectional initial distribution of x, denoted by $P(x;0)$. At this general level the set-up and definition of an impulse response is closely related to the one in Borovicka, Hansen, and Scheinkman (2014). The law of motion for the process $f(x)$, with $x \in X \equiv [\underline{x}, \bar{x}]$, is also Markov and is described using

$$\mathcal{H}(f)(x,t) = \mathbb{E}\left[f\left(x\left(t\right)\right)|x(0) = x\right] \tag{1}$$

where the operator $\mathcal{H}$ computes the t period ahead expected value of the function $f : X \to \mathbb{R}$ conditional on the state $x = x(0)$. Next we describe the initial distribution of x, which we denote by $P(\cdot;0) : X \to \mathbb{R}$. This represents the measure of firms that start with value smaller or equal than x at time $t = 0$, each of them following the stochastic process described in $\mathcal{H}(f)$, with independent realizations. We allow the distribution $P(x;0)$ to have mass points. In particular P has a piecewise continuous derivative (density) which we extend to the entire domain, so that $p(\cdot,0) : [\underline{x}, \bar{x}] \to \mathbb{R}$, where P can have countably many jump discontinuities (mass points), denoting the difference between the right and left limits by

$p_m(\cdot;0) : \{x_k\}_{k=1}^{\infty} \to \mathbb{R}$, so that x_k is the location of the mass points.[3] While the possibility of handling a distribution with mass point is of theoretical interest, most of the economic applications that we discuss have no mass points and hence the density p will suffice. Summarizing, we assume that the "shocks" are idiosyncratic, and that the initial condition is given by a cross sectional distribution $P(\cdot,0)$.

We are interested in the standard impulse response function H defined for each $t>0$ as:

$$H\left(t; f, P-\bar{P}\right) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,t)\ \left[dP(x;0) - d\bar{P}(x)\right] \tag{2}$$

where $\bar{P}$ is the invariant distribution of x. The impulse response H is the expected value of the cross-sectional distribution of f *in deviation from its steady state value*, where each $x(t)$ has followed the Markov process associated with $\mathcal{H}(\cdot)$ and whose cross sectional distribution at time zero is given by $P(\cdot;0)$.[4] In other words, for ergodic processes we are forcing the impulse response to go to zero as t diverges. Since we evaluate H only for the difference between two measures, i.e. only for signed measures, we introduce the convenient notation $\hat{P} \equiv P - \bar{P}$ and likewise for the densities $\hat{p} = p - \bar{p}$. Thus

$$\hat{P}(x) \equiv P(x,0) - \bar{P}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{3}$$

so that $\hat{P}$ defines the "initial condition" for the impulse response, an object that we further discuss below.

We note that the impulse response can also be written in terms of the evolution of the cross sectional distribution $P(\cdot,t)$, namely: $H\left(t; f, P-\bar{P}\right) = \int_{\underline{x}}^{\bar{x}} f(x)\ \left[dP(x;t) - d\bar{P}(x)\right]$. The two definitions are equivalent. To characterize equation (2) requires solving a Kol-

[3]Given our assumption on P we can write the expectations of any function $\nu(x)$ as:

$$\int_{\underline{x}}^{\bar{x}} \nu(x)\, dP(x;0) = \int_{\underline{x}}^{\bar{x}} \nu(x)\, p(x;0) dx + \sum_{k=1}^{\infty} \nu(x_k)\, p_m(x_k;0)\,.$$

[4]The ergodicity of $\{x\}$ implies that we can write $H\left(t; f, P-\bar{P}\right) = \int_{\underline{x}}^{\bar{x}} \mathcal{H}(f)(x,t)\, dP(x;0) - \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x)$.

mogorov backward (KB) equation, while the latter requires solving a Kolmogorov forward (KF) equation. In spite of the equivalence, there are two reasons why equation (2) is sometimes preferred. The first reason depends on how well behaved are $\hat{P}$ vs f. If we start with a distribution with mass points, as will be the case after a large shock, then the initial condition will not be a density. The dynamics of the distribution will then involve Dirac functions. Instead, using the KB we do not need to deal with this. Notice the asymmetry: f are functions, where $\hat{P}$ is a generalized function, i.e. can include Dirac functions. Second, even if we do not deal with generalized functions, so that both f and $\hat{P}$ are functions, the boundary conditions can be trickier in the KF than in the KB. The boundary conditions for the KF are stated in terms of right and left derivatives, and indeed the corresponding eigenfunctions are not differentiable at the reinjection point. While using the KB the boundary conditions are standard -just as the ones in a Bellman equation- and the eigenfunctions are differentiable at the reinjection point.

We define another impulse response function, which uses a stopping time τ, and a modified expectation operator $\mathcal{G}$ defined as:

$$\mathcal{G}(f)(x,t) = \mathbb{E}\left[1_{\{t\leq\tau\}} f\left(x\left(t\right)\right)|x(0) = x\right] \tag{4}$$

The operator $\mathcal{G}$ computes the t period ahead expected value of the function $f : X \to \mathbb{R}$ starting from the value of the state $x = x(0)$, conditional on x surviving. The indicator function $1_{\{t\leq\tau\}}$ becomes zero when the first adjustment following the shock occurs at the stopping time τ.

In the context of the price setting models with sS rules we refer to the operator $\mathcal{H}$ as the one for the problem with "reinjection", i.e. one in which the operator follows a firm forever, i.e. does not stop keeping track of the firm after the first adjustment occurs (at time τ). In contrast, we refer to the operator $\mathcal{G}$ as one for the problem without "reinjection", i.e. tracking the firm until the first adjustment. For example, in the sticky price models discussed below the stopping time τ will be given by the occurrence of a price adjustment.

We define the impulse response function G for each $t > 0$ as:

$$G(t; f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} \mathcal{G}(f)(x,t) \ \left[dP(x;0) - d\bar{P}(x)\right] \tag{5}$$

The interpretation of $G(t)$ is the expected value of the cross-sectional distribution of f, conditional on surviving, where each $x(t)$ follows the Markov process, and as before the cross-sectional distribution at time zero is given by $P(\cdot;0)$ so that the initial condition is $\hat{P} \equiv P(\cdot;0) - \bar{P}$.

While H is the impulse response as commonly defined, it turns out that G is simpler to characterize. Moreover, the next section will discuss two useful results: Proposition 1 establishes conditions under which the impulse response $G(t; f, \hat{P})$ coincides with $H(t; f, \hat{P})$ for all t. Proposition 3 shows that under even weaker conditions the cumulative impulse response *of any moment of interest* computed on $G(t; f, \hat{P})$ coincides with the cumulative response $H(t; f, \hat{P})$.

An equivalent, and perhaps simpler, representation of the impulse response function $G(t)$ is obtained by using the transition function $Q_t(y|x) = \Pr\{x(t) < y, t < \tau \big| x(0) = x\}$, with density function $q_t(y|x) = \partial_y Q_t(y|x)$. The impulse response function is

$$G(t; f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} \int_{\underline{x}}^{\bar{x}} f(y)\, q_t(y|x)\, dy\, d\hat{P}(x) \tag{6}$$

Our analytical characterization of $G(t)$, which will be given in Proposition 4, can be easily understood by using a finite-dimensional version of equation (6), as we will discuss after Corollary 1.

In general our interest is to compute $H(t; f, \hat{P})$ by using the simpler operator $G(t; f, \hat{P})$ which takes as an argument the *signed* measure $\hat{P}$. Nevertheless, we will also use $G(t; f, \hat{P})$ to construct other objects of interest such as the survival function of price changes or the conditional distribution of the state q_t. In these cases we abuse notation and let $\hat{P}$ denote a probability measure.

The initial condition. Our setup encodes the impulse in the initial condition, $\hat{P}(x) \equiv P(x,0) - \bar{P}(x)$, which denotes the distribution of the state variable $x \in (\underline{x}, \bar{x})$ at time zero in deviation from the invariant distribution. In particular $P(\cdot;0)$ describes the cross-sectional distribution of the state immediately after the shock. As time elapses the initial distribution will converge to the invariant distribution $\bar{P}(x)$, tracing out the impulse response for the function of interest $f(x)$. We mentioned above that our method allows the initial distribution to have mass points. This can be useful, for instance, if the initial shock is large enough to displace a non-negligible mass of agents onto the return point x^*, to compute the survival function of price changes, or to compute the conditional distribution of the state q_t. Also notice that our formulation allows us to handle a variety of shocks. Several papers have focussed on a small uniform displacement $\delta > 0$ of the whole distribution relative to the invariant $\bar{P}(x)$, what Borovicka, Hansen, and Scheinkman (2014) label the "marginal response function", a case that has been studied in the literature, as discussed in the introduction, which we formalize next. We stress, however, that our method can handle any type of initial condition, such as one triggered by a large shock, or the one triggered by a higher-order shock.

Marginal impulse response function. Starting from the invariant distribution $\bar{p}(x)$ and considering a small uniform displacement $\delta > 0$ of the whole distribution so that $\hat{p}(x) = \bar{p}(x+\delta) - \bar{p}(x) = \bar{p}'(x)\delta + o(\delta)$ we have the following definition

Definition 1. Let the marginal impulse response function to a monetary shock be:

$$Y(t;f) \equiv \frac{\partial}{\partial \delta} H\left(t; f, \bar{p}(x+\delta) - \bar{p}(x)\right)\Big|_{\delta=0} \text{ for all } t \geq 0 \tag{7}$$

The marginal impulse response function is the first order expansion of $H\left(t; f, \bar{p}(x+\delta) - \bar{p}(x)\right)$ with respect to δ. For future reference notice that since $\bar{p}(x+\delta) - \bar{p}(x) = \delta \bar{p}'(x) + o(\delta)$ we will often use $H(t; f, \bar{p}'(x)\delta)$ instead of $H(t; f, \hat{p}(x))$. Since $H(t; f, \mathbf{0}) = 0$, where $\mathbf{0}$ denotes the zero function, thus the expansion gives:

$$H(t; f, \bar{p}(x+\delta) - \bar{p}(x)) = Y(t; f)\delta + o(\delta)$$

where the marginal impulse response $Y(t)$ is measured per unit of the monetary shock.

2.1 The firm's price setting problem

This section lays out the price setting problem solved by a firm in the "Calvo plus" model. In this model, which can be seen as one with random menu costs, the firm is allowed to change prices either by paying a fixed menu cost or upon receiving a random free adjustment opportunity (a menu cost equal to zero). The setup nests several models of interest, from the canonical menu cost problem to the pure Calvo model.

The firm sS problem in the Calvo plus model. We describe the price setting problem for a firm in steady state. The firm cost follows a Brownian motion with variance σ^2 and drift μ, where the latter is due to inflation. The firm can change its price at any time paying a fixed cost $\psi > 0$. At exogenous random times, which occur with a constant probability per unit of time ζ, the firm faces a zero menu cost. The price gap x is defined as the price currently charged by the firm relative to the price that will maximize current profits, which is proportional to the firm cost (measured as the log of the ratio between these prices). The optimal policy is to change the price when the gap x reaches either of two barriers, $\underline{x} < \bar{x}$, or when the menu cost is zero. In either case, at the time of a price change, the firm sets a new price which determines a price gap x^*, which is the optimal return point after the adjustment. Thus price changes are given by $x^* - x(\tau)$, where τ is the time at which either one of the barriers is hit, or a free adjustment opportunity arrived.

The flow cost of the firm is given by $R(x)$. The firm maximizes expected discounted profits, using a discount rate $r > 0$. Thus, the optimal policy can be describe by three numbers: $\underline{x} < x^* < \bar{x}$, whose values can be found by minimizing the value function in the inaction region:

$$(r+\zeta)v(x) = R(x) + \mu v'(x) + \frac{\sigma^2}{2}v''(x) + \zeta v(x^*) \text{ for } x \in [\underline{x}, \bar{x}] \tag{8}$$

and imposing the relevant boundary conditions: value matching, smooth pasting, and optimality of the return point:

$$v(\bar{x}) = v(\underline{x}) = v(x^*) + \psi, \text{ and } v'(\underline{x}) = v'(\bar{x}) = v'(x^*) = 0 \quad . \tag{9}$$

The density $\bar{p}$ of the invariant distribution for price gaps generated by the policy $\{\underline{x}, x^*, \bar{x}\}$ and the law of motion of x is the solution to the Kolmogorov forward equation:

$$\zeta \bar{p}(x) = -\mu \bar{p}'(x) + \frac{\sigma^2}{2}\bar{p}''(x) \text{ for } x \in [\underline{x}, \bar{x}], x \neq x^* \tag{10}$$

with boundary conditions at the exit points, unit mass, and continuity requirements:

$$\bar{p}(\underline{x}) = \bar{p}(\bar{x}) = 0, \int_{\underline{x}}^{\bar{x}} \bar{p}(x)dx = 1, \text{ and } \bar{p} \text{ continuous at } x = x^* .$$

The boundary conditions at the exit points are immediate in sS models with fixed costs since no mass can accumulate at the boundary of the inaction region as long as $\sigma > 0$.

3 Scope of the analysis

This section presents three results that are useful to frame the scope of the main results of this paper. We start by defining the notion of a *symmetric sS problem.* Informally, this amounts to assuming that the firm problem has zero drift and that the firm's return function is symmetric. The first result shows that, for symmetric sS problems, $H(t) = G(t)$ for all horizons $t \geq 0$. This is useful because while H is the object of interest, the function G, the simple IRF computed stopping after the first adjustment, turns out to be much easier to characterize. The second result shows that the impulse response for symmetric sS problems provides a good approximation to the impulse response for non-symmetric problems. In particular, when the lack of symmetry comes from either the presence of a drift or from an asymmetric objective function, we show that the deviations in the marginal impulse response are of second order in the strength of asymmetries. The third result shows that the *cumulative* impulse response H coincides with the *cumulative* impulse responde G, even if the problem is not symmetric and/or shocks are not marginal.

DEFINITION 2. Symmetric sS Calvo plus problem. Assume that the unregulated state has zero drift, i.e. $\mu = 0$, and that the optimal return point x^* is equidistant from the barriers, i.e. $(\bar{x} + \underline{x})/2 = x^*$. Let $\{x(t)\}$ be the regulated state, $g(x; t, x^*)$ be the density of distribution of $x(t) = x$, conditional on $x(0) = x^*$. In this case the density satisfies $g(z + x^*; t, x^*) = g(-z + x^*; t, x^*)$ for all $z \geq 0$.

One can show that the *optimal* decision rule for the problem in equation (8) have barriers that are equidistant from x^* if the flow return $R(x)$ is symmetric in x and if $\mu = 0$. The symmetry on $g(\cdot; t, x^*)$ follows from the combination of the symmetry of the distribution of a BM without drift, and the symmetry of the boundaries relative to x^*.

While we have chosen the Calvo plus problem as our benchmark case to preserve a simple, yet economically interesting, framework, we know that most of our results apply to a more general class of sS problems featuring analogous symmetric properties. This can be seen by noticing that for the results of the next two subsections, symmetry can be defined purely in term of the conditions on the density g. Examples of more general sS problems where all the results presented in this section will apply are the price-plan model developed in Alvarez and Lippi (2019) and briefly described in Section 6, or the multiproduct price-setting problem developed in Alvarez and Lippi (2014) and described in Appendix B, which is a generalization of the Calvo-plus model that considers a firm controlling a vector of n prices instead of a single one. In Appendix C we consider another extension of the benchmark model that allows for a non-degenerate distribution of random fixed costs, which generalizes the seminal analysis of Dotsey, King, and Wolman (1999).

3.1 Equivalence of IRF's H and G for symmetric sS problems

Next we present a proposition for symmetric problems establishing conditions for the standard IRF $H(t)$ to coincide with $G(t)$. The conditions for this to happen are that either the function of interest f, or that the initial condition $\hat{P}(\cdot)$ are antisymmetric, where a function $\nu(x)$ is antisymmetric about x^* if it satisfies $\nu(x - x^*) = -\nu(x^* - x)$ for all $x \in [\underline{x}, \bar{x}]$.

Proposition 1. Consider, for simplicity, the Calvo plus model. Assume the firm's problem is sS symmetric as in Definition 2. Then if either

(i) the function of interest $f : [\underline{x}, \bar{x}] \to \mathbb{R}$ is antisymmetric and $\hat{P}(\cdot)$ is arbitrary

(ii) the signed measure (initial condition) $p(\cdot, 0) - \bar{p}(\cdot) : [\underline{x}, \bar{x}] \to \mathbb{R}$ as well as its mass points $p_m(\cdot, 0) - \bar{p}_m(\cdot) : \{x_k\}_{k=1}^{\infty} \to \mathbb{R}$ are antisymmetric and $f(\cdot)$ is arbitrary

we have that $G(t; f, \hat{P}) = H\left(t; f, \hat{P}\right)$ for all t.

See Appendix A for the proof. The proposition's requirement that either the function of interest f, or the initial condition $\hat{P}$, is anti-symmetric is not that restrictive for our applications. The main function of interest for the paper, used to compute the IRF for output, for example $f(x) = -x$ in the Calvo-plus model, is anti-symmetric in the class of models we analyze. Also, our benchmark case in this class of models is that the density $\hat{P}$ is anti-symmetric when we consider a small monetary shock..

3.2 Local insensitivity to asymmetries of output's IRF

In this section we show that the output's impulse response to a monetary is locally insensitive to two forms of deviations from the assumptions defining a symmetric sS problem. The first concerns deviations from the zero drift, or zero inflation, assumption. The second concerns the deviation from the symmetry of the firm's objective function. The insensitivity applies to the impulse response of any antisymmetric function f, not just the one for output. This gives the precise sense in which our result for symmetric sS problems extend to a range of small inflation rates.

We let μ be the steady state inflation. We also let a be a coefficient that measures the degree to which the firms's period return function is asymmetric. In particular, we let $R(x, a)$ be the return function which satisfies

$$R(x, a) = R(-x, -a) \text{ for all } x, a \tag{11}$$

Note that $a = 0$ implies that the return function is symmetric. For $a \neq 0$, the function can be interpreted as the sum of a symmetric and antisymmetric function. An example of this is

$$R(x, a) = Bx^2 + ax^3$$

which can represent a third order expansion of the original profit function around the static maximizing choice $x = 0$. In this case a is 6 times the third derivative of the original profit function at $x = 0$. When either $a \neq 0$ or $\mu \neq 0$ the thresholds of the optimal sS rule will not be equidistant from the optimal return point x^*.

The next proposition computes two derivatives of the output's IRF to a small monetary shock at any horizon t and shows that they are both zero. One is the derivative with respect to steady state inflation, and the other one is the derivative with respect to the degree of asymmetry of the objective function. Indeed we prove a more general result, that the IRF of any antisymmetric function f, instead of just $f(x) = -x$, to a monetary shock is locally insensitive to inflation or asymmetries in the objective function.

For this proposition we need to consider the general case with re-injection, since we don't have the required conditions of Proposition 1 that ensure that $H = G$. Also note that the steady state distribution $\bar{p}$ will be, in general, also a function of the drift and the degree of asymmetry (μ, a), and hence we include them as argument on the initial impulse $\hat{p}(x; \delta, \mu, a)$, as well as the function H itself, since this function depends on the process for x, and in particular the value of its drift, optimal return point x^*, and thresholds $\underline{x}, \bar{x}$. We also explicitly include the shock δ as an argument of $\hat{p}$, which will clarify the arguments below. The initial condition for a monetary shock are thus $\hat{p}(x; \delta, \mu, a) \equiv \bar{p}(x + \delta; \mu, a) - \bar{p}(x; \mu, a)$. Thus the marginal impulse response is:

$$Y_{\mu,a}(t; f) \equiv \frac{\partial}{\partial \delta} H\left(t; f, \hat{p}(\cdot, \delta, \mu, a), \mu, a\right)\Big|_{\delta=0} \text{ for all } t \geq 0 \tag{12}$$

The next proposition states that impulse response $Y_{\mu,a}(t)$ at any horizon t is approximately the same as our benchmark case with zero inflation and with symmetric return function. Formally we have:

Proposition 2. Consider, for simplicity, the Calvo plus model. Let μ be the drift of the state (the inflation rate), and a the index of asymmetry in the return function $R(x, a)$. Then:

$$\frac{\partial}{\partial \mu} Y_{\mu,a}(t; f)\Big|_{\mu=0,a=0} = 0 \text{ for all } t \geq 0 \tag{13}$$

$$\frac{\partial}{\partial a} Y_{\mu,a}(t; f)\Big|_{\mu=0,a=0} = 0 \text{ for all } t \geq 0 \tag{14}$$

for any antisymmetric function f.

See Appendix A for the proof. Notice that the proof holds for any function of interest f that is antisymmetric. So this holds true for the output impulse response, given by $f(x) = -x$, as well as for other antisymmetric functions.

While Proposition 2 shows the insensitivity of deviations to inflation and asymmetries in the objective function for infinitesimal changes, combining the results in Appendix D and Appendix E we can analyze the effect on the impulse response for deviations of any size. We relegate these results to appendices because characterizing the entire impulse response function requires extending the techniques used for the baseline case. From the results on these appendices we conclude that for moderate levels of inflation and asymmetries in the objective function, the results are materially the same.

3.3 Equivalence of Cumulative IRF's for H and G.

We use the impulse response function $H(t; f, \hat{P})$ to define the discounted cumulative response, given by the (discounted) area under the impulse response function:

$$\mathbf{H}(r; f, \hat{P}) = \int_0^\infty e^{-rt} H(t; f, \hat{P})\, dt\,. \tag{15}$$

Although the cumulative response is not informative about *the shape* of the impulse response function, which is the main focus of this paper, its characterization gives an informative (scalar!) summary of the economy's response to a permanent shock.

We have introduced the notion of cumulative impulse response for output following a monetary shock in Alvarez, Le Bihan, and Lippi (2016). We have shown that $\mathbf{H}$ can be analytically computed by solving an ordinary differential equation that gives the cumulated impulse response for an agent with initial condition $x \in (\underline{x}, \bar{x})$.[5] Integrating across the different values of x after the shock, using the displaced distribution $\hat{P}$, provides an analytic solution for $\mathbf{H}$. The approach is extremely useful since it does not require to solve (or simulate) the whole impulse response function. Our initial use of the cumulative response was restricted to problems with zero drift (inflation). In Alvarez and Lippi (2019) we have extended the result to the cumulated output response for problems featuring non-zero inflation. In this section we extend our original proposition using a result from Baley and Blanco (2019b), to compute the cumulative response $\mathbf{H}$ for *any function of interest* f. The payoff is to go beyond the analysis of the output effect typically obtained by using $f(x) = -x$. For instance we can now analyze the cumulative effect of the dispersion of markups, $f(x) = x^2$, in economies with inflation or other sources of drift.[6]

[5] The logic behind this differential equation is analogous to the one used to compute expected values using a Bellman (or Kolmogorov) equation.

[6] An earlier draft of this paper restricted the result to $f(x) = -x$, i.e. to the impulse response of output.

Next let us define the cumulative effect as the area under the impulse response $G(t)$, namely the one for the problem without reinjections defined in equation (5):

$$\mathbf{G}(r; f, \hat{P}) = \int_0^\infty e^{-rt}\, G(t; f, \hat{P})\, dt\,. \tag{16}$$

We now discuss a useful result concerning the undiscounted (i.e. $r = 0$) cumulated impulse response function defined in equation (15) and equation (16). The result states that for any moment of interest $f(x)$, and for any parametrization of the Calvo-plus model described in the previous section, the cumulated impulse response for the problem without reinjection coincides with the cumulated impulse response for the problem with reinjection. We state the result as follows (see Appendix A for the proof):

PROPOSITION 3. Consider the Calvo plus problem described above with a rate ζ of free adjustment opportunities, decision rule given by $\{\underline{x}, x^*, \bar{x}\}$, and the law of motion of the state x given by parameters $\{\mu, \sigma\}$. Let the economy be subject to an initial shock $\hat{P} \equiv P(x,0) - \bar{P}$ (displacement of the initial distribution) and the function of interest be f. Let $m(x; f)$ be the expected cumulated value of $f(x)$ in deviation from its mean $\bar{f}$, computed between time zero, where $x(0) = x$, and the first time of adjustment τ:

$$m(x; f) = \mathbb{E}\left(\int_0^\tau \left(f(x(t)) - \bar{f} \right) dt \;\middle|\; x(0) = x \right) \quad \text{where} \quad \bar{f} \equiv \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x)$$

We have

$$\mathbf{H}(0; f, \hat{P}) = \mathbf{G}(0; f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} m(x; f) dP(0, x) \quad .$$

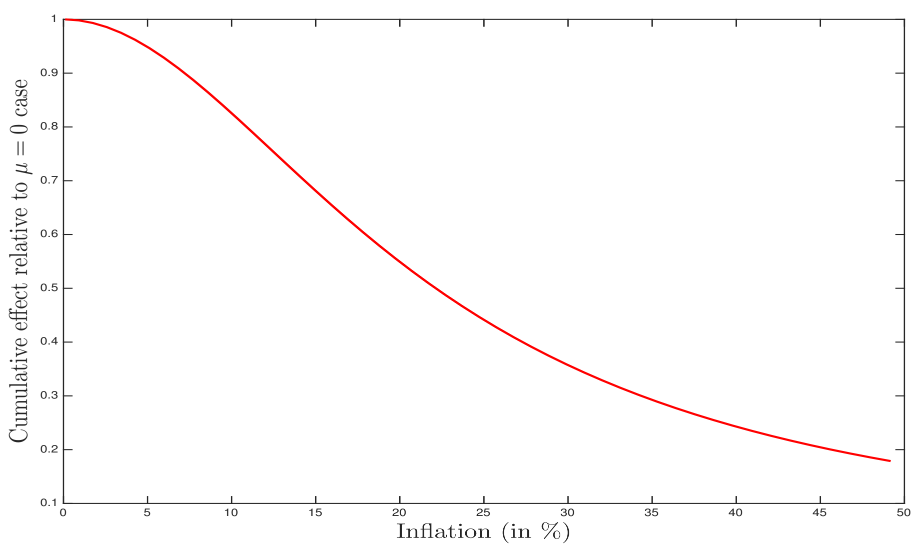
The result is surprising to us. The decision rule $\{\underline{x}, x^*, \bar{x}\}$ need not be the optimal one, for instance it could fail to satisfy value matching and/or smooth pasting. The problem may feature asymmetric sS bands, and the state may have drift $\mu \neq 0$. Yet we can compute

We are indebted with Isaac Baley for pointing out Proposition 2 in Baley and Blanco (2019b).

the cumulative IRF as in the symmetric case with no drift.[7] We make several comments to this result. First, notice that the proposition holds for an arbitrary function f. Second, we stress that the proposition holds for any initial condition $\hat{P}$. This allows us to study either small shocks, such as a marginal displacement of the invariant distribution $\bar{P}$, as well as large shocks that give rise to any type of initial signed mass $\hat{P} = P(x, 0) - \bar{P}$.

Proposition 3 yields a straightforward analysis of an otherwise computationally intensive question. The last equality states that we can obtain the cumulative IRF by simply keeping track of each firm until the time it makes the first adjustment following the shock. This is convenient both for analytical computation as well as for simulations. If one is interested in analytic solutions, the function $m(x; f)$ is readily computed by solving the differential equation $0 = f(x) + m'(x; f)\mu + m''(x; f)\sigma^2/2 - \zeta m(x; f)$, where primes denote derivatives with respect to x, and with boundaries $m(\underline{x}; f) = m(\bar{x}; f) = 0$, as in Alvarez, Le Bihan, and Lippi (2016).

Figure 1: Cumulative output response at different inflation rates



Note: cumulated output response triggered by a marginal monetary shock, at different inflation rates, computed using Proposition 3 with a return function $f(x) = -x$. The cumulated output, measured on the vertical axis, is normalized relative to the one at zero inflation rate ($\mu = 0$).

[7] An identical result holds for the model with price plans discussed in Section 6.

Figure 1 illustrates this point by exploring how the the cumulated output response changes with the inflation rate. For simplicity we focus on a marginal shock by postulating that the distribution right after the shock is equal to a small displacement of the invariant, namely that $p(x, 0) = \bar{p}(x + \delta)$, so that $\hat{p}(x) = p(x, 0) - \bar{p}(x) \approx \bar{p}'(x)\delta$, where δ is a (small) aggregate monetary shock. The figure shows that the cumulated output effect is not responsive to inflation when inflation is low (it is easy to prove that the function has a zero derivative at $\pi = 0$). As inflation increases however the real (cumulated) effect of policy vanishes fast. At an inflation rate equal to 50% per year the effect is about 1/5th of the effect at low inflation.

4 Analytic Impulse Response Functions

This section presents an analytic solution for the operator $\mathcal{G}(f)(x, t)$ defined in equation (4). The main assumption of this section is that the problem is symmetric as defined in Definition 2. By Proposition 1 we then know that the impulse response is equivalent to the one of a problem without reinjections and is given by $G(t, f, \hat{P})$. As argued in Section 3 the symmetric setup is a convenient starting point for the analysis and provides an accurate approximation to problems with small drift or small asymmetries, assumptions that are widely used in sticky-price models.

We assume the process for the firm's price gap is given by a drift-less brownian motion, with instantaneous variance per unit of time σ^2. The stopping time, i.e. the time at which prices are changed, is given by the first time at which $x(t)$ hits either $\underline{x}$ or $\bar{x}$, or that a Poisson counter, with instantaneous rate ζ, changes its value. The definition of $\mathcal{G}(f)(x, t)$ as an expected value implies that this function must satisfy the following partial differential equation:

$$\partial_t \mathcal{G}\left(f\right)(x,t) = \frac{\sigma^2}{2}\partial_{xx}\mathcal{G}(f)(x,t) - \zeta\mathcal{G}(f)(x,t) \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and } t > 0 \tag{17}$$

with boundary conditions

$$\mathcal{G}\left(f\right)(\underline{x}, t) = \mathcal{G}\left(f\right)(\bar{x}, t) = 0 \text{ for all } t > 0 \text{ and} \tag{18}$$

$$\mathcal{G}\left(f\right)(x, 0) = f(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{19}$$

where boundary conditions for $t > 0$ are an implication from $\bar{x}$ and $\underline{x}$ being exit points, and hence close to them the survival rate tends to zero. The boundary condition at $t = 0$ follows directly from the definition of $\mathcal{G}(f)$.

Two key steps, based on the properties of eigenvalues and eigenfunctions proved below, allow us to solve for $\mathcal{G}\left(f\right)$. First, that the function f can be represented by a linear combination of eigenfunctions, with typical member φ_j, as follows $f(x) = \sum_{j=1}^{\infty} b_j[f]\,\varphi_j(x)$, where $b_j[f]$ are coefficients. Second, that the solution for $\mathcal{G}\left(\varphi_j\right)$ for each eigenfunction is multiplicatively separable in (x,t):

$$\mathcal{G}(\varphi_j)(x,t) = e^{\lambda_j\, t}\,\varphi_j(x)$$

for some constant λ_j. Thus the partial differential equation in (17) becomes the following ordinary differential equation:

$$\lambda_j\varphi_j(x) = \varphi_j''(x)\frac{\sigma^2}{2} - \zeta\varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}] \;\text{ with boundary conditions } \varphi_j(\bar{x}) = \varphi_j(\underline{x}) = 0\ .$$

Note that the boundary condition for $\mathcal{G}(\varphi_j)$ at $t = 0$, in equation (19), holds by construction. We will denote the φ_j as eigenfunctions and the corresponding λ_j as eigenvalues. The eigenvalues and eigenfunctions are:

$$\lambda_j = -\left[\zeta + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x} - \underline{x}}\right)^2\right] \;\text{ and }\; \varphi_j(x) = \frac{1}{\sqrt{(\bar{x} - \underline{x})/2}}\sin\left(\frac{[x - \underline{x}]}{[\bar{x} - \underline{x}]}\,j\,\pi\right) \;\text{ for }\; j = 1, 2, 3, \ldots \tag{20}$$

This is the solution to a well known problem which has been studied extensively.[8] A key property of the set of eigenfunctions $\{\varphi_j\}$ is that they form an orthonormal base for the functions $f : [\underline{x}, \bar{x}] \to \mathbb{R}$ and for which $\int_{\underline{x}}^{\bar{x}} [f(x)]^2 dx < \infty$ so that we can write:

$$\hat{f}(x) \equiv \sum_{j=1}^{\infty} b_j[f]\, \varphi_j(x) \text{ for all } x \in [\underline{x}, \bar{x}]$$

where $\int_{\underline{x}}^{\bar{x}} (\hat{f}(x) - f(x))^2 dx = 0$ and the projection coefficients $b_j[f]$ are:

$$b_j[f] = \int_{\underline{x}}^{\bar{x}} f(x)\, \varphi_j(x) dx \text{ for all } j \geq 1\,, \tag{21}$$

which uses that $\{\varphi_j\}$ is an orthonormal base. Next we present a Lemma for the representation of the projections (expectations) of the function of interest f:

LEMMA 1. Assume that $f : [\underline{x}\,,\, \bar{x}] \to \mathbb{R}$ is piece-wise differentiable, with countably many discontinuities, and $\int_{\underline{x}}^{\bar{x}} [f(x)]^2\, dx < \infty$ then:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}\, b_j[f]\, \varphi_j(x) \;\text{ for all } x \in [\underline{x}, \bar{x}]$$

Now we turn to the impulse response. We define the projection coefficients for the initial displaced distribution $\hat{P}(\cdot)$, defined in equation (3), as follows:

$$b_j[\hat{P}] = \int_{\underline{x}}^{\bar{x}} \varphi_j(x) d\hat{P}(x) \equiv \int_{\underline{x}}^{\bar{x}} \varphi_j(x) \hat{p}(x) dx + \sum_{k=1}^{\infty} \varphi_j(x_k) \hat{p}_m(x_k) \tag{22}$$

so that if $\hat{P}$ has no mass points, it coincides with the definition for a function f. Using Lemma 1 and the definition of the projection coefficients $b_j[\hat{P}]$ we can write the impulse response function in equation (5) as follows:

[8] That this is the solution of the o.d.e. follows since $\sin(0) = \sin(j\pi) = 0$ for all $j \geq 1$ and also because $\sin''(x) = -\sin(x)$ for all x. Matching coefficients in the o.d.e. gives the eigenvalues.

Proposition 4. Assume that $f : [\underline{x}\,, \bar{x}] \to \mathbb{R}$ is piece-wise differentiable, with countably many discontinuities, and $\int_{\underline{x}}^{\bar{x}} [f(x)]^2 \, dx < \infty$. Furthermore assume that $\hat{P}$ has a piecewise continuous density and at most countably many mass points, then:

$$G(t; f, \hat{P}) = \sum_{j=1}^{\infty} e^{\lambda_j t} \, \beta_j \qquad \text{where} \qquad \beta_j \equiv b_j[f] \; b_j[\hat{P}] \; . \tag{23}$$

where the projection coefficients $b_j[f]$ and $b_j[\hat{P}]$ were defined in equation (21) and equation (22), respectively.

Proposition 4 provides a tractable analytic representation of the impulse response function that separates the effect of the function of interest f, encoded in the projection coefficient $b_j[f]$, the effect of the initial impulse $\hat{P}$, encoded in the projection coefficient $b_j[\hat{P}]$, and the effect of time, encoded in the eigenvalues λ_j. For instance, we will use $f(x) = -x$ to analyze the impulse response of output, since the contribution of output of each firm is inversely proportional to their price gap. Various shocks $\hat{P}$ will be considered, with the marginal impulse response given by $\hat{P} = \delta \bar{P}'$. It is also straightforward to use the proposition to compute the cumulative impulse response defined in equation (16):

$$\mathbf{G} \equiv \int_0^{\infty} G(t) dt = \sum_{j=1}^{\infty} \frac{\beta_j}{-\lambda_j} \quad .$$

The result in Proposition 4 can be readily understood by considering the discrete-time and discrete-state of the representation in equation (6). In this case $q_t(y|x)$ becomes the t-th power of the 1-period transition matrix. Diagonalizing this matrix delivers the eigenvalue and eigenvectors that can be used to compute projections that are equivalent to the ones in equation (23). It is straightforward to apply Proposition 4 to derive the density for the transition function $q_t(y|x)$ defined in equation (6):

Corollary 1. Let $(x, y) \in [\underline{x}, \bar{x}]^2$, let $f(y, \epsilon) = \mathbf{1}_{\{x \in (y-\epsilon, y+\epsilon)\}}/(2\epsilon)$, and $\hat{P}(\cdot)$ a CDF representing a degenerate random variable concentrated at x so that, by equation (22),

$\hat{p}_m(x) = 1$. Taking the limit as $\epsilon \to 0$, using equation (23) and computing the $b_j[f]$ and $b_j[\hat{P}]$ projections we have

$$q_t(y|x) = \sum_{j=1}^{\infty} \varphi_j(y)\varphi_j(x)e^{\lambda_j t} \quad . \tag{24}$$

We finish this subsection with a discussion of the generality of the representation in Proposition 4, and a brief explanation of why we concentrated on such a case. While we have used the idea of symmetry for our analysis, the applicable more general concept is whether the operator $\mathcal{G}$ is self-adjoint or not. Indeed one can consider an abstract positive operator $\mathcal{G}$ defined in a general vector space and still obtain a result essentially identical to Proposition 4. In particular, without entering in all details, if the law of motion for this state x can be described by a semigroup family (e.g. Anderson, Hansen, and Sargent (2003)), so that we can use the infinitesimal version of its law of motion, and if the resulting operator is compact, then the spectral theorem immediately gives the existence of a version of equation (23). The drawback of such an approach is that one may not have a simple characterization of the eigenfunctions and eigenvalues. We can consider a much simpler case than this abstract one, and yet more general than our benchmark case, in which x follows a general one dimensional diffusion in a closed interval. The Sturm-Louiville theory gives not only the representation in equation (23), but also gives many of the properties for the eigenvalues and eigenfunctions.[9] One requirement in the Sturm-Louiville theory, which is also present in our benchmark case, is the type of boundary condition.[10] This requirement is satisfied for the process of x stopped at the first price change, but it is not satisfied for the problem with re-injection. For the general class of the Sturm-Louiville problems the eigenfuctions and eigenvalues may not be all known in closed form, but there are ways to approximate them and also to efficiently compute them.

Finally there are many cases, besides our benchmark one, where eigenvalues and eigen-

[9]Examples of these properties are indexing the eigenfuctions by the number of zeroes, and having eigenvalues discretely separated, etc.

[10] To be specific, to be of the Neumann or Dirichlet type.

functions are known and are tractable. One example is in Appendix B where we examine the case of the operator representing a firm solving a multiproduct price setting problem, and give the eigenvalues and eigenfunctions on this n-dimensional problem. Another example is given in Appendix C where we consider an extension of the benchmark model that allows for a non-degenerate distribution of random fixed costs, as in e.g. Dotsey, King, and Wolman (1999). In both cases we characterize the eigenfunctions and eigenvalues.

4.1 Application to the canonical menu cost model.

We illustrate Proposition 4 using the canonical menu-cost model, obtained by setting $\mu = 0$ and $\zeta = 0$ in the problem of Section 2.1, which yields the symmetric inaction region $\underline{x} = -\bar{x}$ with optimal return $x^* = 0$. To compute the impulse response of output we use $f(x) = -x$ since the contribution of a firm to the deviation of output (relative to steady state) is inversely proportional to its price gap. Integrating $f(x)$ against $\varphi_j(x)$ we find the projection coefficients $b_j[f]$ in equation (23):

$$b_j[f] = \frac{4\bar{x}^{3/2}}{j\,\pi} \text{ for } j = 2, 4, 6, \ldots, \text{ and } b_j[f] = 0 \text{ otherwise.} \tag{25}$$

In this example we consider a *marginal* monetary shock so that $\hat{p}(x) = \delta \bar{p}'(x)$, as discussed above. The invariant distribution for this model is readily derived from equation (10) and the associated boundary conditions, which gives the triangular density $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$ for $x \in (-\bar{x}, \bar{x})$. It is apparent that $\bar{p}'(x)$ is a step function, equal to $1/\bar{x}^2$ for $x \in [-\bar{x}, 0)$ and equal to $-1/\bar{x}^2$ for $x \in (0, \bar{x}]$. We thus construct the projection coefficients $b_j[\bar{p}']$ by integrating $\bar{p}'(x)$ against $\varphi_j(x)$. This gives

$$b_j[\bar{p}'] = \frac{8}{j\,\pi\bar{x}^{3/2}} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[\bar{p}'] = 0 \text{ otherwise.} \tag{26}$$

Thus the impulse response $b_j[\bar{p}']b_j[f]$ coefficients for equation (23) are:

$$b_j[\bar{p}']b_j[f] = \frac{32}{(j\,\pi)^2} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[\bar{p}']b_j[f] = 0 \text{ otherwise.}$$

and the (marginal) output impulse response as defined in equation (7) is:

$$Y(t) = \sum_{i=0}^{\infty} \frac{32}{((2+4i)\,\pi)^2} e^{-N \frac{((2+4i)\pi))^2}{8} t} \tag{27}$$

where the expression has a single parameter given by $N = \sigma^2/\bar{x}^2$, namely the average number of price changes per period.

5 The selection effect and price dispersion

This section applies Proposition 4 to illustrate why different sticky-price models display different degrees of "selection". The notion of selection, coined by Golosov and Lucas, refers to the fact that firm that adjust prices following a monetary shock are selected from a particular set. For instance, following a monetary expansion, it is more likely to observe price increases (price changes by firms with a low markup) than price decreases. This contrasts with models where adjusting firms are not systematically selected, such as models of rational inattentiveness, or models where the times of price adjustment are exogenously given such as the Taylor or the Calvo model. It is known that different amounts of selection critically affect the propagation of monetary shocks.

We present an analytic result showing that selection creates a wedge between the duration of price changes and that of output. The two durations coincide when there is no selection. In this case the frequency of price changes is a sufficient statistic for the output effect of monetary policy. We show that such wedge is visible in the magnitude of the eigenvalues that control, respectively, the dynamics of the survival function of prices and the dynamics of output. We illustrate this result using the Calvo-plus model, a model that nests several special cases featuring different degrees of selection, from Golosov-Lucas to the pure Calvo model. The result also holds in several other models, featuring multiproduct firms or price plans.

This section also presents an important result, in Proposition 5, showing that the response of even centered moments to a marginal monetary shocks is zero at all horizons. This applies for instance to the variance of price gaps, or cross-sectional price dispersion, the kurtosis of the price changes, as well as to the survival function. The result is relevant because e.g. the cross-sectional price dispersion is the main measure of inefficiency in models with price stickiness. This result implies that cross-sectional price dispersion is not sensitive to small monetary shocks, so that its fluctuations won't be detectable in estimated impulse responses, but that instead it reacts to changes with the steady state level of inflation, or more generally with the monetary policy regime. An advantage of the analytic approach is to clearly illustrate the forces behind this surprising result.

While the remaining of this section works out our benchmark case, the Calvo Plus model, all the results –appropriately restated– hold for the multiproduct model in Appendix B and the general random fixed cost model in Appendix C .

Application to the Calvo-plus model. Next we use the decision problem defined in Section 2.1, and assume zero inflation ($\mu = 0$) and a quadratic profit function $R(x) = x^2$. It is straightforward that $\bar{x} = -\underline{x} > 0$ and that the optimal return is $x^* = 0$. Given the policy parameters $\{-\bar{x}, \bar{x}\}$ and the law of motion of the state $dx = \sigma dW$ it is immediate that the eigenvalues-eigenfunctions of the problem are those computed in equation (20). Since the eigenvalues depend on the speed at which prices are changed, we find it convenient to rewrite them in terms of the average number of price changes per unit of time. To this end we compute the expected number of adjustments per unit of time, the reciprocal of the expected time until an adjustment, $N = \frac{\zeta}{1-\text{sech}(\sqrt{2\phi})}$ where $\phi \equiv \frac{\zeta \bar{x}^2}{\sigma^2}$. Note that as $\bar{x} \to \infty$ then $N \to \zeta$, which is the Calvo model where all adjustment occur after an exogenous poisson shock. As $\zeta \to 0$ then $N \to \sigma^2/\bar{x}^2$ so that the model is Golosov and Lucas. This single parameter

$\phi \in (0, \infty)$ controls the degree to which the model varies between Golosov-Lucas and Calvo. Note that with this parameterization we can distinguish between N and the importance of the randomness in the menu cost ζ vs the width of the barriers, $\bar{x}^2/\sigma^2$. Indeed ζ/N, the share of adjustment due to random free-adjustments, depends only on ϕ. We let $\frac{\zeta}{N} = \ell(\phi)$ where this function is defined as $\ell(\phi) = 1 - \operatorname{sech}\left(\sqrt{2\phi}\right)$. The function $\ell(\cdot)$ is increasing in ϕ, and ranges from 0 to 1 as ϕ goes from 0 to ∞. Using the formula for N and equation (20) we have:

$$\lambda_j = -\zeta - \frac{\sigma^2}{\bar{x}^2}\frac{(j\pi)^2}{8} = -\zeta\left[1 + \frac{(j\pi)^2}{8\phi}\right] = -N\,\ell(\phi)\left[1 + \frac{(j\pi)^2}{8\phi}\right] \quad . \tag{28}$$

Interpretation of the Dominant Eigenvalue. The dominant eigenvalue has the interpretation of the asymptotic hazard rate of price changes. In particular, let $h(t)$ be the hazard rate of price spells as a function of the duration of the price spell t. Let τ be the stopping time for prices, i.e. τ is the first time at which $\sigma W(t)$, which started at $W(0) = 0$, either hits $\bar{x}$ or $\underline{x} = -\bar{x}$, or that the Poisson process changes. Let $S(t)$ be the survival function, i.e.: $S(t) = \Pr\{\tau \geq t\}$. Notice that the function of interest to compute the survival function is the indicator $f(x) = 1$ for all $t < \tau$. The hazard rate is defined as $h(t) = -S'(t)/S(t)$. Application of Proposition 4 gives the following:

Corollary 2. The Survival function $S(t)$ depends only on the odd-indexed eigenvalues-eigenfunctions, i.e. (λ_i, φ_i) for $i = 1, 3, 5, \ldots$. Let $h(t)$ be the hazard rate of price changes. Then, the dominant eigenvalue λ_1 is equal to the asymptotic hazard rate, i.e.

$$S(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j-1} t}\,\beta_{2j-1} \quad \text{and} \quad -\lambda_1 = \lim_{t\to\infty} h(t)$$

where $\beta_j \equiv b_j[1]\,\varphi_j(0)$ where we use equation (22) where $\hat{P}$ represents a degenerate random variable concentrated at $x = 0$ so that $\hat{p}_m(0) = 1$.

Irrelevance of dominant eigenvalue for output IRF. Next we show that the dominant eigenvalue λ_1, as well as all other odd-indexed eigenvalue-eigenfunction pairs, play no role in the output impulse response. Consider the output coefficients in the impulse response, given by equation (25). It is apparent that the coefficients $b_j[f]$ for all the odd-indexed eigenvalues-eigenfunctions ($j = 1, 3, \dots$) are zero, i.e. the loading of these terms are zero. This implies that the coefficient corresponding to the dominant eigenvalue λ_1 is zero. The first non-zero term, which we call the "leading" eigenvalue, involves λ_2. This is because $\varphi_j(\cdot)$ is symmetric around $x = 0$ for j odd, and antisymmetric for j even. Thus:

$$\int_{\underline{x}}^{\bar{x}} \varphi_j(x) f(x) dx = 0 \implies b_j[f] = 0 \text{ for } j = 1, 3, \dots$$

This happens since all the odd-indexed eigenfunctions φ_j ($j = 1, 3, ...$) are symmetric functions, and thus the projection onto them of an asymmetric function, such as $f(x) = -x$, yields a zero b_j coefficient. We summarize this result in the next corollary:

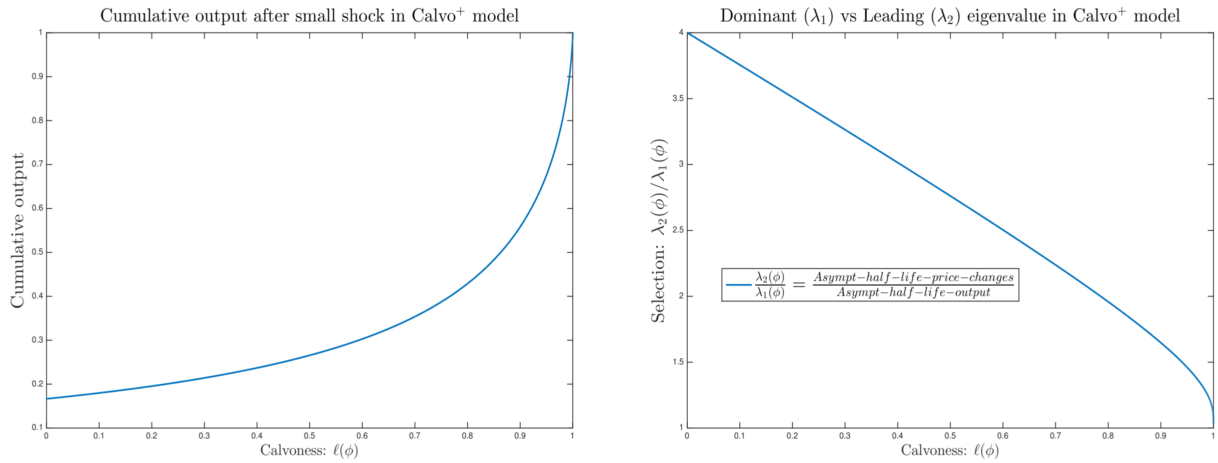
Corollary 3. The output impulse response function for the Calvo-plus model depends only on the even-indexed eigenvalue-eigenfunctions (λ_j, φ_j), and has zero loadings on the odd-indexed ones, such as the dominant eigenvalue. Thus the first leading term corresponds to the second eigenvalue λ_2:

$$G(t) = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} \beta_{2j} \quad \text{and} \quad \lambda_2 = \lim_{t \to \infty} \frac{\log Y(t)}{t}$$

where $\beta_j \equiv b_j[f]\, b_j[\hat{p}]$.

The corollary states that only half of the eigenvalues (those with an even index) show up in the output impulse response function. The largest eigenvalue is λ_2, which we call the "leading" eigenvalue of the output response function. It is interesting to notice that the dominant eigenvalue λ_1 does not appear in the impulse response for output. Notice the difference with the survival function where the only eigenvalues that appear are those with

Figure 2: Selection effect in Calvo$^+$ model



an odd-index. The right panel of Figure 2 plots the ratio between the leading eigenvalue for output λ_2 and the dominant eigenvalue λ_1. It is straightforward to see that the ratio is $\frac{\lambda_2}{\lambda_1} = \frac{8\phi+4\pi^2}{8\phi+\pi^2}$ depends only on ϕ, so that it can be immediately mapped into the "Calvoness" of the problem $\ell(\phi) \in (0,1)$. It appears that the ratio, which can also be interpreted as the ratio between the asymptotic duration of price changes over the asymptotic duration of the output impulse response, is monotonically decreasing in ℓ, and converges to 1 as $\ell \to 1$. The economics of this result is that the shape of the impulse response of output depends on the differential impact of the aggregate shock on price *increases* and price *increases*. Instead the dominant eigenvalue controls the asymptotic behavior of price *changes*, both increases and decreases. As $\ell \to 1$ selection disappears from the model and the two durations coincide. The left panel of the figure uses the particular case of a small monetary shock (developed in detail in the next subsection) to illustrate that as ℓ increases the cumulated output effect becomes larger due to a muted selection effect.

IRF of cross-sectional dispersion of prices to a monetary shock. We conclude this section with a discussion of a relevant topic in monetary models that concerns the welfare effect of shocks. In many monetary models the presence of price stickiness implies that welfare of the representative consumer depends on the dispersion of prices, or markups (i.e. prices relative to a flexible price benchmark where dispersion is nil). It is therefore of interest to analyze how the dispersion of markups behaves following a small monetary shock of size δ.[11] Let $M_k(t;\delta) \equiv \mathbb{E}\left[x(t) - \mathbb{E}(x(t))\right]^k$, for $k = 2, 4, ...$, denote the k-th even centered moment of x, measured t periods after the monetary shock δ hits the economy at the steady state. In particular, $M_2(t;\delta)$ will denote the cross sectional variance of markups t periods after the monetary shock. We have the following result:

Proposition 5. Assume the initial condition $\hat{p}$, the signed mass right after the aggregate shock, is antisymmetric. Then the initial impulse $\hat{p}$ does not have a first-order effect on any even centered moment $M_k(t;\delta)$ with $k = 2, 4, ...$

The proposition implies that a small (marginal) monetary shock does not have a first order impact on the dispersion of markups at all $t > 0$ after the monetary shock. It also shows that a zero first-order effect is predicted for all even centered moments of the distribution of markups (such as Kurtosis). Instead, uneven moments, such as the the mean markup (proportional to total output) or the the skewness of the distribution display a non-zero first order effect following a marginal monetary shock. As mentioned at the beginning of the section this result matters because even moments, such as the dispersion of price gaps, map directly into the efficiency of the economy. In terms of measurement the result implies one should look at the effect of varying the level of inflation on price dispersion as in Alvarez et al. (2019) or Nakamura et al. (2018). Likewise, the hazard rate of price changes should not react to a small monetary shock. This gives theoretical support to the estimation of hazard rates of price changes using time series evidence without controlling for monetary shocks.[12] Finally, we present the result on Kurtosis because elsewhere –Alvarez, Le Bihan, and Lippi (2016) –we have argued that the *steady state* kurtosis and frequency of price changes characterized the

[11]We are thankful to Nobu Kiyotaki for posing this question to us.

[12]In Alvarez et al. (2019) it is shown theoretically that price dispersion and the average frequency of price changes should not be affected by inflation around zero inflation, but that it should be responsive at higher levels. The same paper provides evidence on these two variables.

cumulative IRF to a small monetary shock. This result says that kurtosis does not change with monetary shocks, i.e. that remains approximately constant at its steady state value after a monetary shock.

6 On the shape of the impulse response function

This section discusses whether it is possible to approximate the impulse response function in a parsimonious way, a question that is naturally related to the shape of the impulse response. A natural candidate would be to analyze the impulse response associated to the leading eigenvalue as defined in Section 5, namely the largest eigenvalue associated with non-zero projection coefficient b_j in equation (23), for a case in which the IRF is close to exponential. We analyze this question by focusing on a small monetary shock that causes a marginal displacement of the invariant distribution. We assume a symmetric problem and present results for the baseline Calvo$^+$ model as well as for a model with price plans.

Initial Condition $\hat{p}$ for the impulse response to a monetary shock. The invariant density function $\bar{p}$ solves the Kolmogorov forward $\zeta\bar{p}(x) = \sigma^2/2\bar{p}''(x)$ in the support, except at $x = 0$, integrates to one and it is zero at $\pm\bar{x}$. This gives:

$$\bar{p}(x) = \frac{\theta \left[e^{\theta(2\bar{x}-|x|)} e - e^{\theta|x|} \right]}{2\left(1 - e^{\theta\bar{x}}\right)^2} \text{ for } x \in [-\bar{x}, \bar{x}] \quad \text{where} \quad \theta \equiv \sqrt{2\zeta/\sigma^2} \quad . \tag{29}$$

(note that to simplify notation we define a new parameter θ). The invariant distribution $\bar{p}(\cdot)$ is symmetric with $\bar{p}(x) = \bar{p}(-x)$ and $\bar{p}'(x) = -\bar{p}'(-x)$ for all $x \in [-\bar{x}, 0)$. The density of the distribution is continuous but non-differentiable at the injection point $x = 0$. For concreteness we focus below on the response to a marginal monetary shock δ, starting at the steady state. For such a small shock we can disregard the fraction of firms that change prices on impact, i.e. this effect is of order δ^2. Thus the initial condition is

$$\hat{p}(x) = \bar{p}'(x)\delta + o(\delta) \text{ for all } x \in [-\bar{x}, 0) \text{ and } x \in (0, \bar{x}]$$

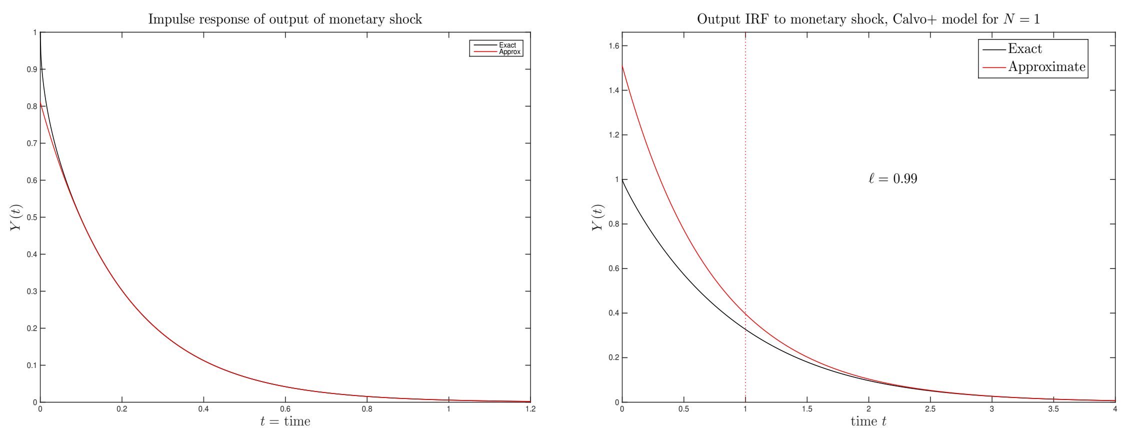
As done above, we find it convenient to express the output IRF $Y(t)$ per unit of the shock δ by using

$$Y(t) \equiv \frac{\partial}{\partial \delta} G(t,\delta)\big|_{\delta=0} = \sum_{j=1}^{\infty} e^{\lambda_{2j} t} b_{2j}[f] b_{2j}[\bar{p}']$$

Application of Proposition 4 gives the following:

Figure 3: Exact vs approximate IRF

Golosov Lucas ($\ell = 0$) Calvo$^+$ model



Proposition 6. The coefficients for the impulse response to a small monetary shock in the Calvo$^+$ model are given by:

$$\beta_j(\phi) \equiv b_j[\bar{p}'] b_j[f] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -2\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{1-\cosh\left(\sqrt{2\phi}\right)}\right]\left[\frac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2\left[\frac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases} \tag{30}$$

Recall that $\phi \equiv \sigma^2\zeta/\bar{x}^2 \in (0,\infty)$ is a single parameter that locates the Calvo$^+$ model between the Golosov-Lucas ($\phi = 0$) and the Calvo ($\phi \to \infty$) model. Each of these cases can be easily computed by simple calculus while keeping N constant. The solution for the first case, $\phi \to 0$, was given in equation (25) and equation (26). The second case, $\phi \to \infty$, is peculiar because in this limit the spectrum is no longer discrete. To see this notice that each of the eigenvalues $\lambda_j \to -\zeta$ and:

$$\lim_{\phi\to\infty} \beta_j(\phi) = \begin{cases} 0 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -2 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

Note that in this case, using just the "leading" eigenvalue, i.e. the term with the first non-zero weight, gives an impulse response $G(t)$ that is twice as large than the true one, for each t. This is in stark contrast with the case when $\phi \to 0$ where the approximation is extremely accurate. The difference is less than 1.5%, to see this note that when $\phi \to 0$ then $\mathbf{G} \to 1/(6N) \approx 0.1677/N$. On the other hand, using only the term corresponding to the second eigenvalue we obtain $b_2/(-\lambda_2) = 16/(\pi^4)/N \approx 0.1643/N$.

The next proposition gives a characterization of the ratio between the true area under the output impulse response and the approximate one, computed using only the leading eigenvalue:

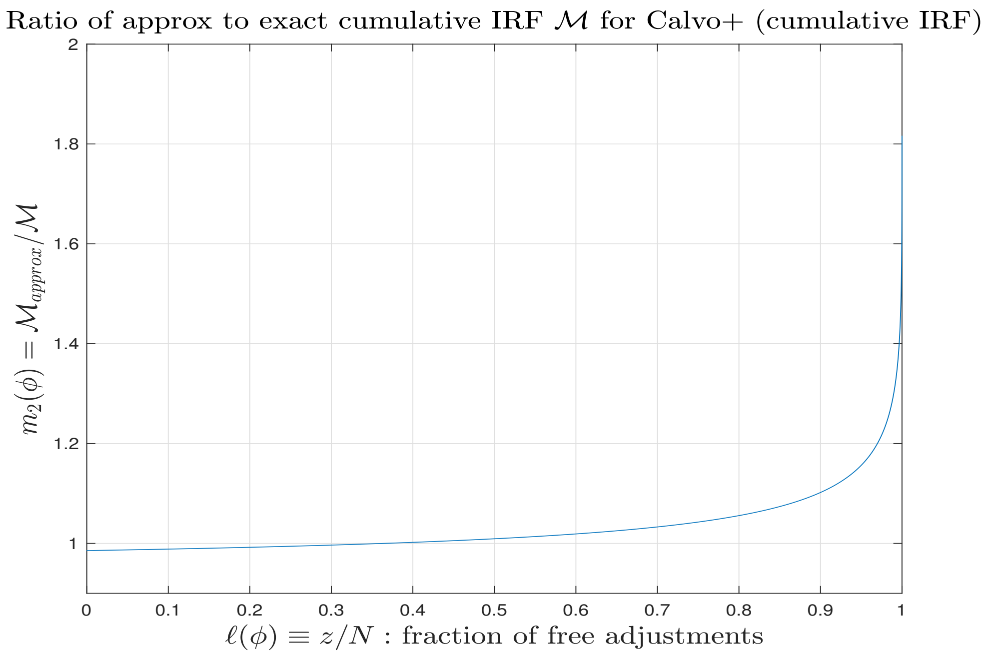
Proposition 7. Consider the marginal impulse response for output, so that $f(x) = -x$ and $\hat{p} = \delta\bar{p}'$. Define the ratio of the approximate cumulative impulse response based on the leading eigenvalue relative to the area under the impulse response as:

$$m_2(\phi) = \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty}\beta_j(\phi)/\lambda_j(\phi)} = 2\frac{\left[1+\cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi})-1-\phi\right]\left[1+\frac{\pi^2}{2\phi}\right]^2}$$

We note that $m_2(0) = \frac{16}{\pi^4}6 \approx 0.98$, $m_2'(\phi) > 0$ and $m_2(\phi) \to 2$ as $\phi \to \infty$.

Proposition 7 shows that the leading eigenvalue provides an accurate approximation of the total cumulative IRF for most variants of the calvo-plus model. At values of $\ell \approx 0.7$ the approximate function is close to 95% of the true effect. Figure 4 shows that accuracy degenerates as the model converges towards a pure Calvo model $\ell \to 1$.

Figure 4: Calvo-plus model



We can also use the expression for the coefficients of the impulse response to show that the slope of Y at $t = 0$ is minus infinity. This is intuitive since, after the shock, there are firms that are just on the boundary of the inaction region where they will increase prices, but there are no firms at the boundary at which they want to decrease prices.

Proposition 8. The derivative of the IRF with respect to t at $t = 0$ is given by:

$$\frac{\partial}{\partial t} Y(t)\Big|_{t=0} = -\infty \quad \text{for} \quad 0 \leq \phi < \infty.$$

Note that when $\phi \to \infty$, so we get the pure Calvo model, then the impulse response becomes $Y(t) = \exp(-Nt)$, and thus $Y'(0)$ is finite.[13]

Price Plans and the hump-shaped Output IRF. The model with price plans assumes that upon paying the menu cost the firm can choose two, instead of one price. At any point in time the firm is free to charge either price within the current plan, but changing to a new plan (another pair of prices) is costly. The idea was first proposed by Eichenbaum, Jaimovich, and Rebelo (2011) to model the phenomenon of temporary price changes (prices that move from a reference value for a short period of time and then return to it). In Alvarez and Lippi (2019) we provide an analytic solution to this problem and characterize the determinants of $\bar{x}$, the threshold where a new plan is chosen, as well as the optimal prices within the plan, named $\tilde{x}$ and $-\tilde{x}$. When $x \in [-\bar{x}, 0]$ the firm charges $-\tilde{x}$ and when $x \in (0, \bar{x}]$ it charges $\tilde{x}$. The invariant density of price gaps is still given by equation (29). For a given threshold $\bar{x}$ the value of $\tilde{x}$ is given by:

$$\tilde{x} = \bar{x} \left[\frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi}\left(e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2\right)} \right] \equiv \bar{x}\, \rho(\phi) > 0 \quad \text{where} \quad \phi = \bar{x}^2 \zeta / \sigma^2$$

and the function $\rho(\phi)$ gives the optimal price within the plan as a function of the adjustment threshold, namely $\tilde{x} = \rho(\phi)\bar{x}$, as a function of ϕ. Simple analysis shows that the images of the function $\rho(\phi)$ lie in the interval $(0, 1/3)$, that it is decreasing, and that it converges to $1/3$ as $\phi \to 0$ (see Alvarez and Lippi (2019)).

In the model with plans the contribution to the aggregate of a firm with output price gap x is, instead of $f(x) = -x$, the following function:

$$\tilde{f}(x) = \begin{cases} -x - \tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ -x + \tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

[13] See the web Appendix **??** analyzes the the output IRF for the pure Calvo model.

By the linearity of Fourier series we can add to the coefficients of the function $f(x) = -x$, shown in equation (25) the ones of the step function:

$$f_0(x) = \begin{cases} -\tilde{x} & \text{if } x \in [-\bar{x}, 0) \\ +\tilde{x} & \text{if } x \in (0, \bar{x}] \end{cases}$$

Importantly, we note the function $\tilde{f}(x) = f(x) + f_0(x)$ is still an asymmetric function. The function f_0 has Fourier sine coefficients equal to:

$$b_j[f_0] = -\frac{8\bar{x}^{3/2}\rho(\phi)}{j\,\pi} \text{ if } j = 2 + i4 \text{ for } i = 0, 1, 2, \ldots, \text{ and } b_j[f_0] = 0 \text{ otherwise}$$

From here we conclude that:

$$\beta_j^0(\phi) = b_j[f_0] b_j[\bar{p}'] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -4\,\rho(\phi) \left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right] \left[\frac{1}{1+\frac{j^2\pi^2}{8\phi}}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ 0 & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases} \tag{31}$$

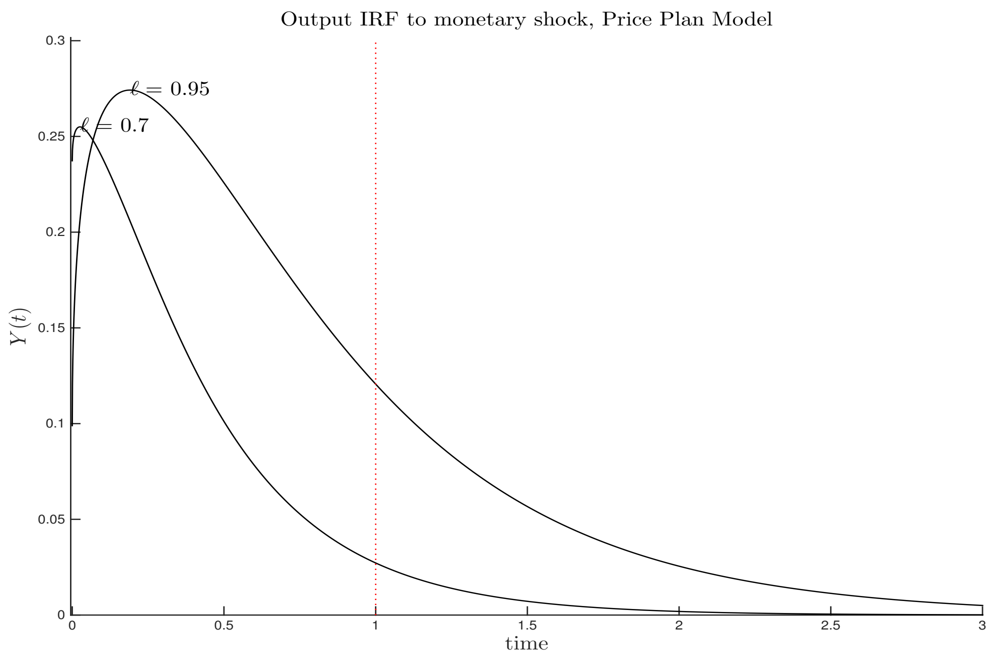
Thus the impulse response is given by:

$$Y_{\text{Plan}}(t) = Y_{\text{Calvo}^+}(t) + \sum_{j=1}^{\infty} \beta_j^0(\phi) e^{\lambda_j(\phi)t}$$

While the impulse response is monotone decreasing in the Calvo$^+$ model, in the price plan model the impulse response can be hump shaped. Indeed, as the ϕ increases, the impulse response goes from decreasing to humped shaped. The reason for this difference is that in the price plan model there is a non-negligible impact effect, due to the non-negligible set of firms that within the plan adjust from one price to the other. The difference across models, as we increase ϕ, is that invariant distribution has more firms with price gaps close to zero, and hence more firms that can change from one price to the other within the plan. This is because the higher is the impact effect on prices, the smaller is the effect on output. The next proposition indeed shows that when $\phi = 1$, so that $\zeta = \sigma^2/\bar{x}^2$, i.e. the number of plan changes in a pure-barrier model equals the one in a pure-random-plan model, are equal. For $\phi = 1$ the impulse response has a "hump", but it is infinitesimal.

Proposition 9. For $0 \leq \phi \leq 1$, the impulse response $Y(t)$ is decreasing. For $\phi > 1$, the impulse response is hump shaped. The value for $t_0 N$ at which the maximum is reached, $Y'(t_0 N) = 0$, increases relative to expected time to adjustment $1/N$.

Figure 5: Price Plan model



Given that the impulse response is hump-shaped any single eigenvalue cannot approximate the impulse response. Even for low values of ϕ, where the impulse response is monotone, using λ_2 gives a very bad approximation.

7 Volatility shocks and the propagation of monetary impulses

This section discusses the effect that changes to the volatility of shocks exert on the propagation of monetary shocks. The issue matters to e.g. the effectiveness of monetary policy in recessions vs boom, when the state of the economy is assumed to feature, respectively, high vs low volatility of shocks as in Vavra (2014). Our method provides a sharp analytic answer to this question.

For concreteness we illustrate the problem by using the pure menu cost model (without Calvo adjustment i.e. $\zeta = 0$ so that $\phi = \ell = 0$), whose output response to a small monetary shock was given in equation (27). We conduct a comparative static exercise to analyze how the propagation is affected by an innovation of the "volatility shocks", namely a permanent change in the common value of the idiosyncratic volatility σ.[14]

We start with a steady state for the model with idiosyncratic volatility σ. We characterize the effect of a small monetary shock, $\delta > 0$, which occurs $s \geq 0$ periods after a change in idiosyncratic volatility from σ to $\tilde{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$. In particular we let $Y(t; s, d\sigma/\sigma)\delta$ denote the output's IRF $t \geq 0$ periods after an unexpected monetary shock of size δ starting with a cross sectional distribution that has evolved s periods since the change in σ.[15]

While we characterize Y for all $t > 0$ and $s \geq 0$, two interesting cases are worthwhile to mention separately: the short-run and the long-run effect of volatility. The short-run effect, defined as $Y(t; 0, d\sigma/\sigma)$ or $s = 0$, consists of considering a simultaneous permanent change of both σ (to $\tilde{\sigma}$) and $\delta > 0$. After the shock the forward looking firm's decision rules adjusts immediately to the new volatility $\tilde{\sigma}$, while the initial distribution of price gaps corresponds to the stationary distribution obtained under the old decision rule. The long-run effect, denoted

[14] For simplicity and clarity of the results we consider here once and for all shocks to volatility. It is simple to modify the setup to consider a two-state Markov switching volatility process and to solve the associated firm's decision rules.

[15] We will keep using the notation of Y as the output's IRF per unit of monetary shock, and then omit the δ in the expressions below.

by $Y(t;\infty, d\sigma/\sigma)$ or $s \to \infty$, is equivalent to computing the effect of a monetary shock δ for a new steady state with volatility $\tilde{\sigma}$. We refer to this as the *long-run* effect since it is the effect of an unanticipated monetary shock once the distribution of price gaps has achieved its new invariant distribution. In this case the firm's decision rule corresponds to the new volatility $\tilde{\sigma}$ and the economy is described by the new invariant distribution of price gaps.

The general case characterizes an IRF whose coefficients are indexed by the parameter $0 < s < \infty$. The key feature of this case is that the monetary shock δ occurs s periods after the volatility shock, thus displacing a cross-section distribution of price gaps that is in a transition towards the new invariant distribution. Our analytic method allows us to exactly compute the evolution of this distribution and hence the effect of a monetary shock.

The next proposition uses the notation introduced above, where $Y(t;0,0)$ denotes the impulse before any change in volatility occurs, which we use as a benchmark. Also, the difference $Y\left(t;s,\frac{d\sigma}{\sigma}\right) - Y\left(t;\infty,\frac{d\sigma}{\sigma}\right)$ is the correction to the long run effect of a volatility shock $d\sigma/\sigma$ due to a finite duration s.

Proposition 10. Let $Y\left(t;s,\frac{d\sigma}{\sigma}\right)$ denote the time-t value of the output marginal impulse response that occurs s-periods after a volatility increase from σ to $\tilde{\sigma} = \left(1+\frac{d\sigma}{\sigma}\right)\sigma$. The *long run* effect ($s \to \infty$) of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right) \quad \text{for all } t \geq 0\ . \tag{32}$$

The *short run* effect ($s \to 0$) of the volatility shock $\frac{d\sigma}{\sigma}$ on the impulse response of output to a monetary shock is:

$$Y\left(t;0,\frac{d\sigma}{\sigma}\right) = \left(1+\frac{d\sigma}{\sigma}\right)\, Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right) \quad \text{for all } t \geq 0 \quad . \tag{33}$$

The deviation from the long run response as a function of s is given by:

$$Y\left(t;s,\frac{d\sigma}{\sigma}\right) - Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k} t} b_{2k}[f]\, b_{2k}\left[\hat{p}'(\cdot,s)\right] \quad \text{for all } t,s \geq 0\,. \tag{34}$$

where $\hat{p}'(\cdot, s)$ is the initial condition (i.e. a displaced cross section) at the time of the monetary shock, s periods after the change in volatility, whose projection coefficients are given by:

$$b_{2k}\left[\hat{p}'(\cdot, s)\right] = \frac{d\sigma}{\sigma}\frac{1}{\bar{x}^{\frac{3}{2}}}\sum_{j=1,3,5,\ldots}^{\infty} e^{\lambda_j s}\left(2\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\,\pi)^2}\right)\left(\frac{4kj}{(4k^2 - j^2))}\right) \quad , \quad k = 1, 2, 3, \ldots \quad (35)$$

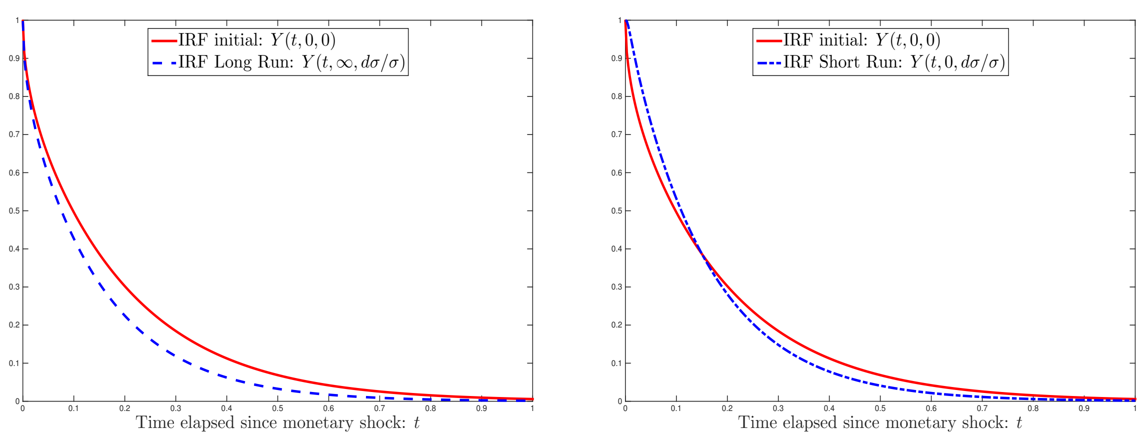
and where $b_{2k}[f] = 2\bar{x}^{3/2}/(k\pi)$ as in equation (25).

A few comments are in order.

(i) Figure 6 illustrates the difference between the short run and long run effect of an increase in volatility on the output's response to a monetary shock. The left panel compares the IRF with no change in volatility, $Y\,(t; 0, 0)$ to the one where the volatility increase has occurred $s \to \infty$ periods ago, i.e. $Y\,(t; \infty, d\sigma/\sigma)$ the long run effect. The right panel compares the IRF with no change in volatility, $Y\,(t; 0, 0)$ to the one where the volatility increase has occurred at the same time as the monetary shock $s = 0$ periods ago, i.e. $Y\,(t; 0, d\sigma/\sigma)$ the short run effect.

Figure 6: Short-run and long-run IRF vs. IRF before volatility increases

Long run ($s \to \infty$) Short run ($s = 0$)



Note: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

(ii) For this proposition we use the form of the decision rules for the threshold $\bar{x}$, which as the discount rate goes to zero is $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$ where ψ is the fixed cost –as fraction of the frictionless profit and B is the curvature of the profit function around the frictionless profit. This implies that the elasticity of $\bar{x}$ to σ is $1/2$. This elasticity is the so called "option value" effect on the optimal decision rules.

(iii) The rescaling of time in $Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right)$ in the expressions for the long and short run effect of volatility reflects the change in the eigenvalues, which depend on the value of N, the implied average number of price changes per unit of time, as $\lambda_j = -N\left(\pi j\right)^2/8$ (see equation (28) for $\zeta = 0$). Recall that $N = (\sigma/\bar{x})^2$, and hence all the eigenvalues change proportionally with σ.

(iv) For the case of the impact effect and in which $\tilde{\sigma} > \sigma$, the invariant distribution just before the monetary shock is narrower than the range of inaction that corresponds to the new wider barriers. This explains the extra multiplicative term level $\left(1+\frac{d\sigma}{\sigma}\right)$ in the impact effect in equation (33): since firms have price gaps that are discretely away from the inaction bands, then prices react more slowly, generating the extra effect on output. The logic for the case where $\tilde{\sigma} < \sigma$ is similar.

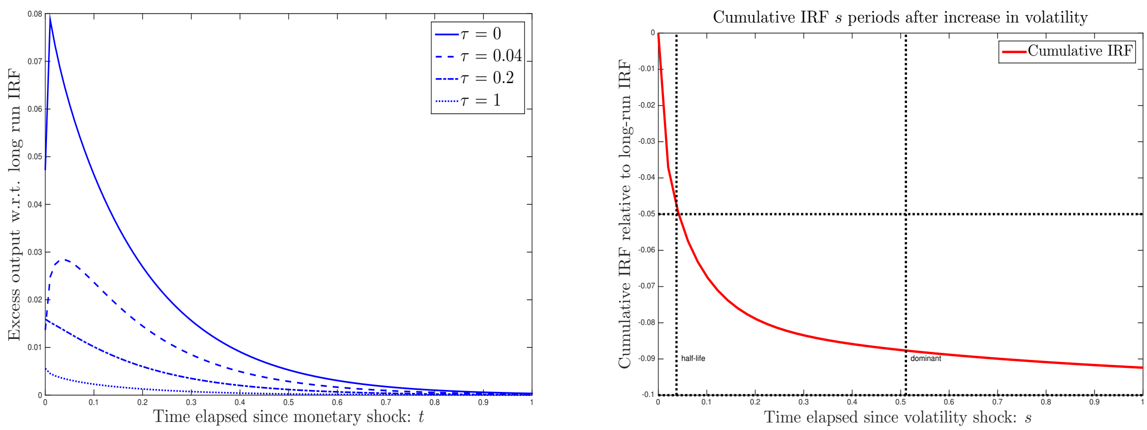
(v) In equation (34) we use only the even terms for the projections, i.e. the index for the projection $b_{2k}[\cdot]$ runs on $2k$ because f is antisymmetric. For these coefficients, as was the case without volatility shocks, the eigenvalues that control the effect of the horizon t in the IRF are the even ones, i.e. $\lambda_2, \lambda_4, \ldots$, starting with the leading one λ_2.

(vi) The expressions in equation (34) and equation (35) show that what governs the difference between the long run and the short run volatility effects are the odd eigenvalues, i.e. $\lambda_1, \lambda_3, \ldots$, since these are the only elements where s affect the expressions. In particular, λ_1 is the dominant eigenvalue.

(vii) We note that the expression for the correction term in equation (34) involves no parameter for the model with the exception of N, which enters only in the eigenvalues $\lambda_j = -N(j\pi)^2/8$. This gives a meaning to the units of t and s, which are measured relative to the (new) steady state duration of price changes $1/N$. This remark is needed to interpret the time units in the horizontal axes of both panels of Figure 7.

Figure 7: The propagation of monetary shocks as s grows

$Y\left(t; s, d\sigma/\sigma\right) - Y\left(t; \infty, d\sigma/\sigma\right)$ Change in Cumulated output: $\mathcal{C}(s, d\sigma/\sigma)$



Note: $N = 1$ (one price adjustment per unit of time, on average) and $d\sigma/\sigma = 0.1$.

(vii) To illustrate the general case of $0 < s < \infty$ in Figure 7 we display two plots. First, the left panel of Figure 7 plots equation (34), evaluated at 4 values of s. It is apparent that as s becomes bigger monetary policy becomes less effective and gradually converges to the long run value. This can be seen by comparing the correction for any given t across the four values of s. Second, the right panel, plots the cumulated IRF of a monetary shock s periods after the volatility shock, relative to the cumulative IRF of a monetary shock when there is no volatility shock. In particular we plot:

$$\mathcal{C}(s, d\sigma/\sigma) \equiv \frac{\int_0^\infty Y(t, s, d\sigma/\sigma)dt}{\int_0^\infty Y(t, 0, 0)dt} - 1$$

We use the cumulated IRF to obtain a simple one-dimensional summary of this effect across all times t. Notice the following properties of $\mathcal{C}$: for all s we have $\mathcal{C}\left(s, d\sigma/\sigma\right) = (d\sigma/\sigma)\mathcal{C}\left(s, 1\right)$, since it is based on a derivative, and for extreme values of s we have $\mathcal{C}\left(\infty, d\sigma/\sigma\right) = -d\sigma/\sigma$, and $\mathcal{C}\left(0, d\sigma/\sigma\right) = 0$. From Figure 7 it is clear that the transition to the higher volatility occurs very fast, a cumulative effect of $\mathcal{C}$ half as large as half of the one in $s \to \infty$ will occur when $s_{1/2} \approx 0.05/N$, a half-life indicated by a vertical bar in the right panel. More precisely, $s_{1/2}$ is defined as $\mathcal{C}(s_{1/2}, d\sigma/\sigma) = -(1/2)(d\sigma/\sigma)$. This effect is much faster than the half-life corresponding to the dominant eigenvalue $\lambda_1 = -N\pi^2/8$, which is given by $t_{1/2} \equiv -8\log(0.5)/(N\pi^2) \approx 0.56/N$, and it is indicated by another vertical bar in the right panel. The ratio of the two times is very large: $t_{1/2}/s_{1/2} \approx 12$, and it is independent of any parameter of the model.[16] From this comparison we conclude that for this particular model using exclusively the dominant eigenvalue λ_1 to approximate the time it takes for the distribution to converge after the change in volatility will be misleading. Summarizing, in the Golosov-Lucas model the short run effect of the volatility change is only relevant when the monetary shock occurs almost immediately after the volatility change.

8 Conclusion and Future Work

We have analytically characterized the impulse response function for a large class of sS models applied to price setting, using eigenfunctions-eigenvalues techniques. We have illustrated the usefulness of this method to study the effect of several shocks of macroeconomic interest. We have relied on assumptions of symmetry and lack of drift to derive many of the results. We established that for price setting problems in economies with low inflation deviating from such assumptions only has second-order consequences for the results. Our characterization also uses general equilibrium environment and price setting problem for the firm which implies that there are no first order strategic complementarities in the firm's optimal decision rules.

Open questions for future research will involve exploring setups featuring some combinations of asymmetries, large drift, and non-negligible strategic complementarities. For instance asymmetries are important in inventory models, such as those applying to the management of liquid assets, a large drift is important in models of capital relocation with frictions (where capital depreciation rates are large), and strategic complementarities can be important depending on the general equilibrium setup faced by firms. Technically, incorporating asym-

[16]The vertical distance on the correction between $s = 0$ and $s \to \infty$ plotted on the right panel is $d\sigma/\sigma$, which is 0.1 for this example. For other values, the vertical axis scales proportionally.

metries and drift requires working out the dynamics of problems with reinjection, for which we already derived some result in this paper (see Appendix D and Appendix E) but more is needed. Incorporating strategic complementarities requires extending the result to endogenous moving boundary problems. We think that the use of similar techniques to analyze such case is feasible, but we believe it should be the subject of a work on its own.

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WEB APPENDIX FOR:

"The Analytic Theory of a Monetary Shock" *

Fernando Alvarez
University of Chicago and NBER

Francesco Lippi
LUISS University and EIEF

May 28, 2019

Abstract

This document contains all the proofs of the paper "The Analytic Theory of a Monetary Shock". The document also contains two applications of the method developed in the paper for Multiproduct firms and for a general random fixed cost problem. Finally the document contains two extensions concerning problems with drift and problems with asymmetries in the return function.

JEL Classification Numbers: E3, E5

Key Words: Menu costs, Impulse response, Dominant Eigenvalue, Selection.

*Appendix to be posted online.

Contents

A Proofs of all propositions in the paper

Proof. (of Proposition 1). Using the definitions of $\mathcal{H}$, $\mathcal{G}$ and τ we have the following recursion:

$$\mathcal{H}(f)(x,t) = \mathcal{G}(f)(x,t) + \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)\middle|x\right] \text{ for all } x \in [\underline{x}, \bar{x}] \text{ and for all } t > 0.$$

Let us begin by defining the following object: $D(x,t) \equiv \mathbb{E}\left[1_{\{t>\tau\}}\mathcal{H}(f)(x^*, t-\tau)\middle|x\right]$. We first consider case (i) and show that $D(x,t) = 0$ for all x and all t. This follows since $\mathcal{H}(f)(x^*, s) = 0$ for all s. This in turn follows because f is antisymmetric, thus we have $\mathbb{E}\left[f(x(t))\,|x(\tau) = x^*\right] = 0$, which follows immediately by the symmetry of the distribution $g(x,t)$ and the antisymmetric property of f. It follows that $\mathbb{E}\left[1_{\{t\geq\tau\}}f(x(t))\,|x(0) = x\right] = 0$. Hence, since $\mathcal{H} = \mathcal{G}$, this implies that $G(t) = H(t)$ for any $p(\cdot,t)$.

Now we turn to case (ii). We note that $D(x,t)$ is symmetric in x around $x^* = (\underline{x} + \bar{x})/2$. This follows since the law of motion of x is symmetric so $g(x,t)$ is symmetric around x^*. This in turn implies that the probability of hitting either barrier at time s, starting with $x(0) = x$, is symmetric in x, which directly implies the symmetry of $D(x,t)$. Now we use that $D(x,t)$ is symmetric and that

$$H(t,f,p) - G(t,f,p) = \int_{\underline{x}}^{\bar{x}} D(x,t)\left(p(x,0) - \bar{p}(x)\right)\,dx \quad .$$

Since $D(x,t)$ is symmetric and $p(x,0) - \bar{p}(x)$ is antisymmetric we have that the right hand side is zero so that $H(t) = G(t)$. □

Proof. (of Proposition 2) We first show equation (13) holds. To simplify the notation we omit a in the expression in this part of the proof. The proof proceeds in several steps. First we analyze properties of the decision rules (optimal thresholds) as a function of μ. Second, we analyze the direct and indirect (i.e. via the decision rules) implications of μ for the transition probabilities of the state at a given horizon t. Third, we establish a symmetry property of the impulse $\hat{p}(\cdot;\delta,\mu)$ as a function of (μ,δ). Fourth, we use the properties of the transition probabilities and decision rules to derive an antisymmetric property of H, viewed as joint function of (δ, μ) for any fixed t. Fifth, we use this antisymmetric property to obtain a zero cross derivative of H, which implies the desired result.

1) We write the boundaries of the inaction rage and the optimal return point as functions of μ. They satisfy

$$x^*(\mu) = -x^*(-\mu)\,,\ \bar{x}(\mu) = -\underline{x}(-\mu) \text{ and } \underline{x}(\mu) = -\bar{x}(-\mu)\,.$$

This can be shown using a guess and verify strategy together with the corresponding guess of the value function $v(x, \mu) = v(-x, -\mu)$.

2) We define $P_t(y|x; \mu)$ to be the transition function for the state starting at $x(0) = x$ to $x(t) = y$, where the state evolves as follows. For $0 < s < t$, then $dx(s) = \mu ds + \sigma dW(s)$ as long as $x(s) \in (\underline{x}(\mu), \bar{x}(\mu))$ and the free adjustment opportunity has not arrived at time s. On the other hand, if $x(s)$ hits either $\bar{x}(\mu)$ or $\underline{x}(\mu)$, or the free adjustment opportunity arrives, then $x_+(s) = x^*(\mu)$, i.e. the firm is re-injected at the optimal return point. Using the properties of the decision rules, and the symmetry of the innovations in BM we have :

$$P_t\left(y|x; \mu\right) = P_t\left(-y| -x; -\mu\right)$$

To see why, write $y = x^*(\mu) + \Delta_y$ and $x = x^*(\mu) + \Delta_x$, so that for (y', x') given by $y' = -y$ and likewise $x' = -x$ we have $P_t\left(y|x; \mu\right) = P_t\left(y'|x' - \mu\right)$. But $y' = x^*(-\mu) - \Delta_y = -x^*(\mu) - \Delta_y = -y$ and likewise $x' = -x'$, establishing the required result.

3) Recall that $\hat{p}(\cdot; \delta, \mu) = \bar{p}(x + \delta; \mu) - \bar{p}(x; \mu)$. Using the properties of the decision rules and of the Kolmogorov forward equation for the steady state density $\bar{p}$, we get that $\bar{p}(x, \mu) = \bar{p}(-x, -\mu)$ is symmetric, which can be proved by a guess and verified strategy.

4) Using P_t we can write the impulse response as:

$$H\left(t; f, \hat{p}(\cdot, \delta, \mu), \mu\right) = \int\int f(y)\, P_t(y|x; \mu)\, \hat{p}(x, \mu, \delta)\, dy\, dx$$

Recall that we define

$$Y_\mu(r; f) = \frac{\partial}{\partial \delta} H\left(t; f, \hat{p}(\cdot, \delta, \mu), \mu\right)\Big|_{\delta=0}$$

and thus

$$\frac{\partial}{\partial \mu} Y_\mu(r; f) = \frac{\partial^2}{\partial \delta \partial \mu} H\left(t; f, \hat{p}(\cdot, \delta, \mu), \mu\right)\Big|_{\delta=0, \mu=0}$$

We will show below that:

$$H\left(t; f, \hat{p}(\cdot, \delta, \mu), \mu\right) = -H\left(t; f, \hat{p}(\cdot, -\delta, -\mu), -\mu\right) \tag{1}$$

for all μ, δ. Using this, we have:

$$\frac{\partial^2}{\partial \mu \partial \delta} H\left(t; f, \hat{p}(\cdot, \delta, \mu), \mu\right) = -\frac{\partial^2}{\partial \mu \partial \delta} H\left(t; f, \hat{p}(\cdot, -\delta, -\mu), -\mu\right)$$

which, evaluated at $(\mu, \delta) = (0,0)$ gives

$$\frac{\partial^2}{\partial \mu \partial \delta} H\left(t; f, \hat{p}(\cdot, 0, 0), 0\right) = -\frac{\partial^2}{\partial \mu \partial \delta} H\left(t; f, \hat{p}(\cdot, 0, 0), 0\right)$$

so the cross derivative has to be zero, establishing the desired results.

To finish the proof we show equation (1) holds. We have

$$\begin{aligned}
&H\left(t; f, \hat{p}(\cdot, \delta, \mu), \mu\right) = \int\int f(y)\, P_t(y|x;\mu)\, \hat{p}(x,\mu,\delta)\, dy\, dx \\
&= -\int\int f(-y)\, P_t(y|x;\mu)\, \hat{p}(x,\mu,\delta)\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \hat{p}(x,\mu,\delta)\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \left[\bar{p}(x+\delta,\mu) - \bar{p}(x,\mu)\right]\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \left[\bar{p}(-x-\delta,-\mu) - \bar{p}(-x,-\mu)\right]\, dy\, dx \\
&= -\int\int f(-y)\, P_t(-y|-x;-\mu)\, \hat{p}(-x,-\mu,-\delta)\, dy\, dx \\
&= -\int\int f(y')\, P_t(y'|x';-\mu)\, \hat{p}(x',-\mu,-\delta)\, dy'\, dx' \\
&= -H\left(t; f, \hat{p}(\cdot, -\delta, -\mu), -\mu\right)
\end{aligned}$$

where we use the definition of H, that f is antisymmetric, that P_t is symmetric (as shown above), the definition of $\hat{p}$, the symmetry of $\bar{p}$ (as shown above), the definition of $\hat{p}$ again, a change of variables of integration, and again the definition of H. This finishes the proof of equation (13).

Now we turn to the proof of equation (14). The proof is almost identical to the one for equation (13), step by step replacing μ by a.

□

Proof. (of Proposition 3). First rewrite the cumulated impulse response function as

$$\mathbf{H}(0, f, \hat{P}) = \int_{\underline{x}}^{\bar{x}} m_\infty(x; f) dP(x, 0)$$

where $m_\infty(x; f)$ is the expected cumulated value of f, in deviation from its mean $\bar{f}$, conditional on $x(0) = x$, computed from the time of the shock until the infinite future, namely:

$$m_\infty(x; f) = \mathbb{E}\left(\int_0^\infty \left(f(x(t)) - \bar{f}\right) dt \;\middle|\; x(0) = x \right) \quad \text{where} \quad \bar{f} \equiv \int_{\underline{x}}^{\bar{x}} f(x) d\bar{P}(x) \ .$$

Note that

$$m_\infty(x; f) = \mathbb{E}\left(\int_0^\tau \left(f(x(t)) - \bar{f}\right) dt \;\middle|\; x(0) = x \right) + m_\infty(x^*; f)$$

So that at x^* we have

$$m_\infty(x^*; f) = \mathbb{E}\left(\int_0^\tau \left(f(x(t)) - \bar{f} \right) dt \;\Big|\; x(0) = x^* \right) + m_\infty(x^*; f)$$

so that

$$0 = \mathbb{E}\left(\int_0^\tau \left(f(x(t)) - \bar{f} \right) dt \;\Big|\; x(0) = x^* \right) \tag{2}$$

Then we can write

$$m_\infty(x; f) = \mathbb{E}\left(\int_0^\tau \left(f(x(t)) - \bar{f} \right) dt \;\Big|\; x(0) = x \right) + \sum_{i=1}^{\infty} \mathbb{E}\left(\int_{\tau_i}^{\tau_{i+1}} \left(f(x(t)) - \bar{f} \right) dt \;\Big|\; x(\tau_i) = x^* \right)$$

where all terms in the summation for $i = 1, 2, ...$ are zero by equation (2). Thus we have

$$m_\infty(x; f) = \mathbb{E}\left(\int_0^\tau \left(f(x(t)) - \bar{f} \right) dt \;\Big|\; x(0) = x \right)$$

which establishes the proposition. □

Proof. (of Lemma 1). The proof uses the linearity of $\mathcal{G}$ to write $\mathcal{G}(\hat{f} + f - \hat{f}) = \mathcal{G}(\hat{f}) + \mathcal{G}(f - \hat{f})$. The projection $\hat{f}$ converges pointwise to f at any point at which f is differentiable. Additionally, by hypothesis, f is not differentiable (at most) at countably many points. Finally, we have defined $\mathcal{G}(f)(x,t) = \mathbb{E}\left[1_{\{t<\tau\}} f(x(t)) | x(0) = x \right]$. We note that this expected value is given by the integral that uses a continuous density, i.e. the density of BM starting at $x(0) = x$ and reaching $x(t) = y$ at t, which is continuous on y for $t > 0$. Hence the function $f - \hat{f}$ is non-zero only at countably many points, and thus its integral with respect to continuous density is zero, i.e. $\mathcal{G}(f - \hat{f})(x,t) = 0$ for all x and $t > 0$. Then we have

$$\begin{aligned}
\mathcal{G}(f)(x,t) &= \mathcal{G}\left(\hat{f}\right)(x,t) + \mathcal{G}(f - \hat{f})(x,t) = \mathcal{G}\left(\hat{f}\right)(x,t) \\
&= \mathcal{G}\left(\sum_{j=1}^{\infty} b_j[f] \varphi_j \right)(x,t) = \sum_{j=1}^{\infty} b_j[f] \mathcal{G}\left(\varphi_j\right)(x,t) = \sum_{j=1}^{\infty} b_j[f] e^{\lambda_j t} \varphi_j(x)
\end{aligned}$$

where we have used the linearity of $\mathcal{G}$, the definition of $\hat{f}$, and the form of the solution for $\mathcal{G}\left(\varphi_j\right)(x,t)$. □

Proof. (of Proposition 4). The result follows by Lemma 1, the definition of the projection coefficients $\{b_j[P]\}$ and the definition of the response function in equation (5).

Proof. (of Proposition 5) Let us define the centered even k-th moment for the variable x: $M_k(t,\delta) \equiv \mathbb{E}_\delta \left(x(t) - \mathbb{E}(x(t)) \right)^k$, where $k = 2, 4....$ and the subscript δ denotes that probabilities are those of an impulse response following a marginal shock δ to the invariant distribution of gaps at zero inflation.

The objective is to show that the $\frac{\partial}{\partial \delta} M_k(t,\delta)\Big|_{\delta=0} = 0$ for all t, i.e. that a marginal shock δ has no first-order effect on the even centered moments at every t. The proof follows two steps. First, to show that the impulse response of any *even* moment is flat at zero. Second, to show that the impulse response of any *centered* moment is well approximated, up to second order terms, by the impulse response of the corresponding non-centered moment.

The first step is readily established since a marginal shock triggers an antisymmetric displaced distribution $\hat{p}(x,0) = \bar{p}'(x)\delta$, whose projection coefficients on all even-indexed eigenfunctions $j = 2, 4, ...$ are zero (since such eigenfunctions are symmetric). Note next that even (non-centered) moments $k = 2, 4, ...$ are symmetric by definition, which immediately implies that their projection coefficients on all odd-indexed eigenfunctions $j = 1, 3, ...$ are zero. It follows that none of the eigenfunctions will have a non-zero coefficient. This proves the first step.

To prove the second step write in terms of the non-centered moments

$$M_k(t,\delta) = B_0 \mathbb{E}_\delta \left(x(t)^k\right) + B_1 \mathbb{E}_\delta \left(x(t)^{k-1}\right) \mathbb{E}_\delta \left(x(t)\right) + ... + B_{k-1} \mathbb{E}_\delta \left(x(t)\right) \left(\mathbb{E}_\delta \left(x(t)\right)\right)^{k-1} + B_k \left(\mathbb{E}_\delta \left(x(t)\right)\right)^k$$

where the B_j are the binomial coefficients. Next, lets us replace each of the moments with its first order expansion in δ, namely let $\mathbb{E}_\delta \left(x(t)^k\right) = a_k \delta + o(\delta)$ where a_k is moment-k first derivative. We get

$$M_k(t,\delta) = B_0(a_k\delta + o(\delta)) + B_1(a_{k-1}\delta + o(\delta))(a_1\delta + o(\delta)) + ... + B_k(a_1\delta + o(\delta))^k$$

It is apparent that the only first order term in δ is a_k, i.e. the coefficient of the non-centered moment. This concludes the proof. □

Proof. (of Proposition 6) Straightforward differentiation of the density function $\bar{p}(x)$ gives

$$\bar{p}'(x) = \begin{cases} -\frac{\theta^2\left[-e^{-\theta x} - e^{2\theta\bar{x}} e^{\theta x}\right]}{2\left[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}\right]} & \text{if } x \in [-\bar{x}, 0] \\ -\frac{\theta^2\left[e^{\theta x} + e^{2\theta\bar{x}} e^{-\theta x}\right]}{2\left[1 - 2e^{\theta\bar{x}} + e^{2\theta\bar{x}}\right]} & \text{if } x \in [0, \bar{x}] \end{cases}$$

where $\theta \equiv \bar{x}^2\zeta/\sigma^2$. The linear projection of $\bar{p}'(x)$ onto φ_j gives the projection coefficients:

$$b_j[\bar{p}'] = \begin{cases} 0 & \text{if } j \text{ is odd} \\ -\left[\frac{2\theta^2 j\pi}{4\theta^2\bar{x}^2 + j^2\pi^2}\right]\left[\frac{1+\cosh(\bar{x}\theta)}{1-\cosh(\bar{x}\theta)}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is odd} \\ -\left[\frac{2\theta^2 j\pi}{4\theta^2\bar{x}^2 + j^2\pi^2}\right] & \text{if } j \text{ is even and } \frac{j}{2} \text{ is even} \end{cases}$$

To see this compute: $\int_{-\bar{x}}^{\bar{x}} \bar{p}'(x)\varphi_j(x)dx = 2\int_{-\bar{x}}^{0} \bar{p}'(x)\varphi_j(x)dx$ for $j = 2, 4, 6, \ldots$. The function $\bar{p}'$ is antisymmetric and φ_j is antisymmetric for j even, with respect to $x = 0$. For $j = 1, 3, 5, ...$ this integral is zero, since φ_j is symmetric, see equation (20). For $j = 2, 4, \ldots$

we thus have:

$$
\begin{aligned}
b_j[\bar{p}'] = 2\int_{-\bar{x}}^{0} \bar{p}'(x)\varphi_j(x)dx &= \frac{\theta^2}{\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]}\int_{-\bar{x}}^{0}\left[e^{-\theta x}+e^{2\theta\bar{x}}e^{\theta x}\right]\frac{1}{\sqrt{\bar{x}}}\sin\left(\frac{(x+\bar{x})}{2\bar{x}}j\pi\right)dx \\
&= \frac{e^{\bar{x}\theta}4\theta^2\bar{x}}{\sqrt{\bar{x}}\left[1-2e^{\theta\bar{x}}+e^{2\theta\bar{x}}\right]}\frac{\left[j\pi\left(1-\cosh\left(\bar{x}\theta\right)(-1)^{j/2}\right)\right]}{4\theta^2\bar{x}^2+j^2\pi^2} \\
&= \frac{8\phi\, e^{\sqrt{2\phi}}}{\bar{x}^{3/2}\left[1-2e^{\sqrt{2\phi}}+e^{2\sqrt{2\phi}}\right]}\frac{\left[j\pi\left(1-\cosh\left(\sqrt{2\phi}\right)(-1)^{j/2}\right)\right]}{8\phi+\pi^2 j^2} \\
&= \frac{j\pi}{4\bar{x}^{3/2}}\,\frac{(-2)}{\left(1+\frac{j^2\pi^2}{8\phi}\right)}\,\frac{1-\cosh\left(\sqrt{2\phi}\right)(-1)^{j/2}}{1-\cosh\left(\sqrt{2\phi}\right)}
\end{aligned}
$$

where we used that $\theta\bar{x} = \sqrt{2\phi}$ and that $\cosh(x) = (1+e^x)/(2e^x)$. Combining it with the expression for $b_j[f]$ in equation (25) gives the desired result. □

Proof. (of Proposition 7) Rewriting the expression for m_2:

$$
\begin{aligned}
m_2(\phi) &= \frac{\beta_2(\phi)/\lambda_2(\phi)}{\sum_{j=1}^{\infty}\beta_j(\phi)/\lambda_j(\phi)} = \frac{\beta_2(\phi)/\lambda_2(\phi)}{Kurt(\phi)/(6\,N)} \\
&= \frac{\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{8(2\phi)}{4(2\phi)+4\pi^2}\right]}{N\,\ell\left(\sqrt{2\phi}\right)\left[1+\frac{\pi^2}{2\phi}\right]}\frac{N\left(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})-2\right)^2}{(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi}))(\exp(\sqrt{2\phi})+\exp(-\sqrt{2\phi})-2-2\phi)} \\
&= 2\frac{\left[1+\cosh\left(\sqrt{2\phi}\right)\right]}{\left[\cosh(\sqrt{2\phi})-1-\phi\right]\left[1+\frac{\pi^2}{2\phi}\right]^2}
\end{aligned}
$$

where the first line follows from the definition, and the first equality from the sufficient statistic result in Alvarez, Le Bihan, and Lippi (2016). The second line uses the expression for β_2, λ_2 derived above, as well as the expression for the Kurtosis derived in Alvarez, Le Bihan, and Lippi (2016). The third line uses the expression for ℓ. The remaining lines are simplifications. □

Proof. (of Proposition 8) First use Proposition 6 to write

$$
\frac{\partial}{\partial t}Y(t)|_{t=0} = \lim_{M\to\infty}\sum_{j=1}^{M}\beta_j(\phi)\lambda_j(\phi) = \lim_{M\to\infty}\sum_{i=0}^{M}\left[\beta_{2+4i}\lambda_{2+4i}+\beta_{4+4i}\lambda_{4+4i}\right]
$$

Using the coefficients for β_j in Proposition 6 and the expression for the eigenvalues in equation (28) we write

$$
\frac{\partial}{\partial t}Y(t)|_{t=0} = -N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M}2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]-1\right) = -2N\ell(\phi)\lim_{M\to\infty}M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]-1\right)
$$

which diverges towards minus infinity for any $0 \le \phi < \infty$. □

Proof. (of Proposition 9) We let $\bar{\phi}$ to be the value of ϕ for which the impulse response, as a function of t, has a local maximum at $t = 0$. For larger values of ϕ the IRF will have an interior maximum at some $t > 0$, and hence the IRF will be hump shaped. For lower values of ϕ it will be monotonically decreasing in t. Thus we characterize the critical value of ϕ for which the IRF has a local maximum, and also verify that the second derivative with respect to time is negative at $t = 0$ for that critical value of ϕ. The slope of the impulse response function at $t = 0$ is given by:

$$\frac{\partial}{\partial t}Y(t)\Big|_{t=0} = \lim_{M\to\infty}\sum_{j=1}^{M}\left(\beta_j(\phi)+\beta_j^0(\phi)\right)\lambda_j(\phi) = \lim_{M\to\infty}\sum_{i=0}^{M}\left[\left(\beta_{2+4i}+\beta^0_{2+4i}\right)\lambda_{2+4i}+\left(\beta_{4+4i}+\beta^0_{4+4i}\right)\lambda_{4+4i}\right]$$

Using the coefficients for b_j in Proposition 6, the expression for $b_j^0(\phi)$ given in equation (31) and the expression for the eigenvalues λ_j in equation (28)

$$\begin{aligned}\frac{\partial}{\partial t}Y(t)\Big|_{t=0} &= -N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M}2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1\right)\\ &= -N\ell(\phi)\lim_{M\to\infty}2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1\right) \quad .\end{aligned}$$

Replacing the value of the value of the optimal decision rule for the plans model $\rho(\phi)$ in the previous expression we have

$$\begin{aligned}\frac{\partial}{\partial t}Y(t)\Big|_{t=0} &= -N\ell(\phi)\lim_{M\to\infty}2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[1-2\left[\frac{\exp\left(\sqrt{2\phi}\right)-\exp\left(-\sqrt{2\phi}\right)-2\sqrt{2\phi}}{\sqrt{2\phi}\left(\exp\left(\sqrt{2\phi}\right)+\exp\left(\sqrt{-2\phi}\right)-2\right)}\right]\right]-1\right)\\ &= -N\ell(\phi)\lim_{M\to\infty}2M\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right]\left[\frac{\sqrt{2\phi}\cosh\left(\sqrt{2\phi}\right)-2\sinh\left(\sqrt{2\phi}\right)+\sqrt{2\phi}}{\sqrt{2\phi}\left(\cosh\left(\sqrt{2\phi}\right)-1\right)}\right]-1\right) \quad .\end{aligned}$$

Thus, this expression is equal to zero for $\bar{\phi}$ solving:

$$\sqrt{2\phi}\left(\cosh\left(\sqrt{2\phi}\right)-1\right)^2 = \left[1+\cosh\left(\sqrt{2\phi}\right)\right]\left[\sqrt{2\phi}\cosh\left(\sqrt{2\phi}\right)-2\sinh\left(\sqrt{2\phi}\right)+\sqrt{2\phi}\right] \quad .$$

Analysis of this function shows that $\bar{\phi} = 1$. Next we check the value of the second derivative at $\phi = 1$; we have:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}Y(t)|_{t=0} &= \left[\left(\beta_{2+4i}+\beta^0_{2+4i}\right)\lambda^2_{2+4i}+\left(\beta_{4+4i}+\beta^0_{4+4i}\right)\lambda^2_{4+4i}\right]\\ &= N\ell(\phi)\lim_{M\to\infty}\sum_{i=0}^{M}2\left(\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))\,|\lambda_{2+4i}|-|\lambda_{4+4i}|\right) = -\infty\end{aligned}$$

since $|\lambda_{2+4i}| > |\lambda_{4+4i}$ and since: $\left[\frac{1+\cosh\left(\sqrt{2\phi}\right)}{\cosh(\sqrt{2\phi})-1}\right](1-2\rho(\phi))-1 = 0$ at $\phi = 1$. To summarize: at $\phi = 1$ the slope of the impulse response at $t = 0$ is zero and the second derivative is negative, thus it is a maximum. □

Proof. (of Proposition 10) First we consider case (i), i.e. the *long run* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$ on the impulse response of output to a monetary shock. We note that the expression for $Y(t)$ for the Golosov Lucas model does not feature $\bar{x}$, which is a function of σ (see equation (27)). Indeed the only place where σ enters in the expression for $Y(t)$ is in the eigenvalues (the parameter $N(j\pi)^2/8$ in equation (27)). Since, $N = \sigma^2/\bar{x}^2$ and $\bar{x} = \left(6\frac{\psi}{B}\sigma^2\right)^{\frac{1}{4}}$, then $d\log\bar{x} = 1/2 d\log\sigma$ and $d\log N = 2\left(d\log\bar{x} - d\log\sigma\right)$, hence $d\log N = d\log\sigma$. Substituting this into the eigenvalue $\lambda_j = -\tilde{N}(j\pi)^2/8 = -(1+\frac{d\sigma}{\sigma})N(j\pi)^2/8$ were N is the average number of price changes before the volatility shock. Using the expression for the impulse response in terms of the post-shock objects we have:

$$Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) = \sum_{j=1}^{\infty} b_j[f]b_j[\bar{p}']e^{-\left(1+\frac{d\sigma}{\sigma}\right)N\frac{(j\pi)^2}{8}t} = Y\left(t\left(1+\frac{d\sigma}{\sigma}\right);0,0\right)$$

and we obtain the desired result.

Now we consider short run effect, i.e. the *impact* effect of a volatility shock $\frac{d\sigma}{\sigma}$, so that $\tilde{\sigma} = \left(1 + \frac{d\sigma}{\sigma}\right)\sigma$ on the impulse response of output to a monetary shock. As in the previous case the eigenvalues can be written as functions of the shock and the old value of the expected number of price changes. Also as the previous case we have $f(x) = -x$. The difference is on the initial distribution $p(x,0)$. The initial condition is given by $p(x,0) = \bar{p}(x+\delta;\bar{x}(\sigma))$ where we write $\bar{x}(\sigma)$ to indicate that the distribution depends on σ. Indeed, since we are using the expression for $Y\left(t,0,\frac{d\sigma}{\sigma}\right)$ in terms of the value of $\bar{x}$ that corresponds to the post-shock value of σ, we need to consider the effect on $\bar{x}$ of a decrease of σ in the proportion $d\sigma/\sigma$. To do this we take a second order expansion of $p(x,0) = \bar{p}(x+\delta;\bar{x}(\sigma))$ with respect to δ and σ evaluated at $\delta = 0$ and $d\sigma = 0$.

$$\begin{aligned} p(x;0) \equiv \bar{p}(x+\delta;\bar{x}(\sigma)) &= \bar{p}(x) + \frac{\partial}{\partial\delta}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta - \frac{\partial}{\partial\bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma \\ &+ \frac{1}{2}\frac{\partial^2}{\partial\delta^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\delta^2 \\ &+ \frac{1}{2}\frac{\partial^2}{\partial\bar{x}^2}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\left(\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\right)^2 d\sigma^2 + \frac{1}{2}\frac{\partial}{\partial\bar{x}}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial^2\bar{x}(\sigma)}{\partial\sigma^2}d\sigma^2 \\ &- \frac{\partial^2}{\partial\bar{x}\partial\delta}\bar{p}(x+\delta;\bar{x}(\sigma))\big|_{\delta=0}\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma\delta + o\left(||(\delta,d\sigma)||^2\right) \end{aligned}$$

for $x \in [\underline{x},\bar{x}]$ and $x \neq 0$. Recall that the invariant distribution for this model is the triangular density $\bar{p}(x) = 1/\bar{x} - |x|/\bar{x}^2$ for $x \in (-\bar{x},\bar{x})$. Using this functional form we have:

$$\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x}) = \begin{cases} +\frac{1}{\bar{x}^2} & \text{if } x \in [-\bar{x},0) \\ -\frac{1}{\bar{x}^2} & \text{if } x \in (0,\bar{x}] \end{cases} \quad , \quad \frac{\partial}{\partial\bar{x}}\bar{p}(\delta+x;\bar{x})\big|_{\delta=0} = \begin{cases} -\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x},0) \\ +\frac{x}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0,\bar{x}] \end{cases}$$

$$\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x}) = \begin{cases} -\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in [-\bar{x},0) \\ +\frac{1}{\bar{x}^2}\frac{2}{\bar{x}} & \text{if } x \in (0,\bar{x}] \end{cases} \quad , \quad \frac{\partial^2}{\partial\bar{x}^2}\bar{p}(\delta+x;\bar{x}) = \begin{cases} +\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in [-\bar{x},0) \\ -\frac{x}{\bar{x}^2}\frac{6}{\bar{x}^2} & \text{if } x \in (0,\bar{x}] \end{cases}$$

Notice that the first order derivatives with respect to δ as well as the cross partial derivative are antisymmetric functions of x around $x = 0$, while the derivatives with respect to $\bar{x}$ are symmetric functions of x. Finally we have $\frac{\partial^2}{\partial \delta^2}\bar{p}(x+\delta;\bar{x}) = 0$.

Now we use the expansion and compute the impulse response coefficients $\beta_j \equiv b_j[f] b_j[p(\cdot,0)]$. The first order term for $d\sigma$ is zero because f is antisymmetric (so that $b_j[f] = 0$ for $j = 2,4,6,\ldots$) and the first derivative with respect to $\bar{x}$ is symmetric (so that $b_j[p(\cdot,0)] = 0$ for $j = 1,3,5,\ldots$) hence the $\beta_j = 0$ for $j = 1,2,3,4,\ldots$. Likewise the second order terms for $d\sigma^2$ are zero since f is antisymmetric and the first and second derivative with respect to $\bar{x}$ are symmetric. The second order term δ^2 is zero because the second derivative with respect to δ is zero. This leaves us with two non-zero terms. The first order term on δ, which is the term for the IRF with respect to a monetary shock, and the second order term corresponding to the cross-derivative. For the cross-partial term we note that, using that $\bar{x}$ has elasticity $1/2$ with respect to σ, we can write

$$
\begin{aligned}
-\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x})\frac{\partial\bar{x}(\sigma)}{\partial\sigma}d\sigma\delta &= -\frac{\partial^2}{\partial\delta\partial\bar{x}}\bar{p}(x+\delta;\bar{x})\bar{x}(\sigma)\left[\frac{\partial\bar{x}(\sigma)}{\partial\sigma}\frac{\sigma}{\bar{x}(\sigma)}\right]\frac{d\sigma}{\sigma}\delta \\
&= -\frac{2}{\bar{x}(\sigma)}\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x})\bar{x}(\sigma)\frac{1}{2}\frac{d\sigma}{\sigma}\delta \quad = -\frac{\partial}{\partial\delta}\bar{p}(\delta+x;\bar{x})\frac{d\sigma}{\sigma}\delta
\end{aligned}
$$

Thus we have that each β_j term is given by the sum of the (non-zero) terms corresponding to the first order term on δ and the second-order term corresponding to the cross-derivative:

$$
b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta + b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta\frac{d\sigma}{\sigma} = b_j[f]b_j\left[\bar{p}'(\cdot)\right]\delta\left(1+\frac{d\sigma}{\sigma}\right)
$$

This gives the projection coefficients for the short run impact that appear in equation (33) in the proposition. In particular it shows that the coefficients for the short run are equal to the ones for the long-run multiplied by the factor $(1 + d\sigma/\sigma)$.

Finally, we consider the case of a monetary shock that occurs s periods after the volatility shock. We proceed in three steps.

Step 1: find initial signed measure $\hat{p}(x,p)$. For a small σ shock, the signed measure $\hat{p}(x,0)$ right after the uncertainty shock is given by

$$
\hat{p}(x,0) \equiv \bar{p}(x;\bar{x}(\sigma)) - \bar{p}(x;\bar{x}(\tilde{\sigma})) = \bar{p}(x;\bar{x}(\sigma)) - \bar{p}(x;\bar{x}(\sigma+d\sigma))
$$

which is by the difference between the original invariant distribution and the new long run distribution. We now take an expansion around the original invariant distribution and write

$$
\hat{p}(x,0) = \bar{p}(x;\bar{x}(\sigma)) - \bar{p}(x;\bar{x}(\sigma)) - \bar{p}_{\bar{x}}(x;\bar{x}(\sigma))\bar{x}'d\sigma + o(d\sigma)
$$

where $\bar{p}_{\bar{x}}$ is the derivative of the density function with respect to $\bar{x}$. For the pure menu cost model we have $\bar{p}(x,\bar{x}) = 1/\bar{x} - (1/\bar{x}^2)|x|$ for $x \in (\bar{x},\bar{x})$ so we have:

$$
\bar{p}_{\bar{x}}(x,\bar{x}) = \frac{1}{\bar{x}^2}\left(-1+\frac{2|x|}{\bar{x}}\right) \quad \text{and} \quad \frac{\partial}{\partial\sigma}\bar{x}(\sigma) = \frac{1}{2}\frac{\bar{x}}{\sigma}
$$

where we use that $\bar{x}(\sigma) = (6\psi/B\sigma^2)^{1/4}$. Replacing into the expression for $\hat{p}$ we have

$$\hat{p}(x,0) = -\bar{p}_{\bar{x}}(x;\bar{x}(\sigma))\bar{x}'d\sigma + o(d\sigma) = \frac{1}{\bar{x}}\left(\frac{|x|}{\bar{x}} - \frac{1}{2}\right)\frac{d\sigma}{\sigma} + o(d\sigma) \text{ for } \; x \in (-\bar{x},\bar{x})$$

Step 2: find signed measure after s periods $\hat{p}(x,s)$. The function $\hat{p}(x,s)$ describes the evolution of this signed measure s periods after the uncertainty shock. We use our characterization of the density of transition function $\sum_j \exp(\lambda_j s)\,\phi_j(x_s)\phi(x_0)$ between time $t=0$ and $t=s$ with eigenfunctions φ_j and eigenvalues λ_j with $\bar{x}(\tilde{\sigma})$ and $N = \tilde{\sigma}^2/\bar{x}(\tilde{\sigma})^2$ to construct the evolution of $\hat{p}(x,s)$. We represent the signed measure (deviation from the invariant distribution) as follows

$$\hat{p}(x,s) = \frac{d\sigma}{\sigma} \sum_{j=1,3,5,...} e^{\lambda_j s}\varphi_j(x)\, b_j[\hat{p}(\cdot,0)] \tag{3}$$

where the projection coefficients $b_j[\hat{p}] = 0$ for $j = 2,4,6,...$ since the function $\hat{p}(x,0)$ is a symmetric function while the even-indexed φ_j functions are antisymmetric. The non zero coefficients are

$$b_j[\hat{p}(\cdot,0)] = \int_{-\bar{x}}^{\bar{x}} \varphi_j(x)\hat{p}(x,0)dx = 2\int_0^{\bar{x}} \frac{1}{\bar{x}}\left(\frac{x}{\bar{x}} - \frac{1}{2}\right)\varphi_j(x)dx \quad \text{for} \quad j = 1,3,5,...$$

Direct calculation gives

$$b_j[\hat{p}(\cdot,0)] = \frac{2}{\bar{x}^{1/2}}\,\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\,\pi)^2} \quad \text{for} \quad j = 1,3,5,... \tag{4}$$

Step 3: Find excess impulse response $Y(t;s,\frac{d\sigma}{\sigma}) - Y(t;\infty,\frac{d\sigma}{\sigma})$. The cross section distribution right after the monetary shock is

$$p(x+\delta,s;\tilde{\sigma}) = \bar{p}(x+\delta;\tilde{\sigma}) + \hat{p}(x+\delta,\tau)$$

the first term is the invariant distribution (under the new variance $\tilde{\sigma}$) which will settle in the long run, the second term is the deviation between the current cross-section distribution and the invariant, discussed above. Then

$$p(x+\delta,s;\tilde{\sigma}) - \bar{p}(x;\tilde{\sigma}) \approx \delta\left(\bar{p}'(x;\tilde{\sigma}) + \hat{p}'(x,s)\right)$$

We let $Y(t;\infty,\frac{d\sigma}{\sigma}) = \sum_{k=1}^{\infty} e^{\lambda_{2k}t}\, b_{2k}[f]b_{2k}[\bar{p}'(\cdot;\tilde{\sigma})]$ be the long run output response to a monetary shock, after the initial uncertainty shock has settled down (i.e. for $s \to \infty$). Our main proposition implies that impulse response to a monetary shock s periods after the uncertainty shock is

$$Y\left(t;s,\frac{d\sigma}{\sigma}\right) = \sum_{k=1}^{\infty} e^{\lambda_{2k}t}\, b_{2k}[f]b_{2k}[\hat{p}'(\cdot,s)] \; + Y\left(t;\infty,\frac{d\sigma}{\sigma}\right) \tag{5}$$

Note that the above summation only uses even-indexed eigenfunctions since the function of interest for the output $f(x) = -x$ is antisymmetric, we know that all $b_{2k+1}[f] = 0$ for $k = 1, 2, 3, ...$

Now we turn to the computation of $b_{2k}[\hat{p}'(\cdot, \tau)]$, given by$b_{2k}[\hat{p}'(\cdot, \tau)] \equiv \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x)\hat{p}'(x, s)dx$. Note that from equation (2) we can write $\hat{p}'(\cdot, \tau)$ as

$$\hat{p}'(x, s) = \frac{d\sigma}{\sigma} \sum_{j=1,3,5,...} e^{\lambda_j s} \varphi_j'(x)\, b_j[\hat{p}(\cdot, 0)]$$

Using equation (3) and the form of the eigenfunctions:

$$b_{2k}[\hat{p}'(\cdot, s)] = \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x)\hat{p}'(x, s)dx = \frac{d\sigma}{\sigma} \sum_{j=1,3,5,...} e^{\lambda_j s} b_j[\hat{p}(\cdot, 0)] b_{2k}[\varphi_j'] \quad , \quad k = 1, 2, 3, ...$$

Direct computation for $k = 1, 2, 3, \ldots$ and $j = 1, 3, 5, \ldots$ gives

$$\begin{aligned} b_{2k}[\varphi_j'] &\equiv \int_{-\bar{x}}^{\bar{x}} \varphi_{2k}(x)\varphi_j'(x)dx \\ &= \frac{j\pi}{2\bar{x}} \int_{-\bar{x}}^{\bar{x}} \sin\left(k\pi\left(\frac{x+\bar{x}}{\bar{x}}\right)\right) \cos\left(j\pi\left(\frac{x+\bar{x}}{2\bar{x}}\right)\right) dx = \frac{4kj}{\bar{x}(4k^2 - j^2))} \end{aligned}$$

and hence

$$b_{2k}[\hat{p}'(\cdot, s)] = \frac{d\sigma}{\sigma} \frac{1}{\bar{x}^{3/2}} \sum_{j=1,3,5,...}^{\infty} e^{\lambda_j s} \left(2\frac{4(-1)^{\frac{j+3}{2}} - j\pi}{(j\,\pi)^2}\right) \left(\frac{4kj}{(4k^2 - j^2))}\right) \quad , \quad k = 1, 2, 3, ...$$

□

B Sticky Price Multiproduct firms

Multiproduct models consider a firm that produces n different products and that faces increasing returns in the price adjustment: if a pays a fixed cost it can adjust simultaneously the n prices. Variations on this model have been studied by Midrigan (2011) and Bhattarai and Schoenle (2014). These models are appealing because they match several empirical regularities: synchronization among price changes within a store and the coexistence of both small and large price changes. Their economic analysis is of interest because in an economy populated by multiproduct firms the monetary shocks have more persistent real effects. In Alvarez and Lippi (2014) we derived results for impulse responses to this multidimensional setup and explore the sense in which such a model is realistic. Here we show that the characterization of the selection effect, as the difference between the survival function and the output IRF holds in this model, with the number of products n serving as the parameter that control selection. We also show that in this case a single eigenvalue gives a poor characterization of the output IRF.

In the multiproduct model the price gap is given by a vector of n price gaps, each of them given by an independently standard BM's $(p_1, p_2, \ldots p_n)$, driftless and with innovation variance σ^2. We are interested only on two functions of this vector, the sum of its squares and it sum:

$$y = \sum_{i=1}^{n} p_i^2 \text{ and } z = \sum_{i=1}^{n} p_i$$

It is interesting to notice that while the original state is n dimensional, (y, z) can be described as a two dimensional diffusion –see Alvarez and Lippi (2014) and Appendix B.1 for details.

We are interested in the sum of its squares y because in Alvarez and Lippi (2014) under the assumption of symmetric demand the optimal decision rule is to adjust the firm time that y hits a critical value $\bar{y}$. We are interested in z, the sum of the price gaps, because this give the contribution of firm to the deviation of the price level relative to the steady state value, and hence $-z$ is proportional to its contribution to output. Note that the domain of (y, z) is $0 \leq y \leq \bar{y}$ and $-\sqrt{ny} \leq z \leq \sqrt{n\,y}$. In Alvarez and Lippi (2014) we show that the expected number of adjustments per unit of time is given by $N = \frac{n\sigma^2}{\bar{y}}$ and also give a characterization of $\bar{y}$ in terms of the parameters for the firm's problem. For the purpose in this paper we find it convenient to rewrite the state as (x, w) defined as

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}.$$

In Lemma 1 in Appendix B.1 we analyze the behavior of the $(x, w) \in [0, \bar{x}] \times [-1, 1]$ process with $\bar{x} \equiv \sqrt{\bar{y}}$. Clearly we can recover (y, z) from (x, w). For instance, $z = w\sqrt{nx}$. Yet with this change on variables, even though the original problem is n dimensional, we define a two dimensional process for which we can analytically find its associated eigenfunctions and eigenvalues for the operator:

$$\mathcal{G}(f)\left((x, w), t\right) = \mathbb{E}\left[f\left(x(t), w(t)\right)\, 1_{y \geq \bar{y}} \,\middle|\, (x(0), w(0)) = (x, w) \right]$$

where $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$. The relevant p.d.e. is defined and its solution via eigenfunctions and eigenvalues, is characterized in Proposition 13 in Appendix B.1. Moreover the eigenfunctions and eigenvalues are indexed by a countably double infinity indices $\{m, k\}$.

Eigenfunctions. The eigenfunctions φ have a multiplicative nature, so $\varphi_{m,k}(x, w) = h_m(w) g_{m,k}(x)$ where for each number of products n then h_m and $g_{m,k}$ are known analytic functions indexed by k and by (k, m) respectively. Indeed h_m are scaled Gegenbauer polynomials, and $g_{m,k}$ are scaled Bessel functions –see Proposition B.1 for the exact expressions and definition.[1]

Eigenvalues. For each n the eigenvalues can be also indexed by a countably double-infinity $\{\lambda_{m,k}\}$. As in the baseline case, the eigenvalues are proportional to N, the expected number of price changes per unit of time:

$$\lambda_{m,k} = -N \, \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\, n} \quad \text{for } m = 0, 1, \ldots, \text{ and } k = 1, 2, \ldots$$

[1]The Gegenbauer polynomials are orthogonal to each other, and so are the Bessel functions when using an appropriately weighted inner product, as defined in Appendix B.1.

$j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

The second sub-index k in the root of the Bessel function denote their ordering, so $k = 1$ is the smallest positive root. Also fixing k the roots $j_{m+\frac{n}{2}-1,k}$ are increasing in m. Thus, the *dominant* eigenvalue is given by $\lambda_{0,1}$. We will argue below that the smallest (in absolute value) eigenvalue that is featured in the (marginal) output IRF is $\lambda_{1,1}$. A very accurate approximation of the eigenvalues consists on using the first three leading terms in its expansion, as is given by: $j_{\nu,k} \approx \nu + \nu^{1/3}2^{-1/3}a_k + (3/20)(a_k)^2 2^{1/3}\nu^{-1/3}$ where a_k are the zeros of the Airy function. [2] Using this approximation into the expression for the eigenvalues, one can see that keeping fixed N, the absolute value both $\lambda_{0,1}$ and $\lambda_{1,1}$ go to infinity, and that the difference between the two decreases and converges to $N/2$. Figure 1 displays the difference between these two eigenvalues.

Impulse response. As before, we want to compute $G(t)$, the conditional expectation of $f : [0, \bar{x}] \times [-1, 1] \to \mathbb{R}$ for (x, w) following equation (6)-equation (7), integrated with respect to $p(w, x; 0)$. We are interested in functions $f : [0, \bar{x}] \times [-1, 1]$ that can be written as:

$$f(x, w) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{m,k}[f] \varphi_{m,k}(x, w)$$

Using the same logic as in the one dimensional case:[3]

$$G(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} e^{\lambda_{m,k}\, t}\, b_{m,k}[f]\, b_{m,k}\left[p(\cdot, 0)/\omega\right] \langle \varphi_{m,k}, \varphi_{m,k} \rangle$$

where the term $\langle \varphi_{m,k}, \varphi_{m,k} \rangle$ appears because the have, as it is customary in this case, use an orthogonal, but not orthonormal base, and where $\omega(w, x)$ is a weighing function appropriately defined – see Appendix B.1. So that $b_{m,k}\left[p(\cdot, 0)/\omega\right]$ are the projections of the ratio of the functions $p(\cdot, 0)$ and ω.

Functions of interest. We analyze two important functions of interest f. The first one a constant, $f(w, x) = 1$ which is used to compute the measure of firms that have not adjusted, or the survival function $S(t)$. The second one is the one that gives the average price gap among the n product of the firm, i.e. $f(w, z) = -z/n = -wx/\sqrt{n}$. This is, as before, the negative of the average across the n products of the price gaps. This is the function f used for the impulse response of output to a monetary shock. An important property of the Gegenbauer polynomials is that the $m = 0$ equals a constant, for $m = 1$ is proportional to w, and in general for m odd are antisymmetric on w and symmetric for even m. Thus for $f = 1$ we can use just the Gegenbauer polynomial with $m = 0$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$. Instead for $f(w, x) = wx/sqrtn = z$ we can use just the Gegenbauer polynomial with $m = 1$ and all the Bessel functions corresponding to $m = 0$ and $k \geq 1$.

Initial shifted distribution for a small shock. We have derived the invariant distribution of (z, y) in Alvarez and Lippi (2014). Using the change in variables (y, z) to $y = x^2$ and $z = \sqrt{yn}\, w =$ we can define the steady state density as $\bar{p}(w, x) = \bar{h}(w)\bar{g}(x)$ – see Appendix B.1

[2] In our case, we are interested in $k = 1$ which is about $a_1 = -2.33811$. See Figure 2 in the APP where we plots both eigenvalues, as well as its approximation for several n.

[3] See Appendix B.1 for a derivation

for the expressions. We perturb this density with a shock of size δ in each of the n price gaps. We want to subtract δ to each component of $(p_1, ..., p_n)$. This means that the density for each $x = ||p||$ just after the shock becomes the density of $x(\delta) = ||(p_1 + \delta, \ldots, p_n + \delta)||$ just before. Likewise the density corresponding to each w becomes th one for $w(\delta) = (z + n\delta)/(\sqrt{n}\, x(\delta))$. We consider the initial condition given by density $p_0(w, x; \delta) = \bar{h}(w(\delta))\bar{g}(x(\delta))$. We will use the first order terms, which are appropriate for the case of a small shock δ. The expressions can be found in Appendix B.1.

Interpretation of dominant eigenvalue, and irrelevance for the marginal IRF. We are now ready to generalize our interpretation of the dominant eigenvalue (as well as those corresponding to symmetric functions of z), as well as its irrelevance for the marginal output IRF.

Proposition 11. The coefficient of the marginal impulse response of output for a monetary shock are a function of the $\{\lambda_{1,k}, \varphi_{1,k}\}_{k=1}^{\infty}$ eigenvalue-eigenfunctions pairs, so that:

$$Y(t) = \sum_{k=1}^{\infty} \beta_{1,k}\, e^{\lambda_{1,k}\, t} \quad \text{and} \quad -\lambda_{1,1} = \lim_{t\to\infty} \frac{\log|Y(t)|}{t}$$

where $\beta_{1,k} = b_{1,k}\,[wx/\sqrt{n}]\; b_{1,k}\,[\bar{p}'(w,x)]$. In particular, the dominant eigenvalue $\lambda_{0,1}$ does not characterize the limiting behavior of the impulse response. Instead the survival function for price changes $S(t)$, can be written in terms of $\{\lambda_{0,k}, \varphi_{0,k}\}_{k=1}^{\infty}$, and hence the asymptotic hazard rate is equal to the dominant eigenvalue $\lambda_{0,1}$, i.e.

$$S(t) = \sum_{k=1}^{\infty} \beta_{0,k}\, e^{\lambda_{0,k}\, t} \quad \text{and} \quad -\lambda_{0,1} = \lim_{t\to\infty} \frac{\log S(t)}{t}$$

where $\beta_{0,k} = b_{0,k}\,[1]\; b_{0,k}\,[\delta_0]$ where δ_0 is the Dirac delta function for $(p_1, \ldots, p_n)$ transformed to the (x, w) coordinates. Recall that $0 > \lambda_{0,1} > \lambda_{1,1}$.

Given the importance of the difference between the eigenvalues $\lambda_{1,k}$ and $\lambda_{0,k}$ we show that for a fixed k they both increase with n, but it difference decreases to asymptote to $1/2$.

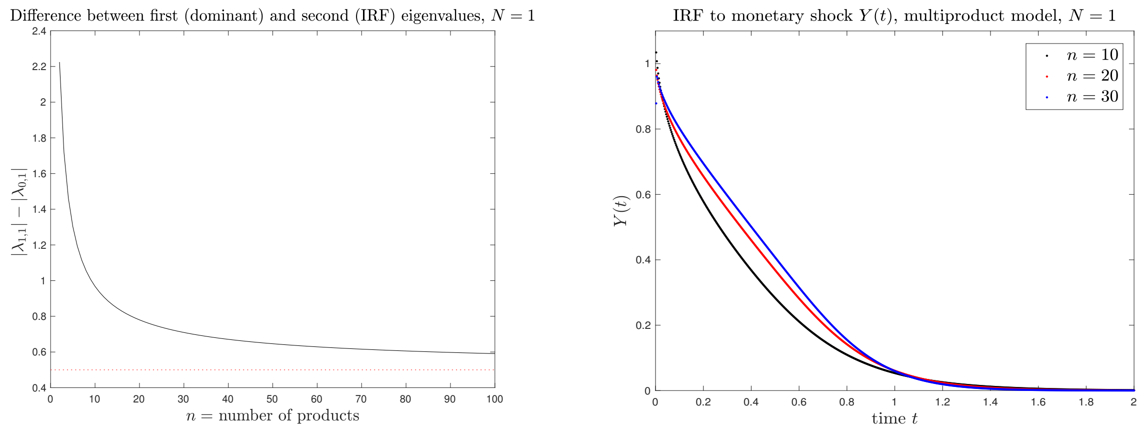
Proposition 12. Fixing $k \geq 1$, the k^{th} eigenvalue for the IRF $Y(\cdot)$ given by $\lambda_{1,k}$ and the k^{th} eigenvalue for the survival function $S(\cdot)$ given by $\lambda_{1,k}$ both increase with the number of products n, diverging towards $-\infty$ as $n \to \infty$. The difference $\lambda_{0,k} - \lambda_{1,k} > 0$ decreases with n, converging to $1/2$ as $n \to \infty$.

Figure 1 illustrates Proposition 12 for the case of $k = 1$, i.e. the eigenvalue that dominates the long run behaviour of the survival and IRF functions. Proposition 12 extends the result for all k. Increasing the number of products n in the multi product model decreases the selection effect at the time of a price change. As n goes to infinity, the eigenvalues that control the duration of the price changes (S) and those that control the marginal output IRF (Y) converge. This result shows that the characterization of selection effect in terms of dynamics controlled by two different types of eigenvalues is present not only in the Calvo$^+$ model, but also in this setup.

In Appendix B.1 we include Proposition 14 which gives a closed form solution for $\bar{p}'(w, x; 0)$

Figure 1: Shock propagation in Multiproduct models

Selection effect | Impulse response



Keeping fixed $N = 1$ for all n

and for the coefficients for $b_{1,k}$ of the output impulse response function. All these expressions depends only of the number of products n. Instead we include a figure of the impulse responses for three values of n. It is clear both the output IRF and the survival function cannot be well described using one eigenfunction-eigenvalue for large n. For instance, as $n \to \infty$ the output's IRF Y becomes a linearly declining function until it hits zero at $t = 1/N$, and the survival function S is zero until it becomes infinite at $t = 1/N$.

B.1 Details of the multiproduct model

Law of motion for y, z.

$$dy = \sigma^2 n \, dt + 2\sigma\sqrt{y}\, dW^a$$
$$dz = \sigma\sqrt{n}\left[\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}}\right)^2}\, dW^b\right]$$

where W^a, W^b are independent standard BM's.

Lemma 1. Define

$$x = \sqrt{y} \text{ and } w = \frac{z}{\sqrt{ny}}$$

so that the domain is $0 \le x \le \bar{x} \equiv \sqrt{\bar{y}}$ and $-1 \le w \le 1$. They satisfy:

$$dx = \sigma^2 \frac{n-1}{2\,x} dt + \sigma dW^a \tag{6}$$

$$dw = \frac{w}{x^2}\left(\frac{1-n}{2}\right) dt + \frac{\sqrt{1-w^2}}{x} dW^b \tag{7}$$

We look for a solution to the eigenvalue-eigenfunction problem (λ, φ) given by equation (6) and equation (7). They must satisfy

$$\lambda\varphi(w,x) = \varphi_x(w,x)\sigma^2\left(\frac{n-1}{2x}\right) + \varphi_w(w,x)\frac{w}{x^2}\left(\frac{1-n}{2}\right) + \frac{1}{2}\varphi_{ww}(w,x)\frac{(1-w^2)}{x^2} + \frac{1}{2}\sigma^2\varphi_{xx}(w,x)$$

for all $(x,w) \in [0,\bar{x}] \times [-1,1]$, with $\varphi(\bar{x},w) = 0$, all w and φ^2 integrable.

Proposition 13. The eigenfunctions-eigenvalues of (w,x) satisfying equation (6)-equation (7) denoted by $\{\varphi_{m,k}(\cdot), \lambda_{m,k}\}$ for $k = 1, 2, \ldots$ and $m = 0, 1, \ldots$ are given by:

$$\begin{aligned} \varphi_{m,k}(x,w) &= h_m(w)\, g_{m,k}(x) \quad \text{where} \\ h_m(w) &= C_m^{\frac{n}{2}-1}(w) \quad \text{for } m = 0,1,2,\ldots \text{ and} \\ g_{m,k}(x) &= x^{1-n/2} J_{\frac{n}{2}-1+m}\left(j_{\frac{n}{2}-1+m,k}\,\frac{x}{\bar{x}}\right) \quad \text{for } k = 1,2,\ldots \text{ and} \\ \lambda_{m,k} &= -N\,\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,n} \quad \text{for } m = 0,1,\ldots, \text{ and } k = 1,2,\ldots \end{aligned}$$

where $C_m^{\frac{n}{2}-1}(\cdot)$ denote the Gegenbauer polynomials, and where $J_{\frac{n}{2}-1+m}(\cdot)$ denote the Bessel function of the first kind, $j_{\nu,k}$ denote the ordered zeros of the Bessel function of the first kind $J_\nu(\cdot)$ with index ν.

Note that the expressions for the eigenfunctions are only valid only for $n > 2$. For $n = 2$ the expression take a different special form, which we skip to save space. The expressions for the eigenvalues are valid for $n \ge 2$.

We remind the reader how the Gegenbauer polynomial and Bessel function, which form an orthogonal base, are defined. The Gegenbauer polynomial $C_m^{\frac{n}{2}-1}(w)$ is given by:

$$C_m^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{\Gamma(m-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(m-2k)!}(2w)^{m-2k} \tag{8}$$

For a fixed n, the polynomials are orthogonal on with respect to the weighting function

$(1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}$ so that:[4]

$$\int_{-1}^{1} C_m^{(\frac{n}{2}-1)}(w)\, C_j^{(\frac{n}{2}-1)}(w)\, (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\, dw = 0 \text{ for } m \neq j \tag{9}$$

and for $m = j$ we get

$$\int_{-1}^{1} \left[C_m^{(\frac{n}{2}-1)}(w)\right]^2 (1-w^2)^{\frac{n}{2}-1-\frac{1}{2}}\, dw = \frac{\pi\, 2^{1-2(\frac{n}{2}-1)}\Gamma(m+2(\frac{n}{2}-1))}{m!(m+\frac{n}{2}-1)[\Gamma(\frac{n}{2}-1)]^2} \tag{10}$$

The Bessel function of the first kind is given by :

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu} \tag{11}$$

For a given ν, the following functions are orthogonal, using the weighting function x^{n-1} so that:[5]

$$\begin{aligned}
&\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right] \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right)\right] x^{n-1}\, dx \\
=&\int_0^{\bar{x}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right) J_\nu\left(j_{\nu,s}\frac{x}{\bar{x}}\right) x\, dx = 0 \text{ if } k \neq s \in \{1,2,3,\dots\} \text{ and} \\
&\int_0^{\bar{x}} \left[x^{1-\frac{n}{2}} J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 x^{n-1}\, dx = \bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu\left(j_{\nu,k}\frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} &(12)\\
=&\frac{1}{2}\left(\bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\right)^2 \text{ for all } k \in \{1,2,3\dots\} &(13)
\end{aligned}$$

where $j_{\nu,k}$ and $j_{\nu,s}$ are two zeros of $J_\nu(\cdot)$.

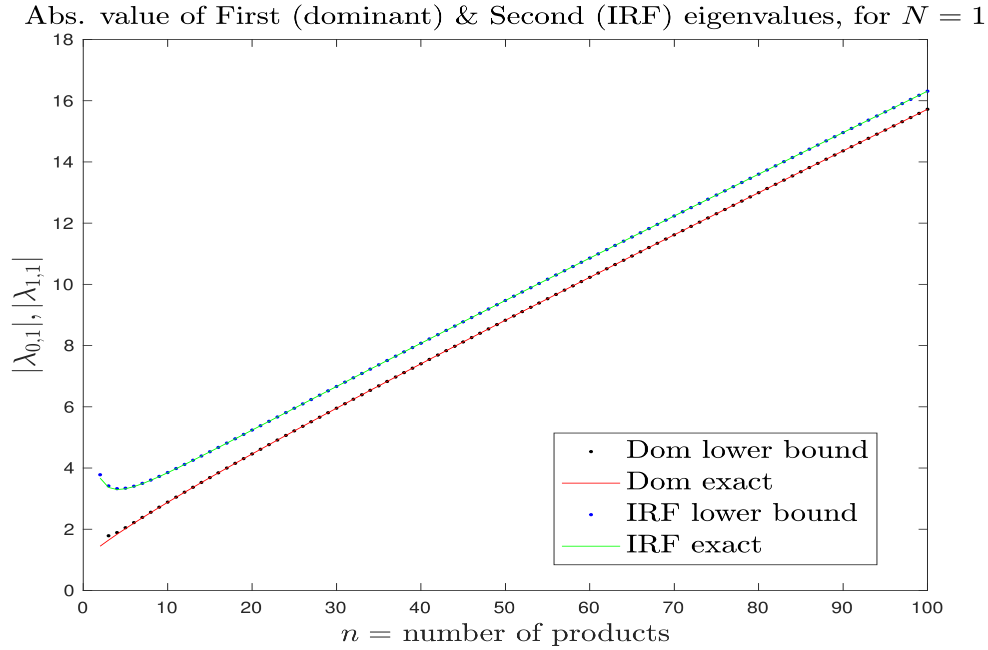
Derivation of IRF. Thus we have

$$G(t) \equiv \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G}(f)(x,w,t)\, p(x,w;0) dw\, dx$$

[4] By this we mean that we define the inner product between functions a, b from $[-1,1]$ to $\mathbb{R}$ as : $\langle a,b\rangle = \int_{-1}^{1} a(w)b(w)\left(1-w^2\right)^{\frac{n}{2}-1-\frac{1}{2}} dw$.

[5] By this we mean that we define the inner product between functions a, b from $[0,\bar{x}]$ to $\mathbb{R}$ as: $\langle a,b\rangle = \int_0^{\bar{x}} a(x)b(x)x^{n-1}dx$.

Figure 2: Eigenvalues for multiproduct model



Kepping fixed $N = 1$ for all n

As in Section 4, we can write this expected value as:

$$\begin{aligned} Y(t) &= \int_0^{\bar{x}} \int_{-1}^{1} \mathcal{G}\left(\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} b_{m,k}[f]\,\varphi_{k,m}\right)(x,w,t)\; p(x,w;0)dw\,dx \\ &= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} b_{m,k}[f]\,\mathcal{G}\left(\varphi_{k,m}\right)(x,w,t)\; p(x,w;0)dw\,dx \\ &= \int_0^{\bar{x}} \int_{-1}^{1} \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} b_{m,k}[f]\,e^{\lambda_{m,k}\,t}\,\varphi_{m,k}(x,w)\,p(x,w;0)dw\,dx \\ &= \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} e^{\lambda_{m,k}\,t}\,b_{m,k}[f]\;\int_0^{\bar{x}} \int_{-1}^{1} \varphi_{m,k}(x,w)\,p(x,w;0)dw\,dx \end{aligned}$$

Then we get:

$$G(t) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} e^{\lambda_{m,k}\,t}\,b_{m,k}[f]\,b_{m,k}\left[p(\cdot,0)/\omega\right]\,\langle\varphi_{m,k},\varphi_{m,k}\rangle$$

Inner product. We let $\omega(w,x) = x^{1-n}\,(1-w^2)^{\frac{n-3}{2}}$. The inner product of functions a, b

from $[0,\bar{x}] \times [-1,1]$ to $\mathbb{R}$ is defied as

$$\langle a,b\rangle = \int_0^{\bar{x}} \int_{-1}^{1} a(x,w)\, b(x,w)\, x^{1-n}\, (1-w^2)^{\frac{n-3}{2}}\, dw\, dx$$

The term $\langle \varphi_{m,k}, \varphi_{m,k}\rangle$ is given by the product of equation (10) and equation (13) found above. Indeed since the polynomials are orthogonal we have:

$$b_{m,k}[f] = \frac{\langle f, \varphi_{m,k}\rangle}{\langle \varphi_{m,k}, \varphi_{m,k}\rangle} = \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) h_m(w)(1-w^2)^{\frac{n-3}{2}} dw\right] g_{m,k}(x)\, x^{n-1}\, dx}{\left[\int_{-1}^{1} (h_m(w))^2\, (1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} (g_{m,k}(x))^2\, x^{n-1}\, dx\right]}$$
$$= \frac{\int_0^{\bar{x}} \left[\int_{-1}^{1} f(x,w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}} dw\right] J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\, \frac{x}{\bar{x}}\right) x^{\frac{n}{2}} dx}{\left[\int_{-1}^{1} \left(C_m^{\frac{n}{2}-1}(w)\right)^2 (1-w^2)^{\frac{n-3}{2}} dw\right] \left[\int_0^{\bar{x}} \left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\, \frac{x}{\bar{x}}\right)\right)^2 x\, dx\right]}$$

Invariant Distribution. After the change in variables we have:

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \left(1-w^2\right)^{(n-3)/2} \quad \text{for} \quad w \in (-1\,,\,1) \tag{14}$$
$$\bar{g}(x) = x\ (\bar{x})^{-n} \left(\frac{2n}{n-2}\right) \left[\,\bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0,\bar{x}] \tag{15}$$

Initial distribution after a small monetary shock.

$$p(w,x;0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w,x;0)\delta + o(\delta) \text{ with}$$
$$\bar{p}'(w,x;0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0)$$

where:

$$\frac{\partial}{\partial\delta} x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\, w \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0)$$
$$\frac{\partial}{\partial\delta} w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0)$$

PROPOSITION 14. The expressions for $\bar{p}'(x,w;0)$ and the coefficients $b_{1,k}(n)$ for the impulse response of output are given by:

$$\bar{p}'(w,x;0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0)$$
$$= \frac{w\,(1-w^2)^{(n-3)/2}}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \sqrt{n} \left(\frac{2n}{n-2}\right) \frac{[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}]}{\bar{x}^n}$$

and the coefficients for the impulse response $b_{1,k}(n) = b_{1,k}[f]\, b_{1,k}[\bar{p}'(\cdot,0)/\omega]\, \langle \varphi_{1,k}, \varphi_{1,k}\rangle$ are given by

$$b_{1,k}(n) = -\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \left[(4-n)\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} - \frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right.$$
$$\left. -(4+n)2^{-1-\frac{n}{2}}(j_{\frac{n}{2},k})^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\,{}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]$$

where ${}_1\tilde{F}_2(a_1; b_1, b_2; z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1; b_1, b_2; z) = {}_1F_2(a_1; b_1, b_2; z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function and $j_{\frac{n}{2},k}$ is the k^{th} ordered zero of the Bessel function $J_{\frac{n}{2}}(\cdot)$.

Note that, as our notation emphasizes, the coefficients $b_j(n)$ depends only on the number of products.

B.2 Proofs for the Multiproduct model of Appendix B.1

Proof. (of Lemma 1) Using Ito's lemma we have: $dx = (1/2)y^{-1/2}dy - (1/2)(1/4)y^{-3/2}dy^2$ which gives

$$dx = \frac{n-1}{2\,x}dt + dW^a$$

and $w = f(y,z) = z/\sqrt{ny}$. We have:

$$dw = f_y dy + f_z dz + \frac{1}{2}f_{yy}(dy)^2 + \frac{1}{2}f_{zz}(dz)^2 + f_{yz}dydz$$

where $f = (z/\sqrt{n})\, y^{-1/2}$, and thus

$$f_y = -\frac{z}{2\sqrt{n}}y^{-3/2}$$
$$f_z = \frac{1}{\sqrt{n}}y^{-1/2}$$
$$f_{yy} = \frac{3z}{4\sqrt{n}}y^{-5/2}$$
$$f_{zz} = 0$$
$$f_{yz} = -\frac{1}{2\sqrt{n}}y^{-3/2}$$

We thus have:

$$
\begin{aligned}
dw = & -\frac{z}{2\sqrt{n}} y^{-3/2} \left(n dt + 2\sqrt{y}\, dW^a \right) \\
& + \frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \left(\frac{z}{\sqrt{ny}}\, dW^a + \sqrt{1 - \left(\frac{z}{\sqrt{ny}} \right)^2}\, dW^b \right) \\
& + \frac{1}{2} \frac{3z}{4\sqrt{n}} y^{-5/2} 4y dt - \frac{1}{2\sqrt{n}} y^{-3/2} 2z dt
\end{aligned}
$$

which we can rearrange as:

$$
\begin{aligned}
dw = & \frac{z}{\sqrt{n}} y^{-3/2} \left(\frac{1-n}{2} \right) dt \\
& + \left(\frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \frac{z}{\sqrt{n}\sqrt{y}} - \frac{z}{2\sqrt{n}} y^{-3/2} 2\sqrt{y} \right) dW^a \\
& + \frac{1}{\sqrt{n}} y^{-1/2} \sqrt{n} \sqrt{1 - \left(\frac{z}{\sqrt{n}\sqrt{y}} \right)^2}\, dW^b
\end{aligned}
$$

or

$$
\begin{aligned}
dw &= \frac{w}{x^2} \left(\frac{1-n}{2} \right) dt + \left(\frac{z}{\sqrt{ny}} - \frac{z}{\sqrt{ny}} \right) dW^a + \frac{1}{x} \sqrt{1 - (w)^2}\, dW^b \\
&= \frac{w}{x^2} \left(\frac{1-n}{2} \right) dt + \frac{1}{x} \sqrt{1 - w^2}\, dW^b
\end{aligned}
$$

□

Proof. (of Proposition 13) We try a multiplicative solution of the form:

$$\varphi(w, x) = h(w)\, g(x)$$

To simplify the proof we set $\sigma^2 = 1$. Thus

$$
\begin{aligned}
\lambda h(w) g(x) = & h(w) g'(x) \left(\frac{n-1}{2x} \right) + h'(w) g(x) \frac{w}{x^2} \left(\frac{1-n}{2} \right) \\
& + \frac{1}{2} h''(w) g(x) \frac{(1-w^2)}{x^2} + \frac{1}{2} h(w) g''(x)
\end{aligned}
$$

Dividing by $h(w)$ in both sides we have:

$$
\begin{aligned}
\lambda g(x) = & g'(x) \left(\frac{n-1}{2x} \right) + \frac{h'(w) w}{h(w)} \frac{g(x)}{x^2} \left(\frac{1-n}{2} \right) \\
& + \frac{1}{2} \frac{h''(w)\,(1-w^2)}{h(w)} \frac{g(x)}{x^2} + \frac{1}{2} g''(x)
\end{aligned}
$$

Collecting terms:

$$\lambda g(x) = \frac{g(x)}{x^2}\left[\frac{h'(w)w}{h(w)}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\left(1-w^2\right)}{h(w)}\right]$$
$$+ g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

Which suggests to try the following separating variable:

$$\mu = \frac{h'(w)w}{h(w)}\left(\frac{1-n}{2}\right) + \frac{1}{2}\frac{h''(w)\left(1-w^2\right)}{h(w)}$$

or

$$0 = -2\mu h(w) + h'(w)w\,(1-n) + h''(w)\left(1-w^2\right)$$

The solution of this equation is given by the Gegenbauer polynomials $C_m^{\alpha}(w)$. The Gegenbauer polynomials are the solution to the following o.d.e.:

$$(1-w^2)h(w)'' - (2\alpha+1)wh'(w) + m(m+2\alpha)h(w) = 0 \text{ for } w \in [-1,1]$$

for integer $m \geq 0$. Matching coefficients we have:[6]

$$-2\mu = m(m+2\alpha) \text{ and } -(2\alpha+1) = (1-n)$$

which gives

$$\alpha = \frac{n}{2} - 1 \text{ and } \mu = -\frac{m}{2}(m+n-2)$$

Then given $\mu = -(m/2)(m+n-2)$ the o.d.e. for g is:

$$\lambda g(x) = \frac{g(x)}{x^2}\mu + g'(x)\left[\frac{n-1}{2x}\right] + \frac{1}{2}g''(x)$$

or

$$0 = g(x)\left(\mu - x^2\lambda\right) + g'(x)\left[\frac{n-1}{2}\right]x + \frac{1}{2}g''(x)x^2$$

or

$$0 = g(x)\left(2\mu - x^2 2\lambda\right) + g'(x)x\,(n-1) + g''(x)x^2$$

with boundary condition $g(\bar{x}) = 0$. The solution of this o.d.e., which does not explode at $x = 0$ is given by a Bessel function of the first kind. This is because the following o.d.e.:

$$g(x)(c + bx^2) + g'(x)x\,a + g''(x)x^2 = 0$$

[6]See https://en.wikipedia.org/wiki/Gegenbauer_polynomials, which is bases on Abramowitz, Milton; Stegun, Irene Ann, eds. (1983) [June 1964], Chapter 22, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematics Series. 55, Dover Publications.

has solution:[7]

$$g(x) = x^{(1-a)/2} J_\nu\left(\sqrt{b}\,x\right) \text{ where } \nu = \frac{1}{2}\sqrt{(1-a)^2 - 4c}$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind. Matching coefficients we have:

$$\begin{aligned} &a = n-1,\ b = -2\lambda,\ c = 2\mu \text{ and} \\ &\nu = \frac{1}{2}\sqrt{(n-2)^2 - 8\mu} = \frac{1}{2}\sqrt{(n-2)^2 + 8(m/2)(m+n-2)} = \frac{n}{2} - 1 + m \end{aligned}$$

We argue that $\nu = n/2 - 1 + m$ to see that note we have

$$\begin{aligned} 4\nu^2 &= (n-2)^2 + 4m(m+n-2) \text{ and} \\ 4\nu^2 &= 4\left(\frac{n-2+2m}{2}\right)^2 = (n-2)^2 + 4m(n-2) + 4m^2 \end{aligned}$$

which verifies the equality. So we have:

$$g(x) = x^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\,x\right)$$

We still have to determine the eigenvalue λ. For this we use the boundary condition $g(\bar{x}) = 0$ and that $J_\nu(\cdot)$ has infinitely strictly orderer positive zeros, denoted by $j_{\nu,k}$ for $k = 1, 2, \ldots$ so that $J_\nu(j_{\nu,k}) = 0$. Thus fixing μ, i.e. $m \geq 0$, we have:

$$0 = g(\bar{x}) = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\lambda}\,\bar{x}\right)$$

so that:

$$0 = (\bar{x})^{1-n/2} J_{\frac{n}{2}-1+m}\left(\sqrt{-2\,\lambda_{m,k}}\,\bar{x}\right)$$

Hence

$$j_{\frac{n}{2}-1+m,k} = \sqrt{-2\,\lambda_{m,k}}\,\bar{x} \quad \text{or} \quad \lambda_{m,k} = -\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Collecting the terms for h, g and λ we obtain the desired result.

Since $\sigma^2 \neq 1$ changes the units of time, we need only to adjust the eigenvalue by its value, so that

$$\lambda_{m,k} = -\sigma^2 \frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2\,\bar{x}^2}$$

Using that $N = n\sigma^2/\bar{x}^2$ we get

$$\lambda_{m,k} = -\frac{n\sigma^2}{\bar{x}^2}\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n} = N\frac{\left(j_{\frac{n}{2}-1+m,k}\right)^2}{2n}$$

□

[7] See http://eqworld.ipmnet.ru/en/solutions/ode/ode0215.pdf which uses Polyanin, A. D. and Zaitsev, V. F., Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition, Chapman & Hall/CRC, Boca Raton, 2003.

Proof. (of Proposition 11) First take $f(w,x) = w\,x/\sqrt{n} = \frac{1}{n}\sum_{i=1}^{n} p_i$. But note that the Gegenbauer polynomial of degree 1 is

$$C_1^{\frac{n}{2}-1}(w) = \sum_{k=0}^{\lfloor 1/2 \rfloor} (-1)^k \frac{\Gamma(1-k+\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-1)k!(1-2k)!}(2w)^{1-2k} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}-1)}(2w) = (n-2)w$$

Thus for $f(w,x) = wx/\sqrt{n}$ we can simply write:

$$f(x,w) = \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\left[\int_{-1}^{1} C_1^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}}dw\right]\left[\int_0^{\bar{x}} x\, J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)x^{\frac{n}{2}}dx\right]}{\left[\int_{-1}^{1}\left(C_m^{\frac{n}{2}-1}(w)\right)^2(1-w^2)^{\frac{n-3}{2}}dw\right]\left[\int_0^{\bar{x}}\left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\,dx\right]}$$

and thus $b_{m,k}[f] = 0$ for all $m \neq 1$, since the polynomials are orthogonal, and

$$b_{1,k}[f] = \frac{1}{\sqrt{n}(n-2)} \frac{\int_0^{\bar{x}} xJ_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)x^{\frac{n}{2}}dx}{\int_0^{\bar{x}}\left(J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\right)^2 x\,dx} \quad \text{for all } k \geq 1$$

Now we argue that fixing x the function $p(w,x;0)$ is odd (antisymmetric) viewed as a function of w. This is because $\bar{h}$ is even and $x'(0)$ is odd, so $\bar{h}'(w)x'(0)$ is odd. Also $\bar{h}'$ is odd and $w'(0)$ is even, hence $\bar{h}'(w)w'(0)$ is odd. Hence $p(w,x;0)$ is not orthogonal to the $C_1^{\frac{n}{2}-1}(\cdot)$. Thus $b_{1,k}[\bar{p}'] \neq 0$.

Finally, to represent the survival function, take $f(w,x) = 1$. Note that this also coincides with a Gegenbauer polynomial for $m = 0$, i.e. $C_0^{\frac{n}{2}-1}(w) = 1$. Thus:

$$f(x,w) = \sum_{k=1}^{\infty} b_{0,k}[f]\varphi_{1,k}(x,w)$$

since

$$b_{m,k}[f] = \frac{\left[\int_{-1}^{1} C_0^{\frac{n}{2}-1}(w) C_m^{\frac{n}{2}-1}(w)(1-w^2)^{\frac{n-3}{2}}dw\right]\left[\int_0^{\bar{x}} J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)x^{\frac{n}{2}}dx\right]}{\left[\int_{-1}^{1}\left(C_m^{\frac{n}{2}-1}(w)\right)^2(1-w^2)^{\frac{n-3}{2}}dw\right]\left[\int_0^{\bar{x}}\left(J_{m+\frac{n}{2}-1}\left(j_{m+\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\,dx\right]}$$

and since the Gegenbauer polynomials are orthogonal, and thus $b_{m,k}[f] = 0$ for all $m > 0$, and

$$b_{0,k}[f] = \frac{\int_0^{\bar{x}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)x^{\frac{n}{2}}dx}{\left[\int_0^{\bar{x}}\left(J_{\frac{n}{2}-1}\left(j_{\frac{n}{2}-1,k}\frac{x}{\bar{x}}\right)\right)^2 x\,dx\right]} \quad \text{for all } k \geq 1$$

□

Proof. (of Proposition 12) Recall that for each $k \geq 1$:

$$\lambda_{1,k} = -N\frac{\left(j_{\frac{n}{2},k}\right)^2}{2n} \text{ and } \lambda_{0,k} = -N\frac{\left(j_{\frac{n}{2}-1,k}\right)^2}{2n}$$

and use $\nu = n/2$ in the first case and $\nu = n/2 - 1$ in the second. It is well know that $j_{\nu,k}$ is strictly increasing in both variables –see Elbert (2001). From here we know that $|\lambda_{1,k}| - |\lambda_{0,k}| > 0$ for all n and k. Also in Elbert (2001) we see that $\frac{\partial}{\partial v} j_{v,k} < 0$ for $\nu > -k$ and $k \geq 1/2$. Thus, the difference between $|\lambda_{1,k}| - |\lambda_{0,k}|$ is decreasing in n.

From Qu and Wong (1999) we have the lower and upper bound for the zeros of the Bessel function $J_\nu(\cdot)$:

$$\nu + \nu^{1/3} 2^{-1/3}|a_k| \leq j_{\nu,k} \leq \nu + \nu^{1/3} 2^{-1/3}|a_k| + \frac{3}{20}|a_k|^2 2^{1/3}\nu^{-1/3}$$

where a_k is the k^{th} zero of the Airy function. Thus, as $n \to \infty$ then $v \to \infty$ and thus both $\lambda_{1,k}$ and $\lambda_{0,k}$ diverge towards $-\infty$. From the same bounds we see that as $n \to \infty$, the difference $\lambda_{0,k} - \lambda_{1,k} \to 1/2$.

□

Proof. (of Proposition 14) We start with the projections for $z/n = f(w,x) = wx/\sqrt{n}$. We are looking for:

$$\begin{aligned} f(x,w) = wx/\sqrt{n} \sim \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w) &= \sum_{k=1}^{\infty} b_{1,k}[f] h_1(w) g_{1,k}(x) \\ &= \sum_{k=1}^{\infty} b_{1,k}[f]\, C_1^{\frac{n}{2}-1(w)}\, J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}} \\ &= w\, x^{1-\frac{n}{2}}\,(n-2) \sum_{k=1}^{\infty} b_{1,k}[f] J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) \end{aligned}$$

We can replace the expression we obtain below for $b_{1,k}[f]$ to get:

$$\begin{aligned} \sum_{k=1}^{\infty} b_{1,k}[f]\varphi_{1,k}(x,w) &= w\, x^{1-\frac{n}{2}}\,(n-2) \sum_{k=1}^{\infty} \frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) \\ &= \frac{w\,x}{\sqrt{n}} \sum_{k=1}^{\infty} \frac{2\,(x/\bar{x})^{-\frac{n}{2}}\, J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)}{j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \end{aligned}$$

To get the coefficients we start by computing

$$\begin{aligned} &\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1} dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \int_0^{\bar{x}} \left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) j_{\frac{n}{2},k}\frac{dx}{\bar{x}} \\ &= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1} J_{\frac{n}{2}}(z)\, dz \end{aligned}$$

Using that

$$\int_a^b z^{\nu+1} J_\nu(z) dz = z^{\nu+1} J_{\nu+1}(z)\Big|_a^b$$

then

$$\int_0^{\bar{x}} J_{\frac{n}{2}}\left(j_{\frac{n}{2},k}\,\frac{x}{\bar{x}}\right) x^{\frac{n}{2}+1} dx = \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \int_0^{j_{\frac{n}{2},k}} (z)^{\frac{n}{2}+1}\, J_{\frac{n}{2}}\,(z)\, dz$$

$$= \left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)$$

Using that

$$\bar{x}^2 \int_0^{\bar{x}} \frac{x}{\bar{x}} \left[J_\nu \left(j_{\nu,k} \frac{x}{\bar{x}}\right)\right]^2 \frac{dx}{\bar{x}} = \frac{1}{2} \left(\, \bar{x}\, J_{\nu+1}\left(j_{\nu,k}\right)\,\right)^2 \text{ for all } k \in \{1, 2, 3 \ldots\}$$

we have

$$\int_0^{\bar{x}} \left(J_{\frac{n}{2}} \left(j_{\frac{n}{2},k}\, \frac{x}{\bar{x}}\right)\right)^2 x\, dx = \frac{1}{2} \left(\, \bar{x}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)\,\right)^2$$

Thus:

$$b_{1,k}[f] = \frac{2}{\sqrt{n}(n-2)} \frac{\left(\frac{\bar{x}}{j_{\frac{n}{2},k}}\right)^{\frac{n}{2}+2} \left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}+1} J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)}{\left(\, \bar{x}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)\,\right)^2}$$

$$= \frac{2\, \bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \text{ for all } k \geq 1$$

Now we turn to compute: $b_{1,k}[\bar{p}'(\cdot,0)]\langle \varphi 1, k, \varphi 1, k\rangle$. We start deriving an explicit expression for $\bar{p}'(\cdot,0)$. We have

$$\bar{h}(w) = \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \left(1 - w^2\right)^{(n-3)/2} \quad \text{for} \quad w \in (-1\,,\,1)$$

$$\bar{g}(x) = x\ (\bar{x})^{-n} \left(\frac{2n}{n-2}\right) \left[\, \bar{x}^{n-2} - x^{n-2}\right] \qquad \text{for } x \in [0, \bar{x}]$$

$$p(w, x; 0) = \bar{h}(w(\delta))\bar{g}(x(\delta)) = \bar{h}(w)\bar{g}(x) + \bar{p}'(w, x; 0)\delta + o(\delta) \text{ with}$$

$$\bar{p}'(w, x; 0) = \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0)$$

where:

$$\frac{\partial}{\partial\delta}x(\delta)|_{\delta=0} = x'(0) = \sqrt{n}\,w \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{h}(w(\delta))|_{\delta=0} = \bar{h}'(w)w'(0)$$
$$\frac{\partial}{\partial\delta}w(\delta)|_{\delta=0} = w'(0) = \frac{\sqrt{n}\,(1-w^2)}{x} \quad \text{and} \quad \frac{\partial}{\partial\delta}\bar{g}(x(\delta))|_{\delta=0} = \bar{g}'(x)x'(0)$$

so:

$$\begin{aligned}
\bar{p}'(w,z;0) &= \bar{g}(x)\bar{h}'(w)w'(0) + \bar{h}(w)\bar{g}'(x)x'(0)\\
&= -\left(\bar{x}\right)^{-n}\left(\frac{2n}{n-2}\right)\left[\bar{x}^{n-2} - x^{n-2}\right]\frac{(n-3)w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\\
&+ \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\,\bar{x}^{-n}\left[\left(\frac{2n}{n-2}\right)\left[\bar{x}^{n-2} - x^{n-2}\right] - 2nx^{n-2}\right]\sqrt{n}\\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\frac{\sqrt{n}}{\bar{x}^n}\left(\frac{2n}{n-2}\right)\left[(4-n)\left(\bar{x}^{n-2} - x^{n-2}\right) - 2nx^{n-2}\right]\\
&= \frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\frac{\left[(4-n)\bar{x}^{n-2} - (4+n)x^{n-2}\right]}{\bar{x}^n}
\end{aligned}$$

We want to compute:

$$b_{1,k}[\bar{p}'(\cdot,0)/\omega]\langle\varphi_{1,k},\varphi_{1,k}\rangle = \int_0^{\bar{x}}\int_{-1}^{1}\bar{p}'(x,w;0)h_{1,k}(x)g_{m,k}(w)dwdx$$

So we split the integral in the product of two terms. The first term involves the integral over w given by:

$$\begin{aligned}
&\int_{-1}^{1}(n-2)w\,\frac{w\left(1-w^2\right)^{(n-3)/2}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\sqrt{n}\left(\frac{2n}{n-2}\right)\,dw\\
&= \frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\int_{-1}^{1}w^2\left(1-w^2\right)^{(n-3)/2}\,dw = \frac{2n\sqrt{n}}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{2\,\Gamma\left(\frac{n}{2}+1\right)}\\
&= \frac{n\sqrt{n}\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\frac{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}{\Gamma\left(\frac{n}{2}+1\right)} = n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}
\end{aligned}$$

where we use that $C_1^{\frac{n}{2}-1}(w) = (n-2)w$, and properties of the $Beta$ and Γ functions.

The second term involves the integral over x and is given by:

$$
\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\,dx \\
&= \frac{\bar{x}^{1-\frac{n}{2}}\bar{x}^{n-2}}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,dx \\
&= \bar{x}^{-\frac{n}{2}}\int_0^{\bar{x}} \left[(4-n)-(4+n)\left(\frac{x}{\bar{x}}\right)^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{dx}{\bar{x}} \\
&= \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{\bar{x}} J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{j_{\frac{n}{2},k}dx}{\bar{x}} \\
&- \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{\bar{x}} \left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{n-2} J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right)\left(\frac{j_{\frac{n}{2},k}x}{\bar{x}}\right)^{1-\frac{n}{2}}\,\frac{j_{\frac{n}{2},k}dx}{\bar{x}} \\
&= \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}(4-n)\int_0^{j_{\frac{n}{2},k}} z^{1-\frac{n}{2}} J_{\frac{n}{2}}(z)\,dz \\
&- \frac{\bar{x}^{-\frac{n}{2}}}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}}(4+n)\int_0^{j_{\frac{n}{2},k}} z^{\frac{n}{2}-1} J_{\frac{n}{2}}(z)\,dz
\end{aligned}
$$

To find an expression for this integrals note that:

$$
\int_0^a z^{1-\frac{n}{2}} J_{\frac{n}{2}}(z)\,dz = -\frac{2^{1-n/2}\left(-1+{}_0F_1\left(n/2,-a^2/4\right)\right)}{\Gamma(n/2)} = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - a^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(a)
$$

and

$$
\int_0^a z^{\frac{n}{2}-1} J_{\frac{n}{2}}(z)\,dz = 2^{-1-\frac{n}{2}} a^n \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{a^2}{4}\right)
$$

where ${}_1\tilde{F}_2(a_1;b_1,b_2;z)$ is the regularized generalized hypergeometric function, i.e. it is defined as ${}_1\tilde{F}_2(a_1;b_1,b_2;z) = {}_1F_2(a_1;b_1,b_2;z)/\left(\Gamma(b_1)\Gamma(b_2)\right)$ where ${}_1F_2$ is the generalized hypergeometric function. Thus

$$
\begin{aligned}
&\frac{1}{\bar{x}^n}\int_0^{\bar{x}} \left[(4-n)\bar{x}^{n-2}-(4+n)x^{n-2}\right] J_{n/2}\left(j_{\frac{n}{2},k}\frac{x}{\bar{x}}\right) x^{1-\frac{n}{2}}\,dx \\
&= \bar{x}^{-\frac{n}{2}}\left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}}\left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
&\left. -\frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}} 2^{-1-\frac{n}{2}} (j_{\frac{n}{2},k})^n \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
& b_{1,k}[f] b_{1,k}[\bar{p}'(\cdot,0)]\langle \varphi_{1,k}, \varphi_{1,k}\rangle \\
&= n\sqrt{n}\,\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2\,\bar{x}^{\frac{n}{2}}}{\sqrt{n}(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
& \bar{x}^{-\frac{n}{2}} \left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} \left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
& \left. - \frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}} 2^{-1-\frac{n}{2}} (j_{\frac{n}{2},k})^{n} \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
& \left[\frac{(4-n)}{\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} \left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} - (j_{\frac{n}{2},k})^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)\right)\right. \\
& \left. - \frac{(4+n)}{\left(j_{\frac{n}{2},k}\right)^{\frac{n}{2}}} 2^{-1-\frac{n}{2}} (j_{\frac{n}{2},k})^{n} \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right] \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{2n}{(n-2)\, j_{\frac{n}{2},k}\, J_{\frac{n}{2}+1}\left(j_{\frac{n}{2},k}\right)} \\
& \left[(4-n) \left(\frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})\left(j_{\frac{n}{2},k}\right)^{2-\frac{n}{2}}} - \frac{J_{\frac{n}{2}-1}\left(j_{\frac{n}{2},k}\right)}{j_{\frac{n}{2},k}}\right)\right. \\
& \left. -(4+n) 2^{-1-\frac{n}{2}} (j_{\frac{n}{2},k})^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) {}_1\tilde{F}_2\left(\frac{n}{2}; 1+\frac{n}{2}, 1+\frac{n}{2}; -\frac{(j_{\frac{n}{2},k})^2}{4}\right)\right]
\end{aligned}
$$

□

C General Random Fixed Cost Model

In the standard Calvo plus model a firm can change its price paying a fixed cost at any time, but it also faces random opportunities to change its price at zero fixed cost. The random opportunities arrive in an iid manner, so that the time between them is exponentially distributed. The optimal decision rule is quite simple: either change prices when the price gaps are large enough, or change them when the free adjustment opportunity arrives.

In the generalized Calvo plus model, at random times–arriving at a Poisson probability per unit of time $\kappa > 0$, the firm draws fixed cost ψ from a distribution. These fixed cost can be zero or positive. If they are zero, the firm decision is the same as in the Calvo plus model. If the realization is positive, the firm will change prices depending on the size of the price gap at that time. Furthermore, when the size of the price gap is large enough, the firm will pay a (large) fixed cost, which is always available, and change prices. This leads to an optimal decision rule of changing prices if the price gap reaches a threshold for a price change, or when a random fixed cost opportunity arrives, depending on the size of the current price gap. This decision rule can be described using thresholds corresponding to each level of fixed cost, and one for the largest fixed cost.

Specifically, let $\kappa > 0$ the probability per unit of time of a low adjustment cost opportunity. Let q_i be the probability of drawing a fixed cost ψ_i for $i = 1, 2, \ldots, n-1$, conditional of drawing a low adjustment cost opportunity. We have $0 = \psi_0 < \psi_1 < \cdots < \psi_{n-1}$. A firm can always pay a fixed cost ψ_n and change its price, with $\psi_n > \psi_{n-1}$. The period return function of the firm is Bx^2 where x is the price gap. The firm discounts costs at rate r. We can then write the value function of the firm with price gap x as:

$$rv(x) = \min \left\{ Bx^2 + \frac{\sigma^2}{2} v''(x) \; + \; \kappa \sum_{j=0}^{n-1} \min \left\{ \psi_j + \min_{x'} v(x') - v(x) \, , \, 0 \right\} q_j \, , \, r \left(\psi_n + \min_{x'} v(x') \right) \right\}$$

It is clear that, given the symmetry of Bx^2 the value function is symmetric around $x = 0$. A proof can be constructed by a simply guess and verify argument. The term $\min_{x'} v(x')$ is the value right after adjustment, and given the symmetry of the return function, we have $v(0) = \min_{x'} v(x')$. The optimal decision rule can be described by n thresholds $0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_n$. The optimal decision rule is that conditional on drawing the adjustment cost ψ_j an adjustment take place if $|x| \geq \bar{x}_i$ for $j = 0, 1, \ldots, n-1$. Note that this implies that: $v(\bar{x}_j) + \psi_j = v(0)$ for $j = 0, 1, 2, \ldots, n$. Moreover, if $|x| \geq \bar{x}_n$ an adjustment will take place. To simplify the notation we let: $\theta_j = \kappa q_j$ for $j = 0, \ldots, n-1$. To summarize: the firm problem is defined by parameters $r, B, \sigma^2, \{\theta_j\}_{j=0}^{n-1}, \{\psi_j\}_{j=1}^{n}$. The optional deicion rule is given by a set of thresholds $\{\bar{x}_j\}_{j=0}^{n}$ with $0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_n$.

The value function v should differentiable for all $x \in \mathbb{R}$, and twice differentiable for all $x \in \mathbb{R}$, except $x = \bar{x}_j$ for $j = 1, \ldots, n$. Thus we have the boundary conditions:

$$v'(\bar{x}_0) = v'(\bar{x}_n) = 0 \tag{16}$$

This includes the optimal return point for $\bar{x}_0 = 0$, smooth pasting at $\bar{x}_n$.

Furthermore, note that if a random fixed cost were to be larger than ψ_n we can, without loss of generality ignore it since it will never be used.

Given the thresholds $\{\bar{x}_j\}_{j=0}^{n}$ we can write a linear o.d.e. valid for x in each segment $[\bar{x}_j, \bar{x}_{j-1}]$ for $j = 1, \ldots, n$. This o.d.e. is parametrized by three constants a_j, b_j, c_j as follows:

$$v_j(x) = a_j + b_j x^2 + c_j \left(e^{\eta_j x} + e^{-\eta_j x} \right) \text{ for } x \in [\bar{x}_{j-1}, \bar{x}_j] \text{ and } j = 1, \ldots, n \tag{17}$$

where η_j is given by:

$$\zeta_j = \sqrt{\frac{(r + \sum_{k=0}^{j-1} \theta_j)}{\sigma^2/2}} \tag{18}$$

Thus we have $4 \times n$ unknowns, namely $\{\bar{x}_j, a_j, b_j, c_j\}_{j=1}^{n}$, and $4 \times n$ equations, namely n equations matching quadratic terms, n equations matching constants, $n-1$ equations enforcing continuity, $n-1$ equations enforcing differentiability, value matching, and smooth pasting. The next proposition states that we can find cost to rationalize any thresholds and probabilities.

Proposition 15. An inverse result. Fix a discount rate, curvature and variance $r, B, \sigma^2 > 0$, and a set of probability rates for costs $\{\theta_j\}_{j=1}^{n-1} \in \mathbb{R}_+^n$ for $n \geq 1$. Given any set of n thresholds $\{\bar{x}_j\}_{j=0}^n$ with $0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_n$, there is a unique set of $n+1$ fixed costs $0 = \psi_0 < \psi_1 < \cdots < \psi_n$ so that the $n+1$ thresholds $\{\bar{x}_j\}_{j=0}^n$ solve the firm's problem defined by $r, B, \sigma^2, \{\theta_j\}_{j=0}^{n-1}, \{\psi_j\}_{j=1}^n$. Moreover, the fixed costs $\{\psi_j\}_{j=1}^n$ and the coefficients for the value function $\{a_j, b_j, c_j\}_{j=1}^n$ solve a system of linear equations.

Now we turn to the eigenvalues and eigenfunctions. By examining the forward and backward Kolmogorov equation in each of its n segments and its boundary conditions, it is easy to see that the Backward and Forward operators are self-adjoint.

Let $f(x,t)$ be the solution of the Kolmogorov backward equation, with terminal condition $f_0(x)$, i.e. $f(x,t) = \mathbb{E}\left[1_{\{\tau > t\}} f_0(x(t)) \,|\, x(0) = x\right]$. We look for pairs of real numbers and functions (λ, φ) which using the method of separation of variables solve for $f(x,t)$ as:

$$f(x,t) = e^{\lambda t}\varphi(x) \text{ for all } t \geq 0 \text{ and } x \in [-\bar{x}_n, \bar{x}_n]$$

where thus $f_0(x) = \varphi(x)$. In this case, φ must solve the o.d.e.:

$$\varphi(x)\left[\lambda + \sum_{k=0}^{i-1}\theta_k\right] = \frac{\sigma^2}{2}\varphi''(x) \text{ for all } x \in (\bar{x}_{i-1}, \bar{x}_i) \text{ and for } i = 1, \ldots, n$$

continuous on $[-\bar{x}_n, \bar{x}]$ and once differentiable for all $x \in [-\bar{x}_n, \bar{x}]$ satisfying the Dirichlet boundary conditions: $\varphi(\bar{x}_n) = \varphi(-\bar{x}_n) = 0$.

The operator that define them is self-adjoint, and have a compact support $[-\bar{x}_n, \bar{x}_n]$. So the eigenvalues λ will be all real, the spectrum discrete, and the eigenfunctions will be orthogonal. Moreover the one-dimensional nature and the symmetry of the equations imply that the eigenfunctions of φ_j are either symmetric or antisymmetric.

Each eigenvalue-eigenfunction pair (λ, φ) can we written as:

$$\varphi(x) = c_i \sin(a_i + b_i x) \text{ for } x \in [\bar{x}_{i-1}, \bar{x}_i] \text{ for } i = 1, \ldots, n$$

with $c_1 = 1$. This eigenvalue-eigenfunction pair is described by $3 \times n$ constants: $\lambda, \{c_i\}_{i=2}^n, \{a_i, b_i\}_{i=1}^n$. We can describe a simple algorithm, involving one non-linear equation and one unknown, to compute each eigenvalue and eigenfunction. The algorithm gives a solution for $\lambda, \{c_i\}_{i=2}^n, \{a_i, b_i\}_{i=1}^n$. This algorithm is indexed by j, which sets $a_n + b_n \bar{x}_n = j\pi$ for each integer $j \geq 1$. Thus we can obtain the set of eigenvalues and eigenfunctions indexed by $j = 1, 2, \ldots$.

D Symmetric Problem with Drift (inflation)

In this section we introduce a non-zero drift μ to the process for x and solve for the eigenfunctions and eigenvalues for $\mathcal{G}(f)(x,t)$, i.e. for the process without reinjection. To lighten the notation we use $g(x,t) = \mathcal{G}(f)(x,t)$.

Proposition 16. Assume that the process has a drift μ, and variance σ^2, so the Kolmogorov backward equation is:

$$\partial_t \mathcal{G}(f)(x,t) = \partial_x \mathcal{G}(f)(x,t)\mu + \partial_{xx}\mathcal{G}(f)(x,t)\frac{\sigma^2}{2} \text{ for all } x \in [\underline{x}, \bar{x}]$$

with boundary conditions $\mathcal{G}(f)(\underline{x},t) = \mathcal{G}(f)(\bar{x},t) = 0$ for all $t > 0$ and $\mathcal{G}(f)(x,0) = f(x)$ for all x. The eigenvalues, eigenfunctions, projections, and inner product are given by:

$$\lambda_j = -\left[\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x}-\underline{x}}\right)^2\right] \text{ for all } j = 1, 2, \ldots$$

$$\varphi_j(x) = \sqrt{\frac{2}{\bar{x}-\underline{x}}}\,\sin\left(\left[\frac{x-\underline{x}}{\bar{x}-\underline{x}}\right]j\pi\right)e^{-\frac{\mu}{\sigma^2}x} \text{ for all } x \in [\underline{x}, \bar{x}] \text{ where}$$

$$b_j[f] = \frac{\langle f, \varphi_j\rangle}{\langle \varphi_j, \varphi_j\rangle} \text{ where } \langle a, b\rangle \equiv \int_{\underline{x}}^{\bar{x}} a(x)b(x)e^{2\frac{\mu}{\sigma^2}x}dx$$

Thus the solution for $\mathcal{G}(f)$ is:

$$\mathcal{G}(f)(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} b_j[f]\,\varphi_j(x) \text{ for all } t \geq 0 \text{ and } x \in [\underline{x}, \bar{x}].$$

Proof. (of Proposition 16) To lighten the notation, denote $g(x,t) = \mathcal{G}(f)(x,t)$. We start by rewriting g as

$$g(x,t) = h(x,t)s(x)$$

for some function s, so that the Kolmogorov backward equation is:

$$s(x)\partial_t h(x,t) = [s'(x)h(x,t) + s(x)\partial_x h(x,t)]\,\mu + [s''(x)h(x,t) + 2s'(x)\partial_x h(x,t) + s(x)\partial_{xx}h(x,t)]\,\frac{\sigma^2}{2}$$

We will take $s(x) = \exp(ax)$ for some constant a to be determined. Replacing this function and its derivatives, and cancelling we get

$$\begin{aligned}\partial_t h(x,t) &= [ah(x,t) + \partial_x h(x,t)]\,\mu + \left[a^2 h(x,t) + 2a\partial_x h(x,t) + \partial_{xx}h(x,t)\right]\frac{\sigma^2}{2} \\ &= h(x,t)\left[a\mu + a^2\frac{\sigma^2}{2}\right] + \partial_x h(x,t)\left[\mu + a\sigma^2\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2}\end{aligned}$$

Setting $a = -\frac{\mu}{\sigma^2}$ into the p.d.e. for h we have:

$$\begin{aligned}\partial_t h(x,t) &= h(x,t)\left[-\frac{\mu^2}{\sigma^2} + \frac{\mu^2}{\sigma^4}\frac{\sigma^2}{2}\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2} \\ &= h(x,t)\left[-\frac{\mu^2}{\sigma^2} + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right] + \partial_{xx}h(x,t)\frac{\sigma^2}{2} \\ &= -h(x,t)\frac{1}{2}\frac{\mu^2}{\sigma^2} + \partial_{xx}h(x,t)\frac{\sigma^2}{2}\end{aligned}$$

Since $s(x) \neq 0$, the boundary conditions for h are the same as for g, namely $h(\underline{x}, t) = h(\bar{x}, t) = 0$. The equation for the eigenvalues-eigenfunctions for h is the same as in the Calvo+ model

$$\left(\lambda_j + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right)\phi_j(x) = \partial_{xx}\phi_j(x)\frac{\sigma^2}{2}$$

so that we have the same expression for the eigenvalues-eigenfuctions:

$$\lambda_j = -\left[\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{\sigma^2}{2}\left(\frac{j\,\pi}{\bar{x} - \underline{x}}\right)^2\right] \text{ for all } j = 1, 2, \ldots$$
$$\phi_j(x) = \sin\left(\left[\frac{x - \underline{x}}{\bar{x} - \underline{x}}\right] j\pi\right) \text{ for all } x \in [\underline{x}, \bar{x}]$$

Thus we can write the solution for h as:

$$h(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}\, \frac{\int_{\underline{x}}^{\bar{x}} h(x', 0)\phi_j(x')dx'}{\int_{\underline{x}}^{\bar{x}} (\phi_j(x'))^2 dx'}\, \phi_j(x)$$

Multiplying both sides by $s(x)$:

$$h(x,t)s(x) = \sum_{j=1}^{\infty} e^{\lambda_j t}\, \frac{\int_{\underline{x}}^{\bar{x}} h(x', 0)\phi_j(x')dx'}{\int_{\underline{x}}^{\bar{x}} (\phi_j(x'))^2 dx'}\, \phi_j(x)s(x)$$

Thus we define

$$\langle a, b \rangle = \int_{\underline{x}}^{\bar{x}} a(x)b(x)\frac{1}{s(x)^2}dx \text{ and}$$
$$\varphi_j(x) = \phi_j(x)s(x) \text{ so that}$$
$$\langle \varphi_j, \varphi_i \rangle = 0 \text{ if } i \neq j \text{ since } \int_{\underline{x}}^{\bar{x}} \phi_j(x)\phi_i(x)dx = 0 \text{ for } i \neq i$$

Note that we can write the solution for g as follows:

$$\frac{\langle g(x,0), \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} = \frac{\int_{\underline{x}}^{\bar{x}} g(x,0)\varphi_j(x)\frac{dx}{s(x)^2}}{\int_{\underline{x}}^{\bar{x}} (\varphi_j(x))^2 \frac{dx}{s(x)^2}} = \frac{\int_{\underline{x}}^{\bar{x}} \frac{g(x,0)}{s(x)}\phi_j(x)dx}{\int_{\underline{x}}^{\bar{x}} (\phi_j(x))^2 dx} = \frac{\int_{\underline{x}}^{\bar{x}} h(x,0)\phi_j(x)dx}{\int_{\underline{x}}^{\bar{x}} (\phi_j(x))^2 dx}$$

where we use that $g(x, 0) = s(x)h(x, 0)$. $\square$

E Asymmetric Problems: Dealing with Reinjection

In this section we study the impulse response function for problems where the symmetry assumptions of Proposition 1 do not hold. In such a case the computation of the impulse response function requires keeping track of firms after their first adjustment, so that the impulse response function $H(t)$ cannot be computed by means of the simpler operator $G(t)$.

The solution to this problem is to compute the law of motion of the cross-sectional distribution using the Kolmogorov forward equation, keeping track of the reinjenctions that occur after the adjustments.

The nature of reinjection in our set up differs from the one in Gabaix et al. (2016) and hence we cannot use their results for the ergodic case. The added complexity of our case originates because the exit points (of our pricing problem) are not independent of where x is, as in the case of poisson adjustments. Rather, prices are changed when either barrier $\underline{x}$ or $\bar{x}$ is hit, and then the measure of products whose prices are changed are all reinjected at single value, the optimal return point x^*.

The set-up consists of an unregulated BM $dx = \mu dt + \sigma dB$, which returns to a single point x^* the first time that x hits either of the barriers $\underline{x}$ or $\bar{x}$ or that a Poisson counter with intensity ζ changes. As implied by Proposition 1 we cannot ignore the reinjenctions at x^* if either $x^* \neq (\bar{x} + \underline{x})/2$ or $\mu \neq 0$. For simplicity consider the case with no drift, so we set $\mu = 0$. We can use the method in Appendix D to modify the result accordingly. This set up can be used to study the problem of a firm with a non-symmetric period return function in an economy without inflation ($\mu = 0$). In this case the optimal decision rule implies $x^* \neq (\bar{x} + \underline{x})/2$, i.e. the reinjection point (after adjustment) is not located in the middle of the inaction region. Note that the number of price adjustment per unit of time is given by $N = \frac{\sigma^2}{(\bar{x}-\underline{x})^2\alpha(1-\alpha)}$.

Let $\hat{p}(x)$ denote the initial condition for the density of firms relative the invariant distribution $\bar{p}(x)$, i.e. $\hat{p}(x) = p(x) - \bar{p}(x)$ for some density p, where $\bar{p}$ is the asymmetric (steady state) tent map. Notice that to analyze small shocks δ, i.e. an initial condition $p(x) = \bar{p}(x+\delta)$ the signed measure is $\hat{p}(x) = \delta \bar{p}'(x)$ by a simple Taylor expansion and mass preservation requires that $\int_{\underline{x}}^{\bar{x}} \hat{p}(x)dx = 0$, so that

$$\hat{p}(x)/\delta = \begin{cases} \frac{2}{(\bar{x}-\underline{x})^2\alpha} & \text{if } x \in [\underline{x}, x^*) \\ \frac{-2}{(\bar{x}-\underline{x})^2(1-\alpha)} & \text{if } x \in (x^*, \bar{x}] \,. \end{cases} \tag{19}$$

We define the Kolmogorov forward operator $\mathcal{H}^*(\hat{p}) : [\underline{x}, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}$ for the process with reinjection, where $\mathcal{H}^*(\hat{p})(x,t)$ denotes the cross-sectional density of the firms t periods after the shock. In this case we have that for all $x \in [\underline{x}, x^*) \cup (x^*, \bar{x}]$ and for all $t > 0$:

$$\partial_t \mathcal{H}^*(\hat{p})(x,t) = \frac{\sigma^2}{2}\partial_{xx}\mathcal{H}^*(\hat{p})(x,t) - \zeta \mathcal{H}^*(\hat{p})(x,t) \tag{20}$$

with boundary conditions:

$$\mathcal{H}^*(\hat{p})(\underline{x},t) = \mathcal{H}^*(\hat{p})(\bar{x},t) = 0 \quad , \quad \lim_{x\uparrow x^*} \mathcal{H}^*(\hat{p})(x,t) = \lim_{x\downarrow x^*} \mathcal{H}^*(\hat{p})(x,t) \tag{21}$$

$$\partial_x^- \mathcal{H}^*(\hat{p})(\bar{x},t) - \partial_x^+ \mathcal{H}^*(\hat{p})(x^*,t) + \partial_x^- \mathcal{H}^*(\hat{p})(x^*,t) - \partial_x^+ \mathcal{H}^*(\hat{p})(\underline{x},t) = \frac{2\zeta}{\sigma^2} \tag{22}$$

$$\mathcal{H}^*(\hat{p})(x,0) = \hat{p}(x) \text{ for all } x \in [\underline{x}, \bar{x}] \tag{23}$$

The p.d.e. in equation (20) is standard, we just note that it does not need to hold at the reinjection point x^*. The boundary conditions in equation (21) are also standard, given that

$\underline{x}$ and $\bar{x}$ are exit points, and that with $\sigma^2 > 0$, the density must be continuous everywhere. The condition in equation (22) ensures that the measure is preserved, or equivalently that there is no change in total mass across time: $\int_{\underline{x}}^{\bar{x}} \mathcal{H}^*(\hat{p})(x,t)dx = \int_{\underline{x}}^{\bar{x}} \hat{p}(x)dx$ for all t. This is a small extension of Proposition 1 in Caballero (1993).

We can use $\mathcal{H}^*$ to compute the Impulse response function defined above as follows:

$$H(t, f, \hat{p}) = \int_{\underline{x}}^{\bar{x}} f(x)\, \mathcal{H}^*(\hat{p})(x,t)\, dx\,. \tag{24}$$

If $\zeta > 0$ and $\bar{x} \to \infty$ as well as $\underline{x} = \infty$, we will have the pure Calvo case, and we can use the ideas in Gabaix et al. (2016), and thus the case without reinjection and with reinjection are quite similar. To highlight the difference, we consider the opposite case, and set $\zeta = 0$ and use an eigenvalue decomposition of $\mathcal{H}^*$.

Proposition 17. Assume that $\zeta = 0$ and that α is not a rational number. The orthonormal eigenfunctions of $\mathcal{H}^*$ are:

$$\varphi_j^m(x) = \sqrt{\frac{2}{(\bar{x}-\underline{x})}} \sin\left(\left[\frac{x-\underline{x}}{\bar{x}-\underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, \bar{x}] \tag{25}$$

$$\varphi_j^l(x) = \sqrt{\frac{2}{(x^*-\underline{x})}} \sin\left(\left[\frac{x-\underline{x}}{x^*-\underline{x}}\right] 2\pi j\right) \text{ if } x \in [\underline{x}, x^*] \text{ and } 0 \text{ otherwise} \tag{26}$$

$$\varphi_j^h(x) = \sqrt{\frac{2}{(\bar{x}-x^*)}} \sin\left(\left[\frac{x-x^*}{\bar{x}-x^*}\right] 2\pi j\right) \text{ if } x \in [x^*, \bar{x}] \text{ and } 0 \text{ otherwise} \quad , \tag{27}$$

with corresponding eigenvalues:

$$\lambda_j^m = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(\bar{x}-\underline{x})^2}\ ,\ \lambda_j^l = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(x^*-\underline{x})^2},\ \text{and}\ \lambda_j^h = -\frac{\sigma^2}{2}\frac{(2\pi j)^2}{(\bar{x}-x^*)^2}\ , \tag{28}$$

for all $j = 1, 2, \ldots$. The eigenfunctions in the set $\{\varphi_j^m\}_{j=1}^\infty$ are orthogonal to each other, and so are those in the set $\{\varphi_j^l, \varphi_j^h\}_{j=1}^\infty$. The eigenfunctions $\{\varphi_j^m, \varphi_j^l, \varphi_j^h\}_{j=1}^\infty$ span the set of functions $g : [\underline{x}, \bar{x}] \to \mathbb{R}$, piecewise differentiable, with countably many discontinuities, and with $\int_{\underline{x}}^{\bar{x}} g(x)dx = 0$.

Proof. (of Proposition 17) The proof consists on checking that the functions described are the only ones that satisfy the sufficient conditions equation (20), equation (21), equation (22) and equation (23) for $\mathcal{H}^*(\varphi_j^k(x)e^{\lambda_j^k t})$ for $k \in l, m, j$ and $j = 1, 2, \ldots$.

First, let's consider the case of eigenvalues $\lambda \neq 0$. In this case the only non-constant real function that satisfy the o.d.e.: $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$ for $\lambda < 0$ is $\varphi(x) = \sin(\phi + \omega x)$ for some ϕ and for $\lambda = -\frac{\sigma^2}{2}\omega^2$. The cases below use this characterization when the function is not constant, to determine the values of ϕ and ω.

Second, consider the case of functions that are differentiable in the entire domain $[\underline{x}, \bar{x}]$. The continuity at $x = x^*$ is satisfied immediately. In this case, the o.d.e. : $\lambda\varphi(x) = \partial_{xx}\varphi(x)\sigma^2/2$, with boundaries equation (21) and equation (22) is satisfied only by $\varphi(x) =$

$\varphi_j^m(x)$ for all $j = 1, 2, ...$ This gives the particular value of ϕ and ω, for $j = 1, 2, \ldots$, and hence no other differentiable function different from zero satisfy all the conditions.

Third, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[\underline{x}, x^*]$. Then $\varphi(x) = 0$, to satisfy the boundary condition equation (21) at $x = \underline{x}$. Then $\varphi(x) = 0$ for all $x \in [\underline{x}, x^*)$, since φ can only be non-differentiable at $x = x^*$. Then, for $x \in (x^*, \bar{x}]$ it has to be differentiable, non-identically equal to zero, satisfy $\varphi(x^*) = 0$ so that it is continuous at $x = x^*$, and also $\varphi(\bar{x}) = 0$, to satisfy the boundary condition equation (21) at $x = \bar{x}$. Finally, to satisfy the measure preserving condition equation (22), it has to be of the form of $\varphi_j^l(x)$ for $j = 1, 2, \ldots$.

Fourth, consider the case of functions $\varphi(x)$ which are constant in an interval of strictly positive length included in $[x^*, \bar{x}]$. Following the same steps as in the previous case, we obtain that $\varphi(x) = \varphi_j^h(x)$ for $j = 1, 2, \ldots$ for this case.

For the fifth and remaining case, we consider the case of functions $\varphi(x)$ which are non-constant for all intervals included in $[\underline{x}, \bar{]}$, and that $\varphi(x)$ is not differentiable at $x = x^*$. To satisfy the o.d.e. in each segment $[\underline{x}, x^*)$ and $(x^*, \bar{x}]$ then we must have $\varphi(x) = \sin\left(\underline{\phi} + \underline{\omega} x\right)$ and $\varphi(x) = \sin\left(\bar{\phi} + \bar{\omega} x\right)$ in each of the respective segments. Since the eigenvalue has to be the same for all segments, then we have that $\underline{\omega} = \bar{\omega} \equiv \omega$. The eigenfunction φ must be measure preserving, so that

$$0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\underline{\phi} + \omega \underline{x}\right) + \cos\left(\bar{\phi} + \omega \bar{x}\right) - \cos\left(\bar{\phi} + \omega x^*\right)$$

To satisfy the boundary conditions equation (21) we require $\sin\left(\underline{\phi} + \omega \underline{x}\right) = \sin\left(\bar{\phi} + \omega \bar{x}\right) = 0$. Thus $\cos\left(\underline{\phi} + \omega \underline{x}\right) = \pm 1$ and $\cos\left(\bar{\phi} + \omega \bar{x}\right) = \pm 1$. Hence, we have that:

$$\text{either } 0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \text{ or } \pm 2 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right)$$

In the first case we have:

$$0 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \text{ and } 0 = \sin\left(\underline{\phi} + \omega x^*\right) - \sin\left(\bar{\phi} + \omega x^*\right)$$

so the function is differentiable at $x = x^*$, which is a contradiction. So we must have the second case, and because eigenfunctions are defined up to sign, must have:

$$2 = \cos\left(\underline{\phi} + \omega x^*\right) - \cos\left(\bar{\phi} + \omega x^*\right) \text{ and } 0 = \sin\left(\underline{\phi} + \omega x^*\right) - \sin\left(\bar{\phi} + \omega x^*\right)$$

Using the properties of cos it must be the case that $\sin\left(\underline{\phi} + \omega x^*\right) = \sin\left(\bar{\phi} + \omega x^*\right) = 0$. Then, φ must be zero in the extremes of each of the two following segment $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$. This requires that $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$ be an in an multiple integer of each other, since in each of the segments φ is a sine function with the same frequency ω which is zero at the two extremes. But this violate that $[\underline{x}, x^*]/[x^*, \bar{x}]$ is not rational.

Now we show that the eigenfunctions span the densities for the signed measures. It suffices to show that if $g : [\underline{x}, \bar{x}] \to \mathbb{R}$ is in the domain of $\mathcal{H}^*$, and $\langle g, \varphi_j^m \rangle = \langle g, \varphi_j^l \rangle = \langle g, \varphi_j^h \rangle = 0$ for all $j = 1, 2, \ldots$, then it must be that $g = 0$.

As a way of contradiction, suppose we have a function $g \neq 0$ that g is orthogonal to all the eigenfunctions. Given that the eigenfunctions can span antisymmetric functions defined in different domains as explained above, it must be that g is a symmetric function as defined in $[\underline{x}, \bar{x}]$, so that it is orthogonal to $\{\varphi_j^m\}_{j=1}^{\infty}$, and also a symmetric in the following restricted domains $[\underline{x}, x^*]$ and $[x^*, \bar{x}]$, so that g is also orthogonal each of eigenfunctions $\{\varphi_j^l\}_{j=1}^{\infty}$ and $\{\varphi_j^h\}_{j=1}^{\infty}$ when defined in the restricted domains.

Now, without loss of generality, assume that $x^* < (\underline{x} + \bar{x})/2$. Below we sketch a proof that for a function g to be even (or symmetric) in these three domains, it must be the case that $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, which contradicts the assumption that $[x^* - \underline{x}]/[\bar{x} - x^*]$ is not a rational number.

Let $L = x^* - \underline{x}$. To arrive to this conclusion we first notice that since g must be symmetric in the entire domain $[\underline{x}, \bar{x}]$, then it must be the case that g has identical symmetric shape in the segment $[\underline{x}, \underline{x} + L]$ than in the segment $[\bar{x} - L, \bar{x}]$. Then using that g is symmetric in the restricted domain $[x^*, \bar{x}]$, it must be that it also has the same symmetric shape in the interval $[x^*, x^* + L]$ than in both intervals $[\underline{x}, \underline{x} + L]$ and $[\bar{x} - L, \bar{x}]$. If it is the case that $x^* + L = \bar{x} - L$, then $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, and find a contradiction. If this is not the case, i.e. if $x^* + L < \bar{x} - L$, we use the g is symmetric in the entire domain, to say that again g must take the same symmetric shape in the interval $[\bar{x} - 2L, \bar{x} - L]$. Now either $x * +L = \bar{x} - 2L$, which gives a contradiction, or we continue using the symmetry of g in either the entire domain $[\underline{x}, \bar{x}]$ or in the restricted domain $[x^*, \bar{x}]$ until we get that $[\bar{x} - x^*]$ is an integer multiple of $[x^* - \underline{x}]$, which is a contradiction with our assumption. Formally, this can be set up as an induction step, but it requires to develop enough notation, which we skip to shorten. □

By defining $\mathcal{H}^*$ for initial conditions given by the differences of a density relative to the density of the invariant distribution, we are excluding the zero eigenvalue and its corresponding eigenfunction, the invariant distribution $\bar{p}$ from the its representation. From the proposition we see what are the first two non-zero eigenvalues.

$$\lambda_1 = -\frac{\sigma^2}{2}\left(\frac{2\pi}{\bar{x} - \underline{x}}\right)^2 > \lambda_2 = -\frac{\sigma^2}{2}\left(\frac{2\pi}{\max\left\{(\bar{x} - x^*), (x^* - \underline{x})\right\}}\right)^2 \tag{29}$$

Notice that the difference between the first and the second eigenvalues depends on the asymmetry of the bands.

The proposition also proves that the eigenfunctions φ_j^k with $k = \{l, h, m\}$ form a base, so that projecting the initial condition onto them is possible. It is however more involved than in the symmetric case since the eigenfunctions are not all orthogonal with each other, e.g. $\langle \varphi_j^m, \varphi_j^h \rangle \neq 0$. Note however that $\{\varphi_j^m, \lambda_j^m\}$ coincide withe the antisymmetric eigenfunction and eigenvalues for the case without reinjection and $\mu = \zeta = 0$. Because of this, any piecewise differentiable function $g : [\underline{x}, \bar{x}] :\to \mathbb{R}$ that is antisymmetric around $(\underline{x} + \bar{x})/2$ can be represented, in a L^2 sense, as a Fourier series using $\{\varphi_j^m\}$.

In spite of the lack of orthogonality, the general logic for constructing the impulse response function is the same. Given the projection of the initial condition on the eigenfunctions

$$\hat{p}(x, 0) = \sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k(x)$$

where $k = \{l, h, m\}$. We use the linearity of $\mathcal{H}^*$ to write the operator in equation (24) as

$$\mathcal{H}^*(\hat{p})(x,t) = \mathcal{H}^*\left(\sum_k \sum_{j=1}^{\infty} a_j^k \varphi_j^k\right)(x,t) = \sum_k \sum_{j=1}^{\infty} a_j^k \mathcal{H}^*(\varphi_j^k)(x,t) = \sum_k \sum_{j=1}^{\infty} a_j^k e^{\lambda_j^k t}\varphi_j^k(x)$$

where the last equality uses that the $\varphi_j^k(x)$ are eigenfunctions. Thus, given the a_j^k coefficients (whose computation is discussed below), we can write the impulse response in equation (24) as

$$H(t,f,\hat{p}) = \sum_{\{k=l,h,m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t}\, a_j^k \int_{\underline{x}}^{\bar{x}} f(x)\, \varphi_j^k(x)\, dx\,.$$

or, computing the inner products $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x)\, \varphi_j^k(x)\, dx$

$$H(t,f,\hat{p}) = \sum_{k=\{l,h,m\}} \sum_{j=1}^{\infty} e^{\lambda_j^k t}\, a_j^k\, b_j^k[f]\,. \tag{30}$$

A straightforward numerical approach to finding the projection coefficients a_j^k requires running a simple linear regression of $\hat{p}(x,0)$ on the basis $\{\varphi_j^h(x), \varphi_j^l(x), \varphi_j^m(x)\}_{j=1}^J$ (up to some order frequency J). For the output impulse response, given the function of interest $f(x) = -x$, the projection coefficients are also readily computed $b_j^k[f] = \int_{\underline{x}}^{\bar{x}} f(x)\varphi_j^k(x)dx$ for $k = \{m, l, h\}$, and $j = 1, 2, 3....$ which gives

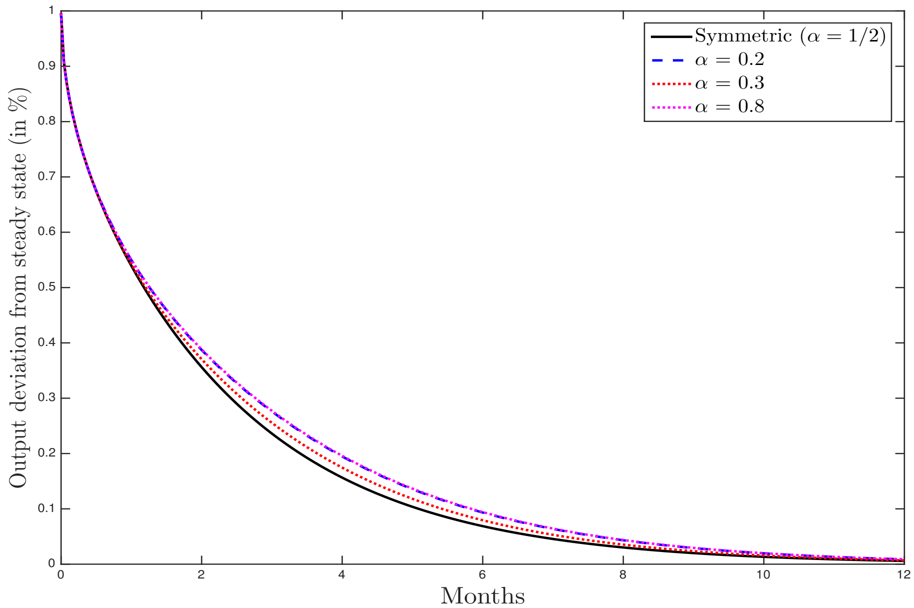
$$b_j^m[f] = \frac{(\bar{x} - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^l[f] = \frac{(x^* - \underline{x})^{3/2}}{\sqrt{2}\pi j} \quad , \quad b_j^h[f] = \frac{(\bar{x} - x^*)^{3/2}}{\sqrt{2}\pi j} \quad . \tag{31}$$

Figure 3 displays some impulse response generated by asymmetric problems where $\alpha \neq 1/2$ and contrasts them to the one produced by the symmetric problem where $\alpha = 1/2$. Two remarks are in order: first, modest degrees of asymmetry do not have a major effect on the impulse response: the impulse response function for $\alpha = 0.4$ would be barely distinguishable from the symmetric impulse response. Second, once quantitatively large asymmetries are considered, such as the small values of α considered in the figure, the impulse response becomes more persistent than the symmetric one. The presence of the asymmetry makes the convergence to the mean x value of the invariant distribution slower; this is intuitive since for symmetric problems the mean of the distribution is obtained right after the first adjustment, while this is not anymore true.

Irrelevance of the sign of the reinjection point. We discuss here the irrelevance of whether the optimal return point x^* is to the left of the interval's midpoint, as when $x^* < 0$, or to the right of it hence with $x^* > 0$. Formally we consider two problems: the first one has $\alpha = 1/2 - z$ where $z \in (0, 1/2)$ and the second problem has $\tilde{\alpha} = 1/2 + z$. We will show that, somewhat surprisingly to us, the sign of the optimal return point x^* is irrelevant for the impulse response which is the same one for the problem with α and for the one with $\tilde{\alpha}$. We have the following result

Proposition 18. Consider the inaction region for x defined by the interval $(-\bar{x}, \bar{x})$, let

Figure 3: Response to monetary shock for asymmetric problem



$z \in (0, 1/2)$ be a non-rational number. Consider a problem with reinjection point $\alpha = 1/2 - z$ and another problem with reinjection point $\tilde{\alpha} = 1/2 + z$. Then the impulse response function is the same for both problems.

Proof. of Proposition 18. Consider the first problem with $\alpha < 1/2$. Normalize (WLOG) the interval width to $2\bar{x} = 1$ and rewrite the initial condition as $\hat{p} = \hat{p}^s + \hat{p}^a$, respectively the symmetric and antisymmetric component as

$$\hat{p}^s(x) = \begin{cases} \frac{1-2\alpha}{\alpha(1-\alpha)} \\ \frac{-1}{(1-\alpha)} \end{cases}, \quad \hat{p}^a(x) = \begin{cases} \frac{1}{\alpha(1-\alpha)} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ 0 & \text{for } x \in (-z, z) \end{cases}$$

Notice that for $\alpha = 1/2 - z$ the slope of the antisymmetric part is either zero or $\frac{1}{1/4 - z^2}$. The same obtains for $\tilde{\alpha} = 1/2 + z$. Thus the asymmetric component of the initial condition $\hat{p}^a(x)$ is the same for α and for $\tilde{\alpha}$. The symmetric component $\hat{p}^s(x)$ is as follows

$$\hat{p}^s(x, \alpha) = \begin{cases} \frac{2z}{1/4 - z^2} \\ \frac{-1}{1/2 + z} \end{cases}, \quad \hat{p}^s(x, \tilde{\alpha}) = \begin{cases} \frac{-2z}{1/4 - z^2} & \text{for} \quad x \in (-\bar{x}, -z) \cup (z, \bar{x}) \\ \frac{1}{1/2 + z} & \text{for } x \in (-z, z) \end{cases}$$

which reveals that the symmetric component of the initial condition for the problem with $\tilde{\alpha}$ is given by -1 times the symmetric component of the initial condition for the problem with α.

Now let's consider the consequences for the output impulse response as defined in equation (24). For the problem with α we use the decomposition $\hat{p} = \hat{p}^s + \hat{p}^a$ and the linearity of

$\mathcal{H}^*$ to write the IRF as

$$H_\alpha(t, f, \hat{p}(\alpha)) = H_\alpha(t, f, \hat{p}^a(\alpha)) + H_\alpha(t, f, \hat{p}^s(\alpha))$$

where we use the subscript to emphasize that this is the impulse response for the problem with reinjection point α. Using the properties for the initial condition associated to the problem with $\tilde{\alpha}$ discussed above we can the write its impulse response as

$$H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha})) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha)) + H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$$

where we used that $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and that $\hat{p}^s(\alpha) = -\hat{p}^s(\tilde{\alpha})$.

It is immediate to see that $H_\alpha(t, f, \hat{p}^a(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}^a(\alpha))$, i.e. that the IRF component triggered by the asymmetric part of the initial condition, is the same in both problems. This follows since $\hat{p}^a(\alpha) = \hat{p}^a(\tilde{\alpha})$ and because both problems share the same identical base for asymmetric functions, given by the eigenfunctions φ_j^m.

Finally, we argue that $H_\alpha(t, f, \hat{p}^s(\alpha)) = H_{\tilde{\alpha}}(t, f, -\hat{p}^s(\alpha))$. To see this notice that the symmetric part of the impulse response function is obtained by projecting the initial condition on the orthogonalized symmetric eigenfunctions v_j^k, where $k = \{l, h\}$, produced by e.g. the Gram-Schmidt algorithm. The key is to notice that the symmetrized eigenfunction for the problem with $\tilde{\alpha}$, equals -1 times the eigenfunctions for the problem with α, formally $v_j^k(\alpha) = -v_j^k(\tilde{\alpha})$. Inspection if the eigenfunctions φ_j^h and φ_j^l reveals that, for all $x \in (-\bar{x}, \bar{x})$ they obey $\varphi_1^h(x; x^* = -z) = -\varphi_1^l(-x; x^* = z)$. It therefore follows that $H_\alpha(t, f, \hat{p}(\alpha)) = H_{\tilde{\alpha}}(t, f, \hat{p}(\tilde{\alpha}))$. □

Figure 3 illustrates the results of the proposition by showing that the impulse response for $\alpha = 0.2$ coincides with the one for $\alpha = 0.8$. An important implication of this property is that the derivative of the impulse response function with respect to α evaluated at $alpha = 1/2$ must be zero, which explains why small deviations from the symmetric benchmark produce results that are essentially almost indistinguishable from those produced by the symmetric case. Overall this result suggests that the symmetric benchmark is an accurate approximation of problems with modest degrees of asymmetry.

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