The Substitution Elasticity, Factor Shares, Long-Run Growth, And The Low-Frequency Panel Model

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Abstract

The value of the elasticity of substitution between labor and capital (σ) is a "crucial" assumption in the study of factor incomes (e.g., Piketty (2014a), Piketty and Zucman (forthcoming), Karabarbounis and Neiman (2014)) and long-run growth (Solow, 1956). This paper begins by examining the role of σ in the analyses of these two issues. It then develops and implements a new strategy for estimating this crucial parameter by combining a low-pass filter with panel data techniques to identify the low-frequency/long-run relations appropriate to production function estimation.

Our low-frequency/long-run approach is in the spirit of Friedman's permanent income theory of consumption and Eisner's related permanent income theory of investment. While their estimation strategies and ours are similar in relying on permanent (long-run) components, we identify these unobservable components with a low-pass filter. Using spectral analysis, we assess the extent to which our choices of the critical periodicity and window defining the low-pass filter are successful in emphasizing long-run variation.

The empirical results are based on the comprehensive panel industry dataset constructed by Dale Jorgenson and his research associates. Our preferred estimate of σ is 0.40. This result is robust to variations in the two spectral parameters that define the low-pass filter, instrumental and split-sample estimates, and disaggregation by industry. Moreover, we document that standard estimation methods, which do not filter-out transitory variation, generate downwardly biased estimates. As medium and high frequency variation is introduced into the model variables, σ declines by over 40% relative to the benchmark value. Despite correcting for this bias, our preferred estimate of σ is less than one, and we find no support for the Cobb-Douglas assumption.

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Keywords: Substitution elasticity, labor income share, long-run growth, low-pass filter, production function parameters.

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Table Of Contents

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> A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic.
>
> Solow (1956, p. 65)

> The crucial question goes to what is technically referred to as the elasticity of substitution. With 1 percent more capital and the same amount of everything else, does the return to a unit of capital relative to a unit of labor decline by more or less than 1 percent?
>
> Summers (2014)

> The relevant question [regarding secular movements in income shares] is whether the elasticity of substitution between labor and capital is greater or less than one.
>
> Piketty (2014a, p. 217)

I. Introduction

The elasticity of substitution between capital and labor, σ, is a "crucial" parameter. It affects the qualitative and quantitative answers to a host of economic questions. This paper examines the role of σ in analyzing factor income shares and long-run economic growth. It then develops and implements a new strategy for estimating this crucial parameter by combining a low-pass filter with panel data techniques to identify the long-run relations appropriate to production function estimation.

Section II discusses several economic issues that depend crucially on the value of σ. The labor share of income has declined markedly over the past two decades. Whether decreases in capital taxation or relative prices or increases in capital/income ratios have contributed to this secular movement depends crucially on σ being greater than one. A variety of issues in growth theory -- the possibility of perpetual growth or decline, the level of steady-state income per capita, the speed of convergence, the plausibility of the Solow growth model, and the role of technical change -- depend on this elasticity. Our discussion highlights the relations among these important economic issues and the precise value of σ.

Section III develops a strategy for estimating σ that applies a low-pass filter to panel data. Production function parameters are recovered by focusing on the long-run relations among arguments appearing in the first-order condition for capital. Our approach is in the spirit of Friedman's (1957) permanent income theory of consumption and Eisner's (1967) related permanent income theory of investment. Friedman observed that the fundamental relation between consumption and income is not stated in terms of the raw observed values of income, but rather in terms of its permanent component. Eisner also emphasized the distinction between the transitory and permanent components of variables affecting investment demand. He isolated the permanent component by grouping firms by industry and then estimating with the group means. Our approach also relies on permanent components, but they are extracted with a low-pass filter (based on the band-pass filter of Baxter and King, 1999) that affords a more general way for identifying these unobservable variables and a more transparent connection to the economic concept of the long-run.

Section IV examines the theoretical properties of the spectral representation of the low-pass filter used in this study. This filter is defined in terms of two parameters -- the critical frequency ($\omega^{\#}$) defining the long-run and a window (q) for the number of lags and leads used to approximate the ideal low-pass filter. We vary these two parameters to assess the sensitivity of the allocation of variance across low, middle, and high frequencies and verify that the transformed data reflect long-run variation that will be useful in estimating production function parameters.

Sections V and VI contain empirical results based on the comprehensive panel of U.S. industry dataset constructed by Dale Jorgenson and his research associates. Our econometric model relates the growth rate in the long-run capital/output ratio to the growth rate in the long-run relative price of capital (conditioned on a set of fixed time effects). The benchmark estimate of σ is 0.406 for a definition of the long-run in terms of a periodicity (inversely related to frequency and defined as the length of time required for a series to repeat a complete cycle) of eight years or greater and a window of three years. This result is robust to variations in the window. Moreover, there is great value in using the low-pass filter to extract permanent components. As the periodicity declines from eight to the minimum value of two years, the estimated σ declines by over 40% owing to the distorting effects of transitory variation. At this minimum value, the low-pass filter is neutral, the raw data are not transformed, low-frequency variation is not emphasized, and estimates of σ are biased downward by standard estimation methods.

Section VI documents that the benchmark result is robust to an endogenous regressor, split-samples, alternative first-order conditions, and heterogeneous σ's across industries.

Section VII summarizes and concludes.

II. A Crucial Assumption: The Value of σ

Several economic issues depend crucially on the value of σ. This section discusses the role played by σ in understanding the decline in the labor share of income and the nature of long-run economic growth.

II.A. Factor Incomes

The elasticity of substitution between capital and labor (σ) was introduced by Hicks (1932) to analyze changes in the income shares of capital and labor.[1] His key insight was that the impact of the capital/labor ratio on the distribution of income (given output) could be completely characterized by the curvature of the isoquant (Blackorby and Russell, 1989, p. 882). Recent important work by Karabarbounis, and Neiman (2014) and by Piketty (2014a) and Piketty and Zucman (forthcoming) have linked the decline in the labor share of income over the past 20 years to a secular decrease in the relative price of investment and a secular increase in the capital/income ratio, respectively. For either explanation, it is crucial (as the authors note) that the elasticity of substitution is greater than one. If this elasticity is unity, relative prices and capital/income ratios have no effect on the labor share. If the elasticity is less than one, then variables highlighted in their analyses should have contributed to a counterfactual increase in the labor share. The value of σ is of first-order importance for analyzing the secular movement of the income distribution.

The returns to factors of production in an international context also depend on σ. Jones and Ruffin (2008) consider a model of international trade where some factors of production are

[1] This elasticity was independently introduced by Robinson (1933, p. 256). Their formulations differed and, in the second edition of his "Theory of Wages" published in 1963, Hicks (Section III, "Notes on the Elasticity of Substitution", sub-section 1) showed that the two formulations are equivalent. Robinson's definition has proven the more convenient and durable.

sector-specific. They show that σ, as well as the factor-intensity ranking between sectors emphasized in the Stolper-Samuelson model, determines whether changes in the terms of trade, perhaps due to trade liberalization, raises or lowers the real wage rate.[2] Liberalization is also studied by Chari, Henry, and Sasson (2012), though, in their case, financial aspects are examined. They find that that, three years after a stock market in an emerging economy is opened to foreign capital, the average annual growth rate in manufacturing real wages increases by a factor of three. As shown in their equation (9), the value of σ, along with the growth rate in and income share of capital, is key to understanding the empirical regularity.

II.B. Long-Run Growth

B.1. From The Harrod-Domar Knife-Edge to the Solow Interval

The neoclassical revolution in growth theory places the burden of equilibrium on the properties of the production function. When σ equals unity, the capital/labor ratio ($k \equiv K / L$) converges to a positive, finite value because, as k moves towards its limiting values of 0 (or ∞), the marginal product of capital and the average product of capital both tend to ∞ (or 0). With the Inada conditions satisfied, capital accumulation, as determined by Solow's fundamental equation of motion for k, converges to zero.

However, when σ departs from unity, the Inada conditions may fail, and perpetual decline or perpetual growth may occur. Values of σ less than one and less than a critical value (σ_{PD}) that depends on other parameters of the Solow growth model results in perpetual decline in k and output. This unbalanced growth path reflects a failure of an Inada condition and the convergence of the marginal and average products of capital to a positive finite limit such that no root exists to satisfy the equation of motion for k. Details are presented in Appendix A. Intuitively, when σ is low, capital and labor are "dissimilar" productive factors (Brown, 1968, p. 50). With limited substitution possibilities, reductions in capital have little positive impact on the marginal productivity of capital. For a value of $\sigma < \sigma_{PD} < 1$, k declines perpetually. Conversely, values of σ greater than one and greater than a critical value (σ_{PG}) lead to perpetual growth, as capital and labor are very "similar" and the marginal product of capital declines very

[2] The value of σ also serves to reconcile their theoretical results with the simulations by Melvin and Waschik (2001) showing that trade uniformly lowers real wages.

slowly.[3] Balanced growth occurs for values of σ between σ_{PG} and σ_{PD}; in this no growth case, capital accumulation converges to zero and k to a positive, finite value. The Solow growth model replaces the Harrod-Domar knife-edge with the "Solow Interval," the latter defined by these two critical σ's.

In their textbook on economic growth, Burmeister and Dobell (1970, p. 34) refer to situations where $\sigma \neq 1$ as "troublesome cases" because they do not yield balanced growth paths. It is far from clear why the requirements for balanced growth paths in a particular theoretical model should dictate the shape of the production function, especially when $\sigma = 1$ is a sufficient but not necessary condition for a balanced growth path. To treat σ as a free parameter determined by the theory runs dangerously close to the fallacy of affirming the consequent. An alternative approach -- represented in the models developed by Acemoglu (2003, 2009), Antrás (2004), Eicher and Turnovsky (1999), and Turnovsky and Smith (2006) -- interprets cases where $\sigma \neq 1$ as quite interesting, suggesting needed modifications to the standard growth model and highlighting the key role played by σ.

B.2 Per Capita Income

The value of σ is linked to per capita income and growth. Klump and de La Grandville (2000) show that, for two countries with identical initial conditions, the country with a higher value of σ experiences higher per capita income at any stage of development, including the steady-state (if it exists).[4] De La Grandville (1989, 2009) argues theoretically that a relative price change (e.g., a decrease in the price of capital) leads to relatively more output the higher the value of σ. (He also identifies a second channel depending on σ -- a higher substitution elasticity permits a greater flow of resources between sectors with different factor intensities.) Yuhn (1991) and Mallick (2012) find empirical support of this hypothesis for South Korea and across 90 countries, respectively.

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[3] Solow (1956, pp. 77-78), Pitchford (1960), and Akerlof and Nordhaus (1967) were the first to note the possibility of perpetual growth. See the papers cited in fn.1 of Appendix A for more recent statements.

[4] Miyagiwa and Papageorgiou (2003) demonstrate, however, that a monotonic relationship between σ and growth does not exist in the Diamond overlapping-generations model.

B.3. Speed of Convergence

The speed of convergence to the steady-state depends on σ through capital accumulation. In a calibrated neoclassical growth model, Turnovsky (2002, pp. 1776-1777) finds that the rate of convergence is sensitive to and decreasing in σ. For a given productivity shock, the speed of convergence is 45.3% (per year) when σ equals 0.1, but drops markedly to 12.2% when σ equals 0.8. The speed of convergence falls further to 8.9%, 6.4%, and 3.5% as σ increases to 1.0, 1.2, and 1.5, respectively.[5] Klump and Preissler (2000, equation 14) show that the speed of convergence is sensitive to the relation between the initial and steady-state capital intensities.

B.4. Other Issues in Growth and Development

The value of σ can play an important role in assessing the plausibility of the neoclassical growth model. King and Rebelo (1993, Section IV) show that, in a Cass-Koopmans model with endogenous saving, the rate of return on capital (R) is sensitive to σ and can be implausibly high when some part of growth is due to transitional dynamics. When transitional dynamics are important, R increases dramatically with σ. Mankiw (1995, p. 287, equation (13)) also investigates the relation between σ and R in terms of the following formula,

$$R = (y)^{-\theta} Q \qquad \theta \equiv (1/\sigma) * ((1-\mu^K)/\mu^K), \tag{1}$$

where μ^K is capital's factor share, y is per capita income, and Q represents other factors affecting the return to capital. Equation (1) links rates of return to income and can be used to evaluate rate of return differentials between poor and rich countries. For example, if $\sigma = 4.0$, the differential is only about 3 percent.[6] But if σ falls to 1.0 or 0.5, the differential becomes implausibly large, increasing to 100 and 10,000 respectively, clearly raising concerns about the standard growth model.

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[5] These figures are based on an intratemporal elasticity of substitution between consumption and leisure of 1.0 and an intertemporal elasticity of substitution for the composite consumption good of 0.4. The pattern of results is robust to variations in the latter parameter.

[6] These computations are based on $\mu^K = 0.33$ and an income level in rich countries that is 10 times larger than in poor countries.

Purported differences in measured rates of return to capital across countries – the Lucas Paradox (Lucas, 1990) -- have posed a persistent puzzle. Caselli and Feyrer (2007) introduce a new method for calculating rates of return and show that the cross-country dispersion in rates of return is eliminated. While these estimates are independent of σ, their counterfactual exercises depend on production function characteristics. A value of $\sigma < 1$ reinforces the paper's general conclusions about the modest welfare benefits of reallocating the world's capital stock and the likely ineffectiveness of foreign aid to poorer countries.

The role played by technical change in growth models is sensitive to σ. If $\sigma = 1$, factor-augmenting technical change becomes indistinguishable from neutral technical change. If $\sigma < 1$ (> 1) factor-augmenting technical change lowers (raises) the intensity of the usage of that factor (see equation (3) below). Acemoglu (2003) examines the relations among technical change, the value of σ, and balanced growth. He develops a model in which technical change is both labor-augmenting and capital-augmenting and shows that, along the balanced growth path, all technical change will be labor-augmenting. If $\sigma < 1$, technical change stabilizes income shares, and the balanced growth path is stable and unique. In his review of developmental accounting (which assesses what percentage of cross-country differences in per capita income are attributable to productive factors and technical efficiency), Caselli (2005, Section 7) shows that the relative roles are very sensitive to σ. When σ is near 0.5, variation in productive factors accounts for almost 100% of the variation in per capita income across countries, and technical efficiency plays no role. The percentage decreases in σ and drops to 40% for $\sigma = 1.0$ (the Cobb-Douglas case) and 25% for $\sigma = 1.5$. Caselli (2005, p. 737) concludes “that the Cobb-Douglas assumption is a very sensitive one for development accounting.”

III. Estimation Strategy

III.A. The First-Order Condition

Our approach focuses on long-run production relations and low-frequency variation in model variables. The long-run is defined by the vector of output and inputs consistent with profit maximization when all inputs can be adjusted without incurring costly frictions. This focus allows us to ignore short-run adjustment issues that are difficult to model and may bias estimates if misspecified. Production for industry i at time t is characterized by the following

Constant Elasticity of Substitution (CES) technology that depends on long-run values denoted by an *,

$$Y^*_{i,t} = Y[K^*_{i,t}, L^*_{i,t}, A_{i,t}, A^K_{i,t}, A^L_{i,t}] \tag{2}$$

$$= A_{i,t} \left\{ \phi (A^K_{i,t} K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^L_{i,t} L^*_{i,t})^{[(\sigma-1)/\sigma]} \right\}^{[\sigma/(\sigma-1)]},$$

where $Y^*_{i,t}$ is long-run real output, $K^*_{i,t}$ is the long-run real capital stock, $L^*_{i,t}$ is the long-run level of labor input, ϕ is the capital distribution parameter, and σ is the elasticity of substitution between labor and capital. Technical progress is both neutral ($A_{i,t}$), and factor-augmenting for capital and labor ($A^K_{i,t}$ and $A^L_{i,t}$, respectively). Equation (2) is homogeneous of degree one in $K^*_{i,t}$ and $L^*_{i,t}$ and has three desirable features for the purposes of this study. First, this production function depends on only two parameters -- ϕ the distribution of factor returns and, most importantly, σ representing substitution possibilities between the factors of production. Second, the CES function is strongly separable and thus can include many additional factors of production (e.g., intangible capital) without affecting the estimating equation derived below. This feature gives the CES specification an important advantage relative to other production functions that allow for a more general pattern of substitution possibilities (e.g., the translog, minflex-Laurent). Third, the Cobb-Douglas production function is a special case of the CES; as $\sigma \to 1$ and factor-augmenting technical change disappears ($A^K_{i,t} = 1 = A^L_{i,t},$), equation (2) becomes $Y^*_{i,t} = A_{i,t} \left\{ K^{*\,[\phi]}_{i,t} \; L^{*\,[1-\phi]}_{i,t} \right\}$.

Constrained by the CES production function (2), a profit-maximizing firm chooses capital so that its marginal product equals the relative price of capital (or the Jorgensonian user cost of capital), defined as the price of capital, $P^{K*}_{i,t}$ (which combines interest, depreciation, and tax rates and the nominal price of capital goods), divided by the tax-adjusted price of output, $P^{Y*}_{i,t}$. (The firm also sets the marginal product of labor equal to the tax-adjusted nominal wage rate, $P^{L*}_{i,t}$, divided by $P^{Y*}_{i,t}$; this condition will be utilized in Section VI.C.) Differentiating equation (2) with respect to capital and rearranging terms (the tedious details are in Appendix B), we obtain the following factor demand equation for the long-run capital/output ratio,

$$(K_{i,t} / Y_{i,t})^{*} = \phi^{\sigma} \; ((P^{K}_{i,t} / P^{Y}_{i,t})^{*})^{-\sigma} \; U^{KY}_{i,t} \, , \tag{3a}$$

$$U^{KY}_{i,t} \equiv A^{[\sigma-1]}_{i,t} \; A^{K[\sigma-1]}_{i,t} . \tag{3b}$$

The impact of both neutral and capital-augmenting technical change are captured in a two-way error component model of the error term,

$$U^{KY}_{i,t} = \exp[u^{KY}_{i} + u^{KY}_{t} + u^{KY}_{i,t}] , \tag{4}$$

where $u^{KY}_{i,t}$ may have a non-zero mean. Taking logs of equation (4) and defining

$ky^{*}_{i,t} \equiv \ln(K_{i,t} / Y_{i,t})^{*}$ and $p^{KY*}_{i,t} \equiv \ln(P^{K}_{i,t} / P^{Y}_{i,t})^{*}$, we obtain,

$$ky^{*}_{i,t} = \sigma \ln(\phi) - \sigma \, p^{KY*}_{i,t} + u^{KY}_{i} + u^{KY}_{t} + u^{KY}_{i,t} \, . \tag{5}$$

Removing fixed industry effects by first-differencing and defining $\tau^{KY}_{t} \equiv \Delta u^{KY}_{t}$ and

$e^{KY}_{i,t} \equiv \Delta u^{KY}_{i,t}$, we obtain the following estimating equation,

$$\Delta ky^{*}_{i,t} = \zeta^{KY} - \sigma \, \Delta p^{KY*}_{i,t} + \tau^{KY}_{t} + e^{KY}_{i,t} \, , \tag{6}$$

where τ^{KY}_{t} is an aggregate fixed time effect and ζ^{KY} is a constant term (included in place of one of the τ^{KY}_{t}'s). Conditional on time fixed effects, identification of σ is achieved by the correlation between the growth rates of the capital/output ratio and its relative price.

Given observations on $\Delta ky^{*}_{i,t}$, and $\Delta p^{KY*}_{i,t}$, equation (6) provides a rather straightforward framework for estimating σ. Consistent estimates are obtained because the factor prices, largely driven by aggregate factors, are assumed exogenous. (This assumption is relaxed in Section VI.A., which contains instrumental variables estimates.) Importantly, in light of the critique and evidence by Antrás (2004), our estimates of σ control for factor-augmenting technical change.[7]

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[7] See his equation (1'), which is comparable to our equation (5) and, with first-differencing, equation (6). The effects of factor-augmenting technical change are removed by a linear time trend in Antrás' framework based on aggregate data and by time effects in our framework based on panel data. If we adopt Antrás' specification of facto-augmenting technical change, $A^{K}_{i,t} = \exp[\lambda t]$, the λ is absorbed in the constant in equation (6). Panel data permit a more general specification for controlling for the effects of factor-augmented technical change.

The key unresolved issue is the unobservability of long-run values denoted by *'s, an issue to which we now turn.

III.B. Low-Pass Filters and Long-Run Values

Previous research on capital formation has addressed this unobservability problem in several ways.[8] The cointegration approach introduced by Caballero (1994, 1999) provides an elegant solution for extracting long-run values from data subject to short-run deviations. While innovative, this estimation strategy faces some econometric difficulties in recovering production function parameters (Chirinko and Mallick, 2011). Karabarbounis, and Neiman (2014) introduce a novel method that relies on the cross-country variation in the exponential trends of income shares and the price of capital.[9] These deterministic trends are "capturing movements from an initial to a final steady state" (p. 86). Their approach, as well as that of Caballero, defines the long-run at only the 0^{th} frequency,[10] as opposed to a broader band of frequencies used in the macroeconomics literature (cf., fn. 12), and thus discards information. Chirinko, Fazzari, and Meyer (2011) also emphasize cross-section variation. For a panel of firms, they divide the sample period in half, average the data in each of the two intervals, difference the interval-averaged data to remove nuisance parameters, and then estimate σ in a cross-section regression. A shortcoming of this estimation strategy is that it discards information in the panel dataset by relying on disjoint intervals, as well as defining the long-run at only the 0^{th} frequency. The most frequently used approach estimates an investment equation that begins with the first-order condition for capital and links changes in the observed capital stock to changes in the unobserved long-run capital stock by assuming that the latter is determined by changes in output and the relative price of capital and that these changes are distributed over time due to various short-run frictions (Chirinko, 1993, Section II). Relying on investment data solves the unobservability problem by imposing a set of assumptions about dynamics and frictions affecting the investment

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[8] Engle and Foley (1975) also use a band-pass filter to study capital formation. They estimate a model relating investment spending to an equity price series (approximately a Brainard-Tobin's Q variable) and use a band-pass filter to emphasize middle frequencies centered at two years.

[9] The Karabarbounis and Neiman strategy for estimating σ also focuses on long-run movements, and hence is similar in spirit to the econometric approach in this paper.

[10] For the frequency domain properties of deterministic and stochastic trends, see Granger (1964, p. 130) and Engle and Granger (1987, p. 253), respectively.

process. These assumptions – for example, convex vs. non-convex adjustment costs or the existence and nature of financial frictions -- can be controversial.

Our approach also focuses on the first-order condition for capital that holds in the long-run but uses a low-pass filter (LPF) to measure the long-run values of variables denoted by *'s.[11] An LPF allows frequencies lower than some critical frequency, $\omega^{\#}$, to pass through to the transformed series but excludes frequencies higher than $\omega^{\#}$. Baxter and King (1999) present two important results regarding band-pass filters for the purpose of the current study. They derive the formulas that translate restrictions from the frequency domain into the time domain. For an input series, x_t, the ideal LPF for a critical value $\omega^{\#}$ produces the transformed series, $x_t^*[\omega^{\#}, q]$, where q represents the "window," the length of the lags and leads used in computing the filter. As $q \rightarrow \infty$, we obtain the ideal LPF for x_t,

$$x_t^*[\omega^{\#}, q] = \lim_{q\rightarrow\infty} \sum_{h=-q}^{q} d_h[\omega^{\#}]\, x_{t-h}\,, \tag{7a}$$

$$d_h[\omega^{\#}] = d'_h[\omega^{\#}] + \theta[\omega^{\#}, q]\,, \tag{7b}$$

$$d'_h[\omega^{\#}] = \omega^{\#} / \pi\,, \qquad h = 0, \tag{7c}$$

$$d'_h[\omega^{\#}] = \sin[|h|\ \omega^{\#}] / (|h|\ \pi)\,, \qquad h = \underline{+}1, \underline{+}2, \ldots, q, \tag{7d}$$

$$\theta[\omega^{\#}, q] = \lim_{q\rightarrow\infty} \left(1 - \sum_{h=-q}^{q} d'_h[\omega^{\#}] \right) \Big/ \ (2q+1)\,, \tag{7e}$$

$$\omega^{\#} = 2\pi / p^{\#} = h[p^{\#}]\,, \qquad p^{\#} = [2, \infty), \tag{7f}$$

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[11] Lucas (1980, esp. fn. 1 and p. 1013) also uses low-frequency filtered data to measure long-run variables. Relative to Lucas' exponentially smoothing filter, the LPF used in our paper can more easily be related to periodicities and readily interpretable magnitudes. Apart from Lucas' filter, there are two additional filters for isolating frequencies of interest (to the best of our knowledge, neither has been used to estimate σ). The Hodrick-Prescott (HP) filter (1997) has been primarily used to study business cycle frequencies in quarterly data. Baxter and King (1999, Section V.C) note two problems with applying the HP filter to annual data: the appropriate value for the smoothing parameter (λ) is not well established and there is a substantial divergence between HP and band-pass filters. The Christiano-Fitzgerald (2003) filter is based on the assumption that the raw data follow a random walk. This assumption is important in deriving the estimator (cf. their fn. 17) and is not appropriate for the data in the present study.

where the $d_h[\omega^{\#}]$'s are weights defined as the sum of two terms – a provisional set of weights denoted by a prime (the $d'_h[\omega^{\#}]$'s in equations (7c) and (7d)) and a frequently imposed normalization that the $d_h[\omega^{\#}]$'s sum to 1 (per the constant $\theta[\omega^{\#},q]$ computed in equation (7e)). Equation (7g) defines the inverse relation between the critical frequency ($\omega^{\#}$) and the critical periodicity ($p^{\#}$), the latter defined as the length of time required for the series to repeat a complete cycle. Since periodicities are somewhat easier to interpret than frequencies, this relation will prove useful when discussing the empirical results.

A difficulty with implementing equations (7) is that the *ideal* LPF requires an infinite amount of data. Baxter and King's second important result is that the *optimal approximate* LPF for a window of finite length q truncates the symmetric moving average at q. Thus, for $|h| \leq q$, the $d_h[\omega^{\#}]$'s are given in equations (7); for $|h| > q$, $d_h[\omega^{\#}] = 0$. The optimal approximate LPF for the critical frequency $\omega^{\#}$ and window q, $LPF[\omega^{\#},q]$, is given by equations (7) for any finite q.

IV. Spectral Properties of the Low-Pass Filter

Our estimation strategy is designed to emphasize long-run variation, and this section uses spectral analysis to evaluate our approach and the choices of $\omega^{\#}$ and q. (Details about the spectral analysis used in this section are provided in Appendix C.) The properties of a time series can be conveniently represented in the frequency domain by the spectrum defined as a Fourier transformation (a series of trigonometric functions) of autocovariances associated with a time series. The spectrum is defined over frequencies from 0 to π, and the integral of the spectrum from 0 to an arbitrary frequency less than or equal to π measures the variance of the series over those frequencies. This property will be particularly useful for assessing how well our econometric model emphasizes long-run frequencies.

Transformations of the data alter the spectra and hence the weights given to long-run, medium-run, and short-run frequencies. The estimating equation is derived by three transformations in the time domain: a) defining long-run values with the $LPF[\omega^{\#},q]$ (equations (7)); b) inserting these long-run values into the first-order condition for optimal capital accumulation and taking logarithms (equation (5)); c) first-differencing this logarithmic equation to remove industry fixed effects (equation (6)). To compute the spectrum of a transformed series, we rely on the fundamental result from spectral analysis linking the spectrum of an output series (e.g., $x^*_t[.]$) to the product of the spectrum of an input series (e.g., x_t) and a scalar that will, in general, be a function of ω, $\omega^\#$, and q. To understand the impact of each step, we need only compute the frequency response scalar associated with each transformation. The scalars for the LPF, logarithmic, and first-difference transformations are represented by $a[\omega, \omega^\#, q]$, b, and $c[\omega]$, respectively, and, per the analysis in Appendix C, are defined as follows,

$$a[\omega,\omega^\#,q] = \alpha[\omega^\#,q] \left\{ \begin{array}{l} (\omega^\# / \pi) + 2\sum_{h=1}^{q} \cos[h\,\omega]\, d'_h[\omega^\#] \\ + \theta[\omega^\#,q]\, \{(1-\cos[\omega(2q+1)])/(1-\cos[\omega])\}^{1/2} \end{array} \right\}^2 , \tag{8a}$$

$$b = \beta\, (\mu_{x*})^{-2} , \tag{8b}$$

$$c[\omega] = \gamma\; 2\; (1-\cos[\omega]) , \tag{8c}$$

where μ_{x*} equals the unconditional expectation of $x^*_t[.]$. To ensure comparability in the analyses to follow that vary $\omega^\#$ and q, the areas under the spectra from 0 to π are normalized to one by an appropriate choice of normalizing scalars, $\alpha[\omega^\#,q]$, β, and γ in equations (8).

With the frequency response scalars defined in equation (8), we are now in a position to examine the extent to which our estimation strategy emphasizes long-run frequencies. Since the spectra for the raw series and the scalars associated with the logarithmic and first-difference transformations (b and $c[\omega]$, respectively) do not depend on $\omega^\#$ or q, their impacts on the data will be absorbed in the normalizing scalars, and hence they will not affect relative comparisons. Alternative values of $\omega^\#$ or q, will only affect the $LPF[\omega^\#,q]$ and the associated scalar, $a[\omega,\omega^\#,q]$.

Our first set of analyses holds the window fixed at q = 3 and examines different values of the critical frequency, $\omega^\#$, that determines which frequencies are passed-through by the $LPF[\omega^\#,q]$. Recall that, per equation (7f), there is a one-to-one inverse relationship between frequencies (ω and $\omega^{\#}$) and periodicities (p and $p^{\#}$), and it is easier to discuss the properties of low-pass filters in terms of the latter. Four values of $p^{\#}$ are considered in Figure 1. We begin with the minimum value of the critical periodicity, $p^{\#} = 2$, which corresponds to a standard investment equation that does not transform the raw data (other than the logarithmic and differencing operations) or an untransformed first-order condition. The associated frequency response is flat, indicating that this estimator does not reweight the variances of the raw series across frequencies. By contrast, our benchmark model represented by a critical periodicity of eight years results in a substantial reweighting. With $p^{\#} = 8$, the benchmark model emphasizes the variances from periodicities greater than or equal to eight years (corresponding to $\omega^{\#} \leq 0.79$ on the horizontal axis), thus allocating a substantial amount of weight to those frequencies appropriate for estimation of production function parameters. The remaining entries in Figure 2 are for the intermediate cases, $p^{\#} = 4$ and $p^{\#} = 6$.

The benchmark model is based on the assumption that periodicities greater than or equal to eight years contain useful information for the parameter estimates. How much additional reweighting occurs when the critical periodicity is greater than eight years? We examine values of $p^{\#}$ equal to 10, 20 and, in the limit, ∞. The $LPF[\omega^{\#}, q]$'s corresponding to these critical values are graphed in Figure 2. The frequency responses for these higher periodicities indicates that they weight the lower frequencies in a manner very similar to the benchmark case of $p^{\#} = 8$.

This analysis suggests two conclusions concerning our choice of the critical periodicity. First, our estimation strategy based on $p^{\#} = 8$ appears to be reasonably successful in emphasizing long-run variation. This result is consistent with what appears to be a well-accepted standard for separating long-run frequencies from short-run and medium-run frequencies.[12] Second, results in Figure 2 suggest that parameter estimates are likely to be insensitive to the critical periodicity for values of $p^{\#} > 8$.

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[12] A critical value of a periodicity of eight years is used by Baxter and King (1999, p. 575), Levy and Dezhbakhsh (2003, p. 1502), Prescott (1986, p. 14), and Stock and Watson, 1999, p. 11). Burns and Mitchell (1946) report that the duration of the typical business cycle in the U.S. is less than eight years.

We can also use the spectral formulas to assess the impact of variations in the window, q, in approximating the ideal low-pass filter. Recall that the ideal LPF is based on the limiting behavior as q --> ∞. This ideal filter, $LPF[\omega^{\#}, q \rightarrow \infty]$, is represented by the rectangle in Figure 3 that only passes-through frequencies less than 0.79; frequencies greater than 0.79 are totally excluded. Our empirical work relies on the optimal approximation based on a finite number of q leads and lags. This approximation introduces error into the analysis because variances associated with frequencies other than those desired enter into the transformation of the model variables. However, increasing q is costly in terms of lost degrees of freedom.

This tradeoff between approximation error and degrees of freedom is assessed in Figure 3, which plots $LPF[\omega^{\#}, q]$ for $\omega^{\#} = 0.79$ (corresponding to $p^{\#} = 8$) and values of q equal to 1, 3 and 5. When q = 1, the LPF is extensively contaminated by the variances associated with frequencies above the critical frequency. We measure this contamination by $\chi[q]$, defined as the area under the frequency response curve for the interval $\omega = [0.79, \pi]$. For q = 1, $\chi[q = 1] = 0.314$. As q increases to 3 and then 5, this contamination is greatly reduced with $\chi[q = 3] = 0.055$ and $\chi[q = 5] = 0.054$, respectively. The $\chi[q]$'s for q = 3 and q=5 are nearly identical. Since using a window of q = 5 is costly in terms of degrees of freedom and roughly the same frequencies are emphasized, we will adopt q = 3 as our preferred window, though robustness will be examined with q = 1 and q = 5.[13]

V. Benchmark Empirical Results

This section estimates the substitution elasticity using our low-frequency panel model defined with various critical periodicities and windows. Data are obtained from the webpage of Dale Jorgenson and represent output, inputs, and prices for 35 industries for the period 1960-2005.[14] Summary statistics are presented in Table 1, which contains statistics for the capital/output, labor/output, and capital/labor models in panels A., B., and C., respectively.

[13] Baxter and King (1999, pp. 581-582) reach a similar conclusion based on their analysis of band-pass filters at medium frequencies.

[14] The data are obtained at http://post.economics.harvard.edu/faculty/jorgenson/data/35klem.html and are described in Jorgenson and Stiroh (2000, especially the appendices). The effective time dimension equals the 46 datapoints contained in the dataset for a given industry less 2q for the construction of the LPF less one for first differencing.

In a given panel, the first and third rows are the model variables; the second and fourth rows are comparable variables that have not been transformed by the LPF. Comparisons of the first two rows and then of the third and fourth rows document that the LPF substantially reduces variation in the model variables. As shown in columns 3 and 4 for the interquartile range and column 5 for the within industry standard deviation, the LPF substantial lowers the variation in the variables. By either measure, the variation is approximately halved due to the reweighing of the lower frequencies. Column 6 reports the variance due to industry-specific time variation as a percentage of the overall variance in the series and confirms that, even with industry and common time variation removed, there remains a great deal of variation within industries with which to estimate the parameter of interest.

The benchmark OLS results from our low-frequency panel model based on $p^{\#} = 8$ and q = 3 are as follows,

$$\Delta ky^{*}_{i,t} = \underset{(0.003)}{0.006} - \underset{(0.034)}{0.406}\ \Delta p^{K*}_{i,t} + \tau^{K}_{t} + e^{K*}_{i,t}, \qquad R^2 = 0.503 . \tag{9}$$

The point estimate for σ is 0.406, and it is precisely estimated. Since the low-pass filter creates overlapping observations, the standard errors are computed with the procedure of Newey and West (1994); we use (T-1) lags, which is equivalent to using Bartlett kernel with a bandwidth of T. This correction is important, and doubles the size of the standard error relative to a White (1980) procedure. These results are robust in several dimensions. When important differences occur, they are due to the presence of high frequency variation in the model variables.

Table 2 examines the sensitivity of estimates of σ to variations in the window (q) and the critical periodicity ($p^{\#}$). For a given $p^{\#}$, estimates of σ are robust to variations in q. For example, when $p^{\#} = 8$, estimates of σ are 0.331, 0.406, and 0.379 for q of 1, 3, and 5, respectively. As the window is increased, more data are used in computing the filters, less data are available for estimation, and the standard errors generally increase. Nonetheless, the standard errors for the σ's remain less than 0.05 for all entries. (The R^2's are not strictly comparable across cells because the dependent variable depends on $p^{\#}$ and q.) These results suggest that little is gained by increasing the size of the window and compromising degrees of freedom above q = 3.

Table 2 also allows us to assess the robustness to variations in $p^{\#}$ for a given q by reading down the columns. For q = 3 in column 2, as $p^{\#}$ increase from 8 to ∞, the estimates of σ hardly change. This robustness is consistent with the theoretical analysis in Figure 2 and suggests that the relevant information about the long-run has been largely captured at $p^{\#} = 8$. However, when $p^{\#}$ is set to its minimum value of 2, the low-pass filter is neutral, and the raw data are transformed only by logarithmic and first-difference operations that do not depend on $p^{\#}$ (cf. equations (7) and (8)). In this case, σ drops by over 40% relative to the benchmark value (0.229 vs. 0.406). Medium and high-frequency transitory variation affects the estimates of σ and, as has been frequently noted with permanent income models (e.g., Hayashi, 2000, pp. 194-195), transitory variation attenuates point estimates. Standard estimation methods, which do not filter-out transitory variation, generate downwardly biased estimates of σ.

VI. Alternative Estimates

This section examines the robustness of our benchmark result in equation (9) to an endogenous regressor, split-samples, alternative first-order conditions, and heterogeneous σ 's across industries.

VI.A. Instrumental Variables

The OLS estimates reported in Section V are valid under the assumption that the growth rate in the long-run relative price of capital is uncorrelated with the error term. This assumption may be compromised because our procedure for measuring the true long-run value is subject to error or a common factor exists that affects the regressor and enters the error term. Since this assumption necessary for consistent estimation may not hold strictly, instrumental variable (IV) estimates provide a useful robustness check. A lagged regressor is a familiar candidate instrumental variable. However, in our model, the low-pass filter depends on leads that may be correlated with the contemporaneous error term. Hence, a lagged regressor is not a suitable instrument unless it is lagged many periods, potentially compromising its relevance. While the model developed in Section III places restrictions on the construction of model variables, it does not place restrictions on the instrument, and thus we create an alternative instrument, $p_{i,t-2}^{KY\&}$, as a one-sided lag of the relative price of capital and lag it two periods (see the note to Table 3 for details). This variable will be uncorrelated with the contemporaneous error term and will deliver consistent estimates.

Table 3 contains IV results for equation (6) and the same combinations of $p^{\#}$ and q reported for the OLS results. However, the R^2 statistic is removed and replaced by an F-statistic for the first-stage auxiliary regression of the regressor on the instrument and time dummies. Instrumental relevance is assessed with the latter statistic proposed by Stock, Wright, and Yogo (2002), which involves the above auxiliary regression and a comparison of the F-statistic for the goodness of fit to a critical value of 8.96 (reported in their Table 1). Based on this critical value, the instrument is not weak for all models except those in row 1, which do not transform the raw data to emphasize lower frequencies. For the other estimates with relevant instruments, the estimated σ's tend to be about 10% larger than the comparable OLS estimates. For example, for our preferred estimate based on $p^{\#} = 8$ and q = 3, $\sigma_{IV} = 0.438$, somewhat larger than $\sigma_{OLS} = 0.406$. The IV standard errors are two to three times larger than their OLS counterparts. Nonetheless, the results continue to decidedly reject the Cobb-Douglas hypothesis that $\sigma = 1$. These IV results document the robustness of the OLS estimates of σ.

VI.B. Split-Samples and Time Fixed Effects

To further assess the robustness of our results, Table 4 contains OLS estimates from the first and second halves of the sample, 1960-1982 and 1983-2005, respectively. The results closely follow those reported previously. For example, for our preferred specification with $p^{\#} = 8$ and q = 3, the estimates of σ from the first and second halves of the sample are 0.336 and 0.460, respectively. These estimates are not economically different from our preferred estimate from the full sample of 0.406. Owing to the precision of our estimates, these deviations from the benchmark are statistically significant; for our preferred estimate based on $p^{\#} = 8$ and q = 3, the null hypotheses for the equality between point estimates is rejected at the 1% level.

The benchmark results are also robust with respect to the inclusion of time fixed effects. When they are removed, σ [standard error] $\{R^2\}$ increases slightly to 0.415 [0.029] {0.482}. This stability of results suggests that our low-pass filter is performing reasonably well at removing short-run and medium-run variation that would be captured by time fixed effects.

VI.C. Other Estimating Equations

The first-order conditions for profit maximization yield two additional estimating equations that contain the labor/output ratio or the labor/capital ratio as the dependent variable.[15] However, the Solow growth model implies that neither series is stationary, an implication consistent with the first two of the stylized facts of growth advanced independently by Kaldor (1961) and by Klein and Kosobud (1961).[16] Hence, the low-pass filter used in this study is not strictly applicable because spectral methods require stationary data. Murray (2003) documents the problems that can arise when band-pass filters are applied to nonstationary data. With this important caveat noted, we nonetheless examine estimates of σ derived from the equations with labor/output, $\ell y^*_{i,t}$, or the capital/labor ratio, $k\ell^*_{i,t}$, as the dependent variable, and estimate the following equations,

$$\Delta \ell y^*_{i,t} = \zeta^{LY} - \sigma \Delta p^{LY*}_{i,t} + \tau^{LY*}_{t} + e^{LY*}_{i,t}, \tag{10}$$

$$\Delta k\ell^*_{i,t} = \zeta^{KL} - \sigma \Delta p^{KL*}_{i,t} + \tau^{KL*}_{t} + e^{KL*}_{i,t}, \tag{11}$$

where $p^{LY*}_{i,t} \equiv \ln(P^{L}_{i,t} / P^{Y}_{i,t})^*$, $p^{KL*}_{i,t} \equiv \ln(P^{K}_{i,t} / P^{L}_{i,t})^*$, and the other elements in equations (10) and (11) parallel those defined in equation (6).

Table 5 contains σ's and R^2's for q = 3 and the usual range of $p^{\#}$'s. Column 1 contains the previously reported estimates (Table 2) for $\Delta ky^*_{i,t}$ and columns 2 and 3 the results for $\Delta \ell y^*_{i,t}$

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[15] Several recent studies have normalized the CES production function and estimated it along with the first-order conditions for labor and capital (Klump, McAdam, and Willman, 2007, 2012). While normalization has been shown to be important in estimating these systems (León-Ledesma, McAdam, and Willman, 2010), normalization does not affect the estimation of first-order conditions. We do not pursue systems estimation because the production function must include all relevant factors of production (a concern avoided with first-order conditions derived from strongly separable production functions), and this and other specification issues may lead to specification errors that are transmitted through the equation system and distort our estimate of σ.

[16] These facts remain well-accepted today. Jones and Romer (2010, p. 225) write that "Kaldor's first five facts have moved from research papers to textbooks." Acemoglu and Guerrieri (2008, p. 467) claim that "[m]ost models of economic growth strive to be consistent with the 'Kaldor facts,'…" As an aside, we would take some exception to whether the capital and labor income shares have remained constant in recent decades.

and $\Delta k\ell^{*}_{i,t}$, respectively. Relative to the results with $\Delta ky^{*}_{i,t}$, the σ's estimated with the $\Delta\ell y^{*}_{i,t}$ equation are higher for all critical periodicities and the standard errors higher by at least a factor of three. For $\Delta k\ell^{*}_{i,t}$, the σ's are uniformly lower and the standard errors about 50% higher. In all cases, the point estimates remain very far from one.

IV.D. Heterogeneous Industry σ_i's

Given our interest in the impact of aggregate σ on growth theory, the homogeneity assumption imposed across industries in the prior two sections is a natural way of obtaining an aggregate substitution elasticity. From an estimation perspective, however, it might be desirable to exploit the panel feature of our dataset and allow the σ's to differ across industries. The aggregated σ (σ_{agg}) is a weighted average of the industry σ's (σ_i's, see Appendix D for details),[17]

$$\sigma_{agg} = \sum_i \sigma_i * \omega_i , \qquad \omega_i \equiv \left(K_i / K_{agg}\right) \tag{12}$$

where K_{agg} is the aggregate capital stock and the ω_i's are industry weights defined in terms of industry capital ratios.[18]

Table 6 contains estimates of the heterogeneous industry σ_i's and the σ_{agg}'s. Panel A shows that the σ_i's are precisely estimated and are all statistically far from zero (save Communications) and far from unity (save Finance, Insurance & Real Estate). Panel B presents several σ_{agg}'s based on different weighting schemes. Column 1 use capital weights for all 35

[17] An alternative aggregation procedure has been developed in a provocative paper by Jones (2005), who formally relates industry and aggregate (global) production functions to the distribution of alternative production techniques (APT's) for combining capital and labor. His striking result is that the industry and aggregate production functions will be Cobb-Douglas in the long-run. This approach has the benefit of developing solid microfoundations for production functions but is sensitive to the assumed distribution of ideas. When APT's are distributed according to a Pareto distribution with independence between marginal APT distributions, the Cobb-Douglas result obtains. However, when APT's are distributed according to a Weibull distribution (Growiec (2008a)) or a Pareto distribution with dependence between marginal APT distributions (Growiec (2008b)), the industry and aggregate production functions are CES. These theoretical results, coupled with the empirical results presented in this paper, suggest the need for further study of aggregation procedures and the underlying distribution of ideas.

[18] It should be noted that the σ from the benchmark model is also effectively a weighted-average estimate. The heterogeneous model weights the σ_i's by industry capital shares, while the homogeneous model effectively weights the σ_i's by relative industry variances.

industries, and the $\sigma_{agg} = 0.657$ is larger than the benchmark value of 0.406. This difference is due to the distribution of capital weights. Column 2 weights the 35 industries equally, and the $\sigma_{agg} = 0.417$ is very close to the benchmark value of 0.406. The estimates presented in Table 6 provide additional support for the conclusion that the σ characterizing substitution possibilities in the U.S. economy is well-below the Cobb-Douglas value of one.

These results reflect historical production patterns. Is σ likely to rise as production shifts toward industries favored in a post-industrial economy?[19] Column 3 of Panel B allows for an informal examination of this question.[20] We identify post-industrial industries as those that are not agriculture, manufacturing, mining, and utilities. Thus, in our dataset, we focus on the following six industries: Construction (6), Transportation (28), Communications (29), Trade (32), Finance, Insurance and Real Estate (33) and Services (34). The σ_{agg} for these selective six industries is 0.857. These estimates support the notion that the σ for the aggregate economy is likely to rise in future years as production and hence capital weights rise in these post-industrial industries.

VII. Summary and Conclusions

The elasticity of substitution between labor and capital (σ) is a "crucial" parameter. The qualitative and quantitative answers to a host of issues in economics depend on the precise value of σ. This crucial production function parameter is estimated by combining a low-pass filter with panel data techniques to identify the long-run relations appropriate to production function estimation. Our preferred point estimate is 0.406, and it proves robust to variations in several directions.

We focused on two sets of economic issues for which σ is a crucial parameter. As noted in the quotation by Piketty and Summers at the beginning of this paper, whether σ is greater than or less than one is of crucial importance in the study of factor incomes. A secular increase

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[19] Arrow, Chenery, Minhas, and Solow (1961, p. 247) speculate that "[g]iven systematic intersectoral differences in the elasticity of substitution and in income elasticities of demand, the possibility arises that the process of economic development itself might shift the over-all elasticity of substitution."

[20] Piketty (2014a, p. 222) asserts that σ is lower in agriculture, a claim confirmed in Table 5, Panel A. Moreover, Piketty (2014b, p. 39) seems to suggest that σ may increase in more recent years: "…the elasticity of substitution between capital and labor in the future may be different (and possibly higher) than the past one."

in the capital/income ratio or a secular decreases in the relative price of investment or capital taxation can explain the recent decline only if $\sigma > 1$. For the United States, we find little support for this assumption.

In the Solow growth model, certain values of σ can lead to unbalanced positive or negative growth. When $\sigma < 1$, it cannot be an engine of perpetual growth. However, this value of the elasticity may be an agent of perpetual decline if it is less than a critical value that depends on the rates of saving, depreciation, population growth and the share of income accruing to labor. As shown in Table 1-A in Appendix A, $\sigma = 0.406$ is much larger than the critical value for the United States. Thus, our estimate of σ suggests that this elasticity is not so small as to create unbalanced negative growth for the U.S. economy. Several other issues concerning long-run growth (discussed in Section II) depend on the precise value of σ.

Our benchmark value of sigma of 0.406 suggests that the convenient assumption of a Cobb-Douglas production function used in many areas of economic analysis needs to be abandoned. DSGE models that maintain that $\sigma = 1$ amplify the true effect of price movements -- including the traditional channel of monetary policy -- relative to a model based on a lower value of this elasticity. Tax simulation models using Cobb-Douglas production functions impart a similar upward bias to the effects of tax cuts. A departure from a Cobb-Douglas production function will force an expansion of the neoclassical growth model to include, among other factors, a central role for directed technical change that affects factor shares and balances growth.

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Appendix A: Perpetual Growth, Perpetual Decline, and σ

This appendix examines the role of σ in generating non-standard growth behavior. Two conditions must hold: (1) $\sigma \neq 1$ and (2) certain relation between σ and the rates of saving (s), depreciation (δ), and population growth (n) and another production function parameter, ϕ, which is related to factor income shares.

We first consider the Solow's equation of motion for the capital/labor ratio ($k \equiv K/L$) and the limiting behavior of the marginal product of capital ($MPK[k:\sigma]$) and the average product of capital ($APK[k:\sigma]$).[21] The well-known equation of motion in the neoclassical growth model is as follows,

$$\dot{k}/k = s * APK[k:\sigma] - (n+\delta), \qquad \text{(A-1)}$$

where s, n, and δ are the rates of saving, population growth, and depreciation, respectively. The $MPK[k:\sigma]$ and the $APK[k:\sigma]$ are derived from the following intensive form of the CES production function (where we have ignored in this appendix the role of technical change),

$$f[k:\sigma] = \{\phi k^{((\sigma-1)/\sigma)} + (1-\phi)\}^{(\sigma/(\sigma-1))} \qquad 0 < \phi < 1, \qquad \text{(A-2)}$$

where $f[k:\sigma]$ is a per capita neoclassical production function depending on σ and ϕ (the capital distribution parameter). The $MPK[k:\sigma]$ and the $APK[k:\sigma]$ follow from equation (A-2) as follows,

$$MPK[k:\sigma] \equiv f_k[k:\sigma] = \phi\left\{\phi + (1-\phi)\, k^{((1-\sigma)/\sigma)}\right\}^{(1/(\sigma-1))}, \qquad \text{(A-3)}$$

$$APK[k:\sigma] \equiv \frac{f[k:\sigma]}{k} = \left\{\phi + (1-\phi)\, k^{((1-\sigma)/\sigma)}\right\}^{(\sigma/(\sigma-1))} = \left\{MPK[k:\sigma]/\phi\right\}^{\sigma}. \qquad \text{(A-4)}$$

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[21] The analysis is this sub-section draws on the presentations of the neoclassical growth model in Barro and Sala-i-Martin (1995, Section 1.3.3), Klump and Preissler (2000), Klump and de La Grandville (2000), de La Grandville (1989, 2009), de La Grandville and Solow (2004, which also discusses how increases in σ expand production possibilities in a manner similar to exogenous technical progress), and especially Mallick (2010).

We are now in a position to assess the impacts of the two conditions. Condition (1) implies that the $\mathrm{MPK}[k:\sigma]$ fails to satisfy one of the Inada conditions and that this positive, finite limit can affect the $\mathrm{APK}[k:\sigma]$ such that no root exists for equation (A-1) for any value of k. The limits for $\mathrm{MPK}[k:\sigma]$ and $\mathrm{APK}[k:\sigma]$ as $k \to \infty$ (equations (A-5)) and as $k \to 0$ (equations (A-6)) are as follows,

$k \to \infty$

$$\sigma > 1: \lim_{k\to\infty} \mathrm{MPK}[k:\sigma] = \lim_{k\to\infty} \mathrm{APK}[k:\sigma] = \phi^{(\sigma/(\sigma-1))} < 1, \tag{A-5a}$$

$$\sigma = 1: \lim_{k\to\infty} \mathrm{MPK}[k:\sigma] = \lim_{k\to\infty} \mathrm{APK}[k:\sigma] = 0, \tag{A-5b}$$

$$\sigma < 1: \lim_{k\to\infty} \mathrm{MPK}[k:\sigma] = \lim_{k\to\infty} \mathrm{APK}[k:\sigma] = 0. \tag{A-5c}$$

$k \to 0$

$$\sigma > 1: \lim_{k\to 0} \mathrm{MPK}[k:\sigma] = \lim_{k\to 0} \mathrm{APK}[k:\sigma] = \infty, \tag{A-6a}$$

$$\sigma = 1: \lim_{k\to 0} \mathrm{MPK}[k:\sigma] = \lim_{k\to 0} \mathrm{APK}[k:\sigma] = \infty, \tag{A-6b}$$

$$\sigma < 1: \lim_{k\to 0} \mathrm{MPK}[k:\sigma] = \lim_{k\to 0} \mathrm{APK}[k:\sigma] = \phi^{(\sigma/(\sigma-1))} > 1. \tag{A-6c}$$

The Inada conditions are met when $\sigma = 1$ (equations (A-5b) and (A-6b)) or, in one direction, when $\sigma > 1$ (equation (A-6a)) or when $\sigma < 1$ (equation (A-5c)). However, they fail in one direction when $\sigma > 1$ (equation (A-5a)) or when $\sigma < 1$ (equation (A-6c)).

The implications of these two failures are that perpetual growth or decline is possible. These non-standard cases depend on a second necessary condition comparing these limiting values to the critical values of the $\mathrm{APK}[k:\sigma]$ that set $\dot{k}/k$ equal to zero in equation (A-1). The critical value of $\mathrm{APK}[k:\sigma]$ can be stated in terms of the associated critical value of σ by setting equation (A-1) to zero and solving for σ_C, which will depend on four other model parameters (s. n, δ, and ϕ),

$$\sigma_C \equiv g[s, n, \delta, \phi] = \frac{\log[s/(n+\delta)]}{\log[(s\phi)/(n+\delta)]} >=< 1 \ . \qquad \text{(A-7)}$$

Whether σ_C is above, equal to, or below 1.0 depends on the values of the four parameters.

To assess the possibility of perpetual growth, perpetual decline, or balanced growth, we will consider only variations in the saving rate, holding the other three parameters fixed. Three cases need to be considered:

- **Case 1, Perpetual Growth:** If the saving rate is sufficiently large so that $s > (n+\delta)/\phi$, then, since $0 < \phi < 1$, the numerator is greater than the denominator, and equation (A-7) defines the critical value defining perpetual growth, $\sigma_{PG} > 1$. When $\sigma > \sigma_{PG} > 1$, no increase in capital accumulation, however large, will be able to push the $MPK[k:\sigma]$ and $APK[k:\sigma]$ sufficiently low to terminate perpetual growth.
- **Case 2, Perpetual Decline:** Alternatively, if the saving rate is sufficiently low so that $s < (n+\delta)$, then, since $0 < \phi < 1$, the numerator is less than the denominator, and equation (A-7) defines the critical value defining perpetual decline, $\sigma_{PD} < 1$. When $\sigma < \sigma_{PD} < 1$, no decrease in capital accumulation (bounded below by zero) will be able to push the $MPK[k:\sigma]$ and $APK[k:\sigma]$ sufficiently high to escape perpetual decline.
- **Case 3, Balanced Growth:** If the saving rate takes on intermediate values relative to Cases 1 and 2 ($(n+\delta)/\phi \geq s \ \geq (n+\delta)$), then the Inada conditions are satisfied, and balanced growth is achieved. We refer to this range of σ's as the Solow Interval.

The critical values depend on four parameters -- $s, n, \delta,$ and ϕ. We assume the following values for three of the parameters,

- $n = 0.01$,
- $\delta = 0.10$ (a standard value for business plant and equipment),
- $\phi = 0.33$ (this production function parameter is not identified in the first-order conditions estimated in this paper. We assume that it equals the capital's income share).

The most interesting variation is with the saving rate, and we evaluate the critical value with alternative values of s in Table A-1,

Table A-1: Critical Values of σ (σ_C) for Alternative Saving Rates

Saving Rate	σ_C		Saving Rate	σ_C
0.01	0.68		0.14	-0.28
0.02	0.61		0.15	-0.39
0.03	0.54		0.16	-0.51
0.04	0.48		0.17	-0.65
0.05	0.42		**0.18**	**-0.80**
0.06	0.35		0.19	-0.97
0.07	0.29		0.20	-1.17
0.08	0.22		0.21	-1.40
0.09	0.15		0.22	-1.67
0.10	0.08		0.23	-1.99
0.11	0.00		0.24	-2.37
0.12	-0.09		0.25	-2.85
0.13	-0.18			

For s = 0.01, the critical value for σ is 0.68, and it declines monotonically for the saving rates considered in Table A-1. (There is a singularity (not shown in the table) for 0.33 < s < 0.34. For s ≥ 0.35, σ_C is positive, declines monotonically in s and, as $s \to 1$, $\sigma_C \to 2.01$.) Over the last twenty years, the average saving rate is 0.18 (the ratio of gross saving (NIPA Table 5.1, Line 1) to GDP (NIPA Table 1.1.5, Line 1) averaged for the period 1993 to 2012). For this value of s, $\sigma_C < 0$. Thus, it is impossible for the economy to enter a state of perpetual decline (since σ cannot be less than zero), and the economy exhibits balanced growth.

Appendix B: Specifying the Marginal Product of Capital with Neutral and Factor-Augmenting Technical Change

This appendix presents the details of the derivation of the marginal product of capital when there is both neutral and biased technical change. We assume that production possibilities are described by the following CES technology that relates output ($Y^*_{i,t}$) to capital ($K^*_{i,t}$), labor ($L^*_{i,t}$), neutral technical progress ($A_{i,t}$), and factor-augmenting technical progress on capital and labor (A^K_t and A^L_t, respectively) for industry i at time t,

$$\begin{aligned} Y^*_{i,t} &= Y[K^*_{i,t}, L^*_{i,t}, A_{i,t}, A^K_t, A^L_t], \\ &= A_{i,t}\left\{\phi(A^K_t K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^L_t L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{[\sigma/(\sigma-1)]} \end{aligned} \quad \text{(B-1)}$$

where ϕ is the capital distribution parameter and σ is the elasticity of substitution between labor and capital.

The derivative of $Y^*_{i,t}$ with respect to $K^*_{i,t}$, $Y'^*_{i,t}$, is computed from equation (B-1) as follows,

$$\begin{aligned} Y'^*_{i,t} &= [\sigma/(\sigma-1)]A_{i,t}\left\{\phi(A^K_t K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^L_t L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{[[\sigma/(\sigma-1)]-1]} \\ &\quad *[(\sigma-1)/\sigma]\phi(A^K_t K^*_{i,t})^{[[(\sigma-1)/\sigma]-1]} A^K_t. \end{aligned} \quad \text{(B-2)}$$

Since $((\sigma-1)/\sigma)-1 = -1/\sigma$, we can rewrite equation (B-2) as follows,

$$\begin{aligned} Y'^*_{i,t} &= \phi\ K^{*\,[-1/\sigma]}_{i,t} \\ &\quad A_{i,t}\left\{\phi(A^K_t K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^L_t L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{[\sigma/(\sigma-1)]} \\ &\quad \left\{\phi(A^K_t K^*_{i,t})^{[(\sigma-1)/\sigma]} + (1-\phi)(A^L_t L^*_{i,t})^{[(\sigma-1)/\sigma]}\right\}^{-1} \\ &\quad A^{K[(\sigma-1)/\sigma]}_t. \end{aligned} \quad \text{(B-3)}$$

In equation (B-3), the second line equals $Y^*_{i,t}$ per equation (B-1), and the third line equals the product of $Y^*_{i,t}$ and $A_{i,t}$ raised to the appropriate powers,

$$Y'^{*}_{i,t} = \phi\ K^{*\ [-1/\sigma]}_{i,t}$$

$$Y^{*}_{i,t}$$

$$Y^{*[(1-\sigma)/\sigma]}_{i,t}\ A^{[(\sigma-1)/\sigma]}_{i,t} \quad , \qquad \text{(B-4)}$$

$$A^{K[(\sigma-1)/\sigma]}_{t}$$

which can be rewritten as follows,

$$Y'^{*}_{i,t} = \phi\ K^{*\ [-1/\sigma]}_{i,t}\ Y^{*[1/\sigma]}_{i,t}\ A^{[(\sigma-1)/\sigma]}_{i,t}\ A^{K\ [(\sigma-1)/\sigma]}_{t} \quad . \qquad \text{(B-5a)}$$

$$= \phi\ ((Y_{i,t}\ /\ K_{i,t})^{*})^{[1/\sigma]}\ U^{KY\ [1/\sigma]}_{i,t}, \qquad \text{(B-5b)}$$

$$U^{KY\ [1/\sigma]}_{i,t} \equiv A^{[\sigma-1]}_{i,t}\ A^{K[\sigma-1]}_{t}. \qquad \text{(B-5c)}$$

A profit-maximizing firm will equate the marginal product of capital in equation (B-5a) to the user cost of capital, the price of capital divided by the price of output,

$$(P^{K}_{i,t}\ /\ P^{Y}_{i,t})^{*} = \phi\ ((Y_{i,t}\ /\ K_{i,t})^{*})^{[1/\sigma]}\ U^{KY\ [1/\sigma]}_{i,t}. \qquad \text{(B-6)}$$

Equation (B-6) can be rearranged to isolate the capital/output ratio on the left-side,

$$(K_{i,t}\ /\ Y_{i,t})^{*} = \phi^{\sigma}\ ((P^{K}_{i,t}\ /\ P^{Y}_{i,t})^{*})^{-\sigma}\ U^{KY}_{i,t}, \qquad \text{(B-7)}$$

which is equation (3a) in the text.

Appendix C: Data Transformations and the Frequency Response Scalars

Our estimation strategy is designed to emphasize long-run variation, and Section IV uses spectral analysis to evaluate our approach and the choices of $\omega^{\#}$ and q. This appendix provides some analytic details underlying the results stated and used in Section IV.

In analyzing the spectral properties of our estimator, it is convenient to recast the LPF transformation (for a finite q), the logarithmic transformation, and the first-difference transformation as follows,

$$x_t^*[\omega^{\#}, q] = \sum_{h=-q}^{q} d_h[\omega^{\#}]\ x_{t-h}, \tag{C-1a}$$

$$y_t^*[\omega^{\#}, q] = \ln[x_t^*[\omega^{\#}, q]], \tag{C-1b}$$

$$z_t^*[\omega^{\#}, q] = \Delta y_t^*[\omega^{\#}, q], \tag{C-1c}$$

where x_t represents the raw data series, either $(K_{i,t} / Y_{i,t})$ or $p_{i,t}^K$. The spectra corresponding to the $x_t^*[.]$, $y_t^*[.]$, and $z_t^*[.]$ output series in equations (C-1) are defined over the interval $\omega = [0, \pi]$ as the product of the spectrum for an input series and a scalar that is nonnegative, real, and may be depend on ω, $\omega^{\#}$, or q,

$$g_{x*}[e^{-i\omega}] = a[\omega, \omega^{\#}, q]\ g_x[e^{-i\omega}], \tag{C-2a}$$

$$g_{y*}[e^{-i\omega}] = b\ g_{x*}[e^{-i\omega}], \tag{C-2b}$$

$$g_{z*}[e^{-i\omega}] = c[\omega]\ g_{y*}[e^{-i\omega}], \tag{C-2c}$$

where $g_x[e^{-i\omega}]$ is the spectrum for the raw series and the scalars are defined as follows,

$$a[\omega, \omega^{\#}, q] = \alpha[\omega^{\#}, q] \begin{Bmatrix} (\omega^{\#} / \pi) + 2\sum_{h=1}^{q} \cos[h\,\omega]\ d'_h[\omega^{\#}] \\ + \theta[\omega^{\#}, q]\ \{(1 - \cos[\omega(2q+1)]) / (1 - \cos[\omega])\}^{1/2} \end{Bmatrix}^2 \tag{C-3a}$$

$$b = \beta\ (\mu_{x*})^{-2}, \tag{C-3b}$$

$$c[\omega] = \gamma\ 2\ (1 - \cos[\omega]), \tag{C-3c}$$

where μ_{x^*} equals the unconditional expectation of $x_t^*[.]$. To ensure comparability in the analyses to follow that vary $\omega^{\#}$ and q, the areas under the spectra from 0 to π are normalized to one by an appropriate choice of normalizing scalars, $\alpha[\omega^{\#},q]$, β, and γ in equations (C-3).

The three scalars -- $a[\omega,\omega^{\#},q]$, b, and $c[\omega]$ -- correspond to the LPF, logarithmic, and first-difference transformations, respectively, and are derived as follows. The $a[\omega,\omega^{\#},q]$ scalar is based on Sargent (1987, Chapter XI, equation (33)),

$$a[\omega,\omega^{\#},q]=\alpha[\omega^{\#},q]\left\{\sum_{h=-q}^{q} e^{-ih\omega}\, d_h[\omega^{\#}]\right\}\left\{\sum_{h=-q}^{q} e^{ih\omega}\, d_h[\omega^{\#}]\right\}. \tag{C-4}$$

The two-sided summations are symmetric about zero and only differ by the minus sign in the exponential terms. Hence, the two sums in braces are nearly identical. The $d_h[.]$'s appearing in the summations are separated into $\theta[.]$ and the $d'_h[.]$'s (cf. equations (7)). For the latter terms, a further distinction is made between the term at h=0 and the remaining terms (h=±1,±q) that are symmetric about h=0. Equation (C-4) can be written as follows,

$$a[\omega,\omega^{\#},q]=\alpha[\omega^{\#},q]\left\{\theta[\omega^{\#},q]\sum_{h=-q}^{q} e^{ih\omega}+(\pi\,\omega)+\sum_{h=1}^{q}(e^{-ih\omega}+e^{ih\omega})\; d'_h[\omega^{\#}]\right\}^2 . \tag{C-5}$$

The first sum of exponential terms is evaluated based on Sargent (1987, p. 275),

$$\begin{aligned}\sum_{h=-q}^{q} e^{ih\omega} &=\left\{\left(\sum_{h=-q}^{q} e^{ih\omega}\right)^2\right\}^{1/2} \\ &=\left\{\left((1-\cos[(2q+1)\omega]/(1-\cos[\omega]\right)\right\}^{1/2}.\end{aligned} \tag{C-6}$$

The second sum of exponential terms is evaluated with the Euler relations,

$e^{\pm ih\omega}=\cos[h\omega]\pm\sin[h\omega]$,

$$\sum_{h=1}^{q}(e^{-ih\omega}+e^{ih\omega})\,d'_h[\omega^{\#}]=2\sum_{h=1}^{q}\cos[h\omega]\; d'_h[\omega^{\#}]. \tag{C-7}$$

The b scalar is based on the approximation in Granger (1964, p. 48, equation 3.7.6), which states that the approximation will be accurate if the mean is much larger than the standard deviation of the input series ($x_t^*[.]$).

The $c[\omega]$ scalar is based on the well-known formula for the first-difference transformation (Hamilton, 1994, equation 6.4.8).

The importance of the above analytical results is that the combined effects of the three transformations are captured by three scalars that multiply the spectrum of the raw series,

$$g_{z*}[e^{-i\omega}] = \left\{a[\omega, \omega^{\#}, q] * b * c[\omega]\right\} * g_x[e^{-i\omega}]. \qquad \text{(C-8)}$$

Equation (C-8) allows us to examine the extent to which our estimation strategy emphasizes long-run frequencies. Since the spectra for the raw series ($g_x[e^{-i\omega}]$) and the scalars associated with the logarithmic and first-difference transformations (b and $c\left[\omega\right]$, respectively) do not depend on $\omega^{\#}$ or q, their impacts on the data will be absorbed in the normalizing scalars, and hence they will not affect relative comparisons. Alternative values of $\omega^{\#}$ or q, will only affect the $LPF[\omega^{\#}, q]$ and the associated scalar, $a[\omega, \omega^{\#}, q]$.

Appendix D: Relating Heterogeneous Industry and Aggregated σ's

This appendix develops the formula for relating the aggregated σ (σ_{agg}) to heterogeneous industry σ's (σ_i). We begin with the definitions of σ_{agg} and σ_i that follow from equation (5) and are stated in terms of percentage changes in the capital/output ratios ((K_{agg} / Y_{agg}) and (K_i / Y_i), respectively) and the aggregate relative price of capital (P_{agg}^{KY}),

$$\sigma_{agg} \equiv \frac{d\left(K_{agg} / Y_{agg}\right) / \left(K_{agg} / Y_{agg}\right)}{d\,P_{agg}^{KY} / P_{agg}^{KY}}, \qquad Y_{agg} = \text{constant}, \tag{D-1}$$

$$\sigma_i \equiv \frac{d\left(K_i / Y_i\right) / \left(K_i / Y_i\right)}{d\,P_i^{KY} / P_i^{KY}}, \qquad Y_i = \text{constant}. \tag{D-2}$$

Note that σ can be defined in terms of the capital/labor ratio (see Section VI.C). In this case, the derivation presented below is unaffected (merely replace Y_{agg} with L_{agg} and Y_i with L_i in equations (D-5) to (D-8) and P_{agg}^{KY} with P_{agg}^{KL} and P_i^{KY} with P_i^{KL} in equation (D-3)), and equation (D-9) continues to link aggregated and industry σ's. While the relative price of capital varies by industry, we consider a percentage change that is equal across all industries (e.g., a change in the nominal or relative price of investment),

$$d\,P_{agg}^{KY} / P_{agg}^{KY} = d\,P_i^{KY} / P_i^{KY} = d\,P / P, \qquad \forall i. \tag{D-3}$$

We begin with identities relating changes in the aggregated to industry capital, and changes in the aggregated and industry capital/output ratios, respectively,

$$\frac{dK_{agg}}{d\,P / P} \equiv \sum_i \frac{dK_i}{d\,P / P}, \tag{D-4}$$

$$\frac{dK_{agg}}{d\,P / P} \equiv \frac{d\left(K_{agg} / Y_{agg}\right)}{d\,P / P} * Y_{agg}, \qquad Y_{agg} = \text{constant}, \tag{D-5}$$

$$\frac{dK_i}{d\,P / P} \equiv \frac{d\left(K_i / Y_i\right)}{d\,P / P} * Y_i, \qquad Y_i = \text{constant}. \tag{D-6}$$

Substituting equations (D-5) and (D-6) into equation (D-4), dividing both sides by K_{agg}, and rearranging, we obtain the following equation,

$$\frac{d\left(K_{agg} / Y_{agg}\right) / \left(K_{agg} / Y_{agg}\right)}{dP / P} = \sum_{i} \frac{d\left(K_i / Y_i\right)}{dP / P} * \left(Y_i / K_{agg}\right). \tag{D-7}$$

The left-side equals σ_{agg} by equation (D-1). Multiplying the right-side by $\left(K_i / K_i\right)$ and rearranging, we obtain the following equation,

$$\sigma_{agg} = \sum_{i} \frac{d\left(K_i / Y_i\right) * \left(K_i / Y_i\right)}{dP / P} * \left(K_i / K_{agg}\right). \tag{D-8}$$

Using the definition of σ_i from equation (D-2) and defining the latter object in equation (D-8) as an industry weight, we obtain the following equation,

$$\sigma_{agg} = \sum_{i} \sigma_i * \omega_i , \qquad \omega_i \equiv \left(K_i / K_{agg}\right). \tag{D-9}$$

In equation (D-9), the aggregated σ (σ_{agg}) is a weighted average of the industry σ's (σ_i's), where the ω_i's are industry weights defined in terms of industry capital ratios.

Table 1: Summary Statistics For Model And Untransformed Variables

Variable	Mean	Median	25th Quantile	75th Quantile	Within Industry Standard Deviation	Industry-Specific Time Variation
	(1)	(2)	(3)	(4)	(5)	(6)
A. Capital/Output						
$\Delta ky^{*}_{i,t}$	0.0108	0.0092	-0.0069	0.0282	0.0301	0.8724
$\Delta ky_{i,t}$	0.0089	0.0056	-0.0246	0.0400	0.0697	0.7763
$\Delta p^{KY*}_{i,t}$	-0.0058	-0.0050	-0.0350	0.0234	0.0504	0.8119
$\Delta p^{KY}_{i,t}$	-0.0011	0.0049	-0.0669	0.0694	0.1403	0.8643
B. Labor/Output						
$\Delta \ell y^{*}_{i,t}$	-0.0189	-0.0192	-0.0326	-0.0029	0.0320	0.9051
$\Delta \ell y_{i,t}$	-0.0200	-0.0209	-0.0535	0.0146	0.0795	0.9420
$\Delta p^{LY*}_{i,t}$	0.0158	0.0157	0.0017	0.0306	0.0294	0.8093
$\Delta p^{LY}_{i,t}$	0.0153	0.0183	-0.0070	0.0431	0.0653	0.8460
C. Capital/Labor						
$\Delta k\ell^{*}_{i,t}$	0.0298	0.0290	0.0084	0.0500	0.0386	0.8596
$\Delta k\ell_{i,t}$	0.0289	0.0280	-0.0142	0.0691	0.0969	0.8824
$\Delta p^{KL*}_{i,t}$	-0.0213	-0.0201	-0.0536	0.0097	0.0569	0.8249
$\Delta p^{KL}_{i,t}$	-0.0164	-0.0120	-0.0917	0.0609	0.1606	0.8721

Table notes are placed after the final table.

Table 2: Ordinary Least Squares Estimates Of Equation (6)
Dependent Variable: Capital/Output Ratio
Various Critical Periodicities ($p^{\#}$) And Windows (q)

		q = 1	q = 3	q = 5
		(1)	(2)	(3)
$p^{\#} = 2$	σ (s.e.) [NW s.e.] {R^2}	0.229 (0.022) [0.020] {0.409}	0.229 (0.023) [0.022] {0.405}	0.224 (0.025) [0.023] {0.398}
$p^{\#} = 4$	σ (s.e.) [NW s.e.] {R^2}	0.329 (0.016) [0.022] {0.502}	0.319 (0.017) [0.024] {0.504}	0.312 (0.017) [0.026] {0.492}
$p^{\#} = 6$	σ (s.e.) [NW s.e.] {R^2}	0.333 (0.016) [0.022] {0.500}	0.367 (0.016) [0.027] {0.502}	0.346 (0.017) [0.026] {0.502}
$p^{\#} = 8$	σ (s.e.) [NW s.e.] {R^2}	0.331 (0.016) [0.022] {0.497}	**0.406** **(0.017)** **[0.034]** **{0.503}**	0.379 (0.017) [0.032] {0.491}
$p^{\#} = 10$	σ (s.e.) [NW s.e.] {R^2}	0.331 (0.016) [0.022] {0.496}	0.417 (0.018) [0.037] {0.495}	0.429 (0.019) [0.040] {0.499}
$p^{\#} = 20$	σ (s.e.) [NW s.e.] {R^2}	0.330 (0.016) [0.022] {0.495}	0.409 (0.018) [0.037] {0.472}	0.510 (0.020) [0.049] {0.511}
$p^{\#} \to \infty$	σ (s.e.) [NW s.e.] {R^2}	0.330 (0.016) [0.022] {0.494}	0.405 (0.018) [0.037] { 0.467}	0.499 (0.020) [0.047] {0.499}

Table notes are placed after the final table.

Table 3: Instrumental Variable Estimates Of Equation (6)
Dependent Variable: Capital/Output Ratio
Various Critical Periodicities ($p^{\#}$) And Windows (q)

		q = 1	q = 3	q = 5
		(1)	(2)	(3)
$p^{\#} = 2$	σ (s.e.) [NW s.e.] {F}	0.374 (0.069) [0.060] {6.75}	0.359 (0.064) [0.055] {7.14}	0.370 (0.066) [0.058] {7.32}
$p^{\#} = 4$	σ (s.e.) [NW s.e.] {F}	0.393 (0.038) [0.053] {11.22}	0.367 (0.038) [0.049] {10.57}	0.396 (0.042) [0.051] {10.65}
$p^{\#} = 6$	σ (s.e.) [NW s.e.] {F}	0.395 (0.037) [0.053] {11.55}	0.412 (0.035) [0.058] {12.99}	0.383 (0.035) [0.056] {13.65}
$p^{\#} = 8$	σ (s.e.) [NW s.e.] {F}	0.395 (0.037) [0.053] {11.56}	**0.438** **(0.033)** **[0.062]** **{14.16}**	0.425 (0.035) [0.063] {14.26}
$p^{\#} = 10$	σ (s.e.) [NW s.e.] {F}	0.395 (0.037) [0.053] {11.56}	0.450 (0.033) [0.063] {14.30}	0.468 (0.033) [0.067] {15.38}
$p^{\#} = 20$	σ (s.e.) [NW s.e.] {F}	0.395 (0.037) [0.053] {11.55}	0.462 (0.034) [0.065] {13.90}	0.536 (0.032) [0.072] {17.74}
$p^{\#} \rightarrow \infty$	σ (s.e.) [NW s.e.] {F}	0.395 (0.037) [0.053] {11.55}	0.464 (0.035) [0.065] {13.77}	0.548 (0.033) [0.072] {18.51}

Table notes are placed after the final table.

Table 4: Ordinary Least Squares Estimates Of Equation (6)
Dependent Variable: Capital/Output Ratio
Various Critical Periodicities ($p^{\#}$) And q = 3
Split-Samples: 1960-1982 and 1983-2005

	Period		q = 1	q = 3	q = 5
			(1)	(2)	(3)
$p^{\#} = 2$	1960-82	σ	0.221	0.219	0.217
		(s.e.)	(0.034)	(0.035)	(0.036)
		[NWs.e.]	[0.034]	[0.035]	[0.036]
		{R^2}	{0.427}	{0.419}	{0.410}
	1983-05	σ	0.238	0.242	0.233
		(s.e.)	(0.027)	(0.030)	(0.033)
		[NWs.e.]	[0.033]	[0.036]	[0.040]
		{R^2}	{0.371}	{0.369}	{0.339}
$p^{\#} = 4$	1960-82	σ	0.292	0.291	0.292
		(s.e.)	(0.021)	(0.022)	(0.023)
		[NWs.e.]	[0.027]	[0.030]	[0.032]
		{R^2}	{0.531}	{0.539}	{0.522}
	1983-05	σ	0.366	0.350	0.338
		(s.e.)	(0.022)	(0.025)	(0.026)
		[NWs.e.]	[0.031]	[0.033]	[0.035]
		{R^2}	{0.459}	{0.446}	{0.407}
$p^{\#} = 6$	1960-82	σ	0.298	0.316	0.296
		(s.e.)	(0.023)	(0.021)	(0.0215)
		[NWs.e.]	[0.029]	[0.029]	[0.032]
		{R^2}	{0.532}	{0.510}	{0.483}
	1983-05	σ	0.366	0.413	0.394
		(s.e.)	(0.022)	(0.024)	(0.024)
		[NWs.e.]	[0.031]	[0.040]	[0.036]
		{R^2}	{0.455}	{0.472}	{0.472}
$p^{\#} = 8$	1960-82	σ	0.298	**0.336**	0.306
		(s.e.)	(0.023)	**(0.020)**	(0.020)
		[NWs.e.]	[0.029]	**[0.034]**	[0.033]
		{R^2}	{0.529}	**{0.472}**	{0.424}
	1983-05	σ	0.364	**0.460**	0.441
		(s.e.)	(0.022)	**(0.024)**	(0.025)
		[NWs.e.]	[0.031]	**[0.047]**	[0.043]
		{R^2}	{0.451}	**{0.496}**	{0.488}

Table 4: (cont.) **Ordinary Least Squares Estimates Of Equation (6)**
Dependent Variable: Capital/Output Ratio
Various Critical Periodicities ($p^{\#}$) And q = 3
Split-Samples: 1960-1982 and 1983-2005

$p^{\#} = 10$	1960-82	σ (s.e.) [NWs.e.] {R^2}	0.298 (0.023) [0.029] {0.528}	0.341 (0.020) [0.037] {0.442}	0.344 (0.020) [0.039] {0.406}
	1983-05	σ (s.e.) [NWs.e.] {R^2}	0.362 (0.022) [0.031] {0.449}	0.472 (0.025) [0.051] {0.499}	0.497 (0.027) [0.053] {0.045}
$p^{\#} = 20$	1960-82	σ (s.e.) [NWs.e.] {R^2}	0.298 (0.023) [0.030] {0.527}	0.335 (0.022) [0.036] {0.407}	0.415 (0.025) [0.053] {0.400}
	1983-05	σ (s.e.) [NWs.e.] {R^2}	0.361 (0.023) [0.031] {0.447}	0.464 (0.026) [0.052] {0.484}	0.575 (0.028) [0.063] {0.535}
$p^{\#} \rightarrow \infty$	1960-82	σ (s.e.) [NWs.e.] {R^2}	0.298 (0.023) [0.030] {0.527}	0.332 (0.022) [0.036] {0.403}	0.404 (0.028) [0.052] {0.391}
	1983-05	σ (s.e.) [NWs.e.] {R^2}	0.361 (0.023) [0.031] {0.447}	0.460 (0.026) [0.052] {0.480}	0.564 (0.027) [0.060] {0.531}

Table notes are placed after the final table.

Table 5: Ordinary Least Squares Estimates of Equations (6), (10), and (11)
Three Different Estimating Equations
Various Critical Periodicities ($p^{\#}$) And q = 3

		Dependent Variable		
		$\Delta ky^*_{i,t}$	$\Delta \ell y^*_{i,t}$	$\Delta k\ell^*_{i,t}$
		(1)	(2)	(3)
$p^{\#} = 2$	σ (s.e.) [NW s.e.] {R^2}	0.229 (0.023) [0.022] {0.405}	0.614 (0.086) [0.135] {0.259}	0.120 (0.035) [0.050] {0.154}
$p^{\#} = 4$	σ (s.e.) [NW s.e.] {R^2}	0.319 (0.017) [0.024] {0.504}	0.541 (0.052) [0.124] {0.274}	0.262 (0.028) [0.051] {0.274}
$p^{\#} = 6$	σ (s.e.) [NW s.e.] {R^2}	0.367 (0.016) [0.027] {0.502}	0.594 (0.049) [0.124] {0.320}	0.304 (0.025) [0.052] {0.315}
$p^{\#} = 8$	σ (s.e.) [NW s.e.] {R^2}	**0.406** **(0.017)** **[0.034]** **{0.503}**	0.629 (0.045) [0.118] { 0.364}	0.333 (0.024) [0.053] { 0.339}
$p^{\#} = 10$	σ (s.e.) [NW s.e.] {R^2}	0.417 (0.018) [0.037] {0.495}	0.644 (0.042) [0.113] {0.382}	0.333 (0.023) [0.051] {0.333}
$p^{\#} = 20$	σ (s.e.) [NW s.e.] {R^2}	0.409 (0.018) [0.037] {0.472}	0.654 (0.040) [0.106] {0.390}	0.310 (0.022) [0.046] {0.301}
$p^{\#} \to \infty$	σ (s.e.) [NW s.e.] {R^2}	0.405 (0.018) [0.037] { 0.467}	0.655 (0.040) [0.105] {0.389}	0.303 (0.022) [0.046] {0.294}

Table notes are placed after the final table.

Table 6: Ordinary Least Squares Estimates Of Equation (6)
Dependent Variable: Capital/Output Ratio
$p^{\#}$=8 and q=3
Heterogeneous Industry σ_i's And Aggregated σ_{agg}

Panel A: Heterogeneous Industry σ_i's

	Industry	σ (s.e.) [NW s.e.]
1	Agriculture	0.289 (0.062) [0.089]
2	Metal mining	0.667 (0.130) [0.167]
3	Coal mining	0.305 (0.073) [0.076]
4	Oil and gas extraction	0.649 (0.040) [0.032]
5	Non-metallic mining	0.586 (0.033) [0.029]
6	Construction	0.410 (0.076) [0.098]
7	Food and kindred products	0.078 (0.021) [0.034]
8	Tobacco	0.312 (0.086) [0.140]
9	Textile mill products	0.204 (0.054) [0.063]
10	Apparel	0.547 (0.081) [0.098]
11	Lumber and wood	0.484 (0.051) [0.067]
12	Furniture and fixtures	0.203 (0.065) [0.081]
13	Paper and allied	0.148 (0.048) [0.054]
14	Printing, publishing and allied	0.484 (0.053) [0.043]
15	Chemicals	0.210 (0.037) [0.051]
16	Petroleum and coal products	0.294 (0.016) [0.019]
17	Rubber and misc plastics	0.300 (0.064) [0.059]
18	Leather	0.425 (0.093) [0.059]
19	Stone, clay, glass	0.371 (0.031) [0.037]
20	Primary metal	0.562 (0.045) [0.045]
21	Fabricated metal	0.401 (0.026) [0.034]
22	Machinery, non-electrical	0.483 (0.071) [0.067]
23	Electrical machinery	0.486 (0.074) [0.100]
24	Motor vehicles	0.365 (0.051) [0.033]
25	Transportation equipment & ordnance	0.419 (0.091) [0.052]
26	Instruments	0.570 (0.074) [0.108]
27	Misc. manufacturing	0.246 (0.065) [0.061]
28	Transportation	0.358 (0.098) [0.078]
29	Communications	0.240 (0.122) [0.112]
30	Electric utilities	0.231 (0.108) [0.146]
31	Gas utilities	0.473 (0.088) [0.080]
32	Trade	0.744 (0.048) [0.054]
33	Finance, Insurance & Real Estate	1.160 (0.178) [0.140]
34	Services	0.633 (0.048) [0.094]
35	Government enterprises	0.272 (0.027) [0.018]

Table 6: (cont.) **Ordinary Least Squares Estimates Of Equation (6)**
Dependent Variable: Capital/Output Ratio
$p^{\#}$=8 and q=3
Heterogeneous σ_i's

Panel B: Aggregated σ_{agg}

	Heterogeneous				Homogeneous
	Capital Weights	Equal Weights	Capital Weights, Selective Industries		
	(1)	(2)	(3)		(4)
σ_{agg}	0.657	0.417	0.857		**0.406**
(s.e.)	(0.055)	(0.067)	(0.070)		**(0.017)**
[NW s.e.]	[0.043]	[0.072]	[0.089]		**[0.034]**

Table Notes:

Table 1: The table contains statistics for the capital/output, labor/output, and capital/labor models in panels A., B., and C., respectively. In a given panel, the first and third rows are the model variables; the second and fourth rows are comparable variables that have not been transformed by the LPF. The sample period is 1960-2005 before applying the LPF and differencing. For a given industry, the effective time dimension equals these 46 data points less 2q for the construction of the LPF less one for first differencing. The statistics reported in this table are based on an LPF of $p^{\#} = 8$ and q = 3. The data underlying the moments in column 1 to 5 have had industry means removed by first-differencing. Column 6 reports the variance due to industry-specific time variation (i.e., where the variance due to industry and common time effects have been removed) as a percentage of the overall variance in the series; this statistic is $(1 - R^2)$ from the following auxiliary regression for each variable, $\Delta x_{i,t}$: $\Delta x_{i,t} = a + b_t + r_{i,t}$, where a and b_t are estimated parameters and $r_{i,t}$ is a residual.

Table 2: OLS estimates of σ are based on panel data for 35 industries. The sample period is 1960-2005 before applying the LPF and differencing. For a given industry, the effective time dimension equals these 46 data points less 2q for the construction of the LPF less one for first differencing. We use the maximum amount of available data, and hence the sample size changes with q. Figures in parentheses (s.e.) are White (1980) heteroskedasticity corrected standard

errors. Figures in brackets [NW s.e.] are Newey-West (1994) heteroskedasticity-autocorrelation corrected (HAC) standard errors with (T-1) lags, which is equivalent to using Bartlett kernel with a bandwidth of T. A constant term and fixed time effects (the number of fixed time effects equals the effective time dimension less one due to the inclusion of the constant) are included in the regression equation but are not reported. The R^2's are not comparable across cells because the dependent variable depends on $p^{\#}$ and q. Our preferred estimate is for the equation for which the LPF parameters are $p^{\#} = 8$ and q = 3.

Table 3: IV estimates of σ. See the notes to Table 2 for further information. The instrument is $P_{i,t-2}^{KY\&}$, which is constructed from equation (7) as a one-sided filter (h = $\{0, -q\}$) and lagged two periods. Figures in braces are F-statistics computed from the first-stage auxiliary regression of $P_{i,t}^{KY*}$ on $P_{i,t-2}^{KY\&}$ (and time dummies) that assess instrument relevance. The null hypothesis of a weak instrument is evaluated at the 5% level and rejected for F's greater than or equal to 8.96 (Stock, Wright, and Yogo, 2002, Table 1). Our preferred estimate is for the equation for which the LPF parameters are $p^{\#} = 8$ and q = 3.

Table 4: OLS estimates of σ for split samples. See the notes to Table 2 for further information. Our preferred estimate is for the equation for which the LPF parameters are $p^{\#} = 8$ and q = 3.

Table 5: OLS estimates of σ for three different estimating equations. See the notes to Table 2 for further information. The window is fixed at q = 3; our preferred estimate is for the equation in column (1) for which the Low-Pass Filter parameter is $p^{\#} = 8$.

Table 6: OLS estimates of industry σ_i's. See the notes to Table 2 for further information. Panel A contains estimates that allow the σ_i's to vary by industry. Panel B contains aggregated σ_{agg}'s that are weighted averages of the σ_i's in Panel A as defined in equation (12). See Appendix D for a derivation of equation (12). Column 1 uses capital weights for all 35 industries; the weight for a given industry (ω_i) is defined as follows, $\omega_i = \sum_t K_{i,t} / \sum_t \sum_i K_{i,t}$. Column 2 uses equal weights for all 35 industries. Column 3 uses capital weights for six selective "post-industrial" industries: Construction (6), Transportation (28), Communications (29), Trade (32), Finance, Insurance and Real Estate (33) and Services (34). Column 4 contains the estimates for a homogeneous model with the restriction that $\sigma_i = \sigma \ \forall i$; that is, the benchmark estimates presented in Table 2 and equation (9). Standard errors are in parentheses or brackets and are computed with the following formula, $\sqrt{\omega' V \omega}$, where ω is a vector of weights and V is the variance-covariance matrix of the estimated coefficients. Standard errors in parentheses or brackets are based on a V computed with White (1980) heteroskedasticity corrected standard errors or Newey-West (1994) heteroskedasticity-autocorrelation corrected (HAC) standard errors with (T-1) lags, respectively.

Figure 1
Frequency Response Of a[ω : ω#, q]
Equation (8a)
Various Critical Periodicities (p#); q=3

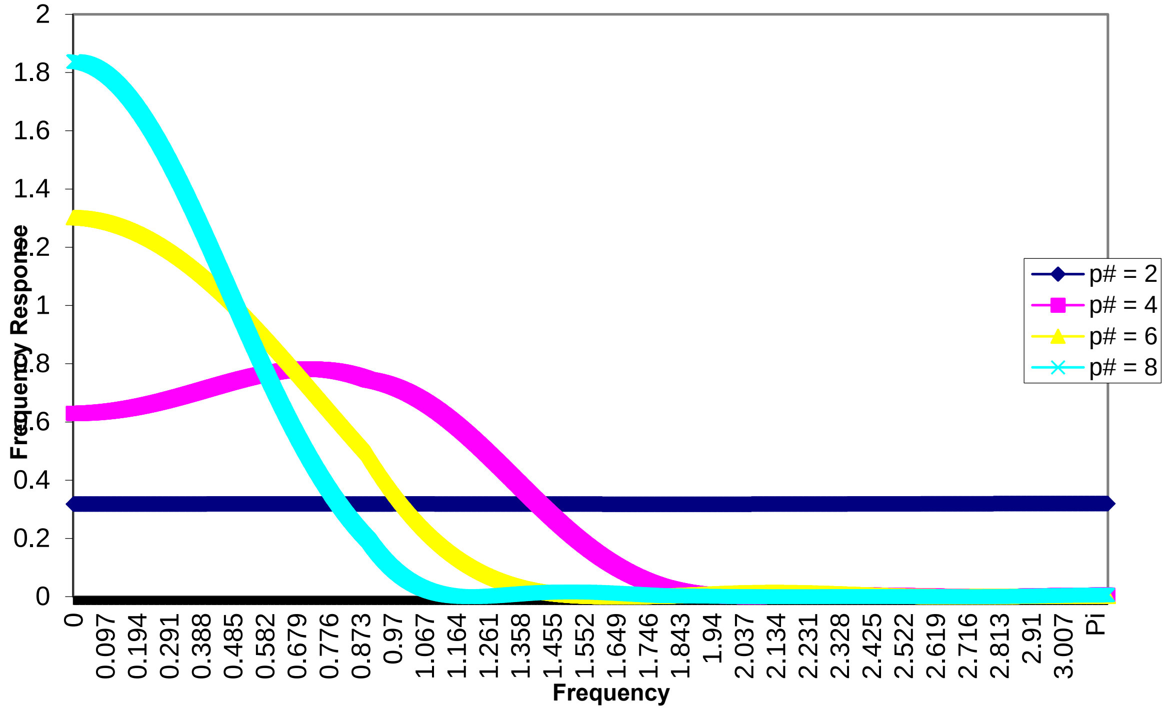

Figure 2
Frequency Response Of a[ω : ω#, q]
Equation (8a)
For Various Critical Periodicities (p#); q=3

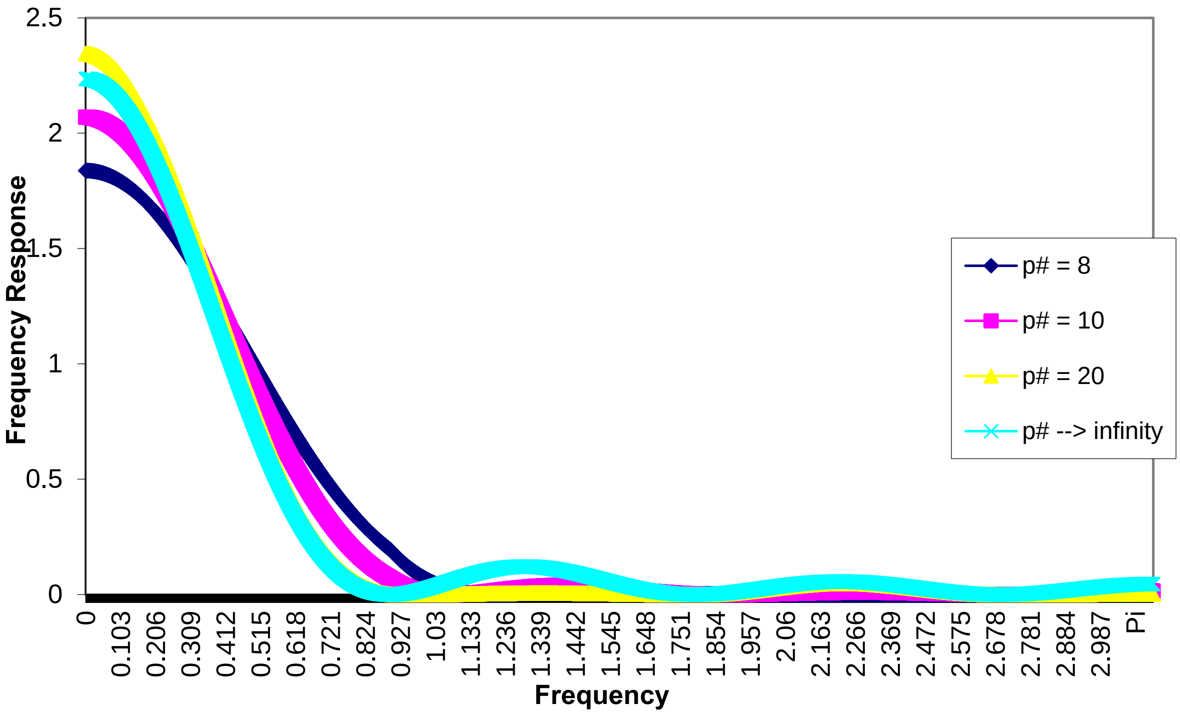

Figure 3
Frequency Response Of a[ω : ω#, q]
Equation (8a)
For The Ideal LPF And Various Windows (q); p# = 8
χ[q] Measures Contamination; See Section IV For Details