

# Trade Delay, Liquidity, and Asset Prices in Over-the-Counter Markets

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## Abstract

In over-the-counter markets, the presence of two frictions is central to determine prices, liquidity, and efficiency: the *search friction* reflected in how long it takes to find a trading opportunity and the *bargaining friction* reflected in how promptly gains from trade are realized once the opportunity is identified. This paper captures both frictions by introducing an asset-specific trade delay into a standard search-and-bargaining model. For both exogenous and endogenous specifications of delay, the set of traded assets and the dependence of asset prices and spreads on default risk, liquidity, and market conditions are determined in equilibrium. The proposed model with endogenous delay has several implications. First, it offers a novel testable prediction: for assets within the same credit rating class, the liquidity is U-shaped in quality. Assets closer to the extremes of the quality range are more liquid, while assets in the middle of the quality range may be not traded at all. This is in contrast with a monotone relation in models with asymmetric information. Second, this model shows that the reduction in search and bargaining frictions may have opposite effects on market liquidity which is reflected in the range of traded assets. Finally, it establishes a connection between market uncertainty about the asset payoff and market liquidity. This link sheds light on the role of transparency in over-the-counter markets and explains the occurrence of dried-up liquidity and flights-to-quality during periods of increased market uncertainty.

**Keywords:** *search friction, trade delay, liquidity, asset prices, over-the-counter markets, transparency, flight-to-quality*

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# 1 Introduction

Trade delay is a salient feature of many economic transactions and is predominant in over-the-counter (OTC) markets for asset-backed securities, derivatives, corporate bonds, sovereign debt and bank loans. In contrast to the stock market, OTC markets feature decentralized trade that occurs through bilateral bargaining. Therefore, the presence of two trade frictions is central in order to determine prices, liquidity, and efficiency: *search friction* which is reflected in the ability of market participants to find a trading opportunity and *bargaining friction* which is reflected in the ability of market participants to promptly realize gains from trade once an opportunity is identified. The literature on search and bargaining successfully captures the notion of search friction by introducing the random matching of agents but does not separately incorporate bargaining friction and assumes that trade is immediate after agents are matched (see Duffie (2012) for a literature review). Search friction is sometimes thought of as a reduced form for both trade frictions. For example, Duffie (2012) states that “[s]earch delays ... proxy for delays associated with reaching an awareness of trading opportunities, arranging financing and meeting suitable legal restrictions, negotiating trades, executing trades, and so on.” This paper argues that modeling both frictions is worthwhile because many forms of trade delay may naturally differ across assets. It analyzes the effect of both trade frictions on prices and liquidity and shows that these effects may differ for these two types of friction.

This study incorporates asset-specific delay into an otherwise standard search-and-bargaining model. To capture a variety of sources of bargaining friction, the paper analyzes both a model with exogenously-specified delay and a model with endogenous delay arising from strategic bargaining. In both models, *market liquidity* proxied by a range of traded assets, and the dependence of asset prices and yield spreads on the default risk, asset liquidity as well as market conditions are determined in equilibrium. The model with endogenous delay has several novel implications. First, it provides a testable prediction about the U-shaped dependence of asset *liquidity*, reflected in the asset-specific trade delay, on default risk for assets within the same credit-rating class. This contrasts with the prediction of a monotone relation in models with asymmetric information. Second, when bargaining friction arises from strategic bargaining, there is a drastic difference between the effect on liquidity of the reduction in search and bargaining frictions, with the former reducing market liquidity and the latter improving market liquidity. Hence, it is important to distinguish in the analysis the two frictions, and restricting attention to only one type of friction is with a loss of generality. Third, the model with endogenous delay provides a link between market liquidity and *market uncertainty*, reflected in the variance of asset payoffs. This link sheds light on the role of transparency in market liquidity and on the decrease in liquidity and flight-to-quality episodes during periods of heightened market uncertainty.

Specifically, I consider an infinite-horizon, steady-state economy with a continuum of assets of varying quality. Asset quality is an index that summarizes various factors affecting the

asset payoff, such as payment and risk structure. Assets of higher quality give a higher flow payoff. Agents are hit by idiosyncratic liquidity shocks and can share risks by trading assets. In order to trade, they search for a counter-party in a market with search frictions. Agents are randomly matched via the search technology commonly used in the search-and-bargaining literature (see Duffie, Gârleanu and Pedersen (2005)). After two agents are matched, they trade following an the asset-specific delay. The price is set so that the surplus is split between parties proportionally. The proportional split of the surplus is a common assumption in the literature, while asset-specific delay is a novel feature of this paper.

There are various reasons why trade may not be immediate after a trade opportunity is identified. First, trade delay is a natural screening/signaling device and many theoretical strategic bargaining models with uncertainty about values feature a significant trade delay (see Ausubel, Cramton, and Deneckere (2002) for a survey).<sup>1</sup> Second, assets traded in OTC markets are less standardized than assets listed on exchanges. Publicly available information about assets, like credit ratings and past quotes, is usually too crude to assess a particular trade, and as a result, pre-trade evaluation of the asset by parties can be time-consuming.<sup>2,3</sup> Another reason for trade delay is the privacy concerns of traders. The lack of liquidity in OTC markets makes prices sensitive to large trades. To minimize price impact, large trades are often split into smaller trades spread over time, causing delay in the realization of gains from trade.<sup>4</sup> Limited mobility of capital can also lead to delayed transactions. Mitchell and Pulvino (2012) show that in 2008, arbitrage opportunities persisted in the corporate bonds market for a long time, as arbitrage hedge funds had trouble raising the capital to invest in those opportunities.

Motivated by these reasons for trade delay in OTC markets, I consider both models of exogenous and endogenous delay. In the model with exogenous delay, each asset quality is associated with a particular exogenously-specified trade delay. This model provides a framework for analyzing asset prices and liquidity for a broad range of specifications of the bargaining friction. However, the exogenous-delay model cannot capture situations in which trade delay itself depends on how easily and at what price the asset can be traded in the future. When bargaining is strategic, big differences in values of assets give market participants additional

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<sup>1</sup>In the bargaining literature, an immediate agreement is obtained under quite restrictive assumptions on the information structure and on the bargaining protocol. These assumptions are often too strong for the description of trade in OTC markets.

<sup>2</sup>In discussions of the 2007-2008 liquidity crisis, the opaqueness of OTC markets is pointed out as one of the main causes of the dried-up liquidity (see for example IMF (2008)). Pagano and Volpin (2012) shows theoretically that the issuers of asset-backed securities have incentives to release only crude public information about the assets, reducing the transparency of secondary markets.

<sup>3</sup>Saunders, Srinivasan, and Walter (2002) provides a case study of trade in the OTC market for corporate bonds and reports that while the trade of more standardized bonds is relatively rapid, for assets with non-standard features, traders often request a research evaluation which delays trade.

<sup>4</sup>Friewald, Jankowitsch, and Subrahmanyam (2012) and Dick-Nielsen, Feldhutter, and Lando (2012) document that during the liquidity crisis of 2007-2008, the number of trades increased while other liquidity measures declined. Their interpretation is that in less liquid environments, large orders are indeed executed over time through smaller trades that have less impact on prices.

incentives to negotiate longer in order to trade at a more favorable price. In turn, the value of an asset depends on how easily the position can be liquidated when the holder is hit by a liquidity shock. Hence, it is natural that trade delay is endogenously determined when strategic bargaining is the key source of bargaining friction.

The model with endogenous delay uses a novel *common screening bargaining solution* to determine the amount of delay due to strategic bargaining in equilibrium. In a companion paper, I show that the common screening bargaining solution can be thought of as the reduced form for the bargaining outcome in an alternating-offer bargaining model where instead of observing asset quality, agents receive almost perfect-signals about the quality that determine their values (see Tsoy (2014)). In a similar fashion, the generalized Nash (1950) bargaining solution commonly used in the literature can be viewed as the reduced form for the bargaining outcome in an alternating-offer bargaining model with perfectly observable quality (see Binmore, Rubinstein, and Wolinsky (1986)).

Unlike the generalized Nash bargaining solution, in the common screening bargaining solution, trade between matched agents is not immediate and exhibits more realistic two-sided screening dynamics: the buyer makes a decreasing sequence of price offers and the seller responds with an increasing sequence of counter-offers, until one of the parties accepts the opponent's offer. Despite vanishing uncertainty of agents about quality, two-sided screening dynamics is possible because of the gap between precise private information and crude public information about quality. Through the two-sided screening process, endogenous public information about the asset's quality is produced. Agents agree as soon as a sufficient amount of public information is produced, and the quality of the asset is either the highest or the lowest quality among the qualities that still remain. Therefore, the initial amount of public information, rather than the precision of the private information, is crucial for both efficiency and trade dynamics of the common screening bargaining solution. This leads to an interpretation of the common screening bargaining solution as the description of bargaining over the price of assets within the asset class as defined by public information, e.g. credit ratings or past quotes.

In both models, equilibrium provides an intuitive decomposition of asset prices into three components: default-risk component, liquidity premium component and average-liquidity component. This decomposition is consistent with the empirical evidence that there is a significant non-default component in corporate spreads which depends both on the liquidity of bond and marketwide liquidity. (see, for example, Longstaff, Mithal, and Neis (2005) and Bao, Pan, and Wang (2011)). The effect of different components on asset prices is better understood through the lens of how they affect agents' outside options of continuing search. Factors that improve the outside option of the seller increase price, while factors that improve the outside option of the buyer decrease price. Naturally, the price of the asset decreases with the increase in default risk. Holding a more risky asset is more costly for the seller and so higher default risk depreciates the outside option of the seller and decreases price. On the contrary, the price increases with the

asset's liquidity, which improves the seller's outside option in the search market. Interestingly, higher average asset liquidity in the market decreases asset prices. In a market with higher average asset liquidity, a buyer is more likely to be matched to a seller of a more liquid asset which improves the buyer's outside option.

Market conditions, like masses of searching buyers and sellers in the market, also affect prices although not directly but through their sensitivity to asset liquidity and average asset liquidity. When the mass of searching buyers in the market is higher, it is easier for a seller to find a counter-party. Hence, the gains for the seller from holding a more liquid asset are higher, which translates into the higher sensitivity of the asset price to asset liquidity. In turn, this leads to an increase in asset prices. On the contrary, when the mass of searching sellers in the market is higher, the buyer can more easily find a seller in the market. Hence, the gains for the buyer from an increase in average asset liquidity are higher, which translates into the higher sensitivity of the asset price to average asset liquidity and a dampening of prices.

The model with endogenous delay gives several insights into asset liquidity in an OTC market. This model gives a novel, testable prediction about the relation between default risk and liquidity for assets within the same credit ratings class. In contrast to adverse selection models in which more risky assets are more liquid (see Guerrieri and Shimer (2014)), the two-sided screening dynamics of the common screening bargaining solution lead to a non-monotone relation. Trade delay is higher for qualities in the middle of the quality range and lower for qualities near the extremes of the quality range. Owners and buyers of assets in the middle of the quality range have incentives to delay trade to hold out for a more favorable price offer instead of trading earlier at very low or very high prices, respectively. As a result, it is possible that a range of asset qualities in the middle may not be traded at all. For such assets, it takes parties too long to agree on price, and buyers prefer to reject such assets and continue their search for an asset whose price takes less time to negotiate.

The model provides a mechanism through which market uncertainty, reflected in the variance of asset payoffs, affects market liquidity. An increase in market uncertainty leads to longer screening during the bargaining stage. As a result, the range of asset qualities traded in the market decreases, as agents prefer to trade fewer assets for which the negotiation times do not increase significantly. More generally, increasing the heterogeneity of assets decreases the set of liquid assets. This brings up another empirical implication of the model: the range of traded assets within the asset class (e.g. assets with the same credit rating) is negatively correlated with the variance of asset payoffs in the asset class.

The link between market uncertainty and market liquidity describes a channel through which market transparency improves liquidity. Improved transparency can be broadly defined as the reduction in either search or bargaining friction, and the effect of the increase in transparency depends crucially on which of the two frictions is reduced. An increase in transparency through better public information leads to assets being traded in the market within more finely defined

classes. The variance of asset payoffs in each class is lower compared to the variance of payoffs in the market as a whole. As a result negotiation times decrease, and a greater variety of assets becomes attractive for risk-sharing. This contrasts with the effect on liquidity of a reduction in search friction, for example, through a better platform for matching agents. When the search friction is lower, buyers can more easily find alternative assets in the market. Because of this, they accept fewer assets for trade that allow for the fastest realization of gains from trade, and market liquidity is reduced. This demonstrates that the effect of search and bargaining frictions on market liquidity can be quite different. Market liquidity increases in response to a reduction in bargaining friction, while it decreases in response to a reduction in search friction.

In the analysis of liquidity, it is important to take into account the fact that different assets act as substitutes for risk-sharing. In the recent financial crisis of 2007-2008, traders reacted to the increase in market uncertainty by a shift in their preferences towards safer assets, a phenomenon known as flight-to-quality (Dick-Nielsen, Feldhutter, and Lando (2012), Friewald, Jankowitsch, and Subrahmanyam (2012)). Similarly, opponents of greater transparency in OTC markets point out that it can result in the migration of trade to certain asset classes, which will hurt the liquidity of the market as a whole. Therefore, it is important to understand how changes in market uncertainty affect the migration of agents.

In a simple multi-class extension of the model, I demonstrate that increased market uncertainty can result in flight-to-quality episodes wherein agents migrate to trading assets that have not suffered from increased uncertainty. The flight-to-quality exacerbates the negative effect of increased uncertainty on liquidity. I also show that while increasing transparency is a potentially useful measure that can increase both market liquidity and efficiency, it can also have the opposite effect. If after the release of public information there is an asset class that is significantly more liquid than the rest of the market, then agents will migrate to trading assets in this class. This adversely affects the liquidity of the rest of the market and can result in an overall decrease in market liquidity and welfare.

The structure of the paper is as follows. Section 2 presents the model. In Section 3, I begin the analysis with a simpler model with exogenous delay, as many steps of the analysis are similar in the two models. I derive the effect of default risk, liquidity and market conditions on asset prices and spreads. Section 4 analyzes the model with endogenous delay determined by the common screening bargaining solution. I first introduce the common screening bargaining solution for a general bargaining problem which I then apply to determine the endogenous delay. I study properties of the model both analytically and via numerical simulations. In Section 5, I introduce a multi-class extension of the model which I use it to analyze flights-to-quality and transparency in OTC markets. Section 6 points out some empirical implications. Proofs are relegated to the Appendix. Before proceeding with the analysis, I next describe the relationship of the current paper to existing literature.

**Related literature** This paper most closely related to the literature on search-and-bargaining in OTC markets represented by Duffie, Garleanu and Pedersen (2005, 2007), Lagos and Rocheteau (2007, 2009). The paper extends the analysis of the search-and-bargaining model for a single asset by Duffie, Garleanu and Pedersen (2007) to the case of multiple asset qualities. In this respect, it is complementary to Vayanos and Weill (2008), and Weill (2008). Many OTC markets such as markets for corporate bonds or derivatives contain a variety of assets, rather than specializing in a single asset. In fact, one of the purposes of OTC asset markets is to provide liquidity for non-standard assets. Extending Duffie, Garleanu and Pedersen (2007) to the case of multiple assets, besides capturing an important feature of OTC markets, allows for the analysis of the liquidity and prices of different assets and for understanding of how they are affected by changes in market conditions.

Previous multi-asset extensions have considered models in which agents choose between asset classes with varying characteristics, like trading shares or short-selling possibilities.<sup>5</sup> In this paper, agents learn/acquire a signal about the quality of an asset only after they find a trade partner which captures how trade happens within asset classes. In OTC markets, traders often look for assets with broadly-defined characteristics, such as a certain credit rating, maturity, or industry. However, there is a great deal of variation in expected returns among assets satisfying these rough criteria. Therefore, both the choice between different asset classes and the trade of assets within a particular class are important issues.

Another related strand of the literature is the literature on the dynamics of asset trading with adverse selection. Guerrieri and Shimer (2014) studies a model with asymmetric information about asset quality in a directed search model. In their model, in order to provide incentives for sellers of lower-quality assets to reveal their quality, such assets should be more liquid. Chang (2014) shows that when private information is multi-dimensional, there exist semi-pooling equilibria in which distressed owners of higher quality assets trade faster at a lower price. In contrast, in this paper, the dependence of liquidity on quality is U-shaped, reflecting the two-sided screening dynamics of the common screening bargaining solution. Another distinction with Guerrieri and Shimer (2014) and Chang (2014) is that in this paper, there is no asymmetric information, but the bargaining solution that I apply is motivated by a model with vanishing private information. In such setting, the range of traded asset qualities becomes important. Hence, the difference with Guerrieri and Shimer (2014) lies in the accent on the role of public information and market uncertainty rather than asymmetric information in the functioning of OTC markets.

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<sup>5</sup>In Weill (2008), assets are homogeneous in quality, but have different trading shares. In equilibrium, assets with larger trading shares are more liquid, as it is easier to find a counter-party for such assets. Vayanos and Weill (2008) analyzes a model with homogeneous agents in which two assets have different endogenous liquidity and prices because one of the assets is easier to short sell than the other. Vayanos and Wang (2007) studies a model with identical assets but agents with heterogeneous horizons. They show that clientèle equilibria are possible in which agents with shorter horizon prefer one asset over another because of its higher endogenous liquidity.

The paper also contributes to the recent literature on the connection between liquidity and default risk in OTC markets represented by He and Milbradt (2014), Chen, Cui, He, and Milbradt (2014). He and Milbradt (2014) analyzes the feedback loop between default and liquidity. Assets closer to default are associated with higher bid-ask spreads and, in turn, higher bid-ask spreads make endogenous default more likely. I focus on trade delay as the measure of asset liquidity instead of bid-ask spreads and show that generally the relationship between this measure of liquidity and default risk is U-shaped.

Most of the theoretical literature on search-and-bargaining pioneered by Rubinstein and Wolinsky (1985) focuses on the case of complete information and hence immediate agreement. Exceptions include work by Satterthwaite and Shneyerov (2007) and Lauer mann and Wolinsky (2014) which analyzes search models with incomplete information and provides conditions for the existence of equilibria and convergence to Walrasian outcomes. In these papers, allocations are determined by auction mechanisms and feature no delay. In contrast, the focus of this paper is on trading delay and how it affects the preferences of agents over assets. Another paper that explicitly incorporates trade delay into a search model is Atakan and Ekmekci (2014). In their model, agents imitate commitment types requesting a fixed share of the surplus, while in my model all agents are rational. Because of the interaction of incomplete information and search, the analysis of search-and-bargaining models with incomplete information presents a great challenge, and existence, let alone uniqueness and clear comparative statics, is difficult to prove. To tackle these complications while maintaining the realism of the model, the current paper employs the common screening bargaining solution that, on the one hand, features the two-sided screening dynamics that is common in bargaining models with two-sided incomplete information, but on the other hand, maintains the tractability of the analysis. The application of the novel common screening bargaining solution constitutes a methodological contribution of this paper to the theoretical literature on search and bargaining.

The paper is also related to the literature on asset pricing with transaction costs. This literature assumes that some assets are associated with exogenous proportional transaction costs (Constantinides (1986), Heaton and Lucas (1996), Vayanos (1998), Huang (2003)), fixed trading costs (Lo, Mamaysky, and Wang (2004)) or exogenous bid-ask spreads (Amihud and Mendelson (1986)). This paper studies the asset pricing and liquidity implications of a different type of costs, the opportunity costs of delayed trade. In the model with endogenous delay, these costs are also determined in equilibrium.

The bargaining solution used in this paper is motivated by the analysis of a bargaining model with private correlated values in Tsoy (2014). While Tsoy (2014) focuses on bargaining over the price of a durable good, this paper focuses on how agents use different assets to share risk. Correspondingly, values of agents, rather than being primitives of the model, are endogenously determined and reflect the transitory nature of liquidity shock and the possibility of future trade.



## 2 Model

There is a continuum of asset qualities  $\theta \in [0, 1]$ . Asset  $\theta$  is supplied in quantity  $f(\theta)$ . I assume that  $f$  is continuous and strictly positive and normalize  $\int_0^1 f(\theta)d\theta = 1$ . Denote by  $F$  the CDF of the distribution of asset qualities.

There is a continuum of agents of mass  $a$ . Time  $t \geq 0$  is continuous. There are two observable intrinsic types of agents which I call in anticipation of their equilibrium behavior buyers ( $b$ ) and sellers ( $s$ ). The intrinsic type of each agent switches independently from  $b$  to  $s$  with Poisson intensity  $y_d$ , and from  $s$  to  $b$  with Poisson intensity  $y_u$ . The initial distribution of types is such that the type distribution is stationary: there is a mass  $\frac{y_u}{y_u+y_d}a$  of buyers and a mass  $\frac{y_d}{y_u+y_d}a$  of sellers in the population.

Agents are risk-neutral and discount the future at the common discount rate  $r$ . The flow payoff from asset  $\theta$  is  $\bar{v}(\theta)$  for the buyer and  $\underline{v}(\theta)$  for the seller. I assume the following specification for  $\bar{v}$  and  $\underline{v}$ :

$$\bar{v}(\theta) = kg(\theta) + d, \tag{2.1}$$

$$\underline{v}(\theta) = kg(\theta) + d - \ell_\theta, \tag{2.2}$$

for some constants  $k > 0$  and  $d$ , measurable, positive function  $\ell_\theta$ , and weakly increasing function  $g$  with  $g(0) = 0$  and  $g(1) = 1$ . The interpretation is that the quality  $\theta$  is an index that aggregates various asset characteristics, and higher asset qualities translate into higher expected flow payoffs for the buyer. Sellers experience a transitory liquidity shock, and holding the asset is associated for them with additional *holding costs*  $\ell_\theta \equiv \bar{v}(\theta) - \underline{v}(\theta)$ . This implies that if trade were frictionless, buyers would purchase assets from sellers. In the endogenous-delay model, I additionally assume that functions  $\bar{v}$  and  $\underline{v}$  are strictly increasing and continuously differentiable.

To interpret payoff functions, consider the following simple model for bond payoffs:  $\bar{v}(\theta) = C - (R_\theta + R)D$  and  $\underline{v}(\theta) = C - (R_\theta + R)D - \ell$  for positive  $C, R, D, \ell$ , and strictly decreasing, continuously differentiable, positive function  $R_\theta$  with values in  $[0, \bar{R}]$ . In this example, an asset is a bond with infinite maturity and coupon  $C$ . In case of default, the bond-holder incurs costs  $D$ , and the bond is immediately reissued to the same holder after the default. The risk of default consists of two components: asset-specific component  $R_\theta$  and systemic component  $R$ . These payoffs are obtained from equations (2.1) and (2.2) by setting  $k = \bar{R}D$ ,  $d = C - (R + \bar{R})D$ ,  $\ell_\theta = \ell$ ,  $g(\theta) = 1 - \frac{R_\theta}{\bar{R}}$ . Keeping this interpretation in mind, I interpret that  $g(\theta)$  reflects the *default risk* associated with the asset,  $k$  reflects the *market uncertainty* about the quality represented by the variance of default risk in the market, and  $d$  reflects the *aggregate default risk*.

Notice that when  $\ell_\theta = \ell$ , holding costs are constant across asset qualities. Then it is a standard result that without loss of generality the distribution of qualities can be assumed to

be uniform.<sup>6</sup> In general, one can argue both that more risky assets are associated with higher gains from trade and  $\ell_\theta$  is decreasing (e.g. agents holding toxic assets are especially eager to sell them), and that higher-quality assets are associated with higher benefits for the holder and  $\ell_\theta$  is increasing (e.g. such assets can be used as collateral for cheaper short-term borrowing).

Each agent is constrained to hold at most one asset. This way, I abstract from agents' portfolio decisions and focus on their risk-sharing motives. Assets are initially randomly distributed among agents.<sup>7</sup> I assume not all agents own assets, and thus  $a > 1$ .

Agents can trade assets in a market with both search and bargaining frictions. There are two stages to the trading process: the search stage and the bargaining stage. In the bargaining stage, agents trade an asset  $\theta$  with delay  $t_\theta$  at price  $q_\theta$ . The positive, asset-specific trade delay represents a bargaining friction. The price of trade  $q_\theta$  splits the (endogenous) surplus between the buyer and the seller in proportion  $\alpha$  to  $1 - \alpha$  where  $\alpha \in (0, 1)$ . While the assumption of proportional split is common in the literature where it is motivated by the generalized Nash (1950) bargaining solution, non-trivial trade delay is a novel feature of this paper. Trade delay  $t_\theta$  reflects the liquidity of the asset: an asset with lower  $t_\theta$  is more liquid as it can be more quickly transferred from the seller to the buyer. This specification incorporates both exogenous and endogenous delay. In the former,  $t_\theta$  is a primitive of the model, while in the latter,  $t_\theta$  is pinned down by additional equilibrium conditions. I assume that once the intrinsic type of one of the matched agents switches or agents complete the trade, the match is destroyed, and agents do not participate in search while matched. If  $t_\theta = 0$  for all qualities  $\theta$ , then the model reduces to that of Duffie, Gârleanu and Pedersen (2005).

Search is costless, and all unmatched agents participate in search. Agents are randomly matched to each other in a market with search friction. The matching process is independent of the evolution of intrinsic types and is given by the quadratic matching technology commonly used in the search-and-bargaining literature (see for example Duffie (2012)). Buyers of mass  $m_b$  contact sellers of mass  $m_s$  with intensity  $\frac{\lambda}{2}m_b m_s$  and so the total meeting rate of these two groups of agents is  $\lambda m_b m_s$ . This matching technology is the same as in Duffie, Garleanu, and Pedersen (2005).<sup>8</sup> The fact that the match is not instantaneous represents the search friction.

Each agent can be either matched ( $m$ ) or unmatched ( $u$ ). I refer to the intrinsic type of the agent and his match status as the *type*  $\tau \in \{bu, su, bm, sm\}$  of the agent. The *asset position* of the agent  $[0, 1] \cup \{\phi\}$  is the quality of the asset that the agent owns or bargains over. I use notation  $\phi$  for agents who do not own an asset and are not matched to a seller.

When a match is found, the agents involved choose whether to participate in bargaining. Agents' strategies condition only on the type and the quality of the asset that the agent owns

<sup>6</sup>Indeed, I can transform function  $g$  so that the distribution of asset payoffs remains the same, but  $F$  is uniform.

<sup>7</sup>One can think of the distribution of assets as follows. Agents are associated with a point  $(\theta, i) \in [0, 1] \times [0, a]$  such that agents that belong to  $\{(\theta, i) | 0 \leq i \leq f(\theta)\}$  own one unit of asset  $\theta$  and a fraction  $\frac{y_u}{y_u + y_d}$  of them is buyers. Agents that belong to  $\{(\theta, i) | i > f(\theta)\}$  do not own any asset.

<sup>8</sup>Duffie and Sun (2007) provides probabilistic foundations for this matching technology.

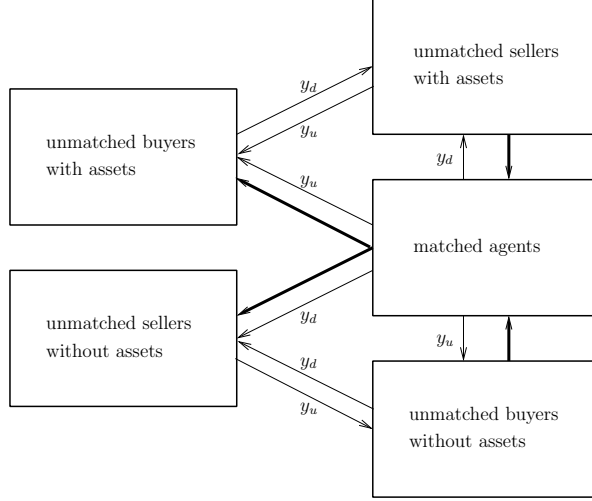


Figure 1: **The evolution of types and asset holdings.** Bold arrows indicate transitions between types and changes in asset holding caused by bargaining, and thin arrows indicate transitions caused by the switching of the intrinsic types (intensities are written next to arrows).

or is matched to. For exogenous delay, asset quality is directly observable. For endogenous delay, the interpretation is that the buyer conditions his strategy on an almost-perfect signal about the asset's quality. The bargaining solution used in Section 4 is the reduced form for bargaining between agents with almost-perfect signals about quality. In what follows, I will not make a distinction between the observed quality and the arbitrary precise signals about it, but the difference in the interpretation should be kept in mind.

I assume that sellers always choose to participate in the bargaining stage. This assumption simplifies the notation and is without loss of generality; as I will show in the derivation of the equilibrium, the seller always derives higher utility from bargaining than from continuing to search. The (mixed) strategy of the buyer  $\sigma_\theta \in [0, 1]$  specifies the probability with which the buyer matched with the seller of asset  $\theta$  participates in the bargaining stage. Denote by  $\Theta_L$  the set of assets such that  $\sigma_\theta = 1$ , and by  $\Theta_M$  the set of assets such that  $\sigma_\theta \in (0, 1)$ . I call assets in  $\Theta_L$  *unconditionally liquid* or simply *liquid*, assets in  $\Theta_M$  *conditionally liquid*, and assets in  $\Theta_I \equiv [0, 1] \setminus (\Theta_L \cup \Theta_M)$  *illiquid*. The evolution of types and asset holdings is depicted in Figure 1. For example, consider a group of matched sellers, each of whom holds an asset of quality  $\theta$ . Then the transition from this group could happen according to three possible scenarios. First, the bargaining stage is completed and the asset changes hands (bold arrows from block of matched agents in Figure 1). Second, a seller in this group recovers from liquidity shock and becomes a buyer (arrow indexed by intensity  $y_u$ ). Finally, the buyer to whom the seller is matched switches intrinsic type and the match is destroyed (arrow indexed by intensity  $y_u$ ).

The economy is in steady state. Denote the steady-state distribution of assets among different

types of agents by  $M = \{M_\tau \in \Delta([0, 1]), \tau \in \{bm, bu, sm, su\}\}$ . For example, for any measurable set  $\Theta \subseteq [0, 1]$ ,  $M_{bu}(\Theta)$  gives the mass of unmatched sellers that own assets in  $\Theta$ , and  $M_{bm}(\Theta)$  gives the mass of matched buyers that bargains over some asset in  $\Theta$ . I consider equilibria such that there exists the mass density function  $\mu_\tau$  of  $M_\tau$ . I also denote by  $\mu_s(\theta) \equiv \mu_{su}(\theta) + \mu_{sm}(\theta)$  the mass of sellers each of whom (inefficiently) owns an assets of quality  $\theta$ .

There are several balance conditions imposed on  $M$ . First, for any asset  $\theta$ , the sum of agent positions is equal to the supply of the asset,

$$\mu_{su}(\theta) + \mu_{bu}(\theta) + \mu_{bm}(\theta) = f(\theta). \quad (2.3)$$

Second, since  $\int_0^1 f(\theta)d\theta = 1$  by normalization, and the total mass of assets is  $a$ , the mass of agents that do not hold any asset is equal to  $a - 1$ ,

$$M_{su}(\phi) + M_{bu}(\phi) + M_{bm}(\Theta_L \cup \Theta_M) = a - 1. \quad (2.4)$$

Third, the number of matched agents of each intrinsic type should coincide with the number of matches,

$$\mu_{sm}(\theta) = \mu_{bm}(\theta). \quad (2.5)$$

Finally, the steady-state assumption requires that there be no changes in the distribution  $M$  over time. I analyze the equilibrium of the model in steady state defined as follows.

**Definition 1.** *A tuple  $(\sigma_\theta, M)$  constitutes an equilibrium if the buyer's strategy  $\sigma_\theta$  is optimal given  $M$ , and  $M$  is the steady-state distribution of assets generated by  $\sigma_\theta$ .*

For the model with exogenous delay, I additionally assume that whenever the buyer is indifferent between proceeding to the bargaining stage and continuing to search, the probability with which he chooses to bargain is independent of asset quality. This assumption will ensure uniqueness of the equilibrium.

### 3 Equilibrium

This section analyzes the model. The steps of the analysis provided in this section do not depend on the specification of the delay, and for the model with exogenous delay, they are sufficient to characterize the equilibrium. In Subsection 3.1, for a given strategy  $\sigma_\theta$  and delay profile  $t_\theta$ , I derive the steady-state distribution  $M$  of assets among different types. In Subsection 3.2, for a given steady-state distribution  $M$ , I derive the buyer's optimal strategy  $\sigma_\theta$ . The resulting strategy takes a simple form. A buyer proceeds to the bargaining stage for asset qualities that satisfy certain criteria on holding costs and trade delay associated with the asset. In Theorem 1, I combine the two relations between  $M$  and  $\sigma_\theta$  to show that the equilibrium in the model with exogenous delay is unique and to describe the properties of prices and spreads.

### 3.1 Steady-State Distribution

In this subsection, I show that for a given strategy profile  $\sigma_\theta$  and trade delay profile  $t_\theta$ , there exists a unique steady-state distribution  $M$ , and I describe its properties. The following characteristics of the distribution  $M$ , which I refer to as *market conditions*, are important for the optimality of the behavior of agents. Let  $\Lambda \equiv \lambda M_{bu}(\phi)$  be the intensity with which a seller is matched to buyers,  $\Lambda_b \equiv \lambda M_{su}(\Theta_L)$  be the intensity with which a buyer is matched to sellers of liquid assets (in  $\Theta_L$ ), and  $F_L \in \Delta(\Theta_L)$  be the steady-state probability distribution of asset qualities in the pool of sellers of liquid assets  $\Theta_L$ . Quantities  $\Lambda$  and  $\Lambda_b$  reflect how easily a seller or a buyer, respectively, can find a trade partner if they continue to search, and  $F_L$  gives the probability distribution over asset qualities that bring the buyer strictly higher utility than any utility gained from searching for another asset in the market. Let  $L \equiv \int_{\Theta_L} f(\theta)d\theta$  be the mass of assets in  $\Theta_L$ . I refer to  $L$  as the *market liquidity*; higher  $L$  means that the buyer accepts a broader range of assets for trade.<sup>9</sup> Notice that when  $F$  is uniform (recall that in the constant-holding-cost case, this is without loss of generality), market liquidity measures the range of liquid asset qualities. The following lemma describes  $\Lambda, \Lambda_b, F_L$  in a unique steady-state distribution corresponding to a given strategy profile and specification of delay.

**Lemma 1.** *For any strategy  $\sigma_\theta$  and delay profile  $t_\theta$ , there exists a unique steady-state distribution  $M$  in which  $\Lambda$  is the unique solution to*

$$\frac{\Lambda}{\lambda} = \frac{y_u}{y_u + y_d}(a - 1) - \frac{y_d}{y_u + y_d} \int_0^1 \frac{\Lambda \sigma_\theta}{y_u + y_d + \Lambda \sigma_\theta} dF(\theta), \quad (3.1)$$

$\Lambda_b$  is given by

$$\Lambda_b = \frac{\lambda y_d L}{y_u + y_d + \Lambda}, \quad (3.2)$$

and  $F_L$  is given by the conditional distribution of  $F$  conditional on  $\theta \in \Theta_L$ .

The key simplifying feature of Lemma 1 is that  $\Lambda, \Lambda_b$ , and  $F_L$  depend only on  $\sigma_\theta$ , but not on  $t_\theta$ .<sup>10</sup> The result that the liquidity characteristic  $t_\theta$  does not affect the distribution of assets  $F_L$  is a bit counter-intuitive at first sight, as one may expect that more liquid assets are traded more quickly and so are more abundant in the market. To see why this is the case, observe that the inflow into the group of sellers of asset  $\theta$  is formed from matched sellers whose counter-party is hit by a liquidity shock and from unmatched buyers owning asset  $\theta$  who are hit by a liquidity shock (see Figure 1). Both these inflows have intensity  $y_d$ . At the same time, the outflow from this group of sellers happens because of the recovery from the shock of sellers and the formation

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<sup>9</sup>Alternatively, one could consider the mass  $\int_{\Theta_L \cup \Theta_M} f(\theta)d\theta$  of both conditionally and unconditionally liquid assets as a measure of market liquidity. I focus on  $L$ , as it allows for clear comparative statics. Moreover, the difference between two measures is insignificant in numerical simulations.

<sup>10</sup>Unlike  $\Lambda, \Lambda_b, F_L$ , steady-state distribution  $M$  derived explicitly in the Appendix depends on the delay profile  $t_\theta$ .

of new matches. The former has intensity  $y_u$  and the latter has intensity  $\Lambda$ , and both are again independent of  $t_\theta$ . Therefore,  $t_\theta$  only changes the distribution of agents between those who have already completed a trade and those still bargaining but does not affect the mass of sellers in the search stage.

Equation (3.1) has a natural interpretation. The left-hand side gives the mass of buyers without an asset, which in the absence of trade, equals  $\frac{y_u}{y_u+y_d}(a-1)$ . When agents are allowed to trade the mass of buyers without an asset decreases, which reflects the fact that ownership of assets becomes more efficient.

An interesting feature that follows from equation (3.1) is that if buyers accept a greater variety of assets this reduces the chances of the seller to be matched. In particular, if  $\sigma_\theta$  weakly increases,<sup>11</sup> then it follows from equations (3.1) and (3.2) that  $\Lambda$  decreases and  $\Lambda_b$  increases. The more assets buyers accept, the more likely it is for the buyer to find a match, however, this implies more competition for sellers and for them the likelihood of forming a match decreases. Notice that this happens despite the fact that the matching technology does not feature externalities, i.e. the fact that additional sellers are searching for buyers does not reduce the chances of others to be matched. The competition between sellers arises, however, for the following reason: the fact that buyers accept a wider variety of assets implies that more buyers find matches. These buyers are either busy in the bargaining stage or have already completed their trades. This reduces the number of buyers searching in the market and reduces the likelihood of a match for unmatched sellers.

In the empirical analysis, the trade delay is often not directly observable, and various proxies are used to measure asset liquidity. Trading volume and turnover are the most relevant characteristics for my analysis, and I next show how they are related to the trade delay  $t_\theta$  and other variables. Denote the trading volume by  $\gamma_\theta$ . It is shown in the Appendix that the distribution  $G(\theta, u)$  of times  $u$  that a matched seller of asset  $\theta$  spends in the bargaining stage is a truncated exponential distribution supported on  $[0, t_\theta]$  with the parameter  $y_u + y_d$ . In time interval  $du$ , matches that have already spent time  $[t_\theta - du, t_\theta]$  in the bargaining stage trade. Therefore, trading volume can be determined from  $\gamma_\theta = \frac{d}{du}G(\theta, t_\theta)$  or, as shown in Appendix,

$$\gamma_\theta = \frac{\Lambda\sigma_\theta y_d}{y_u + y_d + \Lambda\sigma_\theta} f(\theta) e^{-(y_u+y_d)t_\theta}.$$

Trading volume is decreasing in the length of bargaining delay, and it is also affected by the intensity  $\Lambda$  as well as by buyer strategy  $\sigma_\theta$ . In my model, the asset turnover is given by  $\frac{\gamma_\theta}{f(\theta)}$ , and is again decreasing with the trade delay associated with the asset. Both liquidity proxies increase with the decrease in trade delay.

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<sup>11</sup>Here and further on, I say that function  $f_1$  is *greater than*  $f_2$  if  $f_1$  is greater than or equal to  $f_2$  at all points. For a sequence of functions  $f_i$  indexed by  $i$  belonging to some interval, I say that  $f_i$  is *increasing* if for any  $i > i'$ ,  $f_i$  is greater than  $f_{i'}$ . Definitions for a decreasing sequence of functions are analogous.

### 3.2 Equilibrium

In this subsection, given a steady-state distribution  $M$ , I compute the optimal strategy  $\sigma_\theta$ . For  $\tau \in \{bu, su, bm, sm\}$ , let  $V_\tau(\theta)$  be the expected utility of an agent of type  $\tau$  owning (or bargaining over) asset  $\theta$ , and for  $\tau \in \{bu, su\}$ , let  $V_\tau(\phi)$  be the expected utility of an agent of type  $\tau$  owning no asset. Value functions during the search stage are determined by the following Bellman equations,

$$rV_{su}(\phi) = y_u(V_{bu}(\phi) - V_{su}(\phi)), \quad (3.3)$$

$$rV_{bu}(\theta) = \bar{v}(\theta) + y_d(V_{su}(\theta) - V_{bu}(\theta)), \quad (3.4)$$

$$rV_{bu}(\phi) = y_d(V_{su}(\phi) - V_{bu}(\phi)) + \Lambda_b(\mathbb{E}[V_{bm}(\theta)|\theta \in \Theta_L] - V_{bu}(\phi)) \quad (3.5)$$

$$rV_{su}(\theta) = \underline{v}(\theta) + y_u(V_{bu}(\theta) - V_{su}(\theta)) + \sigma_\theta \Lambda(V_{sm}(\theta) - V_{su}(\theta)). \quad (3.6)$$

The depreciation of value functions in the left-hand side of equations (3.3) – (3.6) equals the sum of flow payoffs and changes in value functions due either to switches of intrinsic types or the formation of matches. For example, consider equation (3.5). The flow payoff of the searching buyer without an asset is zero. If the buyer is hit by a liquidity shock, his value function drops to  $V_{su}(\phi)$ , while if he is matched to a seller, then his value function increases to  $\mathbb{E}[V_{bm}(\theta)|\theta \in \Theta_L]$ . Notice that if a buyer is matched to a seller of an asset in  $\Theta_M$ , then his continuation utility is  $V_{bu}(\phi)$  irrespective of whether he starts to negotiate or continues to search. Therefore, in equation (3.5), it is sufficient to consider the case when the buyer is matched to sellers of assets  $\Theta_L$  and the relevant distribution is  $F_L$ .

In the bargaining stage, the match can be exogenously destroyed if the intrinsic type of one of the agents switches, so the *efficient discount factor* is given by  $\rho \equiv r + y_u + y_d$ . To determine the price of trade, I compute the benefits  $v(\theta)$  from trade for the buyer of asset  $\theta$ , and the costs of trade  $c(\theta)$  for the seller of asset  $\theta$ . Let  $\hat{c}(\theta)$  be the value for the seller of asset  $\theta$  from staying in the match but never selling the asset, and  $\hat{v}$  be the value for the buyer from staying in the match but not buying from the current seller. Then  $c(\theta) = -(V_{su}(\phi) - \hat{c}(\theta))$  and  $v(\theta) = V_{bu}(\theta) - \hat{v}$ . By the assumption of the proportional split of the surplus, the price of trade is given by

$$q_\theta = (1 - \alpha)v(\theta) + \alpha c(\theta). \quad (3.7)$$

Given the Bellman equations (3.3) – (3.6), the price of trade (3.7) and the delay profile  $t_\theta$ , one can find value functions and determine optimal strategies. Before describing the optimal strategies, let me introduce the key asset liquidity measure. Denote by  $x_\theta = e^{-\rho t_\theta}$  the loss from delay, the factor by which the surplus from trade of the asset  $\theta$  is dissipated due to delay. I refer to  $x_\theta$  as the *liquidity* of asset  $\theta$ . Let  $z_\theta = \xi_\theta x_\theta$  be the *expected surplus* from trade where  $\xi_\theta \equiv v(\theta) - c(\theta)$  is the trade *surplus*. The interpretation is that with probability  $1 - x_\theta$ , the match is destroyed because of switches of types or discounting and the realized surplus in the

match is zero, and with complementary probability  $x_\theta$ , the surplus  $\xi_\theta$  is realized after agents negotiate for time  $t_\theta$ . The following lemma states that the equilibrium strategy takes the simple threshold form.

**Lemma 2.** *Given a profile of expected surpluses  $z_\theta$ , the asset of quality  $\theta$  is liquid ( $\theta \in \Theta_L$ ) if and only if*

$$z_\theta > \underline{z} \equiv \frac{\Lambda_b}{\rho + \Lambda_b} \bar{z}, \quad (3.8)$$

where  $\bar{z} \equiv \frac{1}{L} \int_{\theta \in \Theta_L} z_\theta dF(\theta)$ .

By Lemma 2, the buyer trades off the trade delay and the surplus from trade. Even when the gains from trade are large, the buyer may reject the asset because of the high delay associated with it. The threshold  $\underline{z}$  is equal to a fraction of the average (over assets in  $\Theta_L$ ) asset liquidity  $\bar{z}$ . The difference between the buyer threshold  $\underline{z}$  and the average liquidity  $\bar{z}$  depends on how easily the buyer can find another seller. If the intensity of a match is low for the buyer (low  $\Lambda_b$ ), then  $\underline{z}$  is much lower than  $\bar{z}$ , and the buyer accepts a greater variety of assets.

Observe that when holding costs are constant across asset qualities, preferences of buyers over assets are driven solely by liquidity considerations. Indeed, in this case,  $z_\theta = \xi x_\theta$  for all  $\theta$  where  $\xi$  is the constant trade surplus. Denoting by  $\bar{x} \equiv \frac{1}{L} \int_{\theta \in \Theta_L} x_\theta dF(\theta)$  the average liquidity for assets in  $\Theta_L$ , I see from Lemma 2 that buyers search for most liquid assets in the market and accept assets with  $x_\theta > \underline{x} \equiv \frac{\Lambda_b}{\rho + \Lambda_b} \bar{x}$ .

For the model with exogenous delay, in the next theorem I combine Lemmas 1 and 2 with the exogenous-delay profile  $t_\theta$  to show the existence and uniqueness of the equilibrium and describe the behavior of prices and spreads. Yield spreads are defined as the difference between the asset yield (flow payoff of the buyer divided by the price) and risk-free rate, i.e.  $s_\theta \equiv \frac{\bar{v}(\theta)}{q_\theta} - r$ .

**Theorem 1.** *In the model with exogenous delay, there exists a unique equilibrium. In equilibrium, the following hold:*

1. *Prices of assets in  $\Theta_L \cup \Theta_M$  are given by*

$$q_\theta = \underbrace{\frac{1}{r} (kg(\theta) + d - (r + y_d)\xi_\theta) + (1 - \alpha)\xi_\theta}_{\text{default-risk}} + \underbrace{(1 - \alpha) \frac{y_d}{r} \frac{\sigma_\theta \Lambda}{\rho + \sigma_\theta \Lambda}}_{\text{liquidity premium}} z_\theta - \underbrace{\alpha \frac{y_u}{r} \frac{\Lambda_b}{\rho + \Lambda_b} \bar{z}}_{\text{average liquidity}}. \quad (3.9)$$

2. *If the yield spread  $s_\theta$  is positive, then the partial derivatives of  $s_\theta$  with respect to  $g(\theta)$ ,  $\sigma_\theta$ ,  $z_\theta$  are negative and with respect to  $d$  are positive.*
3. *The equilibrium strategy and asset distribution  $(\sigma_\theta, M)$  are characterized by the match intensity of the seller and the threshold of the buyer  $(\Lambda, \underline{z})$  and are independent of the aggregate default risk parameter  $d$ .*



An important implication of Theorem 1 is the decomposition of prices in equation (3.9) into three components: the default-risk component, the liquidity premium and the average liquidity component. To interpret these various components, we first focus on the case of constant holding costs. The first component gives the asset's price if there were no market where the buyer and the seller can search for another trade partner. This price captures the value of holding the asset for the seller plus his fraction of the trade surplus. The flow payoff from the asset enters the price only through this component. Given the earlier interpretation of flow payoffs as reflecting the risk of default, I call this component the default-risk component. When holding costs are constant, differences in the first component are driven solely by differences in the default-risk associated with assets.

The other two components reflect how the outside options created by the search market affect prices. The second component of the price depends on  $x_\theta$  and reflects the liquidity premium. The more liquid the asset is, the higher the price the buyer is willing to pay. This effect is driven by the outside option of the seller to search in the market for another buyer. For a more liquid asset, after the new match is formed, less surplus is dissipated due to delay, which increases the outside option of the seller and hence increases the price of asset. Observe that this outside option depends on the ability of the seller to find a buyer ( $\Lambda$ ). The more unmatched buyers in the market, the more valuable the outside option of the seller and the higher the price sensitivity to the asset's liquidity. The fact that the sensitivity of the price to liquidity depends on aggregate market conditions was documented empirically in Bao, Pan, and Wang (2011) and Friewald, Jankowisch, and Subrahmanyam (2012).

The third component is the effect of average (across assets in  $\Theta_L$ ) liquidity in the market. This component accounts for the buyer's outside option of finding another seller. Naturally, the outside option of the buyer is increasing in the average liquidity and pushes the price down. Therefore, the third component has a negative sign. This effect is larger the easier it is for the buyer to find a trade partner (higher  $\Lambda_b$ ).

When  $\ell_\theta$  varies with quality, higher holding costs decrease the default-risk component, but increase the liquidity premium component. The intuition is that for higher level of liquidity shock, the value of holding an asset is lower. However, this drop in asset value because of the default-risk component is partially compensated for by the liquidity premium component. Higher holding costs imply that the surplus from trade is larger, and hence, gains from trade for the seller are larger. This increases the outside option of the seller and hence, increases the asset price.

When  $z_\theta = \bar{z}$  for all  $\theta$ , my model reduces to Duffie, Gârleanu and Pedersen (2007) which already allows us to distinguish between the default and non-default components of asset prices. An important new feature of this paper is that equation (3.9) further separates the liquidity premium component which varies in the cross-section of assets, and the average-liquidity component which will be shown in Section 5 to depend on the liquidity of other asset classes. This

distinction is empirically relevant as demonstrated in Longstaff, Mithal, and Neis (2005).

It should be noted that even though illiquid assets not in  $\Theta_L \cup \Theta_M$  are rejected by buyers, equation (3.9) with  $\sigma_\theta = 0$  also determines the price of these assets when the buyer deviates from his equilibrium strategy and proceeds to the bargaining stage with the seller of an illiquid asset. Also note that by the second conclusion of Theorem 1 whenever spreads are positive, the reaction of yield spreads to the default risk, liquidity, and aggregate default risk is the opposite of the reaction of prices to these same factors.

The last conclusion of Theorem 1 reveals the key simplifying step in the analysis of the model. In general, equilibrium in the search-and-bargaining model with varying delay is a fixed-point  $(M, \sigma_\theta)$  of some functional operator. Distribution  $M$  should be generated by the behavior of agents following strategy  $\sigma_\theta$  and at the same time, given the distribution  $M$ , strategy  $\sigma_\theta$  should be optimal for buyers. By conclusion 3 of Theorem 1, in my setting, a pair  $(\Lambda, \underline{z})$  is sufficient to compute the equilibrium  $(M, \sigma_\theta)$  and hence, the problem of the existence and uniqueness of the fixed-point in the functional space is reduced to the simpler problem of finding a pair of numbers  $(\Lambda, \underline{z})$  that completely specifies the equilibrium.

Finally, the aggregate default risk reflected by parameter  $d$  does not affect the liquidity of assets but does affect the level of asset prices and yield spreads. An increase in the aggregate default risk leads to an increase in spreads, but does not affect the preferences of agents over assets. In subsequent sections, this feature of the model will allow me to show that preferences of agents during periods of heightened uncertainty are driven by liquidity concerns, rather than by quality concerns.

## 4 Endogenous Bargaining Delay.

This section studies the model with endogenous trade delay. A central assumption of the model is that prices and delay are given by the common screening bargaining solution (CSBS), which can be thought of as a reduced form for a bargaining outcome with almost-perfect information about an asset's quality. In Subsection 4.1, I describe this bargaining solution as well as its game-theoretic foundations as provided in Tsoy (2014). In Subsection 4.2, the CSBS is applied to determine the endogenous bargaining delay in a search-and-bargaining model. The analysis is particularly tractable in its linear specification with constant holding costs for which the comparative statics is derived analytically. In Subsection 4.3, the effect of market uncertainty on welfare and liquidity is further analyzed via a numerical simulation.

### 4.1 Common Screening Bargaining Solution.

In this subsection, I define and characterize the CSBS for a general class of bargaining problems. The CSBS exhibits intuitive, two-sided screening dynamics and has game-theoretic foundations

similar to those of the generalized Nash (1950) bargaining solution commonly used in the literature.

Consider the following general bargaining problem described by the tuple  $(\rho, v, c)$ .<sup>12</sup> There is a unit continuum of asset qualities  $\theta \in [0, 1]$  and for each  $\theta$ , the buyer's valuation is  $v(\theta)$  and the seller's cost is  $c(\theta)$ . Suppose that  $v$  and  $c$  are strictly increasing, continuously differentiable, and the trade surplus  $\xi_\theta \equiv v(\theta) - c(\theta)$  is positive for all  $\theta$ . Time is continuous, and parties discount at rate  $\rho$ . If parties trade at time  $t$  at price  $q$ , then the payoff to the buyer is  $e^{-\rho t}(v(\theta) - q)$  and the payoff to the seller is  $e^{-\rho t}(q - c(\theta))$ . The CSBS to this bargaining problem is defined as follows.

**Definition 2.** *The common screening bargaining solution (CSBS)  $(q_\theta, t_\theta, \theta^*)$  to the bargaining problem  $(\rho, v, c)$  with the surplus split  $\alpha \in (0, 1)$  requires the following conditions:*

1. *The price of trade  $q_\theta$  is given by  $q_\theta = (1 - \alpha)v(\theta) + \alpha c(\theta)$ , for all  $\theta \in [0, 1]$ .*
2. *The delay of trade  $t_\theta$  satisfies*

$$\theta \in \operatorname{argmax}_{\theta' \in [\theta^*, 1]} e^{-\rho t_{\theta'}} (v(\theta) - q_{\theta'}), \text{ for } \theta \geq \theta^*, \quad (4.1)$$

$$\theta \in \operatorname{argmax}_{\theta' \in [0, \theta^*]} e^{-\rho t_{\theta'}} (q_{\theta'} - c(\theta)), \text{ for } \theta < \theta^*, \quad (4.2)$$

with  $t_0 = t_1 = 0$  and  $\lim_{\theta \rightarrow \theta^* -} t_\theta = \lim_{\theta \rightarrow \theta^* +} t_\theta$ .

The first condition in Definition 2 states that trade happens at prices that split the surplus proportionally in accordance with the description of the model in Section 2. The last two conditions implicitly define the delay for every asset quality. Before providing an interpretation of the delay in the CSBS, the next lemma gives the explicit equations determining delay  $t_\theta$  as a function of primitives  $(\rho, v, c)$  of the bargaining problem and describes its properties.

**Lemma 3.** *In CSBS  $(q_\theta, t_\theta, \theta^*)$ , trade delay  $t_\theta$  is continuously differentiable, strictly increasing for  $\theta \leq \theta^*$  and strictly decreasing for  $\theta > \theta^*$ , and is characterized as follows:*

$$t_\theta = \begin{cases} \int_0^\theta \frac{c'(\theta) + (1-\alpha)\xi'_\theta}{\rho(1-\alpha)\xi_\theta} d\theta, & \text{for all } \theta \leq \theta^*, \\ \int_\theta^1 \frac{v'(\theta) - \alpha\xi'_\theta}{\rho\alpha\xi_\theta} d\theta, & \text{for all } \theta > \theta^*. \end{cases}$$

Moreover, holding  $\xi'_\theta$  fixed, an increase in  $v'$  and/or  $c'$  results in an increase in  $t_\theta$ .

The CSBS exhibits two-sided screening dynamics. To see this, consider the following related continuous-time bargaining game. For  $t \in [0, t_{\theta^*}]$ , define the path of seller price offers by

<sup>12</sup>With a little abuse of the notation, I use the same notation for the discount factor and agents' values in the description of the bargaining problem in this subsection as for the efficient discount factor and value functions in the description of the bargaining stage in Section 3. In the next subsection, I use the latter as primitives of the bargaining problem to determine the endogenous trade delay.

$q_t^S = \max\{q_\theta : t_\theta = t\}$  and the path of buyer price offers by  $q_t^B = \min\{q_\theta : t_\theta = t\}$ . For  $t > t_{\theta^*}$ , let  $q_t^S = q_t^B = q_{\theta^*}$ . By Lemma 3, such price-offer paths are well-defined, and  $q_t^S$  is decreasing and  $q_t^B$  is increasing.<sup>13</sup> Both sides follow the corresponding path of offers and only choose the time when they accept the offer of the opponent. In the unique Nash equilibrium of this game, buyers of asset qualities  $\theta \geq \theta^*$  accept at time  $t_\theta$ , sellers of asset qualities  $\theta < \theta^*$  accept at time  $t_\theta$ , and the remaining buyers and sellers accept at time  $t_{\theta^*}$ . It is easy to see that the equilibrium outcome of this game coincides with the outcome of the CSBS. Moreover, the bargaining game can be interpreted as the two-sided screening process. The seller makes decreasing offers to screen buyers of asset qualities above  $\theta^*$ , and the buyer makes increasing offers to screen sellers of asset qualities below  $\theta^*$ . By time  $t_{\theta^*}$ , the game ends with one of the parties accepting the offer of the opponent.

A natural concern about the continuous-time bargaining game described above is why parties have incentives to stick to screening offers  $q_t^S$  and  $q_t^B$ . Indeed, if the asset quality were known, then both parties would know that they trade after a specific costly delay, and they would have incentives to make different offers to trade earlier. Tsoy (2014) addresses this concern and shows that this behavior can arise in equilibrium in a standard bargaining model with private correlated values. Specifically, instead of directly observing the quality, both parties receive conditionally independent signals about the quality that determine their values. After parties observe their signals, they determine the price of trade by making alternating offers until one of the parties accepts the opponent's offer. Tsoy (2014) shows that there is a sequence of equilibrium outcomes that converges to the CSBS as the time between offers converges to zero and the correlation between signals becomes perfect.<sup>14</sup>

The two-sided screening dynamics of the continuous-time bargaining game is sustained as follows. There is a gap between arbitrarily precise private information about values given by parties' signals and relatively crude public information about values reflected in the range of values possible before parties get signals about quality. Because of this gap, it is possible to construct continuation equilibrium in which trade is almost immediate and the side that deviates from the prescribed price-offer path gets a very low share of the surplus. As a result, despite the fact there is an efficiency loss due to trade delay, both parties prefer adhering to equilibrium price paths and getting their share ( $\alpha$  or  $1 - \alpha$ ) of the smaller surplus to deviating and getting a lower share of the larger surplus in the continuation equilibrium.<sup>15</sup> The crucial element of the argument is the gap between private and public information. This assumption is realistic in OTC

<sup>13</sup>To see this, observe that since  $t_\theta$  is strictly monotone, continuous on intervals  $[0, \theta^*]$  and  $(\theta^*, 1]$ , and  $v$  and  $c$  are strictly increasing and continuous, for any  $t \in [0, t_{\theta^*})$ , there exist exactly two asset qualities  $\theta_1 < \theta^* < \theta_2$  for which  $t_\theta = t$  and  $q_t^S = q_{\theta_2}, q_t^B = q_{\theta_1}$ . Moreover, the monotonicity of paths follows from the monotonicity of  $t_\theta$  on intervals  $[0, \theta^*]$  and  $(\theta^*, 1]$  and the monotonicity of  $q_\theta$  on  $[0, 1]$ .

<sup>14</sup>Tsoy (2014) puts an additional regularity assumption on  $v$  and  $c$ . The assumption is needed to obtain the characterization of *all* limit equilibrium outcomes of the bargaining game, but can be dispensed with in proving the foundations for the CSBS in which prices split the surplus proportionally.

<sup>15</sup>See Tsoy (2014) for more details.

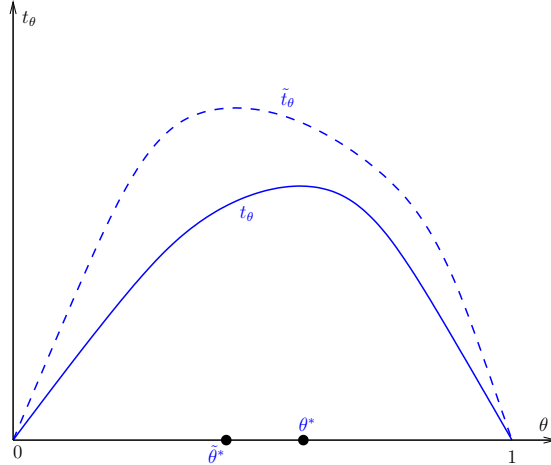


Figure 2: **Trade delay  $t_\theta$  as a function of asset quality  $\theta$ .** Increase in the slope of  $v$  and  $c$  leads to a downward shift in  $t_\theta$  to  $\tilde{t}_\theta$  depicted by solid and dashed lines, respectively.

markets that are known for their opaqueness as well as sophistication of market participants.

In this paper, I do not model the bargaining process explicitly as a game but rather use the CSBS as a reduced form for strategic bargaining. Tsoy (2014) provides the foundations for the use of the CSBS as a limit outcome of the alternating-offer bargaining game as the frequency of offers increases and signals about quality become almost perfect.<sup>16</sup> In the same fashion, the generalized Nash bargaining solution commonly used in the search-and-bargaining literature is motivated as the limit outcome of the complete-information, alternating-offer bargaining game as frequency of offers increases (Binmore, Rubinstein, and Wolinsky (1986)).

I next describe properties of the CSBS that are useful in understanding the liquidity pattern in the model with endogenous delay. The first observation is that the delay is inverse-U-shaped. Figure 2 depicts typical delay times  $t_\theta$ . It follows from Lemma 3 that for asset qualities above  $\theta^*$ , assets of higher quality are traded earlier. This is a standard distortion of the efficient allocation at the bottom known from the screening literature (see for example Rothchild and Stiglitz (1976)). Buyers of higher-quality assets are more impatient and are willing to accept higher prices, while buyers of lower-quality assets wait longer in hopes of more favorable terms of trade. The situation is reversed for asset qualities below  $\theta^*$ . For  $\theta \in [0, \theta^*]$ , the allocation is distorted for higher asset qualities, while it is more efficient for lower qualities. As a result, the delay has an inverse U-shape and is highest closer to the threshold quality  $\theta^*$  and lower closer to the extremes of the quality range.

<sup>16</sup>Empirical evidence suggests that the assumption that agents have only a small amount of private information can be relevant in OTC markets. Downing, Jaffee and Wallace (2009) documents that in primary markets, asymmetric information between the originator of the MBS and the investor is both present and statistically significant, however, the absolute magnitude of its effect on transactions costs and prices is small. A natural conjecture is that a similar pattern is inherited by secondary OTC markets.

Another observation concerns the reaction of  $t_\theta$  to an increase in the variance of values  $v$  and  $c$ . Lemma 3 shows that the absolute value of the slope of  $t_\theta$  is increasing in the slope of  $v$  and  $c$ . Figure 2 depicts with a dashed line the effect of an increase in the slope of  $v$  and  $c$  on trade delay. A higher difference in payoffs gives agents additional incentives to wait for more favorable terms of trade. In the next section, the variance of endogenous values  $v$  and  $c$  will depend on the variance of payoffs  $\bar{v}$  and  $\underline{v}$ . This will provide a link between primitives of the model, like market uncertainty  $k$  and the curvature of  $g$ , and market liquidity  $L$ .

## 4.2 Equilibrium with Endogenous Delay

In this subsection, I apply the CSBS to determine delay endogenously within the search-and-bargaining framework. The equilibrium always exists, and comparative statics can be derived analytically for the linear specification with constant holding costs.

Proceeding as in Subsection 3.2, one can derive the value of trade for the buyer  $v(\theta)$  and the cost of trade for the seller  $c(\theta)$  during the bargaining stage (see equations (8.13) and (8.14) in the Appendix). Together with the efficient discount factor  $\rho$  in the bargaining stage, they define bargaining problem  $(\rho, v, c)$ . If functions  $v$  and  $c$  are strictly increasing and continuously differentiable, then the CSBS to the bargaining problem  $(\rho, v, c)$  determines the endogenous delay (which can be computed from Lemma 3). However, there is one nuance that does not make the application of the CSBS immediate. Because buyers can reject assets or accept them with probability strictly less than one, endogenous value functions  $v$  and  $c$  can have discontinuities or be constant on some intervals. To incorporate these possibilities, I proceed as follows. I first restrict attention to equilibria satisfying the following condition on  $v$  and  $c$  which is maintained throughout the rest of the paper.

**Condition R.** *Functions  $v$  and  $c$  are piecewise continuously differentiable, weakly increasing, and continuously differentiable on  $\Theta_L$  and  $\Theta_M$ .*

For functions satisfying condition R, construct a sequence of strictly increasing and continuously differentiable functions  $v_i$  and  $c_i$  such that functions  $v_i$  and  $c_i$ , as well as their derivatives  $v'_i$  and  $c'_i$ , converge to corresponding limits  $v, c, v', c'$  at all differentiability points of  $v$  and  $c$ . Define the CSBS  $(q_\theta, t_\theta, \theta^*)$  for bargaining problem  $(\rho, v, c)$  as the limit of the CSBSs for bargaining problems  $(\rho, v_i, c_i)$  passing to subsequence if necessary. In general, the limit outcome depends on the choice of sequences  $v_i$  and  $c_i$ . However, under condition R, the limit outcome for assets in  $\Theta_L \cup \Theta_M$  does not depend on the choice of sequences  $v_i$  and  $c_i$  (see Lemma 8 in the Appendix). In other words, for observable trades, the choice of approximating sequence is not consequential. Additionally, to guarantee that  $v$  and  $c$  are, indeed, weakly increasing, I assume that the buyer's share of the surplus is sufficiently large,

$$\alpha \geq \frac{y_d}{r + y_d}. \quad (4.3)$$

Specifying how trade delay is determined in equilibrium allows for more precise predictions about the asset's liquidity compared to those in Lemma 2. The following lemma shows that sets  $\Theta_L$  and  $\Theta_M$  have a particularly simple form.

**Lemma 4.** *In any equilibrium, either  $\Theta_L = [0, 1]$  or there exist  $0 < \check{\theta} < \underline{\theta} \leq \hat{\theta} < 1$  such that  $\Theta_L = [0, \check{\theta}] \cup [\hat{\theta}, 1]$  and  $\Theta_M = (\check{\theta}, \underline{\theta})$ .*

Lemma 4 is intuitive given the discussion of the CSBS in the previous subsection. To see this consider the case of constant holding costs. Asset liquidity  $x_\theta$  is U-shaped (by Lemma 3), and  $x_\theta$  exceeds the cutoff  $\underline{x}$  for asset qualities that are sufficiently high or sufficiently low. This implies the structure of liquid assets as in Lemma 4.

Now I can combine the specification of prices and trade delays given by the CSBS with Lemmas 1 and 2 to show that an equilibrium always exists and that the decomposition of the price of the asset in Theorem 1 holds.

**Theorem 2.** *In the model with endogenous delay, an equilibrium always exists and conclusions 1-3 of Theorem 1 hold.*

The analysis is particularly tractable for a linear model in which the equilibrium is unique and comparative statics are derived analytically.

**Proposition 1.** *For the linear model with constant holding costs, the equilibrium is unique and the following comparative statics obtain. Suppose that a tuple of parameters  $\nu = (k, \ell, a, \lambda, \alpha)$  is such that in equilibrium,  $\Theta_I$  is not empty. Then in some neighborhood of  $\nu$  the following hold:*

- *market liquidity  $L$  is decreasing in market uncertainty  $k$  and contact intensity  $\lambda$ , and increasing in holding cost  $\ell$  and mass of agents  $a$ ;*
- *match intensity for sellers  $\Lambda$  is increasing in market uncertainty  $k$  and mass of agents  $a$ , and decreasing in holding cost  $\ell$ ;*
- *$L$  and  $\Lambda$  are independent of the buyer's share of surplus  $\alpha$ , while  $\hat{\theta}$  and  $\check{\theta}$  are increasing in  $\alpha$ ;*
- *if, in addition, the equilibrium under  $\nu$  is such that  $L = 1$  and  $x_{\theta^*} > e^{-\frac{k}{\xi r}}$ , then  $x_\theta$  is decreasing in market uncertainty  $k$  for all  $\theta$ .*

In the limit  $\lambda \rightarrow \infty$ , the equilibrium is characterized by  $(M_{bu}(\phi), L) = (m, \max\{l, 1\})$  where

$$m = \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\xi r} l} - 1 \right) - l \right)$$

and  $l$  is given by the unique solution of

$$\frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\xi r} l} - 1 \right) - l \right) = \frac{y_u}{y_u + y_d} (a - 1) - \frac{y_d}{y_u + y_d} \max\{l, 1\} - H; \quad (4.4)$$

where  $H$  is a constant depending on the parameters of the model.

In addition to the characterization of prices and spreads in Theorem 1, a particular specification of delay sheds light on the dependence of market liquidity on search and bargaining frictions. The bargaining friction is reflected in the variance of asset payoffs captured by market uncertainty parameter  $k$ . To see this, notice that a greater variance of payoffs  $k$  results in higher slopes of  $v$  and  $c$ , and by Lemma 3, leads to longer a two-sided screening process in the bargaining stage. Therefore, delay caused by bargaining increases.

Proposition 1 shows that higher bargaining friction due to higher market uncertainty  $k$  results in lower market liquidity  $L$ . On the other hand, increasing search friction through the decrease in search efficiency  $\lambda$  leads to higher market liquidity. When it is harder for buyers to find another seller, buyers are willing to accept a wider range of asset qualities for trade. Higher market liquidity, however, does not imply that welfare is increasing. The fact that a wider range of assets is traded leads to a more efficient allocation of these assets which does increase welfare, but at the same time the increase in search times decreases welfare. Therefore, despite the increase in the market liquidity, it is possible that the equilibrium becomes less efficient with the increase in search friction. In the next subsection, I demonstrate this with a numerical example.

The comparative statics with respect to  $k$  and  $\lambda$  shows that only one type of friction cannot serve as a proxy for the other, and in fact, may give rise to misleading predictions. For example, one might conjecture that when bargaining friction increases, agents negotiate longer, and the longer negotiation time simply adds to the search time, increasing search friction. However, this logic does not take into account the fact that this changes the set of asset qualities that are traded.

The increase in bargaining friction increases the match intensity of sellers  $\Lambda$ . This is the effect of competition among sellers for buyers. When bargaining friction is greater, fewer assets are actively traded. Therefore, a larger fraction of unmatched buyers then searches for more scarce trade opportunities, which improves the match intensity for sellers.

Notice that even though the mass of liquid assets and expected search time of the seller do not depend on  $\alpha$ , the composition of traded assets depends on the split of surplus. The greater the share of the buyer, the higher the fraction of high-quality assets (above  $\theta^*$ ) in the set of liquid assets. For high-quality assets, the buyer is screened in the CSBS. A higher fraction of the buyer surplus gives the buyer additional incentives to trade faster, as he bears a larger fraction of costs of trade delay. As a result, the endogenous liquidity of high-quality assets increases. For low-quality assets (below  $\theta^*$ ), the logic is the opposite. The seller bears a smaller fraction of the delay costs, which increases his incentives to wait longer, and hence, decreases the liquidity of such asset qualities.

Observe that even when search friction vanishes ( $\lambda \rightarrow \infty$ ), markets are not perfectly liquid, and prices differ from the Walrasian equilibrium prices. This happens because there is additional bargaining friction in the model, reflected in the trade delay. This contrasts with Duffie, Gârleanu



and Pedersen (2007) where in the limit of vanishing search frictions prices approach competitive levels and equilibrium is efficient.

In the next subsection, I study the exponential convex model via numerical simulations. The next proposition shows that in this model equilibria are ordered by the utility of agents. Hence, in the simulation, I focus on the equilibrium which is either the best or the worst for both sides.<sup>17</sup>

**Proposition 2.** *For the convex exponential model with constant holding costs, all equilibria are ordered by the utility of agents: equilibria corresponding to higher utility of the buyer also correspond to higher utility of the seller conditional on asset liquidity and acceptance probability, i.e. for any two equilibria  $(\sigma_\theta, M)$  and  $(\tilde{\sigma}_\theta, \tilde{M})$ ,  $V_{bu}(\phi) \geq \tilde{V}_{bu}(\phi)$  implies  $V_{sm}(\theta) \geq \tilde{V}_{sm}(\tilde{\theta})$ , for any  $\theta$  and  $\tilde{\theta}$  such that  $x_\theta = \tilde{x}_{\tilde{\theta}}$ ,  $\sigma_\theta = \tilde{\sigma}_{\tilde{\theta}}$ .*

In Proposition 2, I refer to the conditional utility of the seller for the following reason. From Theorem 2 equilibrium is pinned down by two quantities: the lowest expected surplus  $\underline{z}$  accepted by the buyer and the match intensity of the seller  $\Lambda$ . As these quantities vary across equilibria of the model or across equilibria corresponding to different parameters of the model, the set of liquid assets  $\Theta_L$  changes, and in particular, it is possible that in one equilibrium the asset can be traded, while in the other it can be illiquid. For this reason, to compare the utility of the seller in different equilibria, I condition on the liquidity characteristics  $x_\theta$  and  $\sigma_\theta$  of the asset and focus on whether market conditions improved for the seller or not. As Proposition 2 shows, interests of buyers and sellers are aligned in the case of a convex exponential model.

### 4.3 Numerical Simulations

In this section, I further explore properties of the model via numerical simulations. Analytic insights from the analysis of the linear model with constant surplus derived in the previous section are useful in understanding the behavior of other model specifications. The numerical simulations allow one not only to explore the effect of curvature of the payoffs and varying holding costs on liquidity, but also to study how welfare changes with the changes in search and bargaining frictions.

To illustrate the results of simulations, I use the market liquidity  $L$  and the average delay as aggregate indicators of liquidity and the distribution of inefficient asset holdings  $\mu_s$  as the measure of equilibrium efficiency. I also introduce the following welfare measure

$$W = 1 - \frac{\int_0^1 \mu_s(\theta) s_\theta d\theta}{\int_0^1 \frac{y_d}{y_u + y_d} f(\theta) s_\theta d\theta},$$

which shows by how much trade improves efficiency as compared to the absence of trade. Higher

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<sup>17</sup>In fact, in all numerical simulations the equilibrium appears to be unique.

$y_u$	$y_d$	$\lambda$	$r(\%)$	$\alpha$	$a$
70	.1	1500	10	.5	1.5

Table 1: Parameters of the model.

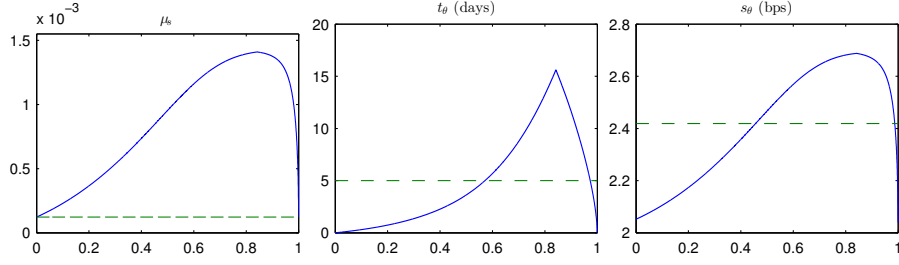
$W$  indicates a larger improvement in the welfare from the introduction of the market and corresponds to more efficient equilibria. The mass of traded assets  $L$  gives only a rough indication of equilibrium efficiency. On the one hand, compared to  $W$ , market liquidity  $L$  overstates the welfare, as it only captures the fact that assets in  $\Theta_L$  are traded eventually, but ignores the fact they are traded with different levels of delay. On the other hand,  $L$  understates welfare, as it ignores assets that are accepted for trade with probability strictly less than one. As I will show below, sometimes efficiency and liquidity move in opposite directions.

Parameters of the model are given in Table 1. In the baseline specification, liquidity shocks are relatively rare, and agents expect to recover from the shock in 3.6 days. Each agent expects to contact nine other agents per day. Despite relatively quick recoveries, because of the high intensity of contact, agents hit by a liquidity shock have strong incentives to search for a counterparty to unload their position (rather than hold the asset and wait for a recovery). In negotiation, the surplus is split equally between the buyer and seller.<sup>18</sup> Agents holding assets constitute 67% of the population. I also assume that each asset quality is in unit supply, i.e.  $f(\theta) = 1$  for all  $\theta \in [0, 1]$ .

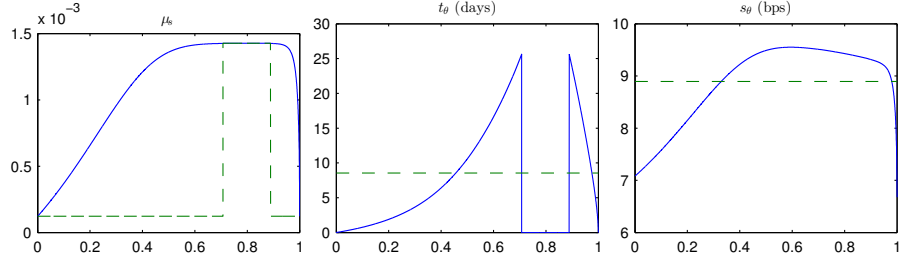
I consider the exponential specification of flow payoffs given by (2.1) – (2.2) with  $g(\theta) = \frac{\exp(\beta\theta)-1}{\exp(\beta)-1}$ . Parameter  $\beta$  controls the curvature of payoff functions and determines in what parts of the quality range the variance of payoffs is the highest. For  $\beta > 0$ , payoffs are convex in quality, and most of the variance in payoffs is concentrated in high-quality assets, while for  $\beta < 0$ , payoffs are concave in quality, and most of the variance in payoffs comes from the low-quality assets. As before,  $\ell$  denotes the constant holding costs,  $k$  controls the variance of asset payoffs and is interpreted as market uncertainty, and  $d$  is interpreted as the aggregate default risk.

I depict results of simulations in Figure 3 and present aggregate measures in Table 2. As a benchmark, consider the model with convex payoffs specified by  $k = .025, \beta = 3, \ell = 4, d = 4$ . Figure 3a depicts the steady-state distribution  $\mu_s$  of assets among inefficient holders, trade delay  $t_\theta$ , and yield spread  $s_\theta$  for this specification. All assets are liquid in equilibrium and buyers do not randomize ( $L = 1$ ). However, each asset is associated with delay, which results from two-sided screening during the bargaining stage. The average delay is 5 days (with mode 3.4 days) and is significant compared to the seller’s search time ( $250/\Lambda \approx 2.4$  hours). Trade delay is highest for assets in the middle of the quality range reaching the maximum of 15.6 days for  $\theta^* = 0.82$ , and

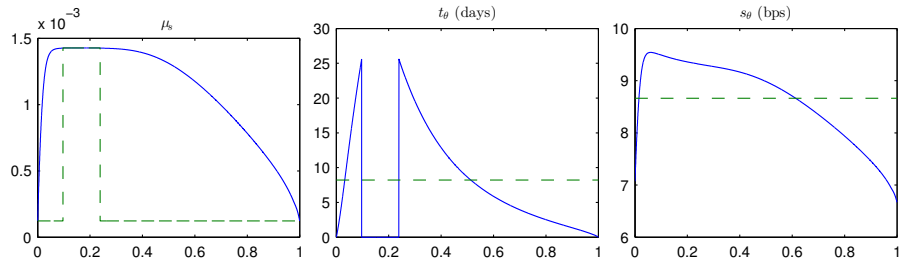
<sup>18</sup>Observe that  $\frac{y_d}{r+y_d} = .5 = \alpha$ , and condition (4.3) holds.



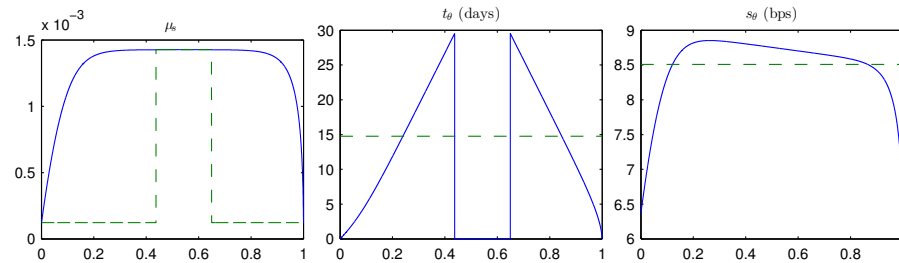
(a) Equilibrium for  $k = .025, \beta = 3, \ell = 4, d = 4$ .



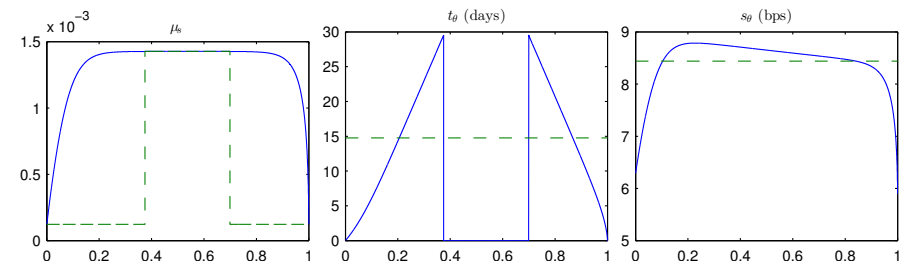
(b) Equilibrium for  $k = .06, \beta = 3, \ell = 4, d = 1$ .



(c) Equilibrium for  $\beta = -5$



(d) Equilibrium for linear model



(e) Equilibrium with increasing surplus

Figure 3: The steady-state distribution  $\mu_s$  of assets among sellers (first column), trade delay  $t_\theta$  (second column), yield spread  $s_\theta$  (third column) for various specifications with parameters as in Table 1. Dashed lines depict  $\mu_{su}$ , average  $t_\theta$ , and average  $s_\theta$ , respectively.

	$W(\%)$	$L(\%)$	average delay (days)	average spread (bps)
convex (low market uncertainty) $k = .025, \ell = 4, \beta = 3$	39.5	100	5	2.4
convex $k = .06, \ell = 4, \beta = 3$	23.4	81.9	8.5	8.9
concave $k = .06, \ell = 4, \beta = -5$	22.1	85.8	8.2	8.7
linear $k = .06, \ell = 4, \beta \rightarrow 0$	8.7	78.8	14.8	8.5
increasing surplus ( $\bar{k} > \underline{k}$ ) $\bar{k} = .06, \underline{k} = .06, \ell = 3.995, \beta \rightarrow 0$	7.5	67.6	14.7	8.4

Table 2: Comparison of welfare  $W$ , range of traded qualities  $L$ , average delay, and average spread for concave, convex and linear models with constant surplus, linear models with increasing surplus. For the first specification  $d = 4$  and  $d = 1$  for the others.

lowest closer to the extremes of the quality distribution.<sup>19</sup> The cutoff  $\theta^*$  is shifted to the right from the middle of the quality range, as payoffs are convex and it takes more time to screen higher-quality assets. Despite a relatively low levels of search friction, inefficiency still remains due to the bargaining friction. Trade improves the efficiency of the equilibrium by  $W = 39.5\%$ . Notice that consistent with Lemma 1,  $\mu_{su}$  is uniform as is the distribution of the supply of assets. The average spread is 2.4 bps with the narrow range between 2 and 2.7 bps.

**Bargaining Friction** I start with an illustration of the effect on liquidity and efficiency of increased bargaining friction. When asset-specific delay arises from strategic bargaining, bargaining friction is determined by market uncertainty reflected in the range  $k$  of asset payoffs. Figure 3b and Table 2 represent results of simulations for higher  $k = .06$ . It is a natural assumption that during a crisis regime, the systemic component in the default risk increases simultaneously with the increase in the market uncertainty. Therefore, I carry out the simulations for lower  $d = 1$ .

An increase in the variance of asset qualities leads to a smaller range of liquid assets: around 18.1% of asset qualities are not traded ( $L = 81.9\%$ ) and the efficiency of the market decreases by 40% ( $W = 23.4\%$ ). Since the difference in payoffs is larger, agents spend more time negotiating, and as a result, fewer assets are liquid enough to be an attractive means of risk-sharing. Because of that, the average negotiation time increases to 8.5 days (and is as high as 25.6 days for the least liquid assets in  $\Theta_L$ ), while the search time for sellers still remains small (2.6 hours).

If  $d$  remained high at  $d = 4$ , then the average yield spread would decrease to 2.2 with the

<sup>19</sup>Trade delay is 2.4 hours for the 10% most liquid assets.

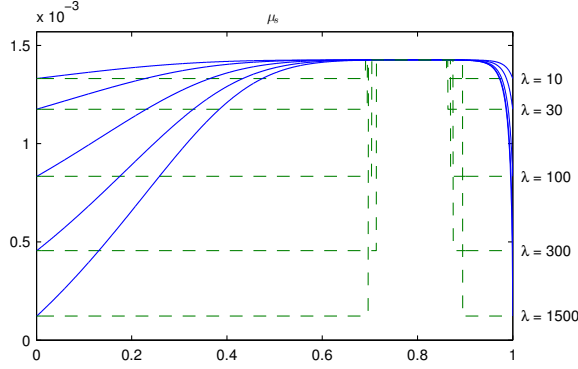


Figure 4: Distribution  $\mu_s$  for  $\lambda = 10, 30, 100, 300, 1500$ .

$\lambda$	10	30	100	300	1500
$L(\%)$	87.2	87.3	87.6	83.9	81.9
$W(\%)$	1.4	3.8	9.5	16.6	23.4
average $t_\theta$ (days)	10.9	10.7	10.4	9.3	8.5
seller search time (days)	50.1	16.7	5	1.7	.3

Table 3: Equilibrium characteristics for various  $\lambda$ .

increase in  $k$ . This happens because the outside options of buyers worsen (the third term in (3.9)). With longer negotiation times, the alternative of continuing the search deteriorates and so the price of assets increases, decreasing in turn the spreads. When  $d$  decreases together with the increase in  $k$ , the average spread more than triples and reaches 8.9bps and there is a higher variance in spreads ranging from 6.7 bps to 9.6 bps. Figure 3b shows that there is a negative correlation between liquidity and yield spreads, a pattern which is confirmed empirically.

**Search Friction** I next demonstrate that increases in search and bargaining frictions have opposite effects on liquidity. Figure 4 illustrates changes in the inefficient asset holdings  $\mu_s$  as the search friction vanishes ( $\lambda$  increases from 10 to 1500) for  $k = 0.06$ . First, notice that equilibria do not converge to the competitive equilibrium, and bargaining friction is sufficient to generate imperfectly liquid markets. For low  $\lambda$  all assets are traded in equilibrium. As  $\lambda$  increases, the range  $L$  of traded assets decreases, which is consistent with the intuition from Proposition 1. At the same time, the allocation for the liquid assets becomes more efficient ( $\mu_s(\theta)$  decreases for  $\theta \in \Theta_L$ ).

Table 3 provides characteristics of equilibria as  $\lambda$  increases. A decrease in search friction increases welfare but decreases the range of traded assets. As search friction vanishes, costs of continuing the search for buyers decrease, and hence, buyers accept only the most liquid assets

for trade. The improvement in efficiency stems from the decreased seller search time (from more than 5 months to less than a day). This suggests that  $L$  can be misleading for estimating efficiency, as in this example it moves in the opposite direction from welfare. Moreover, Figure 4 demonstrates that even though welfare  $W$  increases with an increase in  $\lambda$ , it does not imply that the allocation of all assets becomes more efficient. An increase in  $\lambda$  leads to a decrease in  $L$  and so, for a wider range of assets the allocation becomes more inefficient. However, this inefficiency is compensated by an increase in efficiency of allocation of the remaining assets which results in an increase in  $W$ .

**Curvature and Increasing Holding Costs** I now turn to the analysis of the curvature of payoffs. Table 2 provides a comparison of equilibrium characteristics for different payoff specifications. Instead of convex payoffs, I consider the case of concave payoffs given by  $\beta = -5$ . The results of this simulation are illustrated in Figure 3c. Observe that this equilibrium is close in the efficiency ( $W = 22.1\%$ ), range of traded assets ( $L = 85.8\%$ ), average size of spreads (8.7 bps), as well as average delay (8.2 days) to the model with concave payoffs ( $\beta = 3$ ). However, now the illiquid assets are assets of lower quality. The composition of the liquid assets changes substantially. While for the model with convex payoffs, assets in  $[\underline{.64}, \underline{.92}]$  constitute the 10%-quantile of the delay distribution, and for concave payoffs, it is assets in  $[\underline{.07}, \underline{.24}]$ .

These results are in line with the intuition from the comparative statics with respect to  $k$ . The analyses of the linear and exponential models for different  $k$  suggest that the sharper the slope of  $\bar{v}$  and  $\underline{v}$ , the longer it takes to negotiate the price of assets and the fewer asset qualities are traded. When the payoff function becomes convex instead of concave, most of the payoff variance is concentrated at higher asset qualities. As a result, it takes agents longer to negotiate the price of these assets, which leads to a smaller fraction of high quality assets among traded assets. The situation is reversed for concave payoffs.

Finally, I analyze the effect of increasing holding costs in a linear model  $\bar{v}(\theta) = \bar{k}\theta + d + \ell$  and  $\underline{v}(\theta) = \underline{k}\theta + d$  with  $\bar{k} \geq \underline{k}$ .<sup>20</sup> I take as a benchmark the case with constant surplus depicted in Figure 3d. This case is also a useful illustration of the effect of the curvature as it is the limit of the exponential model as  $\beta \rightarrow 0$ . It follows from Table 2 that compared to models with non-zero curvature, the range of traded assets decreases, mean delay increases and welfare decreases. In the convex model, agents spend large amount of time screening high-quality assets, while trading relatively quickly low-quality assets, and the situation is reversed for the concave model. In the linear model, both high- and low-quality assets are screened for a significant amount of time, which results in a relatively more inefficient outcome.

I compare the linear model with constant surplus with the case of increasing holding costs. As shown above, in this case asset prices and liquidity depend on expected surplus from trade  $z_\theta$  which trades off the time of negotiation and the size of the holding costs. Consider a linear

<sup>20</sup>This functional form is obtained from equations (2.1) – (2.2) by setting  $k = \underline{k}$  and  $\ell_\theta = (\bar{k} - \underline{k})g(\theta)$ .

model with  $\bar{k} = .06, \underline{k} = .05, \ell = 3.995$ , and  $d = 1$ . In this specification, holding costs increase with quality and  $\ell$  is adjusted so that the total surplus from trading all assets remains the same as in the linear model. These results are depicted in Figure 3e. The set of liquid assets decreases in size, average delay increases, and welfare decreases. This effect is similar to the effect of increased uncertainty. Making assets more heterogeneous increases the willingness of parties to wait in negotiation. This reduces the range of traded assets and decreases the scope of risk-sharing. Notice that in Figure 3e, the delay associated with asset  $\hat{\theta}$  is slightly higher than the delay associated with  $\check{\theta}$ . This reflects the fact that buyers decide whether to negotiate the deal or continue searching based on  $z_\theta$ , which incorporates both delay and the size of surplus.

## 5 Transparency and Flights-to-Quality

Often, the increase in market uncertainty or transparency of a particular class of assets can result in an inflow in or outflow from trading this asset class. To study this migration of agents, in this section I consider a simple, multi-class extension of the baseline model in Section 2. Although stylized, the model allows one to illustrate flight-to-quality episodes and study the effect of the increased transparency on liquidity and welfare. As in the previous section, I derive results analytically for the linear specification and give the numerical illustration for the exponential specification.

There are two asset classes indexed by  $i = 1, 2$ , each of mass 1 and a mass  $a > 2$  of agents. For each asset class  $i$ , flow payoffs of the buyer and seller are given by functions  $\bar{v}_i$  and  $\underline{v}_i$ , respectively. The mass  $a_i \geq 1$  of agents trading assets in each class  $i$  is determined in equilibrium so that  $a_1 + a_2 = a$ . Other than that, parameters of the search-and-bargaining model are as in the baseline model in Section 2. The delay is endogenous and is described by the CSBS. I allow that for one of the classes it holds that for some  $\underline{v} < \bar{v}$ ,  $\bar{v}_i(\theta) = \bar{v}$  and  $\underline{v}_i(\theta) = \underline{v}$  for all  $\theta$ .<sup>21</sup> In this case, there is no market uncertainty for asset class  $i$  and all asset qualities in the class are traded immediately. The equilibrium in the multi-class model is defined next. Super-indices indicate equilibrium quantities for the corresponding asset class.

**Definition 3.** *A tuple  $(\sigma_\theta^i, M^i, a_i)_{i=1,2}$  is a multi-class equilibrium if  $(\sigma_\theta^i, M^i)$  is the equilibrium of the baseline model with mass of agents  $a_i$  and the following conditions hold*

$$\begin{cases} \underline{z}^1 = \underline{z}^2, & \text{if } a - 1 > a_1 > 1, \\ \underline{z}^1 \leq \underline{z}^2, & \text{if } a_1 = 1, \\ \underline{z}^1 \geq \underline{z}^2, & \text{if } a_1 = a - 1. \end{cases} \quad (5.1)$$

The interpretation of (5.1) is that sellers cannot switch between asset classes, while buyers can choose what asset class they want to use for the risk-sharing. To see this, recall that buyers'

<sup>21</sup>This case can be analyzed as a limit case of the model with endogenous delay as  $k \rightarrow 0$ .

$\gamma(\%)$	0	48	7.5
$L(\%)$	81.9	90	75.6
$W(\%)$	23.4	30.8	20
average delay (days)	8.5	6.5	7.2

Table 4: Comparison of liquidity and welfare measures for single-class model ( $\gamma = 0$ ), optimal split into classes ( $\gamma = 48\%$ ), and adverse split into classes ( $\gamma = 7.5\%$ ).

preferences over assets within each class are determined by thresholds  $\underline{z}^1$  and  $\underline{z}^2$  (cf. Lemma 2). If both are equal, then buyers are indifferent between the two classes. If one is greater, then all agents migrate to the more preferable (for buyers) class making the other class illiquid. The next theorem shows that equilibrium exists in this model and is unique.

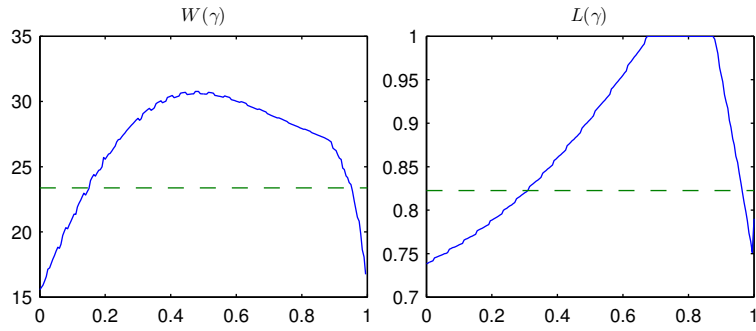
**Theorem 3.** *Suppose that parameters of the model are such that in every asset class, there exists a unique equilibrium of the baseline model for every  $a$ . Then there exists a unique multi-class equilibrium.*

**Transparency** I next study the effect of transparency on market liquidity. There are two ways an increase in transparency can affect variables in the model. First, greater transparency can result in a decrease in search friction  $\lambda$ . The effect of lower  $\lambda$  was studied in the previous section, where it was shown that it reduces market liquidity, but improves welfare. Second, increased transparency through the distribution of past quotes and more accurate or finer credit ratings can lead to assets being traded within more narrowly-defined classes. This results in the switch from the single-class model to a multi-class model, the effect of which I analyze next.

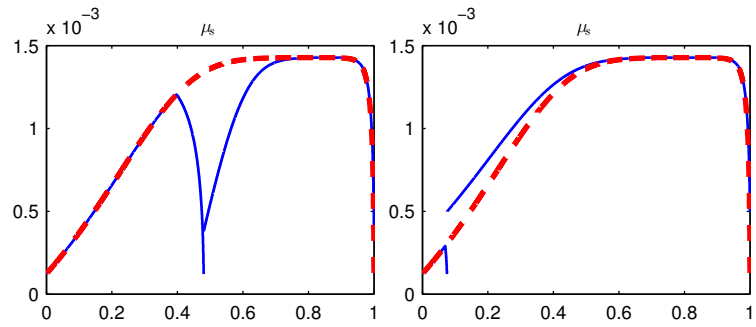
More precisely, suppose that for some  $\gamma \in (0, 1)$ , asset qualities below  $\gamma$  are traded in class 1, and asset qualities above  $\gamma$  are traded in class 2. To develop some intuition let me first abstract from the migration of agents, and a fraction  $\gamma$  of agents trades assets in class 1, and a fraction  $1 - \gamma$  trades assets in class 2. This assumption implies that the ratio of agents to the asset supply is still equal to  $a$ . Then trading within each asset class is described by a model identical to the benchmark model describing trading before the division into classes in all respects but the range of asset payoffs. Specifically, for class 1, the range of payoffs equals  $\gamma k$ , and for class 2, it is  $(1 - \gamma)k$ . As was shown in the previous subsection, in each class  $i$ , the mass of liquid assets  $L_i$  increases and, as a result, market liquidity  $\gamma L_1 + (1 - \gamma)L_2$  after increased transparency exceeds market liquidity  $L$  before the division. I next show that if one allows for migration of agents between asset classes, then the division of one asset class into several asset classes need not always improve liquidity and efficiency.

To explore the effect of the increased number of asset classes consider the exponential model with parameters as in the convex model in Table 2. Suppose that the market is divided into





(a) Welfare measure  $W$  and mass of liquid assets  $L$  as a function of  $\gamma$ . Dashed line represents levels for a single-class model.



(b) Steady-state distribution  $\mu_s$  for  $\gamma = 48\%$  (left panel) and  $\gamma = 7.5\%$  (right panel). Dashed lines plot  $\mu_s$  for  $\gamma = 0$ .

Figure 5: **Effect of increased transparency on the liquidity and welfare.**

two asset classes:  $\theta < \gamma$  and  $\theta \geq \gamma$ . In Figure 5a, I compare changes in welfare and liquidity in the multi-class model as the border  $\gamma$  dividing asset classes varies. Changes in liquidity and welfare do not always go together. In particular, maximum welfare is reached when  $\gamma = 48\%$ , while maximal liquidity is reached for higher  $\gamma$ .

Observe that it is possible that for some split of the market into classes, welfare decreases. For example, for  $\gamma = 7.5\%$  both welfare and market liquidity decrease compared to the single-class model (see Table 4). This happens because many agents migrate from trading assets  $\theta \geq \gamma$  into trading assets  $\theta < \gamma$ , leading to a decrease in efficiency and liquidity in the former segment.

On the other hand, an optimal split of assets into classes ( $\gamma = 48\%$ ) results in greater liquidity and welfare (market liquidity increases by 31.6% and welfare increases by 9.9%). The improvement in the allocation is obtained for every asset quality as can be seen from the right panel of Figure 5b. For comparison, in the left panel of Figure 5b, I depict the distribution of assets  $\mu_s(\theta)$  for suboptimal  $\gamma = 7.5\%$ . Allocation of assets in class 1 is more efficient compared to the no-division case (the solid line is below the dashed line for  $\theta < 7.5\%$ ). This happens both because of the reduced variance of payoffs in the class and the increased mass of agents trading assets in class 1. However, there is a wide range of asset qualities in class 2 for which allocation is less efficient in the multi-class model. The two classes combined result in a less efficient allocation compared to no-division case.

**Flight-to-Liquidity** I next show that a flight-to-quality occurs as a response to the increase in market uncertainty in one of the asset classes. Specifically, in the multi-class model, suppose that  $\bar{v}_1(\theta) = kg(\theta)$  and  $\underline{v}_1(\theta) = kg(\theta) - \ell$ , while  $\bar{v}_2 = 1$  and  $\underline{v}_2 = 1 - \ell$ . After the uncertainty shock,  $k$  increases to  $\tilde{k} > k$ . That is, after the shock there is greater variation in the asset payoffs within assets in class 1. The next proposition describes the effect of such a shock on the distribution of agents between classes and the liquidity in the linear model. I use tildes to refer to equilibrium quantities after the shock.

**Proposition 3.** *Suppose the payoffs in the first asset class are linear ( $g(\theta) = \theta$ ). Suppose that  $k$  increases to  $\tilde{k}$  and let multi-class equilibria corresponding to  $k$  and  $\tilde{k}$  be such that masses of agents in the first class  $a_1$  and  $\tilde{a}_1$  lie strictly between 1 and  $a - 1$ . Then an increase in  $k$  to  $\tilde{k}$  leads to a decrease in the range of traded assets in the first class ( $\tilde{L}_1 \leq L_1$ ), a flight-to-quality ( $\tilde{a}_1 \leq a_1$  and  $\tilde{a}_2 \geq a_2$ ), and a reduction in the buyer utility  $V_{bu}(\phi)$ .*

Proposition 3 shows that after the shock, a flight-to-quality occurs: fewer agents trade assets from class 1 with greater variance of asset payoffs, and agents migrate to trading assets in class 2. The flight-to-quality exacerbates the drop in liquidity. By Proposition 1, both an increase in  $k$  and a decrease in  $a$  lead to a decrease in  $L$ . As a result, as fewer agents are trading assets in class 1, the negative effect on liquidity of the uncertainty shock is amplified.

To illustrate the flight-to-quality numerically, consider a multi-class model in which asset payoffs in asset class 1 are given by the exponential specification with  $k = .025, \ell = 4, d = 1$ ,

	a	$W(\%)$	$L(\%)$	average delay (days)	average spread (bps)
low market uncertainty ( $k = .025, \ell = 4, d = 4$ )	1.57	41	100	5	2.2
high market uncertainty ( $k = .06, \ell = 4, d = 1$ )	1.26	24	80.1	8.2	8.4

Table 5: Equilibrium quantities in a multi-class model before and after the shock to market uncertainty.

and asset class 2 has constant payoffs across assets and the same  $\ell$  and  $d$ . Parameters of the model are as in Table 1 except for now the total mass of agents is  $a = 3.67$ . The total mass  $a$  is chosen so that in the multi-class equilibrium  $a_1 \approx 1.57$  is close to the equilibrium in the baseline specification, and I can use Figure 3a as an illustration of equilibrium in class 1.

Suppose that there is an uncertainty shock and  $k$  increases to .06, and  $d$  decreases to 1 for assets in class 1. Equilibrium quantities before and after the shock are presented in Table 5. After the shock, the mass of agents trading assets in class 1 decreases by 19.7%. These agents migrate into trading assets in class 2 that did not experience the shock and the flight-to-quality takes place. The flight-to-quality is associated with a drop in liquidity in class 1: 19.9% of assets become illiquid. Average bargaining delay increases dramatically from 5 days to 8.2 days. This leads to a loss in welfare ( $W$  decreases from 41% to 24%). Because of the increase in aggregate default risk  $d$ , average spreads increase to 8.4 bps.

I can use the results of the numerical simulation in Table 2 (first line) to compare the effect of flight-to-quality on welfare and liquidity. The flight-to-quality exacerbates the negative liquidity consequences of the uncertainty shock. Namely, the set of liquid assets decreases by an additional 1.8%. It is interesting that this does not change welfare significantly.

Notice that by Theorem 2 the level of aggregate default-risk  $d$  does not affect the distribution of agents across asset classes. In particular, if asset class 1 experienced an increase in market uncertainty ( $k$ ) but at the same time a decrease in the aggregate default-risk ( $d$ ), then the direction and the magnitude of the migration to trading assets in class 2 would not change. This is consistent with the empirical evidence that default risk plays a smaller role than liquidity in flights (see Beber, Brandt, and Kavajecz (2009)).

## 6 Empirical Implications.

This paper develops a theory of asset pricing and liquidity that takes into account trade delay due to both search and bargaining frictions. The search friction reflects the time it takes to identify a gainful trade, while the bargaining friction reflects trade delay after the trade opportunity has been found. Incorporating bargaining friction into a standard search-and-bargaining model

leads to a number of testable empirical predictions. Next, I discuss empirical implications of the model and compare them to existing empirical evidence.

Existing empirical evidence on the behavior corporate spreads suggests that the decomposition in the pricing equation (3.9) captures key components of asset prices in OTC markets. Longstaff, Mithal, and Neis (2005) shows that the default component does not explain entirely corporate spreads. The non-default component varies with liquidity measures in the cross-section of assets and depends on the marketwide liquidity in the time series analysis. While Longstaff, Mithal, and Neis (2005) does not provide a direct test of my theory, my model can be useful in explaining these effects on corporate spreads. In my model the last two components in equation (3.9) correspond to the non-default component. While the liquidity premium component ensures the variation of the non-default component across assets, the average-liquidity component ensures the variation of the non-default component with respect to marketwide liquidity. The analysis of flights-to-quality in Section 5 reveals that in the corporate bond market, the latter component would decrease asset prices of all bonds with the improvement in the liquidity of the Treasury market, a regularity confirmed empirically in Longstaff, Mithal, and Neis (2005). Moreover, Longstaff, Mithal, and Neis (2005) shows that the variance of the non-default component across asset classes with different credit rating is much smaller than the variance of the default component. This is in line with the analysis of the model with endogenous delay. By Theorem 2, the aggregate default risk parameter  $d$  (which varies across credit rating classes) affects the level of prices and hence the default component. However, it does not affect liquidity  $x_\theta$  of assets and it is possible that different asset classes have similar average liquidity.<sup>22</sup>

Next, let me mention that the bargaining dynamics that I obtain possess realistic features of the process of price discovery in OTC markets. The common screening bargaining solution used in this paper to model the bargaining process exhibits two-sided screening dynamics: each side makes offers to screen the opponent. These dynamics, in which parties gradually trade through alternating offers, is a realistic description of actual negotiations in OTC markets.<sup>23</sup>

An important implication of this trade dynamic is that the range of asset payoffs plays an

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<sup>22</sup>Longstaff, Mithal, and Neis (2005) also show that an increase in the supply of debt in the market leads to an increase in the non-default component of corporate spreads. In my model, this translates into a decrease in  $a$ , and it can be shown that in the linear model it results in a decrease in  $\underline{x}$  and decrease in asset prices which is in line with the empirical evidence.

<sup>23</sup>For example, Lewis (2011) (pp. 212-213) describes the negotiation between Morgan Stanley and Deutsche Bank over the price of subprime CDOs:

What do you mean seventy? Our model says they are worth ninety-five, said one of the Morgan Stanley people on the phone call.

Our model says they are worth seventy, replied one of the Deutsche Bank people.

Well, our model says they are worth ninety-five, repeated the Morgan Stanley person, and then went on about how the correlation among the thousands of triple-B-rated bonds in his CDOs was very low, ... he didn't want to take a loss, and insisted that his triple-A CDOs were still worth 95 cents on the dollar.

In this example, both parties begin the negotiation from extreme offers as in the model in this paper until eventually the trade takes place at some compromising price.

important role in asset liquidity, as it determines how quickly two-sided screening ends and parties trade. My interpretation of the range of asset payoffs is that public information about asset quality, like credit ratings, splits assets into several classes, and assets are traded within each class.<sup>24</sup> If public information is crude, then each asset class will contain a variety of asset qualities with a wide range of associated default risks and payoffs. This assumption is relevant in many OTC markets. Many OTC markets, like markets for credit default swaps, credit derivatives, corporate bonds, and asset-backed securities, are opaque and only a limited amount of information about assets is public. In these markets, assets are traded within crudely-defined classes, and traders use public information such as credit ratings as a starting point for negotiation. For example, the Committee on the Global Financial System (2005) gives the following account of the OTC trade:

Interviews with large institutional investors in structured finance instruments suggest that they do not rely on ratings as the sole source of information for their investment decisions ... Indeed, the relatively coarse filter a summary rating provides is seen, by some, as an opportunity to trade finer distinctions of risk within a given rating band. Nevertheless, rating agency ‘approval’ still appears to determine the marketability of a given structure to a wider market.

Hence, credit ratings place only crude restrictions on the price of trade, and the actual price is determined during negotiations between the buyer and the seller. In this paper, the range of assets traded restricts the highest and the lowest prices of assets ( $q_1$  and  $q_0$ , respectively), and agents use delay to arrive at the actual trade price  $q_\theta$  belonging to this range. I show that in asset classes with smaller variance in asset payoffs within the class (smaller  $k$ ), more asset qualities are liquid (higher  $L$ ), and the average trade delay caused by bargaining is smaller (see conclusions 1 and 4 in Proposition 1). This prediction can be tested by splitting assets into classes by the public information available about assets, like credit ratings, maturity and industry, and by exploring the correlation between the variance of the default risk within each class and the liquidity of assets within each class. The model predicts that higher variance results into lower liquidity of the class.

The recent financial crisis of 2007-2008 suggests that there is a negative relationship between the range of asset payoffs and market liquidity, confirming the finding of the paper. Benmelech and Dlugosz (2010) reports that for structured finance products, the amount of downgrades increased from 986 in 2006 to 8109 in 2007 and increased further to 36880 in 2008, while the amount of upgrades either decreased or increased slightly. At the same time, the average downgrade size increased from 2.5 notches to 4.7 in 2007, and to 5.6 in 2008. Ashcraft, Goldsmith-Pinkham, and Vickery (2010) documents a similar spike in downgrades for subprime and Alt-A mortgage-backed securities. The increase in the number of rating downgrades and the fact that this

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<sup>24</sup>See the discussion of the CSBS in the introduction.

increase continued for an extended period of time indicates that in 2007 traders realized that the range of expected payoffs from assets with a particular credit rating would be significantly higher than it was before ( $k$  increased). This paper predicts that in this environment, there will be a drop in market liquidity (see Proposition 1 and Figures 3a and 3b for illustration). Moreover, the liquidity drop will be accompanied by a spike in spreads, if simultaneously the risk of default increases for all assets (increase in  $d$ ). The fact that ratings were mostly downgraded suggests that the default risk indeed increased for all assets during this period. Downgrades of structured products coincided with dried-up liquidity of structured finance products (see Brunnermeier (2009)) which is consistent with predictions of this paper.

This paper provides testable implications about the relationship between liquidity and default risk. The model predicts that asset liquidity depends on the default risk, and this dependence is U-shaped. Assets with the highest and lowest default risk within a particular asset class (defined, for example, by credit rating) are the most liquid, as one of the sides quickly accepts the highest and the lowest prices, respectively. On the other hand, traders negotiating the price of assets with default risk in the middle of the range have incentives to delay trade to get a more favorable price. This prediction differs from the implication of Guerrieri and Shimer (2014). They study the model with asymmetric information and discover an increasing relationship between the liquidity and default risk. This stems from the fact that in order to incentivize owners of assets to reveal their private information, assets of higher quality should be traded at higher prices but with lower probability compared to the lower-quality assets.

The empirical literature so far has not explored the relationship between default risk and liquidity within asset classes with different credit ratings or other publicly-observable characteristics. The reason for this is that in empirical studies, credit ratings themselves serve as proxies for default risk. However, one can use other proxies for default risk, like yield spreads of credit default swaps, to measure a finer distinction of the default risk within the asset class with the same credit rating to test the implications of this paper.

The existing empirical literature agrees that liquidity is the most important factor after default risk for asset prices in OTC markets. However, there is contradicting empirical evidence about the sign of the correlation of default risk and liquidity. Longstaff, Mithal, and Neis (2005), and Ericsson and Renault (2006) document a positive correlation between illiquidity and default risk for corporate bonds. At the same time, Beber, Brandt, and Kavajecz (2009) shows that for Euro-bonds the correlation is reversed: more risky sovereign debt is also more liquid. The model in this paper reconciles this evidence within a single framework. While in general the dependence is U-shaped, the shape can be skewed to either side depending on the specification of the payoff function. For convex payoffs (Figure 3b), for the majority of traded assets, higher quality (lower default risk) is associated with lower liquidity. For concave payoffs (Figure 3c) the situation is reversed: for the majority of assets the correlation between liquidity and default risk is positive.

Another testable prediction of the model is that the direction of flights-to-quality is determined by the liquidity preferences of traders. Proposition 3 shows that an increase in market uncertainty results in the migration of agents to trade in asset classes not affected by this increase. The direction of the migration is orthogonal to the change in the aggregate default risk ( $d$  does not affect market liquidity).

OTC markets are known to be prone to flights-to-quality episodes when, due to increased market uncertainty, agents shift their portfolio preferences to safer and more liquid assets. These phenomena are associated with dried-up liquidity in markets for more risky assets. Friewald, Jankowitsch, and Subrahmanyam (2012) and Dick-Nielsen, Feldhutter, and Lando (2012) show empirically that flight-to-quality episodes were observed during the recent liquidity crisis of 2007-2008. The implication of my model is confirmed empirically by Beber, Brandt and Kavajecz (2009). Using the unique negative correlation between liquidity and default risk for sovereign debt, they show that flights are driven by preferences for liquidity rather than quality.

Finally, the model adds to the debate about the effect of transparency on the liquidity of OTC markets. There is a tendency toward increasing transparency of OTC markets. In July 2002, the Transaction Reporting and Compliance Engine (TRACE) was introduced in the U.S. corporate bond market. Currently, the information on nearly all transactions is publicly available. Recent financial crises increased the pressure for greater transparency of markets for credit derivatives and credit-default swaps. The model reveals how the increased transparency can be beneficial and whether or not it will necessarily lead to more liquid markets.

There are two ways to measure the effect of transparency in my model. On the one hand, transparency can be interpreted as the amount of public information available in the market. When more public information is disseminated, assets are traded within more narrowly-defined classes. Numerical simulations in Figure 5b show that under some choice of the division of the market into several classes by the default risk, both liquidity and welfare improve. Bessembinder, Maxwell, and Venkataraman (2006), and Edwards, Harris, and Piwowar (2007) provide empirical evidence that the introduction of the electronic reporting system TRACE in to corporate bond markets improved liquidity and led to a decrease in transaction costs. However, the division of the market into classes can be detrimental for liquidity and welfare. If one asset class has much smaller variance of payoffs within the class, then a flight-to-quality emerges between classes. Agents migrate into trading assets in the class with less bargaining friction, which hurts the liquidity of the remaining classes.

On the other hand, one can interpret transparency as how difficult it is to find a counter-party in the market. This is reflected by parameter  $\lambda$ . Proposition 1 implies that increasing  $\lambda$  decreases the set of liquid assets. Therefore, facilitating search through providing better information to agents does not lead to improved market liquidity. This does not mean that transparency is bad for welfare. On the contrary, Figure 4 and Table 3 demonstrate that welfare improves with the increase of  $\lambda$ , despite the decrease in the set of liquid assets. This suggests that liquidity might

not always be the best indicator of efficiency and one cannot evaluate the effect of reforms, such as the introduction of public quotes, solely based on the effect on market liquidity.

## 7 Conclusion

This paper captures both bargaining and search frictions in OTC markets by introducing asset-specific trade delay in a standard search-and-bargaining model. I study these frictions in both the model with exogenous delay and the model with endogenous delay arising from strategic bargaining. In both models, the dependence of asset prices on the default risk, liquidity, and market conditions is determined in equilibrium. The asset pricing equation provides a decomposition of prices into three components: default-risk component, liquidity-premium component, and average-liquidity component. Existing empirical literature suggests that these components are important determinants of yield spreads. In equilibrium, the set of traded assets is determined by buyers' optimal strategy: buyers accept for trade only assets with sufficiently large expected surplus from trade.

In the model with endogenous delay, trade delay is determined by the common screening bargaining solution which features realistic two-sided screening dynamics of price discovery. Specifying the mechanism through which trade delay is determined allows for more detailed predictions about asset liquidity. The model has several implications for liquidity of assets within the asset class with the same credit rating that can be tested empirically. First, an increase in the variance of payoffs within the asset class leads to an increase in bargaining friction and decreases the range of traded assets. Second, the relationship between liquidity and default risk is U-shaped within the asset class, which contrasts with the prediction of a monotone relationship in adverse selection models.

The analysis of the model with endogenous delay also reveals that both frictions are important in determining the liquidity of assets. Market liquidity increases with an increase in the search friction, while it decreases with an increase in bargaining friction. The model with endogenous delay provides a channel through which market uncertainty affects market liquidity. I use this channel to explain the effect of transparency on market liquidity and the emergence of flights-to-quality during periods of increased market uncertainty. Flights-to-quality cause trade to migrate from asset classes with increased uncertainty about payoffs to asset classes where the uncertainty about payoffs has not changed. Hence, flights-to-quality exacerbate the negative effect on liquidity of an increase in uncertainty about payoffs within an asset class. Similarly, greater transparency through the division of assets into more narrowly-defined classes can be detrimental to liquidity. If, as a result of the division, an asset class emerges that has very little uncertainty about payoffs within the class, then trade will be concentrated in this asset class leaving the rest of the market relatively illiquid.

I next point out several directions for future research. The empirical literature on asset



liquidity so far has used credit ratings as proxies for default risk, but has not looked at the relationship between liquidity and default risk within credit rating classes. This paper provides a framework for analysis of liquidity within asset classes and gives several novel empirical predictions. Testing these predictions is an exciting topic for future research. Moreover, equation (3.9) gives the decomposition of asset prices into default and non-default components and can be structurally estimated to determine the importance of various components in determining spreads.

Further, one can follow the analysis in Sections 3 and 4 to explore the implications of other endogenous bargaining frictions for liquidity. For example, instead of the CSBS one can use the limit of the bargaining model with interdependent values obtained in Fuchs and Skrzypacz (2014) to determine price and delay. In fact, heuristically the equilibrium of such a model can be obtained from the model in this section by setting  $\alpha = 0$  and specifying in Definition 2 that  $\theta^* = 1$ .<sup>25</sup> Exploring other significant sources of bargaining friction in OTC markets is an important direction for future research.

Finally, there are several further developments of the model in this paper that seem promising. First, it is interesting to introduce market makers in the model which would establish the connection between endogenous bid-ask spreads (a measure commonly used in the empirical research to measure liquidity) and liquidity. Second, the CSBS used in this paper is the reduced form for the sequential bargaining model with private almost-perfectly correlated values. It is exciting and challenging to study the model in which values are imperfectly correlated. Third, the analysis of the model relies on the assumption that the economy is in steady-state. The study of transitional dynamics is another potential direction for future research.

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<sup>25</sup>One also needs to put additional restrictions on payoff functions. In particular, Fuchs and Skrzypacz (2014) assume that  $v(1) = c(1)$  (no-gap assumption).

## 8 Appendix

### 8.1 Steady-State Distribution

*Proof of Lemma 1.* In the proof, I find explicitly the steady-state distribution  $M$  for given  $t_\theta$  and  $\sigma_\theta$ . Before deriving the conditions on  $M$ , let me first derive the steady-state distribution of times spent in the match which will allow me to compute at what rate matched agents complete the trade.

For  $\theta \in \Theta_L \cup \Theta_M$ , let  $G(\theta, u)$  for  $u \in [0, t_\theta]$  be the steady-state cumulative mass distribution of times that the buyer and the seller of asset  $\theta$  have already spent in the match by the current time. Observe that after time  $du$  the mass of agents that have spent in the match time less than  $u$  is  $G(\theta, u - du)$ . In addition, a fraction  $(y_u + y_d)du$  of matches are destroyed during the time  $du$  due to switching of intrinsic types by one of sides, and a mass  $\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma_\theta du$  of agents enter the bargaining stage for asset quality  $\theta$ . By the time-invariance of  $G(\theta, u)$ ,

$$G(\theta, u) = (1 - y_u du - y_d du)G(\theta, u - du) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma_\theta du \quad (8.1)$$

or

$$G'(\theta, u) = -(y_u + y_d)G(\theta, u) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma_\theta. \quad (8.2)$$

The distribution  $G(\theta, u)$  is given by the differential equation (8.2) with the initial condition  $G(\theta, 0) = 0$ :

$$G(\theta, u) = \frac{1 - e^{-(y_u + y_d)u}}{y_u + y_d} \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma_\theta.$$

Moreover, total mass of agents in the bargaining stage for asset  $\theta$  is equal to  $\mu_{bm}(\theta)$  which puts the restriction on the distribution  $M$ ,  $G(\theta, t_\theta) = \mu_{bm}(\theta)$ , or

$$\mu_{bm}(\theta) = \frac{1 - e^{-(y_u + y_d)t_\theta}}{y_u + y_d} \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma_\theta. \quad (8.3)$$

Let  $\gamma_\theta$  be the intensity with which agents leave the match. Then

$$\gamma_\theta = G'(\theta, t_\theta) = \lambda M_{bu}(\phi)\mu_{su}(\theta)e^{-(y_u + y_d)t_\theta}\sigma_\theta. \quad (8.4)$$

Now, I can find distribution  $M$ . For illiquid assets (in  $\Theta_I$ ), it is necessary that  $\mu_{su}(\theta) = \frac{y_d}{y_u + y_d}f(\theta)$ ,  $\mu_{bu}(\theta) = \frac{y_u}{y_u + y_d}f(\theta)$  and I focus on assets in  $\Theta_L \cup \Theta_M$ . The following conditions hold

in the steady state:

$$\begin{cases} y_d \mu_{sm}(\theta) + y_d \mu_{bu}(\theta) = y_u \mu_{su}(\theta) + \lambda M_{bu}(\phi) \mu_{su}(\theta) \sigma_\theta, \\ y_u \mu_{sm}(\theta) + y_u \mu_{su}(\theta) + \gamma_\theta = y_d \mu_{bu}(\theta), \\ y_u M_{bm}(\bar{\Theta}_L) + y_u M_{su}(\phi) = y_d M_{bu}(\phi) + \lambda M_{bu}(\phi) \left( \int_{\bar{\Theta}_L} \mu_{su}(\theta) \sigma_\theta d\theta \right), \\ y_d M_{bm}(\bar{\Theta}_L) + y_d M_{bu}(\phi) + \int_{\bar{\Theta}_L} \gamma_\theta d\theta = y_u M_{su}(\phi). \end{cases} \quad (8.5)$$

The first equation in (8.5) states that in the steady-state mass  $\mu_{su}(\theta)$  is constant. The inflow into  $\mu_{su}(\theta)$  consists of matched sellers of asset  $\theta$  whose counter-party buyer became seller (which happens with intensity  $y_d$ ) and unmatched buyers with asset  $\theta$  who become sellers (again with intensity  $y_d$ ). The outflow from  $\mu_{su}(\theta)$  happens with intensity  $y_u$  (due to sellers of type  $su$  with asset  $\theta$  becoming buyers) and with intensity  $\lambda M_{bu}(\phi) \mu_{su}(\theta) \sigma_\theta$  (due to unmatched sellers of asset  $\theta$  finding a match). All these transitions should be balanced which is reflected in the first equation of (8.5). The interpretation of the rest of the equations in (8.5) is analogous. Combining system (8.5) with the balance conditions (2.3) – (2.5) and (8.3) – (8.4), I get the following system of equations:

$$\begin{cases} y_d \mu_{sm}(\theta) + y_d \mu_{bu}(\theta) - y_u \mu_{su}(\theta) - \lambda M_{bu}(\phi) \mu_{su}(\theta) \sigma_\theta & = 0, \\ y_u \mu_{sm}(\theta) + y_u \mu_{su}(\theta) - y_d \mu_{bu}(\theta) + \lambda M_{bu}(\phi) \mu_{su}(\theta) e^{-(y_u+y_d)t_\theta} \sigma_\theta & = 0, \\ \mu_{su}(\theta) + \mu_{bu}(\theta) + \mu_{sm}(\theta) & = f(\theta), \\ (y_u + y_d) \mu_{sm}(\theta) - (1 - e^{-(y_u+y_d)t_\theta}) \lambda M_{bu}(\phi) \mu_{su}(\theta) \sigma_\theta & = 0, \\ y_u M_{sm}(\Theta_L \cup \Theta_M) + y_u M_{su}(\phi) - y_d M_{bu}(\phi) - \lambda M_{bu}(\phi) \left( \int_{\Theta_L \cup \Theta_M} \mu_{su}(\theta) \sigma_\theta d\theta \right) & = 0, \\ y_d M_{sm}(\Theta_L \cup \Theta_M) + y_d M_{bu}(\phi) - y_u M_{su}(\phi) + \lambda M_{bu}(\phi) \left( \int_{\Theta_L \cup \Theta_M} \mu_{su}(\theta) e^{-(y_u+y_d)t_\theta} \sigma_\theta d\theta \right) & = 0, \\ M_{su}(\phi) + M_{bu}(\phi) + M_{sm}(\Theta_L \cup \Theta_M) & = a - 1. \end{cases}$$

Observe that equations are linearly dependent and the rank of the system is five. Forth and sixth equations are linear combinations of the remaining equations and I eliminate them to make the system have a full rank. First, consider a subsystem involving only asset  $\theta$ ,

$$\begin{cases} y_d \mu_{sm}(\theta) + y_d \mu_{bu}(\theta) - y_u \mu_{su}(\theta) - \lambda M_{bu}(\phi) \mu_{su}(\theta) \sigma_\theta & = 0, \\ y_u \mu_{sm}(\theta) + y_u \mu_{su}(\theta) - y_d \mu_{bu}(\theta) + \lambda M_{bu}(\phi) \mu_{su}(\theta) e^{-(y_u+y_d)t_\theta} \sigma_\theta & = 0, \\ \mu_{su}(\theta) + \mu_{bu}(\theta) + \mu_{sm}(\theta) & = f(\theta); \end{cases}$$

which has the solution

$$\begin{cases} \mu_{su}(\theta) &= \frac{y_d}{y_u + y_d + \lambda M_{bu}(\phi) \sigma_\theta} f(\theta), \\ \mu_{bm}(\theta) &= \frac{\lambda M_{bu}(\phi) \sigma_\theta (1 - e^{-(y_u + y_d)t\theta}) y_d}{(y_u + y_d)(y_u + y_d + \lambda M_{bu}(\phi) \sigma_\theta)} f(\theta), \\ \mu_{bu}(\theta) &= \frac{y_u(y_u + y_d) + \lambda M_{bu}(\phi) \sigma_\theta (y_u + y_d e^{-(y_u + y_d)t\theta})}{(y_u + y_d)(y_u + y_d + \lambda M_{bu}(\phi) \sigma_\theta)} f(\theta). \end{cases} \quad (8.6)$$

Now I can solve for  $M_{bu}(\phi)$  and  $M_{su}(\phi)$  from

$$\begin{cases} y_u M_{sm}(\Theta_L \cup \Theta_M) + y_u M_{su}(\phi) - y_d M_{bu}(\phi) - \lambda M_{bu}(\phi) \int_{\Theta_L \cup \Theta_M} \mu_{su}(\theta) \sigma_\theta d\theta &= 0, \\ M_{su}(\phi) + M_{bu}(\phi) + M_{sm}(\Theta_L \cup \Theta_M) &= a - 1; \end{cases}$$

Subtracting the first equation from the second equation multiplied by  $y_u$ , I get

$$\lambda M_{bu}(\phi) \int_{\bar{\Theta}_L} \mu_{su}(\theta) \sigma_\theta d\theta = y_u(a - 1) - (y_u + y_d)M_{bu}(\phi).$$

Plugging  $\mu_{su}(\theta)$  from the first line of (8.6),

$$M_{bu}(\phi) = \frac{y_u}{y_u + y_d}(a - 1) - \frac{y_d}{y_u + y_d} \int_0^1 \frac{\lambda M_{bu}(\phi) \sigma_\theta}{y_u + y_d + \lambda M_{bu}(\phi) \sigma_\theta} dF(\theta), \quad (8.7)$$

which after the change of variables  $M_{bu}(\phi) = \frac{\Lambda}{\lambda}$  gives equation (3.1). The left-hand side of (8.7) is increasing in  $M_{bu}(\phi)$  and the right-hand side is decreasing in  $M_{bu}(\phi)$ . At  $M_{bu}(\phi) = 0$ , the left-hand side equals zero and the right-hand side equals  $\frac{y_u}{y_u + y_d}(a - 1) > 0$ . Therefore, equation (8.7) has a unique solution that is positive. Notice that for positive  $M_{bu}(\phi)$ , it follows from (8.7) that  $M_{bu}(\phi)$  does not exceed the total mass of buyers in the population.

Quantities  $\mu_{su}(\theta)$ ,  $\mu_{bm}(\theta)$ ,  $\mu_{bu}(\theta)$  can be found from (8.6). The distribution of assets of unmatched sellers of  $\Theta_L$  that are searching on the market is given by  $F_L(\theta) = \frac{\int_0^\theta \mu_{su}(\theta) d\theta}{M_{su}(\Theta_L)} = F(\theta | \theta \in \Theta_L)$ .  $\square$

## 8.2 Analysis of Value Functions

Here, I derive value functions of agents. Denote by  $U_s$  the utility of the seller who does not participate in search and simply holds the asset. Then  $U_s$  can be found from equation (3.6) by setting  $\sigma_\theta = 0$ :

$$U_s(\theta) = \frac{1}{r} \left( \frac{y_u}{r + y_u + y_d} \bar{v}(\theta) + \frac{r + y_d}{r + y_u + y_d} \underline{v}(\theta) \right). \quad (8.8)$$

Therefore,  $U_s$  is a weighted average of the present value of holding the asset as a buyer and as a seller, and weights are given by the long-run fractions of time  $\frac{y_u}{r + y_u + y_d}$  and  $\frac{r + y_d}{r + y_u + y_d}$  an agent is a buyer or a seller, respectively. Notice that the utility of sellers of illiquid assets is given by

$V_{su}(\theta) = U_s(\theta)$  for  $\theta \in \Theta_I$ .

The next lemma simplifies equations (3.3) – (3.6) and shows that  $V_{bu}$  and  $V_{su}(\phi)$  can be expressed through  $V_{bu}(\phi)$  and  $V_{su}$ .

**Lemma 5.** *For all  $\theta \in [0, 1]$ ,*

$$V_{bu}(\theta) = \frac{\bar{v}(\theta) + y_d V_{su}(\theta)}{r + y_d}, \quad (8.9)$$

$$V_{su}(\phi) = \frac{y_u V_{bu}(\phi)}{r + y_u}, \quad (8.10)$$

$$V_{bu}(\phi) = \Lambda_b \frac{r + y_u}{r \rho} (\mathbb{E}[V_{bm}(\theta)|\theta \in \Theta_L] - V_{bu}(\phi)), \quad (8.11)$$

$$V_{su}(\theta) = U_s(\theta) + \sigma_\theta \Lambda \frac{r + y_d}{r \rho} (V_{sm}(\theta) - V_{su}(\theta)). \quad (8.12)$$

I further find  $V_{bu}(\phi)$  and  $V_{su}$ . For this purpose, I next turn to the outcome of the bargaining stage and express value functions of matched agents,  $V_{bm}$  and  $V_{sm}$ , through  $V_{bu}(\phi)$  and  $V_{su}$ . In Subsection 3.2, I introduced functions  $\hat{v}$  and  $\hat{c}(\theta)$  as the value functions of the buyer and the seller who remain in the match and never trade with the current partner. By the definition,  $\hat{c}(\theta)$  is given by the Bellman equation

$$r\hat{c}(\theta) = \underline{v}(\theta) + y_u(V_{bu}(\theta) - \hat{c}(\theta)) + y_d(V_{su}(\theta) - \hat{c}(\theta)),$$

and so, it is given by

$$\hat{c}(\theta) = \frac{1}{\rho}(\underline{v}(\theta) + y_u V_{bu}(\theta) + y_d V_{su}(\theta)) = \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta).$$

Analogously, the value  $\hat{v}$  of the buyer who never buys the asset traded, but stays in the match evolves according to

$$r\hat{v} = y_u(V_{bu}(\phi) - \hat{v}) + y_d(V_{su}(\phi) - \hat{v}),$$

or solving for  $\hat{v}$ ,

$$\hat{v} = \frac{1}{\rho}(y_u V_{bu}(\phi) + y_d V_{su}(\phi)) = \frac{y_u}{r + y_u} V_{bu}(\phi).$$

Then functions  $v$  and  $c$  are given by

$$c(\theta) = \hat{c}(\theta) - V_{su}(\phi) = \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi), \quad (8.13)$$

$$v(\theta) = V_{bu}(\theta) - \bar{v} = \frac{\bar{v}(\theta)}{r + y_d} + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi). \quad (8.14)$$

Observe that  $\xi_\theta \equiv v(\theta) - c(\theta) = \frac{\ell_\theta}{\rho}$ . The next lemma expresses value functions of matched agents through  $z_\theta$ ,  $V_{su}$  and  $V_{bu}(\phi)$ .

**Lemma 6.** For any  $\theta \in [0, 1]$ ,

$$V_{bm}(\theta) = \alpha z_\theta + \frac{y_u}{r + y_u} V_{bu}(\phi), \quad (8.15)$$

$$V_{sm}(\theta) = (1 - \alpha) z_\theta + \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta). \quad (8.16)$$

*Proof of Lemma 6.* Given that the trade at the bargaining stage is not immediate, the utility of matched agents depends on time and I index  $V_\tau^t(\bar{\theta})$  by time for  $\tau \in \{bm, sm\}$ . Observe that  $V_{bm}^{t_\theta}(\theta) = V_{bu}(\theta) - q_\theta$  and  $V_{sm}^{t_\theta}(\theta) = q_\theta + V_{su}(\phi)$ . Moreover, the following Bellman equation holds for  $V_{bm}^t(\theta)$ :

$$rV_{bm}^t(\theta) = y_u(V_{bu}(\phi) - V_{bm}^t(\theta)) + y_d(V_{su}(\phi) - V_{bm}^t(\theta)) + \dot{V}_{bm}^t(\theta).$$

I solve this differential equation to get

$$V_{bm}^t(\theta) = (V_{bu}(\theta) - q_\theta) e^{-\rho(t_\theta - t)} + \frac{y_u V_{bu}(\phi)}{r + y_u} (1 - e^{-\rho(t_\theta - t)}).$$

From  $V_{bm}^0(\theta) = V_{bm}(\theta)$ , I get (8.15). Symmetrically, the Bellman equation for  $V_{sm}^t(\theta)$  is

$$rV_{sm}^t(\theta) = \underline{v}(\theta) + y_u(V_{bu}(\theta) - V_{sm}^t(\theta)) + y_d(V_{su}(\theta) - V_{sm}^t(\theta)) + \dot{V}_{sm}^t(\theta),$$

which has solution

$$V_{sm}^t(\theta) = (q_\theta + V_{su}(\phi)) e^{-\rho(t_\theta - t)} + \frac{1}{\rho} (\underline{v}(\theta) + y_u V_{bu}(\theta) + y_d V_{su}(\theta)) (1 - e^{-\rho(t_\theta - t)}).$$

From  $V_{sm}^0(\theta) = V_{sm}(\theta)$ , I get (8.16). □

It follows from (8.15) that the payoff from the match of the buyer depends only on the trade expected surplus  $z_\theta$ , but does not depend on the asset quality otherwise. In equilibrium buyers trade off liquidity and surplus from trade and choose assets with the highest expected surplus. Combining (8.9) and (8.15), I get

$$V_{bu}(\phi) = \alpha \frac{r + y_u}{r} \frac{\Lambda_b}{\rho + \Lambda_b} \bar{z}. \quad (8.17)$$

The buyer prefers to trade with the seller of asset  $\theta$  if and only if  $V_{bm}(\theta) \geq V_{bu}(\phi)$ , or combining (8.15) and (8.17), I get the condition

$$z_\theta \geq \underline{z} \equiv \frac{\Lambda_b}{\rho + \Lambda_b} \bar{z}. \quad (8.18)$$

The inequality (8.18) is strict for  $\theta \in \Theta_L$ , and it holds as equality for  $\theta \in \Theta_M$ . This proves Lemma 2.

It follows from (8.12) and (8.16) that for  $\theta \in \Theta_L \cup \Theta_M$  function  $V_{su}(\theta)$  is given by

$$V_{su}(\theta) = U_s(\theta) + (1 - \alpha) \frac{r + y_d}{r} \frac{\sigma_\theta \Lambda}{\rho + \sigma_\theta \Lambda} z_\theta \quad (8.19)$$

Equation (8.19) implies that  $V_{su}(\theta) > U_s(\theta)$  whenever  $z_\theta > 0$  and so, sellers always prefer to trade. This implies that the liquidity of the asset is determined solely by the buyer strategy. This completes the derivation of the value functions.

### 8.3 Model with Exogenous Delay

Since  $\sigma_\theta$  takes values in  $[0, 1]$  and  $\Lambda$  is decreasing in  $\sigma_\theta$  (by Lemma 1), the following lemma follows.

**Lemma 7.**  $\Lambda$  takes values in the range  $[\Lambda_{min}, \Lambda_{max}]$  where  $\Lambda_{max} \equiv \lambda \frac{y_u}{y_u + y_d} (a - 1)$  and  $\Lambda_{min}$  is given by the unique positive root of the quadratic equation

$$\frac{\Lambda_{min}}{\lambda} = \frac{y_u}{y_u + y_d} (a - 1) - \frac{y_d}{y_u + y_d} \frac{\Lambda_{min}}{y_u + y_d + \Lambda_{min}}. \quad (8.20)$$

*Proof of Theorem 1.* Denote by  $[z_{min}, z_{max}]$  the range of values of  $z_\theta$ . To find the equilibrium, I solve for equilibrium quantities  $\Lambda > 0$  and  $\underline{z} \in [0, z_{max}]$ . The equilibrium conditions that define these quantities are as follows. First, from Lemma 2, the strategy is given by

$$\sigma_\theta = \begin{cases} 1, & \text{if } z_\theta > \underline{z}, \\ \sigma, & \text{if } z_\theta = \underline{z}, \\ 0, & \text{if } z_\theta < \underline{z}; \end{cases}$$

where  $\sigma$  is some number in  $[0, 1]$  determined in equilibrium. Here, I used the assumption that whenever buyers mix between accepting and rejecting the asset, they do not condition on the quality of the asset.

Second, for  $\underline{z} \in [z_{min}, z_{max}]$ , it follows from (3.1) that

$$\int_{z_\theta \geq \underline{z}} \frac{y_d \Lambda \sigma_\theta}{y_u + y_d + \Lambda \sigma_\theta} dF(\theta) - y_u (a - 1) + (y_u + y_d) \frac{\Lambda}{\lambda} = 0. \quad (8.21)$$

By Lemma 1, for fixed  $\sigma$ , equation (8.21) has a unique solution  $\Lambda_1(\underline{z}, \sigma)$ . Let  $\Lambda_1(\underline{z}) \equiv \cup_{\sigma \in [0, 1]} \Lambda_1(\underline{z}, \sigma)$ . Since the left-hand side of (8.21) is continuous in  $\sigma$ ,  $\Lambda_1(\underline{z})$  is upper hemi-continuous. The left-hand side of (8.21) is weakly decreasing in  $\underline{z}$  and strictly increasing in  $\Lambda$ . Therefore,  $\Lambda_1(\underline{z})$  is increasing in  $\underline{z}$  for intervals on which it is single-valued. I can find  $\Lambda_1(z_{min}) = \Lambda_{min}$ . Since the strategy  $\sigma_\theta$  does not depend on  $\underline{z}$  once it is below  $z_{min}$ , I have that for  $\underline{z} \leq z_{min}$ ,  $\Lambda_1(\underline{z}) = \Lambda_1(z_{min})$ . Therefore, for all  $\underline{z} \in [0, z_{max}]$ ,  $\Lambda_1(\underline{z}) > 0$ .

Third, from (3.2) and (3.8),

$$\rho = \int_{z_\theta > \underline{z}} \left( \frac{z_\theta}{\underline{z}} - 1 \right) \frac{\lambda y_d}{y_u + y_d + \Lambda} dF(\theta). \quad (8.22)$$

The right-hand side of (8.22) is strictly decreasing in  $\Lambda$  and strictly decreasing in  $\underline{z}$ . Therefore, there is a unique solution  $\Lambda_2(\underline{z})$  to (8.22) and it is decreasing in  $\underline{z}$ . Moreover, the solution  $\Lambda_2(\underline{z})$  is upper semi-continuous. As  $\underline{z} \rightarrow z_{max}$ , the right-hand side of (8.22) converges to zero for all  $\Lambda \geq 0$ . Therefore,  $\Lambda_2(z_{max}) < 0$ . As  $\underline{z} \rightarrow 0$ , the right-hand side of (8.22) diverges to infinite. Therefore, for any positive constant  $C$ , there exists  $\underline{z} \in (0, z_{max})$  such that  $\Lambda_2(\underline{z}) > C$ .

Combining the observations about mappings  $\Lambda_1$  and  $\Lambda_2$ :

- $\Lambda_1(\underline{z})$  is strictly increasing and upper hemi-continuous,  $\Lambda_1(\underline{z}) = \Lambda_1(z_{min})$  for  $\underline{z} \in [0, z_{min}]$ ;
- $\Lambda_2(\underline{z})$  is decreasing, upper semi-continuous, and  $\lim_{\underline{z} \rightarrow 0} \Lambda_2(\underline{z}) = \infty$ ;
- $\Lambda_1(z_{max}) > 0 > \Lambda_2(z_{max})$ .

Therefore, there exists a unique solution  $\underline{z}$  to equations (8.21) and (8.22), and corresponding  $\Lambda$  and  $\sigma$ . This completes the proof of existence and uniqueness of the equilibrium.

The fact that equilibrium does not depend on  $d$  follows directly from the derivation of the equilibrium conditions. To derive equation (3.9), I plug functions  $v$  and  $c$  from (8.13) and (8.14) into equation (3.7), and then substitute  $V_{su}(\theta)$  and  $V_{bu}(\phi)$  from (8.17) and (8.19). Then spreads are given by

$$s_\theta = r \left( \frac{kg(\theta) + d}{kg(\theta) + d - (r + y_d)\xi_\theta + (1 - \alpha)\xi_\theta + (1 - \alpha)y_d \frac{\sigma_\theta \Lambda}{\rho + \sigma_\theta \Lambda} z_\theta - \alpha y_u \frac{\Lambda_b}{\rho + \Lambda_b} \bar{z}} - 1 \right). \quad (8.23)$$

The sign of partial derivatives of  $s_\theta$  can be obtained from the formula (8.23).  $\square$

## 8.4 Model with Endogenous Delay

Suppose  $v$  and  $c$  are weakly increasing and piecewise continuously differentiable and a sequence of continuously differentiable functions  $v_i$  and  $c_i$  is such that  $(v_i, c_i, v'_i, c'_i) \rightarrow (v, c, v', c')$  at all differentiability points of  $v$  and  $c$ . Denote by  $(q_\theta^i, t_\theta^i, \theta_i^*)$  the CSBS for bargaining problem  $(\rho, v_i, c_i)$ , and let  $x_\theta^i \equiv e^{-\rho t_\theta^i}$  and  $z_\theta^i \equiv x_\theta^i \xi_\theta$ .

*Proof of Lemma 3.* I show that (4.1) is equivalent to conclusion 1 in Lemma 3, and showing that (4.2) is equivalent to condition 2 is analogous. First, I rewrite the maximization problem in (4.1) as follows

$$\theta \in \operatorname{argmax}_{\theta' \in [\theta^*, 1]} x_{\theta'}(v(\theta) - q_{\theta'}). \quad (8.24)$$

By the envelope theorem (Milgrom and Segal (2002)), function  $x_\theta(v(\theta) - q_\theta)$  is absolutely continuous and at differentiability points satisfies  $(x_\theta(v(\theta) - q_\theta))' = x_\theta v'(\theta)$ . This implies that  $x_\theta$  is



continuously differentiable. Since  $v(\theta)$  is continuously differentiable, the maximized function in (8.24) is continuously differentiable. Therefore, the first-order condition holds for this problem:

$$x'_\theta(v(\theta) - q_\theta) - x_\theta q'_\theta = 0. \quad (8.25)$$

Expression for  $t_\theta$  in condition 1 of Lemma 3 gives the solution to this first-order condition.

I next show that the first-order condition is also sufficient. The maximized function in (8.24) has the smooth single crossing differences property (see Milgrom (2004)). By  $q'_\theta > 0$ ,  $x'_\theta > 0$ , and by (8.25), the envelope formula holds. By Theorem 4.2 in Milgrom (2004),  $x_\theta$  given by (8.25) is the optimum of (8.24). Moreover, since (8.25) is the necessary condition, and the unique solution to (8.25) is the unique optimum of (8.24).

The comparative statics of  $t_\theta$  with respect to  $v'$  and  $c'$  follows directly from the expression for  $t_\theta$  in the lemma.  $\square$

**Lemma 8.** *There exists a subsequence of bargaining problems  $(\rho, v_i, c_i)$  such that  $(q_\theta^i, t_\theta^i, \theta_i^*)$  converges point-wise at all differentiability points of  $v$  and  $c$  to  $(q_\theta, t_\theta, \theta^*)$  that satisfies the following properties.*

1.  $q_\theta = (1 - \alpha)v(\theta) + \alpha c(\theta)$  at all differentiability points of  $v$  and  $c$ .
2.  $t_\theta$  and  $z_\theta$  are piece-wise continuously differentiable, decreasing for  $\theta < \theta^*$  and increasing for  $\theta > \theta^*$ .
3. Consider  $\theta'$  and  $\theta''$  such that  $v$  and  $c$  are continuous on  $[\theta', \theta'']$ . Then

$$t_{\theta''} - t_{\theta'} = - \int_{\theta'}^{\theta''} \frac{v'(\theta) - \alpha \xi'_\theta}{\rho \alpha \xi_\theta} d\theta \text{ and } \ln \left( \frac{z_{\theta''}}{z_{\theta'}} \right) = - \int_{\theta'}^{\theta''} \frac{v'(\theta)}{\alpha \xi_\theta} d\theta, \text{ if } \theta' > \theta^*,$$

$$t_{\theta''} - t_{\theta'} = \int_{\theta'}^{\theta''} \frac{c'(\theta) + (1 - \alpha) \xi'_\theta}{\rho(1 - \alpha) \xi_\theta} d\theta \text{ and } \ln \left( \frac{z_{\theta''}}{z_{\theta'}} \right) = \int_{\theta'}^{\theta''} \frac{c'(\theta)}{(1 - \alpha) \xi_\theta} d\theta, \text{ if } \theta'' < \theta^*.$$

4. For differentiability points  $\theta > \theta^*$  of  $v$ ,  $z'_\theta = 0$  if and only if  $v'(\theta) = 0$ , and for differentiability points  $\theta < \theta^*$  of  $c$ ,  $z'_\theta = 0$  if and only if  $c'(\theta) = 0$ .

*Proof.* By (3.7), point-wise convergence of  $v_i$  and  $c_i$  to  $v$  and  $c$ , respectively, implies convergence of  $q_{i\theta}$  to  $(1 - \alpha)v(\theta) + \alpha c(\theta)$  at all differentiability points of  $v$  and  $c$ .

By the definition of  $\theta^*$ , for every  $\theta' > \theta^*$  and  $\theta'' < \theta^*$ , there exists  $I$  such that for all  $i > I$ ,  $t_\theta^i$  and  $z_\theta^i$  are monotone on  $[\theta', 1]$  and  $[0, \theta'']$ . By Helly's theorem,  $t_\theta^i$  and  $z_\theta^i$  converge over subsequence to monotone functions  $t_\theta$  and  $z_\theta$  on  $[\theta', 1]$  and  $[0, \theta'']$ . Therefore,  $t_\theta^i$  and  $z_\theta^i$  converge over subsequence to monotone functions  $t_\theta$  and  $z_\theta$  on  $(\theta^*, 1]$  and  $[0, \theta^*)$ .

If  $\theta' > \theta^*$ , by Lemma 2, for every  $\theta \in (\theta', \theta'')$ , eventually  $t_{i\theta''} - t_{i\theta'} = - \int_{\theta'}^{\theta''} \frac{v'_i(\theta) - \alpha \xi'_{i\theta}}{\rho \alpha \xi_{i\theta}} d\theta$ , and it converges to  $t_{\theta''} - t_{\theta'} = - \int_{\theta'}^{\theta''} \frac{v'(\theta) - \alpha \xi'_\theta}{\rho \alpha \xi_\theta} d\theta$  by the dominated convergence theorem. By the same reasoning, if  $\theta'' < \theta^*$ ,  $t_{i\theta''} - t_{i\theta'}$  converges to  $t_{\theta''} - t_{\theta'} = \int_{\theta'}^{\theta''} \frac{c'(\theta) + (1 - \alpha) \xi'_\theta}{\rho(1 - \alpha) \xi_\theta} d\theta$ .

By the definition of  $x_\theta$ , I can rewrite condition 1 in Lemma 2 as follows

$$(v_i(\theta) - q_\theta^i)(x_\theta^i)' = (q_\theta^i)'x_\theta^i.$$

for  $\theta > \theta_i^*$ . By the proportional split of the surplus,  $v_i(\theta) - q_\theta^i = \alpha\xi_\theta^i$  and so,

$$\alpha\xi_\theta^i(x_\theta^i)' = (q_\theta^i)'x_\theta^i$$

or

$$\frac{(z_\theta^i)'}{z_\theta^i} = \frac{v_i'(\theta)}{\alpha\xi_\theta^i}.$$

Therefore, for differentiability points  $\theta > \theta^*$  of  $v$ ,

$$\frac{z_\theta'}{z_\theta} = \frac{v'(\theta)}{\alpha\xi_\theta}, \quad (8.26)$$

and so,  $z_\theta' = 0$  if and only if  $v'(\theta) = 0$ .

Analogously, for differentiability points  $\theta < \theta^*$  of  $c$ ,

$$\frac{z_\theta'}{z_\theta} = -\frac{c'(\theta)}{(1-\alpha)\xi_\theta}, \quad (8.27)$$

and so,  $z_\theta' = 0$  if and only if  $c'(\theta) = 0$ . □

**Lemma 9.** *If in equilibrium  $\sigma_\theta = 1$  on an interval  $(\theta', \theta'')$  on which  $v$  and  $c$  are continuously differentiable, and either  $\theta' > \theta^*$  or  $\theta'' < \theta^*$ , then for  $\theta' > \theta^*$ ,  $z_\theta$  is strictly increasing and satisfies*

$$z_\theta' \left( \alpha \frac{\xi_\theta}{z_\theta} - \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} (1 - \alpha) \right) = \frac{\bar{v}'(\theta) + y_d U_s'(\theta)}{(r + y_d)}. \quad (8.28)$$

and for  $\theta'' < \theta^*$ ,  $z_\theta$  is strictly decreasing and satisfies

$$(1 - \alpha) z_\theta' \left( \frac{\xi_\theta}{z_\theta} + \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} \right) = -U_s'(\theta), \quad (8.29)$$

Moreover, functions  $v$  and  $c$  are strictly increasing on  $(\theta', \theta'')$ .

*Proof of Lemma 9.* For  $\theta' > \theta^*$ , plugging  $v'(\theta)$  from (8.14) into (8.26), I get

$$\frac{z_\theta'}{z_\theta} = \frac{\bar{v}'(\theta) + y_d V_{su}'(\theta)}{\alpha\xi_\theta(r + y_d)}.$$

By (8.19),

$$V_{su}'(\theta) = U_s'(\theta) + (1 - \alpha) \frac{r + y_d}{r} \frac{\Lambda}{\rho + \Lambda} z_\theta' \quad (8.30)$$

and so,

$$z'_\theta = \zeta_>(z_\theta, \theta, \Lambda), \quad (8.31)$$

where

$$\zeta_>(z, \theta, \Lambda) = \frac{\bar{v}'(\theta) + y_d U'_s(\theta)}{(r + y_d) \left( \frac{\alpha \xi_\theta}{z} - \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} (1 - \alpha) \right)}. \quad (8.32)$$

From (4.3), the denominator of (8.32) is positive. Indeed,

$$\frac{\alpha \xi_\theta}{z} \geq \alpha \geq \frac{y_d}{r} (1 - \alpha) > \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} (1 - \alpha).$$

Plugging (8.31) into (8.30), I get that

$$V'_{su}(\theta) = U'_s(\theta) + \frac{1}{r} \frac{\Lambda}{\rho + \Lambda} (1 - \alpha) \frac{\bar{v}'(\theta) + y_d U'_s(\theta)}{\frac{\alpha}{x_\theta} - \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} (1 - \alpha)}$$

and so, from (8.14),

$$v'(\theta) = \left( \frac{\bar{v}'(\theta)}{r + y_d} + \frac{y_d}{r + y_d} U'_s(\theta) \right) \left( \frac{\alpha}{\alpha - \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} (1 - \alpha) x_\theta} \right) > 0.$$

Analogously, plugging in  $c'(\theta)$  from (8.13),

$$\frac{z'_\theta}{z_\theta} = - \frac{r U'_s(\theta) + y_d V'_{su}(\theta)}{(1 - \alpha) \xi_\theta (r + y_d)},$$

or using (8.19) to find  $V'_{su}(\theta)$ ,

$$z'_\theta = \zeta_<(z_\theta, \theta, \Lambda)$$

where

$$\zeta_<(z, \theta, \Lambda) = - \frac{U'_s(\theta)}{(1 - \alpha) \left( \frac{\xi_\theta}{z} + \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} \right)}. \quad (8.33)$$

By (8.30),

$$V'_{su}(\theta) = U'_s(\theta) \frac{\frac{\xi_\theta}{z_\theta} - \frac{\Lambda}{\rho + \Lambda}}{\frac{\xi_\theta}{z_\theta} + \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda}} > 0$$

and so, since  $\xi_\theta/z_\theta = 1/x_\theta \geq 1 > \frac{\Lambda}{\rho + \Lambda}$ ,  $c'(\theta) > 0$ . □

I next prove Lemma 4 stated in the text.

*Proof of Lemma 4.* The analysis proceeds in a series of claims.

*Claim 1.* If  $z_\theta = \underline{z}$  for some set  $(\theta', \theta'')$ , then  $\sigma_\theta \in (0, 1)$  for almost every  $\theta \in (\theta', \theta'')$ .

*Proof.* Suppose that  $z_\theta = \underline{z}$ , but  $\sigma_\theta = 0$  for some set  $(\theta', \theta'')$ . Then  $V_{su}$  is strictly increasing

on this set by (8.19) and so, by (8.13) and (8.14),  $v$  and  $c$  are strictly increasing. This contradicts conclusion 4 in Lemma 8. Now suppose that  $z_\theta = \underline{z}$ , but  $\sigma_\theta = 1$  for some set  $(\theta', \theta'')$ . By Lemma 9,  $v$  and  $c$  are strictly increasing. This again contradicts conclusion 4 in Lemma 8. *q.e.d.*

*Claim 2.* If  $\sigma_\theta > 0$  for some  $\theta > \theta^*$ , then  $\sigma_{\theta'} > 0$  for almost every  $\theta' > \theta$ . If for some  $\theta \leq \theta^*$ ,  $\sigma_\theta > 0$ , then for almost every  $\theta' < \theta$ ,  $\sigma_{\theta'} > 0$ . Moreover,  $\Theta_L = [0, \check{\theta}] \cup [\hat{\theta}, 1]$ .

*Proof.* By Lemma 2, buyers accept only asset qualities with  $z_\theta \geq \underline{z}$ . By Lemma 8,  $z_\theta$  has a U-shape and so,  $z_\theta \geq \underline{z}$  on a set  $[0, \theta'] \cup [\theta'', 1]$  and  $z_\theta > \underline{z}$  on a set  $[0, \check{\theta}] \cup [\hat{\theta}, 1]$ . The statement then follows from Claim 1. *q.e.d.*

In the next two claims, I use the following notation. For a set  $A$ , I denote by  $\overline{A}$  the closure of  $A$ .

*Claim 3.*  $\overline{\Theta_L \cap [0, \theta^*]} \cap \overline{\Theta_I} = \phi$ .

*Proof.* Suppose not and there exists  $\theta' = \inf \overline{\Theta_L \cap [0, \theta^*]} \cap \overline{\Theta_I}$ . There is an increasing sequence of  $\{\theta'_i\} \subset \Theta_L \cup [0, \theta^*]$  and a decreasing sequence  $\{\theta''_i\} \subset \Theta_I$  both converging to  $\theta'$ . From (8.19) and (8.13), this implies that for sufficiently large  $i$ ,  $c(\theta'_i) > c(\theta''_i)$  while  $\theta'_i < \theta''_i$ , which contradicts monotonicity of  $c$ . *q.e.d.*

*Claim 4.*  $\overline{\Theta_L \cap [\theta^*, 1]} \cap \overline{\Theta_M} = \phi$

*Proof.* Suppose not and there exists  $\theta' = \sup \overline{\Theta_L \cap [\theta^*, 1]} \cap \overline{\Theta_M}$ . There is a decreasing sequence of  $\{\theta'_i\} \subset \Theta_L \cup [\theta^*, 1]$  and an increasing sequence  $\{\theta''_i\} \subset \Theta_M$  both converging to  $\theta'$ . From (8.19) and (8.14), this implies that for sufficiently large  $i$ ,  $v(\theta''_i)$  is constant and  $\sigma(\theta''_i)$  decreases. This contradicts the continuity of  $v$  at  $\theta'$ . *q.e.d.*

It follows from Claims 1-4 that the only possible order of sets is  $\Theta_L \cap [0, \theta^*], \Theta_M, \Theta_I, \Theta_L \cap [\theta^*, 1]$  which is the desired conclusion.  $\square$

**Lemma 10.** Function  $z_\theta$  is determined by  $\Lambda$  on a set  $\Theta_L$ . For  $\theta \in [\hat{\theta}, 1]$ ,  $z_\theta$  is strictly increasing and given by the unique solution of (8.28) with the initial condition  $z_1 = \xi_1$ , and for  $\theta \in [0, \check{\theta}]$ ,  $z_\theta$  is strictly decreasing and given by the unique solution of (8.29) with the initial condition  $z_0 = \xi_0$ . Moreover, there exists  $z_{min} > 0$  such that for any  $\Lambda$ ,  $z_\theta \geq z_{min}$  for all  $\theta \in \Theta_L$ .

*Proof.* Combining Lemmas 4 and 9, I get that  $z_\theta$  is determined on  $\Theta_L$  as in the statement of the lemma. Since equations (8.28) and (8.29) depend only on  $\Lambda$ ,  $z_\theta$  depends only on  $\Lambda$  on  $\Theta_L$ . Moreover, the existence and uniqueness of the solution to differential equations (8.28) with the initial condition  $z_1 = \xi_1$  and (8.29) with the initial condition  $z_0 = \xi_0$  follow from the Picard-Lindelof theorem.

For given  $\Lambda$  (not necessarily equilibrium  $\Lambda$ ), denote by  $z_{\theta, >}(\Lambda)$  the solution to (8.32) and by  $z_{\theta, <}(\Lambda)$  the solution to (8.33). Then

$$z_\theta > \min_{\Lambda \in [\Lambda_{min}, \Lambda_{max}]} \min_{\theta \in [0, 1]} \{\max \{z_{\theta, >}(\Lambda), z_{\theta, <}(\Lambda)\}\} \equiv z_{min}.$$

Minimized function is continuous in  $\Lambda$ , and  $\Lambda$  belongs to a compact interval. Therefore, the minimum is attained for some  $\Lambda$  and  $z_{min} > 0$ .  $\square$

**Equilibrium Expected Surplus  $z_\theta$  and strategies  $\sigma_\theta$**  I now can describe  $z_\theta$  for given  $\Lambda$  and  $\underline{z}$ . For  $\theta \in \Theta_M$ ,  $z_\theta$  is constant and equal to  $\underline{z}$ . For  $\theta \in \Theta_L$ ,  $z_\theta$  is described by Lemma 10, and it is only left to find conditions to determine thresholds  $\hat{\theta}, \check{\theta}, \underline{\theta}$ . Let  $\theta_<(z)$  and  $\theta_>(z)$  be inverse functions of  $z_\theta$  for  $\theta \in [0, \check{\theta}]$  and  $\theta \in [\hat{\theta}, 1]$ , respectively. By the strict monotonicity of  $z_\theta$  on the respective intervals (Lemma 10), these functions are well-defined and  $\theta_<$  is strictly increasing and  $\theta_>$  is strictly decreasing. Since  $\underline{z} = z_{\check{\theta}} = z_{\hat{\theta}}$ ,

$$\check{\theta} = \theta_<(\underline{z}), \quad (8.34)$$

$$\hat{\theta} = \theta_>(\underline{z}), \quad (8.35)$$

For each  $\theta \in \Theta_M$ ,  $z_\theta = \underline{z}$  and so,  $c(\theta) = c(\check{\theta})$  by Lemma 8. Therefore, for  $\theta \in \Theta_M$ ,

$$V_{su}(\theta) = V_{su}(\check{\theta}) - \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta})) \quad (8.36)$$

or

$$V_{su}(\theta) - U_s(\theta) = V_{su}(\check{\theta}) - \left(1 + \frac{r}{y_d}\right) U_s(\theta) + \frac{r}{y_d} U_s(\check{\theta}) = \frac{r + y_d}{r} \left( \frac{r}{y_d} (U_s(\check{\theta}) - U_s(\theta)) + (1 - \alpha) \frac{\Lambda}{\rho + \Lambda} \underline{z} \right). \quad (8.37)$$

Threshold  $\underline{\theta}$  is determined as the minimum of  $\hat{\theta}$  and the solution to the equation  $U_s(\underline{\theta}) = V_{su}(\underline{\theta})$  and so, from (8.37),

$$\underline{\theta} = \min \left\{ \hat{\theta}, U_s^{-1} \left( U_s(\check{\theta}) + (1 - \alpha) \frac{y_d}{r} \frac{\Lambda}{\rho + \Lambda} \underline{z} \right) \right\}. \quad (8.38)$$

This completes the description of  $z_\theta$  for a given  $\Lambda$  and  $\underline{z}$ . The following monotonicity property of  $\hat{\theta}, \check{\theta}, \underline{\theta}$  is immediate from (8.34),(8.35),(8.38), and the strict monotonicity of  $\theta_>$  and  $\theta_<$ .

**Lemma 11.**  $\hat{\theta}$  is strictly increasing in  $\underline{z}$ ,  $\check{\theta}$  and  $\underline{\theta}$  are strictly decreasing in  $\underline{z}$ .

The next lemma determines equilibrium strategies.

**Lemma 12.** For given  $\Lambda$  and  $\underline{z}$ ,

$$\sigma_\theta = \begin{cases} 1, & \text{if } \theta \in [0, \check{\theta}] \cup [\hat{\theta}, 1], \\ 0, & \text{if } \theta \in [\underline{\theta}, \hat{\theta}), \\ \rho \left( \frac{(1-\alpha) \frac{\Lambda}{\rho+\Lambda} \underline{z} - \frac{r}{y_d} (U_s(\theta) - U_s(\check{\theta}))}{(1-\alpha) \frac{\rho}{\rho+\Lambda} \underline{z} + \frac{r}{y_d} (U_s(\theta) - U_s(\check{\theta}))} \right), & \text{if } \theta \in (\check{\theta}, \underline{\theta}). \end{cases} \quad (8.39)$$

Moreover,  $(\Lambda \sigma_\theta)'_{\underline{z}} > 0$  for  $\theta \in (\check{\theta}, \underline{\theta})$ .

*Proof.* Lemma 4 determines equilibrium strategies for  $\theta \in \Theta_M \cup \Theta_I$ , and it remains to determine equilibrium strategies for  $\theta \in \Theta_M$  from (8.19) with  $z_\theta = \underline{z}$  and (8.37):

$$\sigma_\theta = \frac{\rho}{\Lambda} \frac{\frac{r}{r+y}(V_{su}(\theta) - U_s(\theta))}{(1-\alpha)\underline{z} - \frac{r}{r+y}(V_{su}(\theta) - U_s(\theta))} = \frac{\rho}{\Lambda} \left( \frac{(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\underline{z} - \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta}))}{(1-\alpha)\frac{\rho}{\rho+\Lambda}\underline{z} + \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta}))} \right).$$

To see that  $\frac{d(\Lambda\sigma_\theta)}{d\underline{z}} < 0$  for  $\theta \in (\check{\theta}, \underline{\theta})$ , observe that using (8.33),

$$\begin{aligned} \frac{d}{d\underline{z}} \left( (1-\alpha)\frac{\Lambda}{\rho+\Lambda}\underline{z} - \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta})) \right) &= (1-\alpha)\frac{\Lambda}{\rho+\Lambda} + \frac{r}{y_d}U'_s(\check{\theta})\theta'_<(\underline{z}) = \\ &= (1-\alpha)\frac{\Lambda}{\rho+\Lambda} - \frac{r}{y_d}(1-\alpha) \left( \frac{\xi_\theta}{z_\theta} + \frac{y_d}{r} \frac{\Lambda}{\rho+\Lambda} \right) = -\frac{r}{y_d}(1-\alpha)\frac{\xi_\theta}{z_\theta} < 0, \end{aligned}$$

and using (8.32),

$$\begin{aligned} \frac{d}{d\underline{z}} \left( (1-\alpha)\frac{\rho}{\rho+\Lambda}\underline{z} + \frac{r}{y_d}(U_s(\theta) - U_s(\check{\theta})) \right) &= (1-\alpha)\frac{\Lambda}{\rho+\Lambda} - \frac{r}{y_d}U'_s(\check{\theta})\theta'_<(\underline{z}) = \\ &= (1-\alpha)\frac{\Lambda}{\rho+\Lambda} + \frac{r}{y_d}(1-\alpha) \left( \frac{\xi_\theta}{z_\theta} + \frac{y_d}{r} \frac{\Lambda}{\rho+\Lambda} \right) = 2(1-\alpha)\frac{\Lambda}{\rho+\Lambda} + \frac{r}{y_d}(1-\alpha)\frac{\xi_\theta}{z_\theta} > 0. \end{aligned}$$

□

Therefore, I have expressed strategy  $\sigma_\theta$  and expected surplus  $z_\theta$  through  $\Lambda$  and  $\underline{z}$ . Now I can find equilibrium by solving for  $\underline{z}$  and  $\Lambda$  from conditions (8.21) and (8.22) which I repeat here

$$\frac{\Lambda}{\lambda} = \frac{y_u}{y_u + y_d}(a-1) - \frac{y_d}{y_u + y_d} \int_0^1 \frac{\Lambda\sigma_\theta}{y_u + y_d + \Lambda\sigma_\theta} dF(\theta), \quad (8.40)$$

$$\rho = \int_{z_\theta > \underline{z}} \left( \frac{z_\theta}{\underline{z}} - 1 \right) \frac{\lambda y_d}{y_u + y_d + \Lambda} dF(\theta). \quad (8.41)$$

I next prove that there is always a solution satisfying (8.40) and (8.41).

**Proof of Theorem 2** Equilibrium existence follows from the claim below.

*Claim 5.* For any  $\Lambda \in [\Lambda_{min}, \Lambda_{max}]$ , there exists a unique continuous  $\underline{z}_1(\Lambda)$  satisfying (8.40) and a unique continuous  $\underline{z}_2(\Lambda)$  satisfying (8.41).

*Proof.* Fix  $\Lambda \in [\Lambda_{min}, \Lambda_{max}]$ . By Lemma 10, there exist  $z_{min}$  and  $z_{max} = \max\{z_1, z_0\}$  such that  $z_\theta \in [z_{min}, z_{max}]$ . From (8.39), the right-hand side of (8.40) is continuous and strictly decreasing in  $\underline{z}$  on  $[z_{min}, z_{max}]$ , is  $\Lambda_{min}$  at  $\underline{z} = z_{min}$  and  $\Lambda_{max}$  at  $\underline{z} = z_{max}$ , and so, there exists a unique  $\underline{z}_1(\Lambda)$  satisfying (8.40). The right-hand side of (8.41) is continuous and strictly decreasing in  $\underline{z}$  on  $(0, z_{max}]$ , converges to infinity as  $\underline{z} \rightarrow \infty$ , and is zero at  $\underline{z} = z_{max}$ . Since

$\rho > 0$ , there exists a unique solution  $\underline{z}_1(\Lambda)$  to (8.41). The continuity of the solution follows from the continuity of the respective equations in  $\Lambda$ . *q.e.d.*

Restrict  $\Lambda$  to the interval  $[\Lambda_{min}, \Lambda_{max}]$ . Notice that I can let  $\underline{z}_1(\Lambda_{min}) = (0, z_{min}]$ , since for  $\underline{z} < z_{min}$ ,  $\sigma_\theta = 1$  for all  $\theta$  and so, any  $\underline{z} \in (0, z_{min}]$  is compatible with equilibrium in which  $\Lambda = \Lambda_{min}$ . Also  $\underline{z}_1(\Lambda_{max}) = z_{max}$  and  $\underline{z}_2(\Lambda) \in (0, z_{max}]$ . By the continuity of  $\underline{z}_1$  and  $\underline{z}_2$ , there exists a solution to (8.40)-(8.41).

**Proof of Proposition 1** For the linear model, equations (8.48) and (8.49) take form

$$\begin{aligned}\theta_{>}(x) &= 1 + \frac{r}{k}\alpha\xi \ln x_\theta + \frac{y_d}{k}(1-\alpha)\xi \frac{\Lambda}{\rho+\Lambda}(1-x_\theta), \text{ for } \theta > \theta^*, \\ \theta_{\leq}(x) &= -\frac{r}{k}(1-\alpha)\xi \ln x_\theta + \frac{y_d}{k}(1-\alpha)\xi \frac{\Lambda}{\rho+\Lambda}(1-x_\theta), \text{ for } \theta \leq \theta^*.\end{aligned}$$

Given these expressions, one can explicitly calculate

$$X \equiv \int_{\hat{\theta}}^1 x_\theta d\theta + \int_0^{\check{\theta}} x_\theta d\theta = \int_{\underline{x}}^1 x \frac{d\theta_{>}(x)}{dx_\theta} dx - \int_{\underline{x}}^1 x \frac{d\theta_{\leq}(x)}{dx_\theta} dx = \frac{r\xi}{k}(1-\underline{x}) \quad (8.42)$$

and

$$L = 1 - \hat{\theta} + \check{\theta} = -\frac{r\xi}{k} \ln \underline{x}. \quad (8.43)$$

Then equilibrium conditions (8.40) and (8.41) for  $\Theta_I \neq \phi$  become

$$\Lambda = \frac{\lambda y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\xi r} L} - 1 \right) - L \right) - (y_u + y_d), \quad (8.44)$$

$$L = \frac{y_u + y_d + \Lambda}{y_d \Lambda} \left( y_u(a-1) - (y_u + y_d) \frac{\Lambda}{\lambda} - h(\Lambda) \right); \quad (8.45)$$

where

$$h(\Lambda) = \int_0^1 \frac{(1-s)y_d}{1 + \frac{y_u + y_d}{\Lambda} - \left(1 - \frac{y_u + y_d}{\rho}\right) s} ds.$$

For the case when  $L = 1$  (and hence,  $\Theta_I = \phi$ ), equilibrium is given by  $\Lambda = \Lambda_{min}$

$$\Lambda_{min} \geq \frac{\lambda y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{\xi r}{k}} - 1 \right) - 1 \right) - (y_u + y_d). \quad (8.46)$$

To derive the comparative statics, denote by  $\Lambda_1(L)$ ,  $\Lambda$  as a function of  $L$  expressed from equation (8.44), and by  $\Lambda_2(L)$ ,  $\Lambda$  as a function of  $L$  expressed from equation (8.45). Function  $\Lambda_1$  is increasing and  $\Lambda_2$  is decreasing and so, equilibrium is unique.<sup>26</sup> Since  $\left( \frac{\xi r}{k} \left( e^{\frac{k}{\xi r} L} - 1 \right) \right)'_k =$

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<sup>26</sup>The right-hand of (8.44) is increasing in  $L$ , as  $\left( \frac{\xi r}{k} \left( e^{\frac{k}{\xi r} L} - 1 \right) - L \right)'_L = e^{\frac{k}{\xi r} L} - 1 > 0$ , and the right-hand side

$\frac{\xi r}{k^2} \left(1 + e^{\frac{k}{\xi r} L} \left(\frac{k}{\xi r} L - 1\right)\right) > 0$ ,  $\Lambda_1$  is increasing in  $k$  and so, an increase in  $k$  leads to the upward shift of  $\Lambda_1$  and as a result, to an increase in  $\Lambda$  and a decrease in  $L$ . By the same logic,  $\Lambda$  is decreasing in  $\ell$  and  $L$  is increasing in  $\ell$ . Finally, an increase in  $a$  leads to an increase in  $\Lambda_2$  and so, an increase in  $\Lambda$  and  $L$ . Since (8.44) and (8.45) do not depend on  $\alpha$ ,  $L$  is independent of  $\alpha$ .

To derive the comparative statics in  $\lambda$ , I express equilibrium conditions (8.44) and (8.45) in terms of variables  $L$  and  $M_{bu}(\phi)$  as follows

$$\begin{cases} M_{bu}(\phi) = \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\xi r} L} - 1 \right) - L \right) - \frac{y_u + y_d}{\lambda}, \\ L = \frac{(y_u + y_d)/\lambda + M_{bu}(\phi)}{y_d M_{bu}(\phi)} (y_u(a - 1) - (y_u + y_d)M_{bu}(\phi) - H(M_{bu}(\phi))); \end{cases} \quad (8.47)$$

where

$$H(M_{bu}(\phi)) = \int_0^1 \frac{(1-s)y_d}{1 + \frac{y_u + y_d}{\lambda M_{bu}(\phi)} - \left(1 - \frac{y_u + y_d}{\rho}\right) s} ds.$$

The right-hand side of the first equation in (8.47) is increasing in  $L$  and increasing in  $\lambda$ , while the right-hand side of the second equation in (8.47) is decreasing in  $M_{bu}(\phi)$  and decreasing in  $\lambda$ . Therefore, an increase in  $\lambda$  leads to a decrease in  $L$ . Taking the limits of the equations (8.47) one gets (4.4) and  $H = \frac{\rho}{r} \left(1 + \frac{y_u + y_d}{r} \ln \left(1 - \frac{r}{\rho}\right)\right) = \lim_{\lambda \rightarrow \infty} H(M_{bu}(\phi))$ .

**Proof of Proposition 2** In the proof of Theorem 2, I introduced two functions  $\underline{z}_1$  and  $\underline{z}_2$  whose intersection gives equilibrium  $\underline{z}$  and  $\Lambda$ . Observe that if  $\underline{z}_1$  is increasing in  $\Lambda$ , which is the case for the convex model (see Lemma 13 below), then equilibria with higher  $\Lambda$  also have higher  $\underline{z}$ . From (8.17) the utility of the buyer is increasing in  $\underline{z}$ , while from (8.19), given  $z_\theta$  and  $\sigma_\theta$ , the utility of the seller is increasing in  $\Lambda$ .

I next make several useful observations about the model with constant holding costs. First, recall that it is without loss of generality to take  $F$  as uniform. Second, in the model with constant holding costs only the liquidity  $x_\theta$  of the asset matters for the preferences of the buyer and so, I will analyze the equilibrium value of  $\underline{x}$  instead of  $\underline{z}$ . Third, for assets in  $\Theta_L$  differential equations (8.28) and (8.29) determining  $z_\theta$  can be integrated to get

$$\begin{aligned} g(\theta) &= 1 + \frac{r}{k} \alpha \xi \ln x_\theta + \frac{y_d}{k} (1 - \alpha) \xi \frac{\Lambda}{\rho + \Lambda} (1 - x_\theta), \text{ for } \theta > \theta^*, \\ g(\theta) &= -\frac{r}{k} (1 - \alpha) \xi \ln x_\theta + \frac{y_d}{k} (1 - \alpha) \xi \frac{\Lambda}{\rho + \Lambda} (1 - x_\theta), \text{ for } \theta < \theta^*, \end{aligned}$$

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of (8.45) is decreasing in  $\Lambda$ .



and define functions

$$g_{>}(x) = 1 + \frac{r}{k}\alpha\xi \ln x + \frac{y_d}{k}(1 - \alpha)\xi \frac{\Lambda}{\rho + \Lambda}(1 - x), \quad (8.48)$$

$$g_{<}(x) = -\frac{r}{k}(1 - \alpha)\xi \ln x + \frac{y_d}{k}(1 - \alpha)\xi \frac{\Lambda}{\rho + \Lambda}(1 - x). \quad (8.49)$$

The interpretation is that  $g^{-1}(g_{>}(x))$  gives the asset above  $\theta^*$  with liquidity  $x$  ( $/u g^{-1}(g_{<}(x))$  gives the asset below  $\theta^*$  with liquidity  $x$ ).

**Lemma 13.** *In the convex model,  $z_1(\Lambda)$  is strictly increasing.*

*Proof.* It is sufficient to show that the right-hand side of (8.40) is strictly decreasing in  $\Lambda$ . The term that depends on  $\Lambda$  in the right-hand side of (8.40) can be rewritten as

$$-\int_0^1 \frac{\Lambda\sigma_\theta}{y_u + y_d + \Lambda\sigma_\theta} d\theta = -\frac{\Lambda L}{y_u + y_d + \Lambda} - \int_{\check{\theta}}^{\hat{\theta}} \frac{\Lambda\sigma_\theta}{y_u + y_d + \Lambda\sigma_\theta} d(\theta) \quad (8.50)$$

which I show to be strictly increasing in  $\Lambda$ .

*Claim 6.*  $L$  is strictly increasing in  $\Lambda$  in the neighborhood of  $L < 1$ .

*Proof.* Using (8.48) and (8.49),

$$L = \check{\theta}(\underline{x}) + 1 - \hat{\theta}(\underline{x}) = 1 + g^{-1}(g_{<}(\underline{x})) - g^{-1}(g_{>}(\underline{x})).$$

Suppose that  $\Lambda$  increases. Then both  $g_{<}(\underline{x})$  and  $g_{>}(\underline{x})$  increase by the same amount. Since  $g$  is convex, and  $g_{<}(\underline{x}) \leq g_{>}(\underline{x})$ ,  $g^{-1}(g_{<}(\underline{x}))$  increases by a greater amount than  $g^{-1}(g_{>}(\underline{x}))$  and so,  $L$  increases with the increase in  $\Lambda$ . *q.e.d.*

*Claim 7.*  $\Lambda\sigma_\theta$  is increasing in  $\Lambda$

*Proof.* Since  $\check{\theta}$  is increasing in the nominator of  $\Lambda\sigma_\theta = \rho \left( \frac{y_d(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - (g(\theta) - g(\check{\theta}))}{y_d(1-\alpha)\frac{\rho}{\rho+\Lambda}\xi\underline{x} + g(\theta) - g(\check{\theta})} \right)$  is increasing in  $\Lambda$ , while the denominator is decreasing in  $\Lambda$ . Therefore,  $\Lambda\sigma_\theta$  is increasing in  $\Lambda$ . *q.e.d.*

Notice from (8.38) that

$$g(\underline{\theta}) = \min \left\{ g(\hat{\theta}), g(\check{\theta}) + \frac{y_d}{k}(1 - \alpha)\xi \frac{\Lambda}{\rho + \Lambda} \underline{x} \right\}.$$

I now prove the lemma. There are two cases to consider. First, suppose that  $\underline{\theta} < \hat{\theta}$ . Then the first term in (8.50) is decreasing in  $\Lambda$  by Claim 6, so it remains to prove that the second term is decreasing. It can be written as

$$-\int_{\check{\theta}}^{\hat{\theta}} \frac{\Lambda\sigma_\theta}{y_u + y_d + \Lambda\sigma_\theta} d\theta =$$

$$\begin{aligned}
& - \int_{\check{\theta}}^{\theta} \frac{(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g(\theta) - g(\check{\theta}))}{\frac{y_u+y_d}{\rho}\left((1-\alpha)\frac{\rho}{\rho+\Lambda}\xi\underline{x} + \frac{k}{y_d}(g(\theta) - g(\check{\theta}))\right) + (1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g(\theta) - g(\check{\theta}))} d\theta = \\
& - \int_{\check{g}}^{\check{g} + \frac{y_d}{k}(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x}} \frac{(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g - \check{g})}{\frac{y_u+y_d}{\rho}\left((1-\alpha)\frac{\rho}{\rho+\Lambda}\xi\underline{x} + \frac{k}{y_d}(g - \check{g})\right) + (1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g - \check{g})} \frac{dg}{g'(g^{-1}(g))} = \\
& \int_0^1 \frac{1-s}{1 + \frac{y_u+y_d}{\Lambda} - \left(1 - \frac{y_u+y_d}{\rho}\right) s} \frac{ds}{s} \varphi(s) \tag{8.51}
\end{aligned}$$

where  $\check{g} = g(\check{\theta})$ ,  $\varphi(s) = g'(g^{-1}(\check{g} + s\frac{y_d}{k}(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x}))$ , and in the second equality I use the change of variables  $g = g(\theta)$ , and in the third,  $s = \frac{g-\check{g}}{\frac{y_d}{k}(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x}}$ . Since (8.51) is increasing in  $\Lambda$  and so, the second term in (8.50) is decreasing in  $\Lambda$ .

Second, suppose that  $\underline{\theta} = \hat{\theta}$ . Then the second term in (8.51) can be written as

$$\begin{aligned}
& - \int_{\hat{\theta}}^{\hat{\theta}} \frac{\Lambda\sigma_{\theta}}{y_u + y_d + \Lambda\sigma_{\theta}} d\theta = \\
& - \int_{\check{\theta}}^{\hat{\theta}} \frac{(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g(\theta) - g(\check{\theta}))}{\frac{y_u+y_d}{\rho}(1-\alpha)\frac{\rho}{\rho+\Lambda}\xi\underline{x} + \frac{k}{y_d}(g(\theta) - g(\check{\theta})) + (1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g(\theta) - g(\check{\theta}))} d\theta = \\
& - \int_{\check{g}}^{\check{g} + 1 + \frac{\tau\xi}{k} \ln \underline{x}} \frac{(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g - \check{g})}{\frac{y_u+y_d}{\rho}\left((1-\alpha)\frac{\rho}{\rho+\Lambda}\xi\underline{x} + \frac{k}{y_d}(g - \check{g})\right) + (1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}(g - \check{g})} \frac{dg}{g'(g^{-1}(g))} = \\
& - \int_0^{1 + \frac{\tau\xi}{k} \ln \underline{x}} \frac{(1-\alpha)\frac{\Lambda}{\rho+\Lambda}\xi\underline{x} - \frac{k}{y_d}s}{\left(1 + \frac{y_u+y_d}{\rho}\right) (1-\alpha)\frac{\rho}{\rho+\Lambda}\xi\underline{x} - \left(1 - \frac{y_u+y_d}{\rho}\right) \frac{k}{y_d}s} \frac{ds}{g'(g^{-1}(s))} \tag{8.52}
\end{aligned}$$

where I use the change of variables  $g = g(\theta)$  in the second inequality, the change of variables  $s = g - \check{g}$  in the third inequality. Expression (8.52) is again decreasing in  $\Lambda$  and so, the second term in (8.50) is decreasing in  $\Lambda$ . □

## 8.5 Analysis of the Multi-class Model

*Proof of Theorem 3.* Under the assumption of the theorem, equilibrium quantities  $(\Lambda^1, \underline{z}^1)$  and  $(\Lambda^2, \underline{z}^2)$  are determined by the unique solution to the system (8.40)-(8.41) with  $a = a_1$  and  $a = a_2$ , respectively. The equations (8.40)-(8.41) are continuous in parameters and so, solutions  $(\Lambda^1, \underline{z}^1)$  and  $(\Lambda^2, \underline{z}^2)$  vary continuously with  $a_1$  and  $a_2$ . Moreover, an increase in  $a$  leads to an increase in the right-hand side of (8.40), and so to a decrease in  $\underline{z}$ . Denote by  $\underline{z}(a)$  the equilibrium threshold given that the mass of agents is  $a$ . Then  $a_1$  is determined by  $\underline{z}(a_1) = \underline{z}(a - a_1)$  which has a unique solution. □

Before proving Proposition 3, I first prove the following lemma giving the additional com-

parative statics for the model with endogenous bargaining.

**Lemma 14.** *If  $p = (k, a)$  is such that in the unique equilibrium,  $\Theta_I \neq \phi$ , then in some neighborhood of  $p$ ,  $\underline{x}$  is decreasing in  $k$  and  $a$ . If  $p = (k, a)$  is such that in the unique equilibrium,  $L = 1$  and condition (8.46) holds as a strict inequality, then in some neighborhood of  $p$ ,  $\Lambda_b$  is decreasing in  $a$ .*

*Proof.* I first formulate equilibrium conditions in terms of  $(\Lambda, \underline{x})$ ,

$$\begin{cases} \Lambda = \frac{\xi r \lambda y_d}{k \rho} \left( \frac{1}{\underline{x}} - 1 + \ln \underline{x} \right) - (y_u + y_d), \\ \ln \underline{x} = -\frac{k(y_u + y_d + \Lambda)}{r \xi y_d \Lambda} (y_u(a - 1) - (y_u + y_d) \frac{\Lambda}{\lambda} - h(\Lambda)). \end{cases}$$

Denote the solution to the first equation in the system by  $\Lambda = \gamma(\underline{x})$ , and the solution to the second equation by  $\Lambda = \zeta(\underline{x})$ . Observe that  $\gamma$  is a decreasing function and  $\zeta$  is an increasing function. An increase in  $k$  leads to a downward shift of  $\gamma$  and an upward shift of  $\zeta$  and so, to a decrease in  $\underline{x}$ . At the same time, an increase in  $a$  leads to an increase in  $\zeta$  and so, to a decrease in  $\underline{x}$ .

Now suppose that  $L = 1$  and condition (8.46) holds as a strict inequality. Then equilibrium  $\Lambda$  is given by (8.45) with  $L = 1$ , and so  $\Lambda$  is increasing in  $a$  in some neighborhood of  $p$ . From (3.2),  $\Lambda_b$  is decreasing in  $a$  in this neighborhood of  $p$   $\square$

*Proof of Proposition 3.* It follows from (5.1) that

$$\alpha \frac{r + y_u}{r} \xi \underline{x}^1 = V_{bu}^1(\phi) = V_{bu}^2(\phi) = \alpha \frac{r + y_u}{r} \frac{\Lambda_b^2}{\rho + \Lambda_b^2} \xi$$

and so,

$$\underline{x}^1 = \frac{\Lambda_b^2}{\rho + \Lambda_b^2}$$

and analogous condition holds for equilibrium after the shock. By Lemma 14, a decrease in  $a_2$  leads to an increase in  $\Lambda_b^2$ . Since  $\tilde{k}_i > k_i$ , it follows that  $\tilde{a}_1 \leq a_1$  and  $\tilde{a}_2 \geq a_2$ . Indeed, otherwise  $\tilde{a}_1 < a_1$  and  $\tilde{a}_2 < a_2$ , which by Lemma 14 implies  $\tilde{\underline{x}}^1 < \underline{x}^1$  and  $\tilde{\Lambda}_b^2 > \Lambda_b^2$  which contradicts the fact that the market is in equilibrium after the shock ( $\tilde{\underline{x}}^1 = \frac{\tilde{\Lambda}_b^2}{\rho + \tilde{\Lambda}_b^2}$ ). Therefore, by Proposition 1,  $\tilde{L}^1 \leq L^1$  and  $\tilde{\Lambda}_b^1 \geq \Lambda_b^1$ .  $\square$

## References

- [1] Atakan A., and M. Ekmekci (2014): “Bargaining and reputation in search markets,” *Review of Economic Studies*, 81, 1-29.
- [2] Amihud, Y., and H. Mendelson (1986): “Asset Pricing and the Bid-Ask Spread,” *Journal of Financial Economics*, 17, 223-249.

- [3] Ashcraft, A., P. Goldsmith-Pinkham, and J. Vickery (2010): “MBS Ratings and the Mortgage Credit Boom,” *Working Paper*.
- [4] Ausubel, L., P. Cramton, and R. Deneckere (2002): “Bargaining with Incomplete Information,” *Handbook of Game Theory with Economic Applications*, 3, 1897-1945.
- [5] Bao, J., J. Pan, and J. Wang (2011): “The Illiquidity of Corporate Bonds,” *Journal of Finance*, 66, 911-946.
- [6] Beber, A., M. Brandt, and K. Kavajecz (2009): “Flight-to-Quality or Flight-to-Liquidity? Evidence from the Euro-Area Bond Market,” *Review of Financial Studies*, 22, 925-957.
- [7] Benmelech, E., and J. Dlugosz (2010): “The Credit Rating Crisis,” *NBER Macroeconomics Annual 2009*, 24, 161-207.
- [8] Bessembinder, H., W. Maxwell, and K. Venkataraman (2006): “Market Transparency, Liquidity Externalities, and Institutional Trading Costs in Corporate Bonds,” *Journal of Financial Economics*, 82, 251-288.
- [9] Binmore, K., A. Rubinstein, and A. Wolinsky (1986): “The Nash Bargaining Solution in Economic Modeling,” *RAND Journal of Economics*, 176-188.
- [10] Brunnermeier, M. (2009): “Deciphering the Liquidity and Credit Crunch 2007-2008,” *Journal of Economic Perspectives*. 23, 77-100.
- [11] Chang, B. (2014): “Adverse Selection and Liquidity Distortions,” *Working Paper*, Wisconsin School of Business.
- [12] Chen, H., R. Cui, Z. He, and K. Milbradt (2014): “Quantifying Liquidity and Default Risks of Corporate Bonds over the Business Cycle,” *Working Paper*.
- [13] Chen, L., D. Leshmond, and J. Wei (2007): “Corporate Yield Spreads and Bond Liquidity,” *Journal of Finance*, 62, 119-149.
- [14] Constantinides, G. (1986): “Capital Market Equilibrium with Transactions Costs,” *Journal of Political Economy*, 94, 842-862.
- [15] Committee on the Global Financial System (2005), *The Role of Ratings in Structured Finance: Issues and Implications*, Basel, Switzerland: Bank for International Settlements.
- [16] He, Z., and K. Milbradt (2014): “Endogenous Liquidity and Defaultable Bonds,” *Econometrica*, 82, 1443-1508.
- [17] Nash, J. F. (1950): “The Bargaining Problem,” *Econometrica*, 18, 155-162.

- [18] Dick-Nielsen, J., P. Feldhutter, and D. Lando (2012): “Corporate Bond Liquidity before and after the Onset of the Subprime Crisis,” *Journal of Financial Economics*, 103, 471-492.
- [19] Downing, C., J. Dwight, and N. Wallace (2009): “Is the Market for Mortgage-Backed Securities a Market for Lemons?” *Review of Financial Studies*, 22, 2457-2494.
- [20] Duffie, D. (2012): *Dark Markets: Asset Pricing and Information Transmission in Over-the-Counter Markets*, Princeton University Press.
- [21] Duffie, D., N. Gârleanu, and L. Pedersen (2005): “Over-the-Counter Markets,” *Econometrica*, 73, 1815-1847.
- [22] Duffie, D., N. Gârleanu, and L. H. Pedersen (2007): “Valuation in Over-the-Counter Markets,” *Review of Financial Studies*, 20, 1865-1900.
- [23] Duffie, D. and Y. Sun (2007): “Existence of Independent Random Matching,” *Annals of Applied Probability*, 17, 386-419.
- [24] Edwards, A., L. Harris, and M. Piwowar (2007): “Corporate Bond Market Transaction Costs and Transparency,” *Journal of Finance*, 62, 1421-1451.
- [25] Ericsson, J., and O. Renault (2006): “Liquidity and Credit Risk,” *Journal of Finance*, 61, 2219-2249.
- [26] Friewald, N., R. Jankowitsch, and M. Subrahmanyam (2012): “Illiquidity or Credit Deterioration: A Study of Liquidity in the US Corporate Bond Market during Financial Crises,” *Journal of Financial Economics*, 105, 18-36.
- [27] Fuchs, W., and A. Skrzypacz (2013), “Bridging the Gap: Bargaining with Interdependent Values,” *Journal of Economic Theory*, 148, 1226-1236.
- [28] Guerrieri, V., and R. Shimer (2014): “Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality,” *American Economic Review*, forthcoming.
- [29] Heaton, J., and D. Lucas (1996): “Evaluating the Effect of Incomplete Markets on Risk Sharing and Asset Pricing,” *Journal of Political Economy*, 104, 443-487.
- [30] Huang, M. (2003): “Liquidity Shocks and Equilibrium Liquidity Premia,” *Journal of Economic Theory*, 109, 104-129.
- [31] International Monetary Fund (2008): *Global Financial Stability Report*, Washington.
- [32] Lagos, R., and G. Rocheteau: “Search in Asset Markets: Market Structure, Liquidity, and Welfare,” *American Economic Review*, 198-202.

- [33] Lagos, R., and G. Rocheteau (2009): “Liquidity in Asset Markets with Search Frictions,” *Econometrica*, 77, 403-426.
- [34] Lauer mann, S., and A. Wolinsky (2014): “Search with Adverse Selection,” *Working Paper*.
- [35] Lewis, M., (2011): *The Big Short: Inside the Doomsday Machine*, WW Norton & Company, New York.
- [36] Lo, A., H. Mamaysky, and J. Wang (2004): “Asset Prices and Trading Volume under Fixed Transactions Costs,” *Journal of Political Economy*, 112, 1054-1090.
- [37] Longstaff, S., S. Mithal, and E. Neis (2005): “Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Default Swap Market,” *Journal of Finance*, 50, 2213-2253.
- [38] Milgrom, P. (2004): *Putting Auction Theory to Work*, Cambridge University Press.
- [39] Milgrom, P., and I. Segal (2002): “Envelope Theorems for Arbitrary Choice Sets,” *Econometrica*, 70, 583-601.
- [40] Mitchell, M., and T. Pulvino (2012), “Arbitrage Crashes and the Speed of Capital,” *Journal of Financial Economics*, 104, 469-490.
- [41] Nash, J. (1950): “The Bargaining Problem,” *Econometrica*, 155-162.
- [42] Pagano, M. and P. Volpin (2012): “Securitization, Transparency, and Liquidity,” *Review of Financial Studies*, 25, 2417-2453.
- [43] Rothschild, M., and J. E. Stiglitz (1976): “Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information,” *Quarterly Journal of Economics*, 90, 629–649.
- [44] Rubinstein, A., (1982): “Perfect Equilibrium in a Bargaining Model,” *Econometrica*, 97-109.
- [45] Rubinstein, A., and A. Wolinsky (1985), “Equilibrium in a Market with Sequential Bargaining,” *Econometrica*, 53, 1133-1150.
- [46] Satterthwaite, M., and A. Shneyerov (2007), “Dynamic Matching, Two-Sided Incomplete Information, and Participation Costs: Existence and Convergence to Perfect Competition,” *Econometrica*, 75, 155-200.
- [47] Saunders, A., A. Stinivasan, and I. Walter (2002): “Price Formation in the OTC Corporate Bond Markets: a Field Study of the Inter-dealer Market,” *Journal of Economics and Business*, 54, 95-113.

- [48] Tsoy, A. (2014): “Alternating-offer Bargaining with a Global Games Information Structure,” *Working Paper*, MIT.
- [49] Vayanos, D. (1998): “Transaction Costs and Asset Prices: A Dynamic Equilibrium Model,” *Review of Financial Studies*, 11, 1-58.
- [50] Vayanos, D., and T. Wang (2007): “Search and Endogenous Concentration of Liquidity in Asset Markets,” *Journal of Economic Theory*, 136, 66-104.
- [51] Vayanos, D., and P. Weill (2008): “A Search-Based Theory of the On-the-Run Phenomenon,” *Journal of Finance*, 63, 1361-1398.
- [52] Weill, P. (2008): “Liquidity Premia in Dynamic Bargaining Markets,” *Journal of Economic Theory*, 140, 66-96.