The Structural Transformation of Innovation

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We document the structural transformation of innovation using historical patent data since the 1850s, along with R&D expenditure and TFP growth for the post-war period. Over time, innovation has shifted from agricultural sectors to manufacturing, and, more recently, to services. We develop and quantify a multi-sector semi-endogenous growth model of structural change in innovation and production, incorporating the classical demand-pull and technology-push drivers of innovation. Sectors differ in their innovation technologies, and the extent to which they benefit from knowledge spillovers (technology-push). Nonhomothetic demand shifts the market shares toward income-elastic sectors along the growth process (demand-pull). A calibrated version of our model replicates the structural transformations of innovation and production observed in the US data. Using the model, we evaluate the future impact of Baumol's disease on aggregate productivity and find it to be minimal. Our results suggest that aggregate productivity growth may recover in the coming decades as the service sector becomes increasingly innovation-driven.

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1. Introduction

Modern economic growth drives transformative shifts in output and employment across sectors, transitioning economies from agriculture to manufacturing and then to services. This structural transformation has significant implications for productivity growth. Historically, agriculture and manufacturing have achieved faster rates of productivity growth than services, with manufacturing particularly distinguished by its intensive innovation—it employs the most R&D workers and produces the highest patent volumes. Services, by contrast, generally demonstrate slower productivity growth and lower innovation intensity. As services dominate modern economies, this trend risks slowing aggregate productivity growth—a concern encapsulated by Baumol's cost disease [Baumol, 1967].[1]

This paper contributes three key findings to this debate. First, using three measures of innovation activity—patent filings, R&D expenditures, and sectoral TFP growth—we reveal a structural shift in innovation activity from agriculture to manufacturing, and finally to services. Patent data reflect agriculture's prominent role in 19th-century innovation, manufacturing's rise during 20th-century industrialization, and services' recent ascendancy. Post-war R&D spending similarly shifted from manufacturing to services, accelerating after the 1970s. Sectoral TFP growth patterns reinforce this trajectory: the agriculture-manufacturing gap narrowed steadily, while services—once lagging—began a process of catch up by the 1980s and achieved the fastest growth in the past decade. Crucially, the reallocations in innovation activity parallel shifts in production shares, implying that market size drives where innovation occurs.

Our second and central contribution is to develop a quantitative framework unifying structural transformations in innovation and production. We build a multi-sector *semi*-endogenous growth model with two key features: (1) nonhomothetic preferences, which generate time-varying sectoral market sizes via heterogeneous income elasticities of demand, incentivizing innovators to follow expanding markets (*demand-pull* forces à la Schmookler, 1966); and (2) sectorally heterogeneous idea production functions, where differences in innovation efficiency and cross-sector idea applicability steer technical progress toward sectors with higher returns (*technology-push* forces). These dual mechanisms jointly determine the sectoral allocation of R&D, knowledge accumulation, and productivity growth. Productivity dynamics then shape aggregate income and relative prices, which—through income and price effects—drive the sectoral composition of value-added. To our knowledge, this is the first unified framework quantifying how demand-pull and technology-push interact to explain structural change.

We characterize the evolution of output and R&D intensity across sectors, demonstrating convergence to a constant-growth equilibrium path. Critically, while the growth rate along this path depends on both knowledge spillover elasticities and income elasticities of demand, long-run sectoral innovation gaps are determined solely by income elasticities of demand. High income-elastic sectors (e.g., services) attract disproportionate innovation as rising in-

[1] Baumol [1967] coined the term "cost disease" to describe chronic productivity stagnation in personal services, while Gordon [2016] argues that transformative, early-20th-century manufacturing innovations (e.g., electrification, mass production) were historically exceptional and unlikely to be replicated by modern ICT advancements. Clark [2016] links these dynamics to broader structural transformation trends, reinforcing Gordon's skepticism about future productivity growth.

comes expand their market share, raising returns to R&D and accelerating their productivity growth. This process—where demand-driven market expansion fuels sector-specific technical progress—ultimately governs productivity trajectories in the long run.

Our specification of the ideas production function builds on and extends Kortum [1997] to a multi-sector environment (cf. Buera and Oberfield, 2020 for a multi-country analog). This structure links observable R&D inputs, patents, and cross-sector citations to unobserved variables like sectoral knowledge stocks and spillovers. We leverage this relationship to estimate idea production parameters—including sector-specific spillover elasticities (see Bloom et al., 2020 for aggregate parallels)—and calibrate the model using micro-level preference estimates. Finally, we minimize deviations between empirical and simulated sectoral trajectories (value added, R&D, productivity) across the transition path to pin down the remaining model parameters.

Our quantified model replicates postwar U.S. structural shifts in both innovation and production: services' rise in value-added, patents, and R&D displaces agriculture and manufacturing. These shifts depress TFP growth in agriculture/manufacturing while only modestly boosting services. Services' muted TFP response, despite absorbing substantial R&D, stems from weak innovation spillovers (low elasticity of new ideas to past knowledge). In other words, technology-push forces in services limit the effects of demand-pull forces. Consequently, aggregate productivity slows due to both sectoral reallocation and uneven TFP dynamics. Yet, using the model to make future projections suggests that the TFP slowdown will stabilize by the 2020s, with growth recovering to ~0.9% by 2100 as innovation matures in a service-dominated economy. This rebound challenges Baumol's stagnation thesis, suggesting that service-led growth can revive productivity.

Counterfactual experiments examine how technology-push and demand-pull forces shape structural change. Strengthening the elasticity of new ideas to past innovations in services (technology-push) accelerates both structural transformation and long-run TFP growth, akin to one-sector semiendogenous growth dynamics [Jones, 2005]. Conversely, limiting cross-sector knowledge spillovers suppresses agriculture/manufacturing productivity and aggregate TFP. Uniform R&D efficiency gains boost TFP and accelerate structural change, but service-specific gains yield weaker effects due to offsetting income and price effects. Weakening services' nonhomotheticity (demand-pull) initially curbs innovation towards manufacturing but also makes agents less willing to postpone consumption, accelerating transformation. In sum, our counterfactual experiments suggest that both technology-push and demand-pull forces play important and complementary roles in shaping structural transformation and aggregate productivity growth.

Related Literature Our work bridges structural transformation [Herrendorf et al., 2014] and directed technical change literatures.[2] We extend the findings of the former to document the sectoral trends in innovation investments and patenting. Our model endogenizes sectoral productivity dynamics, which are often treated as exogenous in prior structural models (e.g., Kongsamut et al., 2001, Ngai and Pissarides, 2007, Acemoglu and Guerrieri, 2008, Boppart,

[2]Earlier links between innovation and structural change [Pasinetti, 1981, Quatraro, 2009] lacked general equilibrium foundations.

2014, Comin et al., 2021). Recent work in this literature has studied the implications of structural change on aggregate growth with exogenous sectoral productivity trends [Duarte and Restuccia, 2010, Duernecker et al., 2024].

Some recent work endogenizes industry-level productivity growth assuming *fixed* production shares, emphasizing technology-push forces. Ngai and Samaniego [2011] allow industry-specific ideas production functions assuming all spillovers are within industry. Liu and Ma [2021] further incorporate cross-industry spillovers akin to ours. Here, we structurally estimate industry-specific ideas production functions from data (via theory-derived equations) rather than calibrating them externally—strengthening the theory-data link.[3] Unlike these studies, we also integrate demand-pull forces to endogenize sectoral production shares.

Schmookler [1966] pioneered the idea of demand pull as a driver of the direction of innovation, showing industry-level links between market size and patenting [Griliches and Schmookler, 1963, Scherer, 1982]. However, these early studies lacked general equilibrium foundations, relying on observed correlations rather than causal identification. Modern work addresses this concern: Acemoglu and Linn [2004] exploited pharmaceutical demand shocks, finding innovation responses in drug development but not patents, while Cerda [2007], Jaravel [2017] and Argente and Lee [2021] used barcode-level data to demonstrate that product innovation disproportionately targets high-income consumers.

Several models integrate demand-pull forces with structural change, but critical differences remain. Matsuyama [2002], and more recently Oberfield [2023], have built models in which rising incomes lead to productivity growth in income-elastic industries due to learning-by-doing. Boppart and Weiss [2013] model a two-sector economy (durables/non-durables) where income-elastic demand drives innovation. In their framework, unlike in ours, the sectoral bias of innovation is transitory and long-run sectoral productivity growth remains exogenous. Herrendorf and Valentinyi [2022] study the direction of innovation in a two-sector model (services and goods) where demand-pull forces are only driven by relative prices due to homotheticity. Therefore, long-run productivity growth is solely determined by the technology-push forces. In contrast to these theories, ours is a quantitative, multi-sector model that exhibits rich dynamics both in the short and long run, which we directly discipline by the observed trends in the data.

Our work extends the directed technical change literature by shifting focus from factor- or sector-biased innovation to structural transformation. Early work emphasized factor bias (e.g., skilled vs. unskilled labor) in single-sector models, where market-size effects stemmed from fixed endowments [Acemoglu, 2002, 2007]. More recent studies examine sector bias in climate applications (e.g., clean vs. dirty technologies; Acemoglu et al., 2012, 2023), retaining a two-sector lens. We instead analyze multi-sector structural change, integrating demand-pull (market-size) and technology-push (spillover) forces to explain historical innovation reallocations across agriculture, manufacturing, and services—a dynamic absent in factor- or sector-specific frameworks.

[3] Estimating idea production functions has deep roots [Cohen, 2010]. Early studies proxy spillovers using aggregate "spillover pools" of external R&D [e.g., Bernstein and Nadiri, 1989]. Recent work by Bloom et al. [2013] disentangle knowledge spillovers from product market rivalry, finding spillovers dominate.

Outline The paper is organized as follows. Section 2 documents the structural transformation of innovation. Section 3 presents the model and characterizes its equilibrium dynamics along the transitional path and in the long run. Section 4 describes our calibration strategy and presents our quantitative results to study the drivers of sectoral innovation and productivity growth over the last 60 years, and to project sectoral and aggregate growth over the next century. Section 5 concludes the paper. Most proofs are contained in Appendix A.2.

2. Motivating Facts

This section documents patterns of structural change in innovation. We examine sectoral trends across three key dimensions: innovation inputs, captured by R&D expenditures; innovation outputs, measured by patent filings; and the final outcomes of innovation activity, reflected in sectoral TFP growth.

Our analysis is centered on the standard three-sector division of the economy: agriculture, manufacturing, and services, using the US as benchmark. We observe diverging trends in innovation inputs and outputs over time. Innovation inputs and outputs shares closely follow the evolution of sectoral value added and employment—declining in agriculture, rising in services, and exhibiting a hump-shaped pattern in manufacturing. TFP growth rates vary across sectors but show signs of convergence, especially between manufacturing and services. These empirical patterns also inform the calibration of our theoretical model. Due to data availability constraints, our three measures span different periods. Patent data spans the longest period, extending from the mid-nineteenth century to the present, while R&D expenditures and sectoral TFP growth data are available only for the post-World-War-II period.

2.1. Evidence on Innovation Output: Patent Filings

We use the Comprehensive US Patent (CUSP) database [Berkes, 2018], a near-complete coverage of US patent activity from 1855 to 2010.[4] Each patent is categorized by its patent class, and we map these classes to three broad sectors: agriculture, manufacturing, and services.

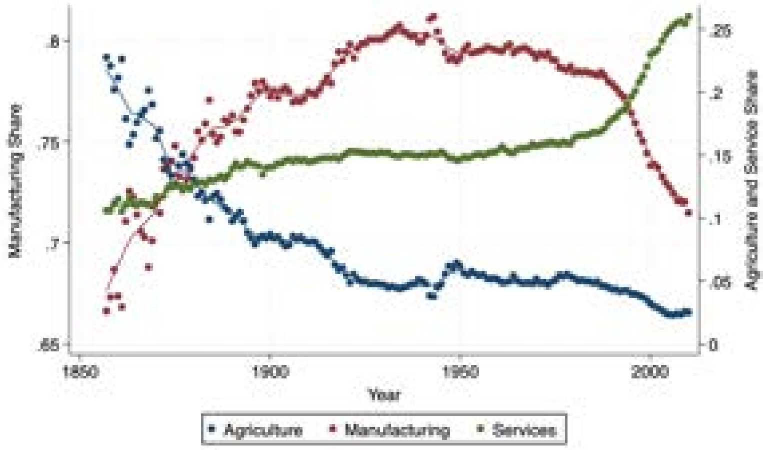
While established crosswalks exist to link patent classes to agricultural and manufacturing industries, there is little guidance for mapping patents to the service sector. To address this gap, we proceed in two steps. First, we use the crosswalk developed by Goldschlag et al. [2017] and used, among others, by Kelly et al. [2018] to assign patents classes to agriculture and manufacturing. Second, we use the Compustat database to calculate the probability that a patent class corresponds to the service sector. To this end, we use the information from Compustat about each firm's patent filings and primary industry to compute the share of patents in each class that originate from firms in service industries.

Figure 1 shows that the evolution of the patent shares follows a pattern similar to the well-known patterns for employment and output. The agricultural share is decreasing , while the manufacturing share is hump-shaped and the services share is increasing.[5] The leading broad

[4] We have 8,116,643 unique patents with corresponding patent classes and granting (or issue) years between 1855 and 2010.

[5] These patterns are in line with the technological waves described in Berkes [2018].

Figure 1: US Patent Shares, 1855-2010



Note: This figure plots the yearly share of granted patents to each broad sector. Data are constructed as described in the text.

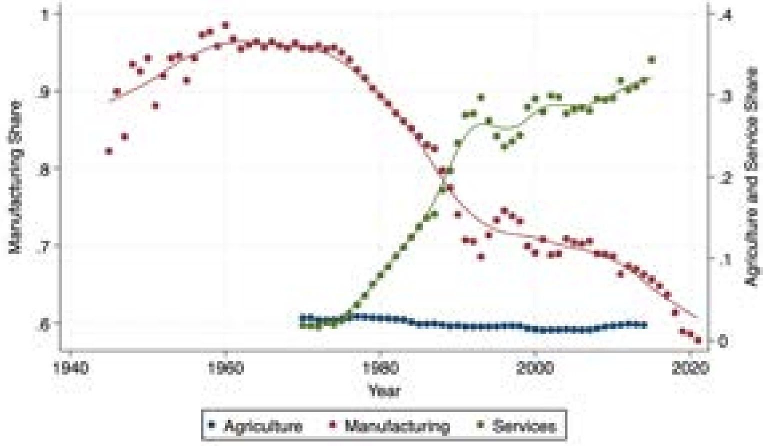
patent class in the US (as a share of total patents) between 1866 and 1875 corresponds to the category "A01 Agriculture; Forestry; Animal husbandry; hunting; trapping; fishing." During the first half of the twentieth century, we observe a surge of transportation and engineering-related patent classes. For example, between 1906 and 1945 the leading broad class is "F16 Engineering Elements or Units". In the last decades of the sample there is a clear shift towards innovations that load relatively more on the service sector. For example, the top broad patenting classes after World War II are "A61 Medical or Veterinary Science; Hygiene" and "G06 Computing; Calculating; Counting." The peak manufacturing share occurs before 1950, around a decade earlier than for value-added shares.[6]

Robustness Checks Our baseline crosswalk from patent classes to sectors assigns patents to sectors based on the sector where the patents originate. A potential concern is that innovations are often applied in sectors different from those where they originate, e.g., fertilizers may be developed in manufacturing but they are used in agriculture. To address this, we use the concordance project correspondence created by the US Patent and Trademark Office and the Commerce Data Service that uses natural language processing to probabilistically match patent classes to industry codes based on text similarity, so as to assign patents to their potential uses. We show in Appendix D that the same structural change pattern emerges with this alternative correspondence.

Appendix D additionally shows that if we directly assign all patents belonging to the patent class that corresponds to agriculture (IPC A01) rather than using our baseline match for this class, we obtain very similar results to our baseline. We also show using the Patstat database from the European Patent Office that France, Great Britain, Germany and Japan follow similar patterns for the sectoral evolution of patents.

[6] Fort et al. [2025] match firm patents to US firms using the US Census Establishment data since 1977. They document a more substantial decline in manufacturing patenting from the 1970s until the 2010. Manufacturing went from representing 78% of all patenting activity to 30%.

Figure 2: Evolution of the Sectoral Shares of R&D Spending



Note: This figure plots the yearly share of private R&D to each broad sector. Data are constructed as described in the text.

2.2. Evidence on Innovation Input: R&D Expenditures

We next document structural changes in the sectoral composition of R&D expenditure in the US. We use data on total private R&D expenses in the post-war period from the fixed asset tables compiled by the US Bureau of Economic Analysis. The fixed asset tables provide a breakdown of R&D spending into manufacturing and non-manufacturing categories. To disaggregate non-manufacturing expenditures into agriculture and services, we use the US Department of Agriculture's series on private R&D in agricultural industries, which cover the period 1970-2015. Figure 2 plots the evolution of the sectoral private R&D shares. The evolution of the manufacturing share in R&D is hump-shaped, while the share devoted to services rises steadily over time since the 1970s. By 1970, the share devoted to agriculture is very small (2.7%), but it continues to decrease and by 2015 it is 1.8% of total private R&D. Therefore, we conclude that, as with patents, there was a reallocation in the share of R&D expenditures from manufacturing to services during the post-war period.

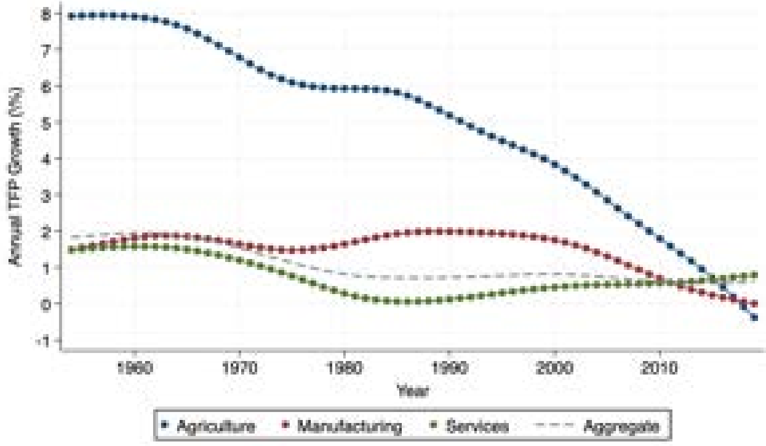
Robustness checks Appendix D shows that the pattern of structural change in R&D is robust to (i) including software investment as part of R&D expenditures, (ii) using NSF data for R&D expenditures and (iii) using data for US publicly traded firms from Compustat.

2.3. Evidence on Sectoral TFP

Since innovation is widely regarded as a key driver of productivity growth, shifts in the sectoral composition of innovation are expected to influence the relative evolution of productivity across sectors. Next, we show that this prediction is consistent with the US evidence. We use data from the BLS to consistently measure sectoral TFP growth based on value-added production functions. For our baseline results, we divide the US private economy into agriculture, manufacturing and services and obtain the sectoral TFP growth for the period 1954-2019.[7]

[7] TFP growth in sector i measures the increases in real output Y_{it} not accounted for by an increase in the combined inputs X_{it}, that is, $g_{it} = \Delta \ln Y_{it} - \Delta \ln X_{it}$. We follow the BLS in constructing the quantity indices of inputs by

Figure 3: Sectoral and Aggregate TFP Growth



Note: The plot shows the trends in sectoral and aggregate trends in TFP after filtering out the cyclical component using a Hodrick-Prescott filter with smoothing parameter $\lambda = 1600$. Data are calculated using value-added sectoral and aggregate production functions as described in the text.

We eliminate high-frequency fluctuations in TFP growth, extracting the low-frequency trends with the Hodrick-Prescott filter ($\lambda = 1600$). Figure 3 plots the evolution of the low-frequency growth rate of TFP in the three sectors. As expected, the average TFP growth rate differs substantially across sectors. It is highest in agriculture (5.0%) and lowest in services (0.7%), with manufacturing standing in between (1.5%). There is a downward trend in the rates of productivity growth in all three broad sectors and also aggregate TFP, which fell from 1.9% in the1950s to 0.6% by 2019 (see figure 16 in the appendix). The largest decline in the sectoral TFP growth is in agriculture (-8.3%), the second in manufacturing (-1.5%) and the smallest is in services (-0.7%), . As a result, the gap in productivity growth between the three broad sectors of the economy has steadily declined and even reversed during the last decade.[8]

An important aggregate consequence of the shift of value added from high-productivity-growth sectors (agriculture and manufacturing) toward services is a slowdown in aggregate TFP growth—Baumol's cost disease. In Appendix D.4, we decompose the change in aggregate TFP growth in two components: the contribution of sectoral reallocation (Baumol's cost disease) and the contribution of changes in sectoral TFP growth. We find that Baumol's cost disease plays a significant role in the first decades of the postwar period, reducing growth by up to 0.37%. Over time, however, as TFP growth declines in agriculture and manufacturing and improves in services, the reallocation effect wans and slower within-sector productivity growth becomes the primary driver of the aggregate TFP slowdown.

sector and the use of the sectoral deflators. Our measures of sectoral productivity cover the private domestic economy and thus exclude the government. We discuss in the appendix the data construction procedure in detail.

[8]Figure 17 in the appendix shows a similar convergence pattern for the farm and non-farm sectors of the US private economy.

Table 1: Model Environment and Key Equilibrium Conditions

	Variable	Definition	Key Equilibrium Equation
Preferences			
Intertemporal Utility	$U(t)$	$U(t) \equiv \int_0^\infty e^{-(\rho-\eta)\tau} \frac{C(t+\tau)^{1-\vartheta}-1}{1-\vartheta} d\tau$	Euler Equation (10)
Intratemporal Consumption	$C(t)$	$\sum_{i=1}^{I} \Xi_i^{\frac{1}{\sigma}} \left(\frac{C_i(t)}{C^{\epsilon_i}(t)}\right)^{\frac{\sigma-1}{\sigma}} = 1$	Sectoral Demand (6)
Production			
Sector-Level Output	$Y_i(t)$	$Y_i(t) = \exp\left(\int_0^1 \log X_{iv}(t)\, dv\right)$	Sectoral Price Index (26)
Variety-Level Intermediate	$X_{iv}(t)$	$X_{iv}(t) = Q_{iv}(t) L_{iv}(t)$	$Q_{iv}(t) \sim F_i(Q,t) \equiv \exp\left(-K_i(t)\, Q^{-\theta}\right)$
Innovation			
Sector-Level R&D Labor	$Z_i(t)$	$\dot{K}_i(t) = \Gamma_i Z_i(t)^{1-\alpha} S_i(t)^{\beta_i}$	Innovation Arbitrage (22)
General Equilibrium			
			Resource Constraint (28)

3. Model

Labor is the only factor of production and is inelastically supplied by households. Households consume goods produced by a fixed set of I distinct industries, which we may interchangeably refer to as sectors. Goods in each industry are produced by perfectly competitive producers who use as their inputs a set of industry-specific intermediate goods.

In the innovation side of the economy, R&D firms hire workers and direct their research and development efforts to a given industry. They develop and patent new ideas leading to improvements in the quality of intermediate goods used in that specific industry. Households save by investing in the equity shares of the firms created by these R&D firms. This simple setup allows us to focus our attention on how the market size implied by preference nonhomotheticities determines the direction of firms' R&D efforts.

Notation. We denote the logarithms of variables with lower case letters, for instance, $p_i(t)$, $q_i(t)$, and $y_i(t)$ denote logarithms of $P_i(t)$, $Q_i(t)$, and $Y_i(t)$, respectively. The only exception to this rule is the real interest rate, which we denote by $r(t)$ to maintain the notation most familiar for the readers. We denote vectors and matrices with bold face notation, for instance $\boldsymbol{p}(t)$ is a vector with elements $[p_i(t)]_{i=1}^{I}$. We utilize the dot notation to indicate time derivatives, e.g., $\dot{p}_i(t) \equiv \frac{d}{dt} \log P_i(t)$. Throughout, we occasionally drop the time argument (t) to simplify notation whenever the dependence on time is evident based on the context.

3.1. Environment

3.1.1. Household Preferences

The mass $L(t)$ of households grows at a constant rate η, such that $\dot{\ell}(t) = \eta$. Household preferences follow the standard dynastic intertemporal utility function

$$U(t) \equiv \int_0^\infty e^{-(\rho-\eta)\tau} \frac{C(t+\tau)^{1-\vartheta}-1}{1-\vartheta} d\tau, \tag{1}$$

where $C(t)$ aggregates a bundle of sectoral consumption goods $\boldsymbol{C}(t) \equiv \{C_i(t)\}_{i=1}^{I}$, according to the implicitly defined function

$$\sum_{i=1}^{I} \Xi_i^{\frac{1}{\sigma}} \left(\frac{C_i(t)}{C^{\epsilon_i}(t)} \right)^{\frac{\sigma-1}{\sigma}} = 1. \tag{2}$$

In Equation (2), $\epsilon_i > 0$ is a parameter specifying the income elasticity of demand for sector i, while $\Xi_i > 0$ is a sector specific taste parameter and σ is the elasticity of substitution. Throughout, we assume $\rho > \eta$.

The aggregator in Equation (2) belongs to the family of nonhomothetic CES preferences (Hanoch, 1975, Sato, 1975, Comin et al., 2021). This aggregator has the unique property that it allows for heterogeneity in income elasticities of sectoral goods for all levels of income, while maintaining the constancy of elasticity of substitution as in the standard CES preferences. Moreover, as we will see below, these preferences give rise to a log-linear demand system that closely parallels recent empirical work documenting robust variations in the income elasticity of demand across sectors [see Young, 2012, 2013, Aguiar and Bils, 2015].[9]

3.1.2. Production

Production in each sector involves two groups of producers. First, a continuum of monopolistically competitive firms produce sector-specific intermediate goods. Second, perfectly competitive final good producers at each sector combine sector-specific intermediate goods to produce output consumed by the households.

Each sector has a unit continuum of varieties of intermediate goods. Final good producers in sector i produce the sectoral output using the production function

$$Y_i(t) \quad = \quad \exp\left(\int_0^1 \log X_{iv}(t)\, dv \right), \tag{3}$$

where $X_{iv}(t)$ is the sector-specific intermediate variety v. The demand for the intermediate good v in sector i is given by $Y_{iv}(t) = P_i(t)\, Y_i(t) / P_{iv}(t)$, where the price index for sector i goods satisfies $\log P_i(t) = \int_0^1 \log P_{iv}(t)\, dv$.

As we will discuss next, there is a time-varying distribution $F_i(Q,t)$ of frontier techniques for production across different varieties of intermediates in each sector i. The firm that has monopoly rights over the frontier technique Q_{iv} in variety v at each point in time produces with production function $X_{iv} = Q_{iv} L_{iv}$. Once a new production technique Q_{iv} arrives in variety v, a slightly less productive vintage with productivity Q_{iv}/χ, with $\chi > 1$, becomes known by all potential fringe firms. Assuming Bertrand competition with these fringe firms, the frontier firm charges a markup χ and sets its price to be $P_{iv}(t) = \chi / Q_{iv}$. Combining everything together, we find the following expression that gives us the price index in sector i at time t:

$$\log P_i(t) = \log \chi - \int \log Q\, dF_i(Q;t). \tag{4}$$

[9] Different features of nonhomothetic preferences have been extensively discussed by Comin et al. [2021].

Throughout we normalize the wage rate to be the numeraire $W(t) \equiv 1$.

3.1.3. Innovation, Patenting, and Sectoral Productivity Growth

R&D firms hire workers to pursue research and develop new ideas in specific sectors. If R&D firm f hires $Z_{if}(t)$ workers to perform R&D, it generates new ideas at a flow rate $\widetilde{\Gamma}_i Z_i(t)^{-\alpha} Z_{if}(t)$, where $Z_i(t)$ denotes the total mass of R&D workers hired by all R&D firms in sector i, that is, $Z_i(t) \equiv \int Z_{if}(t)\, df$. The term $Z_i(t)^{-\alpha}$ stands for a congestion that induces decreasing returns to R&D at the sectoral level.

Each new idea in sector i is combined with an existing technique to produce a novel technique with a productivity Q'. The productivity Q' of this novel technique reflects the productivity of the new idea, Q_n, and of the existing technique, Q_o, as follows: $Q' = Q_n \times Q_o^{\beta_i}$. The parameter $\beta_i \in [0,1)$ reflects decreasing returns to adoption spillovers in sector i. We assume that the productivities of new ideas Q_n follow the same Pareto distribution, $\mathbb{P}(Q_n > Q) = (Q/\underline{Q})^{-\theta}$ in all sectors, where the lower bound $\underline{Q}$ asymptotically approaches zero. We also assume that the shifter of the firm-level innovation production function $\widetilde{\Gamma}_i$ grows proportionally to $\underline{Q}^{-\theta}$so that the product $\widetilde{\Gamma}_i \times \underline{Q}^{\theta}$ remains finite for all sectors i.

To find the existing technique that will be combined with the new idea to produce the new technique, an R&D firm in sector i randomly draws one frontier technique $\widetilde{Q}_{oj}$ from each sector j, and chooses the technique with highest productivity adjusted by the cost of applying a technique from sector j to sector i, Φ_{ij}. That is,

$$Q_o \equiv \max_j \left\{ \frac{\widetilde{Q}_{oj}}{\Phi_{ij}} \right\}, \tag{5}$$

where the applicability cost $\Phi_{ij} \geq 1$ for all i, and where we have $\Phi_{ii} \equiv 1$.

The R&D firm that successfully comes up with a new technique in a given variety takes over the market if it improves upon the current frontier technique,[10] and patents it if the improvement is by a margin greater or equal to $\Psi_i(t) \geq 1$, which can potentially vary over time. Finally, we assume that the patent corresponding to the new technique Q' cites the patent corresponding to the adopted technique $\widetilde{Q}_{o,j} \equiv \Phi_{ij} Q_o$ that has inspired the innovation.

Table 1 summarizes the different building blocks of the environment of the model. We will next discuss the equilibrium allocations in each block.

[10] To be more precise, we need to specify what happens if the improvement between the new frontier technique Q' and the current frontier technique Q is less than the multiplicative gap χ between the new frontier technique Q' and the technique Q'/χ that becomes costlessly available to the fringe firms. We assume that each potential frontier firm has to pay a negligible cost to compete in the market. The current frontier technology anticipates that the new frontier firm will be able to take over the market by charging a price slightly below the marginal cost $1/Q$. As such, it exits the market but its technique Q does not become available to the fringe firms, which can only operate with technique Q'/χ.

3.2. Characterization of the Equilibrium Allocations

3.2.1. Sectoral Composition of Demand and Consumption/Savings Decisions

Solving the expenditure minimization problem for the households, we find that sectoral expenditure shares of households follow

$$\Omega_i\left(t\right) \equiv \frac{P_i(t)C_i(t)}{E(t)} = \Xi_i \left(\frac{P_i(t)}{E(t)} C\left(t\right)^{\epsilon_i}\right)^{1-\sigma}, \tag{6}$$

where $E\left(t\right)$ indicates the total expenditure, which is a function of real consumption $C\left(t\right)$ and sectoral prices. This function is defined by

$$E\left(t\right) = \mathcal{E}\left(C\left(t\right), \boldsymbol{P}\left(t\right)\right) \equiv \left(\sum_i \Xi_i \left(P_i\left(t\right) C\left(t\right)^{\epsilon_i}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \tag{7}$$

Equations (6) and (7) characterize the optimal allocation of consumption across goods from different sectors for aggregate real consumption $C\left(t\right)$ and vector of sectoral prices $\boldsymbol{P}\left(t\right)$.

The next proposition characterizes the allocation of aggregate real consumption $C\left(t\right)$ over time for a given path of real interest rate $\left[r\left(\cdot\right)\right]_{t=0}^{\infty}$ and sectoral goods prices $\left[\boldsymbol{P}\left(\cdot\right)\right]_{t=0}^{\infty}$. Furthermore, it presents a set of conditions that ensure the uniqueness of the solution to the houshold's problem in our economy, which we henceforth assume are satisfied.

Proposition 1. *(Household Intertemporal Problem) Consider the problem of a household that chooses time of per-capita consumption and assets $\left[C\left(\cdot\right), A\left(\cdot\right)\right]_{t=0}^{\infty}$ maximizing (1) for given paths of sectoral prices and real interest rate $\left[\boldsymbol{P}\left(\cdot\right), r\left(\cdot\right)\right]_{t=0}^{\infty}$, where the per-capita stock of assets $A\left(t\right)$ evolves according to*

$$\dot{A}\left(t\right) \leq 1 + r\left(t\right) A\left(t\right) - E\left(t\right) - \dot{\ell}\left(t\right)\; A\left(t\right), \tag{8}$$

where the household wage is normalized to unity,where and $E\left(t\right)$ is given by Equation (7). The household's allocation should further satisfy the No-Ponzi condition

$$\lim_{t\to\infty} A\left(t\right) \exp\left(-\int_0^t \left[r\left(t'\right) - \dot{\ell}\left(t'\right)\right] dt'\right) \geq 0. \tag{9}$$

Assume an interior solution exists for a household starting with initial level of assets $A\left(0\right)$. Then, the path of consumptions and assets should satisfy the following Euler Equation

$$\dot{c}\left(t\right) = \frac{\dot{C}(t)}{C(t)} = \frac{r\left(t\right) - \rho - \iota\left(t\right)\; \overline{\dot{p}}\left(t\right)}{\widetilde{\vartheta}\left(t\right)}, \tag{10}$$

as well as the transversality condition

$$\lim_{t\to\infty} e^{-(\rho-\eta)t} \frac{A(t)}{E(t)} C\left(t\right)^{1-\vartheta} \frac{1}{\bar{\epsilon}(t)} = 0, \tag{11}$$

where we have defined

$$\iota\left(t\right) \equiv 1 + \left(1-\sigma\right) \mathbb{C}_{\boldsymbol{\Omega}(t)}\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}, \frac{\dot{p}_i(t)}{\bar{\dot{p}}(t)}\right), \tag{12}$$

$$\widetilde{\vartheta}(t) \equiv \vartheta + \bar{\epsilon}(t)\left[1 + (1-\sigma)\,\mathbb{V}_{\mathbf{\Omega}(t)}\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}\right)\right] - 1, \tag{13}$$

with $\overline{\dot{p}}(t) \equiv \mathbb{E}_{\mathbf{\Omega}(t)}[\dot{p}_i(t)]$ *and* $\bar{\epsilon}(t) \equiv \mathbb{E}_{\mathbf{\Omega}(t)}[\epsilon_i(t)]$ *denoting of the means of the rates of growth of sectoral price growth* $\dot{p}_i(t)$ *and income elasticity parameters* ϵ_i*, under the distribution over sectors implied by the expenditure shares* $\Omega_i(t)$*, and with* $\mathbb{V}_{\mathbf{\Omega}(t)}$ *and* $\mathbb{C}_{\mathbf{\Omega}(t)}$ *denoting the variance and covariance operators under the same distribution. Furthermore, if we assume that* $\sigma \in (0,1]$ *and* $\epsilon_i > 1 - \vartheta$ *for all* i*, the household problem has a unique solution characterized by the Euler Equation and the transversality condition above.*

Proof. See Appendix A.2. □

Proposition 1 and the static demand allocation Equation (6) jointly characterize the paths of sectoral consumption for the households. First, note that in the special case of homotheticity, where $\epsilon_i \equiv 1$ for all sectors i, the preferences reduce to the standard CES and the Euler Equation (10) reduces to the familiar form of $(\theta - 1)\cdot \dot{c} = r - \rho - \dot{e}$.[11] In the more general case of nonhomothetic CES, where ϵ_i's vary across sectors, Equation (10) deviates from the standard Euler Equations in two important ways. First, the term $\widetilde{\vartheta}(t)$ in the denominator accounts for the fact that in the presence of nonhomotheticity, the concavity of the instantaneous utility function and, correspondingly, the elasticity of intertemporal substitution both vary with time and real income. Second, the term $\iota(t)$ in the numerator shows that consumption grows faster to the extent that prices are falling relatively faster for more income-elastic goods, since households would be more inclined to substitute future consumption with current consumption.

Given the results of Proposition 1, it is easy to see that, along an optimal consumption path for households, the growth rates of aggregate real consumption, and aggregate consumption expenditure satisfy

$$\bar{\epsilon}(t)\,\dot{c}(t) = \dot{e}(t) - \overline{\dot{p}}(t), \tag{14}$$

where $\dot{e}(t)$ denotes the growth rate of consumption expenditure $E(t) \equiv \mathcal{E}(C(t), \boldsymbol{P}(t))$ given by Equation (7). Accordingly, the growth in the share of a given sector i in consumption expenditure is given by:

$$\dot{\omega}_i(t) \equiv \frac{\dot{\Omega}_i(t)}{\Omega_i(t)} = (1-\sigma)\left[(\epsilon_i - \bar{\epsilon}(t))\,\dot{c}(t) + \dot{p}_i(t) - \overline{\dot{p}}(t)\right]. \tag{15}$$

This expression shows two distinct forces that shape the evolution of sectoral shares of consumption expenditure: 1) the difference between the sector's income elasticity parameter and the share-weighted average of all sectors, and 2) the difference between the sector's rate of growth in prices and the share-weighted average rate among all sectors. For instance, assuming gross complementarity across different sectors ($\sigma < 1$), the expenditure share of a sector rises if its income elasticity parameter or rate of price inflation exceed that of the average sector.

[11] Note that the equality $\dot{e}(t) - \dot{c}(t) = \sum_i \Omega_i(t)\,\dot{p}_i(t) = \overline{\dot{p}_i}(t)$, which we used to find $(\theta-1)\cdot\dot{c} = r - \rho - \dot{e}$, only holds in the special case of homothetic preferences where the changes in the aggregate price index are *only* driven by changes in prices.

3.2.2. The Evolution of Sectoral Knowledge and Productivity

Given the assumptions made in Section 3.1.3, the following proposition characterizes the evolution of the distribution of the productivities of frontier sectoral technologies $F_i(Q,t)$.

Proposition 2. *(Evolution of Sectoral Productivity) Assume that the initial distribution of the productivity of the frontier technology across different varieties in each sector is Frechet, characterized by* $F_i(Q,0) \equiv \exp\left(-K_i(0)\, Q^{-\theta}\right)$ *where* $K_i(0)$ *is a measure of the initial stock of knowledge in sector* i*. Then, the distribution of the productivity of the frontier technology across different varieties in each sector remains Frechet for all* t*:*

$$F_i(Q,t) \equiv \exp\left(-K_i(t)\, Q^{-\theta}\right), \tag{16}$$

where the vector of average sectoral productivity $\boldsymbol{K}(t) \equiv \left(K_j(t)\right)_{j=1}^{I}$ *evolves according to*

$$\dot{K}_i(t) = \Gamma_i\, Z_i(t)^{1-\alpha}\, S_i(t)^{\beta_i}, \tag{17}$$

and we have defined $\Gamma_i \equiv \widetilde{\Gamma}_i \underline{Q}^{\theta} \Gamma(1-\beta_i)$ *and the stock of knowledge spillovers* $S_i(t)$ *in sector* i *at time* t *as:*

$$S_i(t) \equiv \sum_j \Phi_{ij}^{-\theta} K_j(t). \tag{18}$$

Proof. See Appendix A.2. □

Proposition 2 characterizes the distribution of frontier productivities across different varieties for each sector i at each time t. Equation (16) shows that this distribution is Frechet, and can be expressed in terms of a measure of the stock of ideas available in that sector $K_i(t)$. As Equations (17) and (18) show, this measure integrates the past investments in R&D in the sector, $Z_i(t)^{1-\alpha}$, along with a measure $S_i(t)$ of the idea spillovers from the stock of all sectors to sector i (including itself). The strength of these intertemporal spillovers is governed by the sector-specific intertemporal knowledge spillover parameter β_i.[12]

Based on our assumptions about the patenting process, and the results of Proposition 2, the next lemma characterizes the patenting process.

Lemma 1. *The number of new patents per unit of time (rate of arrival) in sector* i *at time* t *satisfies*

$$\dot{PAT}_i(t) = \Psi_i(t)^{-\theta}\, \frac{\dot{K}_i(t)}{K_i(t)}, \tag{19}$$

where the rate of change in the stock of ideas of the sector $\dot{K}_i(t)/K_i(t)$ *is given by Equation (17). Moreover, the probability that a given patent in sector* j *at time* t *cites a patent in sector* i *is given by*

$$\Sigma_{i,j}(t) \equiv \mathbb{P}\left(Q_o = \widetilde{Q}_{oi}/\Phi_{ji} \text{ in sector } j, t\right) = \frac{\Phi_{ji}^{-\theta} K_i(t)}{\sum_{i'} \Phi_{ji'}^{-\theta} K_{i'}(t)}. \tag{20}$$

Proof. See Appendix A.2. □

[12] In the special case of a single sector and $\alpha = 0$ and $\beta_i = 1$, the model reduces to that of Kortum (1997).

3.2.3. Pull and Push Forces Driving Innovation Incentives

In this section, we derive the allocation of innovation across sectors, characterizing the mechanisms that drive the incentives of R&D firms in the model to direct their innovation investments across different sectors.

The total profits in sector i at time t is a constant share of revenues in the sector, satisfying

$$\Pi_i(t) = \tfrac{\chi-1}{\chi} P_i(t) Y_i(t) = \tfrac{\chi-1}{\chi} L(t)\; E(t)\; \Omega_i(t), \tag{21}$$

where we have used the fact that the share of a given industry in total revenue equals the household expenditure shares on products from that sector, and where $L(t)\; E(t)$ is the aggregate household expenditure which also equates with the aggregate firm revenue. Accordingly, we can determine the allocation of household expenditure and sectoral profits from the demand Equation (6) and the sectoral price index (4).

The next lemma characterizes the condition that equalizes the returns to innovation and production labor and determines how innovation incentives direct the R&D activities of firms across different sectors.

Lemma 2. *Along any equilibrium path, the marginal product of labor in all R&D firms in sector i cannot exceed the wage rate, which leads to the following condition*

$$\Gamma_i Z_i(t)^{-\alpha} S_i(t)^{\beta_i}\; V_i(t) \leq 1, \tag{22}$$

where the stock of spillovers $S_i(t)$ is given by Equation (18) and where the expected value of producing a variety $V_i(t)$ satisfies

$$r(t) V_i(t) - \dot{V}_i(t) = \tfrac{\Pi_i(t)}{K_i(t)}. \tag{23}$$

Proof. See Appendix A.2. □

Equation (22) implies the following expression for the equilibrium amount of R&D labor in each sector, $Z_i(t)$:

$$Z_i(t) = \left(\Gamma_i S_i(t)^{\beta_i} \times V_i(t)\right)^{\frac{1}{\alpha}}. \tag{24}$$

$Z_i(t)$ results from the product of two terms. The first, $\Gamma_i S_i(t)^{\beta_i}$, reflects the push forces stemming from the differences in R&D productivity of R&D across sectors. This force shifts R&D toward sectors where innovation costs are lower and technological possibilities are more abundant. The second term, $V_i(t)$, reflects the pull forces stemming from differences across sectors in their future market size paths. When firms succeed in their R&D activities and take over a variety market in a given sector, the path of consumer demand in the sector determines the expected profits they will accrue in the future. Substituting for the profit shares from Equations (21) and (6), we find the following expression for the expected flow of profits per idea in sector i relative to sector j

$$\frac{\Pi_i(t)/K_i(t)}{\Pi_j(t)/K_j(t)} = \overbrace{\frac{K_j(t)}{K_i(t)}}^{\text{competition effect}} \times \overbrace{\left(\frac{K_j(t)}{K_i(t)}\right)^{1-\sigma}}^{\text{price effect}} \times \overbrace{C(t)^{(1-\sigma)(\epsilon_i-\epsilon_j)}}^{\text{income effect}}, \tag{25}$$

Equation (25) shows the decomposition of relative sectoral profits into price, income, and competition effects. Innovation and technology affects sectoral prices and the expected lifetime of an innovation. In sectors with higher stock of knowledge, the average cost of intermediate goods is lower, leading to lower relative prices and profits when $\sigma < 1$. This is the price effect. Additionally, faster innovation reflected in higher stocks of knowledge reduce the expected duration of the monopoly of a producer in a given variety, v. This is the competition effect. The income term in Equation (25) distinguishes our model from previous models of directed technical change, by highlighting the force of income elasticities in shaping innovation incentives. If the demand for output of sector i is more income-elastic compared to sector j, the demand for the output of this sector grows relative to sector j as the households' aggregate consumption grows.

3.2.4. General Equilibrium

Substituting the expression for the distribution of frontier technologies from Equation (16) in the expression for the sectoral price index (4) we find[13]

$$P_i(t) = \chi e^{-\gamma/\theta} K_i(t)^{-\frac{1}{\theta}}, \tag{26}$$

where γ denotes the Euler-Mascheroni constant. Equation (26) then, when combined with Equations (6) and (21), helps determine the allocation of sectoral consumption expenditures $\boldsymbol{\Omega}(t) \equiv (\Omega_i(t))_i$ and firm profits $\boldsymbol{\Pi}(t) \equiv (\Pi_i(t))_i$ based on the vector of sectoral knowledge $\boldsymbol{K}(t) \equiv (K_i(t))_i$ given the current level of consumption expenditure and utility, $E(t)$ and $C(t)$. The latter two have to satisfy the condition imposed by the nonhomothetic CES expenditure function in Equation (7).

The economy's sole resource, total labor L, can be employed in R&D, in the form of $Z(t)$, or in production, in the form of $L^p(t)$. Labor markets clear when

$$\sum_{i=1}^{I} \left(L_i^p(t) + Z_i(t)\right) = L^p(t) + Z(t) = L(t), \tag{27}$$

where we have defined total employment of R&D firms as $Z(t) = \sum_i Z_i(t)$ and the total mass of production workers L^p as $L^p(t) \equiv \sum_i L_i^p(t)$. Clearing the goods markets implies that the total consumption expenditure $L(t)\, E(t)$ of households equates the total value of output $\sum_i P_i(t)\, Y_i(t)$. The latter in turn equals the product of markups and total payments to

[13] The two equations together imply: $\log\left(\frac{1}{\chi}P_i(t)\right) = -\int \log Q\, e^{-K_i(t)Q^{-\theta}}\, d\left(K_i(t)\, Q^{-\theta}\right) = -\frac{1}{\theta}\left(\log K_i(t) + \gamma\right).$

production labor $\chi L^p(t)$. Equation (27) then implies the following key equation

$$\tfrac{1}{\chi}E(t) + \tfrac{Z(t)}{L(t)} = 1. \tag{28}$$

Households maintain a balanced portfolio of equity shares in all intermediate goods producers at all times. Therefore, R&D employment is an effective instrument of investment. Equation (28) shows that the employment share of production L^p/L changes linearly with the per-capita nominal expenditure of households. Therefore, in the aggregate, we can think of the decomposition of L into Z and L^p to be reflective of the household's allocation of available resources between investment and consumption.

We define an *allocation* as a collection of the time paths of aggregate and sector consumptions of households $[C(t), \boldsymbol{C}(t)]_{t=0}^{\infty}$, employment in production and R&D in each sector $[\boldsymbol{L}^p(t), \boldsymbol{Z}(t)]_{t=0}^{\infty}$, the stocks of knowledge of intermediate varieties in each sector $[\boldsymbol{K}(t)]_{t=0}^{\infty}$, and innovation value functions $[\boldsymbol{V}(t)]_{t=0}^{\infty}$. An *equilibrium* is an allocation that corresponds to the combination of constraints imposed by household utility maximization, monopolist profit maximization, and the innovation incentives everywhere along the time paths. The sectoral and aggregate consumption of households should satisfy the sectoral demand Equation (6) and the Euler Equations (10) and (11), where household assets satisfy $A(t) = \sum_i K_i(t) V_i(t)$ and sectoral prices indices are given by Equation (26). Employment allocations satisfy $L_i^p = \frac{\chi-1}{\chi} P_i Y_i$ for all i, as well as the labor market clearing condition (27). Finally, the condition on the innovation incentives (22) is satisfied and the growth in the stock of sectoral knowledge is given by $\dot{K}_i = \eta_i Z_i^{1-\alpha} S_i^{\beta_i}$.

3.3. Constant Growth Paths

In this section, we focus our attention on a class of equilibrium allocations that involve asymptotically constant rates of growth of consumption and sectoral technologies. Such equilibria closely parallel the balanced growth paths commonly studied in single sector growth models.

Definition. Constant Growth Path (CGP): An equilibrium path is a CGP if along the allocation path aggregate consumption $C(t)$ and sectoral technologies asymptotically grow at constant rates. That is, if there exist constant nonnegative values $(g^*, \gamma_1, \cdots, \gamma_I)$ such that the following limits exist

$$\lim_{t\to\infty} \dot{c}(t) = g^* > 0, \qquad \lim_{t\to\infty} \dot{k}_i(t) = \gamma_i \theta g^*, \ \text{ for } 1 \le i \le I.$$

Correspondingly, let us define the asymptotic levels of real per-capita consumption and states of sectoral technologies as

$$C^* \equiv \lim_{t\to\infty} C(t)\, e^{-g^* t}, \qquad K_i^* \equiv \lim_{t\to\infty} K_i(t)\, e^{-\gamma_i \theta g^* t}. \tag{29}$$

The following proposition characterizes the constant growth paths in our model.

Proposition 3. *Assume $\sigma \in (0,1)$ and $\Phi_{ij} < \infty$ for all i and j. Then, there exists a unique vector of $\boldsymbol{\gamma} \equiv (\gamma_1, \cdots, \gamma_I) > 0$ such that the asymptotic rate of growth of knowledge spillovers is identical* *across all sectors $\gamma_i^S = \overline{\gamma} = \max_j \{\gamma_j\} > 0$ and is given by the unique solution to the following fixed point problem:*

$$\overline{\gamma} = \frac{\theta}{\theta+(1-\alpha)(1-\sigma)} \left(\max_j \left\{ \left(\frac{1-\alpha}{\theta}\right)(1-\sigma)\,\epsilon_j + \beta_j \overline{\gamma} \right\} + \max_i \left\{ \epsilon_i - \beta_i \overline{\gamma} \right\} \right). \tag{30}$$

A set of sectors $i^* \in \mathcal{I}^*$ have asymptotically nonnegligible shares of production and R&D employment are identical, and is characterized by the condition:

$$i^* = \underset{i}{argmax} \left\{ \epsilon_i - \beta_i \overline{\gamma} \right\}. \tag{31}$$

A set of sectors $i^\dagger \in \mathcal{I}^\dagger$ grow at the fastest rate ($\gamma_{i^\dagger} = \overline{\gamma}$) and make asymptotically nonnegligible contributions to the intertemporal spillovers, and is characterized by the condition

$$i^\dagger = \underset{i}{argmax} \left\{ \left(\frac{1-\alpha}{\theta}\right)(1-\sigma)\,\epsilon_j + \beta_j \overline{\gamma} \right\}. \tag{32}$$

If $\epsilon_{i^*} - \beta_{i^*}\overline{\gamma} > (\vartheta + \overline{\epsilon}^* - 1)(1-\alpha)\frac{\eta}{\rho-\eta}$, then there is a unique constant growth path equilibrium with the asymptotic rate of growth of consumption given by

$$g^* = \frac{1}{\epsilon_{i^*} - \beta_{i^*}\overline{\gamma}} \left(\frac{1-\alpha}{\theta}\right) \eta > 0, \tag{33}$$

with the relative rates of productivity growth across sectors given by

$$\gamma_i = \varepsilon_i + \frac{\theta}{\theta+(1-\alpha)(1-\sigma)} \left[(\varepsilon_{i^*} - \beta_{i^*}\overline{\gamma}) - (\varepsilon_i - \beta_i \overline{\gamma}) \right]. \tag{34}$$

If there is only one sector i^* that achieves the maximum in Equation (31), then the asymptotic real rate of interest satisfies[14]

$$r^* = \rho + \frac{\vartheta - 1 + \left(1 - \frac{1}{\theta}\right)\epsilon_{i^*}}{\epsilon_{i^*} - \beta_{i^*}\overline{\gamma}} \left(\frac{1-\alpha}{\theta}\right) \eta. \tag{35}$$

The transversality condition requires $r^* > \eta$.

Proof. See Appendix A.2. □

As the proof of the proposition in the appendix shows, Proposition 3 combines two distinct sets of constraints on the sectoral rates of productivity growth implied by the demand side and and the innovation sides of the model.

Let us begin with the demand side constraints. Along a CGP, Equation (6) implies that the expenditure shares of different sectors asymptotically fall at the rate

$$\lim_{t\to\infty} \dot{\omega}_i(t) = -(1-\sigma)(\gamma_i - \epsilon_i)\, g^* \leq 0. \tag{36}$$

Equation (31) defines the set $\mathcal{I}^*$ of sectors that asymptotically constitutes a finite share of consumption expenditure and production employment. As we can see in Equation (34), average productivity grows at rate $\gamma_i g^* = \epsilon_i g^*$ for these sectors, and thus Equation (36) implies that the asymptotic rate of growth of the expenditure shares of these sectors is zero. Sectors for

[14] See the proof for the characterization of the asymptotic real interest rate in the more general case.

which the inequality is strict indefinitely shrink along the CGP.[15] When $\sigma \in (0,1)$, productivity in these shrinking sectors grows at rates faster than $\epsilon_i g^*$, while the opposite is the case when $\sigma \in (1,\infty)$. As with the model of Ngai and Pissarides [2007], when $\epsilon_i = 1$ for all sectors i and the preferences are homothetic, $\mathcal{I}^*$ includes only the sectors with the slowest rate of technical growth if $\sigma \in (0,1)$. In contrast, in the presence of nonhomotheticity, the combination of supply and demand (income elasticity) forces together determine the asymptotic sectoral composition of the economy. Lemma 3 in Appendix A.1.1 characterizes the constraints imposed on the CGP by the demand side of the model in more detail

The supply side constraints on the asymptotic rates of productivity growth follow from the conditions that characterize the value of innovation (23), the R&D incentives (24), and the evolution of knowledge stocks (17). Along a CGP, the value of sector-i innovation $V_i(t)$ grows at the same rate as the profits in the sector, which grows as the sum of the rate of growth in population η and the share of the sector in total expenditure given by Equation (36). Equations (17) and (24) then imply two different constraints on the asymptotic rate of growth of R&D labor in sector i, leading to the following result

$$\lim_{t\to\infty} \dot{z}_i(t) = \eta - (1-\sigma)(\gamma_i - \epsilon_i) g^* \leq \eta. \tag{37}$$

Equation (37) provides a parallel to that in Equation (36), showing that the set of sectors that asymptotically constitute a nonnegligible share of R&D investment coincides with those with the set $\mathcal{I}^*$ of sectors which survive in terms of their share of production. Lemma 4 in Appendix A.1.1 provides a full characterization of this result and additional supply side constraints. In particular, it derives constraints on the relative asymptotic rates of technical growth, R&D spillover growth, and income elasticity parameters across different sectors that together lead to Equations (30), (31), and (34).

First, Equation (30) characterizes the asymptotic relative rate of growth of idea spillovers $\overline{\gamma}$, which is identical across all sectors assuming that all sectors generate nonzero spillovers to one another. Having found the value of this rate, the lemma characterizes the set of sectors $\mathcal{I}^*$ with asymptotically nonzero share of production and R&D, as those that achieve the maximum in Equation (31). This maximum, which is guaranteed to be a positive number, also characterizes the asymptotic rate of productivity growth in Equation (33).

Equation (34) summarizes one of the key insights of the model that holds for any intersectoral innovation spillover functions compatible with a constant growth path equilibrium in the empirically relevant case of $\sigma \in (0,1)$. It states that the asymptotic rate of technological growth γ_i rises linearly in the rate of intertemporal knowledge spillovers β_i and in the income dependence parameter ϵ_i of the sector. The difference between the asymptotic rates of productivity growth in any two sectors i and j satisfies:

$$\gamma_i - \gamma_j = \left(\beta_i - \beta_j\right)\overline{\gamma} + \left(\tfrac{1-\alpha}{\theta}\right)(1-\sigma)\left(\epsilon_i - \epsilon_j\right). \tag{38}$$

The asymptotic gap in sectoral productivity growth depends on two forces. First, stronger

[15] Asymptotically, the shares of all other sectors $i \notin \mathcal{I}^*$ in output converges to zero. This result also holds along an equilibrium path for the economy described in the model of Comin et al. [2021], in which the rates of sectoral technical growth are exogenous.

intertemporal spillovers, as implied by higher values of parameter β_i, lead to stronger push effects that raise the rate of productivity growth. Second, higher income elasticities as implied by higher values of $(1-\sigma)\epsilon_i$ lead to stronger pull effects.

Moreover, the proposition implies a starker restriction on the rates of productivity growth for sectors $\mathcal{I}^*$ that asymptotically survive in terms of production and R&D. Since these sectors satisfy Equation (31), Equation (34) now implies that the rate of productivity growth is proportional to the value of the income dependence parameter

$$\gamma_i = \epsilon_i, \qquad i \in \mathcal{I}^*.$$

In other words, among the sectors that do not vanish in the long run, the rate of productivity growth is pinned down only by the degree of the income elasticity of demand for sectoral output. We can already see this result clearly in Equation (36): in order for the sectoral share of consumption not to diminish over time, the rate of decline in sectoral prices should be exactly large enough to compensate for the rise in the sector's share due to income effects.

3.4. Solving for Equilibrium Dynamics

In this section, we discuss our approach to solve for the equilibrium dynamics of the model toward a constant growth path. The state of the economy is characterized by the vector of sectoral (log) stocks of knowledge $\boldsymbol{k}(t) \equiv [k_i(t)]_i$. The corresponding control variables are, first, the (log) consumption $c(t)$, which in turn determines expenditure $E(t)$ and the share of R&D workers $\widehat{Z}(t) \equiv Z(t)/L(t)$ from the market clearing Equation (28), and the distribution of R&D across sectors $\Upsilon_i(t) \equiv Z_i(t)/Z(t)$.

Proposition 4. *Assume that the economy at time t is at a state characterized by the allocation $(\boldsymbol{k}(t), c(t), \boldsymbol{\Upsilon}(t))$. Sectoral prices are given by the vector $\boldsymbol{P}(t) \equiv [P_i(t)]$ where prices are given as a function of sectoral stocks of knowledge by Equation (26). Per capita consumption expenditure $E(t)$ and the vector of consumption expenditures $\boldsymbol{\Omega}(t) \equiv [\Omega_i(t)]_i$ are then given as a function of the economy's state by Equations (6) and (7). The evolution of the state variables is given by*

$$\dot{k}_i(t) = \Gamma_i \widehat{Z}(t)^{1-\alpha} \Upsilon_i(t)^{1-\alpha} \exp\left(\beta_i s_i(t) - k_i(t) + (1-\alpha)\ell(t)\right), \tag{39}$$

where $\ell(t) \equiv \log L_0 + \eta t$, where $s_i(t)$ denotes the log spillover function given by Equation (18), and where the employment share of R&D $\widehat{Z}(t)$ is given by the labor market clearing Equation (28). The evolution of the sectoral shares of R&D is given by

$$\dot{v}_i(t) = \frac{1}{\alpha}\left[\beta_i \dot{s}_i(t) - \mathbb{E}_{\boldsymbol{\Upsilon}(t)}\left[\beta_i \dot{s}_i(t)\right] - \tfrac{\chi-1}{\chi}\tfrac{E(t)}{\widehat{Z}(t)}\left(\dot{k}_i(t) - \mathbb{E}_{\boldsymbol{\Omega}(t)}\left[\dot{k}_i(t)\right]\right)\right], \tag{40}$$

where $\dot{v}_i(t) \equiv \log \Upsilon_i(t)$ denotes the change in log share of R&D in sector i and where the rate of growth of spillovers in sector j is given by $\dot{s}_j(t) = \mathbb{E}_{\boldsymbol{\Sigma}_{\cdot,j}(t)}\left[\dot{k}_i(t)\right]$ where $\boldsymbol{\Sigma}_{\cdot,j}(t)$ is the distribution of patent citations across all other sectors i given by Equation (20). Finally, the evolution of the log consumption *aggregator is given by*

$$\dot{c}\left(t\right)=\frac{\widehat{Z}(t)}{\widetilde{\vartheta}(t)\,\widehat{Z}(t)+\alpha\overline{\epsilon}(t)\,\left(1-\widehat{Z}(t)\right)}\left(\alpha\eta-\rho-\mathbb{E}_{\boldsymbol{\Upsilon}(t)}\left[\beta_i\dot{s}_i(t)\right]\right)+\left(\frac{\iota(t)\,\widehat{Z}(t)+(\chi-1+\alpha)\left(1-\widehat{Z}(t)\right)}{\widetilde{\vartheta}(t)\,\widehat{Z}(t)+\alpha\overline{\epsilon}(t)\,\left(1-\widehat{Z}(t)\right)}\right)\frac{1}{\theta}\mathbb{E}_{\boldsymbol{\Omega}(t)}\left[\dot{k}_i(t)\right], \tag{41}$$

with $\iota\left(t\right)$ *and* $\widetilde{\vartheta}\left(t\right)$ *given by Equations (12) and (13) with* $\overline{p}\left(t\right)=-\frac{1}{\theta}\mathbb{E}_{\boldsymbol{\Omega}(t)}\left[\dot{k}_i(t)\right].$

Proof. See Appendix A.2. □

Starting from each vector of initial vector of log knowledge stocks $\boldsymbol{k}\left(0\right)$, Equations (39)-(41) together describe the dynamical system that characterizes the evolution of the economy toward a constant growth path.[16] The I-dimensional vector of control variables (consumption aggregator c and the $(I-1)$-dimensional distribution of R&D allocations) are chosen to ensure the system evolves along a stable path toward the constant growth path, at which we have

$$\lim_{t\to\infty}\left(\dot{\boldsymbol{k}}\left(t\right),\dot{c}\left(t\right),\dot{\boldsymbol{v}}\left(t\right)\right)=\left(\theta\,\gamma\,g^{*},\,g^{*},\,-\left(1-\sigma\right)\left(\gamma-\boldsymbol{\epsilon}\right)g^{*}\right). \tag{42}$$

Combining the results of Propositions 3 and 4 and Equation (42), we can pin down all the variables at the constant growth path. Note, in particular, that along the constant growth path characterized by Proposition 3, the term inside the exponent on the right hand side of Equation (39) asymptotically converges to a constant. As discussed in Appendix A.1.2, we can renormalize all variables characterizing the state of the economy with respect to their corresponding CGP trends and restate the evolution equations in terms of these normalized variables.

Solving for for the resulting dynamics poses two key challenges relative to the standard problems common in growth models: 1) the multi-dimensional nature of the control variables $\left(c\left(t\right),\boldsymbol{\Upsilon}\left(t\right)\right)$, and 2) the explicit dependence of the evolution operator on time t (non-autonomous, nonlinear ODE) due to the reallocations in the composition of demand along the growth path.[17] We choose to solve the problem using an approach based on functional approximation and optimization. We consider a parameterized family of spline functions for the time paths of the economy's state and control variables $\left(\boldsymbol{k}\left(t\right),c\left(t\right),\boldsymbol{\Upsilon}\left(t\right)\right)$ and minimize the squared error between the discretized approximations of the derivatives in Equations (39)-(41). Appendix A.1.2 provides more details on the implementation of our approach.

4. Calibration

In this section, we quantify the model to examine its ability to explain the patterns of the structural transformation of innovation documented in Section 2. We first discuss our calibration strategy and then proceed to present our quantitative results. We further use the model to study the drivers of the patterns of structural change and to make predictions about the medium and long-run trends in sectoral and aggregate productivity growth.

[16] The model assumptions $\theta>0$ and $\alpha\in(0,1)$ imply that positive R&D expenditures in all sectors along the equilibrium path. Therefore, we focus on an *interior* equilibrium path along which R&D and innovation is carried in all sectors, that is, $Z_i(t),Y_i(t)>0$ for all i and t.

[17] These aspects make it hard to apply standard backward shooting algorithms to solve for the transitional dynamics. In Appendix A.1.2, we outline how the standard approach may be applied to our setting.

4.1. Quantification of the Model

To quantify our model, we need to determine the parameters for household preferences and sector-specific innovation production functions. We calibrate preference parameters and standard production parameters using prior estimates. For key innovation technology parameters, we derive estimation equations from the model's structural equations and infer their values using data on sectoral patents, citations, and R&D spending. For the remaining parameters (initial conditions and constant demand/production shifters), we perform a parameter search to match the evolution of sectoral R&D, TFP growth and production in the data.

4.1.1. Externally Calibrated Parameters

Regarding preference parameters, we set the discount rate to $\rho = 0.02$ and the inverse of the intertemporal elasticity of substitution to $\vartheta = 1.5$ following standard practice (e.g., Buera et al., 2011 and Buera and Shin, 2013). We take the elasticities of the nonhomothetic CES preferences from the estimates in Comin et al. [2021] for sectoral value-added preferences using the US Consumer Expenditure Survey. In particular, we set $\sigma = 0.34$, $\epsilon_a = 0.06$, and $\epsilon_s = 1.64$, while normalizing $\epsilon_m = 1$. We set the population growth rate η to 1.5%, which is close to the US average population growth over the past century, and parameter χ to generate a markup of 33%.

We take the sectoral R&D congestion parameter $\alpha = 0.6$ from Anzoategui et al. [2019] and set the Pareto tail parameter $\theta = 1.45$ which corresponds to the medium-run of the elasticity estimated in Boehm et al. [2023]. The rest of innovation technology parameters are estimated structurally, as we discuss next.

4.1.2. Innovation Parameters: External Estimation Using Patent and Citation Data

We proceed by estimating the remaining parameters governing the innovation production function from patent and citation data. These parameters are the intertemporal knowledge spillovers and patenting cutoffs $\{\beta_i, \Psi_i^\theta(t)\}_{i\in a,m,s}$ and intersectoral applicability costs $\{\Phi_{ij}^{-\theta}\}_{i,j\in a,m,s}$. We first use the equilibrium relationship between cross-sector patent citations and the flow of patents granted in a sector to estimate the time-varying patenting cutoff parameters $\{\Psi_i^\theta(t)\}_{i\in a,m,s}$. These estimates allow us to recover changes in the sectoral knowledge stock from observed patenting flows. Then, we combine these inferred changes with data on R&D expenditures, and the innovation production function to compute the intertemporal knowledge spillovers parameters, $\{\beta_i\}_{i\in a,m,s}$. We finally use the model prediction about the matrix of citation probabilities to estimate the applicability cost matrix parameters, $\{\Phi_{ij}^{-\theta}\}_{i,j\in a,m,s}$.

Combining Equations (19) and (20), it follows that the change in the log probability that sector j cites a patent from sector i at time t is given by

$$\tfrac{d}{dt} \log Pr_{ij}^{CITE}\left(t\right) = \tfrac{d}{dt} \log \Sigma_{i,j}\left(t\right) = \Psi_i^\theta(t)\, P\dot{A}T_i(t) + \delta_j\left(t\right), \tag{43}$$

where $Pr_{ij}^{CITE}\left(t\right)$ is the data counterpart of the probability that sector j cites a patent from sector i, $\Sigma_{i,j}\left(t\right)$, $P\dot{A}T_i(t)$ is the flow of new patents issued in sector i at time t and $\delta_j(t)$ are sector-specific time effects. This equation reflects that the growth rate of citations from j to i is equal to the growth rate of knowledge in i relative to the weighted growth rate of knowledge across sectors, where the weights are given by the probability that sector j cites each sector. Because these weights are specific to the citing sector, they are subsumed in the citing-sector time effects $\delta_j(t)$ in Equation (43).[18] The first term of this equation reflects the fact that the growth rate of knowledge in sector i can be related to the flow of patents in i, $\dot{PAT}_i(t)$, adjusted by a function of the threshold patent rate, $\Psi_i^\theta(t)$. Intuitively, a higher patenting threshold in sector i, $\Psi_i^\theta(t)$, implies that patented innovations have higher quality and hence they have a greater probability of being cited by other sectors. We allow the parameters $\{\Psi_i^\theta(t)\}_{i\in a,m,s}$ to change over time to capture changes in patent quality of the kind documented by Akcigit and Goldschlag [2021].[19]

Given the slow-moving nature of the relationship between citations and patent flows, we estimate Equation (43) using long differences of 20 years. Additionally, we model the time variation in $\Psi_i^\theta(t)$ by allowing the parameters $\{\Psi_j^\theta(t)\}_{j\in a,m,s}$ to vary before and after 1980, but keeping them constant within each subperiod. These assumptions allow us to approximate Equation (43) within each subperiod by the following empirical specification:

$$\Delta \log Pr_{ij}^{CITE}(t) = \Psi_i^\theta \, \Delta PAT_i(t) + \delta_j(t) + e_{ij}(t), \tag{44}$$

where $\Delta \log Pr_{ij}^{CITE}(t)$ is the growth rate between year t and year $t+20$ of the probability that a patent from sector j cites a patent from sector i, $\Delta PAT_i(t)$ is the cumulative flow of patents granted in sector i over the same interval and $e_{ij}(t)$ is the error term. We estimate $\{\Psi_i^\theta\}_{i\in a,m,s}$ separately for the pre-1980 and post-1980 periods. As we have discussed, parameters Ψ_i^θ capture the knowledge content of new patents in sector i. Equation (44) estimates this parameter based on the degree to which a larger flow of patents in sector i lead to relatively more citations for this sector (compared to other sectors) among patents arriving in each sector j. The citing-sector time effects $\delta_j(t)$ absorb aggregate shocks (e.g., changes in R&D policies) as well as citing-sector shocks (e.g., trends in propensity to cite or patent).

We report the estimated parameter values in Table 3 in the appendix.[20] While the magnitude of the parameter is hard to interpret, we note that the average value of Ψ_j^θ in the post-1980 period is an order of magnitude smaller for all three sectors. This is consistent with the interpretation that the quality of patents as measured by forward citations has declined, as documented in Akcigit and Goldschlag [2021].[21]

With the estimates of Ψ_j^θ in hand for each sector in the pre- and post-1980 periods, together with patent and R&D expenditure data, we can use the model structure to compute the intertemporal knowledge spillovers parameters, $\{\beta_i\}_{i\in a,m,s}$. Before explaining the details of this

[18]This term closely parallels the multilateral resistance terms in "gravity" models of trade—our model of knowledge diffusion and patenting shares the structure of Ricardian theories of trade [Eaton and Kortum, 2012].

[19]The proxy for patent quality used by Akcigit and Goldschlag [2021] is forward citations. Other authors have used other proxies for patent quality and have also shown similar declines over time, specially after 1980. For example, Kogan et al. [2017] use the market value of innovations, Bloom et al. [2013] use the fraction of filed patents with few or no citations. The downward trend in patent quality can be caused by multiple reasons including legal changes, a rise in strategic filing, declining examination standards, etc. Regardless of the origin, the exogenous variables $\{\Psi_i^\theta(t)\}_{i\in a,m,s}$ capture the evolution of the patent quality threshold in our model.

[20]Appendix B.1 provides further details on the estimation.

[21]The results are robust to using other lag lengths (e.g., 10, 15, 25 and 30 years). See appendix Table 3.

calculation, we emphasize that while these elasticities are central to growth models, the literature offers limited guidance on their values, either in aggregate or at the sectoral level. We compute β_i by rearranging the sectoral innovation production function, Equation (17), to obtain

$$\beta_i = \frac{1}{\Delta \log S_i(t)} \left(\Delta \log \left(\frac{\dot{K}_i(t)}{K_i(t)} \right) + \Delta \log K_i(t) - (1-\alpha) \Delta \log Z_i(t) \right), \tag{45}$$

where the time operator Δ denotes the difference between any two periods. We can readily compute all the terms in the right-hand-side of Equation (45) using Equation (19) in terms of sectoral patenting flows and the estimated Ψ_i^θ 's (see Appendix B).

After substituting the empirical counterparts to Equation (45), we find the intertemporal knowledge parameters to be $\beta_a = 0.87$, $\beta_m = 0.69$ and $\beta_s = 0.45$. We assess the robustness of the values of β_i by examining the values obtained under different specifications for the length of the time difference operator Δ in Equation (45) and constructing knowledge stocks using different estimates for Ψ_j^θ in Appendix B.1. The estimates we obtain in all these instances consistently paint a picture in which $\beta_a > \beta_m > \beta_s$.[22]

Finally, to estimate the parameters $\{\Phi_{ij}^{-\theta}\}_{i,j \in a,m,s}$, we use empirical counterparts of $\Delta \ln K_j$ to express citation probabilities as

$$\Sigma_{j,i}(t) = \frac{\Phi_{ij}^{-\theta} K_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)} = \frac{\Phi_{ij}^{-\theta} K_j(0) \varkappa_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(0) \varkappa_{j'}(t)}, \tag{46}$$

where $\varkappa_j(t) = \exp\left(\Delta \log K_j(t)\right)$. Under the normalization that $\Phi_{ii}^{-\theta} = 1$ for all i, we estimate the cross-sector applicability costs $\Phi_{ij}^{-\theta}$ with $i \neq j$ through a system of equations (see Appendix B.3). The results, reported in Table 3 in the appendix, indicate that manufacturing has the lowest applicability cost for other citing sectors. This suggests that knowledge originating in manufacturing is relatively easy to adapt in both services and agriculture. In contrast, agriculture exhibits the highest applicability cost, implying that knowledge from this sector is more difficult to transfer to manufacturing and services.

4.1.3. Remaining Model Parameters: Data on Sectoral Trends

To calibrate the remaining model parameters, we rely on the fact that given the initial stocks of sectoral ideas $\boldsymbol{K}(t_o)$ and the model parameters, we can solve for the model dynamics and obtain predictions for the paths of sectoral value-added shares, R&D employment shares, and TFP growth. We compute these paths given a collection of the demand shifters $\Xi_{i \in \mathcal{I}}$, the shifters of the sectoral ideas production functions $\boldsymbol{\Gamma}_{i \in \mathcal{I}}$, and the initial stocks of ideas $\boldsymbol{K}(t_0)$ (taking the initial period to be $t_0 = 1954$). After solving for the paths given the values of these nine parameter values, we compute the sum of squared differences between the predicted paths for sectoral value added shares, R&D expenditure shares, and TFP growth between 1954 and 2010 in the model and in the data. We perform a parameter search over these nine parameters to minimize this distance measure.[23]

[22] The implied values for β_a are above 0.8, β_m takes intermediate values, between 0.65 and 0.85, while for β_s ranges between 0.35 and 0.7.

[23] Our baseline calibration uses external evidence on patenting and citations to estimate the key parameters of the innovation production function via the model's structural equations. An alternative would be to calibrate these

Table 2: Parameter Values in the Quantified Model

Parameter	Description	Value	Source / Notes
Externally Set Parameters			
ρ	Discount rate	0.02	–
η	Population growth rate	0.015	–
ϑ	Intertemporal elasticity of substitution ($1/\vartheta$)	1.5	Buera et al. [2011]
σ	Elasticity of substitution across sectors	0.34	Comin et al. [2021]
$(\epsilon_a, \epsilon_m, \epsilon_s)$	Income elasticity parameters	(0.06, 1, 1.64)	Comin et al. [2021]
χ	Markup	1.33	Loecker et al. [2020]
α	R&D congestion parameter	0.6	Anzoategui et al. [2019]
θ	Dispersion of idea distribution	1.45	Boehm et al. [2023]
Directly Estimated			
$\{\Psi^{\theta}_{j,\text{Pre80}}, \Psi^{\theta}_{j,\text{Post80}}\}_{j\in\{a,m,s\}}$	Semi-elasticity of patent flow to citations	See Appendix B	–
$(\beta_a, \beta_m, \beta_s)$	Intertemporal knowledge spillovers	(0.87, 0.69, 0.45)	–
$\{\Phi^{-\theta}_{ij}\}_{i,j\in\{a,m,s\}}$	Knowledge spillover matrix	See Appendix B	–
Internally Calibrated			
$(k_a(0), k_m(0), k_s(0))$	Initial knowledge stocks	(1.82, 3.45, 6.72)	–
(Ξ_a, Ξ_m, Ξ_s)	Demand shifters	(0.10, 4.00, 62.48)	–
$(\Gamma_a, \Gamma_m, \Gamma_s)$	Idea production function shifters	(0.243, 0.022, 0.456)	–

4.2. Calibrated Model

4.2.1. Constant Growth Path

Before proceeding to simulate the model dynamics, we evaluate the CGP at our selected parameter values. The discussion of the CGP in Section 3 implies that one sector generically dominates the economy in terms of expenditure and R&D shares, while another (potentially distinct) sector asymptotically produces the relevant spillovers for all sectors. This is indeed the case at our selected parameter values, with services asymptotically accounting for all consumption and R&D expenditures ($i^* = s$) and manufacturing dominating sectoral spillovers ($i^\dagger = m$).

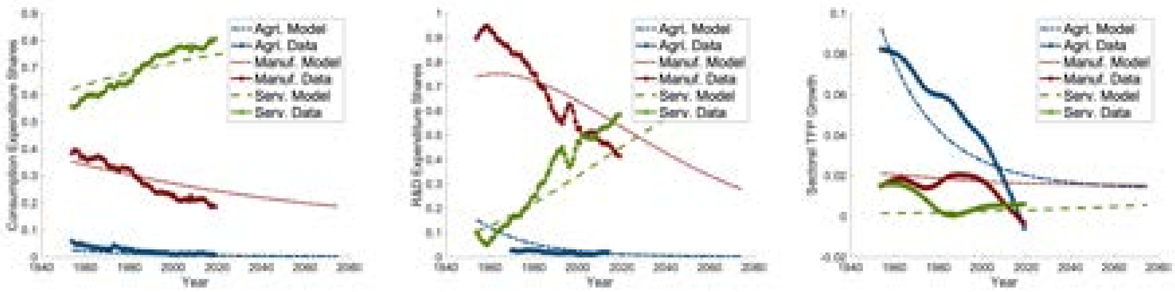
Quantitatively, the asymptotic growth rate of spillovers and aggregate consumption, given by Equations (30) and (33), imply

$$g^* = \frac{(1-\alpha)\eta}{\theta \operatorname*{argmax}_{i\in\mathcal{I}^*} \{\, \epsilon_i - \beta_i \overline{\gamma} \,\}} = \frac{(1-\alpha)\eta}{\theta\,(\epsilon_s - \beta_s \overline{\gamma})} = 0.8\%.$$

Moreover, the rates of productivity growth of the agricultural and manufacturing sectors are faster along the CGP than that of services. From our theory we can directly compute these growth rates and we already know that for the surviving sector, services, the asymptotic growth rate is pinned down by the income elasticity parameter, $\gamma_s = \epsilon_s = 1.64$. Using Equation (34), we find that agriculture and manufacturing have asymptotic growth rates that are 1.78

parameters internally by directly matching observed data. This approach undermines the transparency of the identification of the underlying technological parameters, but we have verified that this alternative calibration approach indeed improves the fit for sectoral value added and R&D expenditure shares.

Figure 4: Transition Path of Sectoral Variables, 1954-2075.



(a) Sectoral Value-Added Shares (b) Sectoral R&D Expenditure Shares (c) Sectoral TFP Growth

Note: Baseline simulation results. Dashed lines correspond to the model simulation. Circles correspond to the data that we target to match with the initial conditions for knowledge stocks, demand shifters and shifters of the innovation production function

and 1.98 multiples of g^*, respectively. Finally, from Equation (35), we have that the asymptotic value of the interest rate is $r^* = 2.4\%$.

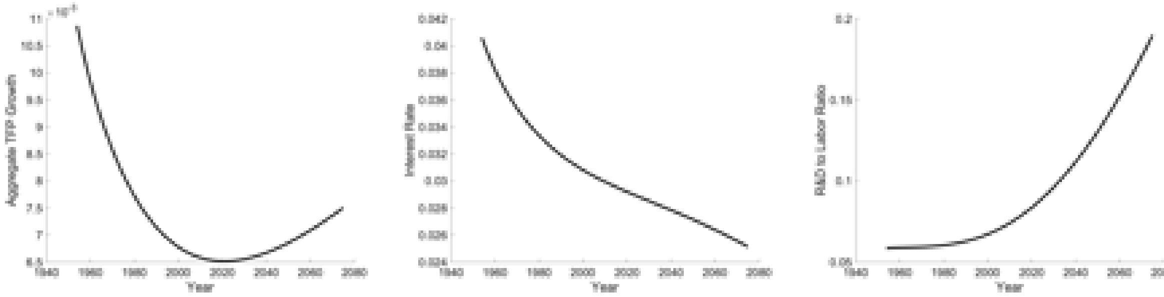
4.2.2. Sectoral Dynamics

Figure 4 reports the evolution of sectoral value added, R&D shares and TFP growth from 1954 to 2019, alongside the model's predicted path, including projections for the next 50 years. The model captures the broad patterns observed in the data both in consumption value-added and sectoral R&D shares. The model further predicts a sustained rise in the shares of services in both production and R&D: by 2050 the shares are predicted to rise to 85.9% and 79.9%, respectively, and by 2100 to 90.5% and 89.6%.

Panel (c) in Figure 4 shows a similar comparison of the model's predictions with those of the data for sectoral rates of productivity growth. Even though here we observe a less precise fit with the data, the model still captures the relative ranking of the sectoral rates of productivity growth. Importantly, in line with the observed data, the model generates a substantial productivity slowdown in agriculture and manufacturing—albeit not of the same magnitude. This slowdown in the model stems from the reallocation of household expenditure away from these sectors, which lowers the innovation incentives of firms in these sectors and redirects their R&D activities toward services. In tandem, the model predicts a slow but sustained rise in the productivity growth in the service sector. Moreover, since the model lacks any shocks, it is only able to capture the long-term trends in productivity and it is not able to generate the observed medium-term fluctuations, which are relevant especially in manufacturing and services [Comin and Gertler, 2006].

Panel (a) of Figure 5 reports the aggregate TFP growth from the model. The calibrated model generates a sustained productivity slowdown of 0.45 percentage points by 2020. We can decompose this drop into the contribution of reallocation of economic activity towards services (Baumol's cost disease), which generates a drop of 0.3pp, and the slowdown of sectoral TFP which generates the remaining 0.15, see Appendix Figure 11. The productivity slowdown is smaller than the 1.3-point drop observed in the data. This gap stems largely from the model's inability to replicate the sharp TFP declines in manufacturing and agriculture in

Figure 5: Transition Path of Aggregate Variables, 1954-2075.



(a) Aggregate TFP growth (b) Interest Rate (c) Share of Labor in R&D

Note: Baseline simulation results for aggregate TFP growth, interest rate and share of employment in R&D.

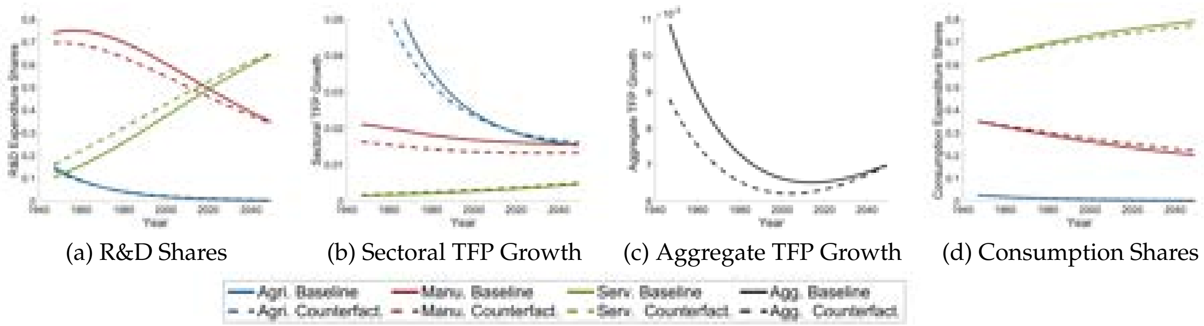
recent decades. While the model closely matches the 2019 productivity growth rate (0.6% in the data vs 0.65% in the model), it leaves much of the within-sector slowdown unexplained, likely due to omitted forces such as slower population growth, rising intangibles, and declining innovation. Looking ahead, the model forecasts a slow but steady recovery in productivity growth, reaching 0.8% by 2075 and 1% growth over the twenty-second century. This recovery is driven by gradually rising TFP growth in services.

Finally, panels (b) and (c) of Figure 5 report the evolution of the aggregate interest rate and the share of labor employed in R&D. The transitional dynamics in our calibrated model generate a sustained decline in interest rate: from 4% in the 1950s, to 2.8% in the 2020s in line with the recent evidence on falling real rates [e.g. Liu et al., 2022]. Going forward, the model implies a further decline in the interest rate—even below to the long run steady state—before converging to 2.4% in the CGP. Interestingly, the model generates an almost hockey-stick pattern for the employment share of R&D [in line with the findings of Bloom et al., 2020]. Prior to the 1990s, the share increases very modestly, from 5.8% in the 1950s to 6.3% in 1990. Subsequently, the growth in the share of R&D employment accelerates, reaching 8.4% by 2020. Going forward, the model suggests that this share will continue to rise to 15% by 2060—since innovation is relatively harder in services due to its low elasticity of intertemporal knowledge spillovers to innovation, β_s.

4.2.3. Understanding the Model Mechanisms

To understand the different mechanisms that generate the observed model dynamics, we investigate how modifying key parameters for the technology-push or demand-pull affects the model dynamics. In particular, we investigate the importance of the technology-push factors modifying the cross-sector knowledge spillovers Φ_{ij}, the intertemporal knowledge spillovers, β_i, and the constant productivity term of R&D, Γ_i. We investigate the importance of demand-pull forces modifying the nonhomotheticities in demand, ϵ_i. In these counterfactuals, we start from the calibrated levels of the stock of knowledge in our initial period (1954), and simulate the model forward keeping the rest of the parameters as in the baseline calibration.

Figure 6: Decreasing Cross-Sector Spillovers, $\Phi_{ij}^{-\theta}$: Baseline and Counterfactual



(a) R&D Shares (b) Sectoral TFP Growth (c) Aggregate TFP Growth (d) Consumption Shares

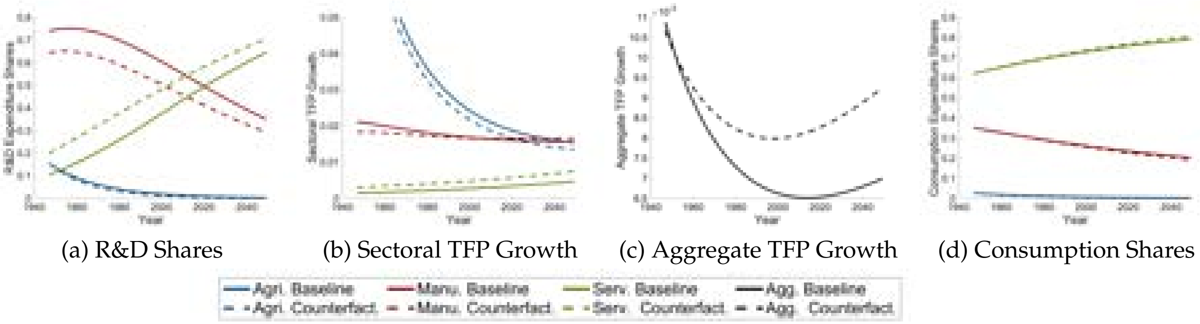
Cross-sector Spillovers In our first experiment, we examine the importance of cross-sector technology knowledge spillovers by modifying the off-diagonal elements of the matrix of applicability costs, $\Phi_{ij}^{-\theta}, i \neq j$. In particular, we reduce the importance of the off-diagonal elements by a third by taking a weighted average of the identity matrix and our baseline matrix, with respective weights of one- and two-thirds. This results in lower spillovers ($S_i^{\beta_i}$) in all sectors relative to the baseline, but especially in sectors with lower knowledge (K_i) and with higher intertemporal knowledge spillover elasticity (β_i). At the initial knowledge stocks, this raises innovation costs in agriculture more than in manufacturing, and in manufacturing more than in services.

Figure 6 shows the evolution of the economy starting at the initial vector of sectoral knowledge stock and the model parameters of our baseline calibration—except for the cross-sector spillovers. Panel (a) shows a strong initial reallocation of R&D from agriculture and manufacturing to services. However, as services absorb the bulk of innovation activity and spillovers become less relevant, the sectoral R&D shares converge to that of the baseline calibration. The lower R&D spending together with the reduction in its productivity reduces significantly TFP growth in agriculture and manufacturing (panel b). By contrast, TFP growth in services is hardly affected since the decline in spillovers from manufacturing is compensated by the increase in R&D expenditures. In the counterfactual, the aggregate TFP growth is substantially lower in the postwar period due to the reduction in sectoral productivity growth in manufacturing and agriculture. Only when services becomes a substantial driver of TFP in the 2000s, aggregate productivity growth picks up in the counterfactual exercise. As a result of the relatively faster TFP growth in services in this counterfactual exercise, structural change in consumption slows down relative to the baseline.

Intertemporal Knowledge Spillovers To explore the role of the intertemporal knowledge spillovers, we increase its value for services, β_s, which is the lowest of the three sectors, to close its gap with manufacturing by a third, from 0.45 to 0.53. At the initial stock of knowledge, this modification increases the productivity of R&D in services leading to a reallocation of R&D from manufacturing towards services (See panel (a) of Figure 7). Note, however, that as a higher β_s also reduces the dynamic diminishing returns to knowledge spillovers, the increase in the services R&D share is more protracted than in our first counterfactual.

The reallocation of R&D together with the higher productivity of innovation activity in ser-

Figure 7: Increasing Services' Intertemporal Knowledge Spillovers, β_s: Baseline and Counterfactual



(a) R&D Shares (b) Sectoral TFP Growth (c) Aggregate TFP Growth (d) Consumption Shares

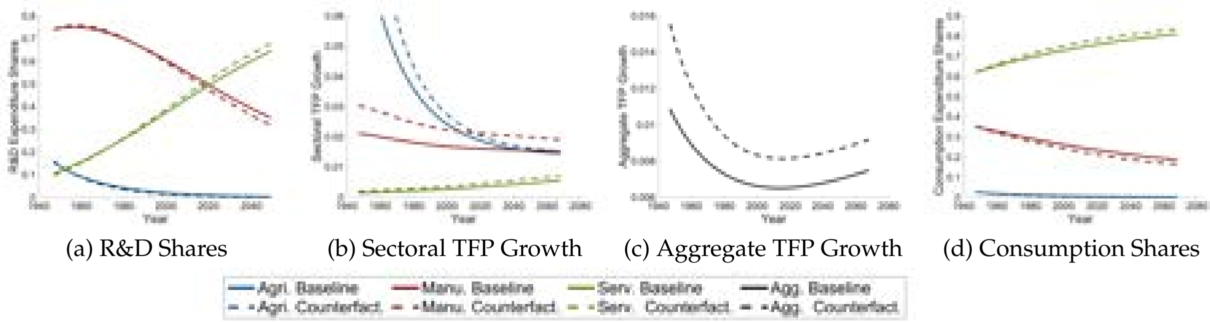
vices leads to an acceleration of TFP growth in services and a reduction in manufacturing and agriculture (panel b). Aggregate TFP growth is not significantly affected on impact but as time goes by and the share of services in the economy grows, the gap in TFP growth between the counterfactual and the baseline increases significantly (see panel c). Although the faster TFP growth in services suggests that in the counterfactual the price effects are stronger than in the baseline, the faster growth rate of consumption also implies stronger income effects. Interestingly, the latter seem to dominate leading to slightly larger share of services in total expenditures (see panel d).

Changes in R&D Productivity Next, we conduct two experiments in which we increase the productivity of R&D through an exogenous and permanent shock to the constant productivity parameter Γ_i in the ideas production function, Equation (17). Even though a normative analysis of the model is beyond the scope of our current analysis, we note that this exercise may shed light also on the potential role that government R&D policy in our enviornment—not only can it affect aggregate TFP growth, as in standard one-sector endogenous growth models, but it can also affect the pace of structural change. We conduct two different experiments. First, we show the effect of setting a uniform 33% increase in R&D productivity across all sectors in the economy. Second, we show the effect of a 50% increase in services R&D productivity.

Perhaps not surprisingly, panel (a) in Figure 8 shows that relative initial R&D investments remain by and large unchanged after a uniform productivity increase of R&D (while the overall level of R&D in the economy decreases slightly from 5.8% in the baseline to 5.4%). But, since overall R&D is substantially more productive, sectoral TFP increases across the board (panel b). This results in sustained faster TFP growth as shown in panel (c) and an accelerated path of structural change as measured by consumption shares which is driven by income rising faster relative to the baseline (panel d).

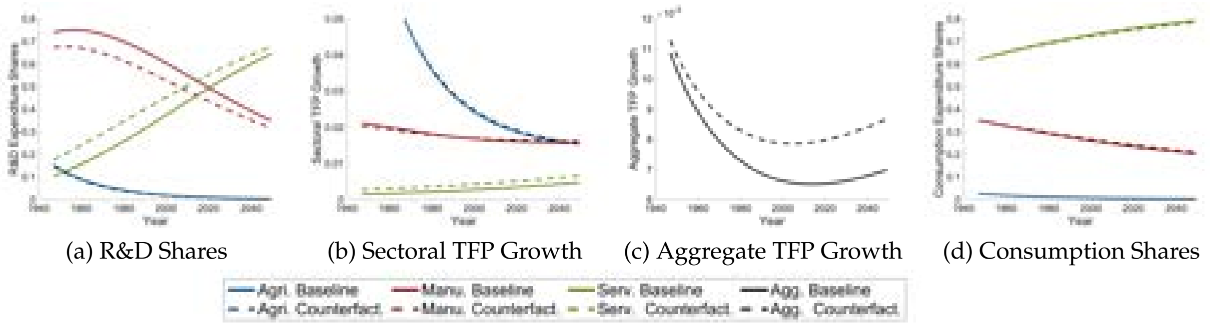
Perhaps more interestingly, if only the service sector innovation experiences a productivity boost, panel (a) in Figure 9 shows that R&D expenditures are reallocated towards services at the expense of manufacturing in the first decades, but as time goes by and the economy becomes servicized the share of R&D devoted to services in the counterfactual scenario becomes similar to that of the baseline. As a result, panel (b) shows that service sectoral TFP growth is higher than in the baseline. While R&D expenditures fall in manufacturing, its sectoral TFP

Figure 8: Uniform Productivity Boost in Innovation Shifters, Γ_i: Baseline and Counterfactual



(a) R&D Shares (b) Sectoral TFP Growth (c) Aggregate TFP Growth (d) Consumption Shares

Figure 9: Productivity Boost to Innovation Shifter in Services, Γ_s: Baseline and Counterfactual



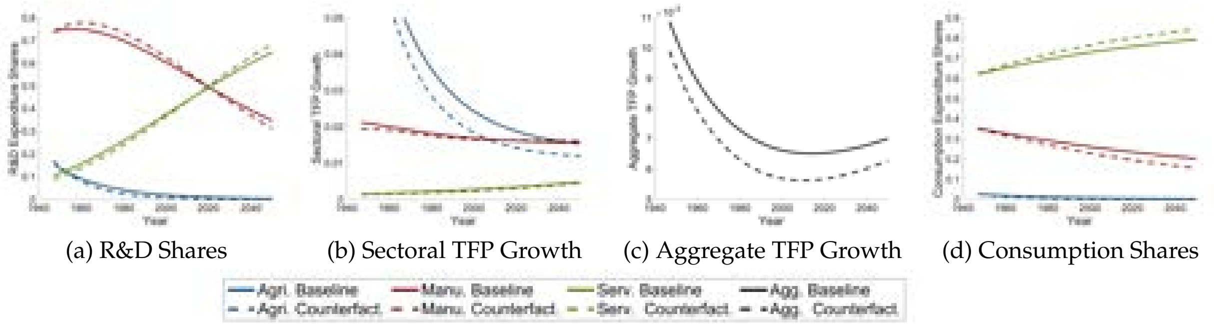
(a) R&D Shares (b) Sectoral TFP Growth (c) Aggregate TFP Growth (d) Consumption Shares

is hardly affected due to the spillovers coming from the increase in TFP growth of services. The aggregate implications for TFP growth are shown in panel (c). Initially the TFP growth is very similar to the baseline but as time goes by, we see an opening gap between the counterfactual TFP growth and our baseline. This reflects the higher productivity growth in services. Panel (d) shows that, quantiatively, the overall effect in structural change as measured in consumption expenditure shares is muted: the decline in service prices is essentially offset by the raising income. As result, the overall pattern of structural change is very similar to that of the baseline.

Nonhomotheticity Finally, we assess the role of demand-pull factors by weakening the strength of nonhomotheticity in services. To do so, we increase the income elasticity parameter of manufacturing, ϵ_m, by one-third of the gap between its baseline value and ϵ_s, raising it from 1 to 1.21. Since the long-run CGP is determined by ϵ_s, this change leaves it unaffected. The increase in ϵ_m reduces the relative market size of services to manufacturing, as the gap $\epsilon_s - \epsilon_m$ becomes smaller (see Equation 25). This leads to a reallocation of R&D resources from services to manufacturing, as shown in panel (a) of Figure 10.

Changing the value of ϵ_m also affects the intertemporal elasticity of substitution (IES). The increase of ϵ_m raises the expenditure-share-weighted average income elasticity parameter, $\bar{\epsilon}$, which in turn reduces the IES (see Equations 10 and 13). As a result, agents become less willing to postpone consumption, leading to a decline in the initial overall R&D investment from 5.9% to 4.9%. This reduces sectoral TFP growth across the board (panel b) and lowers aggregate TFP growth (panel c). Lower IES also increases the growth rate of per capita consumption,

Figure 10: Increasing Manufacturing Nonhomotheticity, ϵ_m: Baseline and Counterfactuals



(a) R&D Shares (b) Sectoral TFP Growth (c) Aggregate TFP Growth (d) Consumption Shares

$\dot{c}$, as agents respond less strongly to falling interest rates (as we also documented in Comin et al. 2021). Since the strength of income effects on sectoral value-added shares depends on the growth of per capita consumption (see Equation 15), this change accelerates structural change. We find that this effect dominates sectoral value-added dynamics, resulting in faster structural change compared to the baseline (panel d).

5. Conclusion

In this paper, we have developed a quantitative multi-sector growth model integrating non-homothetic preferences and sectorally heterogeneous ideas production functions to analyze structural transformation. In an environment featuring income-driven demand shifts, sector-specific elasticities of ideas to past knowledge, and cross-sector knowledge spillovers and applicability costs, we show long-run productivity growth may hinge on income elasticities alone. Calibrated using micro-level estimates, patent citation patterns, and sectoral R&D/value-added/TFP trends, the model explains historical innovation reallocations and predicts post-2020 aggregate TFP recovery (~0.9% by 2100) as service-sector innovation accelerates. This framework resolves the tension between Baumol's stagnation thesis and service-led growth optimism, demonstrating how demand-pull and technology-push forces jointly govern structural change and productivity dynamics.

In this paper, we have consciously focused on studying the positive implications of our framework. However, our quantitative model can also be used to analyze important normative questions, which we intend to pursue in the future. Much of the literature on optimal R&D subsidies has been developed in one-sector environments. Our framework can extend this literature by quantifying the impact on social welfare of asymmetric R&D subsidies across sectors.

The interaction between the size of sectors and innovation activity allows our framework to capture the possibility that industrial policy affects sectoral innovation activity. This channel creates a distinct rationale for industrial policy that has been understudied in the literature but that can rationalize the persistent interest by policy-makers in increasing the size of the manufacturing sector. This line of reasoning may be extended to develop normative theories of the sectoral composition of the economy that complement the extensive positive literature on structural transformations.

Finally, a third normative dimension present in our framework is the knowledge flows across sectors. Certain innovation policies could affect the applicability of ideas developed in one sector across each of the sectors in the economy. Given this possibility, an important normative question is whether there are social gains from altering (in either direction) the difficulty of applying ideas across sectors, and if so, what is the optimal matrix of intersectoral applicability of ideas given the relevant policy constraints.

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A. Theoretical Appendix

A.1. Additional Results and Derivations

A.1.1. Characterization of the Constant Growth Paths

We will now study how the demand side and the innovation technology side, each impose a distinct set of constraints on the allocation of employment and expenditure, and on the rates of technical growth across sectors along any CGP. We discuss these constraints on the rates of growth through two lemmas. Lemma 3 presents the constraints imposed by the demand side and characterizes the set of industries with nonnegligible asymptotic shares in production markets. Lemma 4 presents the constraints imposed by the innovation technology and characterizes the set of industries with nonnegligible asymptotic shares in asset markets. Each lemma is followed by a corollary that discusses the constraints on the asymptotic *levels* in the allocations of sectoral technical states and per-capita consumption. Proposition 3 in the main text combines the two lemmas, shows that the two sets in fact coincide, and characterizes the asymptotic rates of productivity and consumption growth.

Lemma 3. *(Demand Side Constraints on CGPs) Along any CGP, the distribution of of sectoral consumption expenditure, employment, and output in our economy converges to a stationary $\{\Omega_i^*\}_i$, and total production employment converges to a constant, that is*

$$\lim_{t\to\infty} e^{-\eta t} L^p(t) \quad = \quad L^* > 0. \tag{47}$$

Let $\mathcal{I}^*$ denote the set of industries such that $\lim_{t\to\infty}\Omega_i(t) > 0$, and assume $g^* > 0$. Then, for any industry $i \in \mathcal{I}^*$, the asymptotic rate of productivity growth in the sector satisfies

$$\gamma_i = \epsilon_i, \qquad i \in \mathcal{I}^*. \tag{48}$$

For any industry $i \notin \mathcal{I}^*$, we have the condition $\gamma_i > \epsilon_i$ if $\sigma \in (0,1)$ and $\gamma_i < \epsilon_i$ if $\sigma \in (1,\infty)$.

Proof. See Appendix A.2. □

A key implication of Lemma 3 is that, asymptotically, a constant and finite share $L^*/L(0)$ of labor is employed in production and the remainder of labor is employed in the R&D sector. From Equation (47) and market clearing condition (27), total consumption expenditure $E(t)$ and total R&D employment $Z(t)$ asymptotically converge to constant values E^* and $Z^* = 1 - E^*/\chi$, respectively.

The corollary below summarizes the constraints implied by Lemma 3 on the asymptotic levels of variables.

Corollary 1. *The asymptotic shares of consumption expenditure, employment, and output in sector i is given by*

$$\Omega_i^* \equiv \lim_{t\to\infty} \Omega_i(t)\, e^{\xi_i g^* t} = \Xi_i \left(\frac{\chi (C^*)^{\epsilon_i}}{E^* K_i^*}\right)^{1-\sigma}, \tag{49}$$

where $\xi_i \equiv (1-\sigma)(\gamma_i - \epsilon_i)$, C^* and K_i^* are defined by Equations (29), and where the total consumption expenditure of households and the total employment share of production are asymptotically given *by*

$$E^{*}=\chi\frac{L^{*}}{L(0)}=\left[\sum_{i\in\mathcal{I}^{*}}\Xi_{i}\left(\frac{\chi(C^{*})^{\epsilon_{i}}}{K_{i}^{*}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}\in\left(0,\chi\right), \tag{50}$$

as functions of $(C^{*},\boldsymbol{K}^{*})$.

While Lemma 3 characterizes the sectoral composition of the production side of employment based on the demand-side forces, Lemma 4 below characterizes the sectoral composition of the R&D side based on the technological push forces.

Lemma 4. *(Innovation Side Constraints on CGPs) Define the asymptotic rate of growth of innovation spillovers to sector i at* $\lim_{t\to\infty}\dot{s}_{i}\left(t\right)=\gamma_{i}^{S}\theta g^{*}$ *where* $\gamma_{i}^{S}\equiv\max\left\{\gamma_{j}|\,\Phi_{ij}<\infty\right\}$. *The maximum rate of productivity growth* $\overline{\gamma}$ *satisfies*

$$\overline{\gamma}\equiv\max_{j}\left\{\frac{(1-\alpha)(1-\sigma)(\eta/g^{*})+(1-\alpha)(1-\sigma)\epsilon_{j}}{\theta\left(1-\beta_{j}\right)+(1-\alpha)(1-\sigma)}\right\}. \tag{51}$$

Assuming that for all sectors i, there exists some sector j such that $\Phi_{ij}<\infty$ *and* $\gamma_{j}=\overline{\gamma}$, *then the asymptotic rate of productivity growth in sector i satisfies:*

$$\gamma_{i}=\frac{\theta}{\theta+(1-\alpha)(1-\sigma)}\left[\beta_{i}\overline{\gamma}+\left(\frac{1-\alpha}{\theta}\right)\frac{\eta}{g^{*}}+\left(\frac{1-\alpha}{\theta}\right)\left(1-\sigma\right)\epsilon_{i}\right]. \tag{52}$$

For any sector i that asymptotically has nonzero R&D employment $\lim_{t\to\infty}Z_{i}\left(t\right)>0$, *the asymptotic rate of growth of R&D employment satisfies*

$$\lim_{t\to\infty}\dot{z}_{i}\left(t\right)=\eta-\left(1-\sigma\right)\left(\gamma_{i}-\epsilon_{i}\right)g^{*}\leq\eta. \tag{53}$$

The set of sectors that asymptotically constitute a non-negligible share of R&D expenditures, in which the expression (53) is satisfied with equality, coincides with $\mathcal{I}^{*}$ *and are characterized by* $\gamma_{i}\equiv\epsilon_{i}$.

Proof. See Appendix A.2. □

In parallel to Corollary 1, which characterized the asymptotic composition of production employment and consumption in the economy, Corollary 3 below characterizes the asymptotic composition of R&D employment and corporate assets.

Corollary 2. *The asymptotic values of the demand-pull and the technology-push vectors are given by*

$$V_{i}^{*}\equiv\lim_{t\to\infty}V_{i}\left(t\right)e^{-\left[\eta-g^{*}\left(\xi_{i}+\theta\gamma_{i}\right)\right]t}=\frac{(\chi-1)L(0)}{\chi}\frac{E^{*}\cdot\Omega_{i}^{*}}{r^{*}+(\xi_{i}+\theta\gamma_{i})g^{*}-\eta}, \tag{54}$$

$$S_{i}^{*}\equiv\lim_{t\to\infty}S_{i}\left(t\right)e^{(1-\alpha)\left[\eta-\xi_{i}g^{*}\right]t}=\left(\gamma_{i}g^{*}\right)^{\alpha}\left(A_{i}^{*}\right)^{\alpha-1}, \tag{55}$$

and the asymptotic value of the mass of R&D workers in sector i is given by

$$Z_{i}^{*}\equiv\lim_{t\to\infty}Z_{i}\left(t\right)e^{-\left[\eta-(1-\sigma)(\gamma_{i}-\epsilon_{i})g^{*}\right]t}=\left(\Gamma_{i}\left(S_{i}^{*}\right)^{\beta_{i}}V_{i}^{*}\right)^{\frac{1}{\alpha}}.$$

Furthermore, we have

$$\frac{Z^{*}}{L(0)}=\sum_{i\in\mathcal{I}^{*}}\left(\Gamma_{i}\left(S_{i}^{*}\right)^{\beta_{i}}V_{i}^{*}\right)^{\frac{1}{\alpha}}=1-\frac{1}{\chi}E^{*}. \tag{56}$$

Corollary 3. *Let $\overline{\mathcal{I}} \equiv \{i | \gamma_i = \overline{\gamma}\}$ be the set of sectors that asymptotically drive the flow of knowledge spillovers to all other sectors. For any C^*, the vector of asymptotic levels of sectoral knowledge $\boldsymbol{K}^*$ defined by Equation (29) is pinned down by the following constraint*

$$\Gamma_i \left(\sum_{j \in \overline{\mathcal{I}}} \Phi_{ij}^{-\theta} K_j^* \right)^{\beta_i} (Z_i^*)^{1-\alpha} = (\gamma_i g^*)^{\alpha} K_i^*, \tag{57}$$

as well as Equations (34), (54), and (49). The asymptotic share of workers in the R&D sector satisfies

$$\frac{Z^*}{L(0)} = \frac{(1-\alpha)\eta\,\overline{\epsilon}^*}{(1-\alpha)\eta(\theta(\vartheta-1)+\overline{\epsilon}^*)+\theta(\rho-\eta)(\epsilon_{i^*}-\beta_{i^*}\overline{\gamma})}, \tag{58}$$

where we have defined $\overline{\epsilon}^* \equiv \sum_{i\in\mathcal{I}^*} \Omega_i^* \epsilon_i$. Equation (58), Equation (50), and the constraint $\frac{1}{\chi}E^* + \frac{Z^*}{L(0)} = 1$, pin down the asymptotic level of consumption C^*.

If sectors $\mathcal{I}^*$ and $\overline{\mathcal{I}}$ are singletons, that is, there exists a unique sector i^* satisfying condition (31), then Equation (58) pins down the ratio of $C^*/K_{i^*}^*$,

$$\frac{(C^*)^{\epsilon_{i^*}}}{K_{i^*}^*} = \frac{1}{\chi \Xi_i^{\frac{1}{1-\sigma}}} \frac{\theta}{1+\theta} \frac{(1-\alpha)\eta\,\overline{\epsilon}^* L_0}{(1-\alpha)\eta(\theta(\vartheta-1)+\overline{\epsilon}^*)+\theta(\rho-\eta)(\epsilon_{i^*}-\beta_{i^*}\overline{\gamma})},$$

Also, from the asymptotic spillover condition 57, we have that:

$$\Gamma_{\overline{i}} K_{\overline{i}}^{\beta_{\overline{i}}-1} = (\bar{\gamma} g)^{\alpha} \left(\frac{L(0)}{1+\theta} \frac{E^* \cdot \Omega_{\overline{i}}^*}{r^*+(1-\sigma)(\gamma_{\overline{i}}-\epsilon_{\overline{i}})g^*-\eta} \right)^{1-\alpha}$$

$$\Rightarrow \qquad K_{\overline{i}}^{\beta_{\overline{i}}-1+(1-\sigma)(1-\alpha)} = \Gamma_{\overline{i}}^{-1} (\bar{\gamma} g)^{\alpha} \left(\frac{L(0)}{1+\theta} \frac{E^*}{r^*+(1-\sigma)(\gamma_{\overline{i}}-\epsilon_{\overline{i}})g^*-\eta} \Xi_{\overline{i}} \left(\frac{\chi (C^*)^{\epsilon_{\overline{i}}}}{E^* K_{\overline{i}}^*} \right)^{1-\sigma} \right)^{1-\alpha}$$

This equation expresses $K_{\overline{i}}$ in terms of C^* and known variables. Using again (58) for sector i^* we have that:

$$\Gamma_{i^*} \left(\Phi_{i^*\overline{i}}^{-\theta} K_{\overline{i}} \right)^{\beta_{i^*}} = (\bar{\gamma} g)^{\alpha} \left(\frac{L(0)}{1+\theta} \frac{E^*}{r^*+(1-\sigma)(\gamma_{\overline{i}}-\epsilon_{\overline{i}})g^*-\eta} \Xi_{\overline{i}} \left(\frac{\chi (C^*)^{\epsilon_{\overline{i}}}}{E^* K_{\overline{i}}^*} \right)^{1-\sigma} \right)^{1-\alpha} K_{i^*}$$

Combining the last two equations, we have an expression for C^* as

$$C^* = (\text{constant})^{\epsilon_{i^*}(1+\mu) - \frac{\beta_{i^*}\epsilon_{\overline{i}}\mu}{\beta_{\overline{i}}-1+\mu}}$$

where $\mu \equiv (1-\sigma)(1-\alpha)$. Thus there is always unique solution to CGP.

A.1.2. Transitional Dynamics

As discussed in Section 3.4 of the main paper, we perform a transformation of the state and control variables to normalize them with respect to their CGP trends. We define

$$\widehat{k}_i(t) \equiv k_i(t) - \gamma_i \theta g^* t, \tag{59}$$

$$\widehat{z}_i(t) \equiv z_i(t) - z_o(t) + (1-\sigma)\left((\gamma_i - \epsilon_i) - (\gamma_o - \epsilon_o)\right) g^* t, \qquad i \neq o, \tag{60}$$

$$\widehat{c}(t) \equiv c(t) - g^* t, \tag{61}$$

where the vector of normalized log sectoral stocks of knowledge $\widehat{\boldsymbol{k}}\left(t\right) \equiv \left[k_i(t)\right]_i$ characterizes the sectoral states of knowledge, the vector of normalized log relative sectoral R&D $\widehat{\boldsymbol{v}}\left(t\right) \equiv \left[\widehat{v}_i(t)\right]_{i\neq o}$ characterizes the allocation of R&D across sectors (compared to some baseline sector o), and normalized log consumption $\widehat{c}\left(t\right)$ characterizes the size of consumption (and thus the total R&D investment), all relative to their respective CGP trends.

Given the current values of these normalized variables, we can first write per capita expenditure $E\left(t\right)$ as a function of the normalized variables as $E\left(t\right) = \widetilde{E}\left(\widehat{c}\left(t\right), \widehat{\boldsymbol{k}}\left(t\right); t\right)$ where

$$\widetilde{\mathcal{E}}\left(\widehat{c}, \widehat{\boldsymbol{k}}; t\right) = \left(\sum_i \Xi_i \exp\left(\left(1-\sigma\right)\left(\epsilon_i \widehat{c} - \tfrac{1}{\theta}\widehat{k}_i\right)\right) e^{-(1-\sigma)(\gamma_i-\epsilon_i)g^* t}\right)^{\frac{1}{1-\sigma}}.$$

Equation (28) then gives us the R&D share of labor $\widehat{Z}\left(t\right)$. Next, we find sectoral shares of R&D as

$$\widetilde{\Upsilon}_i\left(\widehat{\boldsymbol{z}}; t\right) = \frac{\exp\left(\widehat{z}_i - \left(1-\sigma\right)\left(\left(\gamma_i - \epsilon_i\right) - \left(\gamma_o - \epsilon_o\right)\right) g^* t\right)}{1 + \sum_{i'\neq o} \exp\left(\widehat{z}_{i'} - \left(1-\sigma\right)\left(\left(\gamma_{i'} - \epsilon_{i'}\right) - \left(\gamma_o - \epsilon_o\right)\right) g^* t\right)}.$$

Next, we define the normalized spillover function as

$$\begin{aligned}\widehat{s}_i\left(t\right) &\equiv \log S_i\left(t\right) - \theta\overline{\gamma} g^* t,\\ &= \mathcal{S}_i\left(\widehat{\boldsymbol{k}}\left(t\right); t\right) \equiv \log \sum_j \Phi_{j,i}^{-\theta} \exp\left(\widehat{k}_i + \theta\gamma_i g^* t\right).\end{aligned}$$

Now, note that along a CGP the right hand side of Equation (39) grows at the following rate

$$\beta_i \theta \overline{\gamma} g^* + \left(1-\alpha\right)\left(\eta - \left(1-\sigma\right)\left(\gamma_i - \epsilon_i\right) g^*\right) - \theta\gamma_i g^* = 0,$$

for the value of the CGP growth rate g^* implying that we can can write $\dot{\widehat{k}}_i\left(t\right) = \widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}}\left(t\right), \widehat{\boldsymbol{z}}\left(t\right), \widehat{c}\left(t\right); t\right)$,where we have defined the function $\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right)$ as

$$\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right) \equiv \Gamma_i L_0^{1-\alpha}\left(1 - \tfrac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c}, \widehat{\boldsymbol{k}}; t\right)\right)^{1-\alpha} \widetilde{\Upsilon}_i\left(\widehat{\boldsymbol{z}}; t\right)^{1-\alpha} \exp\left(\beta_i \mathcal{S}_i\left(\widehat{\boldsymbol{k}}; t\right) - \widehat{k}_i\right) - \theta\gamma_i g^*. \quad (62)$$

Using Equation (40), we can write the evolution of the normalized sectoral R&D allocations as $\dot{\widehat{z}}_i\left(t\right) = \widetilde{\mathcal{Z}}_i\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right)$ where we have defined

$$\widetilde{\mathcal{Z}}_i\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right) \equiv \frac{1}{\alpha}\left[\beta_i \dot{\widetilde{\mathcal{S}}}_i\left(\widehat{\boldsymbol{k}}; t\right) - \beta_o \dot{\widetilde{\mathcal{S}}}_o\left(\widehat{\boldsymbol{k}}; t\right) - \frac{\chi - 1}{\chi}\frac{\widetilde{\mathcal{E}}\left(\widehat{c}, \widehat{\boldsymbol{k}}; t\right)}{1 - \frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c}, \widehat{\boldsymbol{k}}; t\right)}\left(\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right) - \widetilde{\mathcal{K}}_o\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right) + \theta\left(\gamma_i - \gamma_o\right) g^*\right)\right], \quad (63)$$

and where we have defined the time derivative of the spillovers as

$$\dot{\widetilde{\mathcal{S}}}_j\left(\widehat{\boldsymbol{k}}; t\right) \equiv \sum_i \frac{\Phi_{ji}^{-\theta} \exp\left(\widehat{k}_i - \theta\left(\overline{\gamma} - \gamma_i\right) g^* t\right)}{\sum_{i'} \Phi_{ji'}^{-\theta} \exp\left(\widehat{k}_{i'} - \theta\left(\overline{\gamma} - \gamma_{i'}\right) g^* t\right)}\left(\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{z}}, \widehat{c}; t\right) + \theta\gamma_i g^*\right).$$

Finally, the evolution of the normalized log consumption variable is given by $\dot{\widehat{c}}\left(t\right) = \widetilde{\mathcal{C}}\left(\widehat{\boldsymbol{k}}(t), \widehat{\boldsymbol{z}}(t), \widehat{c}(t); t\right)$

with the right-hand-side function defined as

$$\widetilde{\mathcal{C}}\left(\widehat{\boldsymbol{k}},\widehat{\boldsymbol{z}},\widehat{c};t\right)=\frac{1-\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)}{\widetilde{\theta}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\left(1-\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)\right)+\alpha\widetilde{\varepsilon}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)}\left(\alpha\eta-\rho-\mathbb{E}_{\widetilde{\boldsymbol{\Upsilon}}\left(\widehat{\boldsymbol{z}};t\right)}\left[\beta_i\dot{\widetilde{\mathcal{S}}}_i\left(\widehat{\boldsymbol{k}};t\right)\right]\right)$$
$$+\left(\frac{\widetilde{\iota}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\left(1-\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)\right)+(\chi-1+\alpha)\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)}{\widetilde{\theta}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\left(1-\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)\right)+\alpha\widetilde{\varepsilon}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\frac{1}{\chi}\widetilde{\mathcal{E}}\left(\widehat{c},\widehat{\boldsymbol{k}};t\right)}\right)\frac{1}{\theta}\mathbb{E}_{\widetilde{\boldsymbol{\Omega}}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)}\left[\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}},\widehat{\boldsymbol{z}},\widehat{c};t\right)+\theta\gamma_i g^*\right]-g^*, \tag{64}$$

where we have defined the following functions

$$\widetilde{\varepsilon}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\equiv\mathbb{E}_{\widetilde{\boldsymbol{\Omega}}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)}\left[\epsilon_i\right], \tag{65}$$

$$\widetilde{\iota}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\equiv 1+(1-\sigma)\,\mathbb{C}_{\widetilde{\boldsymbol{\Omega}}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)}\left(\frac{\epsilon_i}{\widetilde{\varepsilon}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)},\frac{\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}},\widehat{\boldsymbol{z}},\widehat{c};t\right)+\theta\gamma_i g^*}{\mathbb{E}_{\widetilde{\boldsymbol{\Omega}}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)}\left[\widetilde{\mathcal{K}}_i\left(\widehat{\boldsymbol{k}},\widehat{\boldsymbol{z}},\widehat{c};t\right)+\theta\gamma_i g^*\right]}\right), \tag{66}$$

$$\widetilde{\theta}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\equiv\vartheta+\widetilde{\varepsilon}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\left[1+(1-\sigma)\,\mathbb{V}_{\widetilde{\boldsymbol{\Omega}}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)}\left(\frac{\epsilon_i}{\widetilde{\varepsilon}\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)}\right)\right]-1, \tag{67}$$

where the composition of sectoral demand as a function of the state and time is given by

$$\widetilde{\Omega}_i\left(\widehat{\boldsymbol{k}},\widehat{c};t\right)\equiv\frac{\Xi_i\,\exp\left((1-\sigma)\left(\epsilon_i\widehat{c}-\frac{1}{\theta}\widehat{k}_i\right)\right)\,e^{-(1-\sigma)(\gamma_i-\epsilon_i)g^*t}}{\sum_{i'}\Xi_{i'}\,\exp\left((1-\sigma)\left(\epsilon_{i'}\widehat{c}-\frac{1}{\theta}\widehat{k}_{i'}\right)\right)\,e^{-(1-\sigma)(\gamma_{i'}-\epsilon_{i'})g^*t}}.$$

Together the above equations imply the following set of dynamic equations for the vector $\widehat{\boldsymbol{x}}(t)\equiv\left(\widehat{\boldsymbol{k}}(t),\widehat{\boldsymbol{z}}(t),\widehat{c}(t)\right)'$ characterizing the state of the economy

$$\dot{\widehat{\boldsymbol{x}}}(t)=\mathcal{F}\left(\widehat{\boldsymbol{x}}(t);t\right)\equiv\left(\widetilde{\boldsymbol{\mathcal{K}}}\left(\widehat{\boldsymbol{x}}(t);t\right),\widetilde{\boldsymbol{\mathcal{Z}}}\left(\widehat{\boldsymbol{x}}(t);t\right),\widetilde{\mathcal{C}}\left(\widehat{\boldsymbol{x}}(t);t\right)\right)',$$

with functions $\widetilde{\boldsymbol{\mathcal{K}}}$, $\widetilde{\boldsymbol{\mathcal{Z}}}$, and $\widetilde{\mathcal{C}}$ given by Equations (62)-(64). Next, we make the observation that the time-dependence of the vector function $\mathcal{F}$ can be written in terms of a vector of exponential functions $\boldsymbol{\xi}(t)\equiv\left[\exp\left(-\xi_j t\right)\right]_j$ with $\xi_j=\gamma_j-\epsilon_j$ for $1\leq j\leq I$, and $\xi_j=\overline{\gamma}-\gamma_j$ for $j\neq o$. such that we can write $\dot{\widehat{\boldsymbol{x}}}(t)=\widetilde{\mathcal{F}}\left(\widehat{\boldsymbol{x}}(t);\boldsymbol{\xi}(t)\right)$. Let ξ_{min} denote the lowest value of ξ_j characterizing the slowest driver of the time variation in function and let $\widetilde{\xi}\equiv\exp\left(-\xi_{min}t\right)$. We perform a change of variables from time t to $\widetilde{\xi}$and, with slight abuse of notation, write

$$\frac{d}{d\widetilde{\xi}}\widehat{\boldsymbol{x}}\left(\widetilde{\xi}\right)=-\frac{1}{\xi_{min}}\frac{1}{\widetilde{\xi}}\widetilde{\boldsymbol{\mathcal{F}}}\left(\widehat{\boldsymbol{x}}\left(\widetilde{\xi}\right);\boldsymbol{\xi}\left(\widetilde{\xi}\right)\right). \tag{68}$$

The problem now is that of solving the dynamical system in Equation (68) for $\widetilde{\xi}\in[0,1]$ with the following boundary conditions: $\widehat{\boldsymbol{k}}\left(\widetilde{\xi}=1\right)=\log\left(\boldsymbol{K}(0)\right)$ and with the vector $\left(\widehat{\boldsymbol{k}}\left(\widetilde{\xi}=0\right),\widehat{\boldsymbol{z}}\left(\widetilde{\xi}=0\right),\widehat{c}\left(\widetilde{\xi}=0\right.\right.$ approaching $\left(\widehat{\boldsymbol{k}}^*,\widehat{\boldsymbol{z}}^*,\widehat{c}^*\right)'$, where the latter is the vector satisfying $\widetilde{\mathcal{F}}\left(\left(\widehat{\boldsymbol{k}}^*,\widehat{\boldsymbol{z}}^*,\widehat{c}^*\right)';\boldsymbol{0}\right)=\boldsymbol{0}$.

We solve the above boundary value problem by approximating the solution $\widehat{\boldsymbol{x}}\left(\widetilde{\xi}\right)$ with a set of parameterized spline functions. Specifically, we determine the spline parameters by minimizing the discrepancy between the left-hand side (time derivatives) and right-hand side (state-dependent terms) of the ODE in Equation (68) along a number of grid points within the unit interval of $\tilde{\xi}$. This approach transforms the original differential equation into an optimization problem, enabling efficient numerical solution and accurate approximation even in complex nonlinear settings.

Alternative Approach Conceptually, we can compute an equilibrium in an analogous way to a standard shooting algorithm accounting for the multi-sector nature of our model. Given the model parameters and an initial vector for the stock of ideas at time t_0, $\boldsymbol{K}(t_0)$, we can start guessing an initial consumption-savings decision and allocation of researchers across sectors and subsequently determine the path of future consumption following the Euler equation (10) and the allocations of researchers across sectors according to the dynamic equation for the value of sectoral innovation (the Hamilton-Jacobi-Bellman Equation) (23) combined with the free-entry condition, Equation (22). We can then iterate over the guesses until the optimal solution is reached. In practice, the presence of nonhomotheticities makes the model substantially non-linear making convergence to the CGP hard to assess numerically. We have found that to ensure convergence to the CGP, the most efficient procedure is to simulate the model in two regions. For very large values of time, $t \geq T$, we use a log-linear approximation along the stable eigenvectors of the system that takes into account the non-autonomous nature of the dynamic system. This approximation ensures by construction that the system converges to the CGP as time goes forward. For not very large values of time, $t < T$, we use the full dynamic model, which has as initial condition the stock of knowledge at time t_0, $\boldsymbol{K}(t_0)$, and as terminal values those at time T, along with the value function, consumption and employment shares. To solve the transition dynamics between these two points we search over the initial consumption and labor allocations. We simulate the model transition dynamics using a collocation method to solve for the system of differential equations.

A.2. Proofs and Derivations

Proposition. *Proposition 1.*

Proof. The derivations for the intratemporal allocation as given by Equation (6) with the price index as defined by Equation (7) are straightforward [see Comin et al., 2021]. We only present the proof on the intertemporal component of the solution.

For a given path of real interest rate $[r(t)]_{t=0}^{\infty}$ and sectoral good prices $[\boldsymbol{P}(t)]_{t=0}^{\infty}$, the current-value Hamiltonian for the consumer problem (1) may be written as

$$\hat{\mathcal{H}} \equiv \frac{C(t)^{1-\vartheta}-1}{1-\vartheta} + \lambda(t)\left[1 + (r(t)-\eta)A(t) - \mathcal{E}(C(t);\boldsymbol{P}(t))\right],$$

where we have defined the expenditure function $\mathcal{E}$ by Equation (7). Let us start with the necessary conditions. The FOCs for the Hamiltonian are as follows

$$\frac{\partial\hat{\mathcal{H}}}{\partial C} = 0 \qquad \Rightarrow \qquad C^{-\vartheta} - \lambda\frac{\partial\mathcal{E}}{\partial C} = 0, \tag{69}$$

$$\frac{\partial\hat{\mathcal{H}}}{\partial A} = (\rho-\eta)\lambda - \dot{\lambda} \qquad \Rightarrow \qquad -\frac{\dot{\lambda}}{\lambda} = r - \rho. \tag{70}$$

In addition, we impose the transversality condition $\lim_{t\to\infty} e^{-(\rho-\eta)t}\lambda(t)A(t) = 0$. Equations

(69) and (70) together with the law of evolution of assets (8) and the transversality equation (9) characterize paths of per capita real aggregate consumption and asset holdings $[C(\cdot), A(\cdot)]$, and costate $\lambda(\cdot)$ that satisfy necessary conditions for optimality.

Next, we show the conditions that ensure the solution above indeed corresponds to the unique solution to the household utility maximization problem. A standard argument (using (70) and the No-Ponzi constraint) shows that for all feasible pairs $[C(\cdot), A(\cdot)]$, we have that $\lim_{t\to\infty} \exp(-\rho t)\lambda(t) A(t) \geq 0$.Therefore, we can establish that the pair characterized by Equations (69), (70), and the transversality condition indeed correspond to the optimum if the Hamiltonian is concave in C. Furthermore, since the Hamiltonian is separable in (C, A) and linear A, strict concavity in C implies the uniqueness of the optimum for the household problem.

Combining Equations (69), (70), and the transversality condition with the definition of the expenditure function (7) gives the Euler equation (10) and the transversality condition (11). It remains for us to find conditions that ensure the strict concavity of $\mathcal{E}$ in C to ensure the sufficiency of the conditions above and uniqueness of the solution.

The second order condition for C is $0 > -\theta C^{-(\vartheta+1)} - \lambda \frac{\partial^2 \mathcal{E}}{\partial C^2} = -C^{-(\vartheta+1)}(\vartheta + \eta_{\mathcal{E}_C})$,where $\eta_{\mathcal{E}_C} \equiv \frac{C\frac{\partial^2 \mathcal{E}}{\partial C^2}}{\frac{\partial \mathcal{E}}{\partial C}}$ denotes the elasticity of marginal expenditure with respect to real consumption C. In the equality above, we have substituted for $\lambda = \left(C^{\vartheta}\partial\mathcal{E}/\partial C\right)^{-1}$ from Equation (69). The second order condition therefore implies that a sufficient condition for the strict concavity of the Hamiltonian to be $\eta_{\mathcal{E}_C} > -\vartheta$.

Using Equation (7), we can compute the elasticity to find $\eta_{\mathcal{E}_C} = \overline{\epsilon}(t)\left[1 + (1-\sigma)\mathbb{V}_{\boldsymbol{\Omega}(t)}\left(\frac{\epsilon_i}{\overline{\epsilon}(t)}, t\right)\right] - 1$. It then follows that conditions $\sigma \in (0, 1]$ and $\epsilon_i > 1 - \vartheta$ for all i are sufficient to ensure the strict concavity of the Hamiltonian in C. □

Proposition. *Proposition 2.*

Proof. First, we compute the evolution of the distribution $F_i(Q, t)$ of frontier ideas in each sector. Given our assumptions, ideas with productivity exceeding Q arrive in sector i at time t at Poisson rate $\tilde{\Gamma}_i Z_i^{1-\alpha}(1 - \hat{F}_i(Q, t))$, with $\hat{F}_i$ denoting the productivity distribution of ideas arriving in the sector (which we will derive below). Thus, the probability that no new technique in sector i exceeds Q evolves over infinitessimal time interval dt as

$$F_i(Q, t + dt) = F_i(Q, t)e^{-\tilde{\Gamma}_i Z_i^{1-\alpha}(1-\hat{F}_i(Q,t))dt}.$$

Taking the logarithm of this expression and the limit as $dt \to 0$, we find that

$$\frac{\partial}{\partial t}\log F_i(Q, t) = -\tilde{\Gamma}_i Z_i(t)^{1-\alpha}(1 - \hat{F}_i(Q, t)). \tag{71}$$

Next, let us derive the distribution $\hat{F}_i(Q, t)$. As stated in the main text, new techniques come from the combination of a novel component Q_n drawn from a Pareto distribution, $\mathbb{P}[Q_n > Q] = (Q/\underline{Q})^{-\theta}$, and an existing adopted technique Q_o drawn from $\tilde{F}_i(Q_o, t)$. These two components are combined according to $Q = Q_n \times Q_o^{\beta_i}$. Let $\widetilde{F}_i(Q_o, t)$ denote the cumulative distribution function of the productivity of adopted ideas Q_o in sector i at time t with associated density $\widetilde{f}(Q_o, t)$. Thus, the probability that the new idea exceeds Q in productivity by at least a factor χ is given by

$$1-\hat{F}_i(Q,t)=\int_0^\infty \mathbb{P}\left[Q_n>\frac{Q}{Q_o^{\beta_i}}\right]d\widetilde{F}_i\left(Q_o,t\right)=\left(\frac{Q}{\underline{Q}}\right)^{-\theta}\int_0^\infty Q_o^{\theta\beta_i}\,\tilde{f}_i\left(Q_o,t\right)\,dQ_o.$$

Combining the above result with Equation (71), we can show that the cumulative distribution function of the frontier technique in sector i across varieties evolves according to:

$$\tfrac{\partial}{\partial t}\log F_i\left(Q,t\right)=-\widetilde{\Gamma}_i\underline{Q}^\theta\,Z_i\left(t\right)^{1-\alpha}Q^{-\theta}\int_0^\infty Q_o^{\beta_i\theta}\widetilde{f}_i\left(Q_o,t\right)dQ_o. \tag{72}$$

Next, we integrate Equation (72) to find

$$\begin{aligned}\log F_i(Q,t)-\log F_i(Q,0)&=\int_0^t \tfrac{\partial}{\partial\tau}\log F_i(Q,\tau),\\&=-\widetilde{\Gamma}_i\underline{Q}^\theta Q^{-\theta}\int_0^t Z_i\left(\tau\right)^{1-\alpha}\left(\int_0^\infty Q_o^{\beta_i\theta}\widetilde{f}_i\left(Q_o,\tau\right)dQ_o\right)d\tau,\\&=-\left[K_i\left(t\right)-K_i\left(0\right)\right]Q^{-\theta},\end{aligned} \tag{73}$$

where we have defined $K_i\left(t\right)$ as a measure of the stock of knowledge in sector i at time t:

$$K_i\left(t\right)\equiv K_i\left(0\right)+\widetilde{\Gamma}_i\underline{Q}^\theta\chi^{-\theta}\int_0^t Z_i\left(\tau\right)^{1-\alpha}\left(\int_0^\infty Q_o^{\beta\theta}\widetilde{f}_i\left(Q_o,\tau\right)\,dQ_o\right)d\tau, \tag{74}$$

using the boundary condition that at time $t=0$ satisfies $\log F_i(Q,0)=-K_i(0)Q^{-\theta}$. Thus, we arrive at Equation (16).

Using Equation (16), we can now characterize the evolution of the stock of knowledge $K_i\left(t\right)$. From the definition (74), we have

$$\dot{K}_i\left(t\right)=\widetilde{\Gamma}_i\underline{Q}^\theta\chi^{-\theta}Z_i\left(t\right)^{1-\alpha}\left(\int_0^\infty Q_o^{\beta\theta}\widetilde{f}_i\left(Q_o,t\right)\,dQ_o\right). \tag{75}$$

Given that the distribution of the source of ideas from any sector is given by Equation 16, the probability that an adopted technique in sector i is not greater than Q_o follows from computing the maxim of iid draws from each sector, correcting for the applicability cost Φ_{ij} as discussed in Equation (5), leading to

$$\mathbb{P}[Q\leq Q_o]=\widetilde{F}_i\left(Q_o,t\right)=\Pi_{j=1}^I\mathbb{P}[\tilde{Q}_{jo}\leq\Phi_{ij}Q_o]=\exp\left[-\sum_{j=1}^I\left(K_j(t)\Phi_{ij}^{-\theta}\right)Q_o^{-\theta}\right]. \tag{76}$$

Hence, the productivity distribution of adopted techniques Q_o in sector i is given by $\widetilde{F}_i\left(Q_o,t\right)=\exp\left[-S_i\left(t\right)\,Q_o^{-\theta}\right]$, where we have defined the knowledge spillovers $S_i\left(t\right)$ in sector i at time t given by Equation (18). Since this distribution is also Frechet, we compute the expression in Equation (75)

$$\int_0^\infty Q_o^{\beta\theta}\widetilde{f}_i\left(Q_o,t\right)\,dQ_o=\Gamma\left(1-\beta_i\right)\,S_i\left(t\right)^{\beta_i},$$

where $\Gamma\left(\cdot\right)$ is the Gamma function. This result, combined with Equation (75) and $\Gamma_i\equiv\widetilde{\Gamma}_i\underline{Q}^\theta\chi^{-\theta}\Gamma\left(1-\beta_i\right)$ leads to Equation (17). □

Lemma. *Lemma 1.*

Proof. The rate of arrival of patents in sector i at time t thus equals the product of the rate of arrival of new ideas and the probability that the quality of the new idea $Q' = Q_n Q_o^{\beta_i}$ exceeds the current frontier technique in this sector Q by a factor $\Psi_i(t)$, that is,

$$\begin{aligned}
P\dot{A}T_i(t) &= \widetilde{\Gamma}_i Z_i(t)^{1-\alpha} \int_0^\infty \int_0^\infty \mathbb{P}\left(Q_n \geq \tfrac{\Psi_i(t) Q}{Q_o^{\beta_i}}\right) d\widetilde{F}_i(Q_o, t)\, dF_i(Q, t), \\
&= \widetilde{\Gamma}_i Z_i(t)^{1-\alpha} \underline{Q}^\theta \Psi_i(t)^{-\theta} \left(\int_0^\infty Q^{-\theta} dF_i(Q, t)\right) \left(\int_0^\infty Q_o^{\beta_i \theta} d\widetilde{F}_i(Q_o, t)\right), \\
&= \Gamma_i Z_i(t)^{1-\alpha} S_i(t)^{\beta_i} / K_i(t),
\end{aligned}$$

where in the last equation we have used the fact that $\int_0^\infty Q^{-\theta} dF_i(Q,t) = \frac{1}{K_i(t)}$ and $\int_0^\infty Q_o^{\beta_i\theta} d\widetilde{F}_i(Q_o, t) = \Gamma(1-\beta_i) S_i(t)^{\beta_i}$ and $\Gamma_i \equiv \widetilde{\Gamma}_i \underline{Q}^\theta \Gamma(1-\beta_i)$. Using Equation (17) then leads to the desired result in Equation (19). □

Lemma. *Lemma 2.*

Proof. First, we note that the profits of a frontier firm with productivity Q at time t is the same as that of the total sectoral profits $\widetilde{\Pi}_i(Q,t) = \Pi_i(t)$. Next, we compute the probability $\mathbb{P}(Q; s, t)$ that a frontier firm with productivity Q at time t has still not been replaced by another better technology at some time $s \geq t$. From Equation (73), we have

$$\tfrac{\partial}{\partial t} \log \mathbb{P}(Q; s, t) = \dot{K}_i(s)\, Q^{-\theta} \;\Rightarrow\; \mathbb{P}(Q; s, t) = \exp\left(-\left(K_i(s) - K_i(t)\right) Q^{-\theta}\right).$$

Now, let us consider an R&D firm in sector i at time t and compute the return to hiring an additional R&D work for this firm.

$$\begin{aligned}
1 &\leq \widetilde{\Gamma}_i Z_i(t)^{-\alpha} \int_t^\infty \int_0^\infty \int_0^\infty \int_{Q/Q_o^\beta}^\infty \theta \underline{Q}^\theta Q_n^{-(\theta+1)} \widetilde{\Pi}_i\left(Q_n Q_o^{\beta_i}, s\right) \mathbb{P}\left(Q_n Q_o^{\beta_i}; s, t\right) e^{-\int_t^s r(\tau) d\tau} \\
&\qquad \times dQ_n\, d\widetilde{F}_i(Q_o, t)\, dF_i(Q, t)\, ds, \\
&= \widetilde{\Gamma}_i Z_i(t)^{-\alpha} \int_t^\infty \int_0^\infty \int_0^\infty \int_Q^\infty \theta \underline{Q}^\theta Q_o^{(\theta+1)\beta_i} q^{-(\theta+1)} \widetilde{\Pi}_i(q, s)\, \mathbb{P}(q; s, t)\, e^{-\int_t^s r(\tau) d\tau} \\
&\qquad \times \frac{dq}{Q_o^{\beta_i}} d\widetilde{F}_i(Q_o, t)\, dF_i(Q, t)\, ds, \\
&= \widetilde{\Gamma}_i \underline{Q}^\theta Z_i(t)^{-\alpha} \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau) d\tau} \int_0^\infty \int_0^\infty Q_o^{\theta\beta_i} \int_Q^\infty \theta q^{-(\theta+1)} e^{-(K_i(s) - K_i(t)) q^{-\theta}} \\
&\qquad \times dq\, d\widetilde{F}_i(Q_o, t)\, dF_i(Q, t)\, ds, \\
&= \widetilde{\Gamma}_i \underline{Q}^\theta Z_i(t)^{-\alpha} \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau) d\tau} \left(\int_0^\infty \frac{1 - e^{-(K_i(s) - K_i(t)) Q^{-\theta}}}{K_i(s) - K_i(t)} dF_i(Q, t)\right) \\
&\qquad \times \left(\int_0^\infty Q_o^{\theta\beta_i} d\widetilde{F}_i(Q_o, t)\, ds\right), \\
&= \widetilde{\Gamma}_i \underline{Q}^\theta \Gamma(1-\beta_i)\, Z_i(t)^{-\alpha} S_i(t)^{\beta_i} \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau) d\tau} \left(\frac{1 - \int_0^\infty e^{-(K_i(s) - K_i(t)) Q^{-\theta}} dF_i(Q,t)}{K_i(s) - K_i(t)}\right) ds, \\
&= \Gamma_i Z_i(t)^{-\alpha} S_i(t)^{\beta_i} \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau) d\tau} \left(\frac{1 - \int_0^\infty e^{-(K_i(s) - K_i(t)) Q^{-\theta} - K_i(t) Q^{-\theta}} d\left(K_i(t) Q^{-\theta}\right)}{K_i(s) - K_i(t)}\right) ds, \\
&= \Gamma_i Z_i(t)^{-\alpha} S_i(t)^{\beta_i} \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau) d\tau} \left(\frac{1 - K_i(t)/K_i(s)}{K_i(s) - K_i(t)}\right) ds, \\
&= \Gamma_i Z_i(t)^{-\alpha} S_i(t)^{\beta_i} \int_t^\infty \frac{\Pi_i(s)}{K_i(s)} e^{-\int_t^s r(\tau) d\tau} ds,
\end{aligned}$$

$$= \Gamma_i Z_i(t)^{-\alpha} S_i(t)^{\beta_i} V_i(t),$$

where we have defined

$$V_i(t) \equiv \int_t^\infty \tfrac{\Pi_i(s)}{K_i(s)} e^{-\int_t^s r(\tau)d\tau} ds, \tag{77}$$

which satisfied Equation (23).

Next, we compute the value of all assets in sector i, given by

$$\begin{aligned}
A_i(t) &= \int_t^\infty \int_0^\infty \widetilde{\Pi}_i(Q,s)\, \mathbb{P}(Q;s,t)\, e^{-\int_t^s r(\tau)d\tau} dF_i(Q,t)\, ds, \\
&= \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau)d\tau} \int_0^\infty e^{-(K_i(s)-K_i(t))Q^{-\theta}} dF_i(Q,t)\, ds, \\
&= \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau)d\tau} \int_0^\infty e^{-K_i(s)Q^{-\theta}} d\left(K_i(t)\, Q^{-\theta}\right) ds, \\
&= \int_t^\infty \Pi_i(s)\, e^{-\int_t^s r(\tau)d\tau} \tfrac{K_i(t)}{K_i(s)} ds = K_i(t)\, V_i(t).
\end{aligned}$$

This implies the following differential equation that characterizes the evolution of the assets in the sector:

$$\begin{aligned}
\dot{A}_i(t) &= K_i(t)\, \dot{V}_i(t) - \dot{K}_i(t)\, V_i(t), \\
&= K_i(t) \left(r(t)\, V_i(t) - \tfrac{\Pi_i(s)}{K_i(s)} \right) - \Gamma_i Z_i(t)^{1-\alpha} S_i(t)^{\beta_i} V_i(t), \\
&= r(t)\, \dot{A}_i(t) - \Pi_i(t) - Z_i(t).
\end{aligned}$$

□

Proposition. *See Proposition 3.*

Proof. To simplify the expressions, let $\widetilde{\eta} \equiv \left(\frac{1-\alpha}{\theta}\right) \frac{\eta}{g^*}$ and $\widetilde{\alpha} \equiv \left(\frac{1-\alpha}{\theta}\right)(1-\sigma)$. Applying the results of Lemmas 3 and 4, we have the following sets of constraints from both demand and supply sides for the case of $\sigma \in (0,1)$:

$$\gamma_i = \tfrac{\beta_i \overline{\gamma} + \widetilde{\alpha}\epsilon_i + \widetilde{\eta}}{1+\widetilde{\alpha}} \geq \epsilon_i, \qquad \text{with equality for } i^* \in \mathcal{I}^*,$$
$$\overline{\gamma} = \underset{j}{\text{argmax}} \left\{ \tfrac{\widetilde{\eta} + \widetilde{\alpha}\epsilon_j}{1-\beta_j+\widetilde{\alpha}} \right\},$$

From the second equality, we have:

$$\gamma_i - \epsilon_i = \tfrac{\beta_i \overline{\gamma} + \widetilde{\alpha}\epsilon_i + \widetilde{\eta} - \epsilon_i(1+\widetilde{\alpha})}{1+\widetilde{\alpha}} = \tfrac{\beta_i \overline{\gamma} + \widetilde{\eta} - \epsilon_i}{1+\widetilde{\alpha}}.$$

Therefore, to achieve the first inequality, we need to have:

$$\epsilon_i - \beta_i \overline{\gamma} \leq \widetilde{\eta}, \tag{78}$$

with equality for all $i^* \in \mathcal{I}^*$. From the third condition, we have for any sector i and any sector $i^\dagger$, where the latter achieves the maximum in that condition:

$$\gamma_i = \tfrac{\beta_i \overline{\gamma} + \widetilde{\eta} + \widetilde{\alpha}\epsilon_i}{1+\widetilde{\alpha}} \leq \gamma_{i^\dagger} = \overline{\gamma} = \tfrac{\beta_{i^\dagger} \overline{\gamma} + \widetilde{\eta} + \widetilde{\alpha}\epsilon_{i^\dagger}}{1+\widetilde{\alpha}},$$

and thus: $\widetilde{\alpha}\epsilon_{i^\dagger} + \beta_{i^\dagger}\overline{\gamma} \geq \widetilde{\alpha}\epsilon_i + \beta_i\overline{\gamma}, \qquad \forall j$,which leads to the condition in Equation (32).

From the third condition, we now have:

$$\overline{\gamma} = \frac{\beta_{i^\dagger}\overline{\gamma}+\widetilde{\eta}+\widetilde{\alpha}\epsilon_{i^\dagger}}{1+\widetilde{\alpha}} = \frac{\max\left\{\widetilde{\alpha}\epsilon_j+\beta_j\overline{\gamma}\right\}+\widetilde{\eta}}{1+\widetilde{\alpha}} = \frac{\max\left\{\widetilde{\alpha}\epsilon_j+\beta_j\overline{\gamma}\right\}+\max\left\{\epsilon_i - \beta_i\overline{\gamma}\right\}}{1+\widetilde{\alpha}},,$$

where in the second equality, we have used the condition (32) and in the last equality, we have used the condition in Equation (78).

Next, we prove that the uniqueness of the CGP growth rates. Consider the following mapping

$$\mathcal{T}\left(\overline{\gamma}\right) \equiv \tfrac{1}{1+\widetilde{\alpha}}\left(\max_j\left\{\widetilde{\alpha}\epsilon_j + \beta_j\overline{\gamma}\right\} + \max_i\left\{\epsilon_i - \beta_i\overline{\gamma}\right\}\right),$$

such that the solution to the fixed point problem

$$\mathcal{T}\left(\overline{\gamma}\right) = \overline{\gamma}, \tag{79}$$

corresponds to the CGP value of $\overline{\gamma}$. Moreover, consider $\overline{\gamma}_1 > \overline{\gamma}_2$ and define $(\epsilon_1^*, \beta_1^*)$, $(\epsilon_2^*, \beta_2^*)$, $\left(\epsilon_1^\dagger, \beta_1^\dagger\right)$, and $\left(\epsilon_2^\dagger, \beta_2^\dagger\right)$ such that:

$$\begin{aligned}
\chi\epsilon_1^* + \beta_1^*\overline{\gamma}_1 &= \max_j\left\{\widetilde{\alpha}\epsilon_j + \beta_j\overline{\gamma}_1\right\},\\
\chi\epsilon_2^* + \beta_2^*\overline{\gamma}_2 &= \max_j\left\{\widetilde{\alpha}\epsilon_j + \beta_j\overline{\gamma}_2\right\},\\
\epsilon_1^\dagger - \beta_1^\dagger\overline{\gamma}_1 &= \max_i\left\{\epsilon_i - \beta_i\overline{\gamma}_1\right\},\\
\epsilon_2^\dagger - \beta_2^\dagger\overline{\gamma}_2 &= \max_i\left\{\epsilon_i - \beta_i\overline{\gamma}_2\right\}.
\end{aligned}$$

Note that by definition, we have:

$$\chi\epsilon_1^* + \beta_1^*\overline{\gamma}_1 \geq \widetilde{\alpha}\epsilon_2^* + \beta_2^*\overline{\gamma}_1, \qquad \chi\epsilon_2^* + \beta_2^*\overline{\gamma}_2 \geq \widetilde{\alpha}\epsilon_1^* + \beta_1^*\overline{\gamma}_2.$$

Combining the two inequalities, we find:

$$\overline{\gamma}_1\left(\beta_2^* - \beta_1^*\right) \leq \widetilde{\alpha}\left(\epsilon_1^* - \epsilon_2^*\right) \leq \overline{\gamma}_2\left(\beta_2^* - \beta_1^*\right).$$

Since by assumption $\overline{\gamma}_1 > \overline{\gamma}_2$, it follows that $\beta_2^* - \beta_1^* \leq 0$.

Similarly, by definition we have:

$$\epsilon_1^\dagger - \beta_1^\dagger\overline{\gamma}_1 \geq \epsilon_2^\dagger - \beta_2^\dagger\overline{\gamma}_1, \qquad \epsilon_2^\dagger - \beta_2^\dagger\overline{\gamma}_2 \geq \epsilon_1^\dagger - \beta_1^\dagger\overline{\gamma}_2.$$

Combining the two inequalities, we find:

$$\overline{\gamma}_1\left(\beta_1^\dagger - \beta_2^\dagger\right) \leq \epsilon_1^\dagger - \epsilon_2^\dagger \leq \overline{\gamma}_2\left(\beta_1^\dagger - \beta_2^\dagger\right).$$

Once again, since by assumption $\overline{\gamma}_1 > \overline{\gamma}_2$, it follows that $\beta_1^\dagger - \beta_2^\dagger \leq 0$.

Now, we have:

$$\begin{aligned}\mathcal{T}(\overline{\gamma}_1) - \mathcal{T}(\overline{\gamma}_2) &= \chi\left(\epsilon_1^* - \epsilon_2^*\right) + \beta_1^*\overline{\gamma}_1 - \beta_2^*\overline{\gamma}_2 + \epsilon_1^\dagger - \epsilon_2^\dagger - \beta_1^\dagger\overline{\gamma}_1 + \beta_2^\dagger\overline{\gamma}_2,\\ &\leq \overline{\gamma}_2\left(\beta_2^* - \beta_1^*\right) + \beta_1^*\overline{\gamma}_1 - \beta_2^*\overline{\gamma}_2 + \overline{\gamma}_2\left(\beta_1^\dagger - \beta_2^\dagger\right) - \beta_1^\dagger\overline{\gamma}_1 + \beta_2^\dagger\overline{\gamma}_2,\\ &= \left(\beta_1^* - \beta_1^\dagger\right)\left(\overline{\gamma}_1 - \overline{\gamma}_2\right) \leq \overline{\gamma}_1 - \overline{\gamma}_2,\end{aligned}$$

since $\beta_1^*, \beta_1^\dagger \in (0,1)$. Thus, the fixed point problem (79) has a unique solution, which is also positive since we know that $\overline{\gamma}_1 \geq \epsilon_1^* > 0$.

Finally, note that we cannot accommodate any sector i with $\gamma_i = 0$ when $\sigma < 1$, from Equation (36) since $\epsilon_i > 0$ for all i.

Now, from the Euler Equation (10), we find that asymptotically

$$g^* = \lim_{t\to\infty} \frac{r(t) - \rho - \bar{p}(t)\iota(t)}{\widetilde{\vartheta}(t)} = \frac{r^* - \rho + \frac{g^*}{\vartheta}\widetilde{\epsilon}^*}{\vartheta - 1 + \widetilde{\epsilon}^*} = \frac{\vartheta - 1 + \widetilde{\epsilon}^*}{\vartheta - 1 + \widetilde{\epsilon}^*\left(1 - \frac{1}{\vartheta}\right)},$$

where we in the second equality we have used the fact that for all distribution Ω_i^* defined on $\mathcal{I}^*$ we have $\gamma_i = \epsilon_i$, to compute

$$\begin{aligned}\lim_{t\to\infty} \bar{p}(t) &= -\sum_{i\in\mathcal{I}^*} \Omega_i^* \tfrac{1}{\vartheta}\epsilon_i g^*,\\ \lim_{t\to\infty} \iota(t) &= 1 + (1-\sigma)\,\mathbb{C}_{\Omega^*}\left(\tfrac{\epsilon_i}{\bar{\epsilon}^*}, \tfrac{\epsilon_i}{\bar{\epsilon}^*}\right) = 1 + (1-\sigma)\,\mathbb{V}_{\Omega^*}\left(\tfrac{\epsilon_i}{\bar{\epsilon}^*}\right),\\ \lim_{t\to\infty} \widetilde{\vartheta}(t) &= \vartheta + \bar{\epsilon}^*\left[1 + (1-\sigma)\,\mathbb{V}_{\Omega^*}\left(\tfrac{\epsilon_i}{\bar{\epsilon}^*}\right)\right] - 1,\end{aligned}$$

and where we have defined $\widetilde{\epsilon}^* \equiv \bar{\epsilon}^*\left[1 + (1-\sigma)\,\mathbb{V}_{\Omega^*}\left(\frac{\epsilon_i}{\bar{\epsilon}^*}\right)\right]$. Equation (35) then follows. □

Lemma. *See Lemma 3.*

Proof. Along a CGP as characterized by Definition 3.3, the growth in the relative shares of sectors i and j can be found from Equation (6) and (26) to be

$$\lim_{t\to\infty} \dot{\omega}_i(t) - \dot{\omega}_j(t) \quad \equiv \quad (1-\sigma)\,g^*\left[\left(\epsilon_i - \epsilon_j\right) - \left(\gamma_i - \gamma_j\right)\right],$$

which is a constant. Therefore, relative shares asymptotically evolve in a monotonic fashion. Since shares are nonnegative numbers that sum to one, they belong to a compact set and therefore have to converge to constant shares $\left\{\Omega_{i,}^{cgp}\right\}_{i=1}^{I}$ in the limit of $t \to \infty$.

Under the asymptotic distribution $\{\Omega_i^{cgp}\}_{i=1}^{I}$, we can define $\overline{\gamma}^*$ and $\overline{\epsilon}^*$ to be average expenditure-weighted technical growth rates and income elasticity parameters across sectors. From Equation (14), we find that the growth rate of consumption expenditure is given by

$$\lim_{t\to\infty} \dot{e}(t) = \left(\overline{\epsilon}^* - \overline{\gamma}^*\right) g^*,$$

which, again, suggests that $E(t)$ either asymptotically grows or falls at a constant rate. From the market clearing condition (27), we know that $E(t) = \chi\frac{L^p(t)}{L(t)} \leq \chi$ and therefore $E(t)$ also belongs to a closed and bounded set. Therefore, $\lim_{t\to\infty} E(t) = E^* > 0$, where the strict positivity follows from the transversality condition (11). This implies that the production employment also converges to a constant $L^* > 0$. Furthermore, the growth of $E(t)$ has to asymptotically be zero, that is, either $g^* = 0$, or we must have

$$\bar{\epsilon}^* = \bar{\gamma}^*. \tag{80}$$

Since all shares are converge to constants, we have that

$$\begin{aligned}\lim_{t\to\infty} \dot{\omega}_i(t) &= (1-\sigma) g^* ((\epsilon_i - \bar{\epsilon}^*) - (\gamma_i - \bar{\gamma}^*)), \\ &= (1-\sigma) g^* (\epsilon_i - \gamma_i) \leq 0, \end{aligned} \tag{81}$$

where we have used Equation (80) in the second equality. This suggests $\epsilon_i \leq \gamma_i$ in the case of $\sigma \in (0,1)$, and $\epsilon_i \geq \gamma_i$ in the case of $\sigma \in (1,\infty)$.

Let $\mathcal{I}^*$ denote the set of surviving industries, defined as those sectors $i \in \mathcal{I}^*$ with $\lim_{t\to\infty} \dot{\omega}_i(t) = 0$ and $\Omega_i^{cgp} \equiv \Omega_i^* > 0$. Equation (48) and the last statement in the lemma follow from Equation (81). □

Lemma. *See Lemma 4.*

Proof. First, we characterize the asymptotic behavior of the sector-level value function $V_i(t)$ along a CGP, as characterized by Equation (77). Note that the rate of growth of the profits corresponding to the monopoly rights on a variety in sector i are given by

$$\begin{aligned}\lim_{t\to\infty} \frac{\Pi_i(t+\tau)/K_i(t+\tau)}{\Pi_i(t)/K_i(t)} &= \lim_{t\to\infty} \left(\frac{K_i(t)}{K_i(t+\tau)}\right)\left(\frac{\Omega_i(t+\tau)}{\Omega_i(t)}\right)\frac{E^* L(t+\tau)}{E^* L(t)}, \\ &= \exp\left\{\left[-\gamma_i\theta g^* + (1-\sigma)(\epsilon_i - \gamma_i) g^* + \eta\right]\tau\right\}, \\ &= e^{(\eta - (\theta\gamma_i + \xi_i) g^*)\tau}, \end{aligned} \tag{82}$$

where in the first line, we have used the fact that $\lim_{t\to\infty} E(t) = E^*$, and have substituted in the second line for the asymptotic rate of growth of sectoral output shares from Equation (81) with $\xi_i \equiv (1-\sigma)(\gamma_i - \epsilon_i)$. Note that since $\Omega_i(t)$ is a share variable, we have that $\lim_{t\to\infty} \dot{\omega}_i(t) = -\xi_i g^* \leq 0$ with equality being satisfied at least for one sector. Now, we find the asymptotic value of owning a firm in sector i to be

$$\begin{aligned}\lim_{t\to\infty} \frac{V_i(t)}{\Pi_i(t)/K_i(t)} &= \left(\lim_{t\to\infty} \int_0^\infty e^{-\int_0^{\prime\tau} r(t+t')dt'} \left(\frac{\Pi_i(t+\tau)/K_i(t+\tau)}{\Pi_i(t)/K_i(t)}\right) d\tau, \right) \\ &= \int_0^\infty e^{-[r^* + (\theta\gamma_i + \xi_i) g^* - \eta]\tau} d\tau, \\ &= \frac{1}{r^* + (\theta\gamma_i + \xi_i) g^* - \eta}, \end{aligned}$$

where in the first equality we have used the definition of $V_i(t)$ from Equation (23), and in the second equality, we have used Equation (82) and the fact that the Euler equation and Lemma 3 imply asymptotically constant the real interest rate r^*. Since V_i asymptotically grows at the same rate as Π_i/K_i, it then follows:

$$\lim_{t\to\infty} \dot{v}_i(t) = \eta - g^* (\xi_i + \theta\gamma_i). \tag{83}$$

The share of R&D in employment $Z_i(t)/L(t)$ is bounded by 1, so its rate of growth cannot exceed η. Assuming $Z_i(t) > 0$, from condition (24), we find

$$\lim_{t\to\infty} \dot{z}_i(t) = \frac{1}{\alpha}\left[\eta + \left(\theta\left(\beta_i\gamma_i^S - \gamma_i\right) - \xi_i\right)g^*\right] \leq \eta. \tag{84}$$

The equality is satisfied for a sector $i \in \mathcal{I}^\dagger$ such that $\lim_{t\to\infty} e^{-\eta t} Z_i(t) = Z_i^* > 0$.

Given that $\lim_{t\to\infty} \dot{k}_i(t) = \gamma_i g^*$, and using Equation (17), we find

$$\lim_{t\to\infty} \frac{\ddot{k}_i(t)}{\dot{k}_i(t)} = (1-\alpha) \times \lim_{t\to\infty} \dot{z}_i(t) + \theta\left(\beta_i\gamma_i^S - \gamma_i\right)g^* \leq 0,$$

with equality satisfied for any sector with $\gamma_i > 0$ (all sectors in the case of $\sigma < 1$). For such sectors, we then find:

$$\lim_{t\to\infty} \dot{z}_i(t) = -\frac{\theta}{1-\alpha}\left(\beta_i\gamma_i^S - \gamma_i\right)g^*. \tag{85}$$

Combining Equations (84) and (85), we find the following constraint on all sectors with asymptotic productivity growth:

$$\theta\left(\beta_i\gamma_i^S - \gamma_i\right) = -(1-\alpha)\left(\tfrac{\eta}{g^*} - (1-\sigma)(\gamma_i - \epsilon_i)\right). \tag{86}$$

Comparing Equations (86) and (85), we find that for any sector that asymptotically grows in terms of productivity we have:

$$\lim_{t\to\infty} \dot{z}_i(t) = \eta - \xi_i g^* \leq \eta,$$

with $\xi_i \equiv (1-\sigma)(\gamma_i - \epsilon_i) \geq 0$, which is compatible with the inequality in Equation (84). Thus, we have $\xi_i = 0$ if and only if $\lim_{t\to\infty} \dot{z}_i(t) = \eta$, that is, the set of sectors whose share of R&D survives along the CGP $\mathcal{I}^\dagger$ satisfies $\mathcal{I}^\dagger = \mathcal{I}^*$.

From the definition of the spillover function in Equation (18), we have

$$\gamma_i^S \equiv \lim_{t\to\infty} \sum_j \frac{\Phi_{ij}^{-\theta} K_i(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)} \gamma_i = \max_{j:\Phi_{ij}<\infty} \gamma_j.$$

Consider now the sector j such that $\overline{\gamma} \equiv \max_{i'} \gamma_{i'}$. From Equation (86) for any i such that $\Phi_{ij} < \infty$ we find:

$$\begin{aligned}\gamma_i &= \beta_i\overline{\gamma} + \tfrac{1-\alpha}{\theta}\left(\tfrac{\eta}{g^*} - (1-\sigma)(\gamma_i - \epsilon_i)\right), \\ &= \tfrac{\theta}{\theta+(1-\alpha)(1-\sigma)}\left[\beta_i\overline{\gamma} + \tfrac{1-\alpha}{\theta}\left(\tfrac{\eta}{g^*} + (1-\sigma)\epsilon_i\right)\right],\end{aligned}$$

leading to Equation (52). Next, note that since $\Phi_{jj} = 1$, the above equality should hold for sector j as well, leading to

$$\overline{\gamma} = \gamma_j = \frac{\left(\frac{1-\alpha}{\theta}\right)\frac{\eta}{g^*} + \left(\frac{1-\alpha}{\theta}\right)(1-\sigma)\epsilon_j}{1-\beta_j + \left(\frac{1-\alpha}{\theta}\right)(1-\sigma)}. \tag{87}$$

In addition, since $\gamma_i \leq \overline{\gamma} = \gamma_j$ for all i, we also find

$$\frac{\theta}{\theta+(1-\alpha)(1-\sigma)}\left[\beta_i\overline{\gamma}+\frac{1-\alpha}{\theta}\left(\frac{\eta}{g^*}+(1-\sigma)\,\epsilon_i\right)\right]\leq\overline{\gamma}=\frac{1-\alpha}{\theta}\frac{\frac{\eta}{g^*}+(1-\sigma)\epsilon_j}{1-\beta_j+\left(\frac{1-\alpha}{\theta}\right)(1-\sigma)},$$

implying that

$$\frac{\frac{\eta}{g^*}+(1-\sigma)\epsilon_i}{1-\beta_i+\left(\frac{1-\alpha}{\theta}\right)(1-\sigma)}\leq\frac{\frac{\eta}{g^*}+(1-\sigma)\epsilon_j}{1-\beta_j+\left(\frac{1-\alpha}{\theta}\right)(1-\sigma)},\qquad\forall i.$$

This result and Equation (87) together lead to Equation (51). □

Lemma. *See Proposition 4.*

Proof. First, from Equation (17) we find

$$\dot{k}_i\,(t)=\Gamma_i\,\left(L\,(t)\,\frac{Z(t)}{L(t)}\,\frac{Z_i(t)}{Z(t)}\right)^{1-\alpha}\,\frac{S_i(t)^{\beta_i}}{K_i(t)},$$

leading to Equation (39).

Next, define the following auxiliary variable $U(t)$ to ensure that the share of R&D to all workers remains between 0 and 1, as $\mathcal{U}(t)\equiv\frac{1-\widehat{Z}(t)}{\widehat{Z}(t)}$.From Equation (14), we find

$$\overline{\epsilon_i}\,(t)\,\dot{c}\,(t)+\overline{p_i}\,(t)=\dot{e}\,(t)=-\frac{\widehat{Z}(t)}{1-\widehat{Z}(t)}\dot{\hat{z}}(t)=-\frac{1}{\mathcal{U}(t)}\dot{\hat{z}}(t). \tag{88}$$

Equation (23) implies $r(t)=\frac{\dot{V}_i(t)}{V_i(t)}+\frac{\Pi_i(t)}{V_i(t)K_i(t)}$. Substituting for $V_i(t)$ from the free entry condition in Equation (22), assuming an interior solution for innovation in all sectors, this implies

$$\begin{aligned} r(t) &= \alpha\dot{z}_i(t)-\beta_i\dot{s}_i(t)+\frac{\Pi_i(t)}{Z_i(t)}\frac{\Gamma_i Z_i(t)^{1-\alpha_i}S_i(t)^{\beta_i}}{K_i(t)}, \\ &= \alpha\dot{z}_i(t)-\beta_i\dot{s}_i(t)+\frac{\chi-1}{\chi}\frac{E(t)}{\widehat{Z}(t)}\frac{\Omega_i(t)}{Y_i(t)}\,\dot{k}_i(t). \end{aligned} \tag{89}$$

where in the second equality we have substituted for profits from Equation (21). Multiplying both sides of Equation (89) by $\Psi_i(t)$ and summing over i, we find

$$\begin{aligned} r(t) &= \alpha\dot{z}(t)-\mathbb{E}_{\Upsilon(t)}\left[\beta_i\,\dot{s}_i(t)\right]+(\chi-1)\,\frac{1-\widehat{Z}\,(t)}{\widehat{Z}\,(t)}\,\mathbb{E}_{\Omega(t)}\left[\dot{k}_i(t)\right], \\ &= \alpha\left(\eta+\dot{\hat{z}}(t)\right)-\mathbb{E}_{\Upsilon(t)}\left[\beta_i\,\dot{s}_i(t)\right]+(\chi-1)\,\frac{1-\widehat{Z}\,(t)}{\widehat{Z}\,(t)}\,\mathbb{E}_{\Omega(t)}\left[\dot{k}_i(t)\right]. \end{aligned} \tag{90}$$

Now, from Equation (10) and Equation (88), we find

$$\begin{aligned} -\frac{1}{\mathcal{U}(t)}\dot{\hat{z}}(t) &= \frac{\overline{\epsilon}(t)}{\overline{\vartheta}(t)}\left(r\,(t)-\rho-\overline{p}\,(t)\,\iota\,(t)\right)+\overline{p}\,(t)\,, \\ &= \frac{\overline{\epsilon}(t)}{\overline{\vartheta}(t)}\left(r\,(t)-\rho\right)+\frac{1}{\theta}\left(\frac{\overline{\epsilon_i}(t)}{\overline{\vartheta}(t)}\iota\,(t)-1\right)\,\mathbb{E}_{\Omega(t)}\left[\dot{k}_i(t)\right], \\ &= \frac{\overline{\epsilon}(t)}{\overline{\vartheta}(t)}\left(\alpha\left(\eta+\dot{\hat{z}}(t)\right)-\rho-\mathbb{E}_{\Upsilon(t)}\left[\beta_i\,\dot{s}_i(t)\right]+(\chi-1)\,\mathcal{U}(t)\,\mathbb{E}_{\Omega(t)}\left[\dot{k}_i(t)\right]\right) \\ &\quad+\frac{1}{\theta}\left(\frac{\overline{\epsilon}(t)}{\overline{\vartheta}(t)}\iota\,(t)-1\right)\,\mathbb{E}_{\Omega(t)}\left[\dot{k}_i(t)\right], \end{aligned}$$

which, dropping the time arguments to simplify the expression, leads to

$$-\left(\frac{\bar{\epsilon}}{\tilde{\vartheta}}+\frac{1}{\alpha}\frac{1}{\mathcal{U}}\right)\alpha\,\dot{\hat{z}}=\frac{\bar{\epsilon}}{\tilde{\vartheta}}\left(\alpha\eta-\rho-\mathbb{E}_{\Upsilon}\left[\beta_i\,\dot{s}_i\right]+(\chi-1)\,\mathcal{U}\,\mathbb{E}_{\Omega}\left[\dot{k}_i\right]\right)+\left(\frac{\bar{\epsilon}}{\tilde{\vartheta}}\iota-1\right)\frac{1}{\theta}\,\mathbb{E}_{\Omega}\left[\dot{k}_i\right].$$

Substituting the above expression in Equation (90), we find

$$\begin{aligned} r-\rho=&\frac{\frac{1}{\alpha}\frac{1}{\mathcal{U}}}{\frac{\bar{\epsilon}_i}{\tilde{\vartheta}}+\frac{1}{\alpha}\frac{1}{\mathcal{U}}}\left(\alpha\eta-\rho-\mathbb{E}_{\Upsilon}\left[\beta_i\,\dot{s}_i\right]+(\chi-1)\,\mathcal{U}\,\mathbb{E}_{\Omega}\left[\dot{k}_i\right]\right)\\ &-\frac{\frac{\bar{\epsilon}_i}{\tilde{\vartheta}}\iota-1}{\frac{\bar{\epsilon}_i}{\tilde{\vartheta}}+\frac{1}{\alpha}\frac{1}{\mathcal{U}}}\frac{1}{\theta}\,\mathbb{E}_{\Omega}\left[\dot{k}_i\right]. \end{aligned} \tag{91}$$

Finally, we substitute the above expression in Equation (10) to find

$$\begin{aligned} \dot{c}&=\frac{1}{\tilde{\vartheta}}\left(r-\rho+\frac{1}{\theta}\iota\,\mathbb{E}_{\Omega}\left[\dot{k}_i\right]\right),\\ &=\frac{1}{\tilde{\vartheta}+\alpha\bar{\epsilon}\mathcal{U}}\left(\alpha\eta-\rho-\mathbb{E}_{\Upsilon}\left[\beta_i\,\dot{s}_i\right]+(\chi-1)\,\mathcal{U}\,\mathbb{E}_{\Omega}\left[\dot{k}_i\right]\right)\\ &\quad+\left(\frac{\iota}{\tilde{\vartheta}}-\frac{\frac{\iota}{\tilde{\vartheta}}\bar{\epsilon}-1}{\tilde{\vartheta}+\alpha\bar{\epsilon}\mathcal{U}}\,\alpha\,\mathcal{U}\right)\frac{1}{\theta}\mathbb{E}_{\Omega}\left[\dot{k}_i(t)\right],\\ &=\frac{1}{\tilde{\vartheta}+\alpha\bar{\epsilon}\mathcal{U}}\left(\alpha\eta-\rho-\mathbb{E}_{\Upsilon}\left[\beta_i\,\dot{s}_i\right]\right)\\ &\quad+\left(\frac{\iota}{\tilde{\vartheta}}+\frac{\left(\chi-1-\alpha\left(\frac{\iota}{\tilde{\vartheta}}\bar{\epsilon}-1\right)\right)\mathcal{U}}{\tilde{\vartheta}+\alpha\bar{\epsilon}\mathcal{U}}\right)\frac{1}{\theta}\mathbb{E}_{\Omega}\left[\dot{k}_i(t)\right],\\ &=\frac{1}{\tilde{\vartheta}+\alpha\bar{\epsilon}\mathcal{U}}\left(\alpha\eta-\rho-\mathbb{E}_{\Upsilon}\left[\beta_i\,\dot{s}_i\right]\right)\\ &\quad+\left(\frac{\iota+(\chi-1+\alpha)\mathcal{U}}{\tilde{\vartheta}+\alpha\bar{\epsilon}\mathcal{U}}\right)\frac{1}{\theta}\mathbb{E}_{\Omega}\left[\dot{k}_i(t)\right], \end{aligned}$$

leading to Equation 41. Substituting the expression (91) into Equation (89) yields Equation (40). □

B. Calibration and Simulation Details

B.1. Derivations of the Estimating Equation for Ψ_j^{θ} and Estimation Results

Derivation of Equation (43). Use $Pr_{ji}^{CITE}=\Sigma_{i,j}$ and Equation (20), time differentiation and combining with Equation (19) yields

$$\frac{d}{dt}\log Pr_{ij}^{CITE}=\Psi_i^{\theta}P\dot{A}T_i-\sum_{i'}Pr_{i'j}^{CITE}P\dot{A}T_{i'} \tag{92}$$

Noticing that the last term is a function only of j, the desired result follows.

Estimation details We consider two values for $\{\Psi_j^{\theta}(t)\}_{j\in a,m,s}$, by estimating Equation (44) separately in two sub-samples during the postwar period, for the first and second halves of our sample. Since citations take time to occur, there is a natural drop in the number of citations for the latest patents in our sample. To prevent this effect from contaminating our estimation, we conduct our estimation with patents filed prior to 2010, for which we do not observe such a

Table 3: Estimates of Ψ_i^θ and Implied β_i

	20 yr. lag		25 yr. lag		15 yr. lag		10 yr. lag		30 yr. lag	
Panel A: Estimated Ψ_j^θ for two subsamples, 1950-1980 and 1980-2010										
	'50−'80	'80-'10	'50−'80	'80−'10	'50−'80	'80−'10	'50−'80	'80−'10	'50−'80	'80−'10
Agri.	7.79E-05	9.56E-06	1.80E-04	8.41E-06	9.01E-05	8.31E-06	1.40E-04	1.02E-05	7.21E-05	8.65E-06
Manuf.	2.07E-06	1.21E-07	5.39E-06	9.50E-08	2.55E-06	9.33E-08	4.31E-06	1.39E-07	2.02E-06	1.29E-07
Serv.	9.94E-06	2.48E-07	2.65E-05	1.56E-07	1.24E-05	1.28E-07	2.17E-05	3.23E-07	9.78E-06	3.06E-07
Panel B: β_i Values (Pre-Post Difference Calculation)										
β_a	0.87		0.85		0.83		0.86		0.83	
β_m	0.69		0.75		0.65		0.77		0.71	
β_s	0.45		0.63		0.32		0.72		0.53	
Panel C: β_i Values (Average of Yearly Differences)										
β_a	0.97		0.91		0.92		0.93		0.92	
β_m	0.77		0.82		0.83		0.74		0.78	
β_s	0.48		0.68		0.72		0.43		0.53	

: Notes: All estimates are significant at five percent level (standard errors are omitted). Panels B and C use Equation (45) to compute β_i. See the text for a discussion of how the estimates in Panel B and C are estimated

drop (our data on patent citations spans until 2015). We proceed by considering a time interval λ for the time differences, implying that $\Delta \ln Pr_{ji}^{CITE}(t) = \ln Pr_{ji}^{CITE}(t+\lambda) - \ln Pr_{ji}^{CITE}(t)$ and $\Delta PAT_j(t) = \sum_{t'=0}^{\lambda-1} \text{Patents Filed}_j(t+t')$. We set our baseline time difference to $\lambda = 20$ years, and provide results for $\lambda = 10, 15, 25$, as robustness checks.

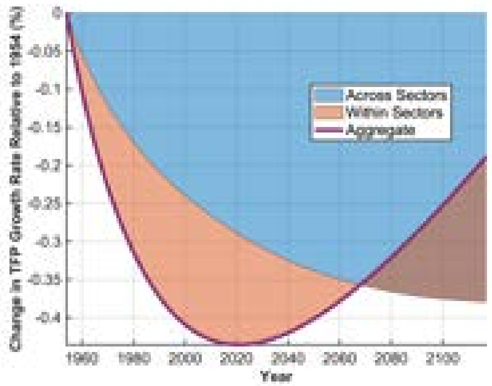
For each subperiod, we estimate Equation (44) for a time window around the the initial year of the subperiod (1950 and 1980, respectively), so that we can smooth out any idiosyncratic shock in citations at time t and $t+\lambda$ in the patenting and citation rates—as is standard in the literature conducting estimation using patent data, e.g., see Manysheva et al. [2022] and the references therein. Specifically, when we estimate $\left\{\Psi_j^\theta(t)\right\}_{j\in a,m,s}$ through Equation (44) for the first subsample, we use a time window between 1950 and 1955 to construct our regressors, so that in our regression we have observations for $\Delta \ln Pr_{ji}^{CITE}(t)$ and $\dot{PAT}_j(t)$ for $t \in [1950, 1955]$. Likewise, for our estimation for the second half we use a time window between 1977 and 1984 for the second, i.e., $t \in [1977, 1984]$. Panel A in Table 3 reports the results. All estimates are significant at five percent level and we omit the standard errors from the table.

B.2. Computation of the Intertemporal Knowledge Spillovers, β_i

Computing the Terms in Equation (45) Equation (19) implies that $\Delta \log\left(\dot{K}_i(t)/K_i(t)\right) = \Delta \log \dot{PAT}_i(t) + \theta \Delta \log \Psi_i(t)$, for which we can use patent data and the estimated values of $\Psi_i^\theta(t)$ to compute its empirical counterpart. In particular the empirical counterpart to $\Delta \log\left(\dot{K}_i(t)/K_i(t)\right)$ corresponds to $\Delta \log\left(\widehat{\Psi_i^\theta(t)}\Delta PAT_i(t)\right)$ where $\widehat{\Psi_i^\theta(t)}$ denotes the estimated values of $\Psi_i^\theta(t)$. Second, approximating the integral of Equation (19) by an annual sum, we also obtain an expression for $\Delta \log K_i(t)$. Approximating the integral by the yearly sum, we obtain $\sum_{t'=t_0}^{t_1-1} \widehat{\Psi_i^\theta(t')}\Delta PAT_i(t')$. Third, to compute the term $(1-\alpha)\Delta \ln Z_{it}$, given our value for α, we need information on real expenditures in R&D—which in the model correspond to workers employed in R&D. We proceed as in Bloom et al. [2020], and use nominal R&D expenditures deflated by the nominal wages of high-skill workers to proxy for $Z_i(t)$.[24] Finally, to compute the change in log

[24] We use mean personal income from the Current Population Survey for males with a bachelor's degree or more of education as a measure of the nominal wage of high-skill workers, as constructed by Bloom et al. [2020].

Figure 11: Aggregate Productivity Decomposition



Note: The within- and across-sectors decomposition of aggregate growth as predicted by the model. See Section 2.3 for a discussion of the decomposition.

spillovers, $\Delta \ln S_i$, we differentiate Equation (18) and use the definition for citation probabilities to obtain

$$\tfrac{d}{dt} S_i(t) = \sum_{j=a,m,s} \Sigma_{ji}(t) \Psi_j^{\theta}(t)\, P\dot{A}T_j(t), \tag{93}$$

which again can be written in terms of observables using citation probabilities, patent flows and our estimates of $\Psi_i^{\theta}(t)$. Analogously to the previous mappings, it follows that the empirical counterpart for $\Delta \log S_i(t)$ corresponds to $\sum_{t'=t_0}^{t_1-1} \sum_{j=a,m,s} Pr_{ij}^{CITE}(t')\, \widehat{\Psi_j^{\theta}(t')} \Delta PAT_j(t')$.

We provide the implied β_i under different scenarios. First, across different columns of Table 3, we report the implied values for β_i when we compute the evolution of knowledge stocks under different estimates for Ψ_j^{θ} (which are estimated through different lags). Second, across different rows, we provide the implied values of β_i when use two different specifications for the length of the time difference operator Δ in Equation (45). In the first rows, we provide a simple two-period comparison between pre-1983 and post-1983 periods.[25] In the bottom rows, we compute first the yearly differences of all variables in appearing in the right-hand-side of (45), then compute their average and use their average values to obtain β_i.[26]

Derivation of $d \ln S_i/dt$, Equation (93) Taking the time derivative of (18) we have that

$$\tfrac{d}{dt} \ln S_i = \sum_j \frac{\Phi_{ij}^{-\theta} K_j(t)}{\sum_{j'} \Phi_{ij'}^{-\theta} K_{j'}(t)} \frac{\dot{K}_j}{K_j} = \sum_j Pr_{ji}^{CITE} \frac{\dot{K}_j}{K_j} = \sum_j Pr_{ji}^{CITE} \psi_j^{-\theta} P\dot{A}T_j$$

where we have used the definition of Pr_{ji}^{CITE}, Equation (20), and how the patenting rate translates into the growth of the knowledge stock, Equation (19).

[25] We do not observe R&D expenditures prior to 1970 for agriculture and services separately. We proceed by assigning the non-manufacturing real R&D expenditures to services and computing average real R&D expenditures in agriculture between 1970 and 1983, and assume that total R&D expenditures in the first period for agriculture are equal to this average times the length of the this subperiod (which is 35 years).

[26] In this case, we do not use information prior to 1970 since we lack R&D data for agriculture and services separately. We therefore compute the average of the yearly changes between 1970 and 2010.

B.3. Estimation of cross-sector applicability costs $\Phi_{ij}^{-\theta}$

The estimating equations are obtained by taking log differences of Equation (46)

$$\log Pr_{ji}^{CITE} - \log Pr_{ii}^{CITE} = \Phi_{ij}^{-\theta} K_j(0) \varkappa_j(t) - K_j(0) \varkappa_j(t) + e_{ij}(t)$$

where we use the empirical counterparts of $\Delta \ln K_j$, $\varkappa_j(t)$, and $e_{ij}(t)$ denotes the error term. We estimate simultaneously the system of equations for all $j \neq i$. The applicability costs we estimate (denoting standard errors in parenthesis) if the receiving sector is agriculture are 1 (by assumption), 0.25 (0.06) and 0.06 (0.02) when the senders are agriculture, manufacturing and services, respectively. When the receiving sector is manufacturing, we have 0.010 (0.003), 1 and 0.21 (0.05) when the senders are agriculture, manufacturing and services. Finally, for services, we have 0.02 (0.005), 1 and 1, where we have imposed the constraint that it is not possible that $\Phi_{ms}^{-\theta} > 1$ in the last term in the estimation (knowledge is at most as transferable as within sector, otherwise we would obtain 1.32).

C. Data Appendix

C.1. Definition of the three broad sectors in the economy

We assign industries to the three broad sectors of the economy following standard practice in the literature, e.g., Herrendorf et al. [2013]. We focus on the private economy and construct the three broad sectors of the economy from based on the BEA industries from the GDP-by-industry data: agriculture comprises "farms" and "forestry, fishing, and related activities," manufacturing include "construction", "manufacturing" and "mining," and services include all other industries.

C.2. Patent Data

We use the CUSP dataset [Berkes, 2018] to construct our figures. CUSP provides patent codes according to the International Patent Classification that are backward compatible throughout the period of study. When a patent is assigned to n patent classes, we assign a weight of $1/n$ to each. We start our analysis in 1856. The years prior to 1856, we observe a substantial jump in agricultural shares such that the vast majority of patents belongs to agriculture (over 90%). Despite being consistent with our theory, to be conservative, we discard the years prior to 1856.

C.3. R&D data

Our main source of data comes from the Bureau of Economic Analysis Fixed Asset Table 2.7 (Net Investment in Fixed Assets). These data separates investment between R&D coming from manufacturing and non-manfuacturing sectors. According to the NSF description and treatment of non-manufacturing, this sector includes industries with NAICS starting from 21-23 and 42-81, but it excludes agriculture, i.e., sectors with NAICS code starting with 1 (See the NSF overview of R&D data). Accordingly, we extend our dataset with data on private R&D

expenditures from the Economic Research Service of the US Department of Agriculture. These data are publicly available here.

C.4. TFP data sources and calculations

We combine series from the BLS Historical series and the BLS/BEA tables (those corresponding to "Major Industries") to construct our series. The BLS Historical Series span 1949-2001. We begin our analysis 1953 due to missing data in 1950-1952. We use the Historical Series until 1987, which is when the BLS/BEA data starts.

The data for the BLS Historical Series contains TFP series computed for value-added production functions for the aggregate economy, private business, non-farm private business and TFP computed for BLS-defined "sectoral-output" production functions for manufacturing—sectoral output is defined as gross output less intra-industry transactions.[27] To convert the manufacturing TFP series we supplement our data with BEA GDP-by-industry data to obtain nominal value added by sector.[28]

C.4.1. Conceptual framework

We construct sectoral TFP series that are consistent with the value-added formulation of our model. To this end, we review the isomorphism between an economy with input-output linkages and its corresponding value-added representation, so that both feature the same aggregate TFP. This analysis underpins the TFP calculations in BLS and BEA analysis based on different output definitions.[29] Hulten's theorem states that in an economy with K sectors, we have $d \log TFP = \sum_{k=1}^{K} \lambda_k \, d \log TFP_k$, where the Domar weights are given by $\lambda_k \equiv \frac{P_k Y_k}{\sum_{k=1}^{K} VA_k}$. Here, $P_k Y_k$ denotes the dollar value of total sales of sector k and $\sum_{k=1}^{K} VA_k$ is the total GDP of the economy (the sum of value added VA in all sectors). Consider a value added representation of an economy with the restriction that it has to generate the same TFP growth. Hulten's theorem in this economy implies that the weights on sectoral TFP growth are $\lambda_k^{VA} = \frac{VA_k}{\sum_{k=1}^{K} VA_k}$. In order to generate an aggregate TFP growth equivalent to the original economy, we have that $d \log TFP = \sum_{k=1}^{K} \lambda_k^{VA} d \log \widetilde{TFP}_k$. Thus, sectoral TFP series in a value-added representation need to be scaled up by the ratio of total sectoral sales over value added to generate sectoral TFP series $\widetilde{TFP}_k$ that are consistent with aggregate TFP.

C.4.2. Computation details

For the private business and private nonfarm business, the output concept used by the BLS Historical Series is "real value-added" from labor and capital. Real value-added output in private business equals gross domestic product less general government, government enterprises, private households (including the rental value of owner-occupied real estate), and non-profit institutions. The private nonfarm business sector further excludes farms (but includes agricultural services). We identify the farm sector with agriculture and use the fact that total private nonfarm business is the weighted sum of TFP growth in agriculutre and

[27] See https://www.bls.gov/news.release/prod5.tn.htm and references therein for further discussion.

[28] These data are available at https://www.bea.gov/itable/gdp-by-industry. Specifically, we use the 1947-2007 historical data table and the 2007, 2017Q2, and 2023Q2 archived data for the later years.

[29] Our approach follows that outlined in the BLS, https://www.bls.gov/opub/hom/msp/concepts.htm

non-agriculture to back out the TFP growth of the agricultural sector. After we convert manufacturing TFP growth from a sectoral-output to a value added formulation, we use the same procedure to decompose non-agriculture in manufacturing and services. Formally, we have $\theta_{Y,t} = \bar{s}_{F,t}\theta_{F,t} + \bar{s}_{NF,t}\theta_{NF,t}$ and $\theta_{NF,t} = \bar{s}_{M,t}\theta_{M,t} + \bar{s}_{NM,t}\theta_{NM,t}$, where $\theta_{X,t}$ is the growth rate of TFP between t and $t-1$ for the private (Y), farm (F), non-farm (NF), manufacturing (M), or non-manufacturing non-farm sectors (NM). $\theta_{Y,t}$ and $\theta_{NF,t}$ are aggregates of component growth rates using Törnqvist aggregation. $\bar{s}_{X,t}$ is the average share in nominal value-added output of component X for t and $t-1$. We convert the sectoral output TFP growth for manufacturing ($\gamma_{M,t}$) into value-added TFP growth by scaling it by the ratio of nominal sectoral output (SO) and value-added output (VA) for manufacturing (using the BEA GDP-by-industry data) according to the discussion above as $\theta_{M,t} = \gamma_{M,t} \times nom_{SO,t}/nom_{VA,t}$.

The computation for the post 1987 sample uses the BLS/BEA tables, which also provide sectoral output measured using the "sectoral output" definition. We derive industry-level TFP growth (sectoral output) using the same logic. In particular,

$$\gamma_{i,t} = \Delta \ln y_{i,t} - \bar{s}^K_{i,t}\Delta \ln k_{i,t} - \bar{s}^L_{i,t}\Delta \ln l_{i,t} - \bar{s}^E_{i,t}\Delta \ln e_{i,t} - \bar{s}^M_{i,t}\Delta \ln m_{i,t} - \bar{s}^S_{i,t}\Delta \ln s_{i,t},$$

where K, L, E, M, and S are capital, labor, and intermediate inputs purchased from firms outside of an industry. We aggregate these growth rates to the sector level using a Törnqvist index of the industry-level input growth and sectoral output growth, then apply the conversion to sector-level TFP growth in value added terms as described above.

D. Additional Empirical Facts and Robustness Checks

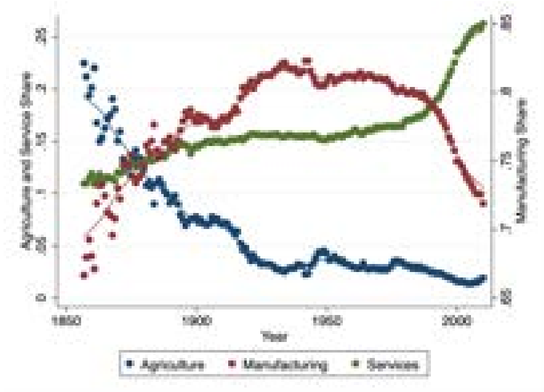
D.1. Patenting

We assign patents to years based on their granting year (or issue year in case the granting year is missing). We have 8,116,643 unique patents with corresponding patent classes and granting (or issue) years between 1855 and 2010. Berkes [2018] provides a consistent classification of patent classes over the entire period.

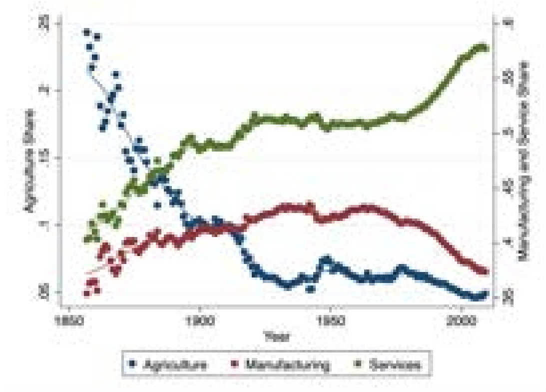
Robustness to alternative mappings Our findings are robust to alternative mappings. First, we show that if we directly assign all patents that belong to the patent class "A01 Agriculture; Forestry; Animal husbandry; hunting; trapping; fishing" to the Agriculture sector instead of using the Goldschlag et al. [2017] correspondence we obtain very similar results. Second, we show that our assignment results to services are robust to using a probabilistic match based on natural language processing similar in spirit to Goldschlag et al. [2017] developed by the US Patent and Trademark Office and the Commerce Data Service.[30] This crosswalk is based on comparing the texts from industry classification descriptions (NAICS) to patent categories (CPC). It gives a probabilistic match where the weight is a function of the text similarity measured as the inner product between keyword strings (also known as a cosine measure)}

[30] The crosswalk is available at https://commercedataservice.github.io/cpc-naics/.

Figure 12: Alternative Correspondences, USA 1855-2010



(a) Agriculture only A01 IPCs



(b) USPTO/CDS Correspondence

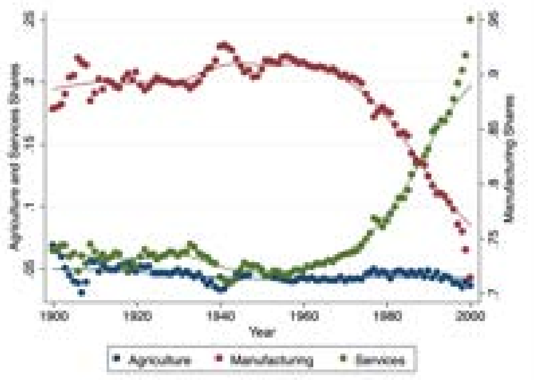
Evidence from Other Advanced Economies We use data from the European Patent Office Patent Statistical Database (PATSTAT, Autumn 2018 version) to extend our analysis to other countries. In particular, this database contains long-run data with patent class information for other leading economies that spans the twentieth century. We focus first on three leading countries for which we have long term data: France, Germany and Great Britain and Japan– one of the major players in patenting in the world. For Japan, our data starts later, in 1970. Figure 13 shows the evolution of patent shares across these economies. For the three European economies economies, we find that the pattern we uncovered for the US also holds for these three countries. There is a steady decline of agriculture, a hump shaped pattern for manufacturing and an increase for services. We note that the peak for manufacturing and the acceleration of patenting shares in services tend to co-move across all these countries. Finally, for Japan, we observe a similar pattern of rising services and decline in manufacturing throughout our sample, very much in line with the US and European economies.

D.2. R&D Expenditures: Including Software and Alternative Sources

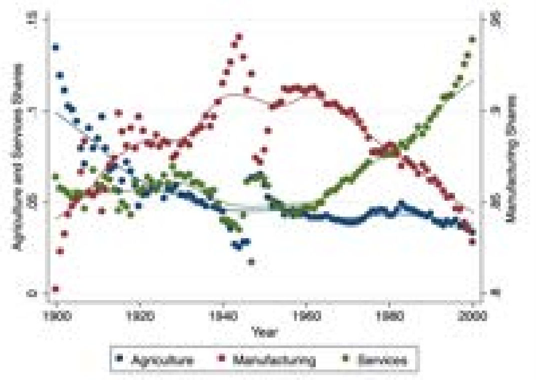
To study the robustnes of our findings in the sectoral composition of R&D expenditures, we use two data sources that cover different notions of what is R&D. The National Science Foundation (NSF) reports domestic private R&D expenditures by all firms, from 1953 onwards and disaggregates them by broad sectors. The NSF definition of R&D is pretty coarse and leaves out expenditures directed at implementing innovations and the development of innovations in fields that are not related to natural sciences or engineering (e.g., management, accounting, marketing, accounting, social sciences).[31] Compustat reports the R&D expenditures of publicly-traded companies, but is much less restrictive in the definition of R&D ultimately reflecting the expenditures companies report in their financial accounts (and has virtually no firm reporting its main sector being agriculture). Since R&D expenditures have fiscal benefits, companies are likely to classify as R&D technology implementation-related expenditures and expenditures associated with innovations that are not scientific which, in principle, may be de-

[31]See Comin and Mulani (2009) .

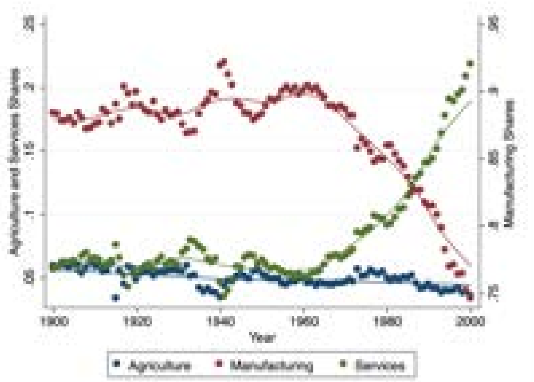
Figure 13: Patenting Shares Other Advanced Economies



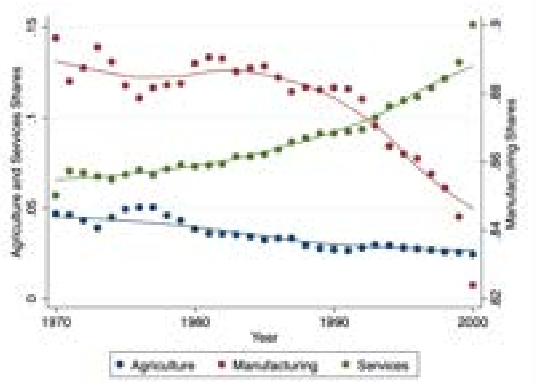
(a) Great Britain



(b) Germany



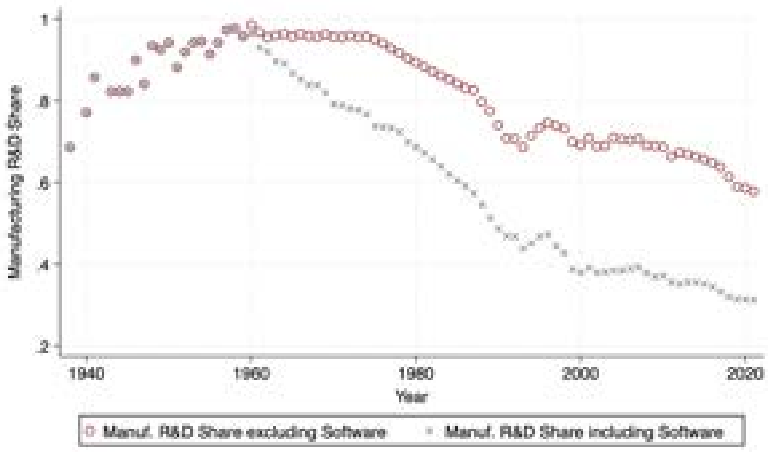
(c) France



(d) Japan

Figure 14: R&D Share in Manufacturing from Alternative Sources (NSF and Compustat)

Figure 15: Manufacturing R&D Shares with and without Software

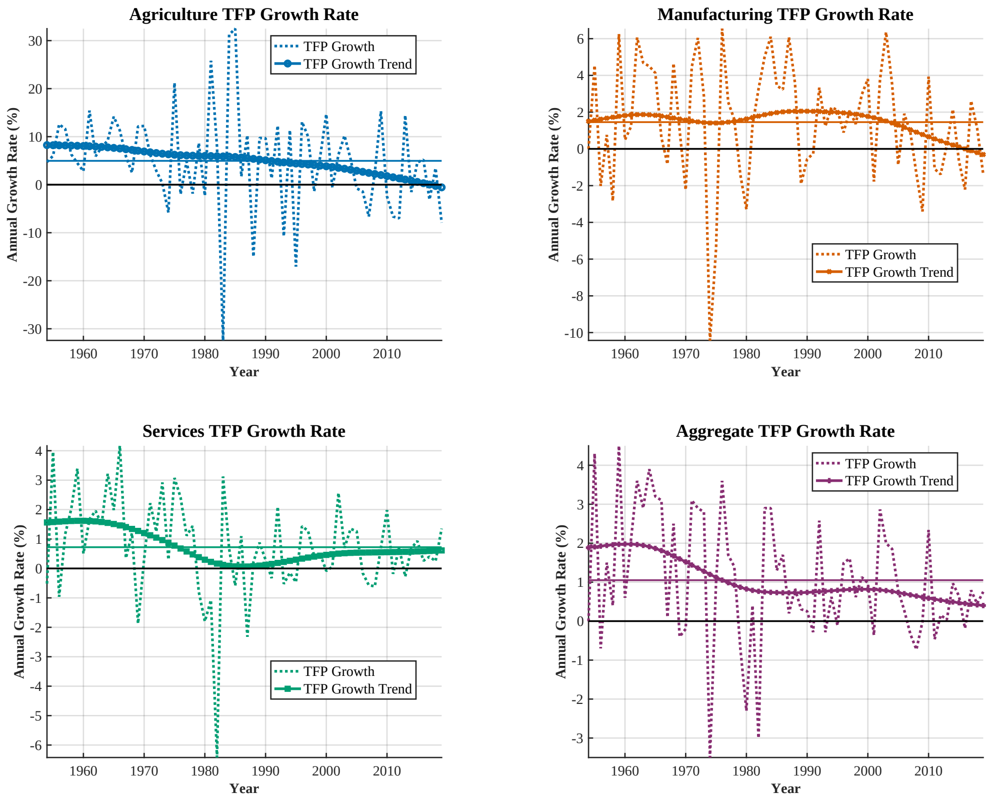


Note: Share of manufacturing R&D without considering software investment (baseline) and including software investment from the BEA fixed asset tables (line 78 in Table 2.7).

sirable to account for as they may also contribute to the growth of the technology frontier and raise TFP. This broader notion of what is R&D in Compustat is evidenced by the fact that, despite covering only publicly-traded companies, total R&D expenditures are significantly greater in Compustat than in the NSF series. Figure 14 plots the share of manufacturing in total R&D expenditures both in the NSF and Compustat series. In both cases, we observe how the share of manufacturing R&D expenditures increased during the 1950s, plateaued during the 60s and 70s, and started to decline around 1980.

Finally, we also report the results of including software investment into services R&D. This investment comes from the corresponding line in the fixed asset table. Not surprisingly, since software expenditures have raised dramatically in the last decades, we find that this implies a steeper decline in manufacturing R&D relative to our baseline, as Figure documents. Including total IP rather than only software exacerbates the result even further.

Figure 16: The Trends in the Aggregate and Sectoral Productivity Growth



D.3. Sectoral TFP growth: Additional Figures

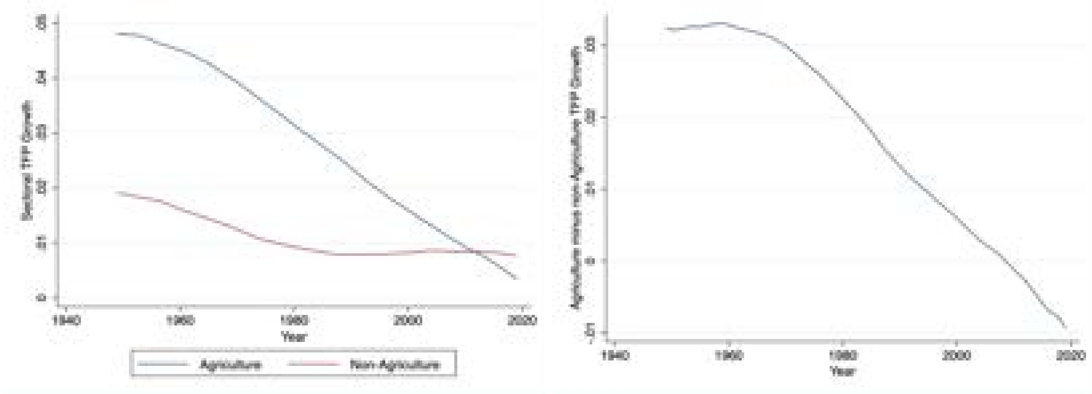
Figure 16 shows the sectoral and aggregate TFP data data we obtain and the resulting trend after using a Hodrick-Prescott filter (with parameter $\lambda = 1600$). Figure 17 shows a non-parametric fit of TFP growth in agriculture and non-agriculture since using the BLS Historical data since 1949. It documents that agriculture has grown, on average, at a faster pace than the rest of the US private economy during the postwar period. However, the gap between agriculture and non-agriculture has been steadily closing at by year 2010 it appears that the sectoral growth rates have converged.

Finally, Figure 18 reports the sectoral value added that we recover from the BEA GDP-by-industry data. In addition to the well known evolution of sectoral shares, we also report the evolution of relative shares of agriculture and services to manufacturing in the second panel, which appear to be remarkably well-approximated by a time linear trend.

D.4. Aggregate TFP Decomposition

The rate of growth g_t in value-added TFP of the aggregate US economy can be written as the weighted average of sectoral TFP growth according to $g_t = \sum_{i \in \{a,m,s\}} S_{it} g_{it}$, where g_{it} and S_{it} denote the rate of growth in the value added TFP of sector i and its share in total value added, respectively. As production shifts away from sectors with faster productivity growth (agriculture and manufacturing) toward services, we should expect a fall in the aggregate rate of TFP growth. Indeed, this is the well-known Baumol cost disease [Baumol, 1967]. However,

Figure 17: Agriculture vs non-Agriculture TFP Growth



(a) Sectoral TFP Growth (b) Difference in TFP Growth

Figure 18: The Evolution of the Sectoral Shares of the Private US Economy

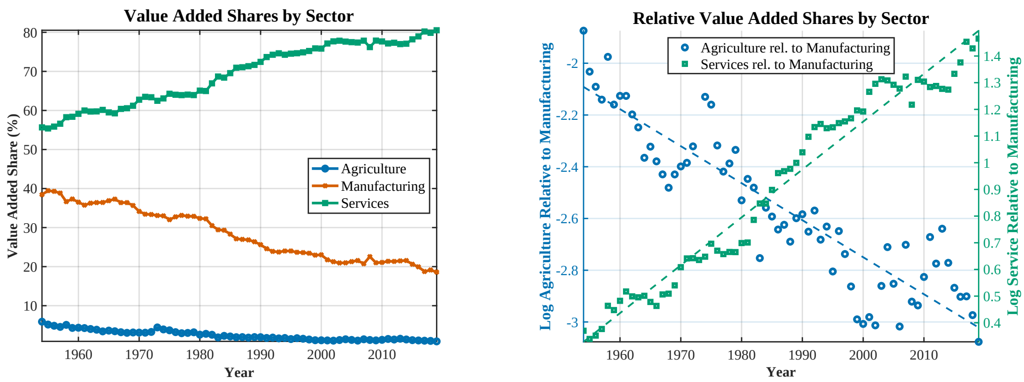
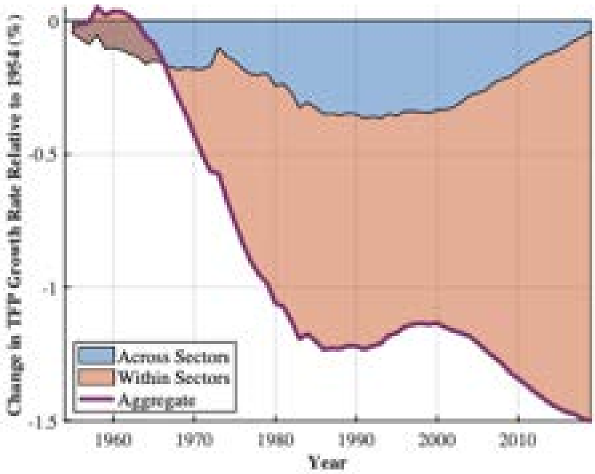


Figure 19: Aggregate TFP: Within and Across Sector Decomposition



Note: The plot shows the aggregate growth decomposition from Equation (94).

as we already saw, TFP growth in services has been catching up relative to manufacturing and services. Thus, a countervailing force emerges since the difference in growth rates across sectors has narrowed over the postwar period.

We decompose the change between the trend rates of TFP growth between year t and an initial year t_0 in our sample according to

$$\Delta g_t = \sum_i \Delta S_{it} \overline{g}_{it} + \sum_i \overline{S}_{it} \Delta g_{it}, \tag{94}$$

where $\overline{g}_{it}$ and $\overline{S}_{it}$ denote the average rate of productivity growth and share of value added for sector i between the two years t and t_0. The first term on the right hand side of Equation (94) accounts for the contribution of sectoral reallocations on the changes in the trends while the second term accounts for the contribution of within sector changes, i.e., changes in sectoral TFP growth.

Figure 19 shows the decomposition of the trends in aggregate TFP growth based on Equation (94) for each year $t > 1954$ in our data and the initial year $t_0 = 1954$. As can be seen, the sectoral reallocations over this period have had a sustained, negative impact on the aggregate rate of productivity growth. Starting in the 1970s, the decline in the sectoral rates of productivity growth have become a major driver of the decline in the aggregate rate of productivity growth. By the end of the period in 2019, the lion's share of the 1.5% decline from 1954 can be attributed to these within sector trends. Moreover, sectoral reallocations increasingly dragged down the rate of TFP growth from the beginning of the period up until mid-1990s, rising to a negative contribution of around −0.37%. Thereafter, the contribution of these structural changes on the aggregate rate of growth began to decline, standing at just −0.04% by 2019.