Generalized Compensation Principle*

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Abstract

We generalize the classic concept of compensating variation and the welfare compensation principle to a general equilibrium environment with distortionary taxes. We show that the problem of designing a tax reform that compensates the welfare gains and losses induced by an economic disruption can be formalized as a solution to a system of differential-algebraic equations (DAEs). We derive its solution in closed form and therefore provide a complete analytical characterization of the welfare-compensating tax reform in general equilibrium. The partial equilibrium compensation consists of adjusting the average tax rate to exactly cancel out the initial wage disruption. We show that in general equilibrium, the compensating tax reform features three primary modifications to this benchmark. First, defining the relevant wage disruption that needs to be compensated requires accounting for the endogenous wage adjustments induced by the initial shock. The other two effects arise because the marginal tax rates, in general equilibrium, impact wages, and hence individual utility. The "progressivity" effect requires adjustments to the tax code that counteract the welfare effects implied by the decreasing marginal product of each skill's labor. This leads to exponentially decreasing or increasing taxes on incomes below those of the disrupted agents. The "compensation of compensation" effect requires adjustments that counteract the welfare effects implied by the complementarities between skills in production. This leads to an inductive procedure to implement compounding rounds of iterative compensation. While we provide a closed form expression for this effect in the general model, in the special case of a CES production function it reduces to a remarkably simple uniform shift of the marginal tax rates. Finally, we derive a closed form formula for the fiscal surplus of the wage disruption and the compensating tax reform, generalizing the traditional Kaldor-Hicks criterion.

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Introduction

In this paper we generalize the classic concept of compensating variation (Mas-Colell, Whinston, and Green [1995], p. 82) and the welfare compensation principle (Kaldor [1939], Hicks [1939, 1940]) to a general equilibrium environment in which only distortionary taxes are available.

Consider a disruption in the economy, for example, an inflow of immigrants or a change in technology, that impacts the distribution of workers' wages. This economic shock generally creates winners and losers, i.e., welfare gains for some individuals and welfare losses for others. The welfare compensation problem consists of designing a reform of the tax-and-transfer system that offsets these losses by redistributing the gains of the winners. The traditional public finance (Kaldor [1939], Hicks [1939, 1940]) literature gives a straightforward answer to the welfare compensation problem. In an economy where type-dependent lump-sum taxes are available policy instruments, the tax reform that redistributes the welfare gains and losses from the economic shock is trivial (and, assuming away income effects, does not distort labor supply decisions). It simply consists of raising (resp., lowering) in a lump-sum way the tax liability of agents whose welfare increases (resp., decreases) from the disruption, up to the point where everyone is exactly as well off as before the economic change. The standard Kaldor-Hicks approach is flawed, however: in practice, because of asymmetric information, the only tax instrument at the disposal of the government, the income tax, is distortionary (Mirrlees [1971]). A compensating tax reform, if it exists, must therefore be designed in such a way that each agent's change in welfare (compensating variation) is equal to zero, taking into account that their labor supply and, in general equilibrium, their wage both respond endogenously to the tax change.

The following considerations highlight the importance of designing a compensating tax reform in an environment that explicitly accounts for the fact that wages are endogenously determined in general equilibrium. Consider for example an immigration inflow, i.e., an exogenous increase in the total labor supply of a given skill. This disruption lowers the wage of agents with the same skill because the marginal product of labor is decreasing and raises the wage of those whose skills are complementary in production. In this situation, therefore, it is clear that the welfare impacts of immigration result only from the general equilibrium forces. Now suppose that the government implements a tax reform that aims at compensating the welfare of agents whose wage is adversely impacted by the immigration inflow. Since the only available policy tools are distortionary taxes, such a reform inevitably impacts the agents' labor supply choices. By the very same general equilibrium forces that led to the welfare implications of immigration, these labor supply adjustments affect individuals' wages, and hence their utility. These themselves need to be compensated, using the distortionary tax code. This leads to an a priori complex fixed point problem.

We show (Section 2) that the problem of designing a compensating tax reform, even when only distortionary taxes are available, is simple in a partial equilibrium environment where wages are fixed. The key insight here is that the changes in marginal tax rates implied by the reform do not matter for welfare, conditional on the average tax rate. This follows from the envelope theorem: the marginal tax rate that the individual faces affects his indirect utility only through his optimal labor supply decision, so that the corresponding welfare effect is second-order. Specifically, we show (Proposition 1) that a suitably designed adjustment in the average tax rate – namely, one that exactly cancels out the exogenous wage disruption – is sufficient to achieve welfare compensation. We moreover derive a simple closed form expression for the fiscal surplus, i.e., the impact on government budget of the disruption and its associated compensation.

The analysis becomes significantly more complicated when distortionary taxes are coupled with the general equilibrium considerations (Section 3). In this case, despite the envelope theorem, the endogenous changes in labor supply matter for welfare, through their impact on wages that result from the decreasing returns and the complementarities in production. Therefore, in general equilibrium, because of the labor supply responses it generates, the marginal tax rate affects directly the agent's utility, even conditional on the average tax rate. In other words, to determine the compensating tax reform, we need to simultaneously solve for the average and the marginal tax rate functions. This is the key difference with the partial equilibrium environment and the key technical challenge of our paper.

We first show that the welfare compensation problem can be formalized mathematically as a system of nonlinear Differential Algebraic Equations (DAEs).[1] This system, the solution of which is the compensating tax reform, is comprised of two

[1] The DAE theory is much more recent than the theory of ordinary differential equations (Ascher and Petzold [1998], p. 231). See Kunkel and Mehrmann [2006] for the first textbook treatment of this topic.

parts: (i) a differential equation, involving the marginal tax rate changes, that arises from the first-order conditions of the agents; (ii) an algebraic component, involving only the average tax rate changes, that arises from the requirement that the indirect utility of agents remains on the level set defined by their pre-disruption utility. The difficulty in the analysis of such a system, relative to a standard system of differential equations, is in the algebraic component that makes the Jacobian of the resulting system of implicit ODEs singular.[2]

We follow Kunkel and Mehrmann [2006] (Chapter 4) to solve the system of non-linear DAEs for marginal wage disruptions and tax reforms. A first-order Taylor expansion of the DAE system around the initial equilibrium shows that the welfare compensation problem reduces to an Integro-Differential Equation (IDE). The difficult part of the analysis is then to use the methods of Vainberg [1964] and Shishkin [2007] to convert the IDE into a non-homogeneous first-order ordinary differential equation, which links the average and the marginal tax rates; this forms the essence of the welfare compensation. The integral part of the original IDE is then fully separated in the nonhomogeneous part of this equation, and we can analyze it using standard tools of the theory of integral equations (e.g., Zemyan [2012]). The main result of this section is Proposition 2 that gives the closed-form solution and thus provides the complete analytical characterization of the compensating tax reform in response to any wage disruption in general equillibrium. This proposition also derives a closed-form formula for the fiscal surplus of the wage disruption and its compensation, which generalizes the traditional Kaldor-Hicks criterion and provides a simple test to determine whether economic shocks or policies are beneficial, in the sense that offseting its associated individual welfare gains and losses using only distortionary tax instruments is budget-feasible.

We then turn to the analysis of the solution and the economic insights of our main result. There are three key terms in the formula for the welfare compensating tax reform: (i) the *modified wage disruption* term, (ii) the *progressivity* term, and (iii) *the compensation-of-compensation* term. We provide a closed form expression for each of them.

First, the "modified wage disruption" variable defines the relevant disruption that needs to be compensated. In addition to the initial shock to wages, the compensation needs to account for the fact that the initial wage disruption induces labor supply

[2]See Lamour, März, and Tischendorf [2013], p. xx.

adjustments that further impact wages, because of both the decreasing marginal product of labor and the complementarities between skills in production.

Second, the "progressivity" term is a correction to the compensating tax reform (relative to the simple partial equilibrium compensation), the role of which is to counteract the welfare effects generated by the compensation because of the decreasing marginal product of labor. The key to understanding this effect is to realize that in general equilibrium, despite the envelope theorem, the marginal tax rate directly affects individual welfare through its impact on wages. Therefore, the compensation needs to be designed in such a way that the welfare effects generated by the marginal tax rates counteract those generated by the average tax rates, in addition to those due to the exogenous disruption. We show that this naturally leads to exponentially decreasing or increasing tax rates on incomes below those of a disrupted agents, and hence a progressive (resp., regressive) tax reform in response to a positive (resp., negative) disruption of a given wage. This effect is easiest to understand in the special case where there is decreasing marginal product of labor for each skill but perfect substitutability with other skills.

Third, the "compensation-of-compensation" term is due to the cross-wage effects originating from the skill complementarities in production. As we have argued above, the progressivity term compensates: (*i*) the individual welfare gains and losses generated by the initial wage disruption, as well as (*ii*) the own-wage effects created endogenously by the compensation itself. Now, if the government implements this tax reform, a lower marginal tax rate at a given income also affects (*iii*) all of the other wages via the cross-wage effects. The welfare impact of this indirect wage adjustment needs to be itself compensated using the tax schedule. However, the marginal tax rates of this second round of compensation generate in turn further wage and welfare changes for all of the agents, and so on. This leads to an a priori complex sequence of compensations. We show, however, that we can solve generally this fixed point problem in closed form by defining inductively a sequence of variables that each capture a given round of iterated compensation. Remarkably, if the production function is CES, we show that each round of iterated compensation is a constant fraction of the previous one. In this case, compensating the welfare gains and losses resulting from the skill complementarities in production simply requires a uniform shift of the marginal tax rates of the compensating tax reform obtained in the absence of cross-wage effects.

We now briefly describe the relationship to the literature. Three sets of papers are closest to our work. First, Hendren [2014] generalizes the Kaldor-Hicks principle and derives an inequality deflator in the partial equilibrium setting. Our results in Section 3 build on his work. Second, Ales, Kurnaz, and Sleet [2015a] studies optimal taxes in response to technical change in general equilibrium. They do not address the compensation problem, which is our main contribution, and the technically and conceptually most difficult part of our paper. Third, Itskhoki [2008] and Antras, de Gortari, and Itskhoki [2016] study taxation and the welfare implications of trade liberalization in an environment with distortionary taxes. Itskhoki [2008] solves for optimal redistribution in a closed and open economy following trade liberalization within a class of distortionary taxes. Antras, de Gortari, and Itskhoki [2016] solve for the optimal welfare and inequality correction following trade liberalization restricting taxes to be of the CRP form (Bénabou [2002], Heathcote, Storesletten, and Violante [2016]). While we do not consider a sophisticated model of trade, we solve the compensation problem allowing both general nonlinear tax schedules and a general production function.

More broadly, our model is within the class of Mirrleesean economies in general equilibrium (Stiglitz [1982], Rothschild and Scheuer [2013, 2014, 2016], Ales, Kurnaz, and Sleet [2015a,b], Scheuer and Werning [2016], Sachs, Tsyvinski, and Werquin [2016]). More specifically, the proof of Lemma 1 uses the results of Sachs, Tsyvinski, and Werquin [2016] to solve for the fixed point response of labor supply to a change in taxes.

DAE equations are also present in a different context in Benhabib, Perla, and Tonetti [2017], however, we do not need to use sophisticated numerical methods to solve the system of DAEs as we obtain our results in closed form.

1 Environment

1.1 Individuals and firms

There is a continuum of measure one of individuals indexed by $i \in [0, 1]$. Preferences over consumption c and labor supply l are given by $u(c - v(l))$,[3] where u and v are twice continuously differentiable and satisfy $u', v' > 0$, $u'' \leq 0$, $v'' > 0$. Each type (or

[3] The assumption that the utility function has no income effects is standard in the taxation literature (e.g., Diamond [1998]).

skill) i is composed of a mass one of identical individuals who are atomistic within their own skill group.

An agent of type i earns a wage $w_i \in \mathbb{R}_+$, which he takes as given. He chooses his labor supply l_i and earns pre-tax income $y_i = w_i l_i$. He pays a tax liability $T(y_i)$ and consumes $y_i - T(y_i)$. The non-linear tax schedule $T : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable. The agent's maximization problem reads

$$\max_{l>0} \; u\left[w_i l - T(w_i l) - v(l)\right].$$

We assume that l_i is the unique solution to this problem. It satisfies the first-order condition

$$0 \;=\; \left[1 - T'(w_i l_i)\right] w_i - v'(l_i). \tag{1}$$

The indirect utility of agent i is denoted by U_i.[4]

There is a continuum of mass one of identical firms that produce output using the labor of each type $i \in [0,1]$. Let L_i denote the aggregate labor supply of type-i agents.[5] The aggregate production function is denoted by $\mathscr{F}(\{L_i\}_{i\in[0,1]})$. We assume that $\mathscr{F}$ has constant returns to scale. In equilibrium, firms earn no profits and the wage w_i is equal to the marginal productivity of type-i labor, i.e.,[6]

$$w_i = \mathscr{F}_i(\{L_j\}_{j\in[0,1]}), \tag{2}$$

where $\mathscr{F}_i \equiv \partial\mathscr{F}/\partial L_i$ denotes the partial derivative of $\mathscr{F}$ with respect to its i^{th} variable.

We assume that wages and labor supplies are bounded so that incomes y belong to a compact interval $[\underline{y}, \bar{y}] \subset \mathbb{R}_+$, that labor incomes $y_i = w_i l_i \equiv y(w_i)$ are strictly increasing in wages w_i, and that the density of wages $f_w(\cdot)$ is continuously differentiable on $[\underline{w}, \bar{w}] \subset \mathbb{R}_+$. We denote by $f_y(\cdot)$ the density of incomes on $[\underline{y}, \bar{y}]$, with

[4] In Appendix B we generalize this model to include heterogenous fixed costs of working, and, hence, a participation decision. Our results of Propositions 1 and 2 are unaffected.

[5] Since the mass of agents with skill i is equal to 1, we have $L_i = l_i$ in the initial equilibrium. Note, however, that each individual agent is atomistic within his skill group, so that his wage changes only if all individuals with the same skill adjust their labor supply (e.g., in response to a tax change). In particular, each agent takes his wage as given and independent of his own choices.

[6] We assume without loss of generality that wages w_i are increasing in the index i. Therefore the agent's skill i can be interpreted as his percentile in the wage distribution.

$f_y(y(w)) = (y'(w))^{-1} f_w(w)$. Finally, we denote by $\boldsymbol{w} = \{w_i\}_{i\in[0,1]}$, $\boldsymbol{l} = \{l_i\}_{i\in[0,1]}$, $\boldsymbol{L} = \{L_i\}_{i\in[0,1]}$, and $\boldsymbol{U} = \{U_i\}_{i\in[0,1]}$ the distributions of wages, individual labor supplies, aggregate labor supplies, and indirect utilities in the baseline economy with the tax-and-transfer schedule T.

1.2 Wage disruptions and tax reforms

Wage disruptions. Consider first a partial equilibrium environment where wages w_i are exogenous. Suppose that an exogenous economic shock impacts the wage distribution $\boldsymbol{w}$ by $\hat{\boldsymbol{w}}^E = \{\hat{w}_i^E\}_{i\in[0,1]}$, so that for each $i \in [0,1]$ the wage of agent i changes from w_i to $w_i + \hat{w}_i^E$. We call $\hat{\boldsymbol{w}}^E$ a *disruption* of the wage distribution $\boldsymbol{w}$.

In the general equilibrium environment where the wage is equal to the (non-constant) marginal product of labor of the corresponding skill, defining a wage disruption is slightly more involved. It can be induced by two possible exogenous shocks: a perturbation of the production function $\mathscr{F}$ (due to, say, technological change), and a perturbation of the distribution of aggregate labor supply $\boldsymbol{L} = \{L_i\}_{i\in[0,1]}$ (due to, say, immigration flows). That is, the production function changes from $\mathscr{F}$ to $\mathscr{F} + \hat{\mathscr{F}}^E$, and/or the aggregate labor supply of type i changes from L_i to $L_i + \hat{L}_i^E$. We then define the wage disruption $\hat{\boldsymbol{w}}^E$ as the impact of these shocks on wages $\{w_i\}_{i\in[0,1]}$, before agents respond to this wage change by adjusting their labor supplies $\{l_i\}_{i\in[0,1]}$.

Definition 1. *A* wage disruption $\hat{\boldsymbol{w}}^E$ *is the change in the wage distribution* $\boldsymbol{w}$ *due to an exogenous shock* $(\hat{\mathscr{F}}^E, \hat{\boldsymbol{L}}^E)$ *to the production function or the distribution of aggregate labor supply, keeping individual labor supplies* $\boldsymbol{l} = \{l_i\}_{i\in[0,1]}$ *fixed:*

$$\hat{w}_i^E = [\mathscr{F}_i + \hat{\mathscr{F}}_i^E](\{L_j + \hat{L}_j^E\}_{j\in[0,1]}) - \mathscr{F}_i(\{L_j\}_{j\in[0,1]}).$$

The wage disruptions that we consider are continuous maps $i \mapsto \hat{w}_i^E$ *on* $[0,1]$, *and we denote by* $\|\hat{\boldsymbol{w}}^E\| = \max_{i\in[0,1]} |\hat{w}_i^E|$ *their infinite norm.*

Tax reforms. In order to compensate the agents for their income losses (or redistribute their gains) due to the disruption $\hat{\boldsymbol{w}}^E$, the government can implement a tax reform $\hat{T}(\cdot)$ of the tax schedule. That is, the statutory tax schedule at income level y changes from $T(y)$ to $T(y) + \hat{T}(y)$. We assume that the tax reforms $\hat{T}$ that the government can implement are continuously differentiable, bounded, with bounded first derivative. This defines a Banach space on which the norm of a function $\hat{T}$ is given by $\|\hat{T}\| = \sup_{y\in\mathbb{R}_+} |\hat{T}(y)| + \sup_{y\in\mathbb{R}_+} |\hat{T}'(y)|$.

Disrupted equilibrium. In response to a wage disruption $\hat{\boldsymbol{w}}^E$ and a tax reform $\hat{T}$, individuals optimally adjust their labor supply. In general equilibrium, this further impacts their wage, which in turn affects their labor supply, and so on. We denote by $\hat{w}_i$ and $\hat{l}_i$ the total endogenous changes in individual i's wage and labor supply following the initial perturbation $(\hat{\boldsymbol{w}}^E, \hat{T})$. That is, the wage and labor supply of an agent with skill i in the equilibrium of the disrupted economy are respectively equal to $\tilde{l}_i = l_i + \hat{l}_i$ and $\tilde{w}_i = w_i + \hat{w}_i^E + \hat{w}_i$.

Formally, $(\hat{w}_i, \hat{l}_i)$ are defined by the perturbed wage equation

$$\tilde{w}_i = [\mathscr{F}_i + \hat{\mathscr{F}}_i^E](\{L_j + \hat{L}_j^E + \hat{l}_j\}_{j\in[0,1]}), \tag{3}$$

and the perturbed first-order condition

$$0 \;=\; [1 - T'(\tilde{w}_i\tilde{l}_i) - \hat{T}'(\tilde{w}_i\tilde{l}_i)]\tilde{w}_i - v'(\tilde{l}_i) \;\equiv\; \Phi[\tilde{w}_i, \tilde{l}_i, T'(\tilde{w}_i\tilde{l}_i) + \hat{T}'(\tilde{w}_i\tilde{l}_i)]. \tag{4}$$

Equation (4) defines a map $\Phi : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$.

Compensating variation. As a result of the wage disruption $\hat{\boldsymbol{w}}^E$ and the tax reform $\hat{T}$, the indirect utility of agent i (transformed into output units by normalizing it by the marginal utility of consumption) changes by

$$\frac{u[\tilde{w}_i\tilde{l}_i - T(\tilde{w}_i\tilde{l}_i) - \hat{T}(\tilde{w}_i\tilde{l}_i) - v(\tilde{l}_i)] - U_i}{u'\left[w_i l_i - T\left(w_i l_i\right) - v\left(l_i\right)\right]} \;\equiv\; \Psi[\tilde{w}_i, \tilde{l}_i, T(\tilde{w}_i\tilde{l}_i) + \hat{T}(\tilde{w}_i\tilde{l}_i), U_i]. \tag{5}$$

Equation (5) defines a map $\Psi : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$.

Definition 2. *The* compensating variation *(Mas-Colell, Whinston, and Green [1995], p. 82) of an individual i from the wage disruption $\hat{\boldsymbol{w}}^E$ and the tax reform $\hat{T}$ is defined by $\Psi[\tilde{w}_i, \tilde{l}_i, T(\tilde{w}_i\tilde{l}_i) + \hat{T}(\tilde{w}_i\tilde{l}_i), U_i]$.*

Intuitively, the compensating variation is the monetary amount that an agent i would be willing to pay, after the wage disruption $\hat{w}_i^E$ and the tax reform $\hat{T}$, in order to be as well off as before these two shocks.[7] A positive (resp., negative) value of $\Psi[\tilde{w}_i, \tilde{l}_i, T(\tilde{w}_i\tilde{l}_i) + \hat{T}(\tilde{w}_i\tilde{l}_i), U_i]$ implies that an individual i benefits (resp., loses) from these shocks.

1.3 The welfare compensation problem

Consider a wage disruption $\hat{\boldsymbol{w}}^E$, as described in the previous section. This economic shock generally creates winners and losers, i.e., welfare gains for some individuals and welfare losses for others. The welfare compensation problem consists of designing a reform $\hat{T}$ of the existing tax code that offsets these losses by redistributing the gains of the winners. Such a tax reform, if it exists, must be designed in such a way that each agent's compensating variation (5) is equal to zero, taking into account that their wage and labor supply both respond endogenously to the tax change, so that equations (3) and (4) remain satisfied.

Definition 3. *The solution to the* welfare compensation problem *in response to a wage disruption $\hat{\boldsymbol{w}}^E$ is a tax schedule $\hat{T}$, a wage distribution $\tilde{\boldsymbol{w}} = \{\tilde{w}_i\}_{i\in[0,1]}$, and a labor supply distribution $\tilde{\boldsymbol{l}} = \{\tilde{l}_i\}_{i\in[0,1]}$ such that the compensating variation of each agent is equal to zero when their labor supply is chosen optimally and their wage is equal to the marginal product of the aggregate labor of their skill type. That is, for all $i \in [0,1]$,*

$$0 = \Psi[\tilde{w}_i, \tilde{l}_i, T(\tilde{w}_i\tilde{l}_i) + \hat{T}(\tilde{w}_i\tilde{l}_i), U_i] \tag{6}$$
$$0 = \Phi[\tilde{w}_i, \tilde{l}_i, T'(\tilde{w}_i\tilde{l}_i) + \hat{T}'(\tilde{w}_i\tilde{l}_i)], \tag{7}$$

where $\tilde{\boldsymbol{w}}$ satisfies (3). The fiscal surplus *is the change in government revenue induced by the wage disruption $\hat{\boldsymbol{w}}^E$ and the corresponding compensating tax reform $\hat{T}$:*

$$\mathcal{R}(\hat{\boldsymbol{w}}^E) = \int_0^1 \left[T(\tilde{w}_i\tilde{l}_i) + \hat{T}(\tilde{w}_i\tilde{l}_i) - T(w_i l_i)\right] di. \tag{8}$$

We say that the welfare gains of the economic disruption $\hat{\boldsymbol{w}}^E$ are redistributable if $\mathcal{R}(\hat{\boldsymbol{w}}^E) \geq 0$.

[7] We could alternatively define the concept of equivalent variation. In the case of marginal disruptions considered in this paper (see below), i.e. as $\|\hat{w}^E\|, \|\hat{T}\| \to 0$, the two concepts coincide.

1.4 Elasticity notations

We start by defining the elasticities of the wage with respect to the aggregate labor supplies of various skills. There are two channels through which labor supply affects wages in general equilibrium. First, since the marginal product of labor is non-increasing, the wage of skill i is decreasing in the labor supply of skill i. We denote the corresponding elasticity by α_i. Second, because different skills are imperfect substitutes in production, the wage of skill i is (positively or negatively) impacted by the labor supply of all the other skills $j \in [0,1]$. We denote the corresponding elasticities by γ_{ij}.

Definition 4. *We define the* structural cross-wage elasticity *of the wage of type i with respect to the labor supply of type $j \neq i$ as*

$$\gamma_{ij} \equiv \frac{\partial \ln w_i}{\partial \ln L_j} = \frac{L_j \mathscr{F}_{ij} (\boldsymbol{L})}{\mathscr{F}_i (\boldsymbol{L})}, \tag{9}$$

where $\mathscr{F}_{ij}$ denotes the second partial derivative of the production function with respect to the variables (i, j). We define the structural own-wage elasticity *of the wage of type i with respect to the labor supply of type i as*

$$-\alpha_i \equiv \frac{\partial \ln w_i}{\partial \ln L_i} - \lim_{k \to i} \frac{\partial \ln w_k}{\partial \ln L_i} = \frac{L_i \mathscr{F}_{ii} (\boldsymbol{L})}{\mathscr{F}_i (\boldsymbol{L})} - \lim_{k \to i} \frac{L_i \mathscr{F}_{ki} (\boldsymbol{L})}{\mathscr{F}_k (\boldsymbol{L})}. \tag{10}$$

We have $\alpha_i \geq 0$ and γ_{ij} can be positive or negative.

Expression (10) captures the effect of L_i on the wage w_i arising purely from the non-constancy of the marginal product of labor of one's own skill.

Next, we define the elasticities of labor supply of agent i with respect to his marginal tax rate $T'(y_i)$ (or his retention rate $1 - T'(y_i)$), and with respect to his wage w_i.

Definition 5. *We define the* labor supply elasticity with respect to the retention rate *of agent i, along the nonlinear budget constraint and along the non-increasing labor* *demand curve, as*[8]

$$\varepsilon_i^r \equiv \frac{\partial \ln l_i}{\partial \ln (1 - T'(y_i))} = \frac{e_i}{1 + p(y_i) e_i + \alpha_i (1 - p(y_i)) e_i}, \tag{11}$$

where $e_i \equiv \frac{v'(l_i)}{l_i v''(l_i)}$ *and* $p(y) = \frac{yT''(y)}{1-T'(y)}$. *We define the* labor supply elasticity with respect to the wage *of agent i as*

$$\varepsilon_i^w = \frac{\partial \ln l_i}{\partial \ln w_i} = \frac{(1 - p(y_i)) e_i}{1 + p(y_i) e_i + \alpha_i (1 - p(y_i)) e_i}. \tag{12}$$

We have $\varepsilon_i^r, \varepsilon_i^w > 0$.

The elasticities (11) and (12) differ from the usual structural elasticity $e_i = \frac{v'(l_i)}{l_i v''(l_i)}$ as they take into account the fact that the labor supply response to a tax change (given by e_i) or a wage change (given by $(1 - p(y_i)) e_i$) impacts (i) the marginal tax rate faced by the agent, by an amount equal to the rate of progressivity $p(y)$ of the nonlinear tax schedule, and (ii) his wage, by an amount equal to the elasticity α_i. These two endogenous effects yield further labor supply adjustments (given by e_i and $(1 - p(y_i)) e_i$, respectively), thus explaining the denominators in (11) and (12).

2 Compensation in Partial Equilibrium

In this section, we show that the solution to the compensation problem takes a simple form in partial equilibrium, even when lump-sum taxes are not available policy instruments (i.e., when taxes are distortionary).[9]

We suppose that there is infinite substitutability between skills in production, i.e.,

$$\mathscr{F}(\{L_i\}_{i \in [0,1]}) = \int_0^1 \theta_i L_i di, \tag{13}$$

where $\theta_i \in \mathbb{R}_+$ for all i. This implies that wages are exogenous and equal to $w_i = \theta_i$

[8]Note that, since we have assumed that there is a one-to-one map between wages w_i (or skills i) and incomes y_i, we can denote the elasticities $\varepsilon_i^r, \varepsilon_i^w, \alpha_i$ equivalently by $\varepsilon_{y_i}^r, \varepsilon_{y_i}^w, \alpha_{y_i}$. In the sequel we use these notations interchangeably depending on the context. On the other hand, the correct change of variables for the wage elasticities γ_{ij} and Γ_{ij} is $\gamma_{y_i, y_j} \equiv \frac{\gamma_{ij}}{y'(j)}$ and $\Gamma_{y_i, y_j} \equiv \frac{\Gamma_{ij}}{y'(j)}$. See Appendix A for further details.

[9]Here we allow the government to implement continuous (not necessarily continuously differentiable) tax reforms.

for all i. This is the standard partial equilibrium assumption made by Mirrlees [1971]. We have in particular $\alpha_i = \gamma_{ij} = 0$ for all $i, j \in [0,1]$.

2.1 General solution to the welfare compensation problem

In the partial equilibrium environment, the initial wage disruption $\hat{\boldsymbol{w}}^E$ generates no further adjustment in the wage, i.e., for all $i \in [0,1]$, $\hat{w}_i = 0$ and $\tilde{w}_i = w_i + \hat{w}_i^E$. We characterize analytically the solution to the welfare compensation problem (6)-(7) for marginal wage disruptions, i.e., as $\|\hat{\boldsymbol{w}}^E\| \to 0$. We construct a compensating tax reform $\hat{T}$ that is continuous in the exogenous disruption $\hat{\boldsymbol{w}}^E$, in the sense that $\|\hat{T}(\hat{\boldsymbol{w}}^E)\| \to 0$ as $\|\hat{\boldsymbol{w}}^E\| \to 0$.

A first-order Taylor expansion of equations (6) and (7) around the initial equilibrium implies that the solution $(\hat{T}, \hat{\boldsymbol{l}})$ to the welfare compensation problem satisfies the following linear system of two equations: for all $i \in [0,1]$,

$$0 = [\Psi_1 + \Psi_3 l_i T'(w_i l_i)]\, \hat{w}_i^E + [\Psi_2 + \Psi_3 w_i T'(w_i l_i)]\, \hat{l}_i + [\Psi_3]\, \hat{T}(w_i l_i) \qquad (14)$$

$$0 = [\Phi_1 + \Phi_3 l_i T''(w_i l_i)]\, \hat{w}_i^E + [\Phi_2 + \Phi_3 w_i T''(w_i l_i)]\, \hat{l}_i + [\Phi_3]\, \hat{T}'(w_i l_i), \qquad (15)$$

where $\Psi_k \equiv \Psi_k(w_i, l_i, T(w_i l_i), U_i)$ and $\Phi_k \equiv \Phi_k(w_i, l_i, T'(w_i l_i))$, for $k \in \{1,2,3\}$, denote the partial derivatives of the functions Ψ and Φ with respect to their k^{th} variable, evaluated at their original (un-disrupted) equilibrium. We analyze these two equations in turn.

Equation (14) imposes that agent i keeps the same level of welfare in the disrupted economy as in the initial equilibrium, once the new tax schedule is implemented. This equation can be simplified by recognizing that $[\Psi_2 + \Psi_3 w_i T'(w_i l_i)] = 0$, which follows immediately from the first-order condition (1), or from the envelope theorem: since individuals choose their labor supply optimally before the perturbation, their response has no first-order effect on welfare. That is, their labor supply adjustment $\hat{l}_i$ does not affect their compensating variation (the right hand side of (14)). We show in Appendix A that equation (14) can then be rewritten as

$$0 = \left[-\frac{\Psi_1}{\Psi_3} - l_i T'(w_i l_i)\right] \hat{w}_i^E - \hat{T}(w_i l_i) = (1 - T'(y_i))\, y_i \frac{\hat{w}_i^E}{w_i} - \hat{T}(y_i). \qquad (16)$$

This equation shows that, in the partial equilibrium framework, the change in the indirect utility of agent i is due to:

(i) the exogenous change $\hat{w}_i^E$ in his wage, weighted by the share $(1 - T'(y_i))$ that he keeps after paying taxes (the first term of (16));

(ii) the change in his tax liability $\hat{T}(y_i)$ (the second term of (16)), which makes him poorer (resp. richer) if $\hat{T}(y_i) > 0$ (resp. < 0).

Crucially, note that the change in the marginal tax rate, $\hat{T}'(y_i)$, does not enter equation (16), and therefore does not matter for welfare (conditional on the average tax rate $\hat{T}(y_i)$). This again follows from the envelope theorem: the marginal tax rate that the individual faces affects his indirect utility only through his (optimal) labor supply decision, so that the corresponding welfare effect is second-order.

Next, we further develop equation (15), which imposes that the labor supply of agent i remains optimal in the disrupted economy. Using the elasticity notations introduced in Section 1.4, we show that this equation can be rewritten as[10]

$$\frac{\hat{l}_i}{l_i} = \frac{-\frac{\Phi_1}{\Phi_3} - l_i T''(w_i l_i)}{\frac{\Phi_2 l_i}{\Phi_3} + w_i l_i T''(w_i l_i)} \hat{w}_i^E - \frac{\hat{T}'(w_i l_i)}{\frac{\Phi_2 l_i}{\Phi_3} + w_i l_i T''(w_i l_i)} = \varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(y_i)}{1 - T'(y_i)}. \quad (17)$$

This equation shows that in response to the perturbation $(\hat{\boldsymbol{w}}^E, \hat{T})$, the agent's labor supply adjusts both because of the change in his wage $\hat{w}_i^E$ (by an amount given by the elasticity ε_i^w defined in (12)), and because of the change in his marginal tax rate $\hat{T}'(y_i)$ (by an amount given by the elasticity ε_i^r defined in (11)).

We now summarize the results obtained so far. Equation (16) immediately gives the tax reform $\hat{T}$ which ensures that, after reoptimizing their behavior, individuals remain as well off before as after the wage disruption $\hat{\boldsymbol{w}}^E$. Equation (17) then gives the corresponding change in the labor supply of agents following the wage disruption and the compensating tax reform. We thus obtained the solution to the welfare compensation problem in closed form for any potential wage disruption $\hat{\boldsymbol{w}}^E$. We gather these results into the following proposition, which moreover gives the impact of the disruption and its compensation on the government budget.

Proposition 1. *Suppose that the production function is given by (13). Consider an exogenous disruption* $\hat{\boldsymbol{w}}^E = \{\hat{w}_i^E\}_{i \in [0,1]}$ *of the wage distribution* $\boldsymbol{w}$. *There exists a*

[10] We assume that $\frac{\Phi_2 l_i}{\Phi_3} + w_i l_i T''(w_i l_i) \neq 0$, which is generically satisfied.

unique tax reform that solves the welfare compensation problem, namely:[11] *for all* $y \in [\underline{y}, \bar{y}]$,

$$\hat{T}(y) = (1 - T'(y))\, y\, \frac{\hat{w}_y^E}{w_y}. \tag{18}$$

Moreover, the labor supply of agent i *changes by (17) following the wage disruption and the tax reform. The fiscal surplus of the wage disruption* $\hat{\boldsymbol{w}}^E$ *and the corresponding compensating tax reform* $\hat{T}$ *is given by*

$$\mathcal{R}(\hat{\boldsymbol{w}}^E) = \int_{\underline{y}}^{\bar{y}} \left[\frac{\hat{w}_x^E}{w_x} + T'(x) \left(\varepsilon_x^w \frac{\hat{w}_x^E}{w_x} - \varepsilon_x^r \frac{\hat{T}'(x)}{1 - T'(x)} \right) \right] x f_y(x)\, dx. \tag{19}$$

Proof. See Appendix A. □

We now sketch the proof and provide the interpretation of formula (19). Consider an individual who earns income $x \in [\underline{y}, \bar{y}]$ before the wage disruption. First, his wage changes by $\hat{w}_x^E$, so that his income (absent any labor supply responses) changes by $x\hat{w}_x^E$. The government keeps a share $T'(x)$ of this income change. Moreover, by equation (18), the government raises the agent's tax liability by $\hat{T}(x) = (1 - T'(x))\, x\, \hat{w}_x^E$. Therefore tax revenue increases by $T'(x)\, x\, \hat{w}_x^E + (1 - T'(x))\, x\, \hat{w}_x^E = x\, \hat{w}_x^E$. This is the first term in the square brackets of expression (19). Second, the wage disruption and the compensating tax reform lead the agent to adjust (say, reduce) his labor supply by $\hat{l}_x$, given by (17). This lowers government revenue by a fraction $T'(x)$ of the corresponding income loss (the term in parenthesis in (19)). This yields the second term in the square brackets of (19). Summing over all incomes $x \in [\underline{y}, \bar{y}]$, weighted by the density f_y, leads to the total impact on government revenue, or fiscal surplus. Note that (19) is a closed-form expression, since it depends only on the exogenous wage disruption $\hat{\boldsymbol{w}}^E$ and on the characteristics of the undisrupted economy (income distribution, tax schedule, labor supply elasticities).

Proposition 1 is our first step in generalizing the standard Kaldor-Hicks criterion to the environment when lump-sum taxes are unavailable. It shows that the compensating tax reform consists of adjusting the average tax rates $(\hat{T}(y)/y)$ up or down by an amount equal to the income gain or loss of agents resulting from the economy's disruption. Our measure of the redistributable aggregate gains $\mathcal{R}(\hat{\boldsymbol{w}}^E)$ of the wage

[11] In this formula, we change variables from the index i to the income level y, as there is a one-to-one map between these two variables. The variable w_y is the wage earned by the agents whose income in the undisrupted economy is y (i.e., $w_y = w_i$ if $y = y_i$).

disruption $\hat{\boldsymbol{w}}^E$ takes into account the fact that redistributing the individual gains through the tax code generates labor supply distortions. The government is able to compensate the gains and losses from the wage disruption in a budget-neutral way if and only if $\mathcal{R}(\hat{\boldsymbol{w}}^E) \geq 0$. It is thus possible that a shock to the economy (say, technological change, an immigration inflow, or opening to international trade) generates strictly positive aggregate gains, but that these gains are not redistributable because the distortions generated by the compensating tax reform $\hat{T}$ outweigh the aggregate income gains of the initial economic shock.

2.2 Graphical representation

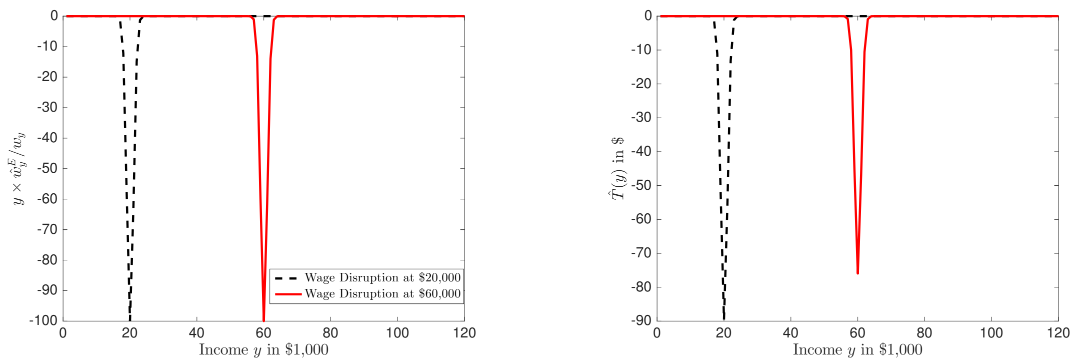
We assume the disutility of labor to be iso-elastic with $\varepsilon = 0.33$ [Chetty et al., 2011]. We assume that incomes are log-normally distributed apart from the top, where we append a Pareto distribution for incomes above \$150,000. We follow Saez [2001] to obtain the underlying wage distribution from the first-order conditions of the agents. We assume a CRP specification, i.e., $y - T(y) \propto y^{1-p}$, and set the parameters as in Heathcote, Storesletten, and Violante [2016].

We focus our numerical analysis on elementary disruptions that consist of a shock at only one income level y^*. Formally, $\hat{\boldsymbol{w}}^E$ is a Dirac delta function at $y = y^*$. Formula (18) then shows that only the average tax rate at income y^* must be adjusted. As we show in Appendix A and discuss further below, these elementary disruptions are particularly useful because the compensating tax reform $\hat{T}(\cdot)$ associated with any general disruption $\hat{\boldsymbol{w}}^E$ can be immediately expressed as the sum of the corresponding elementary compensations $\hat{T}_{y^*}(\cdot)$ at each income level $y^* \in [\underline{y}, \bar{y}]$ that is disrupted.

We specifically consider two illustrative wage disruptions and the respective compensating tax reforms. To approximate the elementary Dirac shocks, we construct smooth wage disruptions that are normally distributed and centered around $y^* = \$20,000$ and $y^* = \$60,000$. We assume that at these points the wage decreases by an amount $\hat{w}^E_{y^*}$ that implies a decrease in pre-tax income of $y^* \times \left(\hat{w}^E_{y^*}/w_{y^*}\right) = \$100$. The resulting gross income disruptions are illustrated in the left panel of Figure 1.

In the right panel of Figure 1 we illustrate the respective compensating tax reforms, formally derived in Proposition 1. The decrease in the agent's average tax rate implied by these reforms mirrors the income loss due to the wage disruptions. Note that the compensation is larger for an income loss that affects lower incomes, because the marginal tax rates in our calibration are increasing with income. The marginal tax rate is 10% at $20,000 and 22% at $60,000. As a consequence, the gross income reduction of $100 translates into after-tax income losses of $90 and $78 respectively. Therefore, the compensating tax reform implies a reduction in tax payment of $90 and $78, respectively.

Figure 1: Wage disruptions centered at $20,000 and $60,000 (left panel) and respective compensating tax reforms (right panel)



3 Compensation in General Equilibrium

In this section we analyze the welfare compensation problem (6)-(7) in the general equilibrium environment laid out in Section 1. As in Section 2, we derive its solution in a closed form as the size of the wage disruption $\|\hat{\boldsymbol{w}}^E\| \to 0$.[12]

3.1 General solution to the welfare compensation problem

Before solving the welfare compensation problem (6)-(7), it is useful to first discuss the mathematical formalism of this system of equations. This is a system of Differential Algebraic Equations (DAE). The unknown function to solve for is $(\hat{T}, \hat{\boldsymbol{l}})$. The system consists of (i) a differential equation (7), which involves the derivative $\hat{T}'$ of the unknown function; and (ii) an algebraic component (6), the level set constraint, which only features the function $\hat{T}$ itself. The difficulty in the analysis of such a system, relative to a standard system of differential equations, is that the Jacobian of

[12] More precisely, we let the exogenous perturbation $\|\hat{\boldsymbol{L}}^E\|, \|\hat{\mathscr{F}}^E\| \to 0$. The wage disruption $\hat{\boldsymbol{w}}^E$ is then given by $\hat{w}_i^E = \hat{\mathscr{F}}_i^E(\boldsymbol{L}) + \int_0^1 \hat{L}_j^E \mathscr{F}_{ij}(\boldsymbol{L})\, dj$.

the implicit ODE is singular due to the presence of the algebraic constraint that does not include $\hat{T}'$ (Ascher and Petzold [1998], p. 231). The DAEs can also be viewed as differential equations on manifolds (Rheinboldt [1984], Hairer and Wanner [1996], Chapters VI and VII, Brunner [2004], Chapter 8). The algebraic constraint forms a manifold, and the literature proceeds by analyzing the behavior of the suitably projected differential equation using a geometric viewpoint.

We follow Kunkel and Mehrmann [2006] (Chapter 4) to linearize the system of nonlinear DAEs.[13] A first-order Taylor expansion of equations (6) and (7) implies that the solution $(\hat{T}, \hat{\boldsymbol{l}})$ to the welfare compensation problem satisfies the following linear system of two equations: for all $i \in [0,1]$,

$$
\begin{aligned}
0 &= [\Psi_1 + \Psi_3 l_i T'(w_i l_i)] \left(\hat{w}_i^E + \hat{w}_i\right) + [\Psi_2 + \Psi_3 w_i T'(w_i l_i)] \, \hat{l}_i + [\Psi_3] \, \hat{T}(w_i l_i) && (20) \\
0 &= [\Phi_1 + \Phi_3 l_i T''(w_i l_i)] \left(\hat{w}_i^E + \hat{w}_i\right) + [\Phi_2 + \Phi_3 w_i T''(w_i l_i)] \, \hat{l}_i + [\Phi_3] \, \hat{T}'(w_i l_i), && (21)
\end{aligned}
$$

where the partial derivatives Ψ_k, Φ_k are evaluated at the original (un-disrupted) equilibrium. These equations are similar to (14)-(15) obtained in the partial equilibrium model of Section 2, except that the adjustment in the wage of agent i now includes the endogenous component $\hat{w}_i$ in addition to the initial disruption $\hat{w}_i^E$. This endogenous wage correction affects directly the agent's indirect utility (equation (20)) and his choice of labor supply (equation (21)).

In order to find the solution $(\hat{T}, \hat{\boldsymbol{l}})$ to the welfare compensation problem, we therefore need to first characterize the endogenous wage change $\hat{w}_i$. We show in Appendix A that a first-order Taylor expansion of the perturbed wage equation (3) leads to

$$
\frac{\hat{w}_i}{w_i} = -\alpha_i \frac{\hat{l}_i}{l_i} + \int_0^1 \gamma_{ij} \frac{\hat{l}_j}{l_j} dj, \tag{22}
$$

where the elasticities γ_{ij} and α_i are defined in (9) and (10). This equation has the following economic interpretation: a one percent increase in the labor supply of an individual of type i leads to a $-\alpha_i$ percent change in the wage of type i, beyond the

[13] Rabier and Rheinboldt [1990, 1994] provide conditions for the local existence and uniqueness of solutions. März [2011] is perhaps the most comprehensive recent analysis of the conditions under which linearizations are valid (see also Campbell [1995]). Campbell and Griepentrog [1995] discuss the computational verification of solutions. However, complications primarily arise in complex systems of higher indices (Campbell and Griepentrog [1995]), while our linearized system is a Hessenberg index-1 DAE (see Hairer and Wanner [1996], p. 374) which poses fewer challenges (see, e.g., a discussion in März [1995]).

initial wage disruption $\hat{w}_i^E$; analogously, a one percent increase in the labor supply of an individual of type $j \neq i$, for any $j \in [0,1]$, leads to a γ_{ij} percent change in the wage of type i.

We now follow the literature (see, e.g., Hairer and Wanner [1996], p. 374) and transform the DAE system (20)-(21) into a differential equation.[14] Substituting for $\hat{w}_i$ using (22), equation (20) can be rewritten as

$$0 \quad = \quad (1 - T'(y_i))\, y_i \left[\frac{\hat{w}_i^E}{w_i} - \alpha_i \frac{\hat{l}_i}{l_i} + \int_0^1 \gamma_{ij} \frac{\hat{l}_j}{l_j} dj \right] - \hat{T}(y_i)\,. \tag{23}$$

This equation generalizes equation (16) (where $\hat{w}_i^E$ is replaced by $(\hat{w}_i^E + \hat{w}_i)$) and shows that, in addition to the two partial equilibrium forces described in Section 2, there is now the third channel through which the compensating variation of the agent is affected, namely:

(iii) the endogenous changes $\hat{l}_i$ and $\{\hat{l}_j\}_{j\in[0,1]}$ in the labor supplies of type-i and type-j agents, by impacting the wage of skill i (through equation (22)), have a first-order impact on the indirect utility of agent i.

This shows that despite the envelope theorem, the endogenous changes in labor supply matter for welfare, through their impact on wages that result from the decreasing marginal productivity and the complementarities in production. Therefore, in general equilibrium, because of the labor supply responses it generates, the marginal tax rate affects directly the agent's utility, even conditional on the average tax rate. This is the key difference with the partial equilibrium environment.

Next, substituting for $\hat{w}_i$ into equation (24) and solving for $\hat{l}_i$ using the labor supply elasticities introduced in Definition 5,[15] we show in Appendix A that we can rewrite equation (21) as

$$\frac{\hat{l}_i}{l_i} \quad = \quad \left[\varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(w_i l_i)}{1 - T'(y_i)} \right] + \int_0^1 \varepsilon_i^w \gamma_{ij} \frac{\hat{l}_j}{l_j} dj. \tag{24}$$

[14] This transfomation is the essence of the relationship between the DAEs and ODEs. Lamour, März, and Tischendorf [2013] (p. xxi) writes: "almost all approaches to DAEs suppose that the DAE is eventually reducible to an ODE as a basic principle. This opinion is summarized in Rabier and Rheinboldt [2000] (p. 191) (...)."

[15] In particular, the effect of the agent's labor supply on his own wage is accounted for by the denominator of the labor supply elasticities along the decreasing labor demand curve, (11) and (12).

This equation gives the relationship between, on the one hand, the wage disruption and the change in the marginal tax rate of agent i, and on the other hand, all agents' changes in labor supply. Equation (24) is more complex, however, than the corresponding equation (17) obtained in partial equilibrium, because the change in labor supply of agent i, $\hat{l}_i$, depends on those of all other agents j through the skill complementarities γ_{ij}. Hence all of the labor supply adjustments $\{\hat{l}_i\}_{i\in[0,1]}$ have to be solved for simultaneously as functions of the whole wage disruption function $\hat{\boldsymbol{w}}^E$ and the tax reform $\hat{T}$. The following lemma derives a closed form solution for $\hat{l}_i$, for all $i \in [0,1]$.

Lemma 1. *The solution to (24) is given by: for all $i \in [0,1]$,*

$$\frac{\hat{l}_i}{l_i} = \left[\varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(y_i)}{1-T'(y_i)}\right] + \int_0^1 \varepsilon_i^w \Gamma_{ij} \left[\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} - \varepsilon_j^r \frac{\hat{T}'(y_j)}{1-T'(y_j)}\right] dj, \tag{25}$$

where Γ_{ij} is given by

$$\Gamma_{ij} = \gamma_{ij} + \sum_{n=1}^{\infty} \Gamma_{ij}^{(n)}, \tag{26}$$

with $\Gamma_{ij}^{(0)} = \gamma_{ij}$ and for all $n \geq 1$,

$$\Gamma_{ij}^{(n)} = \int_0^1 \Gamma_{ik}^{(n-1)} \varepsilon_k^w \gamma_{kj} dk. \tag{27}$$

Proof. This follows from Proposition 1 in Sachs, Tsyvinski, and Werquin [2016]. See Appendix A for details. □

We first describe the interpretation of the term Γ_{ij} defined in (26). This term can be thought of an elasticity of the wage of type i with respect to the labor supply of type j. In contrast to γ_{ij}, it accounts for the infinite sequence of cross-wage effects between different skills that occur in general equilibrium, and thus represents the total impact of L_j on w_i once the economy has fully adjusted to the initial shock. The initial change in type-j labor supply directly affects the wage of type i through the structural cross-wage elasticity γ_{ij} – this is the direct effect $\Gamma_{ij}^{(0)}$. This induces a change in type-i labor supply (by ε_i^w), which in turn feeds back into the wage of type j, and so on. The wages and labor supplies of all of the other types k are analogously affected at each stage, adding their own contribution to the total general equilibrium adjustment in w_i. The closed-form formula (26)-(27) constructs as an infinite sequence of indirect effects of L_j on w_i through the sequence $\{\Gamma_{ij}^{(n)}\}_{n\geq 1}$.[16]

We can now interpret equation (25) as follows. The change in labor supply of agent i is the sum of two terms. The first is the direct effect of agent i's wage disruption $\hat{w}_i^E$ and marginal tax rate change $\hat{T}'(y_i)$ on his labor supply (via the labor supply elasticities ε_i^w and ε_i^r, respectively). The second (the integral term in (25)) accounts for the effects of the changes in labor supply of all other agents $j \in [0,1]$ (themselves driven by the corresponding variables $\hat{w}_j^E$ and $\hat{T}'(y_j)$); they affect the wage of agent i though the empirical wage elasticity Γ_{ij}, leading in turn to a change in labor supply of agent i given by the elasticity ε_i^w.

We summarize the results we obtained so far. The key difference between the system (16)-(17) obtained in partial equilibrium, and the system (23)-(25) obtained in general equilibrium, is that the latter two equations can no longer be solved independently. Indeed, in contrast to (16), equation (23) does not yield directly the solution for the compensating tax reform $\hat{T}$ as a function of the wage disruption $\hat{\boldsymbol{w}}^E$. This is because, as we explained above, the agent's endogenous change in labor supply affects directly his welfare, so that the variable $\hat{l}_i$ enters (23). Moreover, equation (25) implies that $\hat{l}_i$ is in turn a function of the change in the marginal tax rate $\hat{T}'(y_i)$. That is, the marginal tax rates of the compensating tax reform that the government implements generate first-order welfare effects that need to be themselves compensated using the tax code. In other words, in general equilibrium the government affects individual welfare through two distinct instruments, the average tax rate and the marginal tax rate, but these two instruments cannot be chosen independently. This leads to a complex fixed point problem. Mathematically, equations (23)-(25) form a linear differential-algebraic equation (DAE) system.[17]

By substituting for $\hat{l}_i$ into (23) using (25) and changing variables from skills i to incomes y,[18] we obtain the generalization of equation (18) to the general equilibrium environment. The following lemma is the first step toward characterizing analytically

[16] A precise interpretation of each round $n \geq 1$ of feedback effect (captured by the variable $\Gamma_{ij}^{(n)}$) is given in Sachs, Tsyvinski, and Werquin [2016]. For empirical purposes, depending on the data at hand, either γ_{ij} or Γ_{ij} may be the more natural elasticities to estimate, and all our formulas can be equivalently expressed in terms of either of them.

[17] More precisely, this is an integro-differential-algebraic equation (IDAE) system (see Chapter 8 in Brunner [2004]), where the integral part arises from the cross-wage effects between different skills in general equilibrium.

[18] See footnote 8.

the solution to the welfare compensation problem in our general setup.

Lemma 2. *The compensating tax reform $\hat{T}$ satisfies the following functional equation: for all $y \in \left[\underline{y}, \bar{y}\right]$,*

$$
\begin{aligned}
& -\hat{T}(y) + \left(\alpha_y \varepsilon_y^r y\right) \hat{T}'(y) - \left(1 - \alpha_y \varepsilon_y^w\right) \int_{\underline{y}}^{\bar{y}} \frac{(1 - T'(y))\, y}{(1 - T'(x))\, x} \left(\Gamma_{yx} \varepsilon_x^r x\right) \hat{T}'(x)\, dx \\
= & -(1 - T'(y))\, y\, \hat{\Omega}_y^E,
\end{aligned}
\tag{28}
$$

where the modified wage disruption $\hat{\Omega}_y^E$ is defined by:

$$
\hat{\Omega}_y^E \equiv \left(1 - \alpha_y \varepsilon_y^w\right) \left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx} \varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right]. \tag{29}
$$

Proof. See Appendix A. □

The right hand side of equation (28) is (minus) the compensating variation due to the wage disruption $\hat{\boldsymbol{w}}^E$, i.e., the welfare gain or loss induced by the initial economic shock incurred by agents with income y. Note that the relevant disruption variable is now $\hat{\Omega}_y^E$ rather than simply $\hat{w}_y^E / w_y$. The scaling by $1 - \alpha_y \varepsilon_y^w$ in (29) accounts for the endogenous own-wage effects: an initial wage increase by $\hat{w}_y^E / w_y = 1$ percent raises the labor supply of agents y by ε_y^w, which in turn lowers their marginal product of labor (wage) by α_y. The integral term in (29) accounts for the cross-wage effects: the disruption $\hat{w}_x^E / w_x$ at any income $x \neq y$ impacts the wage of agents y via Γ_{yx}, through an infinite sequence of general equilibrium effects induced by the skill complementarities in production. Finally, multiplying agent y's total wage change $\hat{\Omega}_y^E$ by $(1 - T'(y))\, y$ yields the welfare impact of the disruption.

The left hand side of (28) is the compensating variation of agent y resulting from the tax reform $\hat{T}$. The first term, $-\hat{T}(y)$, is the same as in partial equilibrium (see equation (18)): it reflects the fact that a tax increase (resp., decrease) lowers (resp., raises) the agent's welfare. In general equilibrium, however, the marginal tax rate also affects the agent's utility. First, a marginal tax rate increase $\hat{T}'(y) > 0$ lowers his labor supply by ε_y^r, and hence raises his wage by α_y, leading to the second term $\left(\alpha_y \varepsilon_y^r y\right) \hat{T}'(y)$ on the left hand side. Note that an increase in the marginal tax rate, keeping the average tax rate constant, raises the agent's welfare, because it makes him work less and earn a higher wage. Second, a marginal tax rate change $\hat{T}'(x)$ at any other income x affects the labor supply of agents x by ε_x^r and hence the wage of agents y by $\Gamma_{yx}\varepsilon_x^r$, leading to the integral term in (28).

Therefore, equation (28) imposes that the welfare impact of the tax reform $\hat{T}$ cancels out that of the wage disruption $\hat{\boldsymbol{w}}^E$, so that each agent's wefare remains unchanged.

The next proposition gives the closed-form solution to the functional equation (28), thus providing the complete analytical characterization of the compensating tax reform in response to any wage disruption in general equillibrium. This is the main result of the paper.

Proposition 2. *Consider an exogenous disruption $\hat{\boldsymbol{w}}^E = \{\hat{w}_i^E\}_{i\in[0,1]}$ of the wage distribution $\boldsymbol{w}$, and let $\|\hat{\boldsymbol{w}}^E\| \to 0$. The welfare compensating tax reform is given in closed-form by:*

$$\hat{T}(y) = \int_y^{\bar{y}} \mathcal{E}(x,y)\left[(1-T'(x))\,x\,\hat{\Omega}_x^E + \mathcal{C}(x)\right]dx, \tag{30}$$

where the "modified wage disruption variable" $\hat{\Omega}_x^E$ is defined by (29), the "progressivity" variable $\mathcal{E}(x,y)$ is defined by

$$\mathcal{E}(x,y) \equiv \frac{1}{\alpha_x \varepsilon_x^r x}\exp\left[\int_x^y \frac{1}{\alpha_s \varepsilon_s^r s}ds\right] \tag{31}$$

and the "compensation-of-compensation" variable $\mathcal{C}(x)$ is defined by

$$\mathcal{C}(x) \equiv \int_{\underline{y}}^{\bar{y}}\left[\sum_{n=0}^{\infty}\Lambda_{xs}^{(n)}\right](1-T'(s))\,s\,\hat{\Omega}_s^E ds \tag{32}$$

where for all $n \geq 0$, $\Lambda_{xs}^{(n)}$ are given by

$$\frac{\Lambda_{xs}^{(0)}}{1-\alpha_x\varepsilon_x^w} = \frac{(1-T'(x))\,x}{(1-T'(s))\,s}\frac{\Gamma_{xs}}{\alpha_s} - \int_{\underline{y}}^{s}\mathcal{E}(s,u)\frac{(1-T'(x))\,x}{(1-T'(u))\,u}\frac{\Gamma_{xu}}{\alpha_u}du, \quad \Lambda_{xs}^{(n)} = \int_{\underline{y}}^{\bar{y}}\Lambda_{xu}^{(n-1)}\Lambda_{us}^{(0)}du. \tag{33}$$

The changes in wages and labor supplies following the wage disruption and the tax reform are given respectively by (22) and (25). Finally, the fiscal impact of the dis-

ruption and the tax reform is given by

$$\mathcal{R}(\hat{\boldsymbol{w}}^E) = \int_{\underline{y}}^{\bar{y}} \frac{\hat{T}(x)}{1-T'(x)} f_y(x)\,dx + \int_{\underline{y}}^{\bar{y}} \frac{T'(x)}{1-T'(x)} \frac{\alpha_x^{-1}}{1-\alpha_x \varepsilon_x^w} \cdots \times \left[(1-T'(x))\,x\,\hat{\Omega}_x^E + \mathcal{C}(x) - \hat{T}(x)\right] f_y(x)\,dx, \tag{34}$$

with $\hat{T}$ given by (30).

Proof. See Appendix A. The proof consists of two parts. Equation (28) is an integro-differential algebraic equation.[19] That is, it is a functional equation which features both the derivative and an integral of the unknown function $\hat{T}$. The difficult part of the analysis is to use the methods of Vainberg [1964] and Shishkin [2007] and consists of formally representing and solve equation (28) as a non-homogeneous first-order ordinary differential equation. The integral part of the original IDE is contained in the nonhomogeneous part of this equation, so that the solution to the first-order ODE is obtained in terms of an auxiliary function, which we can characterize as the solution to a Fredholm integral equation.[20] We can then directly use the standard theory of integral equations (see Zemyan [2012]) to express this auxiliary function in closed-form as a Neumann series (33). We finally provide a condition ensuring the convergence of this series and show that it is satisfied in several cases of interest.[21] □

We discuss the interpretation of our main formula (30) in the next section. The formula for the fiscal surplus (34) generalizes the traditional Kaldor-Hicks criterion and provides a simple test to determine whether economic shocks or policies are beneficial, in the sense that offseting its associated individual welfare gains and losses using only distortionary tax instruments is budget-feasible.

[19] The analysis of this section from the system of equations (22)-(23) to the main result of (30) echoes that of Lamour, März, and Tischendorf [2013] (p. xxi) who argue that the IDAEs are a special case of abstract differential-algebraic equations (ADAE), so that the methods of analysis of the DAEs are applicable.

[20] Note that, just like a standard differential equations, equation (28) has a multiplicity of solutions, indexed by a constant c_0 (or an initial condition $\hat{T}(\bar{y})$). These solutions all satisfy the individual first-order conditions and keep each agent's welfare constant. This is true even in the absence of an initial wage disruption: the government is able reform the tax code in such a way that everyone's utility remains the same. All of these tax reforms have the same effect on government revenue if the tax code is initially Pareto efficient. We pick the most natural solution, that is the simplest to write formally, which satisfies $\hat{T}(\bar{y}) = 0$.

[21] See also Pachpatte [1986] for conditions ensuring the existence and uniqueness of some classes of mixed Volterra-Fredholm type integral equations.

3.2 Analysis of the compensating tax reform (30)

The compensating tax reform (30) features three important departures from the partial equilibrium compensation (18): (i) the modified wage disruption $\hat{\Omega}^E$ that replaces the exogenous disruption $\hat{w}^E$; (ii) the progressivity term (31); (iii) the compensation-of-compensation term (32). We already discussed the intuition underlying the first novel effect in the context of Lemma 2. In order to understand the second and third novel effects ((31) and (32)), we analyze in more detail two special cases of our general environment.

Understanding the progressivity term (31). In this paragraph, we assume that there is infinite substitutability between skills in production (as in Section 2), but we depart from the partial equilibrium benchmark by letting the marginal product of labor of each type i be decreasing. This reflects, for example, the downward-sloping demand curve for labor when there is a fixed factor of production, say land or capital, for each type.[22] That is, the production function writes

$$\mathscr{F}(\{L_i\}_{i\in[0,1]}) = \int_0^1 \mathcal{F}_i\left(L_i\right) di, \tag{35}$$

where for each $i \in [0,1]$, $\mathcal{F}_i$ is a function of the variable L_i only and satisfies $\mathcal{F}_i' > 0$, $\mathcal{F}_i'' < 0$. The wage of type i is equal to $w_i = \mathcal{F}_i'\left(L_i\right)$, the own-wage elasticities satisfy $\alpha_i > 0$ for all i, and the cross-wage elasticities γ_{ij} and Γ_{ij} are equal to zero for all i, j. The integral term in the functional equation (28) is then equal to zero, and the functional equation that defines the compensating tax reform becomes an ordinary differential equation that can be easily solved. We obtain:

Corollary 1. *Suppose that the production function is given by (35), so that the marginal product of labor of each skill is decreasing and different skills are perfect substitutes. Suppose moreover that only the wage of agents with skill i^* (and corresponding income $y^* \equiv y_{i^*}$) is disrupted by $\hat{w}^E_{i^*}$.[23] The compensating tax reform (30)*

[22] Note that this production function does not have constant returns to scale. It is not difficult to extend our theory to this case, but it would require introducing profits and the potentially unequal ownership of firms. We sidestep these techical complications by assuming that the government taxes profits 100% and redistributes the proceeds to a public good that does not enter the agents' utility.

[23] The compensating tax reform to a disruption that affects the whole wage distribution $\boldsymbol{w}$ is then given by the sum (integral) of these elementary compensations of each skill i^*. See Appendix A for details.

is then given by

$$\hat{T}^{DMP}(y) = \left[\left(1 - T'(y^*)\right) y^* \, \hat{\Omega}^E_{y^*}\right] \times \mathcal{E}(y^*, y) \, \mathbb{I}_{\{y \le y^*\}}, \tag{36}$$

Suppose in addition that $\alpha_i \equiv \alpha$ and $\varepsilon^r_i \equiv \varepsilon^r$ are constant with $\frac{1}{\alpha\varepsilon^r} \geq 1$.[24] *If the wage of agents i^* is positively (resp., negatively) disrupted, then the compensating tax reform $\hat{T}^{DMP}$ is progressive (resp., regressive) on the interval $y \in (\underline{y}, y^*)$.*[25]

Proof. See Appendix A. □

To interpret formula (36), consider a wage disruption that adversely affects agents with skill i^*, i.e., $\hat{w}^E_{i^*} < 0$ and $\hat{w}^E_i = 0$ for all $i \neq i^*$.[26] Suppose for simplicity that $\alpha_i \equiv \alpha$ and $\varepsilon^r_i \equiv \varepsilon^r$ are constant. Recall that in partial equilibrium, the welfare compensating tax reform satisfies

$$\frac{\hat{T}(y)}{y} = \left(1 - T'(y)\right) \hat{\Omega}^E_y,$$

where $\hat{\Omega}^E_y \equiv \hat{w}^E_y / w_y$. That is, the change in the average tax rate must exactly compensate the exogenous wage disruption (weighted by the retention rate). In particular, the partial equilibrium compensation $\hat{T}(y)$ is equal to zero for all incomes $y < y^*$ that are not disrupted. When the marginal product of labor is decreasing, instead, the functional equation (28) defining the welfare compensating tax reform reads

$$\frac{\hat{T}(y)}{y} = \left(1 - T'(y)\right) \hat{\Omega}^E_y + (\alpha\varepsilon^r) \, \hat{T}'(y),$$

where $\hat{\Omega}^E_y \equiv (1 - \alpha\varepsilon^w) \, \hat{w}^E_y / w_y$. That is, the change in the average tax rate must now compensate both the modified wage disruption and, in addition, the wage correction generated endogenously by the change in the marginal tax rates – recall that a reform of the marginal tax rate by $\hat{T}'(y)$ impacts the labor supply of agents y by $\varepsilon^r \hat{T}'(y)$, and hence their wage by $\alpha\varepsilon^r \hat{T}'(y)$. Thus, for agents with income $y \in [\underline{y}, y^*)$, who are not initially disrupted, the compensating tax reform must ensure that the

[24] This is indeed the case if $\mathcal{F}_i \propto L_i^{1-\alpha}$ for all i, the disutility of labor is isoelastic, and the baseline tax schedule has a constant rate of progressivity, i.e. if $y - T(y) \propto y^{1-p}$. The restriction $\frac{1}{\alpha\varepsilon^r} > 1$ is unimportant: the results are reversed in the case $\frac{1}{\alpha\varepsilon^r} < 1$.

[25] Recall that we define the local rate of progressivity of the compensating tax reform $p(y)$ as the elasticity of the retention rate $1 - \hat{T}'(y)$ with respect to income.

[26] Formally, the disruption $\hat{\boldsymbol{w}}^E$ is a Dirac delta function. See Appendix A for details.

relationship $\frac{\hat{T}(y)}{y} = (\alpha\varepsilon^r)\,\hat{T}'(y)$ between the average tax rates and the marginal tax rates is satisfied. In other words, the key insight is that in general equilibrium, the government impacts individual welfare through both the average and the marginal tax rates: an increase in the former (resp., the latter) lowers (resp., raises) the agent's utility. Hence a welfare compensating tax reform must be such that these two forces exactly cancel out, so that an income that incurs an average tax hike must also incur a marginal tax hike. Now, this relationship between the two tax rates implies immediately

$$\hat{T}(y) \propto y^{1/(\alpha\varepsilon^r)}$$

on the interval $[\underline{y}, y^*)$. In particular, if $\alpha\varepsilon^r = 1$, the average and the marginal tax rates must coincide, so that the tax schedule $\hat{T}$ is linear. More generally, the marginal tax rates below y^* decline at the constant rate $1/(\alpha\varepsilon^r)$. Thus, in the empirically relevant case where $\frac{1}{\alpha\varepsilon^r} > 1$, the rate of progressivity $p(y) = \frac{y\hat{T}''(y)}{1-\hat{T}'(y)}$ of the tax reform satisfies $p(y) < 0$ if and only if $\hat{w}_{i^*}^E < 0$. Finally, when α_y and ε_y^r depend on income y, we obtain analogously $\hat{T}(y) \propto \mathcal{E}(y^*, y)$,[27] which explains formula (36) and shows that the marginal tax rates below y^* must be lowered in an exponentially decreasing way.

Summarizing, when the production function implies decreasing marginal product of labor but perfect substitutability between skills, only two effects are present in formula (30) – the "modified wage disruption" (29) and the "progressivity" effect (31). Compensating a positive (resp., negative) modified disruption at skill i^* requires in this case a progressive (resp., regressive) tax reform on the interval $[\underline{y}, y^*)$ of the income distribution.

Understanding the compensation-of-compensation term (33). In this paragraph, we consider a simple technology where skills are no longer infinitely substitutable in production. We assume that the production function has a constant elasticity of substitution (CES), i.e.,

$$\mathscr{F}(\{L_i\}_{i\in[0,1]}) = \left[\int_0^1 \theta_i L_i^{1-\alpha} di\right]^{\frac{1}{1-\alpha}}. \tag{37}$$

[27]It is easy to check that the exponential term $\mathcal{E}(y^*, y)$ is proportional to $y^{1/(\alpha\varepsilon^r)}$ when α and ε^r are constant.

We suppose, moreover, that the agents' disutility of labor is isoelastic, i.e., $v(l) = \frac{l^{1+1/\varepsilon}}{1+1/\varepsilon}$, and that the initial tax schedule has a constant rate of progressivity $p(y)$,[28] i.e., $T(y) = y - \frac{1-\tau}{1-p} y^{1-p}$. These assumptions imply that the elasticities ε_i^r, ε_i^w, and α_i are constant. Moreover, the cross-wage elasticities γ_{ij} are independent of i, so that an increase in the labor supply of a given skill j affects the wage of all other skills $i \neq j$ by the same amount. We show in Appendix A that this allows us to simplify the functional equation (28) and its solution (30).

Corollary 2. *Suppose that the production function is given by (37), that the disutility of labor is isoelastic, and that the rate of progressivity $p(y)$ of the initial tax schedule is constant. The compensating tax reform (30) is then given by*

$$\hat{T}^{CES}(y) = \left[(1 - T'(y^*))\, y^*\, \hat{\Omega}_{y^*}^E\right] \mathcal{E}(y^*, y)\, \mathbb{I}_{\{y \leq y^*\}} + c\,(1 - T'(y))\, y, \tag{38}$$

where $\hat{\Omega}_y^E$ is the modified wage disruption (29), $\mathcal{E}(x, y) = \frac{1}{\alpha\varepsilon^r} \frac{1}{x} \left(\frac{y}{x}\right)^{1/(\alpha\varepsilon^r)}$ is the progressivity term (31), and c is a constant given in closed form in Appendix A. The first term in the right hand side of (38) is analogous to the compensation $\hat{T}^{DMP}$ derived in (18), and the second term is a uniform (in percentage terms) shift of the marginal tax rates.[29]

Proof. See Appendix A. □

The second term in (38) is due to the cross-wage effects originating from the skill complementarities in production. Recall that the tax schedule $\hat{T}^{\text{DMP}}$ given by (18) compensates (i) the individual welfare gains and losses generated by the initial wage disruption, as well as (ii) the own-wage effects created endogenously by the compensation itself (a lower marginal tax rate at income x leads to a lower wage w_x via the elasticity $\alpha_x \varepsilon_x^r$). Now, if the government implements this tax reform, a lower marginal tax rate at income x also affects all of the other wages $\{w_s\}_{s \neq x}$ via the cross-wage elasticities Γ_{xs} defined in (26). The welfare impact of this indirect wage adjustment is given by the term $\Lambda_{xs}^{(0)}$ in (33), and needs to be itself compensated using the tax schedule. However, the marginal tax rates of this second round of compensation generate in turn further wage and welfare changes for all of the agents $u \in [\underline{y}, \bar{y}]$. These

[28] See, e.g., Bénabou [2002] and Heathcote, Storesletten, and Violante [2016], who argue that this tax schedule closely approximates the U.S. tax code.

[29] This corresponds to a change in the parameter τ of the baseline tax schedule $T(y) = y - \frac{1-\tau}{1-p} y^{1-p}$.

again must be compensated (third round of "compensating the compensation"), and so on. This leads to an a priori complex sequence of compensations. We showed, however, that we can solve generally this fixed point problem in closed form by defining inductively the sequence of variables $\Lambda_{xs}^{(n)}$ for $n \geq 1$ (equation (33)), where each $\Lambda_{xs}^{(n)}$ captures one round of iterated compensation.

Remarkably, if the production function is CES (as in (37)), we show that each round of iterated compensation is a constant fraction of the previous one. This drastically simplifies the second integral in the formula (30) for $\hat{T}$. Equation (38) shows that in this case, compensating the welfare gains and losses resulting from the skill complementarities in production simply requires a uniform shift of the compensating tax reform $\hat{T}^{\mathrm{DMP}}$ obtained in the absence of cross-wage effects. In particular, in response to a disruption affecting adversely agents with income y^*, the tax function $\hat{T}^{\mathrm{CES}}$ must feature the same exponentially decreasing marginal tax rates as described in the previous section. In the empirically relevant case where $\frac{1}{\alpha\varepsilon^r} > 1$, the compensating reform is progressive (resp., regressive) for incomes lower than that of the agents that are positively (resp., negatively) disrupted.

Summarizing, in the case where there is a constant elasticity of substitution between skills in production in addition to a decreasing marginal product of labor, there is an additional effect relative to the previous paragraph ("compensation of compensation") that requires a particularly simple – linear – adjustment in the compensating tax reform.

3.3 Graphical representation

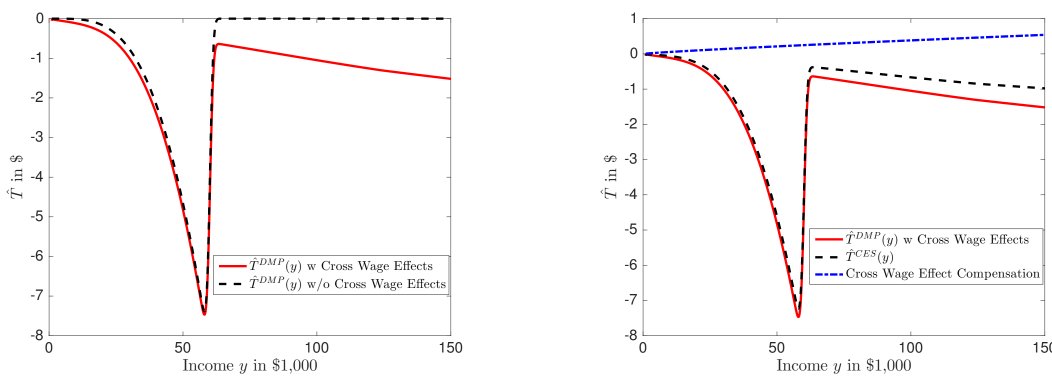
The calibration is the same as in Section 2.2. We assume in addition that the production function is CES with $\sigma = 0.6$. We consider the same normally distributed adverse wage disruptions as in Section 2.2 (see Figure 1), implying a \$100 pre-tax income loss at $y^* = \$20,000$ and $y^* = \$60,000$.

In Figure 2 shows the general equilibrium compensating tax reform, and decomposes it into its two key elements: the progressivity effect (31), and the compensation-of-compensation effect (32). The progressivity effect is represented by the bold red curve. As shown by Corollary 1,[30] the compensation is exponentially decreasing up to

[30] Formula (38) defines the compensating tax reform in response to a Dirac wage disruption at income y^*. In our simulations, the disruptions we consider are normally distributed around y^* and are therefore only (close) approximations to the corresponding Dirac disruptions. As we explained

the income level at which the tax reform peaks. Note that incomes larger than y^* are also compensated because of the cross-wage effects induced by the initial disruption, which is accounted in formula (30) by the modified wage disruption (29). Next, the compensation-of-compensation effect is represented by the dashed-dotted blue curve. As we shown in Corollary 2, in the case of a CES production function this term is proportional to $(1 - T'(y))y$, so that the marginal tax rates of the CRP baseline tax schedule T are uniformly (in percent) shifted upwards.[31]

Figure 2: Decomposition into the progressivity effect (left panel) and the compensation-of-compensation effect (right panel)



4 Conclusion

A classic policy question of compensating winners and losers from an economic disruption becomes quite involved when the environment features both distortionary taxes and general equilibrium. At the same time, both of these considerations are important in many applied and policy questions. We show that this problem can be formalized as a system of Differential Algebraic Equations (DAE). The DAEs are a relatively recent mathematical subject and, while providing a powerful set of tools,

above, the tax reform that compensates a general, non-Dirac, disruption is simply equal to the sum of the reforms that compensate the corresponding Dirac perturbations at each income level.

[31] The average change in tax rates is of the same order of magnitude in partial equilibrium and in general equilibrium. In fact, in the case of the production function (35) with α, ε^r constant, we have $\int \hat{T}^{\text{DMP}}(y)\,dy = \frac{1}{1+\alpha\varepsilon^r}\left(1 - T'(y^*)\right) y^* \hat{\Omega}^E_{y^*} = \frac{1-\alpha\varepsilon^w}{1+\alpha\varepsilon^r} \int \hat{T}^{\text{PE}}(y)\,dy$. However, taking into account the general equilibrium forces requires the compensation to be much more evenly spread out across the income distribution than in partial equilibrium, where only the disrupted agent faces a (larger) compensation.

have been only sporadically used in economics. The main difficulty in the analysis of such systems comes exactly from the central issue of designing compensating tax reforms. The need to ensure that the agents are brought to their pre-disruption utility level results in the algebraic constraint. This algebraic constraint is then coupled with the differential equation due to the requirement that the first order conditions of the agents remain satisfied following both the disruption and the compensating tax reform. We show that the solution to the DAE system leads to a fairly complex implicit integro-differential equation involving the average and the marginal tax rate functions. This equation has a clear economic meaning identifying that in general equilibrium with distortionary taxes both the average and marginal tax rates have to be used in conjunction in the design of the compensation. This is due to the effects of the marginal tax rate of the compensating tax function on the wage that in turn affects welfare. We derive a closed form solution to the problem and show that there are three principal economic effects determining the compensating variation: *(i)* a modified wage disruption term, *(ii)* a progressivity term; *(iii)* a compensation-of-compensation term. All of these effects are derived in a closed form and are easy to compute in practical applications. We highlight intuition behind the general results through a number of special cases, in which these effects become particularly simple and transparent.

References

Laurence Ales, Musab Kurnaz, and Christopher Sleet. Technical change, wage inequality, and taxes. *The American Economic Review*, 105(10):3061–3101, 2015a.

Laurence Ales, Musab Kurnaz, and Christopher M Sleet. Tasks, talents and taxes. *American Economic Review (forthcoming)*, 2015b.

Pol Antras, Alonso de Gortari, and Oleg Itskhoki. Globalization, inequality and welfare. *Working Paper*, 2016.

Uri M Ascher and Linda R Petzold. *Computer methods for ordinary differential equations and differential-algebraic equations*, volume 61. Siam, 1998.

R. Bénabou. Tax and education policy in a heterogenous-agent economy: What of levels of redistribution maximize growth and efficiency? *Econometrica*, 70(2): 481–517, 2002.

Jess Benhabib, Jesse Perla, and Christopher Tonetti. Reconciling models of diffusion and innovation: A theory of the productivity distribution and technology frontier. *Working Paper*, 2017.

Hermann Brunner. *Collocation methods for Volterra integral and related functional differential equations*, volume 15. Cambridge University Press, 2004.

Stephen L Campbell. Linearization of daes along trajectories. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 46(1):70–84, 1995.

Stephen L Campbell and Eberhard Griepentrog. Solvability of general differential algebraic equations. *SIAM Journal on Scientific Computing*, 16(2):257–270, 1995.

R. Chetty, J. Friedman, T. Olsen, and L. Pistaferri. Adjustment costs, firm responses, and micro vs. macro labor supply elasticities: Evidence from danish tax records. *Quarterly Journal of Economics*, 126(2):749, 2011.

P. A. Diamond. Optimal income taxation: An example with a u-shaped pattern of optimal marginal tax rates. *American Economic Review*, 88(1):83–95, 1998. ISSN 0002-8282.

M. Golosov, A. Tsyvinski, and N. Werquin. A Variational Approach to the Analysis of Tax Systems. *Working Paper*, 2014.

Ernst Hairer and Gerhard Wanner. *Solving ordinary differential equations. II, volume 14 of Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1996.

Jonathan Heathcote, Kjetil Storesletten, and Giovanni L Violante. Optimal tax progressivity: An analytical framework. *NBER Working Paper No. 19899*, 2016.

Nathaniel Hendren. The inequality deflator: Interpersonal comparisons without a social welfare function. *NBER Working Paper 20351*, 2014.

John R Hicks. The foundations of welfare economics. *The Economic Journal*, 49 (196):696–712, 1939.

John R Hicks. The valuation of the social income. *Economica*, 7(26):105–124, 1940.

Oleg Itskhoki. Optimal redistribution in an open economy. *Working Paper*, 2008.

Nicholas Kaldor. Welfare propositions of economics and interpersonal comparisons of utility. *The Economic Journal*, pages 549–552, 1939.

Peter Kunkel and Volker Mehrmann. *Differential-algebraic equations: analysis and numerical solution*. European Mathematical Society, 2006.

René Lamour, Roswitha März, and Caren Tischendorf. *Differential-algebraic equations: a projector based analysis*. Springer Science & Business Media, 2013.

Roswitha März. On linear differential-algebraic equations and linearizations. *Applied Numerical Mathematics*, 18(1-3):267–292, 1995.

Roswitha März. Notes on linearization of differential-algebraic equations and on optimization with differential-algebraic constraints. 2011.

Andreu Mas-Colell, Michael Dennis Whinston, and Jerry R Green. *Microeconomic theory*, volume 1. Oxford university press New York, 1995.

J. A. Mirrlees. An exploration in the theory of optimum income taxation. *The Review of Economic Studies*, 38(2):175–208, 1971.

BG Pachpatte. On mixed volterra-fredholm type integral equations. *Indian J. Pure Appl. Math*, 17(4):488–496, 1986.

Patrick J Rabier and Werner C Rheinboldt. A general existence and uniqueness theorem for implicit differential-algebraic equations. Technical report, DTIC Document, 1990.

Patrick J Rabier and Werner C Rheinboldt. A geometric treatment of implicit differential-algebraic equations. *Journal of Differential Equations*, 109(1):110–146, 1994.

Patrick J Rabier and Werner C Rheinboldt. *Nonholonomic motion of rigid mechanical systems from a DAE viewpoint*. SIAM, 2000.

Werner C Rheinboldt. Differential-algebraic systems as differential equations on manifolds. *Mathematics of computation*, 43(168):473–482, 1984.

Casey Rothschild and Florian Scheuer. Redistributive taxation in the roy model. *The Quarterly Journal of Economics*, 128(2):623–668, 2013.

Casey Rothschild and Florian Scheuer. A theory of income taxation under multidimensional skill heterogeneity. *NBER Working Paper No. 19822*, 2014.

Casey Rothschild and Florian Scheuer. Optimal taxation with rent-seeking. *Review of Economic Studies (forthcoming)*, 2016.

D. Sachs, A. Tsyvinski, and N. Werquin. Nonlinear tax incidence and optimal taxation in general equilibrium. *NBER Working Paper 22646*, 2016.

E. Saez. Using Elasticities to Derive Optimal Income Tax Rates. *Review of Economic Studies*, 68(1):205–229, 2001. ISSN 1467-937X.

Florian Scheuer and Iván Werning. Mirrlees meets diamond-mirrlees. *NBER Working Paper No. 22076*, 2016.

Gennady A. Shishkin. *[In Russian] Linear Integrodifferential equations of Fredholm*. Izdatelstvo Buryatskogo Universiteta, Ulan Ude, 2007.

Joseph E. Stiglitz. Self-selection and pareto efficient taxation. *Journal of Public Economics*, 17(2):213–240, 1982.

Mordukhai M. Vainberg. *[In Russian] Integro-differential equations*. Itogi Nauki i Tehniki. Serija "Teorija verojatnostej. Matematicheskaja statistika. Tehnicheskaja kibernetika", 1964.

Stephen M Zemyan. *The classical theory of integral equations: a concise treatment*. Springer Science & Business Media, 2012.

Appendix

A Proofs

We start by deriving the expressions for the functional derivatives used in the text. Further technical details about the definitions of these derivatives and about the construction of a production function with a continuum of inputs are given below.

Proof of equations (22), (23), and (24). Consider an exogenous disruption $(\mu\hat{\mathscr{F}}^E, \mu\hat{\boldsymbol{L}}^E)$ of the initial economy and a tax reform $\mu\hat{T}$, with $\mu > 0$. The corresponding wage disruption is defined by (see Definition 1)

$$\mu\hat{w}_i^E = \frac{\partial[\mathscr{F} + \mu\hat{\mathscr{F}}^E]}{\partial L_i}(\{L_j + \mu\hat{L}_j^E\}_{j\in[0,1]}) - \frac{\partial \mathscr{F}}{\partial L_i}(\{L_j\}_{j\in[0,1]}).$$

A first-order Taylor expansion of the right-hand side as $\mu \to 0$ yields

$$\hat{w}_i^E = \int_0^1 \hat{L}_j^E \frac{\partial^2 \mathscr{F}}{\partial L_i \partial L_j}(\{L_j\}_{j\in[0,1]})dj + \frac{\partial \hat{\mathscr{F}}^E}{\partial L_i}(\{L_j\}_{j\in[0,1]}).$$

Denote by $\mu\hat{w}_i$ and $\mu\hat{l}_i$ the first-order changes in $\mu \to 0$ in the wage (beyond the initial disruption) and labor supply of type i, and let $\tilde{w}_i = w_i + \mu\hat{w}_i^E + \mu\hat{w}_i$ and $\tilde{l}_i = l_i + \mu\hat{l}_i$. To a first order in $\mu \to 0$, the perturbed equilibrium is characterized by the following equations. First, the wage of each type equals its marginal product of labor,

$$\tilde{w}_i = \frac{\partial[\mathscr{F} + \mu\hat{\mathscr{F}}^E]}{\partial L_i}(\{L_j + \mu\hat{L}_j^E + \mu\hat{l}_j\}_{j\in[0,1]}). \tag{39}$$

Second, the first-order condition of each skill holds,

$$0 = [1 - T'(\tilde{w}_i\tilde{l}_i) - \mu\hat{T}'(\tilde{w}_i\tilde{l}_i)]\tilde{w}_i - v'(\tilde{l}_i). \tag{40}$$

Third, we impose that every agent's welfare is the same as in the initial equilibrium, i.e., that his compensating variation is equal to zero,

$$U_i = u[\tilde{w}_i\tilde{l}_i - T(\tilde{w}_i\tilde{l}_i) - \mu\hat{T}(\tilde{w}_i\tilde{l}_i) - v(\tilde{l}_i)]. \tag{41}$$

A first-order Taylor expansion in $\mu \to 0$ of the perturbed wage equation (39) around the initial equilibrium yields the Gateaux (and hence Frechet) derivative of the wage functional:

$$\begin{aligned}\frac{1}{\mu}(\tilde{w}_i - w_i) =& \frac{1}{\mu}\left[\frac{\partial[\mathscr{F} + \mu\hat{\mathscr{F}}^E]}{\partial L_i}(\{L_j + \mu\hat{L}_j^E + \mu\hat{l}_j\}_{j\in[0,1]}) - \frac{\partial \mathscr{F}}{\partial L_i}(\{L_j\}_{j\in[0,1]})\right] \\ =& \int_0^1 (\hat{L}_j^E + \hat{l}_j)\frac{\partial^2 \mathscr{F}(\boldsymbol{L})}{\partial L_i \partial L_j}dj + \frac{\partial \hat{\mathscr{F}}^E(\boldsymbol{L})}{\partial L_i} = \hat{w}_i^E + \int_0^1 \hat{l}_j \frac{\partial^2 \mathscr{F}(\boldsymbol{L})}{\partial L_i \partial L_j}dj.\end{aligned}$$

Therefore, using the definitions of the structural cross-wage and own-wage elasticities (9), (10) (see above for technical details), we obtain

$$\hat{w}_i = w_i \frac{\hat{l}_i}{l_i} \frac{L_i}{w_i} \frac{\partial^2 \mathscr{F}(\boldsymbol{L})}{\partial L_i^2} + \int_0^1 w_i \frac{\hat{l}_j}{l_j} \frac{L_j}{w_i} \frac{\partial^2 \mathscr{F}(\boldsymbol{L})}{\partial L_i \partial L_j} dj = -w_i \frac{\hat{l}_i}{l_i} \alpha_i + \int_0^1 w_i \frac{\hat{l}_j}{l_j} \gamma_{ij} dj.$$

This leads to equation (22).

A first-order Taylor expansion in $\mu \to 0$ of the perturbed first-order condition (40) yields

$$\begin{aligned}
0 = & \left[1 - T'(w_i l_i + \mu w_i \hat{l}_i + \mu \hat{w}_i^E l_i + \mu \hat{w}_i l_i) - \mu \hat{T}'(w_i l_i)\right] \dots \\
& \times (w_i + \mu \hat{w}_i^E + \mu \hat{w}_i) - v'\left(l_i + \mu \hat{l}_i\right) \\
= & \left[1 - T'(w_i l_i) - \mu (w_i \hat{l}_i + l_i \hat{w}_i^E + l_i \hat{w}_i) T''(w_i l_i) - \mu \hat{T}'(w_i l_i)\right] \dots \\
& \times (w_i + \mu \hat{w}_i^E + \mu \hat{w}_i) - v'(l_i) - \mu v''(l_i) \hat{l}_i \\
= & \mu (1 - T'(w_i l_i)) (\hat{w}_i^E + \hat{w}_i) - \mu w_i (w_i \hat{l}_i + l_i \hat{w}_i^E + l_i \hat{w}_i) T''(w_i l_i) \\
& - \mu w_i \hat{T}'(w_i l_i) - \mu v''(l_i) \hat{l}_i.
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{l}_i = & \frac{1 - T'(w_i l_i) - w_i l_i T''(w_i l_i)}{v''(l_i) + w_i^2 T''(w_i l_i)} \left(\hat{w}_i^E + \hat{w}_i\right) - \frac{w_i}{v''(l_i) + w_i^2 T''(w_i l_i)} \hat{T}'(w_i l_i) \\
= & l_i \frac{1 - \frac{y_i T''(y_i)}{1 - T'(y_i)}}{\frac{l_i v''(l_i)}{v'(l_i)} + \frac{y_i T''(y_i)}{1 - T'(y_i)}} \left(\frac{\hat{w}_i^E}{w_i} + \frac{\hat{w}_i}{w_i}\right) - l_i \frac{1}{\frac{l_i v''(l_i)}{v'(l_i)} + \frac{y_i T''(y_i)}{1 - T'(y_i)}} \frac{\hat{T}'(y_i)}{1 - T'(y_i)} \\
= & l_i \frac{(1 - p(y_i)) e_i}{1 + p(y_i) e_i} \left(\frac{\hat{w}_i^E}{w_i} + \frac{\hat{w}_i}{w_i}\right) - l_i \frac{e_i}{1 + p(y_i) e_i} \frac{\hat{T}'(y_i)}{1 - T'(y_i)},
\end{aligned}$$

so that, using (22)

$$\begin{aligned}
\frac{\hat{l}_i}{l_i} = & \frac{(1 - p(y_i)) e_i}{1 + p(y_i) e_i} \left(\frac{\hat{w}_i^E}{w_i} - \alpha_i \frac{\hat{l}_i}{l_i} + \int_0^1 \gamma_{ij} \frac{\hat{l}_j}{l_j} dj\right) - \frac{e_i}{1 + p(y_i) e_i} \frac{\hat{T}'(y_i)}{1 - T'(y_i)} \\
= & \frac{(1 - p(y_i)) e_i}{(1 + p(y_i) e_i)\left(1 + \frac{(1 - p(y_i)) e_i}{1 + p(y_i) e_i} \alpha_i\right)} \left(\frac{\hat{w}_i^E}{w_i} + \int_0^1 \gamma_{ij} \frac{\hat{l}_j}{l_j} dj\right) \\
& - \frac{e_i}{(1 + p(y_i) e_i)\left(1 + \frac{(1 - p(y_i)) e_i}{1 + p(y_i) e_i} \alpha_i\right)} \frac{\hat{T}'(y_i)}{1 - T'(y_i)}.
\end{aligned}$$

Using the definitions of the labor supply elasticities (12), (11) then leads to equation (24).

A first-order Taylor expansion in $\mu \to 0$ of the level set equation (41) yields

$$
\begin{aligned}
0 =& u[(w_i l_i + \mu w_i \hat{l}_i + \mu l_i \hat{w}_i^E + \mu l_i \hat{w}_i) - T(w_i l_i + \mu w_i \hat{l}_i + \mu l_i \hat{w}_i^E + \mu l_i \hat{w}_i) \\
&- \mu \hat{T}(w_i l_i) - v(l_i) - \mu v'(l_i) \hat{l}_i] - U_i \\
=& u[(w_i l_i - T(w_i l_i) - v(l_i)) + \mu (1 - T'(w_i l_i)) (w_i \hat{l}_i + l_i \hat{w}_i^E + l_i \hat{w}_i) \\
&- \mu \hat{T}(w_i l_i) - \mu v'(l_i) \hat{l}_i] - U_i \\
=& u\left[[(w_i l_i - T(w_i l_i) - v(l_i)) + \mu (1 - T'(w_i l_i)) l_i (\hat{w}_i^E + l_i \hat{w}_i) - \mu \hat{T}(w_i l_i)\right] \\
&- u[w_i l_i - T(w_i l_i) - v(l_i)] \\
=& \left[\mu (1 - T'(w_i l_i)) l_i (\hat{w}_i^E + l_i \hat{w}_i) - \mu \hat{T}(w_i l_i)\right] u'(w_i l_i - T(w_i l_i) - v(l_i)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
0 &= (1 - T'(y_i)) y_i \left(\frac{\hat{w}_i^E}{w_i} + \frac{\hat{w}_i}{w_i}\right) - \hat{T}(y_i) \\
&= (1 - T'(y_i)) y_i \left(\frac{\hat{w}_i^E}{w_i} - \alpha_i \frac{\hat{l}_i}{l_i} + \int_0^1 \gamma_{ij} \frac{\hat{l}_j}{l_j} dj\right) - \hat{T}(y_i),
\end{aligned}
$$

where the last equation uses (22). This leads to equation (23).

□

We now use equation (24) to derive the closed-form expression (25) for the labor supply adjustment in response to the disruption and tax reform.

Proof of equation (25). The labor supply adjustments $\{\hat{l}_i\}_{i \in [0,1]}$ satisfy equation (24). This is a linear Fredholm integral equation. Sachs, Tsyvinski, and Werquin [2016] show that its solution is given by (25), and provide conditions under which the resolvent series Γ_{ij} converges. This equation implies that the wage adjustments $\{\hat{w}_i\}_{i \in [0,1]}$ are given by

$$
\begin{aligned}
\frac{\hat{w}_i}{w_i} =& \frac{1 + p(y_i) e_i}{(1 - p(y_i)) e_i} \left[\frac{\hat{l}_i}{l_i} + \frac{e_i}{1 + p(y_i) e_i} \frac{\hat{T}'(y_i)}{1 - T'(y_i)}\right] - \frac{\hat{w}_i^E}{w_i} \\
=& \left\{\varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(y_i)}{1 - T'(y_i)} + \varepsilon_i^w \int_0^1 \Gamma_{ij} \left[\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} - \varepsilon_j^r \frac{\hat{T}'(y_j)}{1 - T'(y_j)}\right] dj\right\} \\
&\times \frac{1 + p(y_i) e_i}{(1 - p(y_i)) e_i} + \frac{1}{1 - p(y_i)} \frac{\hat{T}'(y_i)}{1 - T'(y_i)} - \frac{\hat{w}_i^E}{w_i} \\
=& \left[\frac{1 + p(y_i) e_i}{(1 - p(y_i)) e_i} \varepsilon_i^w - 1\right] \frac{\hat{w}_i^E}{w_i} + \left[\frac{1}{1 - p(y_i)} - \frac{1 + p(y_i) e_i}{(1 - p(y_i)) e_i} \varepsilon_i^r\right] \frac{\hat{T}'(y_i)}{1 - T'(y_i)} \\
&+ \frac{1 + p(y_i) e_i}{(1 - p(y_i)) e_i} \varepsilon_i^w \int_0^1 \Gamma_{ij} \left[\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} - \varepsilon_j^r \frac{\hat{T}'(y_j)}{1 - T'(y_j)}\right] dj,
\end{aligned}
$$

which yields

$$\frac{\hat{w}_i}{w_i} = -\alpha_i \left[\varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(y_i)}{1-T'(y_i)}\right] + (1-\alpha_i\varepsilon_i^w)\int_0^1 \Gamma_{ij}\left[\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} - \varepsilon_j^r \frac{\hat{T}'(y_j)}{1-T'(y_j)}\right] dj. \tag{42}$$

□

We now use derive equation (28), of which the compensating tax reform $\hat{T}$ is a solution.

Proof of Lemma 2. Using equation (42), we can rewrite the level set constraint (23) as

$$\begin{aligned} 0 =& \left(1-T'(y_i)\right) y_i \left(\frac{\hat{w}_i^E}{w_i} + \frac{\hat{w}_i}{w_i}\right) - \hat{T}(y_i) \\ =& \left(1-T'(y_i)\right) y_i \Bigg\{ \frac{\hat{w}_i^E}{w_i} - \alpha_i \left[\varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(y_i)}{1-T'(y_i)}\right] \\ &+ (1-\alpha_i\varepsilon_i^w)\int_0^1 \Gamma_{ij}\left[\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} - \varepsilon_j^r \frac{\hat{T}'(y_j)}{1-T'(y_j)}\right] dj \Bigg\} - \hat{T}(y_i). \end{aligned}$$

Rearranging terms leads to

$$\begin{aligned} 0 =& \left(1-T'(y_i)\right) y_i \left(1-\alpha_i\varepsilon_i^w\right) \left[\frac{\hat{w}_i^E}{w_i} + \int_0^1 \Gamma_{ij}\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} dj\right] \\ &+ \left(\alpha_i\varepsilon_i^r y_i\right) \hat{T}'(y_i) - \left(1-\alpha_i\varepsilon_i^w\right)\int_0^1 \frac{\left(1-T'(y_i)\right) y_i}{\left(1-T'(y_j)\right) y_j} \left(\Gamma_{ij}\varepsilon_j^r y_j\right) \hat{T}'(y_j)\, dj - \hat{T}(y_i). \end{aligned}$$

Changing variables from i to y leads to

$$\begin{aligned} &\hat{T}(y) - \left(\alpha_y\varepsilon_y^r y\right)\hat{T}'(y) + \left(1-\alpha_y\varepsilon_y^w\right)\int_{\underline{y}}^{\bar{y}} \frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x}\left(\Gamma_{yx}\varepsilon_x^r x\right)\hat{T}'(x)\, dx \\ =& \left(1-T'(y)\right) y \left(1-\alpha_y\varepsilon_y^w\right)\left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx}\varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right], \end{aligned}$$

where the assumption of a one-to-one map $i \mapsto y(i)$ ensures that $\alpha_{y(i)} = \alpha_i$, $\varepsilon_{y(i)}^r = \varepsilon_i^r$, $\hat{\Omega}_{y(i)}^E = \hat{\Omega}_i^E$, and $\gamma_{y(i),y(j)} = \frac{\gamma_{ij}}{y'(j)}$, $\Gamma_{y(i),y(j)} = \frac{\Gamma_{ij}}{y'(j)}$. This leads to the functional equation (28).

□

We now derive the closed-form solution to equation (28).

Proof of Proposition 2 (1/2). Following Vainberg [1964], Shishkin [2007], we rewrite the functional equation (28) as a first-order ordinary differential equation

$$-\left(\alpha_y\varepsilon_y^r y\right)\hat{T}'(y) + \hat{T}(y) = \mathcal{A}(y), \tag{43}$$

where the right hand side is an auxiliary function $\mathcal{A}$ that depends on the unknown function $\hat{T}$:

$$\begin{aligned}\mathcal{A}(y) = & \left(1 - T'(y)\right) y \left(1 - \alpha_y \varepsilon_y^w\right) \left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx} \varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right] \\ & - \int_{\underline{y}}^{\bar{y}} \frac{\left(1 - T'(y)\right) y}{\left(1 - T'(x)\right) x} \left(1 - \alpha_y \varepsilon_y^w\right) \left(\Gamma_{yx} \varepsilon_x^r x\right) \hat{T}'(x)\, dx.\end{aligned}$$

Using standard variation of the parameters techniques, we can express the general solution to this differential equation as

$$\begin{aligned}\hat{T}(y) = & \left[c_0 + \int_y^{\bar{y}} \frac{1}{\alpha_x \varepsilon_x^r x} \exp\left(-\int_{\underline{y}}^{x} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) \mathcal{A}(x)\, dx\right] \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) \\ = & c_0 \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) + \int_y^{\bar{y}} \mathcal{E}(x, y)\, \mathcal{A}(x)\, dx,\end{aligned} \tag{44}$$

where c_0 is a constant to be specified, and $\mathcal{E}(x, y)$ is defined by (31). It is easy to verify that this expression for $\hat{T}$ indeed satisfies (28).

Now, using the definition of $\mathcal{A}(y)$, in which we substitute for $\hat{T}'(x)$, we obtain that it satisfies the following equation:

$$\begin{aligned}\mathcal{A}(y) = & \left(1 - T'(y)\right) y \left(1 - \alpha_y \varepsilon_y^w\right) \left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx} \varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right] \\ & - \int_{\underline{y}}^{\bar{y}} \frac{\left(1 - T'(y)\right) y}{\left(1 - T'(x)\right) x} \left(1 - \alpha_y \varepsilon_y^w\right) \left(\Gamma_{yx} \varepsilon_x^r x\right) \frac{1}{\alpha_x \varepsilon_x^r x} \cdots \\ & \times \left[c_0 \exp\left(\int_{\underline{y}}^{x} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) + \int_x^{\bar{y}} \mathcal{E}(u, x)\, \mathcal{A}(u)\, du - \mathcal{A}(x)\right] dx.\end{aligned}$$

This functional equation can be simplified by exchanging the order of the two integrals in last term of the right hand side:

$$\begin{aligned} & \int_{x=\underline{y}}^{\bar{y}} \int_{u=x}^{\bar{y}} \frac{\left(1 - T'(y)\right) y}{\left(1 - T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} \mathcal{E}(u, x)\, \mathcal{A}(u)\, du dx \\ = & \int_{u=\underline{y}}^{\bar{y}} \int_{x=\underline{y}}^{u} \frac{\left(1 - T'(y)\right) y}{\left(1 - T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} \mathcal{E}(u, x)\, \mathcal{A}(u)\, dx du.\end{aligned}$$

We thus obtain

$$\begin{aligned}\mathcal{A}(y) =& \left(1-T'(y)\right) y \left(1-\alpha_y \varepsilon_y^w\right) \left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx} \varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right] \\ & - c_0 \left(1-\alpha_y \varepsilon_y^w\right) \int_{\underline{y}}^{\bar{y}} \frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} \exp\left(\int_{\underline{y}}^{x} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) dx \\ & + \left(1-\alpha_y \varepsilon_y^w\right) \int_{\underline{y}}^{\bar{y}} \frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} \mathcal{A}(x) dx \\ & - \left(1-\alpha_y \varepsilon_y^w\right) \int_{\underline{y}}^{\bar{y}} \left[\int_{\underline{y}}^{x} \frac{\left(1-T'(y)\right) y}{\left(1-T'(s)\right) s} \frac{\Gamma_{ys}}{\alpha_s} \mathcal{E}(x,s) ds\right] \mathcal{A}(x) dx,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\mathcal{A}(y)}{1-\alpha_y \varepsilon_y^w} =& \left(1-T'(y)\right) y \left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx} \varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right] \\ & - c_0 \int_{\underline{y}}^{\bar{y}} \frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} \exp\left(\int_{\underline{y}}^{x} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) dx \\ & + \int_{\underline{y}}^{\bar{y}} \left[\frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} - \int_{\underline{y}}^{x} \frac{\left(1-T'(y)\right) y}{\left(1-T'(s)\right) s} \frac{\Gamma_{ys}}{\alpha_s} \mathcal{E}(x,s) ds\right] \mathcal{A}(x) dx.\end{aligned} \tag{45}$$

But this is a standard linear Fredholm integral equation (see Zemyan [2012]); its solution is therefore known in closed form. Denote its kernel by

$$\Lambda^{(0)}(y,x) \equiv \left(1-\alpha_y \varepsilon_y^w\right) \left[\frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} - \int_{\underline{y}}^{x} \frac{\left(1-T'(y)\right) y}{\left(1-T'(s)\right) s} \frac{\Gamma_{ys}}{\alpha_s} \mathcal{E}(x,s) ds\right],$$

and

$$\begin{aligned}\frac{\mathcal{A}^{PE}(y)}{1-\alpha_y \varepsilon_y^w} \equiv& \left(1-T'(y)\right) y \left[\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx} \varepsilon_x^w \frac{\hat{w}_x^E}{w_x} dx\right] \\ & - c_0 \int_{\underline{y}}^{\bar{y}} \frac{\left(1-T'(y)\right) y}{\left(1-T'(x)\right) x} \frac{\Gamma_{yx}}{\alpha_x} \exp\left(\int_{\underline{y}}^{x} \frac{1}{\alpha_u \varepsilon_u^r u} du\right) dx.\end{aligned}$$

Assume that

$$\int_{[0,1]^2} \left|\Lambda^{(0)}(y_i, y_j)\right|^2 di dj < 1,$$

which ensures the convergence of the series (33). We show below that this condition is satisfied in the case where the production function is CES, the baseline tax schedule T is CRP (i.e., of the form $T(y) = y - \frac{1-\tau}{1-p} y^{1-p}$), and the disutility of labor is isoelastic.

We can then write

$$\begin{aligned}\mathcal{A}(y) &= \mathcal{A}^{PE}(y) + \int_{\underline{y}}^{\bar{y}} \Lambda^{(0)}(y,x) \mathcal{A}(x) dx \\ &= \mathcal{A}^{PE}(y) + \int_{\underline{y}}^{\bar{y}} \left\{\sum_{n=0}^{\infty} \Lambda^{(n)}(y,x)\right\} \mathcal{A}^{PE}(x) dx\end{aligned}$$

where $\sum \Lambda^{(n)}$ is the resolvent kernel defined by

$$\Lambda^{(n)}(y,x) = \int_{\underline{y}}^{\bar{y}} \Lambda^{(n-1)}(y,u)\,\Lambda^{(0)}(u,x)\,du$$

for all $n \geq 1$.

Note that there is a continuum of solutions, indexed by the constant c_0, or equivalently by the initial condition $\hat{T}(\bar{y})$ (see equation (44)). We show below that, if the baseline tax schedule is Pareto efficient, then all of these compensating reforms have the same impact on the government budget. For simplicity, we consider here the solution such that $c_0 = 0$. We obtain the following solution to the compensating tax reform problem:

$$\begin{aligned}\hat{T}(y) &= \int_{y}^{\bar{y}} \mathcal{E}(x,y)\,\mathcal{A}(x)\,dx \\ &= \int_{y}^{\bar{y}} \mathcal{E}(x,y)\left[(1-T'(x))\,x\,\hat{\Omega}_x^E + \int_{\underline{y}}^{\bar{y}}\left\{\sum_{n=0}^{\infty}\Lambda^{(n)}(x,s)\right\}(1-T'(s))\,s\,\hat{\Omega}_s^E ds\right]dx.\end{aligned} \tag{46}$$

This concludes the proof of equation (30).

□

We now derive the budget impact (fiscal surplus) of the wage disruption and its compensation.

Proof of Proposition 2 (2/2). The effect of the wage disruption and the corresponding compensating tax reform on government budget is given by

$$\begin{aligned}\mathcal{R}(\hat{\boldsymbol{w}}^E) &= \lim_{\mu\to 0}\frac{1}{\mu}\left\{\int_0^1\left[T\left(\left(w_i+\mu\hat{w}_i^E+\mu\hat{w}_i\right)\left(l_i+\mu\hat{l}_i\right)\right)+\mu\hat{T}\left(\left(w_i+\mu\hat{w}_i^E+\mu\hat{w}_i\right)\left(l_i+\mu\hat{l}_i\right)\right)\right]di-\int_0^1 T(w_il_i)\,di\right\} \\ &= \int_0^1\hat{T}(y_i)\,di+\int_0^1\left(\frac{\hat{w}_i^E}{w_i}+\frac{\hat{w}_i}{w_i}+\frac{\hat{l}_i}{l_i}\right)w_il_iT'(w_il_i)\,di \\ &= \int_0^1\hat{T}(y_i)\,di+\int_0^1 T'(y_i)\,y_i\frac{\hat{w}_i^E}{w_i}di+\int_0^1 T'(y_i)\,y_i\ldots \\ &\quad\left(-\alpha_i\left[\varepsilon_i^w\frac{\hat{w}_i^E}{w_i}-\varepsilon_i^r\frac{\hat{T}'(y_i)}{1-T'(y_i)}\right]+(1-\alpha_i\varepsilon_i^w)\int_0^1\Gamma_{ij}\left[\varepsilon_j^w\frac{\hat{w}_j^E}{w_j}-\varepsilon_j^r\frac{\hat{T}'(y_j)}{1-T'(y_j)}\right]dj\right. \\ &\quad\quad\left.+\left[\varepsilon_i^w\frac{\hat{w}_i^E}{w_i}-\varepsilon_i^r\frac{\hat{T}'(y_i)}{1-T'(y_i)}\right]+\varepsilon_i^w\int_0^1\Gamma_{ij}\left[\varepsilon_j^w\frac{\hat{w}_j^E}{w_j}-\varepsilon_j^r\frac{\hat{T}'(y_j)}{1-T'(y_j)}\right]dj\right)di \\ &= \int_{\underline{y}}^{\bar{y}}\hat{T}(y)\,f_y(y)\,dy+\int_{\underline{y}}^{\bar{y}}T'(y)\,y\left(1+(1-\alpha_y)\,\varepsilon_y^w\right)\left[\frac{\hat{w}_y^E}{w_y}+\int_{\underline{y}}^{\bar{y}}\Gamma_{yx}\varepsilon_x^w\frac{\hat{w}_x^E}{w_x}dx\right]f_y(y)\,dy \\ &\quad-\int_{\underline{y}}^{\bar{y}}T'(y)\,y\left((1-\alpha_y)\,\varepsilon_y^r\frac{\hat{T}'(y)}{1-T'(y)}+\left(1+(1-\alpha_y)\,\varepsilon_y^w\right)\int_{\underline{y}}^{\bar{y}}\Gamma_{yx}\varepsilon_x^r\frac{\hat{T}'(x)}{1-T'(x)}dx\right)f_y(y)\,dy.\end{aligned}$$

Using equation (28), which can be rewritten as

$$\int_{\underline{y}}^{\bar{y}} \Gamma_{yx}\varepsilon_x^r \frac{\hat{T}'(x)}{1-T'(x)}dx = \left(\frac{\hat{w}_y^E}{w_y} + \int_{\underline{y}}^{\bar{y}} \Gamma_{yx}\varepsilon_x^w \frac{\hat{w}_x^E}{w_x}dx\right) - \frac{\hat{T}(y)}{\left(1-\alpha_y\varepsilon_y^w\right)\left(1-T'(y)\right)y} + \frac{\left(\alpha_y\varepsilon_y^r y\right)\hat{T}'(y)}{\left(1-\alpha_y\varepsilon_y^w\right)\left(1-T'(y)\right)y},$$

to substitute in the left hand side of the previous equality, we get

$$\begin{aligned}
\mathcal{R}(\hat{\boldsymbol{w}}^E) &= \int_{\underline{y}}^{\bar{y}} \hat{T}(y) f_y(y)\, dy + \int_{\underline{y}}^{\bar{y}} \frac{1-\alpha_y\varepsilon_y^w+\varepsilon_y^w}{1-\alpha_y\varepsilon_y^w} \frac{T'(y)}{1-T'(y)}\hat{T}(y) f_y(y)\, dy \\
&\quad - \int_{\underline{y}}^{\bar{y}} T'(y)\, y\left(1-\alpha_y\right)\varepsilon_y^r \frac{\hat{T}'(y)}{1-T'(y)} f_y(y)\, dy - \int_{\underline{y}}^{\bar{y}} \frac{1-\alpha_y\varepsilon_y^w+\varepsilon_y^w}{1-\alpha_y\varepsilon_y^w}\frac{T'(y)}{1-T'(y)}\hat{T}'(y)\left(\alpha_y\varepsilon_y^r y\right) f_y(y)\, dy \\
&= \int_{\underline{y}}^{\bar{y}} \hat{T}(y) f_y(y)\, dy + \int_{\underline{y}}^{\bar{y}} \left(1+\frac{\varepsilon_y^w}{1-\alpha_y\varepsilon_y^w}\right)\frac{T'(y)}{1-T'(y)}\hat{T}(y) f_y(y)\, dy \\
&\quad - \int_{\underline{y}}^{\bar{y}} \frac{\varepsilon_y^r}{1-\alpha_y\varepsilon_y^w}\frac{T'(y)}{1-T'(y)} y\hat{T}'(y) f_y(y)\, dy \\
&= \int_{\underline{y}}^{\bar{y}} \hat{T}(y) f_y(y)\, dy + \int_{\underline{y}}^{\bar{y}} \left(1+\frac{\varepsilon_y^w}{1-\alpha_y\varepsilon_y^w}\right)\frac{T'(y)}{1-T'(y)}\hat{T}(y) f_y(y)\, dy \\
&\quad - \int_{\underline{y}}^{\bar{y}} \frac{1}{\alpha_y}\frac{1}{1-\alpha_y\varepsilon_y^w}\frac{T'(y)}{1-T'(y)}\left[\hat{T}(y)-\mathcal{A}(y)\right] f_y(y)\, dy \\
&= \int_{\underline{y}}^{\bar{y}} \hat{T}(y) f_y(y)\, dy + \int_{\underline{y}}^{\bar{y}} \left(1-\frac{1}{\alpha_y}\right)\frac{T'(y)}{1-T'(y)}\hat{T}(y) f_y(y)\, dy \\
&\quad + \int_{\underline{y}}^{\bar{y}} \frac{1}{\alpha_y}\frac{T'(y)}{1-T'(y)}\frac{\mathcal{A}(y)}{1-\alpha_y\varepsilon_y^w} f_y(y)\, dy.
\end{aligned}$$

where the third equality uses the transformed version (43) of the functional equation (28) defining $\hat{T}$. Substituting for the solution (46) for $\hat{T}$, we thus obtain

$$\begin{aligned}
\mathcal{R}(\hat{\boldsymbol{w}}^E) &= \int_{\underline{y}}^{\bar{y}} \left[\int_{y}^{\bar{y}} \mathcal{E}(x,y)\,\mathcal{A}(x)\, dx\right] f_y(y)\, dy \\
&\quad + \int_{\underline{y}}^{\bar{y}} \frac{T'(y)}{1-T'(y)}\left[\frac{1}{\alpha_y}\frac{\mathcal{A}(y)}{1-\alpha_y\varepsilon_y^w} + \left(1-\frac{1}{\alpha_y}\right)\int_{y}^{\bar{y}} \mathcal{E}(x,y)\,\mathcal{A}(x)\, dx\right] f_y(y)\, dy
\end{aligned}$$

where

$$\mathcal{A}(y) = \left(1-T'(y)\right) y\hat{\Omega}_y^E + \int_{\underline{y}}^{\bar{y}} \left\{\sum \Lambda^{(n)}(x,s)\right\}\left(1-T'(x)\right) x\hat{\Omega}_x^E dx.$$

This concludes the proof of equation (34).

□

We now assume away the cross-wage effects, so that there is perfect susbstitutability between skills in production, but let the marginal product of labor be decreasing.

Proof of Corollary 1. When the production function has the form (35), we have $\Gamma_{ij} = 0$ for all i, j. The functional equation (28) that defines the welfare compensating tax reform becomes a linear first-order ordinary differential equation:

$$-\hat{T}(y)+\left(\alpha_y \varepsilon_y^r y\right)\hat{T}'(y)=-\left(1-T'(y)\right)y\,\hat{\Omega}_y^E.$$

The general solution to this ODE is (following the same steps as in the general setting, and choosing the constant $c_0=0$):

$$\hat{T}(y)=\int_y^{\bar{y}}\mathcal{E}(x,y)\left(1-T'(x)\right)x\hat{\Omega}_x^E dx.$$

Now suppose in particular that the modified wage disruption is a Dirac delta function at income y^*, i.e., $\hat{\Omega}_y^E=\delta(y-y^*)$. More formally, let $\{\hat{\Omega}_y^{E,n}\}_{n\geq 1}$ be a sequence of functions converging to $\delta(y-y^*)$ (this construction is standard), and define the compensating tax reform $\hat{T}$ in response to $\delta(y-y^*)$ as the limit of the sequence $\hat{T}^{(n)}$ of compensating tax reforms in response to $\hat{\Omega}_y^{E,n}$. We then obtain

$$\begin{aligned}\hat{T}(y)\equiv\lim_{n\to\infty}\hat{T}^{(n)}(y)=&\lim_{n\to\infty}\int_{\underline{y}}^{\bar{y}}\mathcal{E}(x,y)\left(1-T'(x)\right)x\hat{\Omega}_x^{E,n}\mathbb{I}_{\{x\geq y\}}dx\\ =&\mathcal{E}(y^*,y)\left(1-T'(y^*)\right)y^*\hat{\Omega}_{y^*}^E\mathbb{I}_{\{y^*\geq y\}},\end{aligned}$$

where the last equality follows from the construction of the Dirac measure. This proves Corollary 1.

□

We now show that the choice of constant c_0 in the solution (44) to the functional equation (28) is innocuous: assuming that the baseline tax schedule T is Pareto optimal,[32] all the constants c_0 lead to the same impact on government revenue. For simplicity, we prove this result in the case of perfect substitutability between skills in production, i.e., the production function (35).

Lemma 3. *Suppose that the tax schedule T is Pareto optimal, and that the production function is given by (35) with profits taxed at 100%. Then all the solutions to (28) have the same impact on the government budget.*

Proof. If the production function is given by (35) and profits are taxed 100%, government revenue is given by

$$\int_0^1 T(y_i)\,di+\int_0^1\left[\mathcal{F}_i(L_i)-w_iL_i\right]di.$$

[32]If the tax schedule is Pareto suboptimal, an obvious tax reform to implement before any compensation consideration is to first make it efficient.

The change in revenue following a tax reform $\hat{T}$ is thus given by[33]

$$\begin{aligned}
&\int_0^1 \hat{T}(y_i)\,di + \int_0^1 T'(y_i)\left(\frac{\hat{w}_i}{w_i}+\frac{\hat{l}_i}{l_i}\right) y_i di + \int_0^1 \left[L_i \mathcal{F}_i'(L_i)\frac{\hat{l}_i}{l_i} - \left(\frac{\hat{w}_i}{w_i}+\frac{\hat{l}_i}{l_i}\right) w_i l_i\right] di \\
=&\int_0^1 \hat{T}(y_i)\,di + \int_0^1 (1-\alpha_i)\, T'(y_i)\left(-\varepsilon_i^r \frac{\hat{T}'(y_i)}{1-T'(y_i)}\right) y_i di + \int_0^1 \alpha_i \left(-\varepsilon_i^r \frac{\hat{T}'(y_i)}{1-T'(y_i)}\right) y_i di \\
=&\int_0^1 \left[\hat{T}(y_i) - (\alpha_i \varepsilon_i^r y_i)\,\hat{T}'(y_i)\right] di - \int_0^1 \varepsilon_i^r \frac{T'(y_i)}{1-T'(y_i)}\hat{T}'(y_i)\, y_i di \\
=&\int_0^1 (1-T'(y_i))\, y_i \hat{\Omega}_i^E di - \int_0^1 \varepsilon_i^r \frac{T'(y_i)}{1-T'(y_i)}\hat{T}'(y_i)\, y_i di,
\end{aligned}$$

where the last equality uses the functional equation (28). Now substituting for the solution

$$\hat{T}'(y) = \frac{1}{\alpha_y \varepsilon_y^r y}\left[c_0 \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_s \varepsilon_s^r s} ds\right) + \int_y^{\bar{y}} \mathcal{E}(x,y)\,(1-T'(x))\, x \hat{\Omega}_x^E dx\right] - \frac{1}{\alpha_y \varepsilon_y^r y}(1-T'(y))\, y \hat{\Omega}_y^E,$$

we obtain that the term that depends on c_0 in the government revenue effect of the tax reform is

$$-\int_{\underline{y}}^{\bar{y}} \varepsilon_y^r \frac{T'(y)}{1-T'(y)} \frac{1}{\alpha_y \varepsilon_y^r y}\left[c_0 \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_s \varepsilon_s^r s} ds\right)\right] y f_y(y)\, dy. \tag{47}$$

Now, the fact that the baseline tax schedule is Pareto optimal implies that there exists a p.d.f. $\tilde{f}_y$ and c.d.f. $\tilde{F}_y$ such that

$$\frac{T'(y)}{1-T'(y)} = \frac{1}{\varepsilon_y^r}\frac{\tilde{F}_y(y) - F_y(y)}{y f_y(y)} + \alpha_y \left(\frac{\tilde{f}_y(y)}{f_y(y)} - 1\right).$$

This result follows from the analysis of Sachs, Tsyvinski, and Werquin [2016].[34]

Thus the term multiplying $-c_0$ in the expression (47) is equal to

$$\begin{aligned}
&\int_{\underline{y}}^{\bar{y}} \varepsilon_y^r \left\{\frac{1}{\varepsilon_y^r}\frac{\tilde{F}_y(y) - F_y(y)}{y f_y(y)} + \alpha_y\left(\frac{\tilde{f}_y(y)}{f_y(y)} - 1\right)\right\} \frac{1}{\alpha_y \varepsilon_y^r y} \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_s \varepsilon_s^r s} ds\right) y f_y(y)\, dy \\
=&\int_{\underline{y}}^{\bar{y}} \left(\tilde{F}_y(y) - F_y(y)\right) \frac{1}{\alpha_y \varepsilon_y^r y} \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_s \varepsilon_s^r s} ds\right) dy + \int_{\underline{y}}^{\bar{y}} \left(\tilde{f}_y(y) - f_y(y)\right) \exp\left(\int_{\underline{y}}^{y} \frac{1}{\alpha_s \varepsilon_s^r s} ds\right) dy \\
=&0,
\end{aligned}$$

[33] Note that the change in revenue following the wage disruption is of course independent of the choice of the constant c_0 in the compensating tax reform (44), so that we can ignore the corresponding terms.

[34] In this paper, we derived the optimal marginal tax rates in the case of a production function with constant returns to scale. However, our analysis of Section B.4.1. easily carries through straightforwardly to the production function considered here. In particular, the envelope condition that guarantees incentive compatibility is unchanged.

where the last equality follows from a simple integration by parts. This concludes the proof of the Lemma.

$\square$

Next we assume that the production function is CES and that the initial tax schedule has a constant rate of progressivity. We derive formula (38) by two different methods. The first follows the same steps as the general derivation of formula (30), the second proceeds by showing that $\hat{T}^{CES}$ is the solution to a simple second-order ODE. We start by stating several useful properties of this environment.

Formulas for the CES technology. All of the following properties are derived formally in Sachs, Tsyvinski, and Werquin [2016]. The CES production function (37) implies that wages are equal to

$$w_i = \theta_i L_i^{\alpha} \left[\int_0^1 \theta_j L_j^{1-\alpha} dj \right]^{\frac{\alpha}{1-\alpha}}.$$

The own-wage elasticities (10) are equal to $\alpha_i = \alpha$ for all i. The cross-wage elasticities (9) are equal to

$$\gamma_{ij} = \alpha \frac{\theta_j L_j^{1-\alpha}}{\int_0^1 \theta_k L_k^{1-\alpha} dk}$$

for all $i, j \in [0,1]$, or after a change of variables,

$$\gamma_{xy} = \alpha \frac{y f_y(y)}{\int_{\underline{y}}^{\bar{y}} s f_y(s) ds} \tag{48}$$

for all $x, y \in [\underline{y}, \bar{y}]$. Moreover, Lemma 1 simplifies in this case:

$$\frac{\hat{l}_i}{l_i} = \left[\varepsilon_i^w \frac{\hat{w}_i^E}{w_i} - \varepsilon_i^r \frac{\hat{T}'(y_i)}{1-T'(y_i)} \right] + \int_0^1 \varepsilon_i^w \frac{\gamma_{ij}}{1 - \int_0^1 \gamma_{kk} \varepsilon_k^w dk} \left[\varepsilon_j^w \frac{\hat{w}_j^E}{w_j} - \varepsilon_j^r \frac{\hat{T}'(y_j)}{1-T'(y_j)} \right] dj,$$

so that $\Gamma_{ij} = \frac{\gamma_{ij}}{1-\int_0^1 \gamma_{kk} \varepsilon_k^w dk}$.

Suppose moreover the disutility of labor is isoelastic (with parameter e) and that the tax schedule is CRP, i.e., it has the functional form

$$1 - T'(y) = (1-\tau) y^{-p}. \tag{49}$$

All of the labor supply elasticities are then constant:

$$\varepsilon^r = \frac{e}{1 + pe + (1-p)\alpha e}$$
$$\varepsilon^w = \frac{(1-p)e}{1 + pe + (1-p)\alpha e},$$

and we get

$$\Gamma_{ij} = \frac{\gamma_{ij}}{1 - \varepsilon^w \int_0^1 \gamma_{kk} dk} = \frac{\gamma_{ij}}{1 - \alpha \varepsilon^w}.$$

Finally, the progressivity term (31) is equal to

$$\mathcal{E}(x,y) = \frac{1}{\alpha\varepsilon^r x}\exp\left(\frac{1}{\alpha\varepsilon^r}\int_x^y \frac{du}{u}\right) = \frac{1}{\alpha\varepsilon^r x}\left(\frac{y}{x}\right)^{\frac{1}{\alpha\varepsilon^r}}. \tag{50}$$

Note that the key simplification allowed by the CES production function is that the cross-wage elasticities γ_{ij}, Γ_{ij} (resp., γ_{xy}, Γ_{xy}) depend only on j (resp., y).

$\square$

We now give the proof of formula (38).

Proof of Corollary 2. If the production function is CES and the initial tax schedule is CRP, all the elasticities are constant and, moreover, the cross-wage elasticities γ_{xy}, Γ_{xy} do not depend on x. Following the same steps as in the proof of Proposition 2 above, we obtain that the kernel $\Lambda^{(0)}(y,x)$ of the Fredholm integral equation (45) satisfied by the auxiliary function $\mathcal{A}$ is separable, i.e., of the form

$$\Lambda^{(0)}(y,x) = \Lambda_1^{(0)}(y) \times \Lambda_2^{(0)}(x).$$

Specifically, we have

$$\begin{aligned}
\Lambda_1^{(0)}(y) &\equiv \left(1-\alpha_y\varepsilon_y^w\right)\left(1-T'(y)\right)y \\
\Lambda_2^{(0)}(x) &\equiv \frac{1}{\left(1-T'(x)\right)x}\frac{\Gamma_{yx}}{\alpha_x} - \int_{\underline{y}}^{x}\frac{1}{\left(1-T'(s)\right)s}\frac{\Gamma_{ys}}{\alpha_s}\mathcal{E}(x,s)\,ds.
\end{aligned}$$

The solution to (45) is then straightforward to obtain, and moreover, the convergence conditions assumed in the proof of Proposition 2 are satisfied in this case. We can indeed write

$$\mathcal{A}(y) = \mathcal{A}^{PE}(y) + \Lambda_1^{(0)}(y)\int_{\underline{y}}^{\bar{y}}\Lambda_2^{(0)}(x)\,\mathcal{A}(x)\,dx$$

which implies

$$\begin{aligned}
\int_{\underline{y}}^{\bar{y}}\Lambda_2^{(0)}(x)\,\mathcal{A}(x)\,dx &= \int_{\underline{y}}^{\bar{y}}\Lambda_2^{(0)}(x)\,\mathcal{A}^{PE}(x)\,dx \\
&\quad + \left(\int_{\underline{y}}^{\bar{y}}\Lambda_1^{(0)}(x)\,\Lambda_2^{(0)}(x)\,dx\right)\left(\int_{\underline{y}}^{\bar{y}}\Lambda_2^{(0)}(u)\,\mathcal{A}(u)\,du\right) \\
&= \frac{\int_{\underline{y}}^{\bar{y}}\Lambda_2^{(0)}(x)\,\mathcal{A}^{PE}(x)\,dx}{1-\int_{\underline{y}}^{\bar{y}}\Lambda_1^{(0)}(x)\,\Lambda_2^{(0)}(x)\,dx}.
\end{aligned}$$

Therefore we get

$$
\begin{aligned}
\mathcal{A}(y) =& \mathcal{A}^{PE}(y) + \Lambda_1^{(0)}(y) \frac{\int_{\underline{y}}^{\bar{y}} \Lambda_2^{(0)}(x) \mathcal{A}^{PE}(x) \, dx}{1 - \int_{\underline{y}}^{\bar{y}} \Lambda_1^{(0)}(x) \Lambda_2^{(0)}(x) \, dx} \\
=& \mathcal{A}^{PE}(y) + \left(1 - \alpha_y \varepsilon_y^w\right) \frac{\int_{\underline{y}}^{\bar{y}} \left[\frac{(1-T'(y))y}{(1-T'(x))x} \frac{\Gamma_{yx}}{\alpha_x} - \int_{\underline{y}}^{x} \frac{(1-T'(y))y}{(1-T'(s))s} \frac{\Gamma_{ys}}{\alpha_s} \mathcal{E}(x,s) \, ds\right] \mathcal{A}^{PE}(x) \, dx}{1 - \int_{\underline{y}}^{\bar{y}} \left(1 - \alpha_x \varepsilon_x^w\right) \left[\frac{\Gamma_{yx}}{\alpha_x} - \int_{\underline{y}}^{x} \frac{(1-T'(x))x}{(1-T'(s))s} \frac{\Gamma_{ys}}{\alpha_s} \mathcal{E}(x,s) \, ds\right] dx}. \\
=& \mathcal{A}^{PE}(y) + \frac{\int_{\underline{y}}^{\bar{y}} \left[\frac{(1-T'(y))y}{(1-T'(x))x} \gamma_{yx} - \int_{\underline{y}}^{x} \frac{(1-T'(y))y}{(1-T'(s))s} \gamma_{ys} \mathcal{E}(x,s) \, ds\right] \mathcal{A}^{PE}(x) \, dx}{\alpha - \int_{\underline{y}}^{\bar{y}} \left[\gamma_{yx} - \int_{\underline{y}}^{x} \frac{(1-T'(x))x}{(1-T'(s))s} \gamma_{ys} \mathcal{E}(x,s) \, ds\right] dx},
\end{aligned}
$$

where the last equality uses the properties of the CES environment derived in the previous proof (in particular, $\Gamma_{xy} = \frac{\gamma_{xy}}{1-\alpha\varepsilon^w}$).

Now, using the definition of the CRP tax schedule (49), the expression (48) for the cross-wage elasticities, and the formula (50) for the progressivity term, we get, letting $\mathbb{E}y$ denote the average income in the economy:

$$
\begin{aligned}
\mathcal{A}(y) =& \mathcal{A}^{PE}(y) + \frac{\int_{\underline{y}}^{\bar{y}} \left[\left(\frac{y}{x}\right)^{1-p} \alpha \frac{x f_y(x)}{\mathbb{E}y} - \int_{\underline{y}}^{x} \left(\frac{y}{s}\right)^{1-p} \alpha \frac{s f_y(s)}{\mathbb{E}y} \frac{1}{\alpha\varepsilon^r x} \left(\frac{s}{x}\right)^{\frac{1}{\alpha\varepsilon^r}} ds\right] \mathcal{A}^{PE}(x) \, dx}{\alpha - \int_{\underline{y}}^{\bar{y}} \left[\alpha \frac{x f_y(x)}{\mathbb{E}y} - \int_{\underline{y}}^{x} \left(\frac{x}{s}\right)^{1-p} \alpha \frac{s f_y(s)}{\mathbb{E}y} \frac{1}{\alpha\varepsilon^r x} \left(\frac{s}{x}\right)^{\frac{1}{\alpha\varepsilon^r}} ds\right] dx} \\
=& \mathcal{A}^{PE}(y) + y^{1-p} \frac{\int_{\underline{y}}^{\bar{y}} \left[x^p f_y(x) - \frac{1}{\alpha\varepsilon^r} x^{-1-\frac{1}{\alpha\varepsilon^r}} \int_{\underline{y}}^{x} s^{p+\frac{1}{\alpha\varepsilon^r}} f_y(s) \, ds\right] \mathcal{A}^{PE}(x) \, dx}{\mathbb{E}y - \int_{\underline{y}}^{\bar{y}} \left[x f_y(x) - \frac{1}{\alpha\varepsilon^r} x^{-p-\frac{1}{\alpha\varepsilon^r}} \int_{\underline{y}}^{x} s^{p+\frac{1}{\alpha\varepsilon^r}} f_y(s) \, ds\right] dx} \\
=& \mathcal{A}^{PE}(y) + y^{1-p} \frac{\int_{\underline{y}}^{\bar{y}} x^p \mathcal{A}^{PE}(x) f_y(x) \, dx - \frac{1}{\alpha\varepsilon^r} \int_{\underline{y}}^{\bar{y}} s^{p+\frac{1}{\alpha\varepsilon^r}} \left[\int_{s}^{\bar{y}} x^{-1-\frac{1}{\alpha\varepsilon^r}} \mathcal{A}^{PE}(x) \, dx\right] f_y(s) \, ds}{\frac{1}{\alpha\varepsilon^r} \int_{\underline{y}}^{\bar{y}} s^{p+\frac{1}{\alpha\varepsilon^r}} \left[\int_{s}^{\bar{y}} x^{-p-\frac{1}{\alpha\varepsilon^r}} \, dx\right] f_y(s) \, ds} \\
=& \mathcal{A}^{PE}(y) + y^{1-p} \frac{\mathbb{E}\left[y^p \mathcal{A}^{PE}(y)\right] - \frac{1}{\alpha\varepsilon^r} \mathbb{E}\left[y^{p+\frac{1}{\alpha\varepsilon^r}} \left(\int_{y}^{\bar{y}} x^{-1-\frac{1}{\alpha\varepsilon^r}} \mathcal{A}^{PE}(x) \, dx\right)\right]}{\frac{1}{\alpha\varepsilon^r} \frac{1}{1-p-\frac{1}{\alpha\varepsilon^r}} \mathbb{E}\left[y\left(\left(\frac{\bar{y}}{y}\right)^{1-p-\frac{1}{\alpha\varepsilon^r}} - 1\right)\right]},
\end{aligned}
$$

where the third equality exchanges the order of the integrals in the numerator and the denominator. But we have

$$
\begin{aligned}
\mathcal{A}^{PE}(y) &= (1-\tau) y^{1-p} \hat{\Omega}_y^E - c_0 \int_{\underline{y}}^{\bar{y}} \frac{y^{1-p}}{x^{1-p}} \frac{\gamma_{yx}}{\alpha} \left(\frac{x}{\underline{y}}\right)^{\frac{1}{\alpha\varepsilon^r}} dx \\
&= (1-\tau) y^{1-p} \hat{\Omega}_y^E - \left(\frac{c_0}{\mathbb{E}y} \underline{y}^{-\frac{1}{\alpha\varepsilon^r}} \int_{\underline{y}}^{\bar{y}} x^{p+\frac{1}{\alpha\varepsilon^r}} f_y(x) \, dx\right) y^{1-p}.
\end{aligned}
$$

We thus get (replacing the constant $\frac{c_0}{\mathbb{E}y}\underline{y}^{-\frac{1}{\alpha\varepsilon^r}}\mathbb{E}\left[y^{p+\frac{1}{\alpha\varepsilon^r}}\right]$ with simply c_0)

$$\begin{aligned}\mathcal{A}(y) =& (1-\tau)\,y^{1-p}\hat{\Omega}^E_y - c_0 y^{1-p} \\ &+ y^{1-p}\frac{\mathbb{E}\left[(1-\tau)\,y\hat{\Omega}^E_y - c_0 y\right] - \frac{1}{\alpha\varepsilon^r}\mathbb{E}\left[(1-\tau)\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}^E_x dx - \frac{c_0}{1-p-\frac{1}{\alpha\varepsilon^r}}y\left(\left(\frac{\bar{y}}{y}\right)^{1-p-\frac{1}{\alpha\varepsilon^r}}-1\right)\right]}{\frac{1}{\alpha\varepsilon^r}\frac{1}{1-p-\frac{1}{\alpha\varepsilon^r}}\mathbb{E}\left[y\left(\left(\frac{\bar{y}}{y}\right)^{1-p-\frac{1}{\alpha\varepsilon^r}}-1\right)\right]} \\ =& (1-\tau)\,y^{1-p}\hat{\Omega}^E_y + c_1 y^{1-p},\end{aligned}$$

where the constant c_1 is defined as

$$c_1 \equiv (1-\tau)\frac{\mathbb{E}\left[y\hat{\Omega}^E_y\right] - \frac{1}{\alpha\varepsilon^r}\mathbb{E}\left[\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}^E_x dx\right]}{\frac{1}{\alpha\varepsilon^r}\frac{1}{1-p-\frac{1}{\alpha\varepsilon^r}}\mathbb{E}\left[y\left(\left(\frac{\bar{y}}{y}\right)^{1-p-\frac{1}{\alpha\varepsilon^r}}-1\right)\right]} - \frac{c_0\mathbb{E}[y]}{\frac{1}{\alpha\varepsilon^r}\frac{1}{1-p-\frac{1}{\alpha\varepsilon^r}}\mathbb{E}\left[y\left(\left(\frac{\bar{y}}{y}\right)^{1-p-\frac{1}{\alpha\varepsilon^r}}-1\right)\right]}.$$

Therefore the compensating tax reform is given by

$$\begin{aligned}\hat{T}(y) =& \int_y^{\bar{y}}\frac{1}{\alpha\varepsilon^r x}\left(\frac{y}{x}\right)^{\frac{1}{\alpha\varepsilon^r}}\mathcal{A}(x)\,dx + c_0\left(\frac{y}{\bar{y}}\right)^{\frac{1}{\alpha\varepsilon^r}} \\ =& \int_y^{\bar{y}}\frac{1}{\alpha\varepsilon^r x}\left(\frac{y}{x}\right)^{\frac{1}{\alpha\varepsilon^r}}\left[(1-\tau)\,x^{1-p}\hat{\Omega}^E_x + c_1 x^{1-p}\right]dx + c_0\left(\frac{y}{\bar{y}}\right)^{\frac{1}{\alpha\varepsilon^r}} \\ =& (1-\tau)\,y^{-p}\frac{1}{\alpha\varepsilon^r}\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}^E_x dx \\ &+ c_1\frac{1}{\alpha\varepsilon^r}\frac{1}{1-p-\frac{1}{\alpha\varepsilon^r}}\left(\left(\frac{\bar{y}}{y}\right)^{1-p-\frac{1}{\alpha\varepsilon^r}}-1\right)y^{1-p} + c_0\left(\frac{y}{\bar{y}}\right)^{\frac{1}{\alpha\varepsilon^r}}.\end{aligned}$$

Letting $\bar{y}\to\infty$ and choosing as above the constant $c_0 = 0$, we obtain

$$\begin{aligned}\hat{T}(y) =& (1-\tau)\,y^{-p}\frac{1}{\alpha\varepsilon^r}\int_y^{\infty}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}^E_x dx \\ &+ (1-\tau)\frac{1}{\mathbb{E}[y]}\left(\mathbb{E}\left[y\hat{\Omega}^E_y\right] - \frac{1}{\alpha\varepsilon^r}\mathbb{E}\left[\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}^E_x dx\right]\right)y^{1-p}.\end{aligned} \tag{51}$$

Since $T(y) = y - \frac{1-\tau}{1-p}y^{1-p}$ and $\mathcal{E}(x,y) = \frac{1}{\alpha\varepsilon^r y}\int_y^{\infty}\left(\frac{y}{x}\right)^{1+\frac{1}{\alpha\varepsilon^r}}dx$, this expression can be rewritten as

$$\hat{T}(y) = \int_y^{\infty}\mathcal{E}(x,y)\left[(1-T'(x))\,x\hat{\Omega}^E_x\right]dx + c\,(1-T'(y))\,y,$$

where the constant c is given by

$$c = \frac{1}{\mathbb{E}[y]}\left(\mathbb{E}\left[y\hat{\Omega}^E_y\right] - \frac{1}{\alpha\varepsilon^r}\mathbb{E}\left[\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}^E_x dx\right]\right).$$

This concludes the proof of Corollary 2.

□

We finally give an alternative proof of formula (38).

Alternative proof of Corollary 2. In the CES-CRP environment, the functional equation solved by $\hat{T}$ reads: for all $y \in [\underline{y}, \bar{y}]$,

$$y^{p-1}\hat{T}(y) - \alpha\varepsilon^r y^p \hat{T}'(y) + \int_{\underline{y}}^{\bar{y}} \gamma_{yx}\varepsilon^r x^p \hat{T}'(x)\,dx = (1-\tau)\,\hat{\Omega}_y^E. \tag{52}$$

Since γ_{yx} depends only on x, we can differentiate this equation with respect to y to obtain, after rearranging:

$$-\alpha\varepsilon^r \hat{T}''(y) + (1 - p\alpha\varepsilon^r)\frac{\hat{T}'(y)}{y} + (p-1)\frac{\hat{T}(y)}{y^2} = (1-\tau)\,y^{-p}\hat{\Omega}_y^{E\prime}. \tag{53}$$

where $\hat{\Omega}_y^{E\prime} = (1-\alpha\varepsilon^w)\left(\hat{w}_y^E/w_y\right)'$. This is a second order linear (Euler) ODE which can be easily solved in closed form.

Consider first the homogeneous equation,

$$-\alpha\varepsilon^r \hat{T}''(y) + (1 - p\alpha\varepsilon^r)\frac{\hat{T}'(y)}{y} + (p-1)\frac{\hat{T}(y)}{y^2} = 0.$$

The general solution to this equation has the form

$$\hat{T}_H(y) = c_1 y^{r_1} + c_2 y^{r_2},$$

where r_1, r_2 are the roots of the characteristic polynomial, i.e., the solutions to

$$\begin{aligned} 0 &= -(\alpha\varepsilon^r)\,r\,(r-1)\,y^{r-2} + (1-p\alpha\varepsilon^r)\frac{ry^{r-1}}{y} + (p-1)\frac{y^r}{y^2} \\ \text{i.e., } 0 &= (\alpha\varepsilon^r)\,r^2 - (1 + (1-p)\,\alpha\varepsilon^r)\,r + (1-p)\,. \end{aligned}$$

We find

$$\begin{aligned} r_1, r_2 =& \frac{[1 + (1-p)\,\alpha\varepsilon^r] \pm \sqrt{[1 + (1-p)\,\alpha\varepsilon^r]^2 - 4\,(1-p)\,\alpha\varepsilon^r}}{2\alpha\varepsilon^r} \\ =& \frac{[1 + (1-p)\,\alpha\varepsilon^r] \pm \sqrt{[1 - (1-p)\,\alpha\varepsilon^r]^2}}{2\alpha\varepsilon^r}, \end{aligned}$$

hence we obtain that $0 < r_1 < r_2$ are given by

$$\begin{aligned} r_1 =& (1-p) \\ r_2 =& \frac{1}{\alpha\varepsilon^r} \end{aligned}$$

where the last equality follows from $\varepsilon^r = \frac{e}{1+pe+(1-p)\alpha e}$, which implies $r_2 = (1-p) + \frac{1+pe}{\alpha e}$.

Next, a particular solution to the initial second-order ODE (53) is given by the method of the variation of parameters. We seek the general solution $\hat{T}(y)$ to the differential equation of the form

$$\hat{T}(y) = c_1(y)\, y^{r_1} + c_2(y)\, y^{r_2},$$

where r_1, r_2 are given by the expressions above and $c_1(y), c_2(y)$ are the unknown functions to solve for. Impose

$$c_1'(y)\, y^{r_1} + c_2'(y)\, y^{r_2} = 0.$$

We thus have

$$\begin{aligned}
\hat{T}'(y) =& r_1 c_1(y)\, y^{r_1-1} + r_2 c_2(y)\, y^{r_2-1} \\
\hat{T}''(y) =& r_1(r_1-1)\, c_1(y)\, y^{r_1-2} + r_2(r_2-1)\, c_2(y)\, y^{r_2-2} + r_1 c_1'(y)\, y^{r_1-1} + r_2 c_2'(y)\, y^{r_2-1}.
\end{aligned}$$

Substituting into the ODE (53) yields

$$\begin{aligned}
(1-\tau)\, y^{-p}\hat{\Omega}_y^{E\prime} =& \left\{\left[(1-p\alpha\varepsilon^r)\, r_1 c_1(y) + (p-1)\, c_1(y)\right] y^{r_1-2} - \alpha\varepsilon^r\left[r_1(r_1-1)\, c_1(y)\, y^{r_1-2} + r_1 c_1'(y)\, y^{r_1-1}\right]\right\} \\
&+ \left\{\left[(1-p\alpha\varepsilon^r)\, r_2 c_2(y) + (p-1)\, c_2(y)\right] y^{r_2-2} - \alpha\varepsilon^r\left[r_2(r_2-1)\, c_2(y)\, y^{r_2-2} + r_2 c_2'(y)\, y^{r_2-1}\right]\right\} \\
=& -\alpha\varepsilon^r r_1 c_1'(y)\, y^{r_1-1} - \alpha\varepsilon^r r_2 c_2'(y)\, y^{r_2-1}
\end{aligned}$$

where the second equality follows from the fact that r_1 and r_2 are the roots of the characteristic polynomial. We therefore obtain the system of equations

$$\begin{aligned}
& c_1'(y)\, y^{r_1} + c_2'(y)\, y^{r_2} = 0 \\
& r_1 c_1'(y)\, y^{r_1-1} + r_2 c_2'(y)\, y^{r_2-1} = -\frac{1}{\alpha\varepsilon^r}(1-\tau)\, y^{-p}\hat{\Omega}_y^{E\prime}
\end{aligned}$$

which is linear in the two unknowns $c_1'(y), c_2'(y)$. Its solution is

$$\begin{aligned}
\begin{pmatrix} c_1'(y) \\ c_2'(y) \end{pmatrix} =& \begin{pmatrix} y^{r_1} & y^{r_2} \\ r_1 y^{r_1-1} & r_2 y^{r_2-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -\frac{1}{\alpha\varepsilon^r}(1-\tau)\, y^{-p}\hat{\Omega}_y^{E\prime} \end{pmatrix} \\
=& \frac{1}{W}\begin{pmatrix} r_2 y^{r_2-1} & -y^{r_2} \\ -r_1 y^{r_1-1} & y^{r_1} \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{\alpha\varepsilon^r}(1-\tau)\, y^{-p}\hat{\Omega}_y^{E\prime} \end{pmatrix},
\end{aligned}$$

where W is the Wronskian of the two functions y^{r_1} and y^{r_2}, equal to

$$W = \begin{vmatrix} y^{r_1} & y^{r_2} \\ r_1 y^{r_1-1} & r_2 y^{r_2-1} \end{vmatrix} = (r_2 - r_1)\, y^{r_1+r_2-1}.$$

Thus we obtain, using the expressions of r_1, r_2,

$$\begin{aligned}
c_1'(y) =& \frac{y^{r_2}\frac{1}{\alpha\varepsilon^r}(1-\tau)\,y^{-p}\hat{\Omega}_y^{E\prime}}{(r_2-r_1)\,y^{r_1+r_2-1}} = \frac{\frac{1}{\alpha\varepsilon^r}(1-\tau)}{\frac{1}{\alpha\varepsilon^r}-(1-p)}\hat{\Omega}_y^{E\prime} \\
c_2'(y) =& -\frac{y^{r_1}\frac{1}{\alpha\varepsilon^r}(1-\tau)\,y^{-p}\hat{\Omega}_y^{E\prime}}{(r_2-r_1)\,y^{r_1+r_2-1}} = -\frac{\frac{1}{\alpha\varepsilon^r}(1-\tau)}{\frac{1}{\alpha\varepsilon^r}-(1-p)}y^{1-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_y^{E\prime},
\end{aligned}$$

i.e.,

$$\begin{aligned}
c_1(y) =& a_1 + \frac{1-\tau}{1-(1-p)\,\alpha\varepsilon^r}\hat{\Omega}_y^E \\
c_2(y) =& a_2 + \frac{1-\tau}{1-(1-p)\,\alpha\varepsilon^r}\int_y^{\bar{y}} x^{1-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^{E\prime}dx \\
=& a_2 + \frac{1-\tau}{1-(1-p)\,\alpha\varepsilon^r}\left\{-y^{1-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_y^E - \left(1-p-\frac{1}{\alpha\varepsilon^r}\right)\int_y^{\bar{y}} x^{-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx\right\},
\end{aligned}$$

where a_1, a_2 are constants to be determined, and where the last equality assumes that as $\bar{y}\to\infty$, we have $\bar{y}^{1-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_{\bar{y}}^E \to 0$).

The general solution to the ODE (53) is therefore finally given by

$$\begin{aligned}
\hat{T}(y) =& \left[a_1 + \frac{1-\tau}{1-(1-p)\,\alpha\varepsilon^r}\hat{\Omega}_y^E\right]y^{1-p} \\
&+ \left[a_2 - \frac{1-\tau}{1-(1-p)\,\alpha\varepsilon^r}y^{1-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_y^E + \frac{1-\tau}{\alpha\varepsilon^r}\int_y^{\bar{y}} x^{-p-\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx\right]y^{\frac{1}{\alpha\varepsilon^r}} \\
=& \left[a_1 y^{1-p} + a_2 y^{\frac{1}{\alpha\varepsilon^r}}\right] + (1-\tau)\,y^{-p}\left[\frac{1}{\alpha\varepsilon^r}\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx\right].
\end{aligned}$$

This expression implies

$$\begin{aligned}
\hat{T}'(y) =& \left[a_1(1-p)\,y^{-p} + a_2\frac{1}{\alpha\varepsilon^r}y^{\frac{1}{\alpha\varepsilon^r}-1}\right] - \frac{1-\tau}{\alpha\varepsilon^r}y^{-p}\hat{\Omega}_y^E \\
&+ \frac{1-\tau}{\alpha\varepsilon^r}y^{-1-p}\left[\frac{1}{\alpha\varepsilon^r}\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx\right].
\end{aligned}$$

To find a relationship between the two constants a_1, a_2, we plug these expressions into the functional equation (52). Tedious but straightforward algebra leads to:

$$0 = a_1 + a_2\frac{1}{\mathbb{E}y}\mathbb{E}\left[y^{p+\frac{1}{\alpha\varepsilon^r}}\right] - \frac{1-\tau}{\mathbb{E}y}\mathbb{E}\left[y\hat{\Omega}_y^E\right] + \frac{1-\tau}{\mathbb{E}y}\frac{1}{\alpha\varepsilon^r}\mathbb{E}\left[\int_x^{\bar{y}}\left(\frac{y}{u}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_y^E dy\right].$$

Note that this equation alone does not allow us to identify separately a_1 and a_2, so that there is a continuum of tax reforms (indexed by, say, a_2) that are distribution-neutral. Therefore we obtain

$$\hat{T}(y) = \frac{1-\tau}{\mathbb{E}y}\left(\mathbb{E}\left[y\hat{\Omega}_y^E\right] - \mathbb{E}\left[\frac{1}{\alpha\varepsilon^r}\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx\right] - \frac{a_2}{1-\tau}\mathbb{E}\left[y^{p+\frac{1}{\alpha\varepsilon^r}}\right]\right)y^{1-p}$$
$$+ a_2 y^{\frac{1}{\alpha\varepsilon^r}} + (1-\tau)\, y^{-p}\frac{1}{\alpha\varepsilon^r}\int_y^{\bar{y}}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx.$$

As $\bar{y}\to\infty$, the term $y^{\frac{1}{\alpha\varepsilon^r}}$ explodes unless $a_2 = 0$. We thus take $a_2 = 0$ and obtain the following compensating reform:

$$\hat{T}(y) = (1-\tau)\, y^{-p}\frac{1}{\alpha\varepsilon^r}\int_y^{\infty}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx$$
$$+ \frac{1-\tau}{\mathbb{E}y}\left(\mathbb{E}\left[y\hat{\Omega}_y^E\right] - \mathbb{E}\left[\frac{1}{\alpha\varepsilon^r}\int_y^{\infty}\left(\frac{y}{x}\right)^{p+\frac{1}{\alpha\varepsilon^r}}\hat{\Omega}_x^E dx\right]\right)y^{1-p}.$$

This is the same expression as (51).

□

We finally provide technical details for several elements of the paper.

Technical details for the general equilibrium environment. We refer to Sachs, Tsyvinski, and Werquin [2016] for the rigorous definition of the production function with a continuum of inputs, the marginal product of labor (2), and the corresponding elasticities (9), (10).

□

The following technical details clarify the construction of the Gateaux and Frechet derivatives used in the text.

Technical details for the functional derivatives. We refer to Golosov, Tsyvinski, and Werquin [2014] for the existence of the Gateaux derivatives in the direction of any tax reform $\hat{T}(\cdot)$, namely (in the case, say, of the labor supply functional $T \mapsto l_i(T)$):

$$\hat{l}_i \equiv \mathrm{d}l_i(T)\cdot\hat{T} \equiv \lim_{\mu\to 0}\frac{l_i(T+\mu\hat{T}) - l_i(T)}{\mu}.$$

In particular, we show in Golosov, Tsyvinski, and Werquin [2014] that this derivative exists as soon as the optimal labor supply l_i is unique in the initial economy (e.g., this is the case if the initial tax schedule is convex), and a (generically satisfied) technical condition holds, ensuring that the labor supply elasticities are well defined.

The initial tax schedule T is bounded and twice continuously differentiable on $\mathbb{R}_+$, with bounded first and second derivatives, and marginal tax rates bounded away from 1.[35] We assume that the

[35] In Section 3.2, in order to interpret our main result, we use a tax schedule that is unbounded, and such that the marginal tax rates converge to 1. Since we assume that incomes are bounded, however, we can work on a large enough compact interval so that the appropriate restrictions on the tax function hold.

tax reforms $\hat{T}$ that the government can implement are continuously differentiable, bounded, with bounded first derivative. This defines a Banach space X on which the norm of a function $\hat{T}$ is given by $\|\hat{T}\|_X = \sum_{k=0}^{1} \sup\limits_{y\in\mathbb{R}_+} |\hat{T}^{(k)}(y)|$ where $\hat{T}^{(k)}$ is the k^{th} derivative of $\hat{T}$.

Below we compute the Gateaux derivatives $\mathrm{d}f(T)\cdot\hat{T}$ of several functionals in any direction $\hat{T}$. Assume that the Gateaux derivatives $\mathrm{d}f(\tilde{T})$ exist for all $\tilde{T}$ in a neighborhood $U \subset X$ of T. Let $\mathcal{B}(X,\mathbb{R})$ denote the space of bounded linear operators on X. It is endowed with the norm $\|\cdot\|_B$ defined by the smallest $M > 0$ such that for all $\hat{T} \in X$,

$$\|\mathrm{d}f(\tilde{T})\cdot\hat{T} - \mathrm{d}f(T)\cdot\hat{T}\| \leq M\|\hat{T}\|.$$

We can the check that the map $\mathrm{d}f : U \to \mathcal{B}(X,\mathbb{R})$ is continuous at T, in the sense that

$$\|\tilde{T} - T\|_X \to 0 \;\Rightarrow\; \|\mathrm{d}f(\tilde{T}) - \mathrm{d}f(T)\|_B \to 0.$$

That is, the derivative depends continuously on the initial tax schedule T. This implies that f is also Frechet differentiable at T, and that the corresponding derivative coincides with the Gateaux derivative.

□

B Extension: participation decisions

We now set up an extension of our model to include an extensive margin, i.e., participation decisions. We require that the compensating tax reform offsets not only the welfare gains and losses of the agents who are employed both before and after the wage disruption, but also those of all of the other agents in the economy (the non-employed and those who switch employment status). Remarkably, and despite the additional channel of response to wage disruption and tax reforms, we show that formula (30) still characterizes the compensating tax reform in that case. We now briefly describe this extension and the formal argument.

Heterogeneity is now two-dimensional: individuals are indexed by their skill $i \in [0,1]$ and by their fixed cost of working $\kappa \in \mathbb{R}_+$. The utility function is given by

$$U(c,l) = u[c - v(l) - \kappa\mathbb{I}_{\{l>0\}}],$$

where $\mathbb{I}_{\{l>0\}}$ is an indicator function equal to 1 if the agent is employed (i.e., $l > 0$).

An individual of type (i,κ) chooses both whether to participate in the labor force at wage w_i, and if so, how much effort to provide. If he decides to stay non-employed, his labor supply and income are equal to zero and he consumes the government-provided transfer $-T(0)$. Thus agent (θ,χ) solves the maximization problem

$$U_i(\kappa) \equiv \max\left\{\sup_{l>0} u\left[w_i l - T(w_i l) - v(l) - \kappa\right] \,;\, u\left(-T(0)\right)\right\}.$$

Due to the lack of income effects, the labor supply l_i that an agent (i, κ) chooses conditional on participation is independent of κ, and it is the solution to the first order condition

$$v'(l_i) = [1 - T'(w_i l_i)] w_i.$$

Moreover, an agent with skill i decides to participate if and only if his fixed cost of work κ is smaller than a threshold $\bar{\kappa}_i$, given by

$$\bar{\kappa}_i = w_i l_i - T(w_i l_i) - v(l_i) + T(0). \tag{54}$$

Note that both l_i and $\bar{\kappa}_i$ are endogenous to the tax schedule: the intensive margin choice of labor effort l_i depends on the marginal tax rate $T'(y_i)$, while the extensive margin choice of participation depends on the average tax rate relative to transfers, $T(y_i) - T(0)$.

Denote by $f_i(\kappa)$ the density of κ conditional on skill i, by

$$\pi_i = \frac{\int_0^{\bar{\kappa}_i} f_i(\kappa)\, d\kappa}{\int_0^{\infty} f_i(\kappa)\, d\kappa}$$

the employment rate within the population of skill i, and by

$$L_i = l_i \int_0^{\bar{\kappa}_i} f_i(\kappa)\, d\kappa$$

the total amount of labor supplied by workers of skill i. The rest of the environment is identical to that of Section 1.

We define the participation elasticity η_i^T of the population with skill i with respect to their average tax rate as

$$\eta_i^T \equiv \frac{\partial \ln \pi_i}{\partial \ln [y_i - T(y_i) + T(0)]} = [y_i - T(y_i) + T(0)] \frac{f_i(\bar{\kappa}_i)}{\pi_i \int_0^{\infty} f_i(\kappa)\, d\kappa}.$$

This elasticity is determined by the reservation density $f_i(\bar{\kappa}_i)$ of agents with skill i who are close to indifference between participation and non-participation in the baseline tax system. We also define the participation elasticity η_i^w with respect to the wage as

$$\eta_i^w \equiv \frac{\partial \ln \pi_i}{\partial \ln w_i} = (1 - T'(y_i))\, y_i \frac{f_i(\bar{\kappa}_i)}{\pi_i \int_0^{\infty} f_i(\kappa)\, d\kappa}.$$

Note that these elasticities are partial equilibrium concepts: they ignore the feedback impact of these initial adjustments in participation on individual wages and, in turn, labor supply.

It is not difficult to extend the formulas leading to equation (28) to this more general environment. The details of the derivations are left to the reader and are available upon request.

Adding the participation decisions does not affect formula (30). To understand this claim, note that keeping unchanged the welfare of those who are employed neither before nor after the perturbation requires leaving the unemployment transfer $T(0)$ unaffected. Moreover, by construction (in order to also keep the welfare of the always-employed agents unchanged), the combination of the wage disruption inflow and the tax reform must leave the disposable income net of the disutility of labor, $[y_i - T(y_i) - v(l_i)]$, unchanged for all i. Now, since the participation decision of an individual with skill i depends only on the difference between these values (see equation (54)), we obtain that the participation threshold $\bar{\kappa}_i$ must also remain constant. That is, in order to keep everyone's welfare constant, the compensating tax reform must ensure that the individuals who were employed (resp., non-employed) before the immigration inflow remain so, i.e., that no one ends up switching participation status. This implies in turn that the values of the participation elasticities η_i^T and η_i^w (which otherwise would appear in the variables Γ_{yx} and the endogenous wage adjustments $\hat{w}_i$) are irrelevant for the construction of the compensating tax reform, and that formula (30) continues to apply.