# Syllabus for "Introduction to Auctions"
# Prof. Andy Skrzypacz, September 2009

Books that cover a lot of auction topics that I recommend are:

Vijay Krishna "Auction Theory"
Paul Milgrom "Putting Auction Theory to Work"
Paul Klemperer "Auctions: Theory and Practice" (this book is available online here: )
http://www.paulklemperer.org/index.htm

The first day will focus on describing standard auctions, equilibria in first-price and second-price auctions and then Revenue Equivalence Theorem.

The second day will focus on optimal auctions, and the paper "Auctions vs Negotiations" to highlight the value of additional bidders.

The third day will focus on auctions with contingent payments, with correlated values, winner's curse, and the Linkage Principle method.

Here are some papers that are related to these topics:

"*" in the list of papers indicate the first-order reading.

## 1. Introduction to Auctions, Revenue Equivalence.

* William Vickrey (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders", Journal of Finance, XVI, 8-37.

*Vijay Krishna "Auction Theory" Chapters 2, 3.

Paul Milgrom "Putting Auction Theory to Work" Chapters 1, 2.

R. Preston McAfee; John McMillan (1987) "Auctions and Bidding" Journal of Economic Literature, Vol. 25, No. 2. (Jun., 1987), pp. 699-738.

Paul Milgrom "Auctions and Bidding (1989): A Primer" The Journal of Economic Perspectives, Vol. 3, No. 3. (Summer, 1989), pp. 3-22.

Paul Klemperer (1999). "Auction Theory: A Guide to Literature." Journal of Economic Surveys 13(3): 227-286.

## 2. Mechanism Design, Optimal Auctions, Bargaining (Independent Private Values);

*Roger B. Myerson (1981), "Optimal Auction Design", Mathematics of Operations Research, 6 (1), February 58-73.

*Roger B. Myerson and Mark A. Satterthwaite (1983), "Efficient Mechanisms for Bilateral Trading", Journal of Economic Theory, 29 (2), April 265-81.

*Jeremy Bulow and John Roberts (1989), "The Simple Economics of Optimal Auctions", Journal of Political Economy 97 (5), October 1060-90.

*Jeremy Bulow and Paul Klemperer (1996), "Auctions versus Negotiations", American Economic Review, 86 (1), March 180-94.

*Vijay Krishna “Auction Theory” Chapter 5.

Paul Milgrom “Putting Auction Theory to Work” Chapters 3,4

Peter Cramton, Robert Gibbons, and Paul Klemperer (1987), "Dissolving a Partnership Efficiently", Econometrica, 55 (3), May, 615-32.

John G. Riley and William F. Samuelson (1981), "Optimal Auctions", American Economic Review, 71 (3), June 381-92.

**3. The Linkage Principle, Affiliation, Common Value, Auctions with Contingent Payments.**

*Paul R. Milgrom and Robert J. Weber (1982a), "A Theory of Auctions and Competitive Bidding", Econometrica 50 (5), September, 1089-122.

*Paul R. Milgrom and Robert J. Weber (1982b), "The Value of Information in a Sealed-Bid Auction", Journal of Mathematical Economics, 10 (1), June, 105-14.

*Robert G. Hansen (1985) , "Auctions with Contingent Payments", American Economic Review, 75 (4) (September, pp. 862-865.

*DeMarzo, P., Kremer, I. and Skrzypacz, A. (2005), "Bidding with Securities: Auctions and Security Design," American Economic Review 95 (4) September, 936 – 959.

Vijay Krishna “Auction Theory” Chapters 6, 7 and 10.

Paul Milgrom “Putting Auction Theory to Work” Chapter 5.

Mathew Jackson and Ilan Kremer (2006) “The Relevance of a Choice of Auction Format in a Competitive Environment” forthcoming in Review of Economic Studies

Jacques Cremer and Richard P. McLean (1985) "Optimal Selling Strategies Under Uncertainty for a Discriminating Monopolist When Demands are Interdependent", Econometrica 53 (2), March 345-61.

John G. Riley (1988), "Ex Post Information in Auctions", Review of Economic Studies LV (3), No. 183, July, 409-29.

R. Preston McAfee and John McMillan, "Competition for Agency Contracts", RAND Journal of Economics, 18 (2) (Summer, 1987), pp. 296-307.

Richard Engelbrecht-Wiggans, Paul R. Milgrom, Robert J. Weber (1983) “Competitive Bidding and Proprietary Information,” Journal of Mathematical Economics, 11, 161-169.

Kenneth Hendricks and Robert H. Porter (1988) “An Empirical Study of an Auction with Asymmetric Information”, American Economic Review, 78, (5), pp. 865-883.

Kenneth Hendricks; Robert H. Porter; Charles A. Wilson (1994) “Auctions for Oil and Gas Leases with an Informed Bidder and a Random Reservation Price,” Econometrica, 62, (6). pp. 1415-1444.

QUESTIONS for E602

1. Assume that there is an auction involving three bidders, each with a value drawn independently and identically from the distribution [0,100].

(a) What is the expected value of the highest bidder?

(b) What would be the expected price in an English or Vickrey auction?

(c) What would be the seller's expected revenue if he established a minimum price of 50, again in the English or Vickrey auction?

(d) How much would a bidder with a value of 80 bid in a first price auction, assuming (i) no minimum bid and (2) assuming a required minimum bid of 50?

2. In an all-pay auction each bidder pays the amount she bids regardless of whether she wins or loses the auction. (An example might be political lobbying where both sides have expenses but only one side gets their way.) Using the Revenue Equivalence Theorem, calculate the following for an auction with two bidders with values drawn from the uniform distribution on [0,100]:

(a) Using the Revenue Equivalence Theorem, what is the expected revenue in the all-pay auction?

(b) Using the revenue equivalence theorem, which states that each bidder's expected payments will be the same regardless of auction type (so long as the two types of auctions lead to the same equilibrium winner) determine the equilibrium bid in the all-pay auction of a buyer with a value of 50.

3. *Buy it now.* Some internet auction sales work the following way: The seller sets a *Buy It Now* price at which any buyer can end the auction by agreeing to pay. So for example if the buy it now price is 50 an auction will start out as an English auction with a price of 0 and then gradually move up until either the bidding ends or one bidder jumps his bid to 50, thereby ending the auction.

Assume that there are two bidders in an auction, each with a value drawn from a uniform distribution on [0,100]. There is a Buy it Now price of 50. Note that the Revenue Equivalence Theorem will apply, so that in equilibrium the person with the higher value should always win and his expected payments will be the same as in a more conventional auction.

(a) What will be the bidding strategy of a buyer in the Buy it Now auction, assuming that her value is less than 50?

(b) A buyer with a value $v > 50$ will tend to wait until the price reaches some level $p$ and then jump to a price of 50.

(i)What will the buyer's expected price be, conditional on the other buyer having a value less than $p$?

(ii) What will the buyer's expected price be conditional on the other buyer having a value between $p$ and $v$?

(iii) What must $p$ be as a function of $v$ for the Revenue Equivalence Theorem to hold? For example, if the buyer has a value of 70, at what price will he jump to 50?

4. Wine auction. Two identical cases of wine are being sold. There are three potential bidders. Each is interested in buying one case. The values of the bidders are drawn from a uniform distribution on [500,900]. Therefore from the seller's perspective the expected highest value is 800, the expected second highest value is 700, and the expected third highest value is 600.

(a) Assume that the seller has a Vickrey auction in which everyone submits a sealed bid and the top two bidders pay the third highest bid. What is the expected price per case that the seller will receive? By the Revenue Equivalence Theorem, what is the expected price in virtually any other auction format?

(b) Assume that the seller instead chooses a format where the top two bidders pay their own bid. Calculate the equilibrium bid for someone with a value of \$800.

(c) Now assume that the seller uses the following format, similar to that used in some oil auctions and a Brazilian telecoms auction: The sellers submit sealed bids for the first case of wine. The winner pays its bid. Then the last two bidders submit a bid in the second round. Again the winner pays its bid. Using the revenue equivalence theorem, how much should the bidder with a value of \$800 bid in the first round? How much should he bid in the second round if he loses in the first round?

(d) Alternatively, assume that the seller uses the following format, similar to that used at Sotheby's to sell wine (largely to absentee bidders): Each bidder gives the auctioneer a maximum price he is willing to pay for the first case of wine, and a maximum price for the second case, conditional on losing out on the first case. The auctioneer awards the first case to the bidder who puts in the highest maximum for the first case, at a price equal to the second highest bid. Then the second case is awarded to the one of the remaining two bidders who puts in the higher bid for the second case, at a price equal to the bid made for the second case by the third bidder.

What would be the equilibrium strategy for a bidder with a value of \$800, using the Revenue Equivalence Theorem? To do this, note that a bidder's value determines the price in a given round if his bid is the second highest, so one's bid must correspond to the expected

Vickrey auction price if in fact the bidder does end up being second highest among the remaining bidders.

5. In the movie "Groundhog Day" there was an auction in which women bid to go out on dates with men.

Assume there were three women bidding in the auction, competing for three prizes or dates. No one could go on more than one date. Each bidder's "type" is independently and identically distributed uniformly on [0,1]. If a buyer makes the highest bid in the auction then she wins first prize. If her type is $v$ then first prize will be worth $3v$ to her. If she makes the second highest bid then she will receive second prize, which she values at $2v$. The buyer would value the third prize at $v$. All the women agree on who would be the most (and least) desirable date.

The other assumptions necessary for the Revenue Equivalence Theorem (risk neutrality, common knowledge) hold.

(a) Assume an English auction where the price starts at zero and rises. When the person with the third highest value (call the jth highest value $v_j$) drops out she is awarded third prize, at a price of zero. When the second highest bidder drops out she is awarded second prize, at the price where the third highest bidder dropped out. The remaining bidder gets first prize, and pays the price at which the second highest bidder drops out.

(i) Conditional on the values of the three bidders turning out to be $v_3 < v_2 < v_1$ what are the prices that will be paid for the second and first prizes?

(ii) What is the expected total revenue in this auction?

(b) For this part, it will be useful to recall that the revenue equivalence theorem implies that each bidder's expected payments are the same in any two mechanisms that lead to the same equilibrium allocations.

Consider the same circumstances as in part (a) except that the seller chooses to run a sealed bid, all-pay auction. In this case, all three bidders will pay whatever amount they bid, with the highest bidder getting the first prize, the second highest getting second prize, etcetera.

(i) What would be the equilibrium bid in this auction for a bidder with a value of $v$?

(ii) What would be the expected amount paid for the first prize? The second prize? The third prize?

6. Splitting the market.

Consider a situation of four bidders that have independent private values drawn from uniform $[0, 1]$. the seller has two units for sale.

a) consider the following two scenarios:

i) suppose that the seller sells both units in one auction.

ii) assume that the seller divides bidders into two groups of two and then runs two auctions, selling one item to each of the groups.

Calculate and compare expected bidder profits and seller's revenues in the two scenarios.

b) Now change the assumption about the distribution of values. In particular, assume that values are distributed *i.i.d.* according to $F(v)$ that satisfies $1 - F(v) = v^{-2}$. Consider again the two scenarios. Calculate and compare expected bidder profits and seller's revenues in the those two scenarios.

7. Optimal Auctions. Two contractors are bidding for a project. One is a local firm whose costs are uniformly distributed between 30 and 60. The other is a foreign firm whose costs are uniformly distributed between 20 and 80. The city needs to accept the bid from one or the other. What rules will minimize its expected cost of the contract?

8. Palo Alto Development Corp. has a number of lots it would like to put up for auction. It has identified two distinct types of potential bidders, whose values are drawn from different distributions. One type is computer executives, who have a distribution of $F(v) = \frac{v}{800}$. Note: This translates into a "demand curve" of $v = 800 - 800q$. The second type is professors, who are considerably less well to do and therefore have a distribution of $F(v) = \frac{v}{400}$, which of course translates into a "demand curve" of $v = 400 - 400q$.

(a) How would you run an auction to maximize expected total revenue, assuming that you could discriminate between these two groups? What kinds of things are we assuming to make implementing such a mechanism feasible?

(b) Now assume that we know these distributions but are unable to discriminate between the two groups, perhaps because of a reason that you came up with in (a). However, we do know that there are 50% more faculty as there are executives. (OK, this example would be more realistic if I said "staff" instead of faculty....). This means that the aggregate demand curve will have a kink in it.

One half of executives and no faculty, or 1/5 of the total population, have values between 400 and 800, so the distribution in this upper range is $F(v) = \frac{4}{5} + \frac{v-400}{2000} = \frac{v+1200}{2000}$. In the range of prices below 400 we have all the faculty and half of the executives, or 4/5 of the total population, so in this range $F(v) = \frac{v}{500}$.

In this case the optimal mechanism will involve (a) a minimum reservation price $R$, so no buyers below this value will be considered even if there is adequate supply, and (b) a range of values for bidders from $v_L$ to $v_H$ in which a bidder is treated the same regardless of his value. For example, the mechanism might say to bidders, "Announce your true value".If it is below $R$ we won't consider you. For those above $R$ we will have a modified $k+1^{\text{st}}$ price auction where $k$ is the number of units available for sale. The modifications are as follows:

(1) If there are $k+1$ or more bidders with values above $v_H > v_L$ then we proceed in the normal way.

(2) If there are fewer than $k+1$ bidders with values greater than or equal to $v_L$ we proceed in the normal way.

(3) If there are $m < k+1$ bidders with values above $v_H$ but $n \geq k+1$ with values above $v_L$ we award units to the buyers with values above $v_H$ with probability 1, at price $\mathsf{p}_H$. We hold a fair lottery among those with values between $v_L$ and $v_H$ and award the units to the lottery winners at a price $v_L$. The price paid by a buyer with a value greater than $v_H$ is $v_H - \frac{k-m+1}{n-m+1}(v_H - v_L)$, so that a bidder with value $v_H$ will have the same expected surplus regardless of whether he elects to buy for sure at the higher price or risks entering the lottery to get the lower price.

Your job is to calculate $R, v_L$, and $v_H$ .

(c) Now consider the analogous monopoly problem. Assume that the monopolist's demand curve is $p = 800 - 10q$ in the range from $p = 400$ to $p = 800$. Assume that the demand curve is $p = 500 - 2.5q$ for prices below 400. Now assume that the monopolist is constrained to having 40 units available for sale. It will announce a price at which people can buy for sure, and then offer the remainder at a low discount price, at which there will be excess demand and whether someone is able to purchase will be determined randomly. What prices will it charge? How would the prices vary with the total quantity available for sale?

9. Consider a two-player auction in which the players values are:

$$\begin{aligned} v_1 &= x + z \\ v_2 &= y + z \end{aligned}$$

where $x, y, z$, are all independent random variables distributed uniformly over $[0, 1]$. The bidders learn their $v_i$ but not the particular $x, y, z$.

Assuming that all the following auctions have symmetric equilibria in increasing strategies do the following:

a) Calculate equilibrium bidding strategies in a sealed-bid second-price auction, expected surplus of each type (i.e. payoff of a bidder given any $v_i$) and expected revenue of the seller.

b) Do the same for a sealed-bid first-price auction.

c) Do the same for an all-pay auction.

How would an optimal auction/mechanism compare to any of these?

10. Consider a modification of the previous model to allow some common value. In particular, assume that the players observe signals:

$$\begin{aligned} s_1 &= x + z \\ s_2 &= y + z \end{aligned}$$

where $x, y, z$, are i.i.d. Uniform over $[0, 1]$. The values of the players are:

$$\begin{aligned} v_1 &= \alpha s_1 + (1-\alpha) s_2 \\ v_2 &= \alpha s_2 + (1-\alpha) s_1 \end{aligned}$$

where $\alpha \in [\frac{1}{2}, 1)$.

Assuming that a symmetric equilibrium in increasing strategies exists in all-pay auction, calculate the bidding strategies.

Calculate then the expected surplus of each type. What can you conclude about the equilibrium?

11. Using a reasoning similar to the proof of linkage principle, prove the following:

Consider a general symmetric model with strict affiliation.

Suppose a symmetric equilibrium in increasing strategies exists in an all-pay auction.

Then FPA generates smaller expected revenue than an all-pay auction.

(Hint: to write the revelation game, use unconditional expected payment, not conditional on winning).

12. Consider a second-price auction with independent private values, $N$ bidders and all values distributed according to the same (well-behaved) distribution $F(v)$ over a range $[v_L, v_H]$. The auctioneer sets a reserve price $R$.

a) Derive the following expression for expected revenues of the seller (these are ex-ante, unconditional of the realized values):

$$REV = N \int_{R}^{v_H} \left(vf(v) + F(v) - 1\right) F^{N-1}(v) dv$$

b) Using this formula derive the optimal reserve price and prove that it is independent of the number of bidders

13. (Jackson-Kremer) There are two states of nature: $S = \{0, 1\}$, each happens with probability $\frac{1}{2}$. Bidder's (each of them demands exactly one unit) do not know the state of nature. Their values are distributed:

if $S = 0$ then values are uniform on $[0, 1]$

if $S = 1$ then $F(v) = v^2$, $f(v) = 2v$

Consider auctions with lots of bidders. The seller has enough units to sell them to $\frac{7}{16}$ of the total number of bidders. So that in equilibrium, if $S = 0$ everyone with value $v \geq \frac{9}{16}$ will get a unit and if $S = 1$ everyone with value $v \geq \frac{3}{4}$ will get a unit.

What are approximately the bidding strategies in a discriminatory-price auction? (we ask you for the limit of equilibrium strategies as $N \to \infty$).

Calculate expected revenue in discriminatory-price and in uniform-price auctions.

HINT: In the discriminatory price auction the limit of the equilibrium bidding strategies will be a simple step function: there will be two cut-offs $v_L$ and $v_H$ such that all types below $v_L$ will always lose, types in between will win only if $S = 0$ and types above $v_H$ will always win. Types in between will bid one number $b_L$ and types above $v_H$ will bid another $b_H > b_L$. The key is to try to figure out the cut-off types and those bids. Assume throughout that the equilibrium is efficient.

14. Consider the following auction environment. The $N$ bidders have private values that are affiliated. In particular, there is a state variable $Q \in [0, 1]$ (for quality) that they do not know, but conditionally on $Q$ the values are distributed independently according to $F(v|Q)$ which satisfies strict MLRP (so that $Q$ and $v$ are affiliated). The seller knows $Q$ but cannot reveal it credibly. All players are risk neutral.

Consider the following game: The nature chooses $Q$ according to a uniform distribution. Then all bidders learn their private values and the seller learns $Q$. Then the seller chooses whether to run a first-price or an English auction, both without a reserve price (he cannot commit to one). Then the bidders submit bids.

Prove that in equilibrium the seller chooses the English auction with probability 1. (If necessary, make additional assumptions).

15. Suppose there are $N$ bidders that have independent private values distributed uniformly over $[0, 1]$.

The seller will not set a reserve price and will run a simple English auction.

The bidders consider two possible ways of colluding:

M1) Have a random bidder be the sole bidder in the true auction and get the good for 0. Then make the good a common property and run the following insider auction (know as a knockout auction): all submit sealed bids. The highest bid wins (gets the good) and pays the second-highest bid. The payment is divided equally among all bidders: i.e. if the second-highest bid is $b$ then all bidders, including the winner, get $\frac{b}{N}$.

M2) One of the bidders is chosen randomly to be the sole bidder in the true auction (which he wins at price 0). Call the chosen player 'ONE'. Then in the knockout auction everybody (including ONE) submits a sealed bid. The good goes to the highest bidder. If the highest bidder is ONE, there are no additional payments. If the winner is some other player, he pays the second-highest bid to ONE.

(In the question assume that the cartel has ways of making sure that no bidder has incentives to show up in the true auction if he was not selected by the cartel to do so).

a) First consider mechanism M1. Denote by $U(v)$ the expected surplus of player with type $v$. Assuming that in the knockout auction bidders will bid according to strictly increasing strategies (keep this assumption for M1 for all the parts of Question 1):

(i) what is $U'(v)$?

(ii) what is the total expected surplus of the cartel? (ex-ante before values are realized)

(iii) what would be the total expected revenue if they bid competitively?

(iv) what is the expected surplus of the lowest type ($U(0)$) [Hint: the answers to the three previous questions can be used to figure it out].

b) Suppose $N = 2$ and still consider M1:

(i) What will be the bid of the lowest type in the knockout auction, $b(0)$?

(ii) Find the equilibrium bidding strategy? (Hint: it is linear - it should help you in solving the FOC).

c) In the second mechanism (M2):

(i) what is the dominant bidding strategy of the players not chosen to be ONE? (note: ONE does not have a dominant stretegy).

(ii) Suppose that $N = 2$. What is the best response of ONE if he has value $v = 0$? How does it change with $N$?

d) Which of the two mechanisms achieves higher total expected surplus for the cartel (or are they the same)? (You can answer it for $N = 2$. The argument for $N > 2$ is analogous).

e) As a bonus you can think about the following complications:

(i) Suppose the seller does not allow the winner of the true auction to resell the object (for example the seller is a government, it sells land lease and does not allow for sub-leasing). Can you modify M1 or propose another mechanism that would maximize the cartels total expected payoffs?

(ii) Suppose the seller sets reserve $R = \frac{1}{2}$ Can you improve upon M1?

(iii) Suppose that there are some bidders that are not in the cartel: additionally to the N bidders in the cartel there are $M$ bidders that will bid competitively in the true auction. There is no reserve and no resale allowed. How should M1 be modified?

(iv) Can you find the equilibrium in M1 for a general $N$?

16. There are $N = 20$ bidders that are planning to bid in an auction for an item. Their private values are distributed $i.i.d.$ over a range $[100, 200]$ according to a distribution $F(v) = \left(\frac{v}{100} - 1\right)^2$.

The seller considers a choice between the following two selling mechanisms:

1) Run a SPA sealed-bid auction with a reserve prive $R = \frac{400}{3}$

2) Run a two-stage auction as follows: In the first stage all bidders submit sealed bids. The seller reveals only the lowest bid. The lowest bidder pays nothing and exits the game. The other 19 bidders pay the lowest bid an move on to the second stage. (You can think of the seller selling 19 tickets for the right to participate in the auction. The bidders know their valuations from the very beginning. If there is a tie for the lowest bid, the seller picks a winner randomly.) The second stage has therefore only 19 bidders. In the second stage the seller sells the item in a sealed Second-Price auction with a reserve price $R = \frac{400}{3}$.

a) Proof that the first mechanism is at least as good as the second one.

b) Proof that the second mechanism cannot have a strictly increasing equilibrium bidding strategies (for types above $\frac{400}{3}$) in the first stage.
(Of course, we are interested only in the bidding strategy of the types above $\frac{400}{3}$, since all lower types never win in the second stage and hence will bid 0 in the first stage.).

17. Now, there are only $N = 2$ bidders and the values are distributed uniformly on $[0, 1]$. The auction is to buy a target firm. The bidding firms have stand-alone values $X = 1$ (that means that if they lose they get payoff $X$ and if they win they get payoff $v + X$ less the payment).

The seller considers the following two selling mechanisms:

1) Run a SPA in cash with reserve price $R = \frac{1}{2}$

2) Run a SPA in equity with no reserve: the rules of the game are that bidders submit share offers $\alpha$. The winner is the one with higher offer and the winner pays the second-higest $\alpha^{(2)}$ (for $N = 2$ it is the lower offer) The revenue to the seller is then

$$REV\left(\alpha^2\right) = \alpha^{(2)}\left(v^{(1)} + X\right)$$

where $v^{(1)}$ is the realization of the winner. The payoff to the winner is

$$v^{(1)}\left(1 - \alpha^{(2)}\right) - X\alpha^{(2)}$$

Which mechanism will yield a higher expected revenue?

Depending on the method of you solution, you may need the formulas:

$$\begin{aligned} \int \frac{t}{t+1} dt &= t - \ln(t+1) \\ \int \left(\frac{(v+1)}{v}(v - \ln(v+1))\right) 2v dv &= (v+1)^2 \left(\frac{2}{3}v + \frac{1}{6} - \ln(v+1)\right) \end{aligned}$$

18. A seller has 2 units available to sell to 5 bidders. Each bidder has a demand for one unit, drawn i.i.d. on [0,1]. The seller chooses the following mechanism:

A price $p$ is announced. All bidders announce whether they are willing to pay $p$ or not. If the number who are willing to pay are less than or equal to $p$ then those bidders receive a unit and pay $p$ . The auction then continues with the seller announcing another price.

If on the other hand there are more bidders willing to pay $p$ than the number of units left then no units are awarded, all bidders who did not agree to pay $p$ are eliminated from the auction, and the seller announces a new price. Bidders do not use weakly dominated strategies (i.e. never offer to pay $p$ if it exceeds their value).

(a) Will the Revenue Equivalence Theorem hold in this mechanism?

(b) In the first round of the auction what is the maximum price $p$ at which any demand will possibly be obtained in equilibrium?

(c) What is the maximum price $p$ that a bidder with a value of .5 would agree to pay in the first round?

(d) Assume that the seller chooses the price you calculated in part (c) as the first price, and no one makes an offer. What is the maximum second round price that has any chance of attracting offers?

19. (a) Consider an English auction with three bidders. Each has a value drawn i.i.d. from the distribution $F(v)$ . Using the Revenue Equivalence Theorem characterize bidding strategies in a jump bidding equilibrium in which the first bidder announces a bid, then the second bidder either announces a bid or announces that he will not participate, and then the third bidder either announces a bid or announces he will not participate, such that after the third bidder's announcement there will be no further bids.

(b) Again assume that there are three bidders with values drawn i.i.d., this time on the uniform distribution [0,1]. The way the auction works is as follows: Each bidder must simultaneously decide whether to pay a fixed fee of .027. If only one bidder chooses to pay the fee then he receives the object with no further payment. If more than one bidder agrees to pay the fee then there is a second round among those who agreed to pay, in which each bidder submits a first price sealed bid.

(i) What is the value of the type that is just indifferent to paying the initial fee?

(ii) Assume that the answer to part (i) is $R$ and that your value is $v > R$ . How much should you bid in the second round if you face one opponent?

(iii) What is the expected payment of a bidder with a value of 1 in this auction?

(iv) Assume that the rules are changed so that there is a minimum price in the second round (even if there is only one bidder) of .183. Now what will the equilibrium look like?

(v) What do you predict would happen if the seller is unable to commit to a minimum second round price?

20. A seller has an object that he does not value himself, but there are two potential bidders that can invest $x = 1$ to obtain value added $v$. $v$ is common across the two bidders and they have the same information about it: they think that with probability $\frac{1}{2}$, $v = 2$ and with probability $\frac{1}{2}$ ,$v = 4$. The seller knows $v$. The meaning of value added is that after investing $x$ the buyers will obtain revenue $z$ which is equal to $z = v + x$

Suppose that the seller allows the bidders to submit any bid that is a combination of cash and equity, so that a bid is a pair

$$(b, \alpha)$$

The feasible bids are restricted to: $b \geq 0$, $\alpha \in [0, 1]$.

After obtaining the two bids the seller chooses the one that maximizes his profit:

$$\Pi_s = b + \alpha (v + x)$$

and allocates the object to the highest bidder (in case of ties he chooses the winner randomly). The winner pays his bid and obtains payoff

$$\Pi_b = (1 - \alpha)(v + x) - x - b$$

Find an equilibrium of this game. (You can focus on pure strategy equilibria).

21. A corrupt government official is in charge of awarding a contract. It is common knowledge that the value of the contract to firm A is 5 and to firm H is 10. These are the only two firms in the competition.

(a) Assume the official asks each bidder to write him a check. He will cash both checks, but will award the contract to the one who writes him the bigger check. The two bids will be opened simultaneously and in the presence of the buyers, so there is no problem of a bidder being concerned that the seller is colluding with the other buyer. Calculate the (mixed strategy) equilibrium.

(b) Now assume there are two contracts that the official is offering, and he is selling them sequentially. H has the capacity to do both jobs. A, however, only has the capacity to do one job. H values each job at 10. A values either job at 6.

(i) What is the equilibrium for the second contract bidding if H wins the first of the two contracts? If A wins the first of the two?

(ii) What will be the equilibrium in the first round?

22. Two risk-neutral bidders are going to compete in an auction. Their values are private and independent, both distributed uniformly over $[0, 1]$. The auction will be a second-price sealed bid auction and the seller will announce the participants (i.e. the identities of bidders that submitted a bid) and the winner, but not the bids. There will be no reserve price. In case the bidders submit identical bids, the seller will chose the winner randomly (with equal probabilities).

Before learning their valuations (i.e. learning exactly what is for sale and learning their outside options), the players try to design a collusive scheme of the following kind: we are both supposed to bid (i.e. submit a bid at least 0). After we observe the winner, after the auction he will pay $t \in \left[0, \frac{1}{2}\right]$ to the loser, $t$ being set before we learn our values and not renegotiable. If a player does not participate, then not only the winner does not have to pay him anything, but also the non-participating deviator has to pay a penalty of 1. Assume that these payments will be enforceable (for example, via leaving some money with a third party or via some continuation game).

The key element of this collusive scheme is that the players will not communicate after learning their values and before the auction (they may do so to minimize the probability of detection).

a) Given a $t$, describe the equilibrium strategies of the bidders

b) Calculate the expected profits of a bidder (before he knows $v$) for a given $t$.

c) Prove that $t = 0$ or $t = \frac{1}{2}$ are not optimal. What is the optimal $t$ for the cartel? From the point of view of the seller, will he prefer the optimal $t$ or $t = \frac{1}{2}$?

d) If the seller runs a FPA instead, how would the optimal $t$ change?

23. Three firms are competing for a contract to provide 100 units of a product to a buyer. The firms each have costs that are drawn i.i.d. from a range with a uniform distribution. The firm agrees that whoever wins the auction will receive a list price equal to the cost associated with the top of this distribution, but it also asks the sellers to offer a rebate per unit in return for the contract. In an auction for the business, the firms bid a number between 0 and 1, representing the amount of discount they are willing to offer the buyer below the maximum price but above the minimum price. For example, if the cost range is between 80 and 120 then a bid of .25 would correspond to a rebate of 10, or a price of 110. Thus, the bidding is equivalent to an auction in which there are three buyers, each with values uniformly distributed on [0,1].

(PART I) Assume that the seller will award the entire contract to the bidder who makes the best bid, at the rebate bid by that bidder.

a. Characterize the bidding strategies of the participants. You may solve the problem considering the bidders to have values between 0 and 1, equivalent to being able to produce at discounts from the maximum price of between 0 and 1, and making bids as a function of their values.

b. What is the expected discount that the buyer will receive from the winning bidder?

(PART II) Now assume that the seller chooses to use two suppliers, one of whom will get a contract to produce 75 units and one of whom will get a contract to produce 25 units. Again all three firms will submit bids. The firm that makes the best bid will get to sell 75 units at the price it bids while the firm that makes the second highest bid will get to produce 25 units.

a. Assume that the auction rules were that the firm that made the largest rebate per unit would get to sell 50 units at the rebate of the second best bidder and 25 units at the rebate of the third best bidder. The second best bidder got to sell 25 units at the rebate of the third best bidder.Characterize firm bidding strategies as a function of the firm's true costs.

b. Given the rules laid out directly above, calculate the expected production and profits of each bidder as a function of its value (the maximum rebate it can offer and still break even).

c. Now assume that the rules are that the winning firms each receive a price per unit equal to the amount they bid. However, the best bidder still gets to sell 75 units and the second best 25. How do firms bid as a function of their values? (Hint: you may use the Revenue Equivalence Theorem in combination with your answers to the previous two parts.)

d. What is the expected revenue (in terms of discounts) from such an auction? If you wish you may approach this part of the problem by noting that the expected revenue is equal to the expected marginal revenues associated with the bidders who win the auction.

24. Consider the following bargaining game. There is a seller that has two units for sale and there are two buyers that are interested in exactly one item each. The seller will make all the offers, in even periods making a price offer to buyer 1 and in odd periods to buyer 2. All players discount payoffs by a per-period factor $\delta$. For example, if the seller sells the first unit in period 0 to buyer 1 for a price $p_1$ and in period 1 to buyer 2 for a price $p_2$, then the payoff are:

$$\begin{aligned}\pi_s &= p_1 + \delta p_2 \\ \pi_{b1} &= v_1 - p_1 \\ \pi_{b2} &= \delta\left(v_2 - p_2\right)\end{aligned}$$

The values of the buyers are not known by the seller and they are either $h$ or $l$, with $h > l > 0$ and $l/h < \delta^2$.The cost to the seller is 0.

The new feature of the model is that the values of the buyers are correlated: they either have both value $h$ (with probability $\alpha$) or both have value $l$ (with probability $1-\alpha$). All offers and accept/reject decisions are publicly observable.